A Bi-Lipschitz Coorbit Embedding of Point Clouds

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Introduction

Point clouds – unordered collections of vectors – may be identified with matrices up to a permutation of columns. In other words, a point cloud is the orbit of a matrix under an action of the symmetric group. Point clouds play an increasingly important role in autonomous driving, robotics, and 3D computer vision.

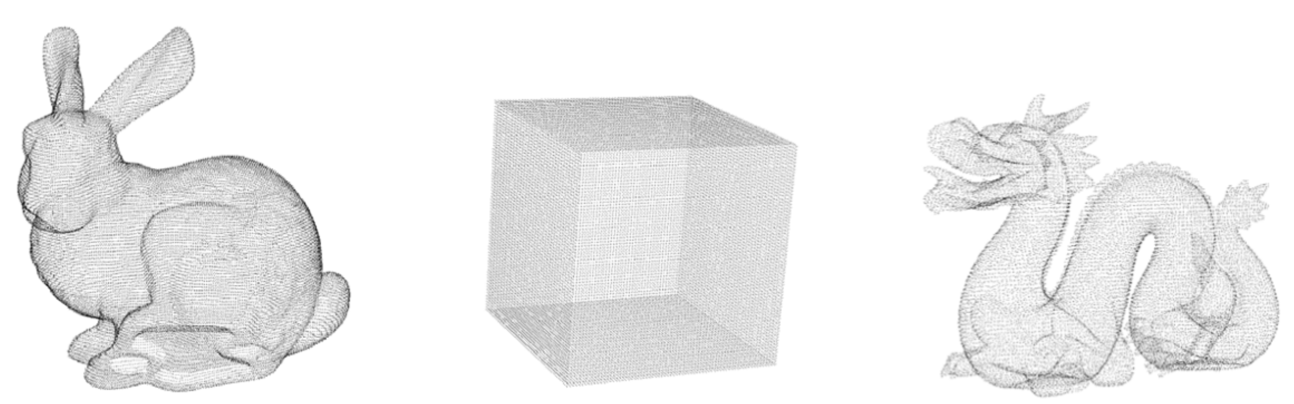


Figure 1: Example Point Clouds [1].

Motivated by the analysis of point cloud data, we construct a bi-Lipschitz embedding of the orbit space of point clouds $\mathbb{R}^{d\times n}/S_n$ into a Euclidean space of small dimension.

Background

Let G be a finite group acting orthogonally on inner product space V by linear representation $\rho : G \to \mathrm{O}(V)$. We construct a metric on the **orbit space** $V/G = \{G \cdot v : v \in V\}$ by defining

$$d_{V/G}(G\cdot x, G\cdot y) = \min\{d_V(g\cdot x, h\cdot y) : g, h \in G\}.$$

This metric is costly to compute, so to analyze data in an orbit space, we would like to embed it into another metric space. A map $f : X \to Y$ between metric spaces is called **bi-Lipschitz** if there exist positive constants α, β such that, for all $x_1, x_2 \in X$

$$\alpha \cdot d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le \beta \cdot d_X(x_1, x_2).$$

Notably, the existence of a positive lower Lipschitz constant α implies that f is injective. If f is bi-Lipschitz, we quantify the extent to which f distorts distances in X by looking at the **distortion**:

$$\mathrm{dist}(f) = \inf\left\{\frac{\beta}{\alpha} : \alpha, \beta \text{ satisfy the above}\right\}.$$

We desire embeddings with small distortion since these maps best preserve the distance in the orbit space.

Our Goal is to construct a bi-Lipschitz embedding $\Phi : \mathbb{R}^{d\times n}/S_n \to \mathbb{R}^m$ of point clouds which has small distortion, and small embedding dimension m.

Invariant Theory

The functions on V/G are precisely those on V which are constant on the orbits. Hence, a natural starting place for our investigation is constructing an invariant $f : V \to \mathbb{R}^m$ which induces an injective map $V/G \to \mathbb{R}^m$ by $G \cdot v \mapsto f(v)$.

