Recitation 2

Relations, Functions, and the Infinite

Part 1: Relations

Operations on Sets

- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (Union)
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ (Intersection)
- $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$ (Set Difference)
- $\overline{A} = A^c = \{x \mid x \notin A\}$ (Set Complement)
- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ (Cartesian Product)

Definitions

Defn 1: A *relation* R on the sets A and B is a subset of the Cartesian product $A \times B$.

A relation R on the set A is a subset of the Cartesian product $A \times A$.

Notationally, if an ordered pair (a, b) is in the relation R, we can write $(a, b) \in R$ or aRb.

Defn 2: An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

Defn 3: A *partition* of a set A is a collection of subsets $B_1, \ldots, B_k$ of A s.t. every element of A is in some subset B_i, but no two subsets share an element.

Defn 4: Let R be an equivalence relation on A. Then the *equivalence class* of $a \in A$, denoted $[a]_R$, is $\{x \mid x \in A, (x, a) \in R\}$.

Proposition: The equivalence classes of a relation R on A form a partition of A.

Relations Quick Guide and Common Mistakes

Discuss the following definitions and common mistakes before your proceed.

Reflexive A relation R on set A is reflexive if $(a, a) \in R$ **for every** $a \in A$.

Common mistake: Consider the relation R on the set of students at Brown where two students are related if they took CS15 at the same time. You might think that this relation is reflexive since a student is clearly took CS15 at the same time as themself. However, there is at least one student s who hasn't taken CS15 and therefore $(s, s) \notin R$. As a result, R is not reflexive. For a relation to be reflexive, $(s, s) \in R$ for every s in the set.

Symmetric A relation R on A is *symmetric* if $\forall a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$.

Common mistake: Consider the relation $R = \{(1, 1), (2, 2)\}$ on the set $\{1, 2\}$. This relation **is symmetric**. Since $(1, 2) \notin R$, it is not required that $(2, 1) \in R$.

Transitive A relation R on A is *transitive* if $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Common mistake: Consider the relation $R = \{(1, 2), (1, 1)\}$ on the set $\{1, 2\}$. This relation **is transitive**. Can you see why?

Warm-Up

a. Consider the set $A = \{1, 2\}$.

i. What is the Cartesian product $A \times A$?

$\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

ii. Is $R_1 = \{(1, 1), (1, 2), (2, 2)\}$ a valid relation on A?

Yes

iii. Is R_1 reflexive? Why or why not?

Yes

iv. Is R_1 symmetric? Why or why not?

No, needs $(2, 1)$.

v. Is $R_0 = \{\}$ a valid relation on A?

Yes

vi. Is R_0 symmetric? Why or why not?

Yes

vii. Is R_0 transitive? Why or why not?

Yes

viii. R_0 is not an equivalence relation because it is not reflexive. Can you see why?

No. Not reflexive.

Checkpoint - Call a TA over

b. Consider the set B of all students at Brown. For each of the following relations on B, state if they are reflexive, symmetric, or transitive. If it is an equivelance relation then list the equivalence classes. **No formal proof needed, just discuss with your group**.

i. Two students are related if they are the same age (e.g. 21).

Reflexive, symmetric, and transitive. Therefore equivelance relation. Equivelance classes are students of each age.

ii. s_1 and s_2 are students and $(s_1, s_2) \in R$ if s_1 is younger than s_2.

Transitive but not reflexive or symmetric.

iii. Two students are related if they are studying anthropology.

Symmetric and transitive but not reflexive

iv. Two students are related if they go to Brown.

Reflexive, symmetric, and transitive. Therefore equivelance relation. One equivelance class which consists of all students at Brown.

Checkpoint - Call a TA over

Part 2: Functions and The Infinite

Definitions

Defn 1: $f : X \to Y$ is an **injection** from set X to set Y if for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

An **injection** is often called *one-to-one* since you are mapping each element in X to a unique element in Y. This guarantees that Y must have at least as many elements as X, so $|X| \leq |Y|$.

Defn 2: $f : X \to Y$ is a **surjection** from set X to set Y if for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

A **surjection** is often called *onto* since every single element in Y is mapped to by f. This guarantees that X must have at least as many elements as Y, so $|X| \geq |Y|$.

Defn 3: $f : X \to Y$ is a **bijection** if it is both an injection and surjection. Since an injection implies $|X| \leq |Y|$ and a surjection implies $|X| \geq |Y|$, a bijection guarantees $|X| = |Y|$.

Defn 4: $\mathcal{P}(S)$ is the set of all subsets of S. It is called the **power set** of S.

Warm-Up

For each of the following function, state if f is an injection, surjection, or neither. Also state if it is a bijection.

Discuss your solutions.

a. $f : \{0, 1\} \to \mathbb{N}$

   $f(0) = 1, f(1) = 0$

   (Injective but not surjective.)

b. $f : \{0, 1\} \to \{0, 1\}$

   $f(0) = 1, f(1) = 0$

   (Injective and surjective. Therefore bijective.)

c. $f : \{0, 1\} \to \{0, 1\}$

   $f(0) = 1, f(1) = 1$

   (Not injective or surjective.)

d. $f : \mathbb{Z} \to \mathbb{Z}$
$f(x) = x^2$
(Not injective or surjective.)

e. f : First Year Students $\to$ First Year Dorms
f(student) = dorm that student lives in
(Not injective. Surjective.)

f. f : Students $\to$ Countries in the World
f(student) = country where student is from
(Not injective or surjective.)

g. $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x$
(Injective and surjective. Therefore bijective.)

h. *Challenge* $f : \mathbb{R} \to \mathbb{R}$
$f(x) = \frac{x}{2}$
(Injective and surjective. Therefore bijective.)

