



# Time Series Analysis & Forecasting Using R

## 7. Exponential smoothing



# Outline

- 1 Exponential smoothing
- 2 Trend methods
- 3 Lab Session 14
- 4 Seasonal methods
- 5 ETS taxonomy
- 6 Lab Session 15
- 7 Non-Gaussian forecast distributions

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- 1 Exponential smoothing
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# Pharmaceutical Benefits Scheme



# Pharmaceutical Benefits Scheme

**The Pharmaceutical Benefits Scheme (PBS) is the Australian government drugs subsidy scheme.**

- Many drugs bought from pharmacies are subsidised to allow more equitable access to modern drugs.
- The cost to government is determined by the number and types of drugs purchased. Currently nearly 1% of GDP.
- The total cost is budgeted based on forecasts of drug usage.

# Pharmaceutical Benefits Scheme



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### POLITICS

## Opp demands drug price restriction after PBS budget blow-out

The Federal Opposition has called for tighter controls on drug prices after the Pharmaceutical Benefits Scheme (PBS) budget blew out by almost \$800 million.

The money was spent on two new drugs including the controversial anti-smoking aid Zyban, which dropped in price from \$220 to \$22 after it was listed on the PBS.



#### FEATURES



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# Pharmaceutical Benefits Scheme

- In 2001: \$4.5 billion budget, under-forecasted by \$800 million.
- Thousands of products. Seasonal demand.
- Subject to covert marketing, volatile products, uncontrollable expenditure.
- Although monthly data available for 10 years, data are aggregated to annual values, and only the first three years are used in estimating the forecasts.
- All forecasts being done with the FORECAST function in MS-Excel!

# Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”:  $\alpha$ ,  $\beta$  and  $\gamma$  respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.



# A model for levels, trends, and seasonalities

We want a model that captures the level ( $\ell_t$ ), trend ( $b_t$ ) and seasonality ( $s_t$ ).

**How do we combine these elements?**

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**Additively?**

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

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**Multiplicatively?**

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

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**Multiplicatively?**

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

**Perhaps a mix of both?**

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

# A model for levels, trends, and seasonalities

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**Multiplicatively?**

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

**Perhaps a mix of both?**

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

**How do the level, trend and seasonal components evolve over time?**

# ETS models

General notation    **E T S : Exponential Smoothing**



**Error   Trend   Season**

**Error:** Additive ("A") or multiplicative ("M")

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**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

# ETS models

General notation    **E T S : Exponential Smoothing**



**Error   Trend   Season**

The diagram shows three arrows pointing from the words 'Error', 'Trend', and 'Season' below to the letters 'E', 'T', and 'S' respectively in the 'ETS' part of the notation above.

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")



## ETS(A,N,N): SES with additive errors

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T$$

Measurement equation

$$y_t = \ell_{t-1} + \varepsilon_t$$

State equation

$$\ell_t = \ell_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

# ETS(A,N,N): SES with additive errors

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T$$

Measurement equation

$$y_t = \ell_{t-1} + \varepsilon_t$$

State equation

$$\ell_t = \ell_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- “innovations” or “single source of error” because equations have the same error process,  $\varepsilon_t$ .
- Measurement equation: relationship between observations and states.
- Transition/state equation(s): evolution of state(s) over time.

# ETS(M,N,N): SES with multiplicative errors

Forecast equation

$$\hat{y}_{T+h|T} = l_T$$

Measurement equation

$$y_t = l_{t-1}(1 + \varepsilon_t)$$

State equation

$$l_t = l_{t-1}(1 + \alpha\varepsilon_t)$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

# ETS(M,N,N): SES with multiplicative errors

Forecast equation

$$\hat{y}_{T+h|T} = l_T$$

Measurement equation

$$y_t = l_{t-1}(1 + \varepsilon_t)$$

State equation

$$l_t = l_{t-1}(1 + \alpha\varepsilon_t)$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

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# Holt's linear trend

## Additive errors: ETS(A,A,N)

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T + hb_T$$

Measurement equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

# Holt's linear trend

## Additive errors: ETS(A,A,N)

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T + hb_T$$

Measurement equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

## Multiplicative errors: ETS(M,A,N)

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T + hb_T$$

Measurement equation

$$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$$

State equations

$$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$$

# Example: Australian population

```
aus_economy <- global_economy |>
  filter(Country == "Australia") |>
  mutate(Pop = Population / 1e6)
fit <- aus_economy |> model(AAN = ETS(Pop))
report(fit)
```

Series: Pop

Model: ETS(A,A,N)

Smoothing parameters:

alpha = 1

beta = 0.327

Initial states:

l[0] b[0]

