Linear Algebra

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1 Vector Spaces

A vector space (V) is a collection of objects called vectors, which may be added together and multiplied by numbers, called scalars in this context.

2 Trace

$$tr : \mathbb{R}^{n\times n} \to \mathbb{R}$$
$$tr(A) = \sum_{i=1}^{n} A_{ii}$$

- $tr(A) = tr(A^T)$
- $tr(A + B) = tr(A) + tr(B)$
- $tr(AB) = tr(BA)$

3 Norms

A vector norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ with:

- $f(x) \geq 0$ and $f(x) = 0 \Leftrightarrow x = 0$
- $f(ax) = |a|f(x)$ for $a \in \mathbb{R}$
- $f(x + y) \leq f(x) + f(y)$

Norms of vectors:

- l_2:

$$||x||_2 = \sqrt{x^T x} = \sqrt{\sum x_i^2}$$

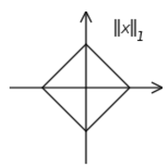
- l_1:

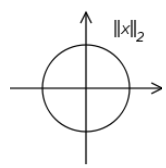
$$||x||_1 = \sum |x_i|$$

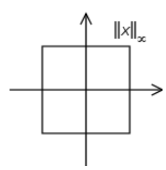
- l_∞:

$$||x||_\infty = \max(|x_i|)$$

Geometric interpretation:







Norms of Matrix:

$$||A||_p = \sup_{x \neq 0} \frac{||A\vec{x}||_p}{||x||_p}$$

- $||A_{m\times n}||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$, which is simply the maximum absolute column sum of the matrix.
- $||A_{m\times n}||_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$, which is simply the maximum absolute row sum of the matrix.
- $||A_{m\times n}||_2 = \sqrt{\lambda_{max}(A^*A)} = \sigma_{max}(A)$ (spectral norm)
- $||A_{m\times n}||_2 \leq (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2)^{1/2} = ||A||_F$ (Frobenious norm, $l_{2,2}$)

Generally, the subordinate matrix norm on $K_{m\times n}$ induced by $||\cdot||_\alpha$ on K_n, and $||\cdot||_\beta$ on K_m as:

$$||A||_{\alpha,\beta} = \max_{x\neq 0} \frac{||A\vec{x}||_\beta}{||\vec{x}||_\alpha}$$

4 The Matrix Inverse

$$AA^{-1} = I$$

$$A^{-1} \text{ exists } \Leftrightarrow Ax \neq 0 \text{ for all } x \neq 0$$

Properties:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

5 Linear Independence and Rank

$$\text{iff } \alpha_i = 0, \sum_i \alpha_i \vec{x}_i = 0 \Rightarrow \text{ linear independence}$$

RREF (Reduced Row Echlon Form) to get rank of a matrix.
column rank = row rank

6 Orthogonality

$$\vec{x}^T \vec{y} = 0$$

Orthonormal:$||\vec{x}||_2 = 1$
A matrix is orthogonal if all it's columns are orthonormal: $U^T U = I$, and its column vectors are linearly independent.

7 Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda \vec{x}$$

$$det(\lambda I - A) = 0$$

8 Diagonalization

For all eigenvectors and eigenvalues, construct:

$$AX = X\Lambda \Rightarrow A = X\lambda X^{-1}$$

If X is invertible, then A is diagonalizable.
Properties of eigenvectors and eigenvalues:

- $tr(A) = \sum_i \lambda_i$
- $det(A) = \prod_i \lambda_i$
- $rank(A) =$ number of non-zero eigenvalues.
- Eigenvalues of A^{-1} are $1/\lambda_i$, and the eigenvectors keep same.