A unifying framework for several cutting plane methods for semidefinite programming *

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Abstract. Cutting plane methods provide the means to solve large scale semidefinite programs (SDP) cheaply and quickly. They can also conceivably be employed for the purposes of re-optimization after branching, or the addition of cutting planes. We give a survey of various cutting plane approaches for SDP in this paper. These cutting plane approaches arise from various perspectives, and include techniques based on interior point cutting plane approaches, non-differentiable optimization, and finally an approach which mimics the simplex method for linear programming (LP).

We present an accessible introduction to various cutting plane approaches that have appeared in the literature. We place these methods in a unifying framework which illustrates how each approach arises as a natural enhancement of a primordial LP cutting plane scheme based on a semi-infinite formulation of the SDP.

Keywords: semidefinite programming, non-differentiable optimization, interior point cutting plane methods, active set approaches

1. Introduction

Semidefinite Programming (SDP) has been one of the most exciting and active areas in optimization recently. Some excellent references for SDP include the survey papers by Todd (2001) and Vandenberghe and Boyd (1996), the SDP handbook edited by Wolkowicz et al. (2000), and the web site maintained by Helmberg (1996). The tremendous activity in SDP was spurred by the discovery of efficient interior point algorithms for solving SDP, and its important applications in control, in developing approximation algorithms for combinatorial optimization problems, finance, and statistics. However these applications require effective techniques for solving large SDPs quickly. Although interior point algorithms are a great theoretical tool, they are fairly limited in the size of problems they can handle. Another drawback of interior

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point methods is that no good warm start techniques are available for re-optimization, after branching, or the addition of cutting planes. We discuss cutting plane approaches for SDP in this paper, which address these shortcomings. Our aim is to provide a unifying framework for the cutting plane approaches for SDP which have appeared in the literature.

Consider the semidefinite programming problem

$$\begin{array}{lrcll} \min & C \bullet X & & & \\ \text{subject to} & \mathcal{A}(X) & = & b & (SDP) \\ & X & \succeq & 0, & \end{array}$$

with dual

$$\begin{array}{lrcll} \max & b^T y & & & \\ \text{subject to} & \mathcal{A}^T y + S & = & C & (SDD) \\ & S & \succeq & 0 & \end{array}$$

where $X, S \in \mathcal{S}^n$, the space of real symmetric $n \times n$ matrices. We define

$$C \bullet X \;=\; \text{trace}(C^T X) \;=\; \textstyle\sum_{i,j=1}^{n} C_{ij} X_{ij}$$

where $\mathcal{A} : \mathcal{S}^n \to I\!R^m$ and $\mathcal{A}^T : I\!R^m \to \mathcal{S}^n$ are of the form

$$\mathcal{A}(X) \;=\; \begin{bmatrix} A_1 \bullet X \\ \vdots \\ A_m \bullet X \end{bmatrix} \text{ and } \mathcal{A}^T y \;=\; \textstyle\sum_{i=1}^{m} y_i A_i$$

with $A_i \in \mathcal{S}^n, i = 1, \ldots, m$, $C \in \mathcal{S}^n$, and $b \in I\!R^m$. Here m denotes the number of primal constraints. The matrix $X \in \mathcal{S}^n$ is constrained to be positive semidefinite (*psd*) expressed as $X \succeq 0$. This is equivalent to requiring that $d^T X d \geq 0$, $\forall d \in I\!R^n$. On the other hand $X \succ 0$ denotes a positive definite (*pd*) matrix, i.e. $d^T X d > 0$, for all nonzero vectors $d \in I\!R^n$. Hereafter we use $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ to denote the space of symmetric *psd* and *pd* matrices respectively. A good reference source is Horn and Johnson (1990).

We will make the following assumptions.

ASSUMPTION 1. *The matrices A_i, $i = 1, \ldots, m$ are linearly independent in $\mathcal{S}^n$.*

ASSUMPTION 2. *Both (SDP) and (SDD) have strictly feasible points, namely the sets $\{X \in \mathcal{S}^n : \mathcal{A}(X) = b, X \succ 0\}$ and $\{(y, S) \in I\!R^m \times \mathcal{S}^n : \mathcal{A}^T y + S = C, S \succ 0\}$ are nonempty.*

Assumption 2 guarantees both (SDP) and (SDD) attain their optimal solutions X^* and (y^*, S^*), and their optimal values are equal, i.e. $C \bullet X^* = b^T y^*$. Thus the duality gap $X^* \bullet S^* = 0$ at optimality.

ASSUMPTION 3.

$$\mathcal{A}(X) \;=\; b \;\Rightarrow\; trace(X) \;=\; a \tag{1}$$

for some constant $a \geq 0$.

Assumption 3 enables one to rewrite (SDD) as an eigenvalue optimization problem, and also ensures the existence of the following $\hat{y}$.

PROPOSITION 1. *Under Assumption 3, there exists a unique* $\hat{y} \in I\!R^m$ *satisfying*

$$\mathcal{A}^T\hat{y} = I.$$

Moreover this $\hat{y}$ *satisfies* $b^T\hat{y} = a$.

Proof: Since $\mathrm{trace}(X) = a$ is satisfied for every feasible X in (SDP), it can be expressed as a linear combination of the other primal constraints $A_i \bullet X \;=\; b_i,\; i \;=\; 1,\ldots,m$. Letting the components of $\hat{y}$ to be the coefficients in this linear combination we get the desired result. Uniqueness follows from Assumption 1. □

We can write down the Lagrangian dual to (SDP) transferring all the equality constraints into the objective function via Lagrangian multipliers y_i, $i = 1,\ldots,m$, to give (2):

$$\max_y \;\; b^Ty + \;\min_{X:\mathrm{trace}(X)=a, X\succeq 0} \;\; (C - \textstyle\sum_{i=1}^m y_iA_i) \bullet X \tag{2}$$

Assumption 2 ensures that this problem is equivalent to (SDP). Using the variational characterization of the minimum eigenvalue function, the quantity in the inner minimization can be expressed as $a\lambda_{min}(C - \mathcal{A}^Ty)$. We can then rewrite (2) as

$$\max_y b^Ty + a\lambda_{min}(C - \mathcal{A}^Ty) \tag{3}$$

This is an eigenvalue optimization problem. We shall return to the formulation (3), when we discuss cutting plane approaches for the SDP. Without loss of generality, and for the ease of exposition, we shall assume that $a = 1$ in the succeeding sections. We must also emphasize that although we are dealing with $\lambda_{min}(S)$ which is a concave function, we shall continue to use terms like subgradients, subdifferential etc. These terms should be understood to mean the corresponding analogues for a concave function. We also fix some notation here

$$\begin{aligned} f(y) \;&=\; b^Ty + \lambda_{min}(C - \mathcal{A}^Ty) \\ &=\; \lambda_{min}(C - \textstyle\sum_{i=1}^m y_i(A_i - b_iI)) \end{aligned}$$

This function is non-differentiable, precisely at those points, where the smallest eigenvalue of $(C - \mathcal{A}^Ty)$ has a multiplicity greater than one.