10.1 0.222

sigma^2: 0.0041

AIC AICc BIC

-77.0 -75.8 -66.7



# Example: Australian population

```
components(fit)
```

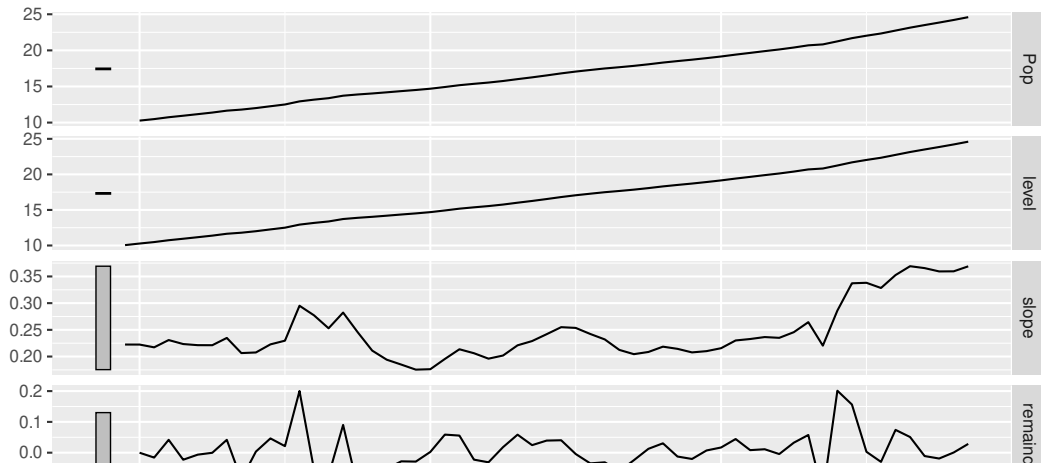
```
# A dable: 59 x 7 [1Y]
# Key:      Country, .model [1]
# :        Pop = lag(level, 1) + lag(slope, 1) + remainder
  Country   .model Year   Pop level slope remainder
  <fct>     <chr>  <dbl> <dbl> <dbl> <dbl>      <dbl>
1 Australia AAN    1959  NA    10.1 0.222  NA
2 Australia AAN    1960  10.3  10.3 0.222 -0.000145
3 Australia AAN    1961  10.5  10.5 0.217 -0.0159
4 Australia AAN    1962  10.7  10.7 0.231  0.0418
5 Australia AAN    1963  11.0  11.0 0.223 -0.0229
6 Australia AAN    1964  11.2  11.2 0.221 -0.00641
7 Australia AAN    1965  11.4  11.4 0.221 -0.000314
8 Australia AAN    1966  11.7  11.7 0.235  0.0418
9 Australia AAN    1967  11.8  11.8 0.206 -0.0869
10 Australia AAN   1968  12.0  12.0 0.208  0.00350
# i 49 more rows
```

# Example: Australian population

```
components(fit) |> autoplot()
```

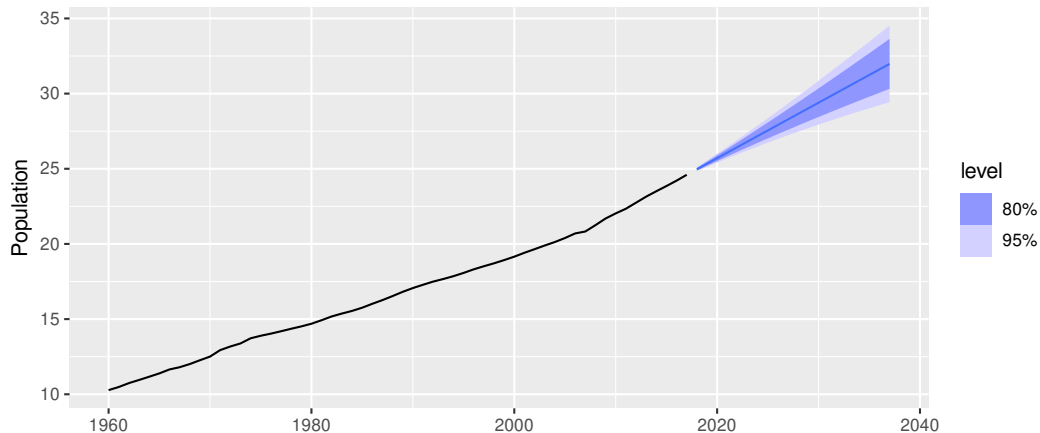
ETS(A,A,N) decomposition

Pop = lag(level, 1) + lag(slope, 1) + remainder



# Example: Australian population

```
fit |>  
  forecast(h = 20) |>  
  autoplot(aus_economy) +  
  labs(y = "Population", x = "Year")
```



# ETS(A,Ad,N): Damped trend method

## Additive errors

|                      |   |
|----------------------|---|
| Forecast equation    | $\hat{y}_{T+h T} = \ell_T + (\phi + \dots + \phi^{h-1})b_T$   |
| Measurement equation | $y_t = (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t$   |
| State equations      | $\ell_t = (\ell_{t-1} + \phi b_{t-1}) + \alpha \varepsilon_t$<br>$b_t = \phi b_{t-1} + \beta \varepsilon_t$ |

