Interacting Particle Systems and Efficient Approximations for Large Sparse Graphs

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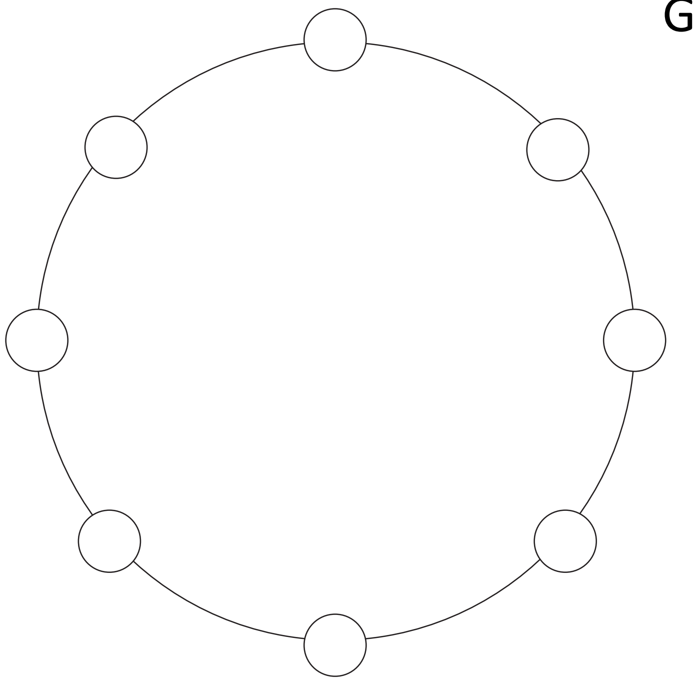
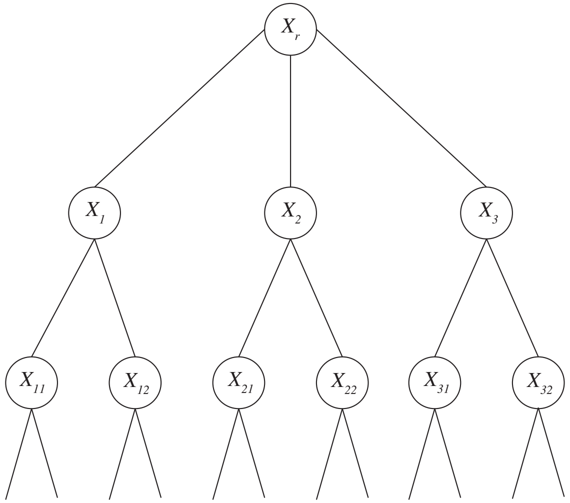
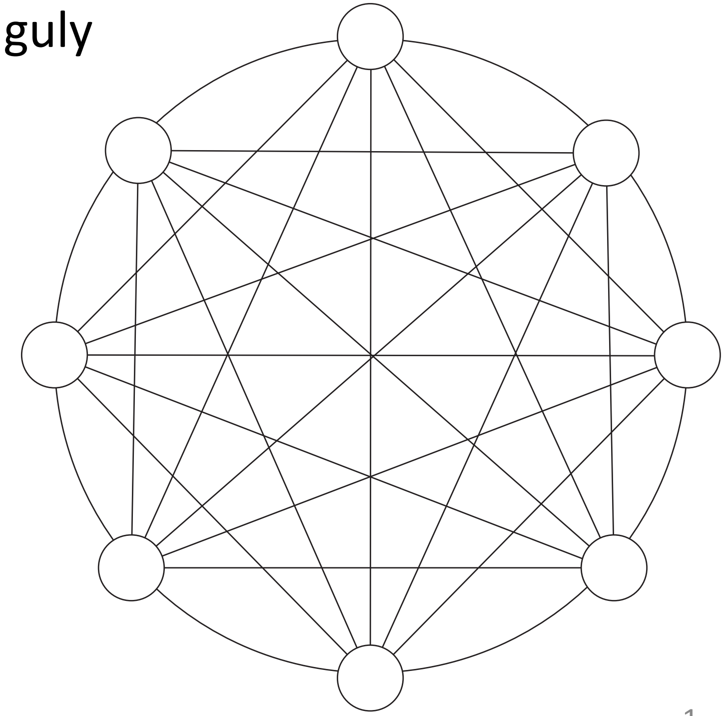
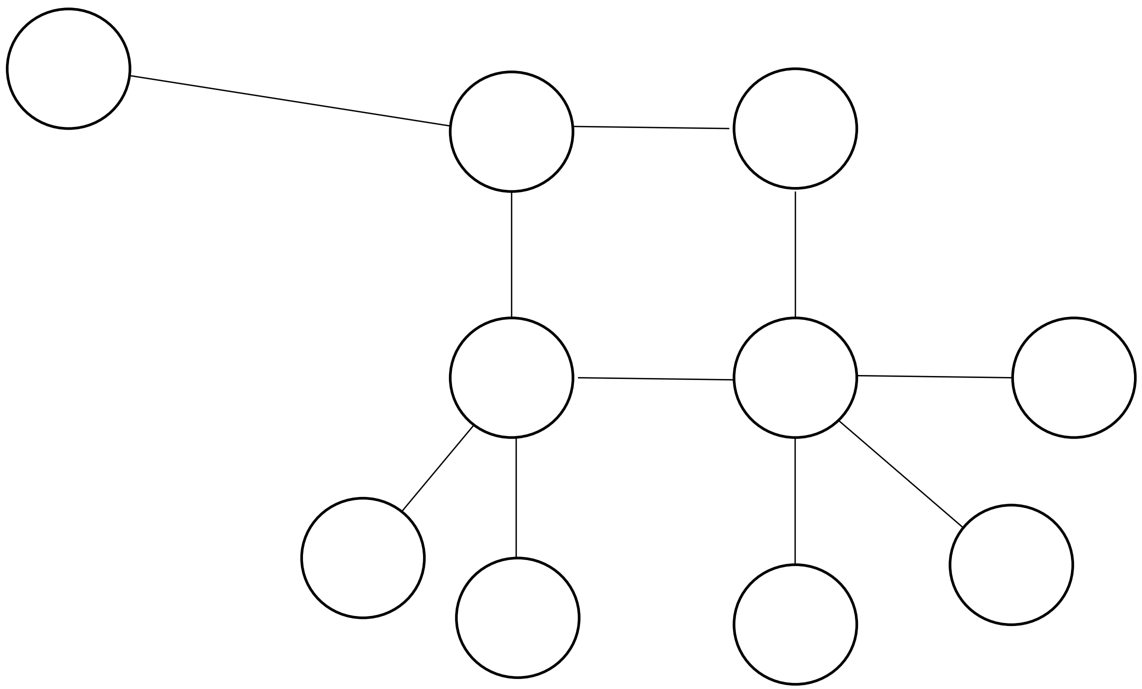




