



Time Series Analysis & Forecasting Using R

8. ARIMA models



Outline

- 1 ARIMA models
- 2 Lab Session 14
- 3 Seasonal ARIMA models
- 4 Lab Session 15
- 5 ARIMA vs ETS
- 6 Forecast ensembles

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ARIMA models

- AR:** autoregressive (lagged observations as inputs)
- I:** integrated (differencing to make series stationary)
- MA:** moving average (lagged errors as inputs)

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An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s , the distribution of (y_t, \dots, y_{t+s}) does not depend on t .

Stationarity

Definition

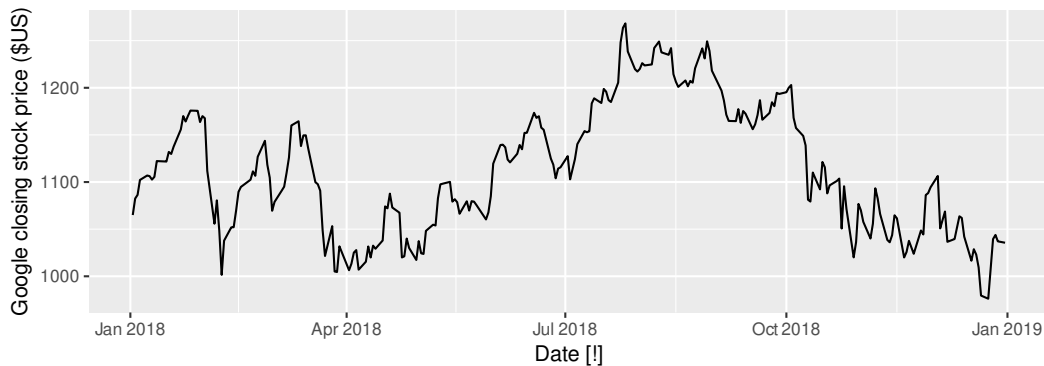
If $\{y_t\}$ is a stationary time series, then for all s , the distribution of (y_t, \dots, y_{t+s}) does not depend on t .

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

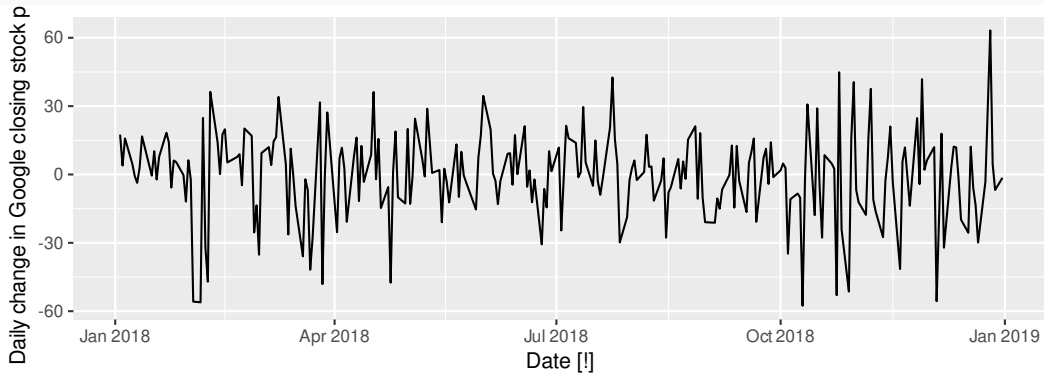
Stationary?

```
gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018) |>  
  autoplot(Close) +  
  labs(y = "Google closing stock price ($US)")
```



Stationary?

```
gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018) |>  
  autoplot(difference(Close)) +  
  labs(y = "Daily change in Google closing stock price")
```



Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series.
- Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time.
- In practice, it is almost never necessary to go beyond second-order differences.

