Fixing Variables and Generating Classical Cutting Planes when using an Interior Point Branch and Cut Method to solve Integer Programming Problems

John E. Mitchell[1]
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180
mitchj@rpi.edu

R.P.I. Technical Report No. 216

October 12, 1994

Abstract

Branch and cut methods for integer programming problems solve a sequence of linear programming problems. Traditionally, these linear programming relaxations have been solved using the simplex method. The reduced costs available at the optimal solution to a relaxation may make it possible to fix variables at zero or one. If the solution to a relaxation is fractional, additional constraints can be generated which cut off the solution to the relaxation, but do not cut off any feasible integer points. Gomory cutting planes and other classes of cutting planes are generated from the final tableau. In this paper, we consider using an interior point method to solve the linear programming relaxations. We show that it is still possible to generate Gomory cuts and other cuts without having to recreate a tableau, and we also show how variables can be fixed without using the optimal reduced costs. The procedures we develop do not require that the current relaxation be solved to optimality; this is useful for an interior point method because early termination of the current relaxation results in an improved starting point for the next relaxation.

[1]Research partially supported by ONR Grant number N00014–94–1–0391. Some of this research was performed thanks to the support of an Obermann Fellowship to attend the 1994 Faculty Research Seminar "Optimization in Theory and Practice" at the University of Iowa Center for Advanced Studies.

1 Introduction

Branch and cut methods for integer programming problems solve a sequence of linear programming problems. Traditionally, these linear programming relaxations have been solved using the simplex method. The reduced costs available at the optimal solution to a relaxation may make it possible to fix variables at zero or one. If the solution to a relaxation is fractional, additional constraints can be generated which cut off the solution to the relaxation, but do not cut off any feasible integer points. Gomory cutting planes [8] and other classes of cutting planes are generated from the final tableau. In this paper, we consider using an interior point method to solve the linear programming relaxations. We show that it is still possible to generate Gomory cuts and other cuts without having to recreate a tableau, and we also show how variables can be fixed without using the optimal reduced costs.

We are interested in solving integer programming problems using an interior point method within a branch and cut algorithm. Obviously, a simplex tableau is not available when using an interior point method, so it is necessary to modify the classical techniques for fixing variables and generating cutting planes from the optimal tableau. It is the aim of this paper to show that this can be done. When using an interior point method in a cutting plane algorithm, it is usually not desirable to solve the current relaxation to optimality, because then the initial iterate for the next relaxation will be an extreme point, which is not a good starting point for an interior point method (see, for example, [14, 13, 4]). The procedures we develop do not require that the current relaxation be solved to optimality.

If a variable has value zero at the optimal solution to a relaxation and if its reduced cost is sufficiently large, then it must have value zero in the optimal solution to the integer programmig problem, so it can be *fixed* at zero. A similar test can be performed to determine whether a variable can be fixed at one. (See, for example, Nemhauser and Wolsey [15].) If the simplex algorithm is used to solve the relaxation, the reduced costs can be read directly from the optimal tableau. We observe in section 2 that the reduced costs are simply the dual slacks and that these are available when using an interior point method. It is thus possible to fix variables when using an interior point method, and this can actually be done before solving the current relaxation to optimality, provided the current dual solution is feasible.

If the solution to the linear programming relaxation is not integral, the relaxation can be tightened by adding an extra constraint which is satisfied by all feasible integral solutions but violated by the optimal solution to the relaxation. The new relaxation is then solved, and the procedure can be repeated as often as is necessary or desired. Such a constraint is called a *cutting plane*.

Gomory cutting planes were one of the first cutting planes proposed in the literature. These cutting planes are derived either from the reduced costs or from the rows of the tableau. We show in section 3.1 how to generate an objective function cutting plane by using the dual variables, and in section 3.2 we present a method for generating cutting planes from the constraints when the optimal tableau is not available. Both of these constructions give the corresponding Gomory cutting planes if the current relaxation is nondegenerate and if the relaxation is solved to optimality. These methods can be used even if the current iterate is not optimal for the relaxation.

There have been other cutting plane methods proposed in the literature which use the optimal tableau for the linear programming relaxation. The lift and project technique [3] requires solving a linear programmng problem over a slice through a cone. Different cutting planes result depending on the slice or *normalization* used. It was shown by Balas *et al.* that for one particular normalization the cutting plane can be obtained directly from the optimal tableau. This cut is an *intersection cut*, as introduced by Balas [2]. In section 4, we show how this cut can be derived if a interior point algorithm is used to solve the relaxations.

We restrict attention in this paper to cutting plane methods which explicitly use the simplex tableau. Many of the more successful cutting plane methods use more sophisticated procedures for generating cuts. For example, polyhedral theory is applied to generate facet defining cutting planes for particular problems such as the traveling salesman problem (see, for example, [9, 11, 16]) or the linear ordering problem (see, for example, [10]), and cuts based on the structure of the knapsack polyhedron are used to generate cutting planes for general 0-1 integer programming problems (see, for example, [6, 12]). The lift-and-project method with different normalizations requires the solution of a linear programming problem to generate a cut. Fenchel cutting planes [5] require the solution of a knapsack subproblem to find a cutting plane. These methods generally do not use the simplex tableau, requiring only the primal and perhaps the dual optimal solutions. Hence, these methods can be used when an interior point algorithm is used to solve the linear programming relaxations. In fact, many of these cutting plane generation routines work well with approximately optimal primal and dual solutions; therefore, they are very suitable for use with an interior point method.