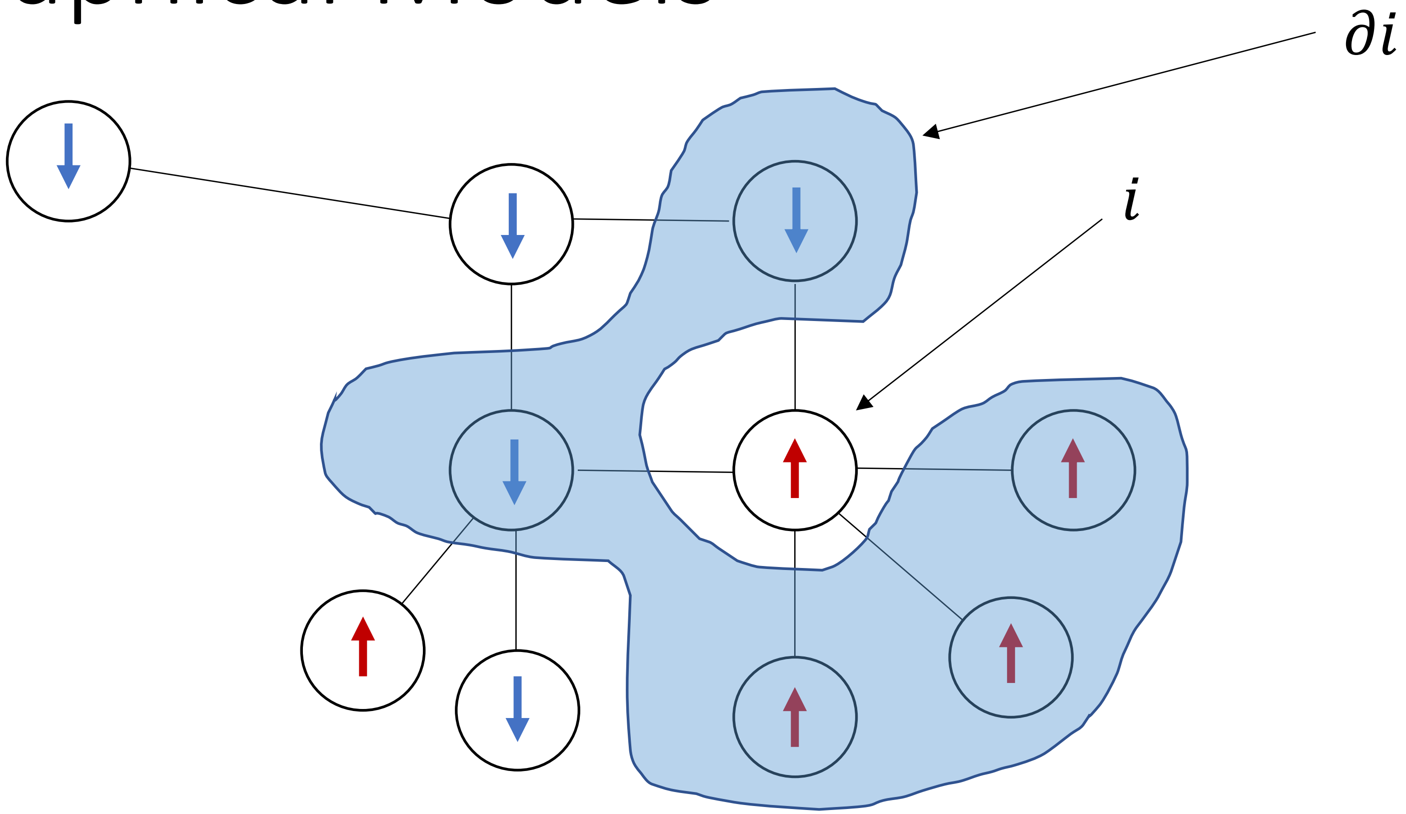


Step 1: Graphs



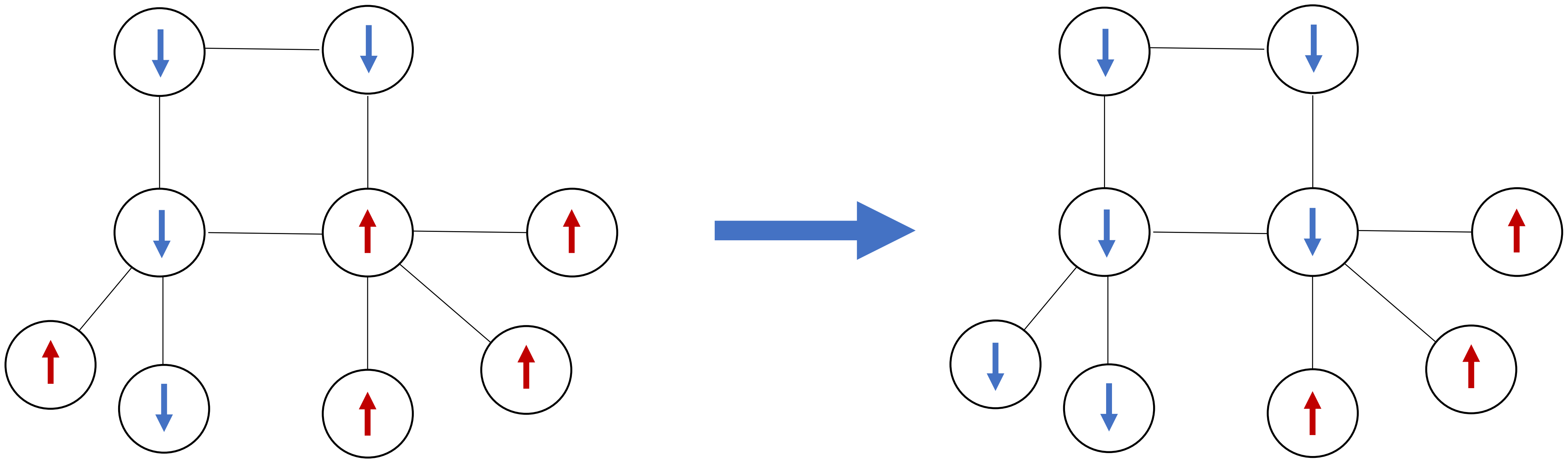
- $G = (V, E)$
  - V is the set of particles.
  - E is the set of connections.

Step 2: Graphical Models



- Each particle has a value.
- The probability that particle i has value x_i depends only on the state of its neighbors.

Step 3: Interacting Particle Systems



$t = T$ $\qquad$ $t = T + 1$

- Each particle has a state which evolves through time.
- Particle i evolves based only on the current state of its neighbors (and some independent noise).

Step 4: Formally…

- The state of particle i at time t is given by $\mathrm{X}_i(\mathrm{t})$
- The random vector $X(t)$= $(X_0(t), X_1(t), \dots)$ represents the state of **all** particle at time t.
- The system satisfies the following properties:

1. The Markov Property

$$\boxed{\begin{gathered}\mathbb{P}(X(t+1) = \vec{x} \mid X(0), \dots, X(t)) \\ = \mathbb{P}(X(t+1) = \vec{x} \mid X(t))\end{gathered}}$$

- If we know the present, knowing the past does not give us any more information about the future.