Let us consider a point y, where $\lambda_{min}(C - \mathcal{A}^T y)$ has a multiplicity r. Let p_i, $i = 1, \ldots, r$ be an orthonormal set of eigenvectors at this point. Also, $P \in I\!R^{n \times r}$ with $P^T P = I_r$ is the matrix, whose ith column is p_i. Any normalized eigenvector p corresponding to $\lambda_{min}(C - \mathcal{A}^T y)$ can be expressed as $p = Px$, where $x \in I\!R^r$, with $x^T x = 1$. The subdifferential of $f(y)$ at this point is then given by

$$\begin{aligned} \partial f(y) &= \text{conv}\{b - \mathcal{A}(pp^T) : p = Px, x^T x = 1\} \\ &= \{b - \mathcal{A}(PVP^T) : V \in \mathcal{S}^r, \text{trace}(V) = 1, V \succeq 0\} \end{aligned} \tag{4}$$

Here conv denotes the convex hull operation. The equivalence of the two expressions can be found in Overton (1992). Each member of $\partial f(y)$ is called a subgradient. The cutting plane approaches for SDP fall in the following categories:

- **Interior point cutting plane methods:** The SDP is a convex optimization problem with a weak polynomial time separation oracle (Grötschel et al. (1993)), and hence can be solved within an interior point cutting plane framework. Good surveys of such methods appear in Goffin and Vial (2002) and Mitchell (2003). In particular, Algorithms 1 and 3 discussed in this survey fall within this framework.

- **Bundle methods for non-differentiable optimization:** An SDP with some additional restrictions (see Assumption 3) can be written as an eigenvalue optimization problem. These are convex but non-smooth optimization problems, that can be handled by bundle methods for non-differentiable optimization. A survey on bundle methods appears in Lemarechal (1989). In particular, Algorithms 2 and 4 discussed in the survey fall in this class.

- **Active set approaches for SDP:** These approaches generate iterates which are on the boundary of the SDP feasible region. In particular, the simplex-like approach developed in Pataki (1996) is a special case of such an approach. Algorithm 5 discussed in this survey belongs to this class.

We are interested in cutting plane methods which deal directly with the dual problem (SDD); in particular the eigenvalue optimization problem (3). In this regard, at the time of writing, we are aware of at least four distinct cutting plane approaches namely : an LP cutting plane scheme for (SDD) due to Krishnan and Mitchell (2001) (see also Goldfarb (2002)), a variant of analytic center cutting plane methods (ACCPM) incorporating semidefinite cuts due to Oskoorouchi and Goffin (2003), the spectral bundle method due to Helmberg and Rendl (2000), and

a non-polyhedral primal active set approach due to Krishnan et al. (2003). These approaches are discussed in this survey.

We must also mention that there are two ACCPM schemes for SDP due to Toh, Zhao, and Sun (2002) and Sun, Toh, and Zhao (2002), but these deal with the SDP problem, with a large number of linear constraints, in the primal formulation. The methods use a cutting plane method to approximate the linear constraints, while the SDP constraint is always explicitly included in the relaxations. We will not discuss these approaches in this paper.

2. Cutting plane models for SDP

In this section we shall discuss polyhedral and non-polyhedral cutting plane models for SDP. Consider the following *semi-infinite* formulation of (SDD).

$$\begin{array}{ll} \max & b^T y \\ \text{subject to} & dd^T \bullet \mathcal{A}^T y \ \leq \ dd^T \bullet C \ \ \forall ||d||_2 = 1 \qquad (LDM) \end{array}$$

We consider (SDD) instead of (SDP) because this gives a problem with m variables. In contrast, a semi-infinite formulation of (SDP) would involve $\binom{n+1}{2} = \frac{n(n+1)}{2} = O(n^2)$ variables.

Note that if $m = O(n^2)$ it may be advantageous to proceed in the following manner. Let $q = \frac{n(n+1)}{2} - m$. It is possible to reformulate SDP as a semi-infinite programming problem in q variables. This is advantageous if q is smaller than m, in particular if $q = O(n)$. Let $\mathcal{B} : \mathcal{S}^n \rightarrow I\!R^q$ be the null space operator corresponding to $\mathcal{A}$, so the kernel of $\mathcal{B}^T$ is exactly the range of $\mathcal{A}$. From Assumption 1, we can regard $\mathcal{B}$ as being composed of q linear functions, each represented by a matrix $B_i \in \mathcal{S}^n$, and these matrices are linearly independent in $\mathcal{S}^n$. Let X^0 be a feasible solution to $\mathcal{A}(X) = b$. The set of feasible solutions to $\mathcal{A}(X) = b$ is the set of all matrices of the form $X = X^0 - \mathcal{B}^T(u)$ for some $u \in I\!R^q$. The problem SDP can then be written equivalently as

$$\begin{array}{lll} \min_{u,X} & C \bullet X_0 - C \bullet \mathcal{B}^T(u) & \\ \text{s.t.} & \mathcal{B}^T(u) + X \ = & X^0 \\ & X \ \succeq & 0 \end{array}$$

This problem is exactly in the form of (SDD), so we can construct a linear programming relaxation of it in the form (LDR) (see below) with q variables.

We consider a discretization of (LDM). Given a finite set of vectors $\{d_i, i = 1, \ldots, k\}$, we obtain the relaxation

$$\begin{array}{ll} \max & b^T y \\ \text{subject to} & d_i d_i^T \bullet \mathcal{A}^T y \ \leq \ d_i d_i^T \bullet C \ \text{ for } i = 1, \ldots, k. \end{array} \qquad (LDR)$$

The linear programming dual to (LDR) can be written

$$\begin{array}{lll} \min & C \bullet (\sum_{i=1}^k x_i d_i d_i^T) & \\ \text{subject to} & \mathcal{A}(\sum_{i=1}^k x_i d_i d_i^T) \ = \ b & (LPR) \\ & x \ \geq \ 0. & \end{array}$$

The problem (LPR) is a constrained version of (SDP); this is reflected in the following proposition.