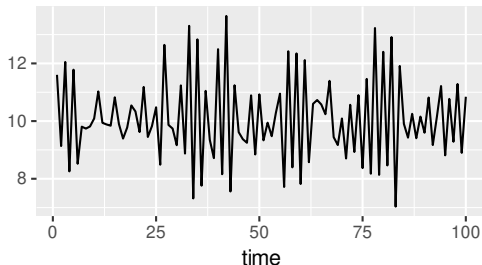
Autoregressive models

Autoregressive (AR) models:

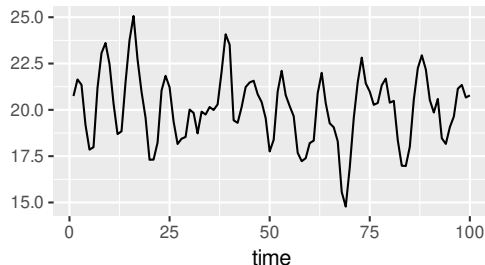
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. A multiple regression with **lagged values** of y_t as predictors.

AR(1)



AR(2)

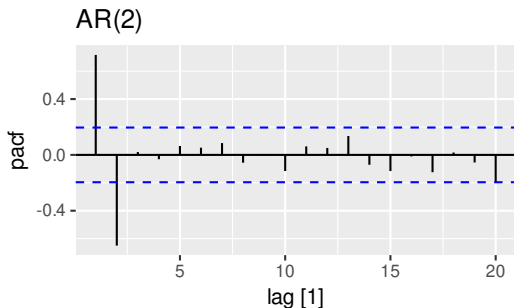
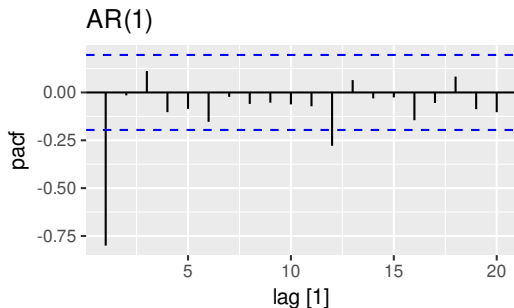


Autoregressive models

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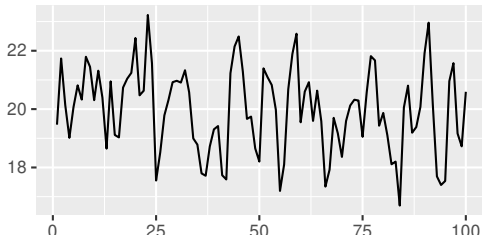
Moving Average (MA) models

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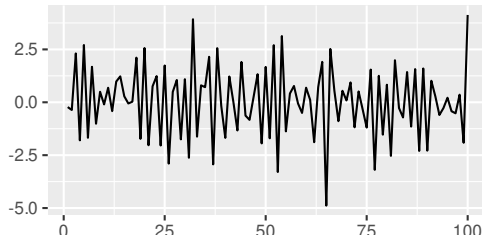
$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise. A multiple regression with **lagged errors** as predictors. *Don't confuse with moving average smoothing!*

MA(1)



MA(2)

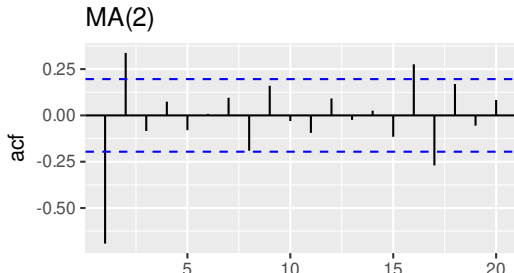
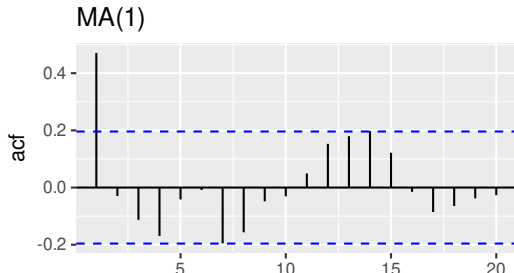


Moving Average (MA) models

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ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

Autoregressive Moving Average models:

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- Predictors include both **lagged values of y_t and lagged errors.**

ARIMA models

Autoregressive Moving Average models:

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- Predictors include both **lagged values of y_t and lagged errors.**

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- d -differenced series follows an ARMA model.
- Need to choose p, d, q and whether or not to include c .