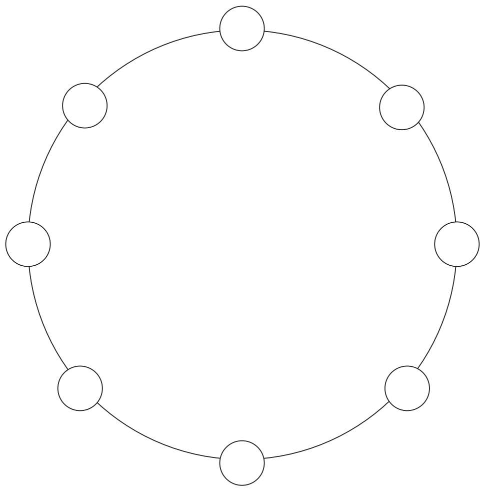
2. The Local Interactions Property

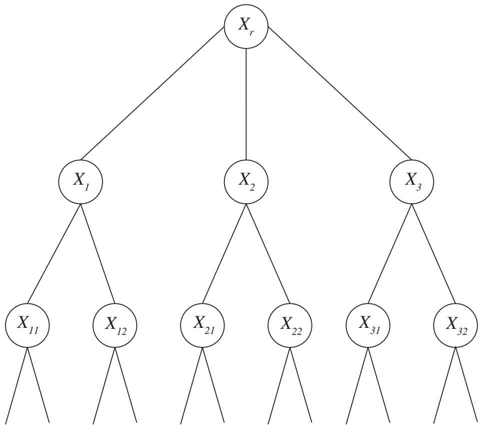
$$\boxed{\begin{gathered}\mathbb{P}(X_i(t+1) = v \mid X(t)) \\ = f\big(v, X_i(t), X_{\partial i}(t)\big)\end{gathered}}$$

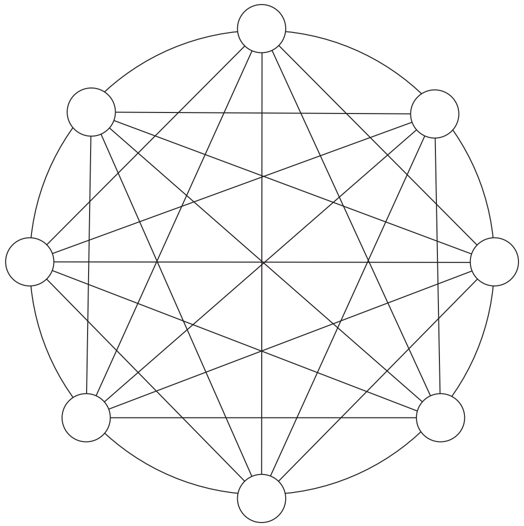
- Particle i evolves based only on the current state of its neighbors.

Step 5: The Goal

- Characterize the dynamics of a typical particle.
- Particle dynamics?
  - What is the probability that particle i is in state x_i at time t?
- Typical particle?
  - Assume that graph is symmetric and so all particles are identical.







Step 6: Why is this a challenge?

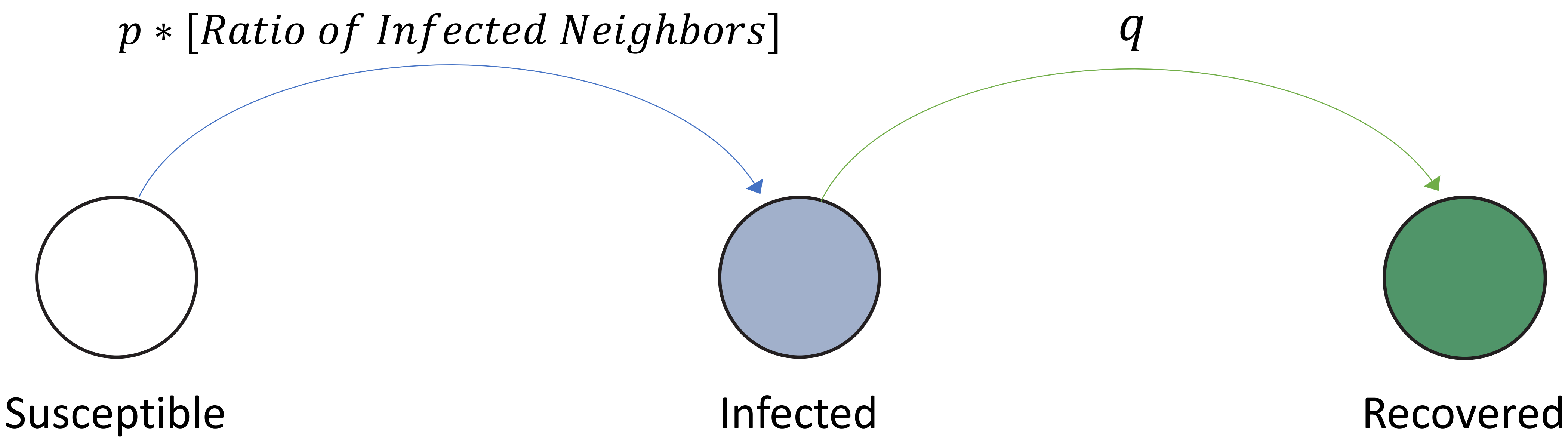
- **We can simulate the full system for a small number of particles N, but what about as N grows large and is perhaps infinite?**

- **The standard Mean Field Approximation works very well, but only for *dense* graphs.**

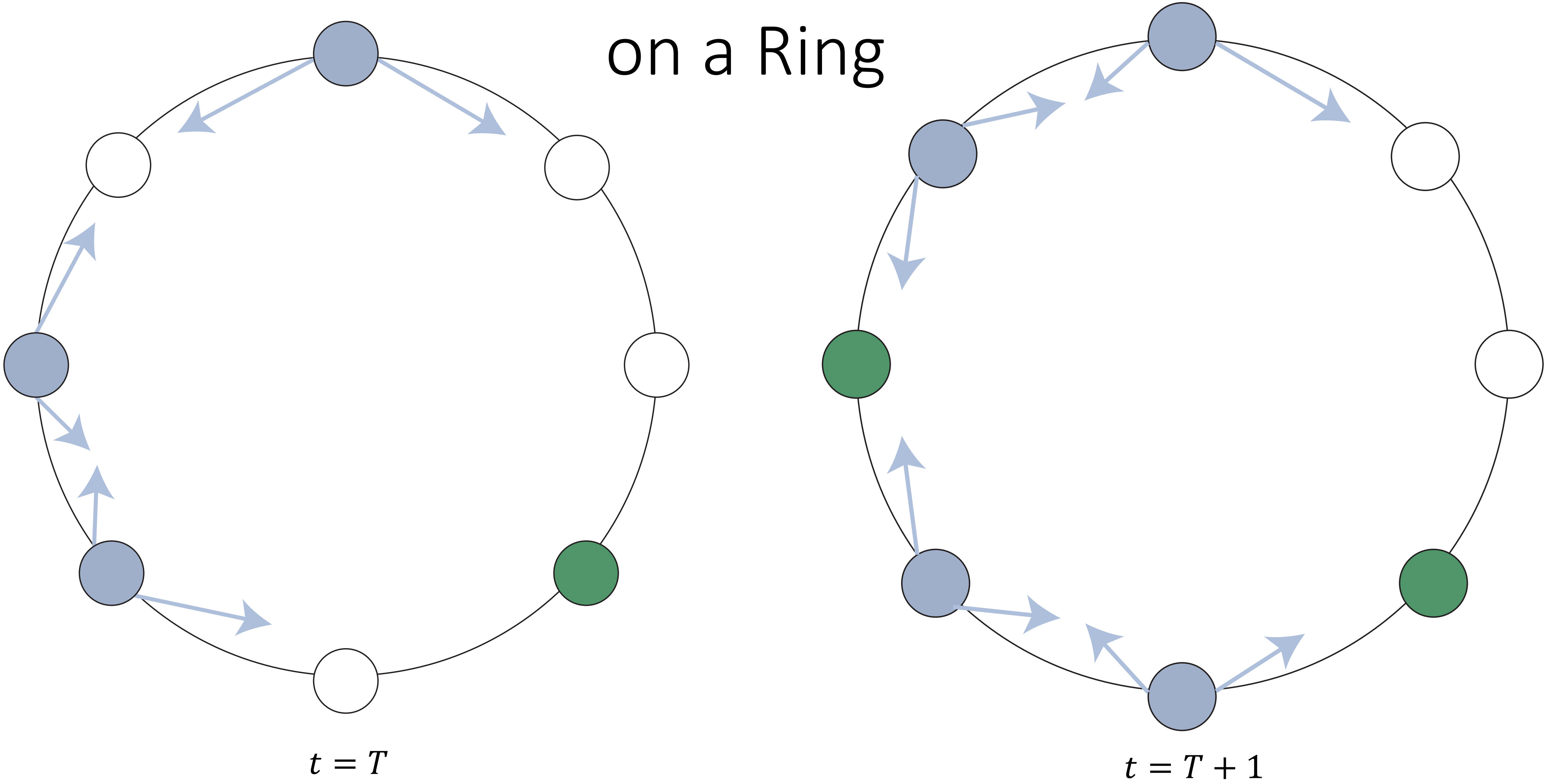
Step 7: Why are we interested?