# ETS(A,Ad,N): Damped trend method

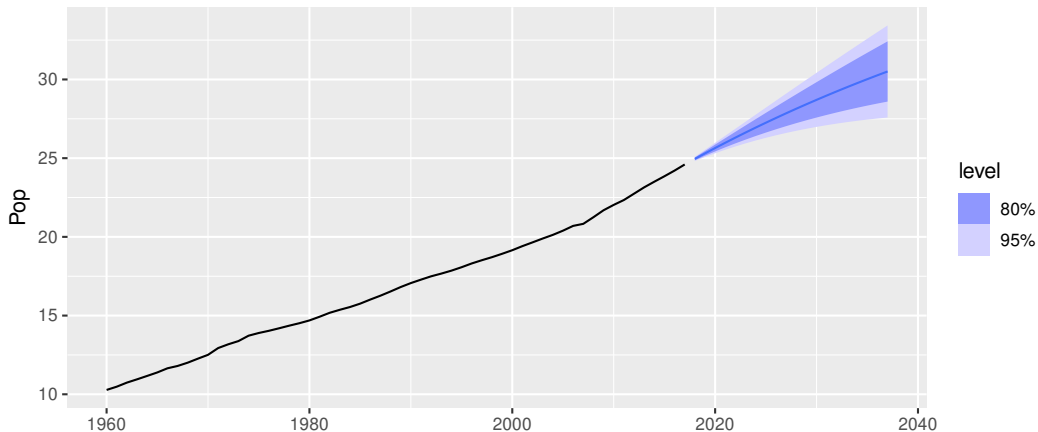
## Additive errors

|                      |   |
|----------------------|---|
| Forecast equation    | $\hat{y}_{T+h T} = \ell_T + (\phi + \dots + \phi^{h-1})b_T$   |
| Measurement equation | $y_t = (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t$   |
| State equations      | $\ell_t = (\ell_{t-1} + \phi b_{t-1}) + \alpha \varepsilon_t$<br>$b_t = \phi b_{t-1} + \beta \varepsilon_t$ |

- Damping parameter  $0 < \phi < 1$ .
- If  $\phi = 1$ , identical to Holt's linear trend.
- As  $h \rightarrow \infty$ ,  $\hat{y}_{T+h|T} \rightarrow \ell_T + \phi b_T / (1 - \phi)$ .
- Short-run forecasts trended, long-run forecasts constant.

# Example: Australian population

```
aus_economy |>  
  model(holt = ETS(Pop ~ trend("Ad"))) |>  
  forecast(h = 20) |>  
  autoplot(aus_economy)
```



# Example: National populations

```
fit <- global_economy |>
  mutate(Pop = Population / 1e6) |>
  model(ets = ETS(Pop))
fit
```

```
# A mable: 263 x 2
```

```
# Key:      Country [263]
```

|    | Country             | ets          |
|----|---------------------|--------------|
|    | <fct>               | <model>      |
| 1  | Afghanistan         | <ETS(A,A,N)> |
| 2  | Albania             | <ETS(M,A,N)> |
| 3  | Algeria             | <ETS(M,A,N)> |
| 4  | American Samoa      | <ETS(M,A,N)> |
| 5  | Andorra             | <ETS(M,A,N)> |
| 6  | Angola              | <ETS(M,A,N)> |
| 7  | Antigua and Barbuda | <ETS(M,A,N)> |
| 8  | Arab World          | <ETS(M,A,N)> |
| 9  | Argentina           | <ETS(A,A,N)> |
| 10 | Armenia             | <ETS(M,A,N)> |

```
# i 253 more rows
```

# Example: National populations

```
fit |>  
  forecast(h = 5)
```

```
# A fable: 1,315 x 5 [1Y]  
# Key:      Country, .model [263]  
  Country      .model  Year      Pop .mean  
  <fct>        <chr>   <dbl>      <dist> <dbl>  
1 Afghanistan ets     2018      N(36, 0.012) 36.4  
2 Afghanistan ets     2019      N(37, 0.059) 37.3  
3 Afghanistan ets     2020      N(38, 0.16) 38.2  
4 Afghanistan ets     2021      N(39, 0.35) 39.0  
5 Afghanistan ets     2022      N(40, 0.64) 39.9  
6 Albania      ets     2018      N(2.9, 0.00012) 2.87  
7 Albania      ets     2019      N(2.9, 6e-04) 2.87  
8 Albania      ets     2020      N(2.9, 0.0017) 2.87  
9 Albania      ets     2021      N(2.9, 0.0036) 2.86  
10 Albania     ets     2022      N(2.9, 0.0066) 2.86  
# i 1,305 more rows
```



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## Lab Session 14

Try forecasting the Australian GDP from the `global_economy` data set using an ETS model.