ARIMA models

ARIMA(p, d, q) model

AR: p = order of the autoregressive part

I: d = degree of first differencing involved

MA: q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA($p,0,0$)
- MA(q): ARIMA(0,0, q)

Example: National populations

```
fit <- global_economy |>  
  model(arima = ARIMA(Population))  
fit
```

```
# A mable: 263 x 2
```

```
# Key:      Country [263]
```

Country	arima
<fct>	<model>
1 Afghanistan	<ARIMA(4,2,1)>
2 Albania	<ARIMA(0,2,2)>
3 Algeria	<ARIMA(2,2,2)>
4 American Samoa	<ARIMA(2,2,2)>
5 Andorra	<ARIMA(2,1,2) w/ drift>
6 Angola	<ARIMA(4,2,1)>
7 Antigua and Barbuda	<ARIMA(2,1,2) w/ drift>
8 Arab World	<ARIMA(0,2,1)>
9 Argentina	<ARIMA(2,2,2)>

Example: National populations

```
fit |>  
  filter(Country == "Australia") |>  
  report()
```

Series: Population

Model: ARIMA(0,2,1)

Coefficients:

ma1

-0.661

s.e. 0.107

sigma^2 estimated as 4.063e+09: log likelihood=-699

AIC=1401 AICc=1402 BIC=1405

Example: National populations

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Series: Population

Model: ARIMA(0,2,1)

Coefficients:

ma1
-0.661
s.e. 0.107

$$y_t = 2y_{t-1} - y_{t-2} - 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 4 \times 10^9)$$

sigma^2 estimated as 4.063e+09: log likelihood=-699

AIC=1401 AICc=1402 BIC=1405

Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

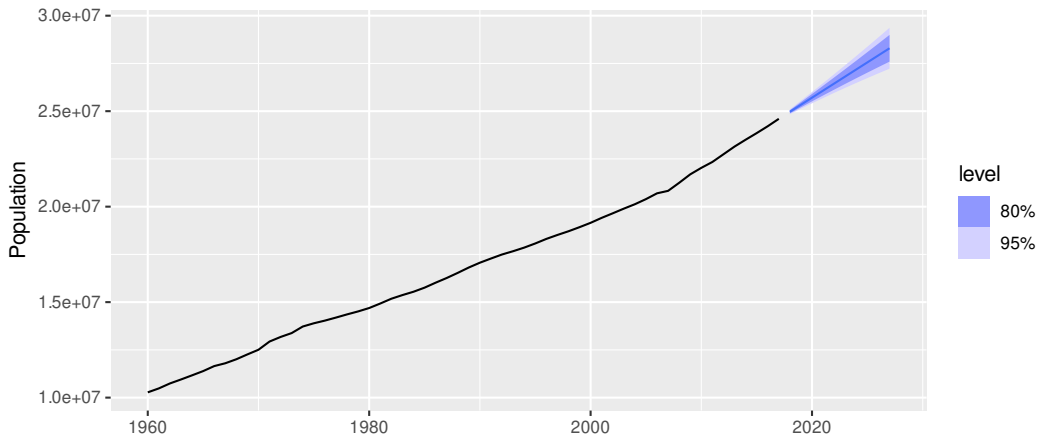
Understanding ARIMA models

Forecast variance and d

- The higher the value of d , the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Example: National populations

```
fit |>  
  forecast(h = 10) |>  
  filter(Country == "Australia") |>  
  autoplot(global_economy)
```



How does ARIMA() work?

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

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$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[1 + \frac{(p + q + k + 2)}{T - p - q - k - 2} \right]$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

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where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Note: Can't compare AICc for different values of d .

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

ARIMA(2, d , 2)

ARIMA(0, d , 0)

ARIMA(1, d , 0)

ARIMA(0, d , 1)

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

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ARIMA(1, d , 0)

ARIMA(0, d , 1)

Step 2: Consider variations of current model:

- vary one of p , q , from current model by ± 1 ;
- p , q both vary from current model by ± 1 ;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

How does ARIMA() work?