- Epidemiology
- Statistical Physics
- Load Balancing
- Information Theory
- Genetics
- Machine Learning

Susceptible-Infected-Recovered (SIR) Process

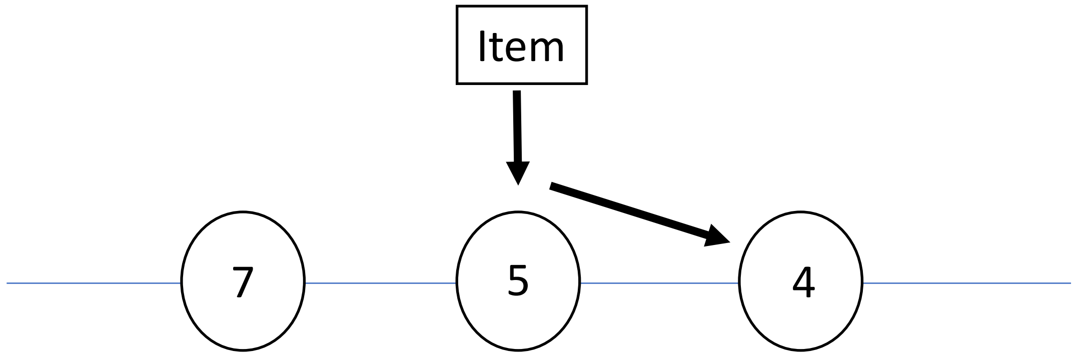


SIR Process on a Ring



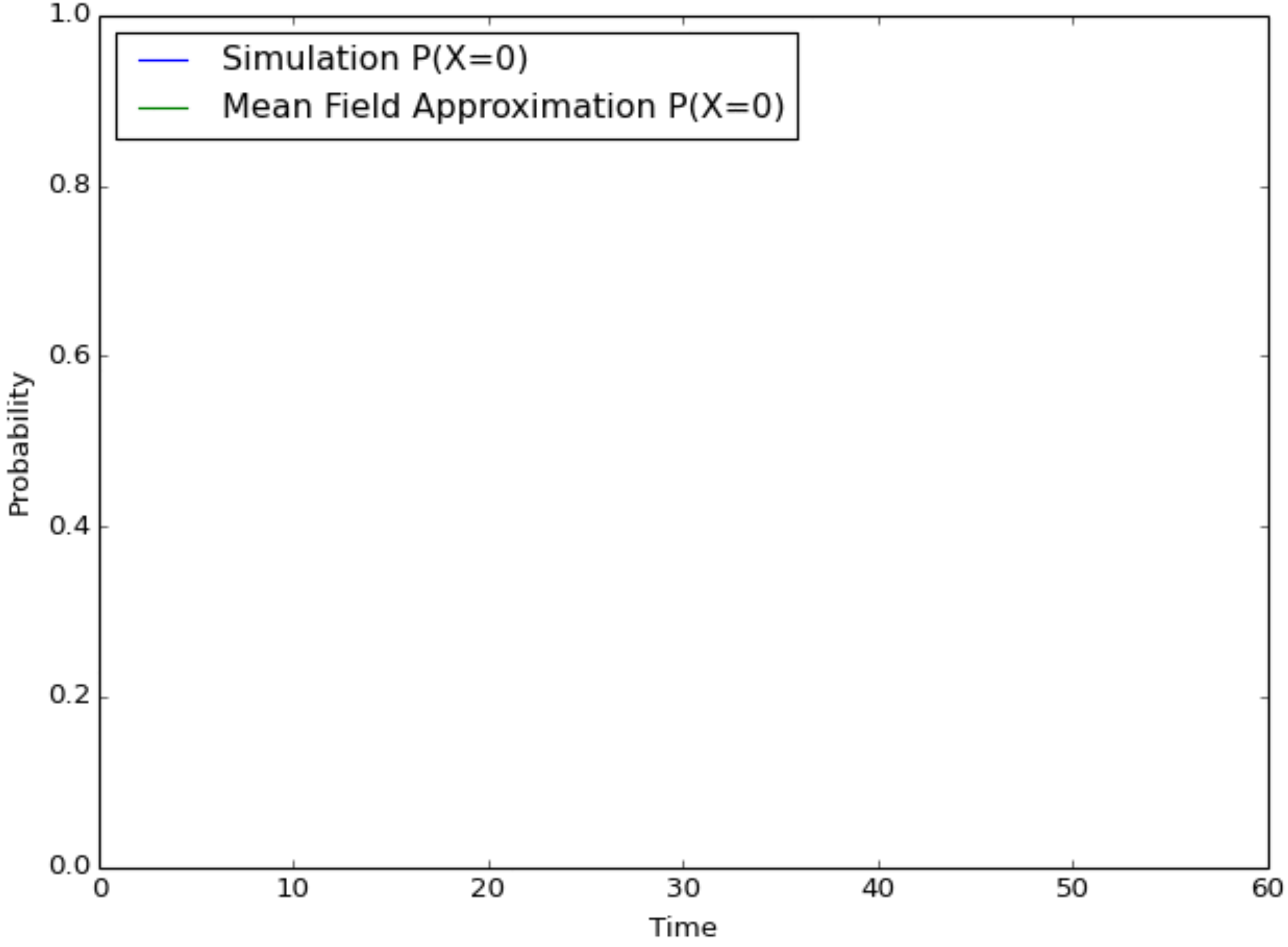
Load Balancing

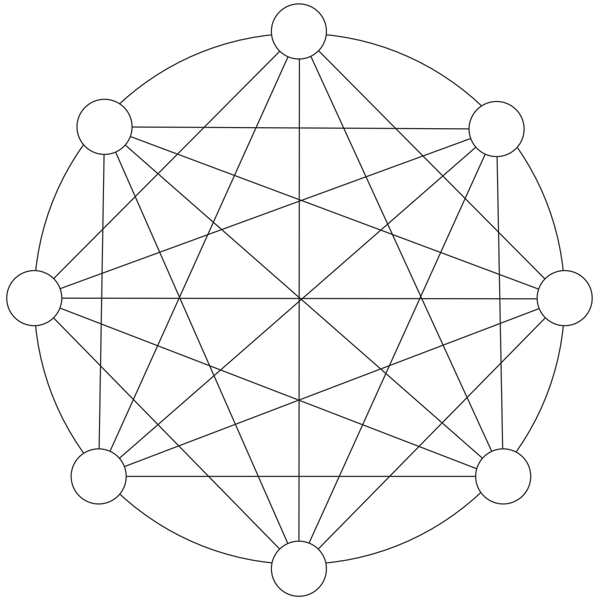
- The state of each particle is the number of items in a queue.
- When an item arrives at queue i, it is routed to the shortest queue among i and its neighbors.