The polynomial invariants form a subring $\mathbb{R}[V]^G \subseteq \mathbb{R}[V]$. Given a finite set, say $f_1, \cdots, f_m$, which generate $\mathbb{R}[V]^G$, we can construct an invariant $\Phi : V \to \mathbb{R}^m$ by $\Phi(v) = (f_1(v), \cdots, f_m(v))$ with the property that $\Phi(v) = \Phi(u)$ if and only if $G \cdot v = G \cdot u$. This yields an embedding of V/G, but Φ will rarely be bi-Lipschitz.

Theorem [4, Thrm. 21]. If $x \in V$ is fixed by some non-identity element of G, and $f : V \to \mathbb{R}^m$ is differentiable at x, then the induced $f : V/G \to \mathbb{R}^m$ is not bi-Lipschitz.

Therefore, to construct a bi-Lipschitz embedding of V/G, we desire a family of invariants which are not differentiable everywhere. One example is obtained by coorbit embedding, recently introduced for this purpose in [2, 3].

Coorbit Embedding

If G is a finite group of order N acting orthogonally on $V = \mathbb{R}^d$, we construct a coorbit embedding as follows. Let $[n] = \{1, 2, \cdots, n\}$, and choose a collection of **templates** $\zeta = (z_1, \cdots, z_k) \in V^k$ and an index set $S \subseteq [k] \times [N]$. For $(i, j) \in S$, let $\phi_{ij} : V \to \mathbb{R}$ be the map which sends $v \in V$ to the jth coordinate of the vector $\mathtt{sort}(\langle v, \sigma\cdot z_i\rangle : \sigma \in G)$ where $\mathtt{sort}$ puts the components of a vector in non-increasing order. The **coorbit embedding** associated to parameters ζ and S is the map

$$\Phi_\zeta : V \to \mathbb{R}^{|S|}, \quad \Phi_\zeta(v) = (\phi_{ij}(v) : (i,j) \in S).$$

Since G acts orthogonally we have $\langle \pi\cdot v, \sigma\cdot z_i\rangle = \langle v, \pi^{-1}\sigma\cdot z_i\rangle$, and therefore the action of $\pi \in G$ on the input simply permutes the vector to be sorted. It follows that the component map ϕ_{ij} is G-invariant, and hence, the coorbit embedding Φ_ζ is too. Therefore, we may regard Φ_ζ as a function on V/G.

Theorem [2, Thrm. 2.1]. If the coorbit embedding $\Phi_\zeta : V/G \to \mathbb{R}^m$ is injective, then Φ_ζ is bi-Lipschitz.

This property reduces the problem of constructing a bi-Lipschitz embedding to finding parameters which make Φ_ζ injective. For finite groups acting orthogonally, it is known that such parameters exist (see [5, Thrm. 12] for example) and in this case, we may take $m = 2nd$. However, relatively little is known about how to determine these parameters.

Point Cloud Embedding

We specify a coorbit embedding of point clouds, that is V/S_n where $V = \mathbb{R}^{d\times n}$ and the symmetric group acts by column permutation. We then provide sufficient conditions on the templates ζ which make this map bi-Lipschitz.

Definition. For z in $\mathbb{R}^d$, define $\phi_z : V \to \mathbb{R}^n$ by $\phi_z(M) = \mathtt{sort}(M^T z)$. For a collection $\zeta = (z_1, \cdots, z_k)$ in $(\mathbb{R}^d)^k$, define

$$\Phi_\zeta : V \to \mathbb{R}^{nk}, \quad \Phi_\zeta(M) = (\phi_{z_1}(M), \cdots, \phi_{z_k}(M)).$$

This map is a coorbit embedding. To see this, let z_i' be the matrix in V whose first and only non-zero column is z_i and notice that, for $M \in V$ with columns $m_1, \cdots, m_n$, we have $\langle M, \sigma\cdot z_i'\rangle = \langle m_{\sigma(i)}, z_i\rangle$. Since the stabilizer of z_i' in S_n is isomorphic to S_{n-1}, we see that Φ_ζ is the coorbit embedding associated to parameters $\zeta = (z_1, \cdots, z_k)$ and $S = [k] \times \{(n-1)!, 2(n-1)!, \cdots, n!\}$. Hence, if we can find templates for which Φ_ζ is injective on the orbit space, then we will have achieved our goal. To begin, let $R = \mathbb{R}[x_1, \cdots, x_d]$, and for $m \in \mathbb{R}^d$, let $m^* = \sum_{i=1}^d m_i x_i \in R$. To a matrix $M \in V$ with columns $m_1, \cdots, m_n$, associate the polynomial

