# CPSC-354 Report

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**Abstract**

## Contents



# 1 Introduction

# 2 Week by Week

## 2.1 Week 1: HW1

The $MU$ puzzle is a puzzle created by Douglas Hofstadter. It consists of four rules that can be applied to a string $MI$.

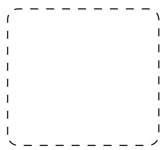
1. $xI \rightarrow xIU$
2. $Mx \rightarrow Mxx$
3. $xIIIy \rightarrow xUy$
4. $xUUy \rightarrow xy$

When first approaching this puzzle, the first strategy that came to mind was to take advantage of rule number 2 to keep duplicating the I's until there is a multiple of three, then using rules 3 and 4 to get rid of the I's and leave a remaining U.

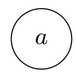
The issue with this is that $2^n \bmod 3$ will never equal 0, it infinitely cycles between equaling 1 and 2, and without being able to get rid of all the I's, which would require them being a multiple of 3, you will never be able to get MU.

Thus, the puzzle is not solvable.

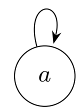
## 2.2 Week 2: HW2



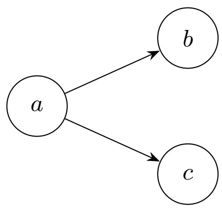
1. $A = \varnothing,\ R = \varnothing$



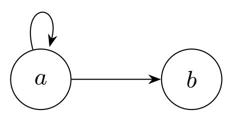

2. $A = \{a\},\ R = \varnothing$



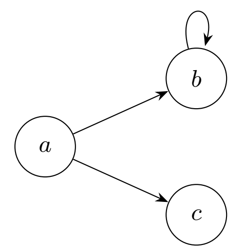

3. $A = \{a\},\ R = \{(a, a)\}$



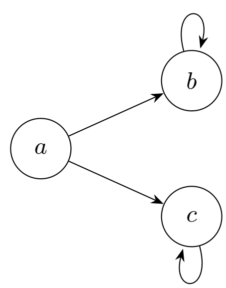

4. $A = \{a, b, c\},\quad R = \{(a, b), (a, c)\}$




5. $A = \{a, b\},\quad R = \{(a, a), (a, b)\}$




6. $A = \{a, b, c\},\quad R = \{(a, b), (b, b), (a, c)\}$




7. $A = \{a, b, c\},\quad R = \{(a, b), (b, b), (a, c), (c, c)\}$

| # | Terminating | Confluent | Unique NFs |
|---|---|---|---|
| 1 | Yes | Yes | Yes |
| 2 | Yes | Yes | Yes |
| 3 | No | Yes | No |
| 4 | Yes | No | No |
| 5 | No | Yes | Yes |
| 6 | No | No | No |
| 7 | No | No | No |

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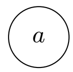
**Confluent True, Terminating True, Unique NFs True**



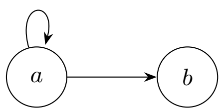

$A = \{a\},\ R = \varnothing$

**Confluent True, Terminating True, Unique NFs False**

*impossible*

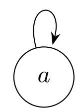
(no ARS exists)

**Confluent True, Terminating False, Unique NFs True**




$A = \{a, b\},\ R = \{(a, a), (a, b)\}$

**Confluent True, Terminating False, Unique NFs False**



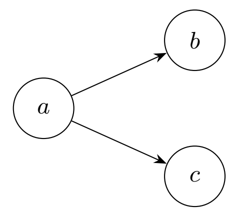

$A = \{a\},\ R = \{(a, a)\}$

**Confluent False, Terminating True, Unique NFs True**

*impossible*

(no ARS exists)

**Confluent False, Terminating True, Unique NFs False**



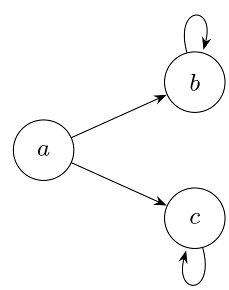

$A = \{a, b, c\},\ R = \{(a, b), (a, c)\}$

**Confluent False, Terminating False, Unique NFs True**

*impossible*

(no ARS exists)