An item incident at the queue with 5 items will be routed to the queue with 4 items.

The SIR Process on a Complete Graph



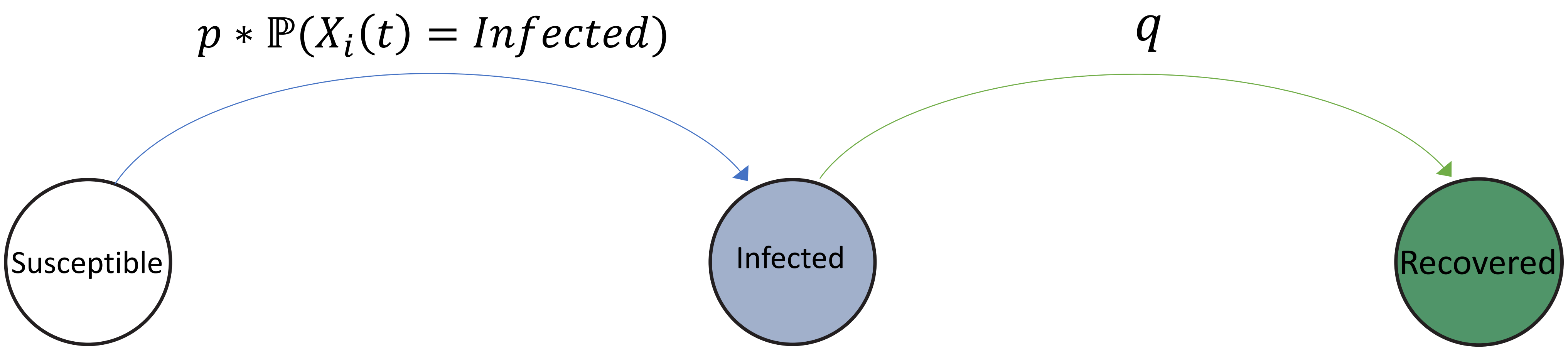


Interaction Network

$n = 50$

The Mean Field Approximation

Mean Field Approximation

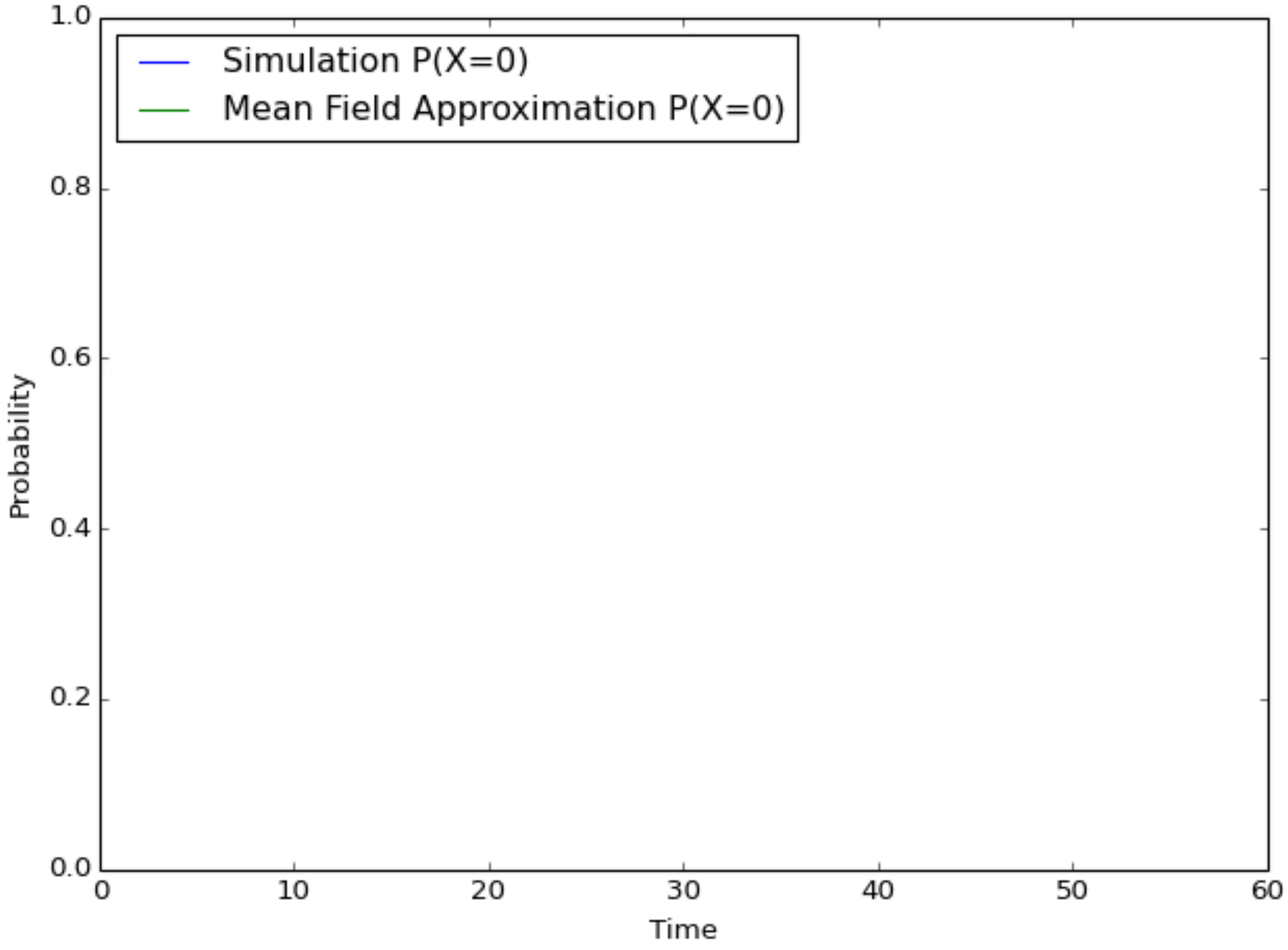


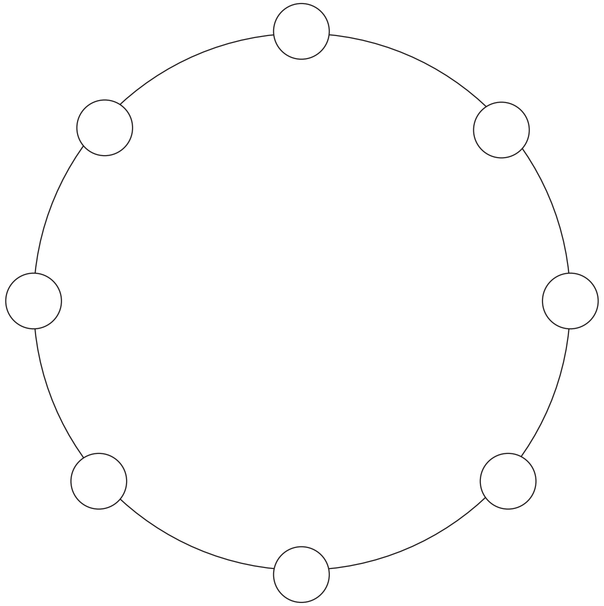
$$\mathbb{P}(X_i(t+1) = v \mid X_i(t)\,) = f\left(v\,, X_i(t), Law\big(X_i(t)\big)\right)$$

Assume that the neighbors of particle i are independent copies of i

- This approximation is **exact** for the complete graph as the population size tends to infinity (Oelschläger, 1984).
- Recently been shown to be asymptotically accurate for sequences of dense graphs whose degree goes to infinity (Bhamidi, Budhiraja, Wu, 2017).
- In many applications the mean-field approximation is used for any graph.