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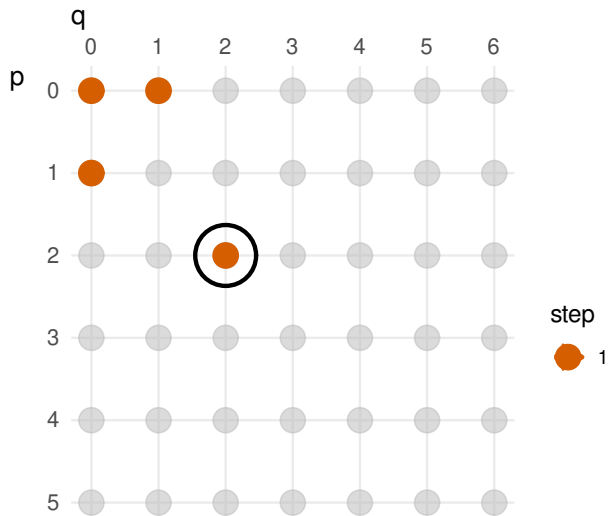
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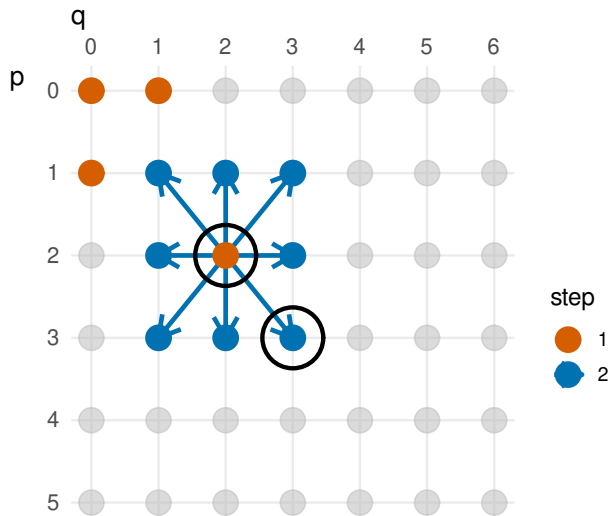
Model with lowest AICc becomes current model.

Repeat Step 2 until no lower AICc can be found.

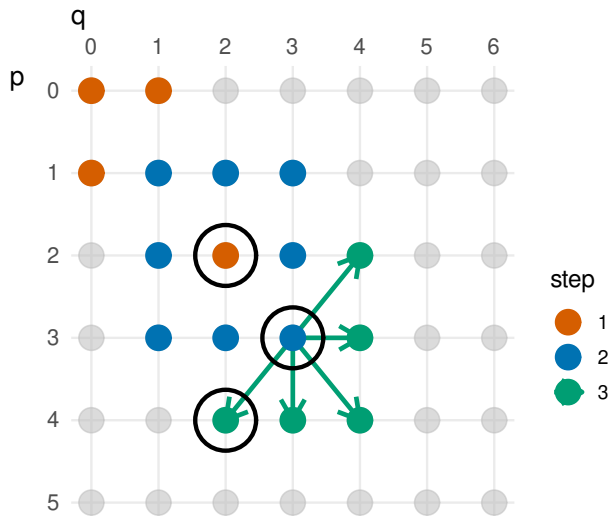
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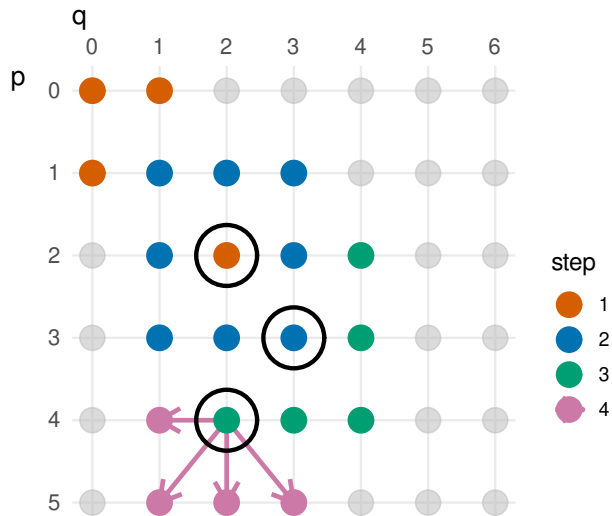
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Lab Session 14

