Noname manuscript No.
(will be inserted by the editor)

On QPCCs, QCQPs and Completely Positive Programs

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January 26, 2014 Received: date / Accepted: date

Abstract We derive equivalent convex completely positive reformulations for several classes of nonconvex optimization problems defined over convex cones, including rank-constrained semidefinite programs and quadratically constrained quadratic programs (QCQPs). The first stage in the reformulation is to cast the problem as a conic QCQP with just one nonconvex constraint $q(x) \leq 0$, where $q(x)$ is nonnegative over the (convex) conic and linear constraints, so the problem fails the Slater constraint qualification (CQ). A quadratic program with (linear) complementarity constraints (or QPCC) has such a structure; we prove the converse, namely that any QCQP can be expressed as an equivalent QPCC. The second stage of the reformulation lifts the problem to a completely positive program, and exploits and generalizes a result of Burer. We also show that a Frank-Wolfe type result holds for a subclass of this class of QCQPs. Another subclass is equivalent to a class of QPCCs. Our results do not make any boundedness assumptions on the feasible regions of the various problems considered.

The work of Bai and Mitchell was supported by the Air Force Office of Sponsored Research under grant FA9550-11-1-0260 and by the National Science Foundation under Grant Number CMMI-1334327. The work of Pang was supported by the National Science Foundation under Grant Number CMMI-1333902 and by the Air Force Office of Scientific Research under Grant Number FA9550-11-1-0151.

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Keywords QCQP · QPCC · Completely Positive Representation · rank-constrained SDP · Local Optimality

1 Introduction

The quadratically constrained quadratic program, abbreviated QCQP, is a constrained optimization problem whose objective and constraint functions are all quadratic. In this paper, we also allow conic convex constraints on the variables. Recent works [3,9,8,15,17,30,36] on the QCQP have developed algorithms of various kinds for solving such a program. Of particular relevance to our work herein are the most recent papers [3,15,17] that lift a QCQP satisfying a boundedness assumption to an equivalent completely positive program [16], which is a problem of topical interest. Also relevant to this paper is the observation that many of the early studies of the QCQP addressed the issue of existence of optimal solutions [11,35,44] via the well-known Frank-Wolfe theorem originally proved for a quadratic program [25].

Quadratic programs with (linear) complementarity constraints (QPCCs) are instances of QCQPs and can be written using a single quadratic constraint to capture the complementarity restriction. Such a complementarity constraint renders the QPCC a nonconvex disjunctive program even if the objective function is convex. A notable feature of a QPCC is that the quadratic constraint cannot be satisfied strictly. We focus on QCQPs that possess this property; namely, they may have several convex quadratic constraints but they have just one nonconvex quadratic constraint $q(x) \leq 0$, and further $q(x) \geq 0$ for any x that satisfies the linear and conic constraints (see problem (3) below). We show that any QCQP can be expressed in this form and it will have a linear objective function, generalizing a result for bounded QCQPs in [17]. Thus, the considered class of QCQPs is broad.

Playing an analogous role to that of a quadratic program in the class of nonlinear programs, the class of QPCCs [29,18] is a subclass of the class of mathematical programs with complementarity constraints (MPCCs) [34,37]. With increasingly many documented applications in diverse engineering fields, the MPCC provides a broad framework for the treatment of such problems as bilevel programs, inverse optimization, Stackelberg-Nash games, and piecewise programming; see [39]. Being a recent entry to the optimization field, the QPCC and its special case of a linear program with (linear) complementarity constraints (LPCC) have recently been studied in [6,28] wherein a logical Benders scheme [19] was proposed for their global resolution. We generalize the definition of QPCC in this paper to replace the nonnegative orthant by a convex cone. Ding, Sun and Ye [22] investigate MPCCs over the semidefinite cone, and in particular they show that a semidefinite program with a rank constraint can be cast as an equivalent LPCC over the semidefinite cone.

This paper addresses several topics associated with the QPCC and QCQP: existence of an optimal solution to a QCQP, the formulation of a QCQP as a QPCC, the local optimality conditions of a class of quadratically constrained nonlinear programs failing constraint qualifications, and the formulation of a QCQP as a completely positive program. Proved for a convex polynomial program as early as in the 1977 book [10] and reproved subsequently in [44, 35], the existence of an optimal solution to a convex QCQP over the nonnegative orthant is fully resolved via the classical Frank-Wolfe theorem, which states that such a minimization program, if feasible, has an optimal solution if and only if the objective function of the program is bounded below on the feasible set. The situation with a nonconvex QCQP is rather different; indeed, while there has been extensive work [4, 5, 12, 38] published on the existence of optimal solutions to nonconvex programs in general, the sharpest Frank-Wolfe type existence results for a feasible QCQP with a nonconvex (quadratic) objective are obtained in [35] and summarized as follows:

(a) if there is at most one nonlinear (but convex) quadratic constraint, then the QCQP attains its minimum if the objective function is bounded below on the feasible set;
(b) if the objective function is quasiconvex on the feasible region and all the constraints are convex, then the same conclusion as in (a) holds.

The authors in the latter reference also give an example of a QCQP with two convex quadratic constraints satisfying a Slater condition and a nonconvex objective function bounded below on the feasible set for which the infimum objective value is not attained. Since a Frank-Wolfe type existence result holds for a QPCC, it is natural to ask whether there is a class of QCQPs, broader than the class of QPCCs, for which a Frank-Wolfe type existence result holds. It turns out that the answer to this question is affirmative; interestingly, within this subclass of QCQPs, those which fail the Slater condition but are solvable will have rational optimal solutions if the data are rational numbers to begin with. These topics are addressed in Section 2.

Extending the QCQP, in Section 3 we consider a class of (nonconvex) quadratically constrained optimization problems that fail CQs, and show that checking the local optimality of this class of problems is equivalent to checking the global optimality of a mathematical program with a linearized objective function subject to the same constraints plus an imposed linear constraint. Checking the latter optimality condition is still hard; in the simplest case when the non-quadratic constraints are all linear, the linearized condition is equivalent to an LPCC.

Lastly, we show in Section 4 that a QCQP of the type considered in this paper, if the objective matrix is copositive, can be lifted to an equivalent completely positive program. This extends the recent papers [15, 17] that address the completely positive representations of binary nonconvex quadratic programs, certain types of quadratically constrained quadratic programs, and a number of other NP-hard problems. An introduction of basic concepts and a summary of recent developments in copositive programming can be found in a survey paper [23] and a book chapter [16]. Computational approaches for solving completely positive programs can be found in [13, 14, 20, 42, 45]. It follows from our result that a copositive QPCC is equivalent to its completely positive relaxation in terms of feasibility, boundedness, attainability as well as solvability.

The results of Section 4 are specialized to binary quadratic programs, QPCCs, and general QCQPs in Section 5. In particular, we show that any QCQP is equivalent to a convex completely positive program, if we first reformulate it as a QCQP with a linear objective function and just the one nonconvex constraint that doesn't have a Slater point. Further, a rank constrained SDP is also equivalent to a convex completely positive program, since it is equivalent to a QPCC over the semidefinite cone, with a linear objective function.