The SIR Process on a Ring





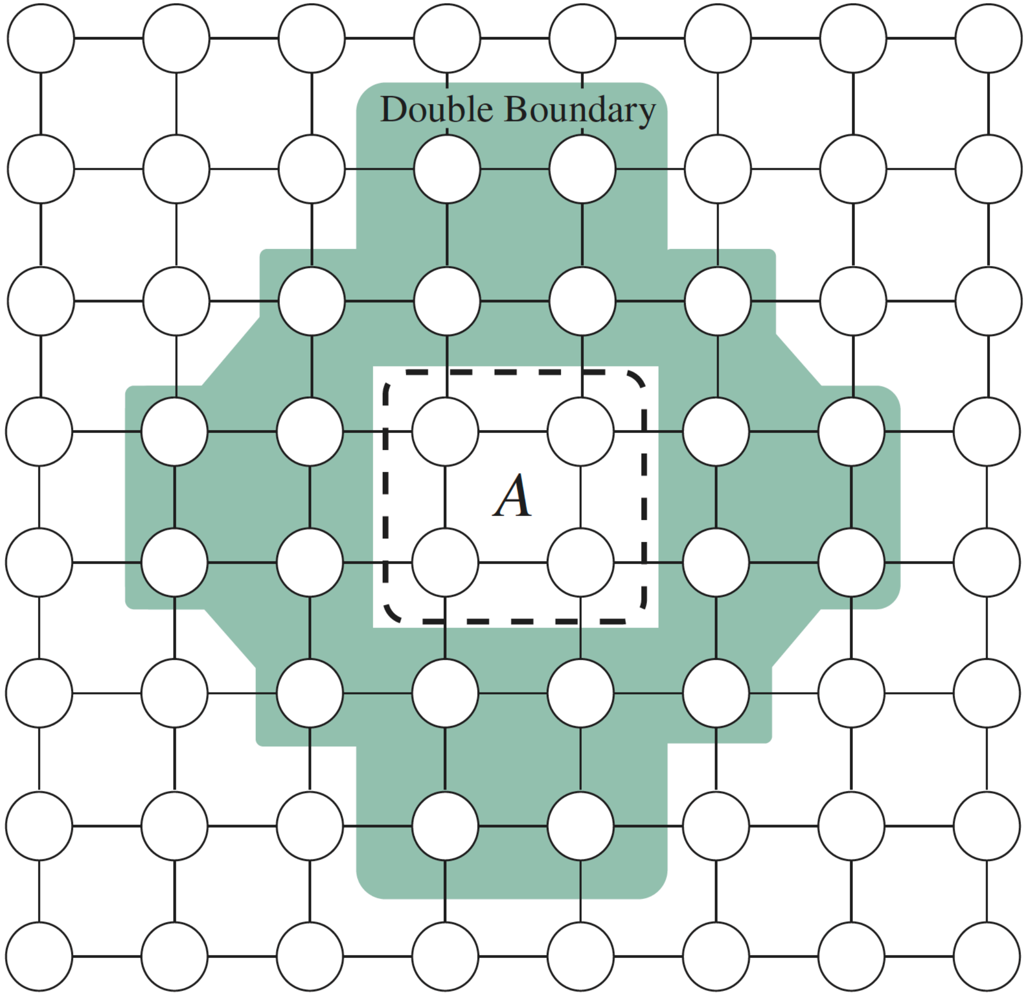
Interaction Network

$n = 50$

Step 8: Our Local Recursions

- **We build on recent theoretical results by Lacker, Ramanan, and Wu (2018) to construct a novel '*Local Recursion*'.**

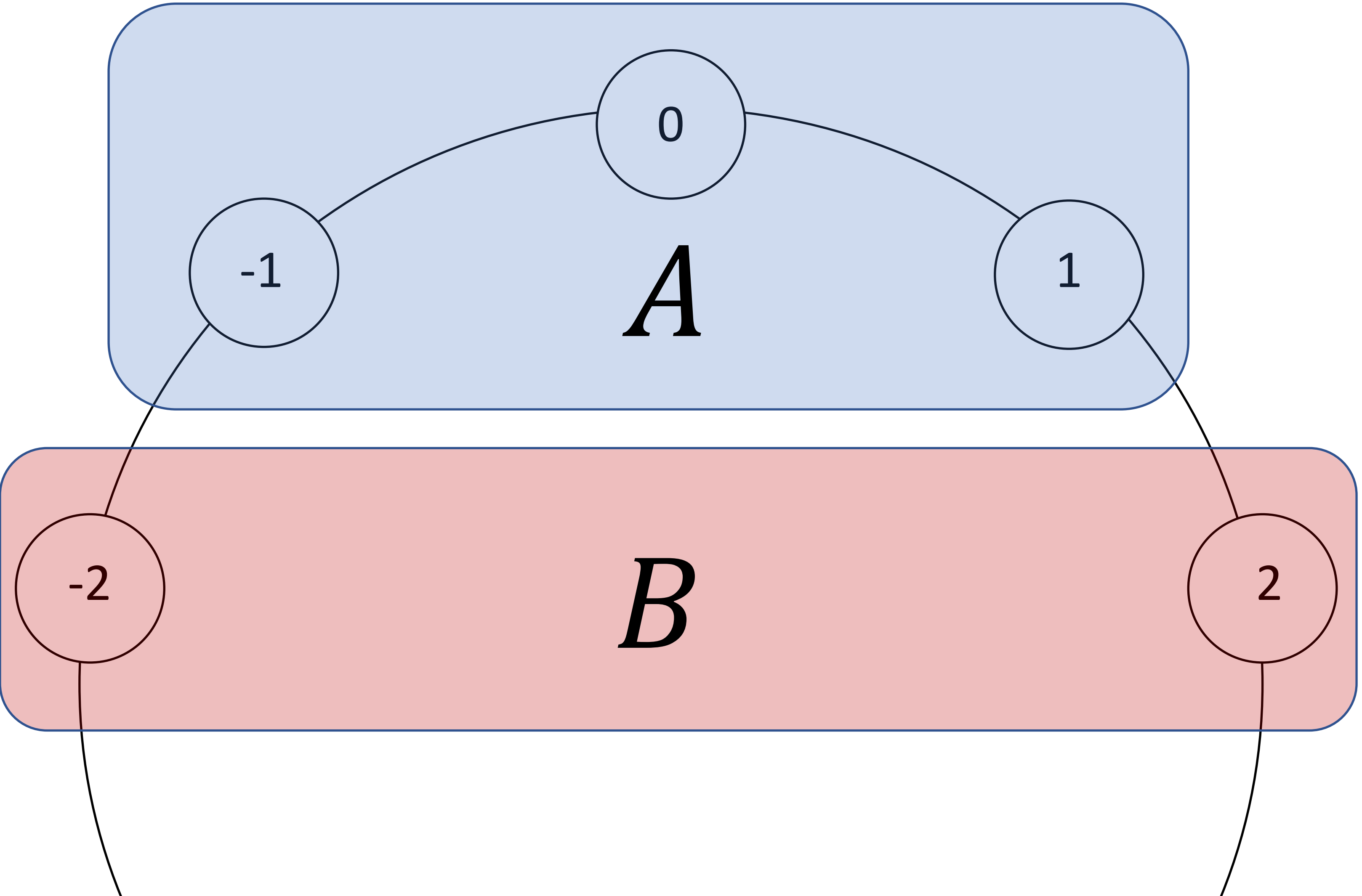
Dynamic Conditional Independence (Lacker, Ramanan, Wu 2018)