For the total number of students in Government, Catholic, and Independent schools:

```
enrolment_affiliation <- students |>  
  group_by(`Affiliation (Gov/Cath/Ind)`) |>  
  summarise(enrolments = sum(`All Full-time and Part-time Student count`))
```

- Fit a suitable ARIMA model (including possible transformation) for the data.
- Produce forecasts of your fitted model. Do the forecasts look reasonable?

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Seasonal ARIMA models

ARIMA	(p, d, q)	$(P, D, Q)_m$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

- m = number of observations per year.
- d first differences, D seasonal differences
- p AR lags, q MA lags
- P seasonal AR lags, Q seasonal MA lags

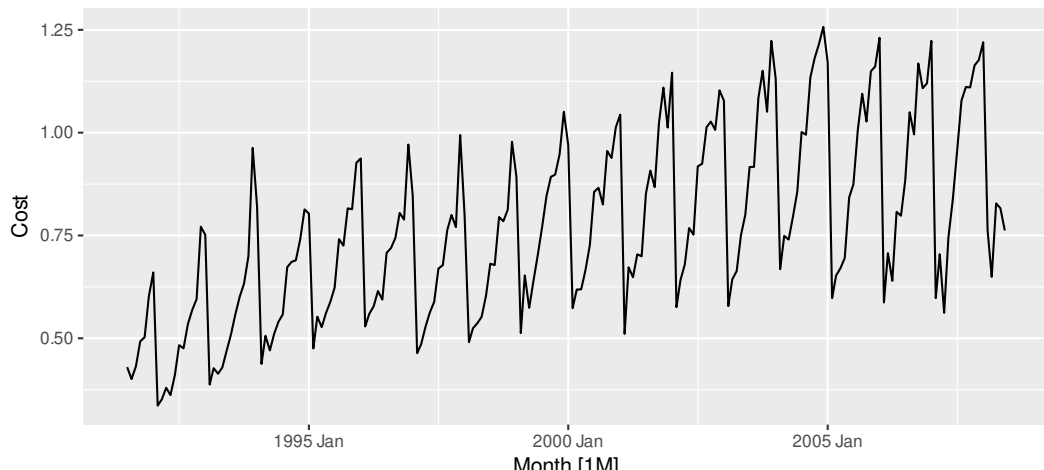
Seasonal and non-seasonal terms combine multiplicatively

Corticosteroid medication subsidies

```
corticosteroid_subsidies <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

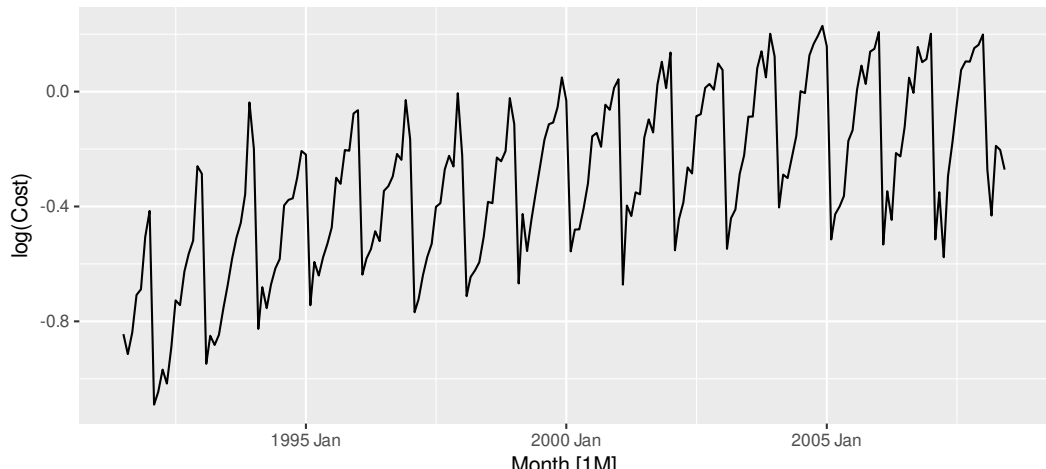
Corticosteroid medication subsidies

