Recitation 1

Numbers and Proofs

Review

Definitions

Defn 1: A **set** is a collection of objects with no repitition.

Defn 2: B is a **subset** of A if every element in B is also in A. This is written as $B \subseteq A$.

Defn 3: The **natural numbers** are the set $\mathbb{N} = \{0, 1, 2, ...\}$. The **integers** $\mathbb{Z}$ are $\{..., -2, -1, 0, 1, 2, ...\}$.

Defn 4: A number n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. A number n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Defn 5: A number n is **rational** if $n = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.

Defn 6: $\mathcal{P}(A)$, called the power set of A, is the set of all subsets of A. Alternate notation for the power set of A is 2^A.

Warm Up

Answer true or false to the following problems. Discuss your solutions.

a. A is any set. Answer true only if the statement is always true.

   i. $A \subseteq A$ (T)
   ii. $\{\} \subseteq A$(T)
   iii. $\{\} \in A$ (F)
   iv. $B := \{A\}$. B is a set. The notation ":=" means "is defined as". (T)
   v. $C := \{A, A\}$. C is a set. (F, repition)

b. A is any set and $\mathcal{P}(A)$ is the set of all subsets of A.

   i. $A \in \mathcal{P}(A)$ (T)
   ii. $A \subseteq \mathcal{P}(A)$ (F)
   iii. $\emptyset \in \mathcal{P}(A)$ (T)

iv. $\emptyset \subseteq \mathcal{P}(A)$ (T)

v. $\{A, \emptyset\} \subseteq \mathcal{P}(A)$ (T)

c. If $A = \{1, 2, 4\}$ then $\{2, 4\} \in \mathcal{P}(A)$ (T)

d. $\mathbb{N} \subseteq \mathbb{Z}$ (T)

e. $\{0, 1, 9\} \subseteq \mathbb{N}$ (T)

f. $\{-1.5, 9\} \subseteq \mathbb{Z}$ (F)

g. S is the set of students in CS22. B is the set of students at Brown. Jerry is a student in CS22.

i. $S \subseteq B$ (T)

ii. Jerry $\subseteq S$ (F)

iii. Jerry $\in S$ (T)

iv. $\{\text{Jerry}\} \subseteq B$ (T)

h. Let $\mathbb{Q}$ be the set of rational numbers.

i. $\mathbb{Q} \cap \mathbb{N} = \mathbb{N}$ (T)

ii. $\mathbb{Q} \cup \mathbb{N} = \mathbb{R}$ (F)

i. *Challenge:* If $B \subseteq A$ and $\exists\ x \in A$ such that $x \notin B$ then $|B| < |A|$. (T if finite, F if infinite)

Checkpoint - Call a TA over.

Section Lesson - Proof Techniques and Examples

Direct Proof

A direct proof occurs when you start with what you know, follow a series of steps, and end up with what you are trying to prove.

Here is an example.

Claim: If n is odd, then n^2 is odd.

Proof (direct): We know that n is odd, so $n = 2k + 1$ for some $k \in \mathbb{Z}$.

So $n^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where $m = 2k^2 + 2k$.

Since m is an integer, n^2 is odd. $\square$

a. Prove that the product of an even number and odd number is even.

> Consider n odd and m even. $n = 2a + 1$ and $m = 2b$.
>
> Their product is $(2a + 1)(2b) = 4ab + 2b = 2(2ab + b)$.
>
> Since $2ab + b$ is an integer their product is even.

b. Prove that the product of two rational numbers is rational.

Hint: The product of two integers is an integer.

> Consider $n = \frac{a}{b}$ and $m = \frac{c}{d}$.
>
> Their product is $nm = \frac{ac}{bd}$. Since the product of two integers is an integer, we have just expressed nm as the ratio of two integers. Therefore nm is rational.

Counterexample

Counterexamples help us prove that something is not true.

For example, suppose Ben makes the claim that if xy is rational then x and y are rational.

Jerry can disprove his claim by coming up with a counterexample. For example, if $x = \sqrt{2}$ and $y = \sqrt{2}$, then $xy = 2$, which is rational.

However, you **cannot** prove a claim by showing one example of it. Jerry has not proven that x and y are irrational, he has just shown that they are not always rational.

For example, the claim "all CS22 students like ice cream" can be disproved by finding a student who does not like ice cream. Finding this counterexample, however, will not prove that no students like ice cream.

Your turn. Disprove the following statements with a counterexample.

c. If xyz is rational, then x, y, and z are rational.

> Consider $x = \sqrt{2}$, $y = \sqrt{3}$, $z = \sqrt{6}$. $xyz = 6$ which is rational.

d. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

Consider $A = \{1\}$ and $B = \{2\}$.