- *Notation*
  - *The Full Trajectory:* $X_A^T = \left(X_i(t)\right)_{i \in A,\, t \leq T}$
  - *The Double Boundary:* $D = \partial^2 A$

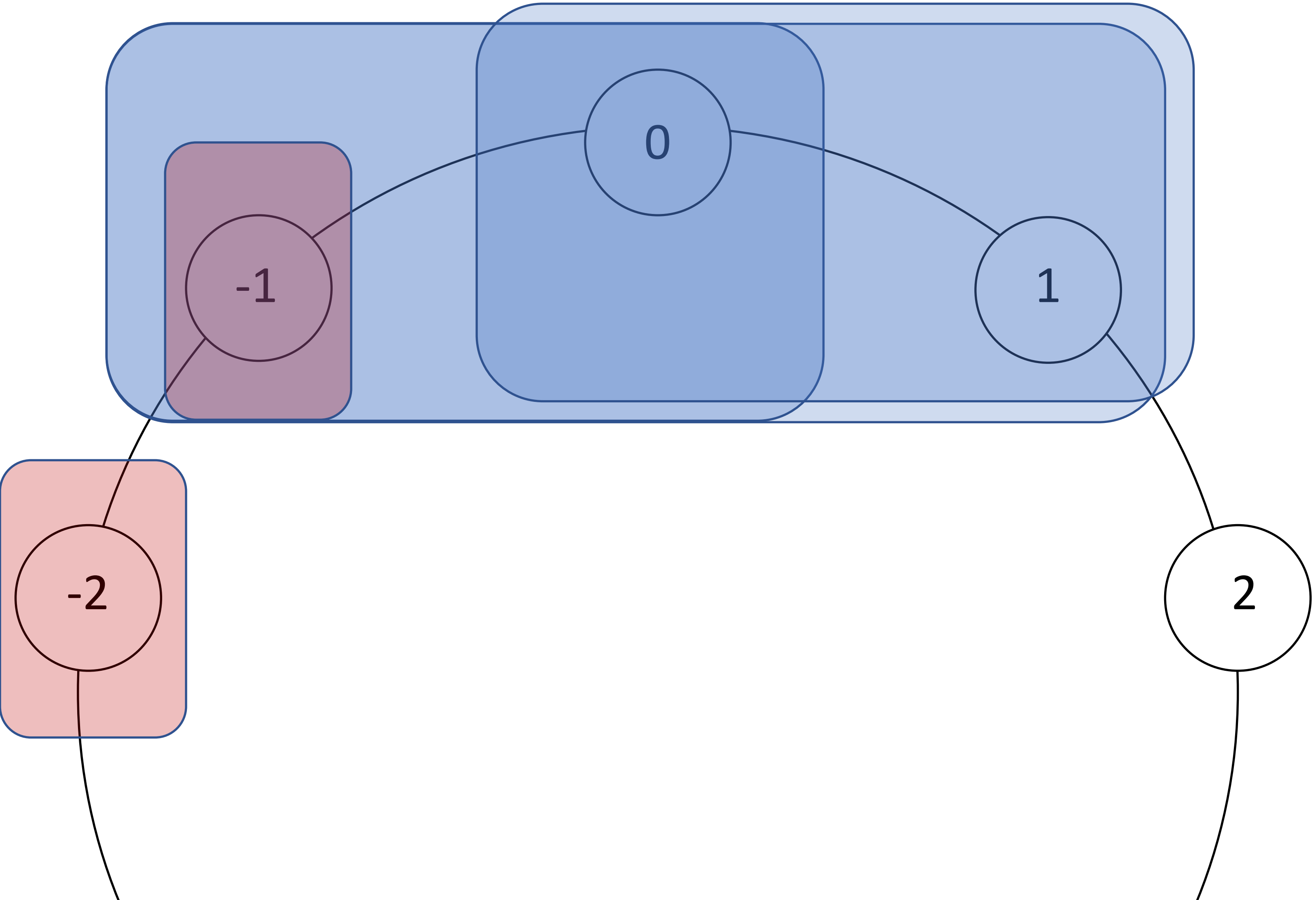
$$X_A^T \perp X_{V \setminus (D \cup A)}^T \mid X_D^T$$

Our Local Recursion



- Consider local region A.
- We would like a *self-contained* equation for A.
- Assume that we know the joint distribution of local region A up to time t.
- To compute the joint distribution of region A up to time $t+1$ we need to know the distribution of region B at time t (conditioned on the region A).

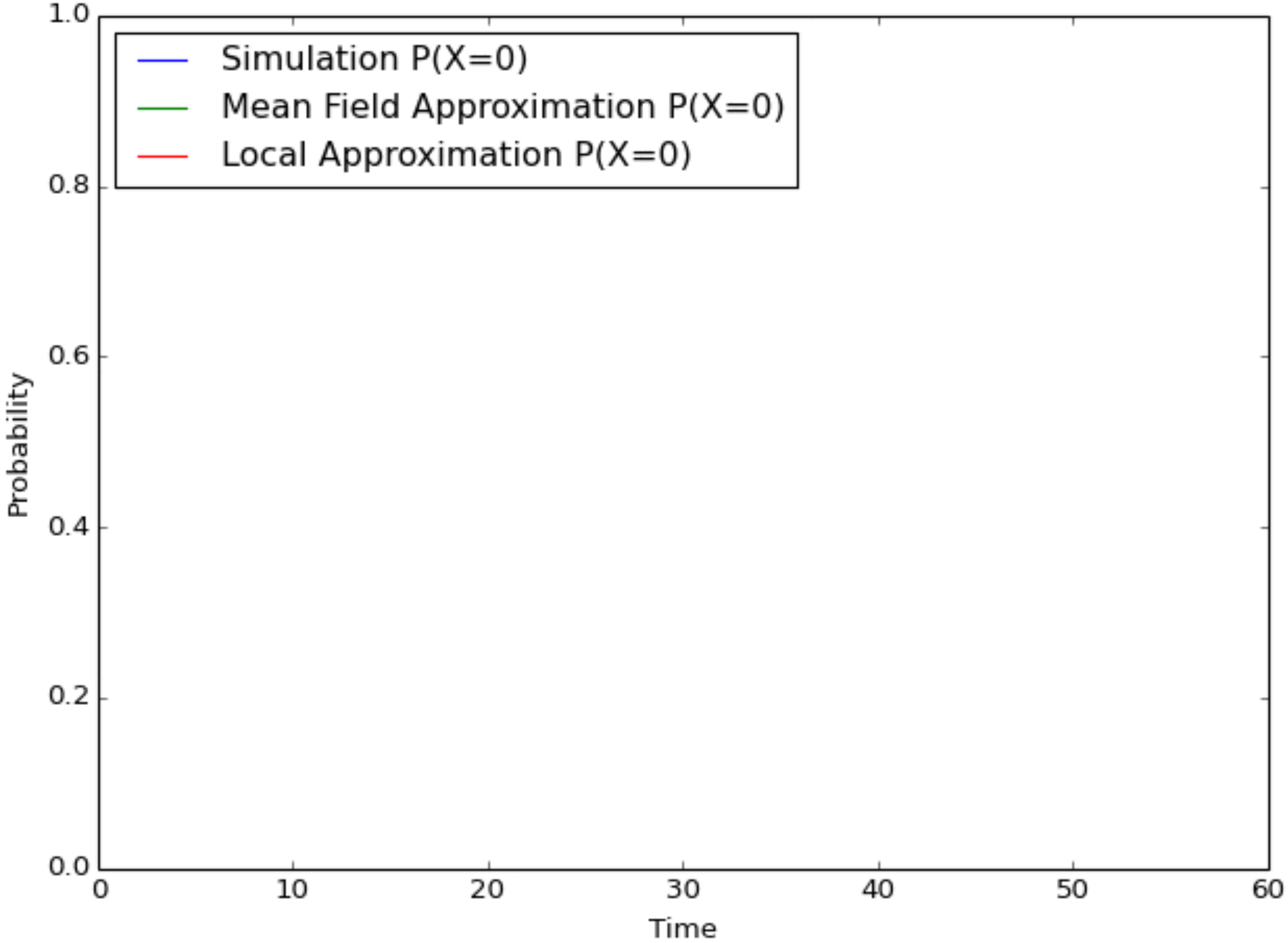
Our Local Recursion (2)