PROPOSITION 2. *Any feasible solution x to (LPR) will give a feasible solution X to (SDP).*

The optimality conditions for SDP can be summarized in the following theorem (Alizadeh et al., 1997):

THEOREM 1. *Let X and (y, S) be primal and dual feasible respectively. Then they are optimal if and only if there exists $Q \in I\!R^{n\times r}$, $R \in I\!R^{n\times(n-r)}$, with $Q^TQ = I_r$, $R^TR = I_{n-r}$, $Q^TR = 0$, and Λ, Ω, diagonal matrices in $\mathcal{S}_+^r$, and $\mathcal{S}_+^{n-r}$, such that*

$$X \ = \ [Q \ \ R] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q^T \\ R^T \end{bmatrix} \qquad (5)$$

$$S \ = \ [Q \ \ R] \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} Q^T \\ R^T \end{bmatrix} \qquad (6)$$

hold.

The diagonal matrices Λ, Ω contain the nonzero eigenvalues of X and S in the spectral decompositions (5) and (6) respectively. Also $P = [Q \ \ R]$ is an orthogonal matrix that contains the common set of eigenvectors.

We get an upper bound on r, using the following theorem 2, due to Pataki (1998) (also see Alizadeh et al. (1997)), on the rank of extreme matrices X in (SDP).

THEOREM 2. *There exists an optimal solution X^* with rank r satisfying the inequality $\frac{r(r+1)}{2} \leq m$, where m is the number of constraints in (SDP).*

It follows that $\sqrt{2m}$ is an overestimate of the upper bound on the rank of at least one optimal solution. We now present the perfect constraints that are needed in the LP relaxations if the optimal solution to (SDP) is unique with distinct positive eigenvalues.

THEOREM 3. *Let $X^* = Q\Lambda Q^T$ be an optimal solution to (SDP), and let q_i, $i = 1, \ldots, r$ be the columns of Q. If the constraints of (LDR) include the constraints $q_i q_i^T \bullet \mathcal{A}^T y \leq q_i q_i^T \bullet C$ for $i = 1, \ldots, r$ then any optimal solution x^* to (LDR) gives an optimal solution to (SDP) by taking $X = \sum_{i=1}^{k} x_i^* d_i d_i^T$.*

Proof: Reordering the constraints if necessary, we can assume that $d_i = q_i$ for $i = 1, \ldots, r$. We have $X^* = Q\Lambda Q^T = \sum_{i=1}^{r} \lambda_i q_i q_i^T$, where $\lambda_i > 0$, $i = 1, \ldots, r$, and q_i, $i = 1, \ldots, r$ are the corresponding eigenvectors. This gives a feasible solution to (LPR) with $x_i = \lambda_i$ for $i = 1, \ldots, r$ and $x_i = 0$ otherwise. Further, this feasible solution is optimal, since (LPR) is a constrained version of (SDP). Thus the optimal values of (LPR) and (SDP) are identical, so any optimal solution to (LPR) gives an optimal solution to (SDP). □

We note that (LPR) can be rewritten as

$$\begin{array}{lrcl} \min & C \bullet (DMD^T) & & \\ \text{s.t.} & A_j \bullet (DMD^T) & = & b_j \;\; j = 1, \ldots, m \\ & M & \succeq & 0 \\ & M & \text{diagonal} & \end{array} \tag{7}$$

Here $M \in \mathcal{S}^m$, and $D \in I\!R^{n \times m}$ with jth column d_j. Theorem 3 suggests that if the columns of D contain eigenbases for all the strictly positive eigenvalues of X^*, then the solution to (7) is an exact solution to (SDP). In other cases, a solution provides an upper bound on this objective value. This polyhedral cutting plane model is the basis for Algorithms 1 and 2 of Section 4.

We can consider alternate nonlinear models, which are simple extensions of this polyhedral cutting plane model in the following manner. Consider a relaxation of (7) dropping the requirement that M be diagonal.

$$\begin{array}{lrcl} \min & C \bullet (DMD^T) & & \\ \text{s.t.} & A_j \bullet (DMD^T) & = & b_j \;\; j = 1, \ldots, m \\ & M & \succeq & 0 \end{array} \tag{8}$$

with $M \in \mathcal{S}^m$, and $D \in I\!R^{n \times m}$. If $m = n$, then (8) is essentially (SDP). In fact, if Range$(D) \supset$ Range(X^*), then a solution to (8) is an exact solution to (SDP). This is a less stringent requirement than the polyhedral cutting plane model, where we require the exact eigenvectors of X^*, and forms the basis of Algorithms 4 and 5 in Section 5.

Finally, another relaxation of (7) which is more restrictive than (8) is the following.

$$\begin{array}{lllll} \min & C \bullet (DMD^T) & & & \\ \text{s.t.} & A_j \bullet (DMD^T) & = & b_j & j = 1, \ldots, m \\ & M & \succeq & 0 & \\ & M & \text{block diagonal} & & \end{array} \tag{9}$$

This is the cutting plane model employed in Algorithm 3 in Section 5.

3. Generic cutting plane approach for SDP

Having considered the various cutting plane models, we now present the generic cutting plane approach below. These approaches work directly with the eigenvalue optimization model (3), i.e. we assume the redundant constraint $trace(X) = 1$ is added to the formulation (SDP). A generic cutting plane approach can be stated as follows:

1. Choose an initial point $\hat{y}$, an initial finite set $\mathcal{D} = \{D_i\}$, and a scalar $u \geq 0$.
2. Solve the following subproblem

$$\begin{array}{ll} \max & \lambda + b^T y - \frac{u}{2}||y - \hat{y}||^2 \\ \text{s.t.} & D_i^T(C - \mathcal{A}^T y)D_i \ \succeq \ \lambda I, \ i \in \mathcal{D} \end{array} \tag{10}$$

   to get (y^*, λ^*).
3. If $S^* = (C - \mathcal{A}^T y^*) \succeq 0$, we are optimal; STOP. Else find $D \in I\!R^{n \times r}$, with $r \leq \sqrt{2m}$ such that $D^T(C - \mathcal{A}^T y^*)D \not\succeq 0$.
4. Either add D to $\mathcal{D}$, or aggregate D into $\mathcal{D}$.
5. Set $\hat{y} = y^*$, and return to Step 2.