2 Two Classes of Quadratic Problems

We begin with the formal definitions of the classes of QPCCs and QCQPs. Specifically, given a symmetric matrix $Q \in \mathbb{R}^{N\times N}$, where $N \triangleq n+2m$, a vector $c \in \mathbb{R}^N$, a matrix $A \in \mathbb{R}^{k\times N}$, two closed convex cones $\mathcal{K}^0 \subseteq \mathbb{R}^n$ and $\mathcal{K}^1 \subseteq \mathbb{R}^m$, and a nonzero vector $b \in \mathbb{R}^k$, the QPCC is the minimization problem:

$$
\begin{array}{lll}
\underset{x \triangleq (x^0,x^1,x^2)}{\text{minimize}} & c^T x + \frac{1}{2}\, x^T Q x & \\
\text{subject to} & Ax = b \quad \text{and} \quad \langle\, x^1,\, x^2 \rangle = 0 & \qquad (1) \\
\text{with} & x^0 \in \mathcal{K}^0,\; x^1 \in \mathcal{K}^1,\; x^2 \in \mathcal{K}^{1*} &
\end{array}
$$

where $\mathcal{K}^{1*}$ is the dual cone to $\mathcal{K}^1$. The special features of this formulation are as follows: (a) the vector x is composed of 3 subvectors: $x^0 \in \mathcal{K}^0 \subseteq \mathbb{R}^n_+$, $x^1 \in \mathcal{K}^1 \subseteq \mathbb{R}^m$ and $x^2 \in \mathcal{K}^{1*} \subseteq \mathbb{R}^m$; the latter two variables are a complementary pair satisfying the complementarity condition: $\mathcal{K}^1 \ni x^1 \perp x^2 \in \mathcal{K}^{1*}$; (b) the other constraints on x are expressed as linear equations; and (c) the matrix Q is *not* necessarily positive semidefinite. For consideration of QPCCs with $\mathcal{K}^0 \times \mathcal{K}^1 \times \mathcal{K}^{1*} = \mathbb{R}^N_+$, see [6,7], for example. For mathematical programs with complementarity constraints with other cones, see [22], for example.

We may state the general QCQP as:

$$
\begin{array}{lll}
\underset{x \in \mathcal{K}}{\text{minimize}} & f_0(x) \triangleq (\, c^0\,)^T x + \frac{1}{2}\, x^T Q^{\,0} x & \\
\text{subject to} & Ax = b & \qquad (2) \\
\text{and} & f_i(x) \triangleq h_i + (\, c^i\,)^T x + \frac{1}{2}\, x^T Q^{\,i} x \,\le\, 0, \quad i = 1, \cdots, I, &
\end{array}
$$

for some closed convex cone $\mathcal{K} \subseteq \mathbb{R}^N$ and for some positive integer I, where each $h_i \in \mathbb{R}$, $c^i \in \mathbb{R}^N$, and $Q^{\,i} \in \mathbb{R}^{N\times N}$ for $i = 0, 1, \cdots, N$ are symmetric matrices. The bilinear equation $\langle x^1, x^2\rangle = 0$ in the QPCC is equivalent to the quadratic inequality $\langle x^1, x^2\rangle \le 0$ due to the conic constraints. We're particularly interested in problems where (some of) the quadratic constraints have no Slater point in $\{x \in \mathcal{K} : Ax = b\}$. Multiple constraints of this type can be aggregated into a single constraint, as we state in the following lemma, the proof of which is left to the reader:

Lemma 1 *Let $\mathcal{S} \subseteq \mathbb{R}^N$ and let $f_i(x), i = 1, \dots, I$ be quadratic functions with $f_i(x) \geq 0$ for all $x \in \mathcal{S}$. Define $q(x) := \sum_{i=1}^I f_i(x)$. Then*

$$\{x \in \mathcal{S} \,:\, f_i(x) \,\leq\, 0,\, i = 1, \dots, I\} \,=\, \{x \in \mathcal{S} \,:\, q(x) \,\leq\, 0\}.$$

In what follows, we exploit this lemma and restrict attention to problems of the form

$$\begin{aligned} &\underset{x \in \mathcal{K}}{\text{minimize}} \quad f_0(x) \,\triangleq\, (\,c^0\,)^T x + \tfrac{1}{2}\, x^T Q^{\,0} x \\ &\text{subject to } Ax \,=\, b \\ &\qquad\qquad\quad q(x) \,\triangleq\, \mathbf{h} + \mathbf{q}^T x + \tfrac{1}{2} x^T \mathbf{Q} x \,\leq\, 0 \\ &\text{and} \qquad\quad\ f_i(x) \,\triangleq\, h_i + (\,c^i\,)^T x + \tfrac{1}{2}\, x^T Q^{\,i} x \,\leq\, 0, \quad i \,=\, 1, \cdots, I, \end{aligned} \tag{3}$$

where $x \in \mathcal{P} := \{x \in \mathcal{K} : Ax = b\}$ implies $q(x) \geq 0$, and where the constraints $f_i(x) \leq 0$, $i = 1 \dots, I$, are convex. This framework includes binary quadratic programs (see §5.1).

Burer and Dong [17] showed that any QCQP with a bounded feasible region has an equivalent representation in the form (3). In the following theorem, we show that this result holds even if the feasible region is unbounded. Thus, any QCQP can be written in the form (3).

Theorem 1 *A generic quadratically constrained quadratic program*

$$\begin{aligned} &\underset{x \in \bar{K}}{\text{minimize}} \quad g_{k+1}^T x + \tfrac{1}{2}\, x^T M_{k+1} x \\ &\text{subject to } Ax \,=\, b \\ &\qquad\qquad\quad h_i + g_i^T x + \tfrac{1}{2} x^T M_i x \leq 0 \qquad i = 1, \dots, k, \end{aligned}$$

where $\bar{K}$ is a convex cone, is equivalent to a QCQP of the form (3) with $I = 0$ and with a linear objective function.

Proof First note that without loss of generality, we can assume the objective function is linear, so M_{k+1} is the zero matrix. If not, we could introduce an extra variable z and an additional constraint

$$g_{k+1}^T x + \tfrac{1}{2}\, x^T M_{k+1} x \,-\, z \,\leq\, 0,$$

and rewrite the objective function as minimize z. In order to simplify the notation of the proof, we assume the objective function is linear in the original formulation in the statement of the theorem.

Introduce a nonnegative variable t with $t = x^T x$. Let λ_i be greater than or equal to the negative of the largest eigenvalue of M_i. The matrix $M_i + \lambda_i I$ can then be factored as

$$R_i^T R_i \,=\, M_i \,+\, \lambda_i I.$$

Each quadratic constraint is then equivalent to the pair of constraints

$$\begin{aligned} &||R_i x||_2^2 \,+\, \tfrac{1}{4}(1 - \lambda_i t + 2 g_i^T x + 2h_i)^2 \,\leq\, \tfrac{1}{4}(1 + \lambda_i t - 2 g_i^T x - 2h_i)^2 \\ &1 \,+\, \lambda_i t \,-\, 2 g_i^T x \,-\, 2h_i \,\geq\, 0 \end{aligned} \tag{4}$$

using the construction of Alizadeh and Goldfarb [2]. This is equivalent to a second order cone constraint, and we let K_i denote the set of vectors $(t, x) \in \mathbb{R}^{n+1}$ satisfying this constraint for $i = 1, \ldots, k$.