Checkpoint - Call a TA over

Section Lesson: Infinite Sizes of Infinity

Introduction: Functions as Tables

It is sometimes helpful to think of a function as a table where the left column contains all elements in the domain. For example, the function $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = x^2$ can be represented as follows:

x	$f(x)$
0	0
1	1
2	4
3	9
4	16
$\vdots$	$\vdots$

We can now redefine injectivity and surjectivity for a function $f : X \to Y$ as follows:

- f is injective if each element in Y appears in the right column at most once.
- f is surjective if all elements of Y appear in the right column at least once.

This gives us better intuition for the important result:

If there is a bijection from X to Y then $|X| = |Y|$.

If we have a unique mapping from each element in X to each element in Y, and all elements of Y appear in the mapping, it must be the case that $|X| = |Y|$.

Extending to the Infinite

The same definition applies to infinite sets. If A and B are infinite sets and there exists a bijection $f : A \to B$ then A and B have the same cardinality.

Consider the following infinite sets:

- The natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$
- The even natural numbers $E = \{0, 2, 4, 6, 8, ...\}$
- The odd natural numbers $O = \{1, 3, 5, 7, 9, ...\}$

Claim: $|E| = |O|$. There are as many even numbers as odd numbers.

Proof: This is intuitive, but we can prove it by giving a bijection $f : E \to O$.

x	$f(x) = x + 1$
0	1
2	3
4	5
6	7
$\vdots$	$\vdots$

$\square$

However, what's more suprising is that $|E| = |\mathbb{N}|$.

This brings us to our first problem:

a. Show that there are just as many even numbers as there are natural numbers by giving a bijection $f : \mathbb{N} \to \mathbb{E}$. You do not need to prove that this is a bijection.

$f(n) = 2n$

Challenge - Different Sizes of Infinity

We can use a similar method to show that there are **different "sizes" of infinity**. You are going to show this by proving that for any infinite set S the following is always true:

$$|S| < |\mathcal{P}(S)|$$

b. First, prove that $|S| \leq |\mathcal{P}(S)|$ by giving an injection $g : S \to \mathcal{P}(S)$.

$f(S) = \{S\}$

Now you will show that $|S| \neq |\mathcal{P}(S)|$

This can be proved by contradiction. Assume, for sake of contradiction, that the two sets are of equal cardinality and therefore there exists a bijection $f : S \to \mathcal{P}(S)$.

The table below depicts one such bijection. (It is just being used as an example, and is not relevant to your answer to this problem.)

$s_i \in S$	$f(s_i) \in \mathcal{P}(S)$
s_1	$\{s_2, s_3, s_5, ...\}$
s_2	$\{s_2, s_{8769}, ...\}$
s_3	$\{s_4, s_9, ...\}$
s_4	$\{\}$
s_5	$\{s_5\}$
$\vdots$	$\vdots$

Now consider the following set:

$$B = \{s_i \in S \mid s_i \notin f(s_i)\}$$

In other words, B is the set of all elements in S that are not a member of the set that they are mapped to by f.

In the sample bijection provided above, $B = \{s_1, s_3, s_4, ...\}$

c. Prove that there does not exist an element $s_i \in S$ such that $f(s_i) = B$ and therefore there is no bijection between the two sets. Given the previous part, what does this say about the cardinalities of S and $\mathcal{P}(S)$?

   Hint: Assume for sake of contradiction that there exists an element $s_i \in S$ such that $f(s_i) = B$. Is $s_i \in B$?

If s_i were in B, then by the definition of B, s_i is not mapped to an element that contains it. Since s_i maps to B, we have reached a contradiction.

If s_i were not in B, then s_i does not map to an element that contains it. Therefore B should contain s_i. t

If you have shown that $|S| \neq |\mathcal{P}(S)|$ and $|S| \leq |\mathcal{P}(S)|$, you have now shown that $|S| < |\mathcal{P}(S)|$ and therefore there are different "sizes" of infinity.

Checkpoint - Call a TA over

Infinite Sizes of Infinity

d. Prove that there are infinitely many different "sizes" of infinity.

Assume for sake of contradiction that there is a largest size of infinity. Now take the power set

Extra Challenging Problems

e. Let B be the set of infinite binary strings. Prove that $|\mathbb{N}| \neq |B|$. (Hint: you have already done this! Think back to class.)

   (Think about subsets.)

f. Let C be the set of real numbers between 0 and 1. Prove that $|C| = |B|$.

   (Nothing special about base 10.)

g. Prove by drawing a picture that $|C| = |\mathbb{R}|$. Conclude that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

   (A function with asymptotes at 0 and 1.)

h. Prove that the unit line (all real numbers between 0 and 1) has the same cardinality as the unit square (all coordinates (a, b) where a and b are real numbers between 0 and 1).

   (Alternating numbers.)