$$P_M = \prod_{j=1}^{n}(t - m_j^*) \in R[t].$$

Two matrices lie in the same S_n-orbit iff they have the same associated polynomial. Expanding P_M and letting e_j denote the jth elementary symmetric polynomial in n indeterminates, we find that $S_n \cdot M = S_n \cdot W$ exactly when

$$e_j(m_1^*, \cdots, m_n^*) = e_j(w_1^*, \cdots, w_n^*) \quad \text{for all } 1 \le j \le n.$$

Notice that $\phi_z(M) = \phi_z(W)$ iff the polynomials above agree at $z \in \mathbb{R}^d$ for all $j \in [n]$, moreover, these polynomials are homogeneous of degree j, belonging to $R_{(j)}$. We say a set $X \subseteq \mathbb{R}^d$ is **unisolvent** for $R_{(j)}$ if the zero polynomial is the only element of $R_{(j)}$ which vanishes at all $x \in X$.

Theorem. If ζ is unisolvent for $R_{(j)}$ for all $1 \le j \le n$, then Φ_ζ is injective on the orbit space.

A non-zero bivariate homogeneous polynomial of degree up to n may have at most n distinct roots on the projective line. Therefore, any collection of $n+1$ vectors in $\mathbb{R}^2$ which lie on distinct lines through the origin are unisolvent for $\mathbb{R}[x_1, x_2]_{(j)}$ when $1 \le j \le n$. To generalize beyond the case $d = 2$, define $\psi : \mathbb{R}^2 \to \mathbb{R}^d$ by

$$(x, y) \mapsto (\omega_1, \cdots, \omega_d) \quad \text{where } \omega_i = \binom{d-1}{i-1}^{1/2} x^{d-i} y^{i-1}.$$

Coefficients are chosen so that $||\psi(v)|| = ||v||^{(d-1)}$ and in particular, unit vectors in $\mathbb{R}^2$ are sent to unit vectors in $\mathbb{R}^d$. To continue, let $p_j = e_j(m_1^*, \cdots, m_n^*)$, and note that evaluating p_j at $\psi(z)$ is equivalent to evaluating $\eta(p_j)$ at z where $\eta : R \to \mathbb{R}[x, y]$ is the ring homomorphism obtained by substituting $x_i \mapsto \omega_i$. Homomorphism η is injective on linear forms, so $P_M = P_W$ if and only if

$$\prod_{j=1}^{n}(t - \eta(m_j^*)) = \prod_{j=1}^{n}(t - \eta(w_j^*)).$$

Now, $\eta(p_j) = e_j(\eta(m_1^*), \cdots, \eta(m_n^*))$ is bivariate and homogeneous of degree $j(d-1)$, so it is determined up to being homogeneous of degree $\le n$ by any collection of $n(d-1)+1$ vectors in $\mathbb{R}^2$ corresponding to distinct projective points.

Theorem. Let $k = n(d-1)+1$. If $z_1, \cdots, z_k \in \mathbb{R}^2$ lie on distinct lines through the origin, then $\zeta = (\psi(z_1), \cdots, \psi(z_k))$ makes Φ_ζ bi-Lipschitz.