Experiment with the various options in the `ETS()` function to see how much the forecasts change with damped trend, or with a Box-Cox transformation. Try to develop an intuition of what each is doing to the forecasts.

[Hint: use `h=20` when forecasting, so you can clearly see the differences between the various options when plotting the forecasts.]

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## ETS(A,A,A): Holt-Winters additive method

|                      |  |
|----------------------|--|
| Forecast equation    | $\hat{y}_{t+h t} = \ell_t + hb_t + s_{t+h-m(k+1)}$   |
| Observation equation | $y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$   |
| State equations      | $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$<br>$b_t = b_{t-1} + \beta\varepsilon_t$<br>$s_t = s_{t-m} + \gamma\varepsilon_t$ |

- $k = \text{integer part of } (h - 1)/m$ .
- $\sum_i s_i \approx 0$ .
- Parameters:  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta \leq 1$ ,  $0 \leq \gamma \leq 1 - \alpha$  and  $m = \text{period of seasonality (e.g. } m = 4 \text{ for quarterly data)}$ .

## ETS(M,A,M): Holt-Winters multiplicative method

|                      |   |
|----------------------|---|
| Forecast equation    | $\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$   |
| Observation equation | $y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$  |
| State equations      | $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$<br>$b_t = b_{t-1}(1 + \beta\varepsilon_t)$<br>$s_t = s_{t-m}(1 + \gamma\varepsilon_t)$ |

- $k$  is integer part of  $(h - 1)/m$ .
- $\sum_i s_i \approx m$ .
- Parameters:  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta^* \leq 1$ ,  $0 \leq \gamma \leq 1 - \alpha$  and  $m$  = period of seasonality (e.g.  $m = 4$  for quarterly data).

# Example: Australian holiday tourism

```
holidays <- tourism |>
  filter(Purpose == "Holiday")
fit <- holidays |> model(ets = ETS(Trips))
fit
```

```
# A mable: 76 x 4
```

```
# Key:      Region, State, Purpose [76]
```

|    | Region                     | State              | Purpose | ets          |
|----|----------------------------|--------------------|---------|--------------|
|    | <chr>                      | <chr>              | <chr>   | <model>      |
| 1  | Adelaide                   | South Australia    | Holiday | <ETS(A,N,A)> |
| 2  | Adelaide Hills             | South Australia    | Holiday | <ETS(A,A,N)> |
| 3  | Alice Springs              | Northern Territory | Holiday | <ETS(M,N,A)> |
| 4  | Australia's Coral Coast    | Western Australia  | Holiday | <ETS(M,N,A)> |
| 5  | Australia's Golden Outback | Western Australia  | Holiday | <ETS(M,N,M)> |
| 6  | Australia's North West     | Western Australia  | Holiday | <ETS(A,N,A)> |
| 7  | Australia's South West     | Western Australia  | Holiday | <ETS(M,N,M)> |
| 8  | Ballarat                   | Victoria           | Holiday | <ETS(M,N,A)> |
| 9  | Barkly                     | Northern Territory | Holiday | <ETS(A,N,A)> |
| 10 | Barossa                    | South Australia    | Holiday | <ETS(A,N,N)> |

# Example: Australian holiday tourism

```
fit |>  
  filter(Region == "Snowy Mountains") |>  
  report()
```

Series: Trips

Model: ETS(M,N,A)

Smoothing parameters:

alpha = 0.157

gamma = 1e-04

Initial states:

l[0] s[0] s[-1] s[-2] s[-3]

142 -61 131 -42.2 -27.7

sigma^2: 0.0388

AIC AICc BIC

852 854 869

# Example: Australian holiday tourism

```
fit |>
  filter(Region == "Snowy Mountains") |>
  components(fit)
```

```
# A dable: 84 x 9 [1Q]
```

```
# Key:      Region, State, Purpose, .model [1]
```

```
# :      Trips = (lag(level, 1) + lag(season, 4)) * (1 + remainder)
```

|    | Region<br><chr> | State<br><chr> | Purpose<br><chr> | .model<br><chr> | Quarter<br><qtr> | Trips<br><dbl> | level<br><dbl> | season<br><dbl> | remainder<br><dbl> |
|----|-----------------|----------------|------------------|-----------------|------------------|----------------|----------------|-----------------|--------------------|
| 1  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1997 Q1          | NA             | NA             | -27.7           | NA                 |
| 2  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1997 Q2          | NA             | NA             | -42.2           | NA                 |
| 3  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1997 Q3          | NA             | NA             | 131.            | NA                 |
| 4  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1997 Q4          | NA             | 142.           | -61.0           | NA                 |
| 5  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1998 Q1          | 101.           | 140.           | -27.7           | -0.113             |
| 6  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1998 Q2          | 112.           | 142.           | -42.2           | 0.154              |
| 7  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1998 Q3          | 310.           | 148.           | 131.            | 0.137              |
| 8  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1998 Q4          | 89.8           | 148.           | -61.0           | 0.0335             |
| 9  | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1999 Q1          | 112.           | 147.           | -27.7           | -0.0687            |
| 10 | Snowy Mountai~  | New            | ~ Holiday        | ets             | 1999 Q2          | 103.           | 147.           | -42.2           | -0.0199            |