The five methods to be discussed in §4 and §5 can then be distinguished as in Table I.

Note that the variable λ in (10) corresponds to the redundant constraint $\text{trace}(X) = 1$.

Algorithms 1 and 2 are polyhedral because only $D \in I\!R^{n \times 1}$ appear in $\mathcal{D}$ so the constraints are actually linear inequalities of the form $D_i^T(C - \mathcal{A}^T y)D_i \geq 0$, $\forall i$. Likewise, Algorithm 3 is non-polyhedral, since not all the constraints $D_i^T(C - \mathcal{A}^T y)D_i \succeq 0$ can be reduced to a finite set of linear inequalities. Finally, Algorithms 4, and 5 have just one semidefinite block of constraints.

Table I. Cutting plane methods for SDP

Algorithm	Model	u	Form of $\mathcal{D}$	Add or aggregate?
1	Polyhedral	zero	$D \in I\!R^{n\times 1}$ only	add, letting $\|\mathcal{D}\|$ grow
2	Polyhedral bundle	positive	$D \in I\!R^{n\times 1}$ only	add, letting $\|\mathcal{D}\|$ grow
3	Nonlinear Block-Diag	zero	$D \in I\!R^{n\times r}$ with $r \leq \sqrt{2m}$	add, letting $\|\mathcal{D}\|$ grow
4	Spectral bundle	positive	$D \in I\!R^{n\times r}$	aggregate, keeping $\|\mathcal{D}\| = 1$
5	Primal active set	zero	$D \in I\!R^{n\times r}$	aggregate, keeping $\|\mathcal{D}\| = 1$

For Algorithms 1, 3 and 5, the weight $u = 0$. In these cases, the dual to the subproblem being solved in each iteration is

$$\begin{array}{lrcl} \min & C \bullet (\sum_{i\in\mathcal{D}} D_i V_i D_i^T) & & \\ \text{s.t.} & \mathcal{A}(\sum_{i\in\mathcal{D}} D_i V_i D_i^T) & = & b \\ & \sum_{i\in\mathcal{D}} \mathrm{trace}(V_i) & = & 1 \\ & V_i & \succeq & 0, \ i \in \mathcal{D} \end{array}$$

This takes the forms (7), (8), and (9), which correspond to Algorithms 1,3 and 5 respectively. Hence, these algorithms are solving a relaxation of (SDD) in every iteration.

For Algorithms 2 and 4, the weight $u > 0$, since the proximal bundle idea is being used. In this case, the Lagrangian dual to the subproblem is a quadratic SDP, and has the form

$$\begin{array}{lrcl} \min & \frac{1}{2u}||b - \mathcal{A}(\sum_{i\in\mathcal{D}} D_i V_i D_i^T)||^2 - b^T\hat{y} & & \\ & -(C - \mathcal{A}^T\hat{y}) \bullet (\sum_{i\in\mathcal{D}} D_i V_i D_i^T) & & \\ \text{s.t.} & \sum_{i\in\mathcal{D}} \mathrm{trace}(V_i) & = & 1 \\ & V_i & \succeq & 0, \ i \in \mathcal{D} \end{array}$$

Due to proximal bundle term in Algorithms 2 and 4, these algorithms may not always be solving a relaxation of (SDD).

4. Polyhedral cutting plane algorithms for SDP

We present two polyhedral cutting plane algorithms for SDP in this section. A discussion on Algorithm 1 appears in Section 4.1, while Algorithm 2 appears in Section 4.2. Both these algorithms are based on the polyhedral model (7).

4.1. Algorithm 1 : Polyhedral cutting plane algorithm

The method is originally due to Cheney and Goldstein (1959), and Kelley (1960). It was introduced by Krishnan and Mitchell (2001) and Goldfarb (2002) in the context of SDP. The complete algorithm appears in Figure 1.

1. Choose an **initial set** of constraints for (LDR). Choose termination parameters $\epsilon_1, \epsilon_2 > 0$. Set the current upper and lower bounds to be UB $= \infty$, and LB $= -\infty$ respectively. Choose an appropriate approximate optimality criterion for the solution to (LPR) and (LDR)

2. In the kth iteration, obtain a solution y^k to the discretization (LDR) and its dual (LPR) satisfying the approximate optimality criterion. Update the upper bound: UB $= \min\{\text{UB}, b^T y^k\}$.

3. Compute $\lambda = |\lambda_{min}(C - \mathcal{A}^T y^k)|$, and a corresponding eigenvector d. Update the lower bound : $LB = \max\{LB, b^T y^k + \lambda a\}$, where a is as in Assumption 3. If $|\text{LB} - \text{UB}| \leq \epsilon_1$, or $\lambda \leq \epsilon_2$, go to step 5.

4. Add the constraint

$$dd^T \bullet \mathcal{A}^T y \ \leq \ dd^T \bullet C$$

   to (LDR). Set $k = k + 1$, update the approximate optimality criterion for (LDR) and (LPR) if desired, and return to step 2.

5. The current solution (x^k, y^k) for (LDR) and (LPR) give an optimal solution (X, y) for (SDP), and (SDD) respectively.

Figure 1. Algorithm 1: Polyhedral cutting plane algorithm

There are several choices for an appropriate approximate optimality criterion for the solution of (LDR) and (LPR) in Algorithm 1. The simplest is to require that the linear programs are solved to optimality, that is, to within a relative duality gap of 10^{-8}, at each stage. Alternatively, a dynamically modified tolerance on the duality gap could be used, with the tolerance being tightened as k increases, so that an

optimal solution to (SDP) is eventually obtained. Another alternative is to solve a feasibility problem that explicitly includes multiple copies of a constraint corresponding to the objective function, and to require that an approximate analytic center or volumetric center be found for this problem. In this case, the number of copies of the objective function constraint and the depth of this constraint are modified in order to push the solution to optimality as k is increased. Finally, the ellipsoid algorithm could be used, with the criterion being that the new iterate is the updated center of the new ellipsoid.