Then $\{1,2\} \in \mathcal{P}(A \cup B)$ but $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

e. *Challenge - Outside the scope of this class:* All true sentences have proofs.

Hint: Consider the sentence "No proof exists for this sentence."

No proof exists for the sentence:

If a proof existed for the sentence "No proof exists for this sentence" then the sentence would be true. Thereofre, no proof would exist for it. This is a contradiction.

The sentence is true:

If the sentence were false then a proof would exist for the sentence and the sentence would therefore be true. This is also a contradiction.

Conclusion:

We have therefore shown that there are no proofs for the sentence but the sentence is true.

Checkpoint - Call a TA over.

Set Element Method

How do you prove that $A = B$? First show that $A \subseteq B$ and then you show that $B \subseteq A$. If every element in A is also an element in B and every element in B is also an element of A, then A must equal B.

To show that $A \subseteq B$ you consider an arbitrary element in A and show it is also in B.

Here is an example.

Claim: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: We will first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Consider an arbitrary element x which is in the set $A \cap (B \cup C)$.

$$
\begin{aligned}
& x \in A \cap (B \cup C) \\
\Rightarrow & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Rightarrow & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
\Rightarrow & x \in (A \cap B) \cup (A \cap C)
\end{aligned}
$$

Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now we will show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Consider an arbitrary element x in the set $(A \cap B) \cup (A \cap C)$.

$$\begin{aligned}
& x \in (A \cap B) \cup (A \cap C) \\
\Rightarrow & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
\Rightarrow & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Rightarrow & x \in A \cap (B \cup C)
\end{aligned}$$

Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ and by the set element method we have proved our claim.

f. Prove $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

$$\begin{aligned}
& x \in \mathcal{P}(A \cap B) \\
\iff & x \subseteq A \cap B \\
\iff & x \subseteq A \text{ and } x \subseteq B \\
\iff & x \in \mathcal{P}(A) \text{ and } x \in \mathcal{P}(B) \\
\iff & x \in \mathcal{P}(A) \cap \mathcal{P}(B)
\end{aligned}$$

□

Proof by contradiction

Say we have some statement T that we are trying to prove. Here is how we prove it by contradiction:

1. Assume T is not true.
2. If T is not true, we arrive at a contradiction.
3. Since T being false leads us to a contradiction, T must be true.

Here is an example.

Claim: $\mathbb{N}$ is an infinite set.

Proof: Assume for sake of contradiction that there are a finite number of natural numbers. Then there must be a largest natural number. Say this largest number is m.

However, $m + 1$ is still a natural number, and $m + 1$ is larger then m.

This is a contradiction, as m is the largest natural number.

Assuming $\mathbb{N}$ was finite led to a contradiction, and therefore $\mathbb{N}$ is infinite. $\square$

Often the claim that you are trying to prove will be of the form "If p then q." If this is the case, then you assume that q is not true and show that if q is not true then p is not true. This is called the contrapositive.

Claim: If n^2 is even, then n is even.

Proof: Assume for sake of contradiction that n^2 is even but n is odd. Since n is odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$.

So $n^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$.

Where $m = 2k^2 + 2k$.

Since m is an integer, n^2 is odd. This is a contradiction since n^2 is even, and therefore if n^2 is even then n must also be even. $\square$

Your turn. Prove the following by contradiction:

g. 2 is an even number. (Use proof by contradiction by assuming 2 is odd.)

> Assume for sake of contradiction that 2 was odd. Then $2 = 2k + 1$ for some integer k. Then $k = \frac{1}{2}$ which is a contradiction since k is an integer.

h. *Challenge - Outside the scope of this class:* There is no integer between 0 and 1. **Hint**: Use the fact that every subset of the natural numbers has a smallest element, and that a natural number squared is still a natural number.

> Consider the set S which is the set of integers between 0 and 1. Assume for sake of contradiction that this set were non-empty. Then S has a smallest element, which we can call n. However, n^2 is still between 0 and 1 and is still an integer (since the integers are closed under multiplication). Moreover, n^2 is smaller than n which is a contradiction as we assumed that n was the smallest integer between 0 and 1.

i. *Challenge - Outside the scope of this class:* Consider "the smallest positive integer not definable in fewer than twelve words". Show that this integer cannot exist.

> Say there was some smallest positive integer not definable in fewer than twelve words. Call this integer s. We could then define s in fewer than twelve words

by saying “the smallest positive integer not definable in fewer than twelve words.” This is a contradiction.

Checkpoint - Call a TA over.