```
corticosteroid_subsidies |> autoplot(  
  Cost  
)
```



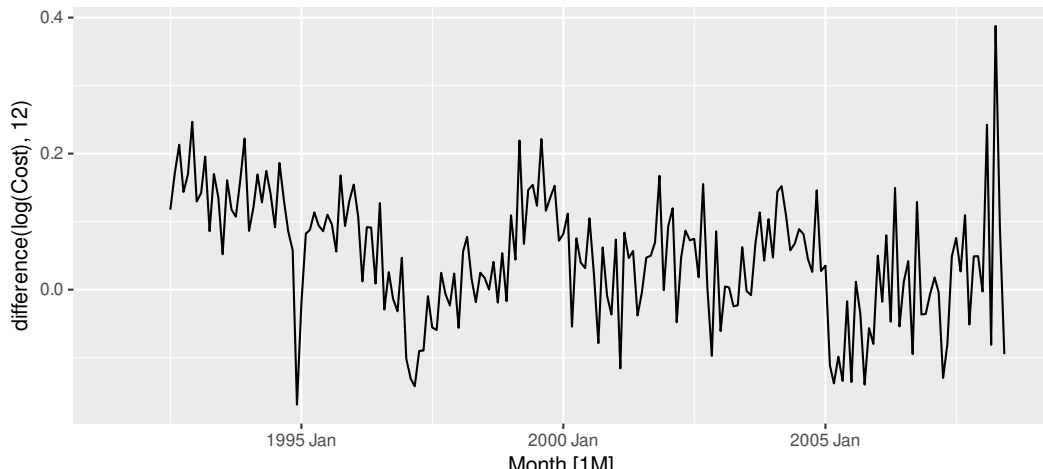
Corticosteroid medication subsidies

```
corticosteroid_subsidies |> autoplot(  
  log(Cost)  
)
```



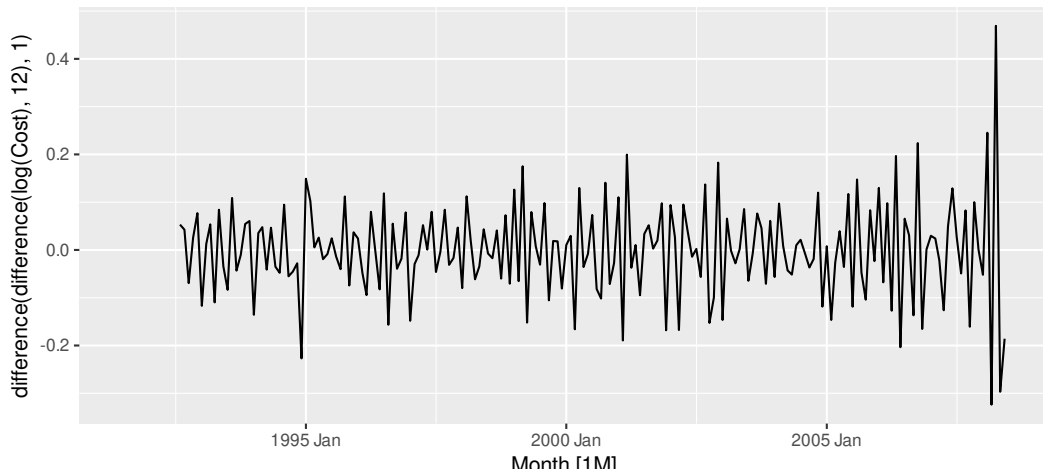
Corticosteroid medication subsidies

```
corticosteroid_subsidies |> autoplot(  
  log(Cost) |> difference(12)  
)
```



Corticosteroid medication subsidies

```
corticosteroid_subsidies |> autoplot(  
  log(Cost) |> difference(12) |> difference(1)  
)
```



Cortecosteroid medication subsidies