An initial set of constraints is obtained by requiring that the diagonal entries of S be non-negative. This amounts to setting $d = e_i$, $i = 1, \ldots, n$. Typically, the most negative eigenvalue $\lambda_{min}(S)$, and its associated eigenvector are estimated by an iterative method like the *Lanczos* scheme. The algorithm employs the weak polynomial separation oracle for SDP in Step 3. This oracle can be implemented in $O(n^3 + mn^2)$ arithmetic operations (Krishnan (2002), Krishnan and Mitchell (2003)). If the algorithm converges, both optimality criteria will eventually be satisfied; one may be preferred to the other in certain situations.

There is another way to motivate this cutting plane approach, which is based on the eigenvalue optimization model (3). Assume that we have a set of points $y = y^1, \ldots, y^k$, and we know the function values $f(y^i)$, $i = 1, \ldots, k$, and subgradients $(b - \mathcal{A}(d_i d_i^T))$, $i = 1, \ldots, k$ (where d_i is a normalized eigenvector corresponding to $\lambda_{min}(C - \mathcal{A}^T y^i)$) at these points. We can construct the following overestimate $\hat{f}_m(y)$ for $f(y)$.

$$\hat{f}_m(y) = \min_{i=1,\ldots,k} d_i d_i^T \bullet (C - \mathcal{A}^T y) + b^T y \geq f(y).$$

To see this note that since the d_i are normalized, we have

$$\begin{array}{rcl} \lambda_{min}(C - \mathcal{A}^T y) & \leq & d_i^T (C - \mathcal{A}^T y) d_i, \quad i = 1, \ldots, k \\ & = & d_i d_i^T \bullet (C - \mathcal{A}^T y), \quad i = 1, \ldots, k \end{array}$$

We now maximize this overestimate instead, i.e.

$$\max_y \hat{f}_k(y) \;=\; \max_y \{ b^T y + \min_{i=1,\ldots,k} \{ d_i d_i^T \bullet (C - \mathcal{A}^T y) \} \}$$

which can be recast as the following linear program

$$\begin{array}{ll} \max & b^T y + v \\ \text{s.t.} & d_i d_i^T \bullet \mathcal{A}^T y + v \;\leq\; d_i d_i^T \bullet C, \quad i = 1, \ldots, k \end{array} \tag{11}$$

with dual

$$\begin{array}{ll} \min & C \bullet (\sum_{i=1}^k x_i d_i d_i^T) \\ \text{s.t.} & \mathcal{A}(\sum_{i=1}^k x_i d_i d_i^T) \;=\; b_j, \;\; j = 1, \ldots, m \\ & \sum_{i=1}^k x_i \;=\; 1 \\ & x \;\geq\; 0 \end{array} \tag{12}$$

This is exactly the problem obtained by considering a discretization of (SDD). Here v is the dual variable corresponding to the redundant constraint $\sum_{i=1}^{k} x_i = 1$, which is implicitly satisfied by any solution x to (LPR). Thus, we can set $v = 0$ without any loss of generality. The solution (v, y) with $v = 0$ is the one corresponding to (LDR).

Unfortunately Algorithm 1 has a very poor rate of convergence in practice. For instance we observed that a simplex implementation that solved each linear program to optimality performed very badly (Krishnan and Mitchell, 2001; Krishnan, 2002). Primarily, minimizing $\hat{f}_k$ to find y^{k+1} makes sense only if $\hat{f}_k \approx f$, near y^k, this is one of the reasons for the slow convergence for the cutting plane scheme. Lemarechal (1989) discusses some convergence estimates for such an algorithm.

On the other hand, we could solve the above cutting plane algorithm within an interior point cutting plane framework. There are two noteworthy advantages to solving the relaxations approximately using an interior point cutting plane method: there is less oscillation in the sequence of iterates, and the cutting planes are generated at points that are more central so they tend to be deeper. The volumetric center algorithm for the convex feasibility problem (Anstreicher (1999)) is such an approach; using this framework for the cutting plane approach of Algorithm 1 gives an algorithm that requires $O(m \log(\frac{1}{\epsilon}))$ calls to the oracle, and $O(m^4 \log(\frac{1}{\epsilon}))$ other arithmetic operations. The overall complexity is better than employing the ellipsoid method for SDP, and also compares favorably with interior point methods for SDP (Krishnan and Mitchell (2003), Krishnan (2002)).

4.2. Algorithm 2 : Polyhedral bundle scheme

One way to improve the convergence of the algorithm is to utilize the proximal bundle idea discussed in Lemarechal (1989), Kiwiel (1985), Hiriart-Urruty and Lemarechal (1993), and Makela and Neittaanmaki (1992). This leads naturally to Algorithm 2.

Before discussing the actual algorithm, we present a short discussion on the proximal bundle scheme. The rough idea here is to maximize $\hat{f}_k(y) - \frac{u}{2}||y - y^k||^2$ (for some chosen $u > 0$). The second term acts as a regularization term which penalizes us from going too far from the current iterate y^k. The idea is to lower u if we are making progress, i.e. taking serious steps, and actually increase u if we perform a null step. As Lemarechal (1989) remarks, choosing this parameter u is an art in itself. The regularization penalty term $\frac{u}{2}||y-y^k||^2$ acts as a trust region constraint $||y-y^k||^2 \leq \sigma_k$, and helps to keep the solution bounded. Thus we can dispense with choosing an initial set of constraints to keep the

subproblems bounded, as in Algorithm 1. For numerical reasons, it is better to introduce the regularization term into the objective function, rather than as a trust region constraint. This keeps the feasible region polyhedral, but we now have a quadratic objective.

Consider adding this quadratic term in the objective function of (11) giving

$$\begin{array}{lrcl} \max & b^T y + v & - & \frac{u}{2}||y - \hat{y}||^2 \\ \text{s.t.} & v & \leq & d_i d_i^T \bullet (C - \mathcal{A}^T y), \quad i = 1, \ldots, k \end{array} \tag{13}$$

with Lagrangian dual

$$\begin{array}{lrcl} \min & \frac{1}{2u}||b - \mathcal{A}(X))||^2 - b^T \hat{y} & & \\ & -(C - \mathcal{A}^T \hat{y}) \bullet X & & \\ \text{s.t.} & X & = & \sum_{i=1}^k x_i d_i d_i^T \\ & \sum_{i=1}^k x_i & = & 1 \\ & x_i & \succeq & 0, \qquad i = 1, \ldots, k. \end{array} \tag{14}$$

Setting $u = 0$ in (13) gives (11). Due to strong duality, (13) and (14) have the same objective value. Their solutions y and X satisfy

$$y \;=\; \hat{y} + \tfrac{1}{u}(b - \mathcal{A}(X)) \tag{15}$$

The complete algorithm appears in Figure 2. The formal proof of convergence of the algorithm can be found in Kiwiel (1990).