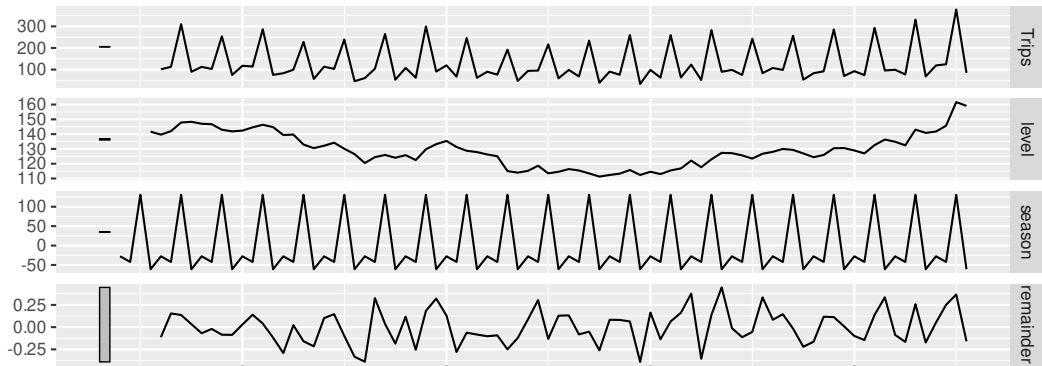


# Example: Australian holiday tourism

```
fit |>  
  filter(Region == "Snowy Mountains") |>  
  components(fit) |>  
  autoplot()
```

## ETS(M,N,A) decomposition

Trips = (lag(level, 1) + lag(season, 4)) \* (1 + remainder)



# Example: Australian holiday tourism

```
fit |> forecast()
```

```
# A fable: 608 x 7 [1Q]
```

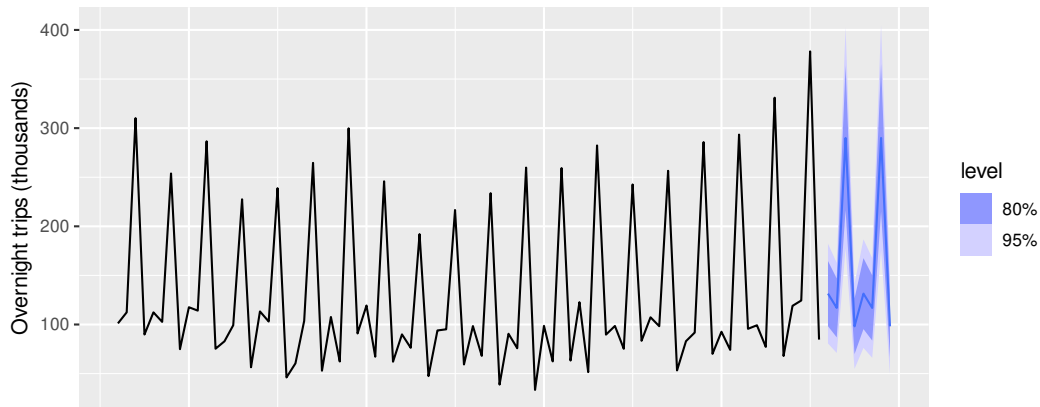
```
# Key:      Region, State, Purpose, .model [76]
```

|    | Region         | State           | Purpose | .model | Quarter | Trips       | .mean |
|----|----------------|-----------------|---------|--------|---------|-------------|-------|
|    | <chr>          | <chr>           | <chr>   | <chr>  | <qtr>   | <dist>      | <dbl> |
| 1  | Adelaide       | South Australia | Holiday | ets    | 2018 Q1 | N(210, 457) | 210.  |
| 2  | Adelaide       | South Australia | Holiday | ets    | 2018 Q2 | N(173, 473) | 173.  |
| 3  | Adelaide       | South Australia | Holiday | ets    | 2018 Q3 | N(169, 489) | 169.  |
| 4  | Adelaide       | South Australia | Holiday | ets    | 2018 Q4 | N(186, 505) | 186.  |
| 5  | Adelaide       | South Australia | Holiday | ets    | 2019 Q1 | N(210, 521) | 210.  |
| 6  | Adelaide       | South Australia | Holiday | ets    | 2019 Q2 | N(173, 537) | 173.  |
| 7  | Adelaide       | South Australia | Holiday | ets    | 2019 Q3 | N(169, 553) | 169.  |
| 8  | Adelaide       | South Australia | Holiday | ets    | 2019 Q4 | N(186, 569) | 186.  |
| 9  | Adelaide Hills | South Australia | Holiday | ets    | 2018 Q1 | N(19, 36)   | 19.4  |
| 10 | Adelaide Hills | South Australia | Holiday | ets    | 2018 Q2 | N(20, 36)   | 19.6  |

```
# i 598 more rows
```