The constraint $x^T x \leq t$ is equivalent to the quadratic constraint

$$||(x_0, x)||_2^2 + \tfrac{1}{4}(1-t)^2 \leq \tfrac{1}{4}(1+t)^2 \tag{5}$$

with $t \geq 0$, which is equivalent to a second order cone constraint. Let K_0 denote the set of vectors $(t, x) \in \mathbb{R}^{n+1}$ satisfying this second order cone constraint. Let $K = (\cap_{i=0}^{k} K_i) \cap \bar{K}$. The QCQP is then equivalent to the problem

$$\begin{aligned} \underset{(t,x_0,x) \in K \subseteq \mathbb{R}^{n+2}}{\text{minimize}} \quad & g_{k+1}^T x \\ \text{subject to} \quad & Ax = b \\ & t - x^T x \leq 0 \end{aligned} \tag{6}$$

which is a problem of the form (3) with $I = 0$, since $(t, x) \in K_0 \supseteq K$ implies $t - x^T x \geq 0$. □

A Frank-Wolfe (FW) result for either (1) or (3) states that the program attains a finite minimum objective value if and only if it is feasible and the quadratic objective function is bounded below on the feasible set. We exploit the following proposition to derive a Frank-Wolfe result. We first define

$$\widehat{\mathcal{P}} \triangleq \{ x \in \mathcal{P} \mid q(x) \leq 0 \}.$$

Proposition 1 *If $I = 0$ then the QCQP (3) can be expressed as an equivalent QPCC of the form (1).*

Proof Under the constant-sign assumption on $q(x)$, it follows that, if $\widehat{\mathcal{P}} \neq \emptyset$, then

$$\widehat{\mathcal{P}} = \left[\underset{x \in \mathcal{P}}{\text{argmin}}\, q(x) \right] = \{ x \in \mathcal{P} \mid q(x) = 0 \}.$$

The condition that $-\nabla q(x)$ be in the normal cone to $\mathcal{P}$ at a point $x \in \mathcal{P}$ can be expressed as:

$$\begin{aligned} \mathcal{K} \ni x \perp \mathbf{q} + \mathbf{Q}x + A^T \lambda \in \mathcal{K}^* \\ 0 = Ax - b. \end{aligned}$$

Note further that if these conditions hold then $q(x) = \frac{1}{2} \left(\mathbf{q}^T x - b^T \lambda \right) + \mathbf{h}$ (see [26,27]). Thus, we obtain an equivalent QPCC

$$\begin{aligned} \underset{(\lambda,x,w)}{\text{minimize}} \quad & (c^0)^T x + \tfrac{1}{2}\, x^T Q^0 x \\ \text{subject to} \quad & Ax = b \\ & \mathbf{q} + \mathbf{Q}x + A^T \lambda - w = 0 \\ & \tfrac{1}{2} \left(\mathbf{q}^T x - b^T \lambda \right) + \mathbf{h} = 0 \\ & x^T w = 0 \\ \text{and} \quad & \lambda \in \mathbb{R}^k,\, x \in \mathcal{K},\, w \in \mathcal{K}^* \end{aligned}$$

of the required form. □

The following corollary follows directly from Theorem 1 and Proposition 1.

Corollary 1 *Any QCQP can be reformulated as an equivalent QPCC, by first using the construction of Theorem 1.*

We have a second corollary to the proposition when $\mathcal{K}$ is a polyhedral cone. Since the feasible set of the QPCC is the union of finitely many polyhedra and since the FW theorem holds for general quadratic programs, it follows readily that such an attainment result holds for the QPCC. The following results extend this conclusion to a class of QCQPs.

Corollary 2 *Assume $\mathcal{K}$ is a polyhedral cone. The feasible region of the QPCC constructed as in the proof of Proposition 1 consists of a union of a finite number of polyhedra,* $\bigcup_{j=1}^{J} P^j$.

Based on the above piecewise representation of the set $\widehat{P}$, we have the following result for the QCQP (3) when $\mathcal{K}$ is a polyhedral cone.

Theorem 2 *Assume the cone $\mathcal{K}$ is polyhedral and $I \leq 1$. Then the FW attainment result holds for the QCQP (3).*

Proof As a result of Corollary 2, it follows that the feasible region of (3) is equal to $\bigcup_{j=1}^{J} \widehat{P}^j$, where each $\widehat{P}^j \triangleq \left\{ x \in P^j \mid f_1(x) \leq 0 \right\}$ is the zero-level set of the convex quadratic function f_I over the polyhedron P^j. By [35, Theorem 2], the FW attainment result holds for each of the convex singly-quadratically constrained quadratic programs $\underset{x \in \widehat{P}^j}{\text{minimize}}\, f_0(x)$. Thus, the same result holds for the QCQP (3). □

QCQPs satisfying the assumption of Theorem 2 are nonconvex programs that fail the Slater constraint qualification; yet, without the convex quadratic constraints, these QCQPs have an interesting property that we highlight in the following result.

Proposition 2 *Suppose that $I = 0$ and $\mathcal{K}$ is a polyhedral cone. If the QCQP (3) has an optimal solution, then it has a rational optimal solution, provided that the input vector $\mathbf{c}$, scalar $\mathbf{h}$ and matrix $\mathbf{Q}$, and the pair (A, b) have rational components.*

Proof By the proof of Theorem 2, it follows that the QCQP (3) has an optimal solution that is an optimal solution of a (possibly nonconvex) quadratic program. Thus, it suffices to show that if the input data of a solvable quadratic program are all rational, then the program has a rational optimal solution. In turn, this claim follows from two facts: (a) the set of global minima of a quadratic program is the union of finitely many polyhedra, a result [24, Exercise 2.9.31] derived from the fact [33] that the objective value of a quadratic program attains finitely many values on the set of stationary points, and (b) defined by finitely many linear inequalities and equations, any nonempty polyhedron must contain at least one rational point, provided that the data of its linear constraints are rational numbers. □

Remark 1 Proposition 2 is false if the QCQP has a quadratic inequality constraint that has a Slater point. The scalar problem: minimize x subject to $x^2 \leq 2$ provides a simple counterexample that illustrates the failure of the proposition under the Slater assumption. □

3 Local Optimality for QC-Problems Failing CQs

Although Theorem 2 has resolved the issue of solvability of the QCQP (3) satisfying the assumptions of this theorem, the question of how to characterize the optimality of a solution to such a problem is not addressed by this theorem, nor is it treated in the current literature, due to the lack of a suitable constraint qualification. In what follows we deal with this issue for the following problem:

$$\begin{aligned} &\underset{x \,\in\, C \,\cap\, \mathcal{P}}{\text{minimize}} \;\; \theta(x) \\ &\text{subject to } q(x) \,\triangleq\, \mathbf{h} + \mathbf{q}^T x + \tfrac{1}{2} x^T \mathbf{Q} x \;\leq\; 0, \end{aligned} \tag{7}$$

where C is a closed convex set, $\mathcal{P} \triangleq \{x \in \mathcal{K} \mid Ax = b\}$ where $\mathcal{K}$ is a convex cone, and $\theta : \mathbb{R}^n \to \mathbb{R}$. In (7), the two sets C and $\mathcal{P}$ play distinctive roles. The set C is an arbitrary closed convex set not required to be a polyhedron; e.g., C can be the intersection of the level-sets $\{x \mid f_i(x) \leq 0\}$ for the convex function $f_i(x)$ for $i = 1, \ldots, I$. The set $\mathcal{P}$ is a polyhedron on which $q(x)$ is nonnegative. The feasible region of (7), denoted S, is in general nonconvex. We have the following necessary and sufficient condition of local optimality for (7); in particular, the last assertion of the following theorem extends the conclusion for a "convex" MPCC, that is, a convex program with additional linear complementarity constraints.