```
fit <- cortecosteroid_subsidies |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 8
  ))
report(fit)
```

Series: Cost

Model: ARIMA(4,1,1)(2,1,1)[12]

Transformation: log(Cost)

Coefficients:

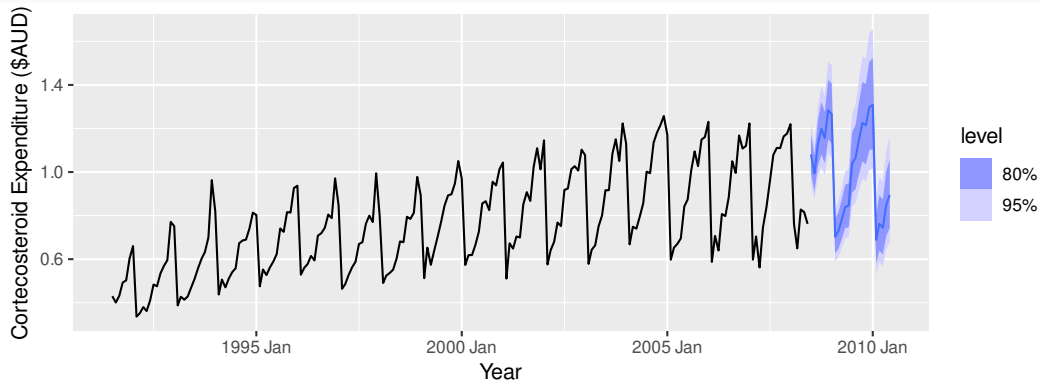
	ar1	ar2	ar3	ar4	ma1	sar1	sar2	sma1
	-0.0099	0.248	0.225	-0.2429	-0.776	0.110	-0.2022	-0.671
s.e.	0.1760	0.150	0.103	0.0805	0.165	0.125	0.0995	0.110

sigma^2 estimated as 0.004098: log likelihood=253

AIC=-488 AICc=-487 BIC=-459

Corticosteroid medication subsidies

```
fit |>  
  forecast() |>  
  autoplot(corticosteroid_subsidies) +  
  labs(y = "Corticosteroid Expenditure ($AUD)", x = "Year")
```



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Lab Session 15

Find an ARIMA model for the each school attendance type for people aged 15-24 in Australia (`student_labour`).

```
youth_learning <- student_labour |>  
  group_by(attendance) |>  
  summarise(persons = sum(persons))
```

- Fit suitable ARIMA models for each time series.
- Produce forecasts of your fitted models.
- Check the forecasts for each of the school attendance category, do they look reasonable?

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Is ARIMA or ETS best?

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It depends!

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It depends!

i Capturing patterns

- Both handle time varying trends and seasonality.
- ETS directly captures multiplicative patterns.
- ARIMA can forecast cyclical patterns.

Is ARIMA or ETS best?

It depends!

 Evaluate accuracy

We can determine which model works best on a specific dataset using accuracy evaluation.

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 Evaluate accuracy

We can determine which model works best on a specific dataset using accuracy evaluation.

Or, combine them with ensembles!

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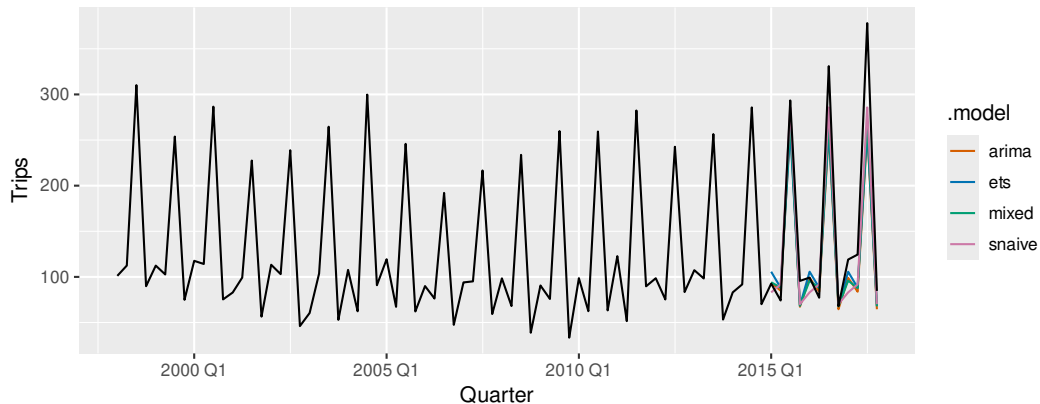
Forecast ensembles