Let

$$\hat{W} \;=\; \{\textstyle\sum_{i=1}^k x_i d_i d_i^T \,|\, \sum_{i=1}^k x_i = 1, x_i \geq 0, i = 1, \ldots, k\} \tag{16}$$

which is the feasible region of (14). It appears that the size of $\hat{W}$ grows indefinitely with iteration count in the above algorithm. We can however choose to keep the number of subgradients no larger than an a priori bound l. We retain the earlier $l - 2$ subgradients $d_i d_i^T$ corresponding to the largest values of x, along with the new subgradient $p^{k+1} p^{k+1^T}$ in the bundle, and aggregate the rest of the subgradients in a subgradient matrix $\bar{W}^l$. The set $\hat{W}^{k+1}$ is then the convex hull of $\bar{W}^l$ and the $l - 1$ subgradients in the bundle, so

$$\begin{array}{rcl} \hat{W}^{k+1} & = & \{\alpha \bar{W}^l + \sum_{i=1}^{l-1} x_i d_i d_i^T | \alpha + \sum_{i=1}^{l-1} x_i = 1, \\ & & \qquad \alpha \geq 0, x_i \geq 0, i = 1, \ldots, r\} \end{array} \tag{17}$$

and so the feasible region of (14) is modified to require that X be in this set $\hat{W}^{k+1}$. Alternatively, in the kth iteration we can choose $\hat{W}^{k+1}$ to be the convex hull of X^{k+1} and $p^{k+1} p^{k+1^T}$, in which case $l = 2$. It can be shown that the algorithm converges in these situations too.

1. Let $y^1 \in I\!R^m$, let $p^1 \in I\!R^n$ be a normalized eigenvector corresponding to $\lambda_{min}(C - \mathcal{A}^T y^1)$. Also choose the weight $u > 0$, an improvement parameter $\nu \in (0, 1)$, and finally a termination parameter $\epsilon > 0$.

2. At iteration k, compute X^{k+1} from (14), and y^{k+1} from (15), where $\hat{y} = y^k$. Also, let $f_{X^{k+1}}(y^{k+1}) = b^T y^{k+1} + (C - \mathcal{A}^T y^k) \bullet X^{k+1}$.

3. If
$$f_{X^{k+1}}(y^{k+1}) - f(\hat{y}^k) \ \leq \ \epsilon$$
stop.

4. Compute $f(y^{k+1})$, and the eigenvector p^{k+1} corresponding to $\lambda_{min}(C - \mathcal{A}^T y^{k+1})$.

5. If the actual increase is not much smaller than the increase predicted by the model (sufficient increase), i.e.
$$f(y^{k+1}) - f(\hat{y}^k) \ \geq \ \nu(f_{X^{k+1}}(y^{k+1}) - f(\hat{y}^k))$$
then perform a serious step, i.e. $\hat{y}^{k+1} = y^{k+1}$. Else, perform a null step, i.e. $y^{k+1} = y^k$.

6. Return to step (2).

Figure 2. Algorithm 2: Polyhedral bundle method for SDP

5. Non-polyhedral cutting plane algorithms for SDP

In this section, we discuss three non-polyhedral cutting plane algorithms for SDP. The three algorithms appear in Sections 5.1, 5.2, and 5.3 respectively. The first algorithm is based on the non-polyhedral block diagonal model (9), while the latter two schemes work with the model (8).

5.1. Algorithm 3 : Non-polyhedral block diagonal cutting plane scheme

The next algorithm we consider is based on the non-polyhedral, block diagonal SDP model. This is employed in Oskoorouchi and Goffin (2003), and an additional reference includes Oskoorouchi (2002). This algorithm is identical to Algorithm 1 except when the multiplicity of $\lambda_{min}(C - \mathcal{A}^T y)$ is greater than one. Since we are essentially minimizing this quantity in (SDD), during the course of Algorithm 1, the smaller eigenvalues generally tend to coalesce together thereby increasing the

multiplicity of this eigenvalue. In fact at optimality, this multiplicity is bounded by $\sqrt{2m}$ (from Theorems 1, and 2).

When this number is r (say), we could instead add the following semidefinite constraint

$$\sum_{i=1}^m y_i(D^T A_i D) \ \preceq \ (D^T C D)$$

where $D \in I\!R^{n\times r}$, with $D^T D = I_r$, whose columns form an eigenbasis for the eigenspace of $C - \mathcal{A}^T y$ with eigenvalue $\lambda_{min}(C - \mathcal{A}^T y)$. We note that this is much stronger than

$$\sum_{i=1}^m y_i(d_j^T A_i d_j) \ \leq \ (d_j^T C d_j), \ j = 1, \ldots, r$$

This leads to the following subproblem to be solved in every iteration.

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & \sum_{i=1}^m y_i(D_j^T A_i D_j) \ \preceq \ (D_j^T C D_j), \ j = 1, \ldots, k \end{array} \tag{18}$$

whose dual is

$$\begin{array}{lll} \min & C \bullet (\sum_{i=1}^k D_i V_i D_i^T) & \\ \text{s.t.} & \mathcal{A}(\sum_{i=1}^k D_i V_i D_i^T) \ = \ b & \\ & V_i \ \succeq \ 0, \ i = 1, \ldots, k & \end{array} \tag{19}$$

where the number of columns in D_j, $j = 1, \ldots, k$ reflect the multiplicities of $\lambda_{min}(C - \mathcal{A}^T y^i)$, $i = 1, \ldots, k$, and some of these could conceivably be 1.

The entire algorithm can be implemented in the ACCPM framework with semidefinite cuts (Oskoorouchi and Goffin (2003)) in fully polynomial time; in particular this can be done in $O(\frac{r^2 m^3}{\mu^2 \epsilon^2})$ calls to the oracle, where r is an upper bound on the number of columns in D_i, $\mu > 0$ is a condition number on the cuts, and $\epsilon > 0$ is the tolerance to which one wants to solve the SDP. The algorithm is detailed in Figure 3. Discussion of an appropriate approximate optimality criterion can be found in Oskoorouchi and Goffin (2003).

5.2. Algorithm 4 : Spectral bundle scheme

We now discuss the spectral bundle method for SDP due to Helmberg and Rendl (2000). Other references include Helmberg (2000), Helmberg and Kiwiel (2002), Helmberg and Oustry (2000), and Oustry (2000).