Theorem 3 *Let $C \subseteq \mathbb{R}^n$ be closed convex and $\theta : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let $S = \{\, x \in C \cap \mathcal{P} \mid q(x) = 0 \,\}$. Let x^* be a feasible solution of (7). Consider the following statements:*

(a) *x^* is a locally optimal solution of (7);*
(b) *x^* is a globally optimal solution of (8):*

$$\begin{aligned} &\underset{x \,\in\, S}{\text{minimize}} \;\; (\, x - x^* \,)^T \nabla\theta(x^*) \\ &\text{subject to } (\, x - x^* \,)^T \, \nabla q(x) \;\leq\; 0; \end{aligned} \tag{8}$$

(c) *x^* is a locally optimal solution of the following:*

$$\underset{x \,\in\, S}{\text{minimize}} \; (\, x - x^* \,)^T \nabla\theta(x^*). \tag{9}$$

It holds that (a) $\Rightarrow$ (b). If $\mathcal{K}$ is a polyhedral cone then (b) $\Rightarrow$ (c). If $\mathcal{K}$ is a polyhedral cone and θ is convex on $C \cap \mathcal{P}$, then (b) $\Rightarrow$ (a).

Proof (a) $\Rightarrow$ (b). Assume by way of contradiction that x^* is not a globally optimal solution to (8). Then there exists $\bar{x}$ feasible to (8) such that $d^T \nabla \theta(x^*) < 0$, where $d \triangleq \bar{x} - x^*$. Note that

$$q(x^* + \lambda d) \, = \, q(x^*) + \lambda(\, \mathbf{q} + \mathbf{Q} x^* \,)^T d + \frac{\lambda^2}{2} \, d^T \mathbf{Q} d. \tag{10}$$

Since $q(x) \geq 0$ for all $x \in \mathcal{P}$, we must have $(\mathbf{q} + \mathbf{Q} x^*)^T d \geq 0$; therefore $(\mathbf{q} + \mathbf{Q} x^*)^T d = 0$. Since $q(x^*) = 0$ and $q(\bar{x}) = 0$ we must have $d^T \mathbf{Q} d = 0$. Therefore $x^* + \lambda d$ is feasible in (8) for all $\lambda \in [0, 1]$; hence $\theta(x^* + \lambda d) \geq \theta(x^*)$ for all $\lambda \geq 0$ sufficiently small, implying that $d^T \nabla \theta(x^*) \geq 0$. This is a contradiction.

(b) $\Rightarrow$ (c) if $\mathcal{K}$ is polyhedral. We show the contrapositive. Assume x^* is not a locally optimal solution to (9). Then a sequence $\{x_k\} \subset S$ converging to x^* exists such that $(x_k - x^*)^T \nabla \theta(x^*) < 0$ for all k. By Corollary 2, we have

$$S \, = \, \{ \, x \, \in \, C \, \cap \, \mathcal{P} \, \mid \, q(x) \, = \, 0 \, \} \, = \, C \, \cap \, \bigcup_{j=1}^{J} P^{\,j} \, .$$

There must exist a point x_{k_0} in the sequence such that x_{k_0} and x^* belong to the same piece $P^{j_0} \cap C$. Let $d \triangleq x_{k_0} - x^*$. Since $P^{j_0} \cap C$ is convex, $x^* + \lambda d \in S$ for all $\lambda \in [0, 1]$. From the same argument as in the previous part, we again obtain $(\mathbf{q} + \mathbf{Q} x^*)^T d = 0$, or $\nabla q(x^*)^T d = 0$, so $x^* + \lambda d$ is feasible in (8) for all $\lambda \in [0, 1]$. Therefore x^* is not a globally optimal solution of (8).

(b) $\Rightarrow$ (a) if $\mathcal{K}$ is a polyhedral cone and θ is convex on $C \cap \mathcal{P}$. Assume by way of contradiction that x^* is not a local minimum to (7). There exists a sequence $\{x_k\}$ feasible to (7) converging to x^* such that $\theta(x_k) < \theta(x^*)$ for all k. As in the proof of (b) implying (c), there exists an x_{k_0} such that $x^* + \lambda d$ is feasible in (7) for all $\lambda \in [0, 1]$, where $d \triangleq x_{k_0} - x^*$. Thus $q(x^* + \lambda d) = 0$ for all such λ. The expansion (10) then implies that $(\mathbf{q} + \mathbf{Q} x^*)^T d = 0$. Thus $x^* + \lambda d$ is feasible in (8) for all $\lambda \in [0, 1]$; in particular, so is x_{k_0}. As $\theta(x)$ is convex, we have

$$\theta(x_{k_0}) \, \geq \, \theta(x^*) + \nabla \theta(x^*)^T (x_{k_0} - x^*),$$

which implies

$$\nabla \theta(x^*)^T (x_{k_0} - x^*) \, \leq \, \theta(x_{k_0}) - \theta(x^*) < 0,$$

which means x^* is not a global optimum for (8). This completes the contrapositive proof. □

The example below shows that if x^* is a locally optimal solution to (7), it is not necessarily a *globally* optimal solution of

$$\underset{x \, \in \, S}{\text{minimize}} \; (\, x - x^* \,)^T \nabla \theta(x^*); \tag{11}$$

thus, to obtain a characterization of x^* as a globally optimal solution of a "linearized problem", it is essential that we add the extra constraint $\nabla q(x^*)^T(x - x^*) \leq 0$, yielding the problem (8). The example also shows that the implication "(c) $\Rightarrow$ (b)" does not hold for the elements of the theorem, even when $\theta(x)$ is convex on $C \cap \mathcal{P}$.

Example 1 Consider the following simple 2-variable QPCC:

$$\begin{array}{ll} \underset{x_1,x_2}{\text{minimize}} & (x_1 - 2)^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ \text{and} & 0 \leq x_1 \perp x_2 \geq 0. \end{array}$$

The optimal solution to this "convex" QPCC is $x^* = (3, 0)$. The corresponding problem (11) is

$$\begin{array}{ll} \underset{x_1,x_2}{\text{minimize}} & 2(x_1 - 3) \\ \text{subject to} & x_1 + x_2 \geq 3 \\ \text{and} & 0 \leq x_1 \perp x_2 \geq 0, \end{array}$$

whose optimal solutions are all points of the form $(0, x_2)$ with $x_2 \geq 3$, none of these is the point $x^* = (3, 0)$. Adding the linearized complementarity constraint, we obtain the problem:

$$\begin{array}{ll} \underset{x_1,x_2}{\text{minimize}} & 2(x_1 - 3) \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & 0 \leq x_1 \perp x_2 \geq 0 \\ \text{and} & 3x_2 \leq 0, \end{array}$$