# Example: Australian holiday tourism

```
fit |>  
  forecast() |>  
  filter(Region == "Snowy Mountains") |>  
  autoplot(holidays) +  
  labs(x = "Year", y = "Overnight trips (thousands)")
```



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# Exponential smoothing models

## Additive Error

|                 |                                  | Seasonal Component  |                     |                              |
|-----------------|----------------------------------|---------------------|---------------------|------------------------------|
|                 |                                  | N<br>(None)         | A<br>(Additive)     | M<br>(Multiplicative)        |
| Trend Component | N (None)                         | A,N,N               | A,N,A               | <del>A,N,M</del>             |
|                 | A (Additive)                     | A,A,N               | A,A,A               | <del>A,A,M</del>             |
|                 | A <sub>d</sub> (Additive damped) | A,A <sub>d</sub> ,N | A,A <sub>d</sub> ,A | <del>A,A<sub>d</sub>,M</del> |

## Multiplicative Error

|                 |                                  | Seasonal Component  |                     |                       |
|-----------------|----------------------------------|---------------------|---------------------|-----------------------|
|                 |                                  | N<br>(None)         | A<br>(Additive)     | M<br>(Multiplicative) |
| Trend Component | N (None)                         | M,N,N               | M,N,A               | M,N,M                 |
|                 | A (Additive)                     | M,A,N               | M,A,A               | M,A,M                 |
|                 | A <sub>d</sub> (Additive damped) | M,A <sub>d</sub> ,N | M,A <sub>d</sub> ,A | M,A <sub>d</sub> ,M   |

# Estimating ETS models

- Smoothing parameters  $\alpha, \beta, \gamma$  and  $\phi$ , and the initial states  $\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1}$  are estimated by maximising the “likelihood” = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

# Model selection

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## Corrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).



# Model selection

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which is the AIC corrected (for small sample bias).

## Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

## AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

## From Hyndman et al. (IJF, 2002):

- 1 Apply each model that is appropriate to the data.  
Optimize parameters and initial values using MLE.
  - 2 Select best method using AICc.
  - 3 Produce forecasts using best method.
  - 4 Obtain forecast intervals using underlying state space model.
- Method performed very well in M3 competition.
  - Used as a benchmark in the M4 competition.

# Outline

- 1 Exponential smoothing
- 2 Trend methods
- 3 Lab Session 14
- 4 Seasonal methods
- 5 ETS taxonomy
- 6 Lab Session 15**
- 7 Non-Gaussian forecast distributions

# Lab Session 15

Find an ETS model for the Gas data from `aus_production`.

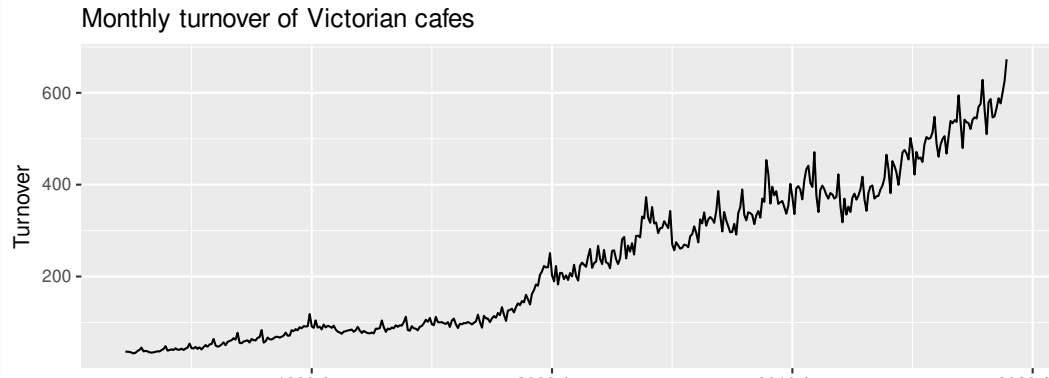
- Why is multiplicative seasonality necessary here?
- Experiment with making the trend damped. Does it improve the forecasts?