We will motivate the spectral bundle scheme in this section, by considering the second aggregation scheme employed by the polyhedral bundle method of Section 4.2, as in (17). The spectral bundle instead chooses the following expression for $\hat{W}$:

$$\hat{W} \ = \ \{\alpha \bar{W} + PVP^T | \alpha + \text{trace}(V) = 1, \alpha \geq 0, V \succeq 0\}$$

1. Choose an **initial set** of constraints for (18); this can be done in the same way as Algorithm 1. Choose an appropriate approximate optimality criterion for the solution to (18) and (19). Set $k = 1$.
2. Find an approximate solution (X^k, y^k) to (19) and (18).
3. Compute $\lambda_{min}(C - \mathcal{A}^T y^k)$, and an orthonormal matrix $D^k \in I\!R^{n \times r^k}$, where r^k is the multiplicity of this eigenvalue. Update the lower and upper bounds as in Algorithm 1. If $\lambda_{min}(C - \mathcal{A}^T y^k)$ is small, or the difference in bounds is small, go to step 4. Else, add the following constraint to (18)
$$\textstyle\sum_{i=1}^m y_i (D_k^T A_i D_k) \ \preceq \ D_k^T C D_k$$
Set $k = k + 1$, update the approximate optimality criterion if desired, and return to step 2.
4. The current solution (X^k, y^k) is optimal for (SDP), and (SDD) respectively.

Figure 3. Algorithm 3: Non-polyhedral cutting plane algorithm for SDP

where $P \in I\!R^{n \times r}$, with $P^T P = I_r$, whose columns are the p_i, $i = 1, \ldots, r$. The columns of the matrix P constitute the bundle. Here r refers to the size of the bundle, and inspired by Theorem 2, this number is chosen to be no greater than $\sqrt{2m}$. It is clear from these two expressions for $\hat{W}$ that the spectral bundle method does a better job in approximating the subdifferential of $\lambda_{min}(S)$.

The spectral bundle method solves the following pair of subproblems in every iteration.

$$\begin{array}{lrl} \max & \lambda + b^T y & \\ & -\frac{u}{2}||y - \hat{y}||^2 & \\ \text{s.t.} & P^T(C - \mathcal{A}^T y)P \ \geq & \lambda I \\ & (C - \mathcal{A}^T y) \bullet \bar{W} \ \geq & \lambda \end{array} \tag{20}$$

with dual

$$\begin{array}{lrcl} \min & \frac{1}{2u}||b - \mathcal{A}(X)||^2 - (C - \mathcal{A}^T y^k) \bullet X - b^T \hat{y} & & \\ \text{s.t.} & X & = & \alpha \bar{W} + P V P^T \\ & \alpha + \text{trace}(V) & = & 1 \\ & \alpha & \geq & 0 \\ & V & \succeq & 0 \end{array} \tag{21}$$

The problem (21) is a quadratic SDP, and can be solved efficiently for X using interior point methods, if r is small. More details can be found

in Helmberg and Rendl (2000), and Helmberg (2000). The solution y to (20) is then given by

$$y \;=\; \hat{y} + \tfrac{1}{u}(b - \mathcal{A}(X)) \tag{22}$$

The algorithm can be found in Figure 4.

1. Start with $y^1 \in I\!R^m$, let $p^1 \in I\!R^n$ a normalized eigenvector corresponding to $\lambda_{min}(C - \mathcal{A}^T y^1)$. Choose parameters $u > 0$, an improvement parameter $\nu_1 \in (0,1)$, and a termination parameter $\epsilon > 0$. Let $P^1 = p^1$, and $\bar{W}^1 = p^1 p^{1^T}$.
2. In the kth iteration, solve (20) and (21) for (α^{k+1}, V^{k+1}), with $X^{k+1} = \alpha^{k+1}\bar{W}^k + P^k V^{k+1} P^{k^T}$. Compute y^{k+1} from (22). Let $f_{X^{k+1}}(y^{k+1}) = b^T y^{k+1} + (C - \mathcal{A}^T y^{k+1}) \bullet X^{k+1}$.
3. If $f(y^{k+1}) - f(\hat{y}^k) \leq \epsilon$, then stop with optimality.
4. Compute p^{k+1} a normalized eigenvector corresponding to $\lambda_{min}(C - \mathcal{A}^T y^{k+1})$.
5. If
$$f_(y^{k+1}) - f(\hat{y}^k) \;\geq\; \nu(f_{X^{k+1}}(y^{k+1}) - f(\hat{y}^k))$$
then set $\hat{y}^{k+1} = y^{k+1}$. Else $\hat{y}^{k+1} = \hat{y}^k$.
6. Update the parameters of set $\hat{W}^{k+1}$ using (23). Set $k = k+1$, and return to Step 2.

Figure 4. Algorithm 4: The spectral bundle method for SDP

We present a short discussion on update rules for $\bar{W}$, and P. Other updates are possible, and these are discussed in length in Helmberg (2000). In the kth iteration, one solves (21) for V^k, and computes its spectral decomposition $V^k = Q\Lambda Q^T$. We then split $Q = [Q_1, Q_2]$, where Q_1, and Q_2 contain the eigenvectors corresponding to the large (Λ_1), and small (Λ_2) eigenvalues of V^k respectively. This distinction is based on keeping the size of the bundle P below the above mentioned bound of $\sqrt{2m}$. Finally, P and $\bar{W}$ are updated as follows:

$$\begin{array}{rcl} P^{k+1} & = & \mathrm{orth}([P^k Q_1, v^{k+1}]) \\ \bar{W}^{k+1} & = & \frac{1}{\alpha^k + \mathrm{trace}(\Lambda_2)}(\alpha^k \bar{W}^k + P^k Q_2 \Lambda_2 (P^k Q_2)^T) \end{array} \tag{23}$$

The proof of convergence of the spectral bundle scheme is along the same lines as the polyhedral bundle method, and can be found in Helmberg and Rendl (2000), and Helmberg (2000). In the extreme case

of aggregation, i.e. when there are two subgradients in the bundle, as considered in Section 4.2, the two methods are exactly the same. The spectral bundle method is only a first order method, but variants of the proximal bundle scheme which converge, and enjoy asymptotically a quadratic rate of convergence, were recently developed by Oustry (2000).