whose unique globally optimal solution is precisely x^*. □

In [40], four types of stationary points are defined for MPCCs, among which the so-called Bouligand or B-stationarity [34] yields the strongest conclusions; see also [41] where the concept of stationarity is generalized to nonlinear programs with "structurally nonconvex" feasible sets that include MPCCs. Checking B-stationarity is equivalent to solving an LPCC, therefore it is hard. To date, there is no clear understanding of the stationarity condition for a general nonconvex mathematical program that fails constraint qualifications (CQs). The problem (8) generalizes the idea of B-stationarity for a MPCC. Specifically consider the special case of (7) where $x \triangleq (x^0, x^1, x^2)$, $\mathcal{P}$ is a polyhedron that is a subset of $\mathbb{R}^n \times \mathbb{R}^{2m}_+$, and $q(x) \triangleq (x^1)^T x^2$. For a feasible vector $x^* \triangleq (x^{*,0}, x^{*,1}, x^{*,2})$ of (7), define the 3 index sets pertaining to the complementarity condition: $0 \leq x^1 \perp x^2 \geq 0$:

$$\begin{aligned} \alpha_* &\triangleq \{ i \in 1, \ldots, m \mid x_i^{*,1} > 0 = x_i^{*,2} \} \\ \beta_* &\triangleq \{ i \in 1, \ldots, m \mid x_i^{*,1} = 0 = x_i^{*,2} \} \\ \gamma_* &\triangleq \{ i \in 1, \ldots, m \mid x_i^{*,1} = 0 < x_i^{*,2} \}. \end{aligned}$$

These 3 index sets play a key role in the B-stationarity of the MPCC:

$$\begin{array}{ll} \underset{x \triangleq (x^0, x^1, x^2) \in C \cap \mathcal{P}}{\text{minimize}} & \theta(x) \\ \text{subject to} & 0 \leq x^1 \perp x^2 \geq 0. \end{array} \tag{12}$$

It is not difficult to show that with S denoting the feasible region of (12) and $q(x) \triangleq (x^1)^T x^2$, we have

$$\begin{array}{l} \{ x \in S \mid \nabla q(x^*)^T (x - x^*) \leq 0 \} = \\ \quad \{ x \in C \cap \mathcal{P} \mid x_i^2 = 0 \, \forall \, i \in \alpha_*; \; x_i^1 = 0 \, \forall \, i \in \gamma_*; \; 0 \leq x_i^1 \perp x_i^2 \geq 0 \}. \end{array}$$

The above expression gives 2 structurally different representations of the same set; the left-hand representation expresses the set as defined by the closed set C intersected with the linearly-quadratically constrained set $\{x \in \mathcal{P} \mid q(x) \leq 0, \, \nabla q(x^*)^T (x - x^*) \leq 0\}$, whereas the right-hand representation reveals the disjunctive structure of the latter set with reference to the given point x^* and shows that it is the union of finitely many closed convex sets. When in addition C is a polyhedron, then the problem (8) is a LPCC with a linear objective function. The upshot of this development is that in this case, the latter LPCC yields the optimality conditions of the quadratically constrained optimization problem (7) that fails the Slater constraint qualification.

4 Completely Positive Representation of QCQPs

The class of QCQPs that we consider in this section is of the form given in (3) with $I = 0$, with $q(x) \geq 0$ if $x \in \mathcal{P} = \{x \in \mathcal{K} : Ax = b\}$, so there is no Slater point in the constraint $q(x) \leq 0$. Denote the recession cone of $\mathcal{P}$ by

$$\mathcal{L}_\infty \triangleq \{ d \in \mathcal{K} \mid A \, d = 0 \}. \tag{13}$$

It follows from the lack of a Slater point that the matrix $\mathbf{Q}$ is copositive on $\mathcal{L}_\infty$, as we prove in the following lemma.

Lemma 2 *If $q(x) \geq 0$ for all $x \in \mathcal{P} \neq \emptyset$ then $d^T \mathbf{Q} d \geq 0$ for all $d \in \mathcal{L}_\infty$.*

Proof Let $\bar{d} \in \mathcal{L}_\infty$ with $\bar{d}^T \mathbf{Q} d < 0$ and let $\bar{x} \in \mathcal{P}$. Then $\bar{x} + \alpha \bar{d} \in \mathcal{P}$ for all $\alpha \geq 0$, but $q(\bar{x} + \alpha \bar{d}) < 0$ for sufficiently large α. □

Note that a QPCC satisfies these assumptions. In addition, a binary restriction on a variable x_j represented by the quadratic inequality $x_j - x_j^2 \leq 0$ also satisfies the assumptions, provided $0 \leq x_j \leq 1$ for all $x \in \mathcal{P}$. Further, the QCQP constructed in Theorem 1 also satisfies this assumption, from the construction of the cone K_0 in the proof. In this section, we investigate a completely positive relaxation of (3).

A completely positive program is a linear optimization problem in matrix variables in the form of:

$$\begin{aligned} &\underset{X\in\mathcal{S}^{1+n}}{\text{minimize}} \;\; < A_0, X > \\ &\text{subject to} < A_i, X > = b_i, \quad i = 1, \cdots, \ell \\ &\text{and} \qquad\;\; X \in \mathcal{C}^*_{1+n}(\mathcal{K}), \end{aligned}$$

where $\mathcal{S}^{1+n}$ is the space of symmetric $(1+n)\times(1+n)$ matrices and $\mathcal{C}^*_{1+n}(\mathcal{K})$ is the cone of completely positive matrices over $\mathcal{K}$,

$$\mathcal{C}^*_{1+n}(\mathcal{K}) \triangleq \text{conv}\left\{ M \in \mathcal{S}^{1+n} \mid M = xx^T,\, x \in \mathbb{R}_+ \times \mathcal{K} \right\},$$

whose dual $\mathcal{C}_{1+n}(\mathcal{K})$ is the cone of copositive matrices over $\mathcal{K}$,

$$\mathcal{C}_{1+n}(\mathcal{K}) \triangleq \left\{ M \in \mathcal{S}^{1+n} \mid x^T M x \geq 0,\, \forall x \in \mathbb{R}_+ \times \mathcal{K} \right\}.$$

We also use the notation $\mathcal{C}^*_{1+n} := \mathcal{C}^*_{1+n}(\mathbb{R}^n_+)$ and $\mathcal{C}_{1+n} := \mathcal{C}_{1+n}(\mathbb{R}^n_+)$.

The QCQP (3) with $I = 0$ is equivalent to the following problem:

$$\begin{aligned} &\underset{x\in\mathcal{K}}{\text{minimize}} \;\; (c^0)^T x + \tfrac{1}{2} \left\langle Q^0,\, X \right\rangle \\ &\text{subject to } A_i\, x = b_i, \quad i = 1, \cdots, k, \\ &\qquad\qquad q(x) \leq 0 \quad \text{and} \quad X = xx^T. \end{aligned} \tag{14}$$

By relaxing the rank-1 constraint over the matrix X to the condition that matrix $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ is in the cone of completely positive matrices over $\mathcal{K}$, we get the following completely positive program:

$$\begin{aligned} &\underset{x,\,X}{\text{minimize}} \;\; (c^0)^T x + \tfrac{1}{2} \left\langle Q^0,\, X \right\rangle \\ &\text{subject to } A_i\, x = b_i \quad \text{and} \quad A_i\, X\, A_i^T = b_i^2, \quad i = 1, \cdots, k, \\ &\qquad\qquad \mathbf{h} + \mathbf{c}^T x + \tfrac{1}{2} \left\langle \mathbf{Q}, X \right\rangle = 0 \\ &\text{and} \qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{1+n}(\mathcal{K}). \end{aligned} \tag{15}$$

Note that we have exploited the assumption regarding the lack of a Slater point in this formulation.