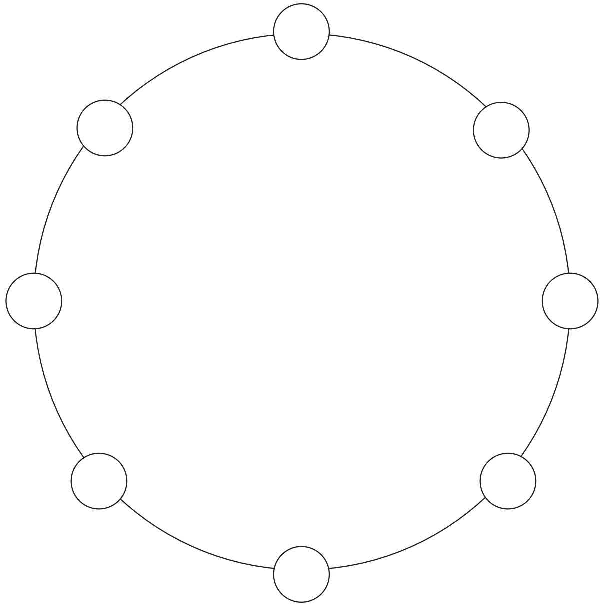


- CDB = Conditional Independence Given the Double Boundary.
- By CDB, the state of particles -2 and 2 at time t are independent given the trajectory of A.
- First we consider particle -2.
- By CDB, conditioning on A is equivalent to conditioning on the trajectory of -1,0.
- Since all particles are identical, this is equal in distribution to the state of particle -1 at time t conditioned on the trajectory of 0,1.
- Which we know by our inductive assumption.

How well does our local approximation work?

The SIR Process on a Ring (with our local approximation)





Interaction Network

$n = 50$

Local Recursions: Time Complexity

- As an approximation to the exact recursions, we may consider only the last τ time steps.
    - We call this **The $\boldsymbol{\tau}$ Approximation.**

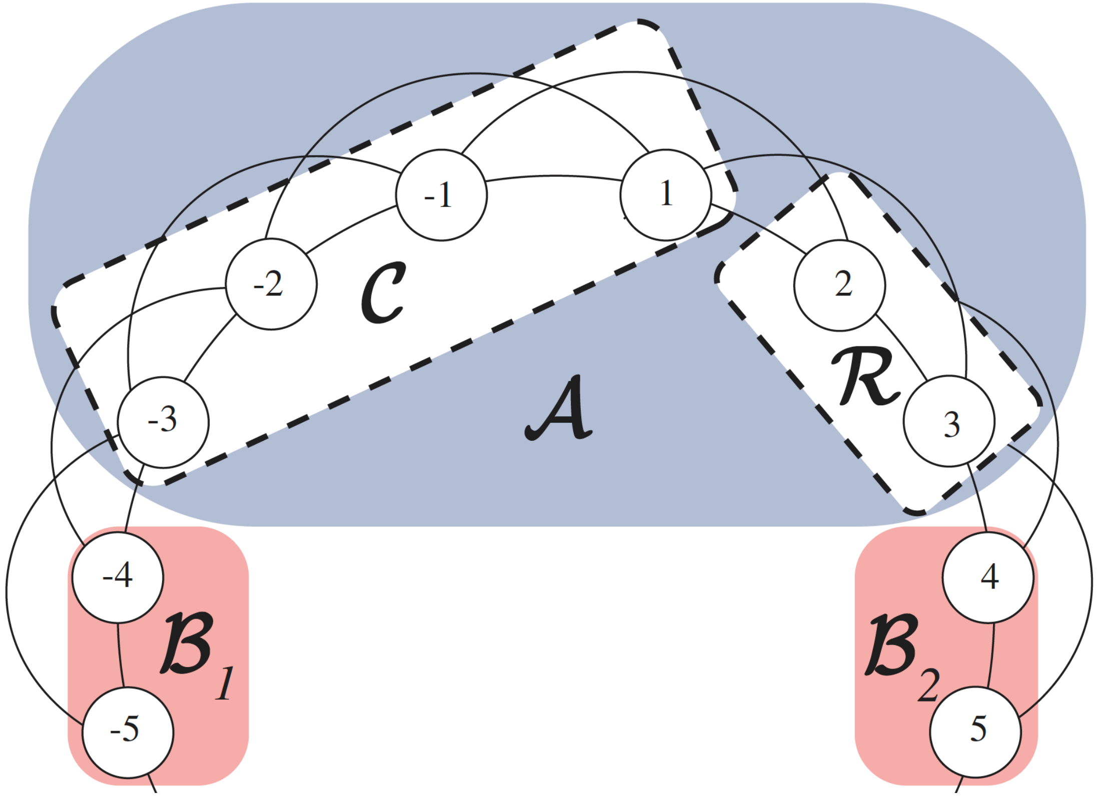
Other Existing Approximations

- Dynamic Cavity Method
  - Only exact for trees.
  - Requires consideration of the dynamics of all particles in the graph (slow).
  - Not clear how to implement for continuous state space.
  - No fast approximation.

- Moment closure methods (i.e. Pair Approximation):
  - Makes a huge assumption (that the time marginals satisfy the spatial Markov property).
  - For many models (e.g. SIR) this approximation is very inaccurate (Gast, 15).

Generalizations to Other Graphs

- Our method generalizes to non locally-tree-like graphs.
  - For example, the graph which arises in Load Balancing on a ring.



Summary

In this presentation we have…

- Defined Interacting Particle Systems and given examples.
- Discussed existing approximations for understanding the behavior of these systems (Mean Field) and highlighted cases where they are not applicable.
- Discussed recent theory which reveals a new conditional independence structure.
- Building off this recent theory we have demonstrated the effectiveness of our new Local Recursions.

Questions?