1. Let s_j be the start time of job j

and v_j be the end time of job j

Let $x_{ij} = \begin{cases} 1 & \text{if job } i \text{ is scheduled before job } j \\ 0 & \text{otherwise.} \end{cases}$

Let z_j be the tardiness of job j

$$\min \sum_j w_j z_j$$

$$\text{s.t.} \quad z_j \geq v_j - t_j \qquad \forall j$$

$$s_j \geq v_i - M(1 - x_{ij}) \qquad \forall i, j$$

$$x_{ij} = 1 - x_{ji} \qquad \forall i, j$$

$$v_j \geq s_j + p_j \qquad \forall j$$

$$x_{ij} \text{ binary} \qquad \forall i, j$$

$$v_j, s_j, z_j \geq 0 \qquad \forall j$$

Here, M is a constant, eg $M = \sum_i p_i$

2. Let $S^1 = \{x \in \mathbb{Z}^3_+ : 7x_1 + 9x_2 + 12x_3 \geq 17\}$

$S^2 = \{x \in \mathbb{Z}^3_+ : 4x_1 + 5x_2 + 6x_3 \geq 10\}$

$S^3 = \{x \in \mathbb{Z}^3_+ : x_1 + x_2 + x_3 \geq 2,\ 2x_1 + 3x_2 + 4x_3 \geq 6\}$

The smallest points in S^1 are $(3,0,0)$, $(0,2,0)$, $(0,0,2)$, $(2,1,0)$, $(1,0,1)$, and $(0,1,1)$.
Every other point in S^1 is $\geq$ at least one of these points.

All of these points are also in S^2 and S^3.
They are the smallest points in those sets.

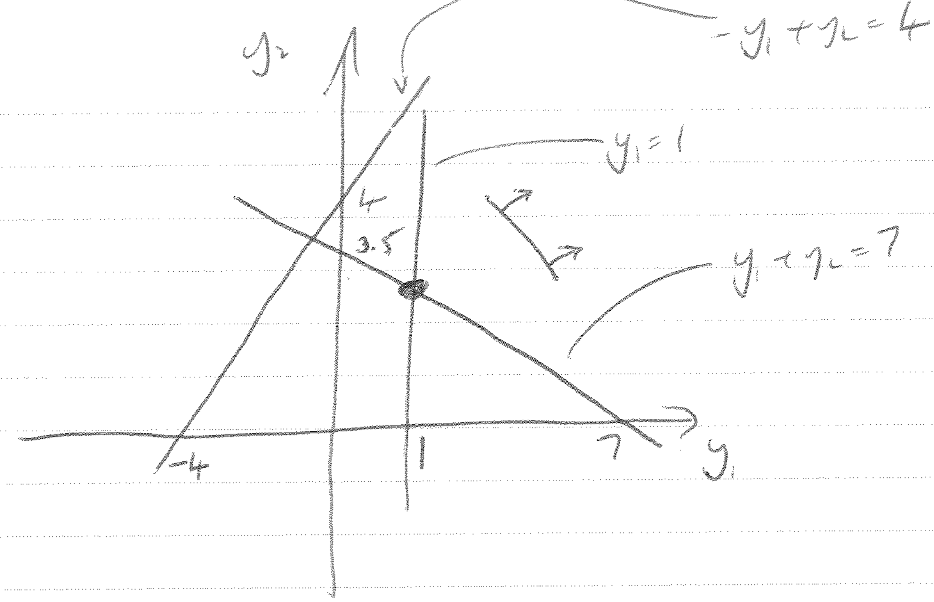
The Gomory rounding procedure can also be used to argue that the three formulations are equivalent.

S^3 is the tightest formulation, in that the LP relaxation has the smallest feasible region.

Eg: $(0, 0, \frac{17}{12}) \in S^1_{LP}$ and $(0, 0, \frac{5}{3}) \in S^2_{LP}$,

but neither of these points ~~—~~ is in S^3_{LP}.

3.



(a) Optimal soln is $y=(1,3)$, with value 11.

(b) Dual is $\min\ 7x_1 + 4x_2 + x_3$

$$\begin{aligned} \text{s.t.}\quad x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 \qquad &= 3 \\ x_i \geq 0 \qquad & i=1,2,3 \end{aligned}$$

(c)
$$\left.\begin{aligned} x_1\,(7-y_1-y_2) &= 0 \\ x_2\,(4+y_1-y_2) &= 0 \\ x_3\,(1-y_1) &= 0 \end{aligned}\right\} \Rightarrow x_2 = 0 \Rightarrow x_1 = 3/2,\ x_3 = \tfrac{1}{2}$$
by primal feasibility.

Dual value: $\frac{22}{2} = 11$. ✓

(d) $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, ie, columns 1 & 3 of A.

$N = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 0 & +\frac{1}{2} \\ +1 & -\frac{1}{2} \end{bmatrix}$ $\quad B^{-1}N = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$ $\quad c_B^T B^{-1} N = 2$

reduced cost $= c_N^T - c_B^T B^{-1} N = 4 - 2 = 2$.

$$c_B^T B^{-1} b = [7\ 1] \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = [1\ 3] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 11.$$

4. Let $y_i = \begin{cases} 1 & \text{if include vertex } i \\ 0 & \text{otherwise} \end{cases}$

$x_{ij} = \begin{cases} 1 & \text{if include both } i \text{ and } j, \\ 0 & \text{otherwise} \end{cases}$ for edges $(i,j) \in E$.

$$\min \sum_{i \in V} w_i y_i$$

$$\text{s.t.} \quad \sum_{j \in U_i} y_i \geq 1 \qquad i = 1, \ldots, k$$

$$x_{ij} \leq y_i \qquad \forall (i,j) \in E$$

$$x_{ij} \leq y_j \qquad \forall (i,j) \in E$$

$$z_W \geq y_j \qquad \forall j \in W, \quad \forall \text{ subsets } W \subset V$$

$$\sum_{\substack{i \in W \\ j \in V \setminus W}} x_{ij} \geq z_W + z_{V \setminus W} - 1 \qquad \forall \text{ subsets } W \subseteq V$$

$$x, y, z \text{ all binary.}$$

ensures connectedness between a set W and its complement, provided both are nonempty.