Burer [15] addressed the copositive representations of a binary or continuous nonconvex QCQP of the form (3), where $\mathcal{K} = \mathbb{R}^n_+$ and $x \in \mathcal{P}$ implies $0 \leq x_i \leq 1$ for the binary variables x_i and with restrictions on the quadratic constraints. He proved that a completely positive relaxation is equivalent to the original QCQP. Burer [16] showed that the results of [15] extend to the situation of a general convex cone $\mathcal{K}$, and showed that the completely positive relaxation is equivalent to the original problem under the assumptions that $d^T\mathbf{Q}d = 0$ for all $d \in \mathcal{L}_\infty$ and that $q(x)$ is bounded above and below on $\{x \in \mathcal{K} : Ax = b\}$. These two assumptions are not satisfied by either a general QPCC of the form (1) or by the QCQP constructed in Theorem 1. Burer and Dong [17] extended the results of [15] to general QCQP's where $\mathcal{P}$ is bounded. Dickinson et al. [21] have developed similar results for more general sets, under certain assumptions.

As a special instance of a QCQP albeit with bilinear quadratic constraints, can a QPCC be cast as a completely positive program? The answer to this question is affirmative if the objective function of the QPCC is copositive over a particular subset of $\mathcal{L}_\infty$. This is true even if the feasible set of the QPCC is unbounded. In what follows, we prove a more general result. In particular, the equivalence will be established between a QCQP whose single constraint has no Slater point and whose objective function is copositive on a particular subset of $\mathcal{L}_\infty$, and its completely positive representation. Our proof borrows from that in [16].

In general, a QPCC with a nonconvex objective function is not guaranteed to have an equivalent copositive representation without using the construction of Theorem 1, except under limited conditions such as bounded complementarity variables, as we show in the following example.

Example 2

$$\begin{array}{ll} \underset{x \geq 0}{\text{miminize}} & -x_2^2 \\ \text{subject to} & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_4 \ = \ 2 \\ \text{and} & x_1 \ \perp \ x_2. \end{array}$$

The optimal objective value of the above QPCC is 0 and the only feasible ray is $d \triangleq (1, 0, 1, 1)$. Below is the completely positive representation of this QPCC:

$$\begin{array}{ll} \underset{x,\,X}{\text{minimize}} & -X_{2,2} \\ \text{subject to} & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_4 \ = \ 2 \\ & X_{1,1} + 2X_{1,2} + X_{2,2} - 2X_{1,3} - 2X_{2,3} + X_{3,3} = 1 \\ & X_{1,1} + X_{4,4} - 2X_{1,4} = 4 \\ & X_{1,2} = 0 \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_5^*. \end{array}$$

Denote $\bar{d} \triangleq (0, 1, 1, 0)$ and it is easy to show that $\begin{pmatrix} 0 & 0 \\ 0 & \bar{d}\,\bar{d}^T \end{pmatrix}$ is a feasible ray of the above completely positive program, therefore this completely positive problem is unbounded below. $\square$

The completely positive program (15) is not only a relaxation problem but also an equivalent form of the QCQP. To prove the equivalence, we define the following sets:

$$
\begin{aligned}
\mathit{\Gamma} &\triangleq \{\, x \in \mathcal{K} \mid Ax = b \text{ and } q(x) \le 0 \,\} = \{ x \in \mathcal{P} \mid q(x) \le 0 \,\} \\
L &\triangleq \left\{\, d \in \mathcal{K} \mid A_i\, d = 0,\ i = 1, 2, \cdots, k, \text{ and } d^T \mathbf{Q} d = 0 \,\right\} \subseteq \mathcal{L}_\infty \\
\mathit{\Gamma}^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \mid x \in \mathit{\Gamma} \right\} \\
L^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & dd^T \end{pmatrix} \mid d \in L \right\} \\
\mathcal{L}_\infty^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & dd^T \end{pmatrix} \mid d \in \mathcal{L}_\infty \right\} \\
\mathit{\Sigma}^+ &\triangleq \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \mid (x, X) \text{ feasible for (15)} \right\}.
\end{aligned}
\tag{16}
$$

Among the above sets, $\mathit{\Gamma}$ is the set of feasible solutions of QPCC (1), $\mathit{\Gamma}^+$ is isomorphic to the convex hull of the set of feasible solutions of program (14), and $\mathit{\Sigma}^+$ is the set of feasible solutions of the completely positive program (15). In general, the set $\mathit{\Gamma}^+$ might not be closed, but all the other listed sets are closed. The following result is due to Burer [16].

Theorem 4 *(Theorem 8.3, [16]) Assume $d^T Q d = 0$ for all $d \in \mathcal{L}_\infty$ and $q(x)$ is bounded above on $\mathcal{P}$. Then*

$$\mathit{\Gamma}^+ + \mathcal{L}_\infty^+ = cl(\mathit{\Gamma}^+) = \mathit{\Sigma}^+.$$

Neither of the assumptions in Theorem 4 is valid for QPCCs (1) or for the QCQP constructed in Theorem 1. As a result of the cone constraint $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_{1+n}^*(\mathcal{K})$, it is clear that $\mathit{\Gamma}^+ \subseteq \mathit{\Sigma}^+$ and L^+ is contained in the recession cone of $\mathit{\Sigma}^+$. Therefore $\mathit{\Gamma}^+ + L^+ \subseteq \mathit{\Sigma}^+$. We establish the converse inclusion in the proof of the following proposition.

Proposition 3 *If $q(x) \ge 0$ for all $x \in \mathcal{P}$, then $\mathit{\Sigma}^+ = \mathit{\Gamma}^+ + L^+$.*

Proof Assume (x, X) is feasible to (15). As shown in Proposition 8.2 in [16], we have two finite index sets J_+ and J_0 with

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{j \in J_+} v_j^2 \begin{pmatrix} 1 \\ \xi_j \end{pmatrix} \begin{pmatrix} 1 \\ \xi_j \end{pmatrix}^T + \sum_{j \in J_0} \begin{pmatrix} 0 \\ d_j \end{pmatrix} \begin{pmatrix} 0 \\ d_j \end{pmatrix}^T \in \mathcal{C}_{1+n}^*(\mathcal{K})$$

where

1. $\sum_{j \in J_+} v_j^2 = 1$ and $v_j \neq 0$ for all $j \in J_+$;
2. $\xi_j \in \mathcal{P}$ for all $j \in J_+$;
3. $d_j \in \mathcal{L}_\infty$ for all $j \in J_0$.