# Outline

- 1 Exponential smoothing
- 2 Trend methods
- 3 Lab Session 14
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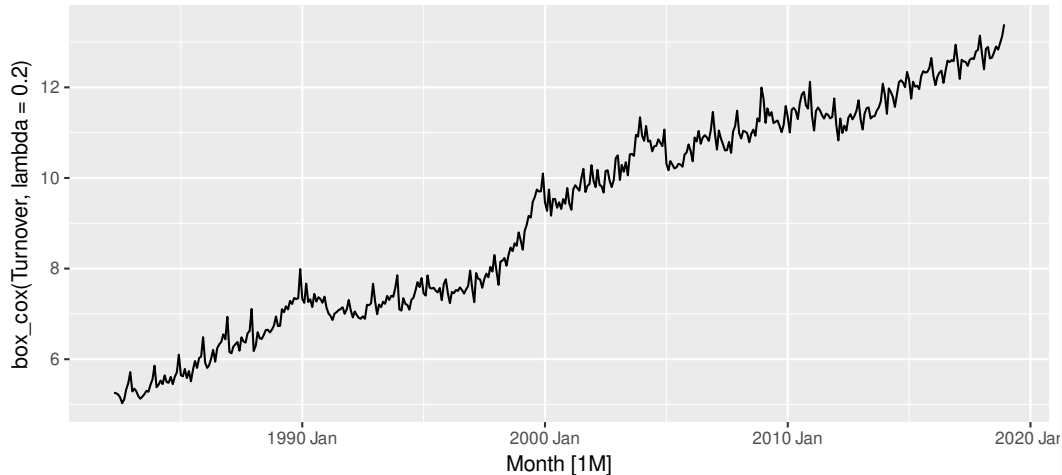
# Non-Gaussian forecast distributions

```
vic_cafe <- tsibbledata::aus_retail |>
  filter(State == "Victoria",
         Industry == "Cafes, restaurants and catering services") |>
  select(Month, Turnover)
vic_cafe |>
  autoplot(Turnover) + labs(title = "Monthly turnover of Victorian cafes")
```



# Forecasting with transformations

```
vic_cafe |> autoplot(box_cox(Turnover, lambda = 0.2))
```





# Forecasting with transformations

```
fit <- vic_cafe |>  
  model(ets = ETS(box_cox(Turnover, 0.2)))  
fit
```

```
# A mable: 1 x 1  
      ets  
  <model>  
1 <ETS(A,A,A)>
```

```
(fc <- fit |> forecast(h = "3 years"))
```

```
# A fable: 36 x 4 [1M]  
# Key:      .model [1]  
  .model    Month      Turnover .mean  
  <chr>     <mth>      <dist> <dbl>  
1 ets      2019 Jan   t(N(13, 0.02)) 608.  
2 ets      2019 Feb   t(N(13, 0.028)) 563.  
3 ets      2019 Mar   t(N(13, 0.036)) 629.  
4 ets      2019 Apr   t(N(13, 0.044)) 615.  
5 ets      2019 May   t(N(13, 0.052)) 613.  
6 ets      2019 Jun   t(N(13, 0.061)) 593.  
7 ets      2019 Jul   t(N(13, 0.069)) 624.
```

# Forecasting with transformations

```
fit <- vic_cafe |>
  model(ets = ETS(box_cox(Turnover, 0.2)))
fit
```

```
# A mable: 1 x 1
      ets
  <model>
1 <ETS(A,A,A)>
```

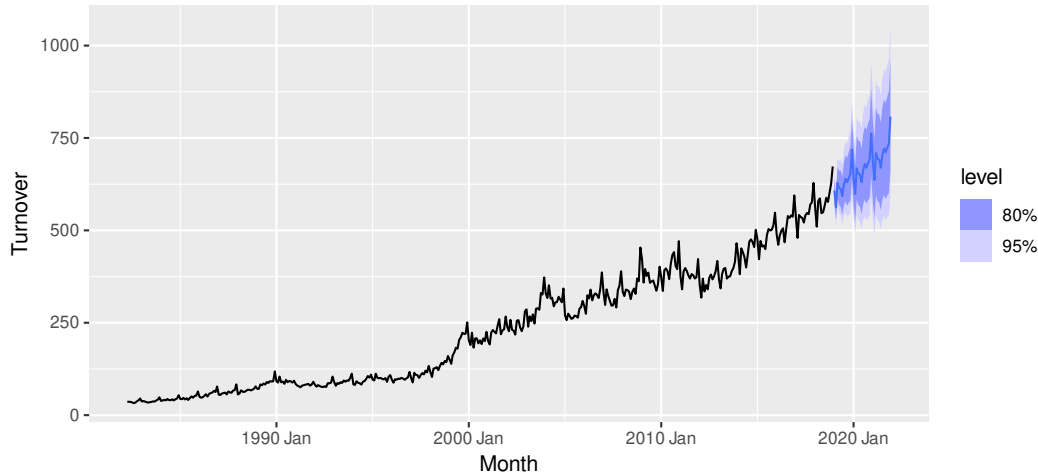
```
(fc <- fit |> forecast(h = "3 years"))
```

```
# A fable: 36 x 4 [1M]
# Key:      .model [1]
      .model      Month      Turnover .mean
   <chr>      <mth>      <dist> <dbl>
1 ets    2019 Jan  t(N(13, 0.02))  608.
2 ets    2019 Feb  t(N(13, 0.028))  563.
3 ets    2019 Mar  t(N(13, 0.036))  629.
4 ets    2019 Apr  t(N(13, 0.044))  615.
5 ets    2019 May  t(N(13, 0.052))  613.
6 ets    2019 Jun  t(N(13, 0.061))  593.
7 ets    2019 Jul  t(N(13, 0.069))  624.
```