```
train <- tourism |>
  filter(year(Quarter) <= 2014)
fit <- train |>
  model(
    ets = ETS(Trips),
    arima = ARIMA(Trips),
    snaive = SNAIVE(Trips)
  ) |>
  mutate(mixed = (ets + arima + snaive) / 3)
```

- Ensemble forecast `mixed` is a simple average of the three fitted models.
- `forecast()` will produce distributional forecasts taking into account the correlations between the forecast errors of the component models.

Forecast ensembles

```
fc <- fit |> forecast(h = "3 years")  
fc |>  
  filter(Region == "Snowy Mountains", Purpose == "Holiday") |>  
  autoplot(tourism, level = NULL)
```



Forecast ensembles

```
accuracy(fc, tourism) |>
  group_by(.model) |>
  summarise(
    RMSE = mean(RMSE),
    MAE = mean(MAE),
    MASE = mean(MASE)
  ) |>
  arrange(RMSE)
```

A tibble: 4 x 4

	.model	RMSE	MAE	MASE
	<chr>	<dbl>	<dbl>	<dbl>
1	mixed	19.8	16.0	0.997
2	ets	20.2	16.4	1.00
3	snaive	21.5	17.3	1.17
4	arma	21.9	17.8	1.06

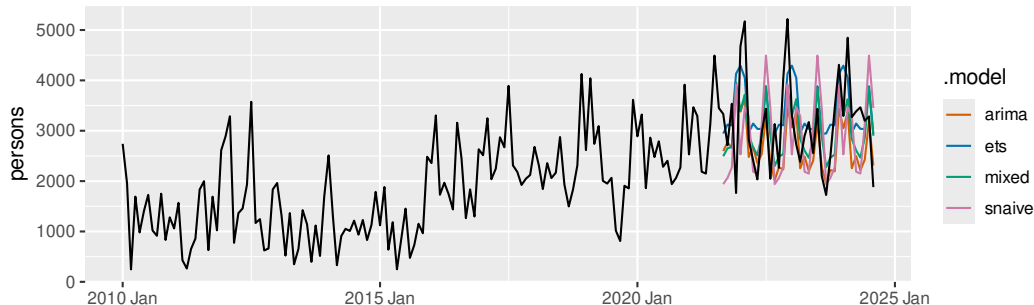
Forecast ensembles

```
train <- student_labour |>
  filter(month <= yearmonth("2021 Aug"))
fit <- train |>
  model(
    ets = ETS(persons),
    arima = ARIMA(persons),
    snaive = SNAIVE(persons)
  ) |>
  mutate(mixed = (ets + arima + snaive) / 3)
```

- Ensemble forecast `mixed` is a simple average of the three fitted models.
- `forecast()` will produce distributional forecasts taking into account the correlations between the forecast errors of the component models.

Forecast ensembles

```
fc <- fit |> forecast(h = "3 years")
fc |>
  filter(
    state == "Australian Capital Territory",
    attendance == "Attending full-time education",
    status == "Employed full-time"
  ) |>
  autoplot(student_labour |> filter(month >= yearmonth("2010 Jan")), level = NULL)
```



Forecast ensembles

```
accuracy(fc, student_labour) |>
  group_by(.model) |>
  summarise(
    RMSE = mean(RMSE),
    MAE = mean(MAE),
    MASE = mean(MASE)
  ) |>
  arrange(RMSE)
```

A tibble: 4 x 4

	.model	RMSE	MAE	MASE
	<chr>	<dbl>	<dbl>	<dbl>
1	mixed	6783.	5835.	1.54
2	ets	6979.	5975.	1.57
3	arima	7135.	6222.	1.61
4	snaive	7729.	6485.	1.72