From points 2 and 3, we have that $q(\xi_j) \geq 0$ for all $j \in J_+$, and $d_j^T \mathbf{Q} d_j \geq 0$ for all $j \in J_0$ from Lemma 2. In particular, we have

$$0 \leq \mathbf{h} + \mathbf{c}^T \xi_j + \frac{1}{2} \xi_j^T \mathbf{Q} \xi_j \quad \forall j \in J_+.$$

Multiplying by v_j^2 and adding over J_+, we obtain

$$\begin{aligned} 0 &\leq \mathbf{h} \sum_{j \in J_+} v_j^2 + \sum_{j \in J_+} v_j^2 \, \mathbf{c}^T \xi_j + \tfrac{1}{2} \sum_{j \in J_+} v_j^2 \xi_j^T \mathbf{Q} \xi_j \\ &= \mathbf{h} + \sum_j v_j^2 \, \mathbf{c}^T \xi_j + \tfrac{1}{2} \sum_{j \in J_+} v_j^2 \xi_j^T \mathbf{Q} \xi_j + \tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j - \tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j \\ &= -\tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j \\ &\leq 0 \end{aligned}$$

since $\sum_{j \in J_+} v_j^2 = 1$, $d_j^T \mathbf{Q} d_j \geq 0$ for all $j \in J_0$, and the matrix is feasible in (15). It follows that

$$\mathbf{h} + \mathbf{c}^T \xi_j + \frac{1}{2} \xi_j^T \mathbf{Q} \xi_j = 0 \quad \forall j \in J_+$$

$$\text{and} \quad d_j^T \mathbf{Q} d_j = 0 \quad \forall j \in J_0.$$

Therefore $\xi_j \in \Gamma$ for all $j \in J_+$ and $d_j \in L$ for all $j \in J_0$. Therefore, $\Sigma^+ \subseteq \Gamma^+ + L^+$. □

Based on Proposition 3, we can now establish the claimed equivalence between the QCQP (3) formulated as (14) and the completely positive program (15).

Theorem 5 *Assume that any point $x \in \mathcal{P}$ satisfies $q(x) \geq 0$. Assume further that Q^0 is copositive over conv(L). The QCQP (3) and the completely positive program (15) are equivalent in the sense that*

1. *The QCQP (3) is feasible if and only if the completely positive program (15) is feasible.*
2. *Either the optimal values of the QCQP (3) and the completely positive program (15) are finite and equal, or both of them are unbounded below.*
3. *Assume both the QCQP (3) and the completely positive program (15) are bounded below, and $(\bar{x}, \bar{X})$ is optimal for the completely positive program, then $\bar{x}$ is in the convex hull of the set of optimal solutions of the QCQP.*

4. The optimal value of the QCQP (3) is attained if and only if the same holds for the completely positive program (15).

Proof 1. Assume (x, X) is a feasible solution of the completely positive program (15). Then

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{j\in J_+} \alpha_j \begin{pmatrix} 1 \\ \xi_j \end{pmatrix} \begin{pmatrix} 1 \\ \xi_j \end{pmatrix}^T + \sum_{j\in J_0} \begin{pmatrix} 0 \\ d_j \end{pmatrix} \begin{pmatrix} 0 \\ d_j \end{pmatrix}^T,$$

where $\alpha_j > 0$ for all $j \in J_+$ and $\sum_{j\in J_+} \alpha_j = 1$. According to the proof of Proposition 3, $\{\xi_j, \forall j \in J_+\}$ are feasible for the QCQP (3). On the other hand, if x is feasible for the QCQP (3), then $\begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ is feasible for the completely positive program (15).

2. Denote the optimal values of the QCQP (3) and the completely positive program (15) as Opt(3) and Opt(15) respectively. Since (3) is equivalent to (14) and since (15) is a relaxation of (14), it is immediate that Opt(3) $\geq$ Opt(15).

Since Q^0 is copositive over L,

$$\begin{aligned} \mathrm{Opt}(15) &\triangleq \underset{Y\in\Sigma^+=\Gamma^++L^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle \\ &\geq \underset{Y\in\Gamma^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle = \mathrm{Opt}(3). \end{aligned}$$

Therefore, Opt(15) $\geq$ Opt(3). As a result, the two optimum objectives are equal.

3. Assume that $(\bar{x}, \bar{X})$ is optimal for the completely positive program (15). Then there exist $\bar{\alpha}_j$, $\bar{\xi}_j$, $\bar{d}_j$, J_+ and J_0 such that

$$\begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} = \sum_{j\in J_+} \bar{\alpha}_j \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix}^T + \sum_{j\in J_0} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix}^T.$$

Here $\bar{\alpha}_j > 0$ for all j and $\sum_{j\in J_+} \bar{\alpha}_j = 1$. In addition, $\{\bar{\xi}_j, \forall j \in J_+\} \subset \Gamma$ and $\{\bar{d}_j, \forall j \in J_0\} \subset L$. It follows from the optimality of $(\bar{x}, \bar{X})$ that $\langle Q^0, \bar{d}_j \bar{d}_j^T \rangle = 0$ for all $j \in J_0$. Therefore

$$\begin{aligned} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, \begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} \right\rangle &= \sum_{j\in J_+} \bar{\alpha}_j \left[(c^0)^T \bar{\xi}_j + \tfrac{1}{2} \bar{\xi}_j^T Q^0 \bar{\xi}_j \right] \\ &\geq \sum_{j\in J_+} \bar{\alpha}_j \, \mathrm{Opt}(3) = \mathrm{Opt}(3). \end{aligned}$$

As Opt(3) = Opt(15), $(c^0)^T\bar{\xi}_j + \frac{1}{2}\bar{\xi}_j^T Q^0 \bar{\xi}_j = \text{Opt}(3)$ for all $j \in J_+$. In other words, $\{\bar{\xi}_j, \forall\, j \in J_+\}$ are all optimal for QCQP (3). Therefore, $\bar{x} = \sum_{j\in J_+} \bar{\alpha}_j \bar{\xi}_j$ is in the convex hull of the optimal solutions of the QCQP.

4. If $\hat{x}$ is an optimal solution to the QCQP (3), the completely positive program (15) is solved by $\begin{pmatrix} 1 \\ \hat{x} \end{pmatrix}\begin{pmatrix} 1 \\ \hat{x} \end{pmatrix}^T$. On the other hand, if the completely positive program (15) is bounded below and solved by

$$\begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} = \sum_{j\in J_+} \bar{\alpha}_j \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix}\begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix}^T + \sum_{j\in J_0} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix}\begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix}^T,$$

then the QCQP is solved by any $\bar{\xi}_j$ such that $j \in J_+$ and $\bar{\alpha}_j > 0$. $\square$

Our results require a copositivity assumption over the recession cone of the linear and conic constraints. In the paper by Sturm and Zhang [43], the authors discuss cones of nonnegative quadratic functions and impose a copositivity restriction on a more general set. More recently, the papers [31, 32] have addressed quadratically constrained quadratic programs by examining cones of nonnegative quadratic functions, based on the programs' KKT conditions that are lifted along with the feasible region of the QCQPs to define a certain cone of symmetric matrices. It would be interesting to investigate the detailed connections of the latter papers with our work; this investigation is left for a future study.