Results and Analysis

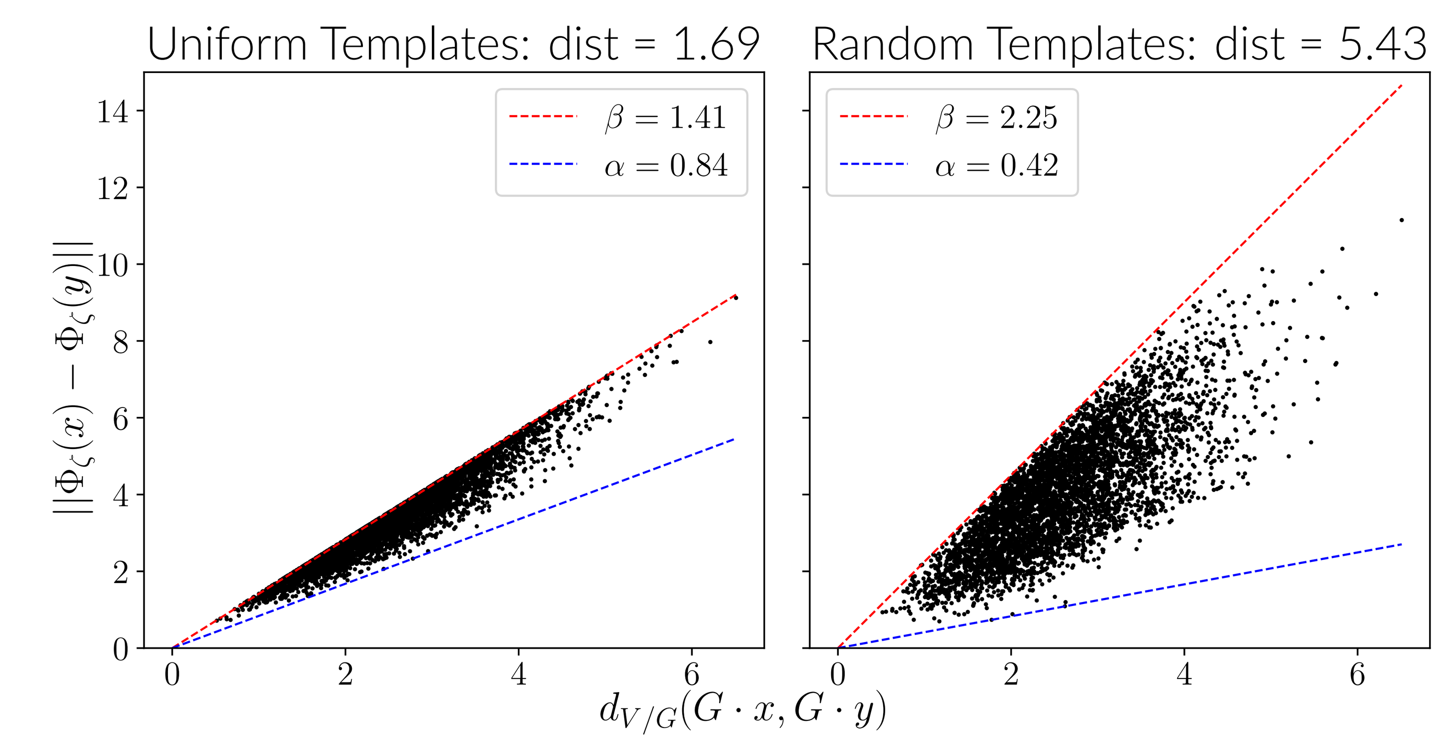
For applications, we would like the dimension of the co-domain to be small. In addition, we must be able to evaluate our map efficiently.

Complexity. Our embedding requires $k = n(d-1)+1$ templates and embeds into $\mathbb{R}^{kn}$. Hence, the embedding dimension is $m = n^2(d-1)+n = O(dn^2)$.

For fixed d, the complexity of $\mathtt{sort}(M^T z)$ is dominated by the sorting step, which can be performed in $O(n \log n)$ time. It follows that the time complexity of Φ_ζ is $kO(n\log n) = O(n^2 \log n)$ for constant d.

Distortion. The distortion of Φ_ζ depends on our choice of ζ. We would like to know, how can we choose ζ which achieves small distortion in practice?

One method is to choose $k = n(d-1)+1$ uniformly spaced vectors from the upper half of the unit circle in $\mathbb{R}^2$ and apply ψ to each to obtain ζ. To evaluate this method, we performed a numerical experiment comparing the distortion attained with uniform templates to that attained with random templates.



Future Work

We would like to find an upper bound on the distortion of Φ_ζ. Recent developments [7] may provide the right tools for this analysis. In addition, it would be useful to study the distortion as n and d vary.

Finally, we would like to apply the techniques introduced here to determine injective templates for other coorbit embeddings. Progress on this will likely factor through [6, Thrm. 3.9.13].

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