**Confluent False, Terminating False, Unique NFs False**




$A = \{a, b, c\},\ R = \{(a, b), (a, c), (b, b), (c, c)\}$

## 2.3 Week 3: HW3

### 2.3.1 Exercise 5

Consider rewrite rules:

$$\begin{aligned} ab &\to ba \\ ba &\to ab \\ aa &\to \\ b &\to \end{aligned}$$

**Example Reductions**

**Reducing `abba`:**

$$\begin{aligned} abba &\to baba \quad (\text{using } ab \to ba) \\ baba &\to bbaa \quad (\text{using } ba \to ab) \\ bbaa &\to baa \quad (\text{using } b \to \varepsilon) \\ baa &\to aba \quad (\text{using } ba \to ab) \\ aba &\to baa \quad (\text{using } ab \to ba) \end{aligned}$$

There is an infinite loop between `aba` and `baa`.

**Reducing `bababa`:**

$$\begin{aligned} bababa &\to ababab \quad (\text{using } ba \to ab) \\ ababab &\to baabab \quad (\text{using } ab \to ba) \\ baabab &\to ababab \quad (\text{using } ba \to ab) \end{aligned}$$

This is an infinite loop between `ababab` and `baabab`.

**Why the ARS is not terminating** The ARS is not terminating because the rules $ab \to ba$ and $ba \to ab$ create infinite cycles. These rules allow us to swap adjacent $a$ and $b$ characters indefinitely, leading to non-terminating reduction sequences.

**Non-equivalent strings** Two strings that are not equivalent: `a` and `aa`.

The string `a` cannot be reduced further, while `aa` reduces to nothing using the rule $aa \to$. Since $nothing \neq a$, these strings are in different equivalence classes.

**Equivalence classes** The equivalence relation $\leftrightarrow^*$ has infinitely many equivalence classes. Each equivalence class can be characterized by the number of $a$'s modulo 2 and the number of $b$'s modulo 1.

The equivalence classes are:

- $[\varepsilon]$: strings with even number of $a$'s and no $b$'s
- $[a]$: strings with odd number of $a$'s and no $b$'s
- $[b]$: strings with any number of $a$'s and at least one $b$

The normal forms are: $\varepsilon$, $a$, and $b$.

**Modifying the ARS to be terminating** To make the ARS terminating without changing equivalence classes, we can remove the symmetric rules and keep only one direction:

$$\begin{aligned} ba &\to ab \\ aa &\to \varepsilon \\ b &\to \varepsilon \end{aligned}$$

This eliminates the infinite cycles while preserving the same equivalence relation.

### 2.3.2 Semantic question

**Parity of $a$'s:** "Does this string contain an odd number of $a$'s?"
Answer: Yes if the normal form is $a$, No if the normal form is $\varepsilon$.

**Exercise 5b**

Consider rewrite rules:

$$\begin{aligned} ab &\to ba \\ ba &\to ab \\ aa &\to a \\ b &\to \end{aligned}$$

**Reducing `abba`:**

$$\begin{aligned} abba &\to baba \\ baba &\to abba \end{aligned}$$

**Reducing `bababa`:**

$$\begin{aligned} bababa &\to ababab \\ ababab &\to bababa \end{aligned}$$

**Why not terminating.** The symmetric swaps $ab \leftrightarrow ba$ allow infinite rewriting.

**Non-equivalent strings.** $a$ and $\varepsilon$ are not equivalent.

**Equivalence classes.** Exactly two: $[\varepsilon]$ (no $a$'s; all $b$'s delete) and $[a]$ (at least one $a$; since $aa \sim a$). Normal forms (under a terminating orientation): $\varepsilon$ and $a$.