5 Specializations of Theorem 5

5.1 Binary Quadratic Programs

Burer [15] proved that a mixed binary nonconvex quadratic program,

$$\begin{array}{lll} \underset{x\geq 0}{\text{minimize}} & \frac{1}{2}x^TQ^0x + {c^0}^T x & \\ \text{subject to} & a_i x = b_i, \quad \forall i = 1, 2, \cdots, k, & \\ \text{and} & x_j \in \{0, 1\}, \quad \forall j \in B, & \end{array} \tag{17}$$

is equivalent to its copositive representation,

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} < Q^0, X > + {c^0}^T x \\ \text{subject to} & a_i x = b_i \quad \text{and} \quad a_i X a_i^T = b_i^2, \quad \forall i = 1, 2, \cdots, k, \\ & x_j = X_{jj}, \quad \forall j \in B, \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{1+n} \end{array} \tag{18}$$

under Burer's key assumption that the linear constraints imply $0 \leq x_j \leq 1$ $\forall j \in J$. The binary constraints in (17),

$$x_j \in \{0,1\}, \forall j \in B,$$

can simply be represented as a single quadratic constraint

$$q(x) = \sum_{j \in B} (x_j - x_j^2) \leq 0.$$

Note that $q(x) \geq 0$ for all x satisfying the linear constraints, under Burer's key assumption. Since the linear constraints imply the binary variables are bounded and since the quadratic constraint only involves the binary variables, our two cones $\mathcal{L}_\infty$ and L are identical. The equivalence between (17) and (18) is generalized in Theorem 5 in the present paper when Q^0 is copositive over $\mathcal{L}_\infty$.

It is also shown in [15] that if the objective function matrix Q^0 is not copositive over $\mathcal{L}_\infty$ then both the binary nonconvex quadratic program (17) and its completely positive relaxation (18) are unbounded. Thus problem (17) is either unbounded or satisfies the assumptions of Theorem 5. Therefore, the equivalence of (17) and (18) can be derived from our result.

5.2 Quadratic programs with complementarity constraints

A QPCC is in the general framework of (3), so the results of Theorem 5 apply directly. If the QPCC has a bounded objective function value then Q^0 may be copositive over L, but as illustrated in Example 2 it may not be copositive over $\mathrm{conv}(L)$.

Lemma 3 *If there exists $\bar{d} \in L$ with $\bar{d}^T Q^0 \bar{d} < 0$ and a feasible solution $\bar{x}$ to a QPCC (1) satisfying $\bar{x} \perp \bar{d}$ then the QPCC has an unbounded optimal value.*

Proof The point $\bar{x} + \alpha \bar{d}$ is feasible in (1) for any $\alpha \geq 0$. The objective function value of this set of points is unbounded below as $\alpha \to \infty$. □

Ding et al. [22] contains several examples of problems of the form (1) where $\mathcal{K}^1$ is the positive semidefinite cone and the objective function is either linear or a convex quadratic function. For all these problems, we have an equivalence between (1) and the corresponding completely positive formulation (15), in the sense of Theorem 5. These problems include the rank constrained nearest correlation matrix problem, bilinear matrix inequality problems, and problems in the electric power market with uncertain data. In particular, it is shown in [22] that an SDP with an additional rank constraint is equivalent to an SDP problem with a complementarity constraint. That is, consider a problem of the form

$$\begin{aligned} \min_X \quad & \mathrm{trace}(CX) \\ \text{subject to } & \mathrm{trace}(A_i X) = b_i \quad i = 1, \ldots, k \\ & \mathrm{rank}(X) \leq p \\ & X \in S^n_+ \end{aligned} \tag{19}$$

where S_+^n denotes the set of $n \times n$ symmetric positive semidefinite matrices, C and each A_i are symmetric $n \times n$ matrices, and $p \leq n$. This problem is then equivalent to the problem

$$\begin{array}{lll} \min_{X,W} & \mathrm{trace}(CX) & \\ \text{subject to} & \mathrm{trace}(A_i X) = b_i & i = 1, \dots, k \\ & \mathrm{trace}(W) = p & \\ & S_+^n \ni X \perp I - W \in S_+^n & \\ & W \in S_+^n. & \end{array} \tag{20}$$

Since the objective function is linear, it follows from Theorem 5 that this problem is equivalent to a completely positive program. Thus we have the following theorem:

Theorem 6 *A rank constrained semidefinite program of the form (19) is equivalent to a convex completely positive program, in the sense of Theorem 5.*

The completely positive program is a lifting of (20), so it will have $O(n^4)$ variables.

Algorithms for finding global optima to QPCCs can be found in [6], for example. These algorithms can be strengthened through the use of improved lower bounds coming from relaxations. It may be possible to obtain good bounds in reasonable computational time by examining relaxations of (15), as in [23] for example.

5.3 General QCQPs

The construction of Theorem 1 shows that any QCQP can be represented in the form (3) with $I = 0$ and with $Q^0 = 0$. It follows that the assumption of Theorem 5, namely that Q^0 be copositive on $\mathrm{conv}(L)$, is vacuously satisfied. Thus we have the following theorem.

Theorem 7 *A quadratically constrained quadratic program is equivalent to a convex conic optimization problem in the sense of Theorem 5.*

Note that this theorem makes no assumptions about the problem. In particular, it is not required that any of the quadratic functions be convex, or that the feasible region be bounded. If the QCQP has a bounded optimal value, but this value is not attained, then it follows from part 3 of Theorem 5 that the optimal value of the completely positive program is also not attained. Thus, a Frank-Wolfe result would not hold for the completely positive program in this setting.

Sensor network localization [1] is an example of a nonconvex quadratically constrained quadratic program that can be expressed in the form (1) without introducing the parameter λ and the variable t of Theorem 5. In this problem, the locations of some sensors and the distances between some pairs of sensors are given. It is then desired to determine the locations of all the sensors. If distances are known exactly, the model can be expressed as a quadratically constrained feasibility problem, with constraints of the form

$$||x^i - x^j||^2 - d_{ij}^2 = 0$$

where x^i and x^j are the locations of two sensors and d_{ij} is the distance between them. We can generalize this structure to a problem with a linear objective and quadratic equality constraints:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to } \; & p_i(x) = 0 \quad i = 1, \dots, k \\ & Ax = b \end{aligned} \tag{21}$$

where each quadratic function $p_i(x)$ has a positive semidefinite Hessian. This is then equivalent to the problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & p_i(x) \le 0 \quad i = 1, \dots, k \\ & \textstyle\sum_{i=1}^m p_i(x) \ge 0 \\ & Ax = b. \end{aligned} \tag{22}$$

Each of the convex quadratic constraints $p_i(x) \le 0$ can then be represented by a second order cone, so we obtain a problem of the form (3) with $I = 0$. Since the objective is linear, the assumption of Theorem 5 is vacuously satisfied. Thus, we can construct a convex completely positive problem that is equivalent to (21) without needing to introduce a bound constraint of the form (5).

6 Conclusions and Future Work

We have extended the literature to show that certain classes of nonconvex quadratically constrained quadratic programs are guaranteed to satisfy the Frank-Wolfe property. The nonconvex constraints in these problems violate the Slater constraint qualification. In deriving these results, we have exploited a relationship between these problems and quadratic programs with complementarity constraints. Further, we have related local optimality conditions for these QCQPs to stationarity conditions for mathematical programs with equilibrium constraints. We have also shown that any QCQP (convex or nonconvex, with bounded or unbounded feasible region) is equivalent to a completely positive program, a convex optimization problem. This result applies in particular to semidefinite programs with a rank constraint.

By combining Theorems 2 and 5, it follows that (15), which is a special copositive program, has the Frank-Wolfe property when $\mathcal{K}$ is a polyhedral cone, via the equivalence with the QCQP (3) failing the Slater condition with respect to its quadratic inequality constraints. A natural question to ask is whether there is a broader class of copositive programs for which the Frank-Wolfe result holds. At this time, we do not have an answer to this question.

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