- $t(N)$  denotes a transformed normal distribution.
- back-transformation and bias adjustment is done automatically.

# Forecasting with transformations

```
fc |> autoplot(vic_cafe)
```



# Bootstrapped forecast distributions

```
sim <- fit |> generate(h = "3 years", times = 5, bootstrap = TRUE)
sim
```

```
# A tsibble: 180 x 5 [1M]
```

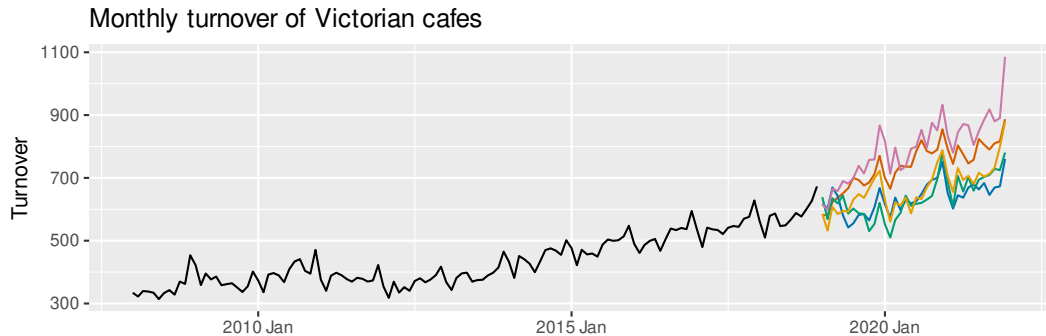
```
# Key:           .model, .rep [5]
```

|    | .model | .rep  | Month    | .innov  | .sim  |
|----|--------|-------|----------|---------|-------|
|    | <chr>  | <chr> | <mth>    | <dbl>   | <dbl> |
| 1  | ets    | 1     | 2019 Jan | -0.156  | 582.  |
| 2  | ets    | 1     | 2019 Feb | 0.215   | 581.  |
| 3  | ets    | 1     | 2019 Mar | -0.0682 | 623.  |
| 4  | ets    | 1     | 2019 Apr | 0.141   | 637.  |
| 5  | ets    | 1     | 2019 May | 0.139   | 650.  |
| 6  | ets    | 1     | 2019 Jun | 0.264   | 667.  |
| 7  | ets    | 1     | 2019 Jul | 0.0903  | 700.  |
| 8  | ets    | 1     | 2019 Aug | -0.0924 | 693.  |
| 9  | ets    | 1     | 2019 Sep | -0.0752 | 676.  |
| 10 | ets    | 1     | 2019 Oct | -0.0403 | 686.  |

```
# i 170 more rows
```

# Bootstrapped forecast distributions

```
vic_cafe |>  
  filter(year(Month) >= 2008) |>  
  ggplot(aes(x = Month)) +  
  geom_line(aes(y = Turnover)) +  
  geom_line(aes(y = .sim, colour = as.factor(.rep)), data = sim) +  
  labs(title = "Monthly turnover of Victorian cafes") +  
  guides(col = FALSE)
```



# Bootstrapped forecast distributions

```
fc <- fit |> forecast(h = "3 years", bootstrap = TRUE)
fc
```

```
# A fable: 36 x 4 [1M]
# Key:      .model [1]
  .model      Month      Turnover .mean
  <chr>      <mth>      <dist> <dbl>
1 ets       2019 Jan sample[5000] 608.
2 ets       2019 Feb sample[5000] 563.
3 ets       2019 Mar sample[5000] 629.
4 ets       2019 Apr sample[5000] 615.
5 ets       2019 May sample[5000] 613.
6 ets       2019 Jun sample[5000] 593.
7 ets       2019 Jul sample[5000] 624.
8 ets       2019 Aug sample[5000] 640.
9 ets       2019 Sep sample[5000] 631.
10 ets      2019 Oct sample[5000] 642.
# i 26 more rows
```

# Bootstrapped forecast distributions

```
fc |> autoplot(vic_cafe) +  
  labs(title = "Monthly turnover of Victorian cafes")
```