5.3. Algorithm 5 : Primal active set approach

The final algorithm is a primal active set approach due to Krishnan et al. (2003), which solves an SDP as a sequence of smaller SDP's in an active set framework. The method relies on the notions of extreme point solutions, and non-degeneracy in SDP. The notion of extreme point solutions in SDP, and a crossover algorithm to generate them can be found in Pataki (1996). A good overview of non-degeneracy in the context of SDP can be found in Alizadeh et al. (1997), while an alternative characterization appears in Pataki (2000).

The method generates extreme point solutions X. The ranks r of these extreme point solutions satisfy the inequality $\frac{r(r+1)}{2} \leq m$ (Theorem 2). There are two steps performed in each iteration, analogous to the primal simplex method for LP.

1. Construct a complementary dual solution (y, S). If $S \succeq 0$, then we are optimal.

2. Else if the previous extreme point solution was non-degenerate, we construct another extreme point solution, whose objective value is strictly lower than the previous one.

Given an extreme point iterate $X = [P_1 \quad P_2] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1{}^T \\ P_2{}^T \end{bmatrix}$ of rank r, a complementary dual solution (y, S) satisfies $XS = 0$. This requires

$$\begin{array}{rcl} \bar{S}_{11} & := & P_1^T(C - \mathcal{A}^T y)P_1 \; = \; 0 \\ \bar{S}_{12} & := & P_1^T(C - \mathcal{A}^T y)P_2 \; = \; 0. \end{array} \tag{24}$$

Setting the symmetric matrix $\bar{S}_{11}$ to zero gives $\frac{r(r+1)}{2}$ equations in y, and requiring $\bar{S}_{12} = 0$ gives another $r(n-r)$ equations. If X is nondegenerate, then the coefficient matrix in (24) has full column rank, so it is typically overdetermined. Since X is also an extreme point, the first $\frac{r(r+1)}{2}$ equations in (24) are linearly independent. Since $m \geq \frac{r(r+1)}{2}$, one can choose m linearly independent equations from (24), including the first $\frac{r(r+1)}{2}$. The resulting system is hereafter denoted as $\bar{S}_B = 0$,

whose unique solution is y^*. We update P to include all the columns p_j present in $\bar{S}_B = 0$; in particular this contains all the columns in P_1. If $S^* = (C - \mathcal{A}^T y^*)$ is not yet psd, we also add the eigenvector corresponding to $\lambda_{min}(S^*)$, suitably orthonormalized, to P.

The update of X is based on the non-polyhedral cutting plane model (8). The new $X = PVP^T$ is obtained by solving the following pair of subproblems.

$$\begin{array}{lll} \min & (P^T C P) \bullet V & \\ \text{s.t.} & (P^T A_i P) \bullet V = b_i, & i = 1, \ldots, m \\ & V \succeq 0 & \end{array} \tag{25}$$

with dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & P^T(C - \mathcal{A}^T y)P \succeq 0 \end{array} \tag{26}$$

If the resulting X is not an extreme point, one can use the crossover algorithm in Pataki (1996) (Algorithm 1 in Section 4.1) to generate an extreme point iterate, whose objective value is no worse than X.

The proof for strict decrease under non-degeneracy conditions can be found in Krishnan et al. (2003). The columns of P contain bases for the positive eigenspaces of the extreme point iterates X. This constitutes the active set in the algorithm, and is analogous to the basis matrix in the simplex method for LP. Once the active set contains the eigenspace of the optimal solution X^*, the algorithm terminates.

The complete algorithm appears in Figure 5. The size of P grows in Step 3, and diminishes in Steps 4, and 5 of the algorithm. The latter two steps can be regarded as constituting the necessary aggregation in the algorithm. We must mention that the convergence and computational aspects of Algorithm 5 are currently under investigation.

6. Conclusions

We present an accessible and unified introduction to various cutting plane methods that have appeared in the literature. These five methods are all roughly solving relaxations of the dual semidefinite program (SDD). Each of the algorithms arise as natural enhancements of the primordial LP cutting plane algorithm (Algorithm 1).

Algorithm 1 can be implemented in polynomial time, if one employs the volumetric center method to approximately solve the LP relaxations. In fact, this complexity compares favorably with interior point methods for SDP. Similarly, Algorithm 3 can be implemented in the ACCPM framework with SDP cuts in fully polynomial time.

1. Consider an extreme point solution $X^1 = P^1 V^1 {P^1}^T$ with $V^1 \succ 0$.
2. In the kth iteration choose a subset of m linearly independent equations from (24). Solve the resulting system $\bar{S}_B = 0$ for a unique y^{k+1}.
3. If $S^{k+1} = (C - \mathcal{A}^T y^{k+1}) \succeq 0$, stop; else update $\bar{P}^{k+1}$ to be all the columns in P_1^k, and those in ${P_2}^k$ included in $\bar{S}_B = 0$. Compute the normalized eigenvector p^{k+1} corresponding to $\lambda_{min}(S^{k+1})$, and set $\bar{P}^{k+1} = \text{orth}\left[\bar{P}^{k+1}, p^{k+1}\right]$. Update the lower and upper bounds as discussed in Algorithm 1. If the difference between these bounds is small, stop.
4. Solve (25), with $P = \bar{P}^{k+1}$. Let $V^{k+1} = {R_1}^{k+1} M^{k+1} {R_1^{k+1}}^T$ with $M^{k+1} \succ 0$. Set ${P_1}^{k+1} = \bar{P}^{k+1} {R_1}^{k+1}$, and $X^{k+1} = {P_1}^{k+1} M^{k+1} {P_1}^{k+1^T}$.
5. If X^{k+1} is not an extreme point, run the crossover algorithm on X^{k+1}, and return to step 1.

Figure 5. Algorithm 5 : Primal active set method for SDP

On the other hand, the spectral bundle method (Algorithm 4) appears to be the most efficient of all the algorithms described in this survey. Excellent computational results have been obtained using the method (Helmberg and Rendl (2000) Helmberg (2003)) for problems that are inaccessible to IPM's due to the high demand for computer time and storage requirements.

The primal active set approach (Algorithm 5) mimics the primal simplex method for LP, and, like the dual simplex method for LP, dual variants of these approaches could be used for re-optimization after the addition of cutting planes.

Highlighting these conclusions, one can say that there are variants of cutting plane methods with good polynomial complexity, warm start capabilities, and ones that are very efficient in practice especially on large SDP's.

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