**Terminating variant.**

$$\begin{aligned} ba &\to ab \\ aa &\to a \\ b &\to \end{aligned}$$

This gives a complete semantics to the ARS: it computes the invariant for any input string.

## 2.4 Week 4: HW4

### 2.4.1 HW4.1: Termination Proof for GCD

Consider the following algorithm:

```
while b != 0:
    temp = b
    b = a mod b
    a = temp
return a
```

**Assume:** Work over integers with Euclidean division: inputs $a \in \mathbb{Z}$, $b \in \mathbb{N}$; if $b \neq 0$ then $0 \leq a \bmod b < b$.

**Model:** States $A = \mathbb{Z} \times \mathbb{N}$. One step $(a, b) \to (a', b')$ is one loop iteration.

**Measure:** $\phi : A \to \mathbb{N},\ \phi(a, b) = b$.

**Show:** $(a, b) \to (a', b') \Rightarrow \phi(a', b') < \phi(a, b)$.

If $b \neq 0$, the update sets $b' = a \bmod b$ with $0 \leq b' < b$; hence $\phi(a', b') = b' < b = \phi(a, b)$.

$\phi$ strictly decreases in $\mathbb{N}$, so there is no infinite $\to$-chain; eventually $b = 0$ and the loop stops. Thus the algorithm terminates under the stated conditions.

### 2.4.2 HW4.2: Termination Proof for Merge Sort

Consider the following fragment of an implementation of merge sort:

```
function merge_sort(arr, left, right):
    if left >= right:
        return
    mid = (left + right) / 2
    merge_sort(arr, left, mid)
    merge_sort(arr, mid+1, right)
    merge(arr, left, mid, right)
```

Prove that

$\phi(\texttt{left}, \texttt{right}) = \texttt{right} - \texttt{left} + 1$

is a measure function for merge_sort.

**Show:** For

```
merge_sort(arr, left, right):
  if left >= right: return
    mid = floor((left + right)/2)
  merge_sort(arr, left, mid)
  merge_sort(arr, mid+1, right)
  merge(arr, left, mid, right)
```

the function $\phi(\text{left}, \text{right}) = \text{right} - \text{left} + 1$ is a strictly decreasing measure, hence `merge_sort` terminates.

**Assume:** `left,right`$\in \mathbb{Z}$ with $0 \leq \text{left} \leq \text{right} < |arr|$. Division for `mid` is integer. If `left < right`, then $\text{left} \leq \text{mid} < \text{right}$. `merge` makes no recursive calls.

**Model:** States are intervals $A = \{(l, r) \in \mathbb{Z}^2 \mid l \leq r\}$. A "step" is a recursive edge from $(l, r)$ (with $l < r$) to each child $(l, \text{mid})$ and $(\text{mid} + 1, r)$.

**Measure:** $\phi : A \to \mathbb{N}, \quad \phi(l, r) = r - l + 1.$

Let $l < r$ and $\text{mid} = \lfloor (l + r)/2 \rfloor$ so $l \leq \text{mid} < r$.

First child: $\phi(l, \text{mid}) = \text{mid} - l + 1 \leq (r - 1) - l + 1 = r - l = \phi(l, r) - 1 < \phi(l, r)$.

Second child: $\phi(\text{mid} + 1, r) = r - \text{mid} \leq r - l = \phi(l, r) - 1 < \phi(l, r)$.

Every recursive edge strictly decreases $\phi$ in $\mathbb{N}$, which is well-founded. Thus no infinite recursion is possible; the calls bottom out at states with $\phi \in \{0, 1\}$ (i.e., `left` $\geq$ `right`), where the function returns. Therefore $\phi(\text{left}, \text{right}) = \text{right} - \text{left} + 1$ is a valid measure and `merge_sort` terminates under the stated conditions.

# 3 Essay

# 4 Evidence of Participation

# 5 Conclusion

## References


[BLA] Author, Title, Publisher, Year.