1.1 Notation

Throughout, all vectors are column vectors. In particular, we represent the rows of a matrix as column vectors. The transpose of a matrix A is denoted by A^T. The ith component of a vector u is denoted by u_i. The (i,j)th component of a matrix A is denoted by A_{ij}. Given a scalar a, the largest integer which is no bigger than a is denoted by $\lfloor a \rfloor$. If u is a vector, then $\lfloor u \rfloor$ denotes the vector with $\lfloor u \rfloor_i = \lfloor u_i \rfloor$. The fractional part of a number a is denoted by $f(a) := a - \lfloor a \rfloor$. If u is a vector, then $f(u)$ denotes the vector with $(f(u))_i = f(u_i)$. Given a scalar a, we define $a_+ := \max\{a, 0\}$. If u is a vector, then u_+ denotes the vector with $(u_+)_i = (u_i)_+$.

2 Fixing variables

In this section, we show how the dual slacks can be used to fix the primal integral variables at a particular integer value. These methods are particularly suitable when the integer variables are restricted to be zero or one; therefore, in this section, we restrict attention to this case. We assume that the current relaxation of the 0-1 integer programming problem is

$$\begin{array}{llcl} \min & c^T x & & \\ \text{subject to} & Ax & = & b \\ & 0 \le x & \le & u \end{array} \qquad (P)$$

where A is an $m \times n$ matrix, c, u, and x are n-vectors, and b is an m-vector. The upper bounds u_i need not all be finite. We further assume that the first k ($k \le n$) components of x are the original $0-1$ variables, so they each have upper bound $u_i = 1$. The dual problem to (P) is

$$\begin{array}{lccccccc} \max & b^T y & - & u^T w & & & & \\ \text{subject to} & A^T y & - & w & + & z & = & c \\ & & & w, z & \ge & 0 & & \end{array} \qquad (D)$$

where y is an m-vector and w and z are n-vectors.

In order to fix variables, it is necessary to know a feasible solution to the integer programming problem. The value of this feasible solution then provides an upper bound v_{UB} on the optimal value of the integer programming problem. When using the simplex algorithm, the relaxation (P) is solved to optimality, giving a solution with value $\hat{v}$. If the integer variable x_i has value zero at the optimal solution to (P) and if the reduced cost r_i of this variable is greater than $v_{UB} - \hat{v}$ then the variable x_i must take value zero in any optimal solution to the integer programming problem, because any feasible solution to (P) with $x_i = 1$ must have value at least $r_i + \hat{v}$. Thus, if the reduced cost is sufficiently large, the variable can be fixed at zero. A similar test can be developed for determining whether a variable can be fixed at one.

Now assume that a primal-dual interior point method is used to solve the linear programming relaxation (P), and that this method generates feasible iterates to both (P) and (D). Let $(\bar{y}, \bar{w}, \bar{z})$ be the current dual solution. By complementary slackness, the primal value $c^T x$ can be written as

$$\begin{aligned} c^T x &= b^T \bar{y} - u^T \bar{w} + \bar{z}^T x + \bar{w}^T (u - x) \\ &= \bar{v} + \bar{z}^T x + \bar{w}^T (u - x), \end{aligned}$$

where $\bar{v}$ is the current dual value. It follows that if $\bar{z}_i > v_{UB} - \bar{v}$ then $c^T x > v_{UB}$ for any solution x with $x_i = 1$. Similarly, for $i = 1, ..., k$, if $\bar{w}_i > v_{UB} - \bar{v}$ then $c^T x > v_{UB}$ for any solution x with $x_i = 0$. We can summarize this in the following theorem.

Theorem 1 *Let $(\bar{y}, \bar{w}, \bar{z})$ be the current feasible dual solution with value $\bar{v}$. Let v_{UB} be a known upper bound on the optimal value of the integer programming problem.*

- *If $\bar{z}_i > v_{UB} - \bar{v}$ then x_i is equal to zero in any optimal solution to the integer programming problem.*

- *If $\bar{w}_i > v_{UB} - \bar{v}$ then x_i is equal to one in any optimal solution to the integer programming problem.*

This test does not depend on the value of x_i; in particular, it is not necessary that x_i be 0 or 1, as required by the simplex test. However, when using an interior point method, the values of the quantities $x_i z_i$ and $(u_i - x_i)w_i$ are usually close to their average $(c^T x - b^T y + u^T w)/2n$, so it is unlikely that z_i will be large unless x_i is close to zero, or that w_i will be large unless x_i is close to one.

3 Gomory Cutting Planes

Gomory cutting planes [8] are generated from the optimal simplex tableau for the current relaxation. They are violated by the optimal solution to the relaxation, but they are satisfied by all feasible integer points. A Gomory cutting plane can be generated from the objective function if the optimal value of the relaxation is fractional. If the optimal value of the relaxation is integer but one of the variables is fractional, a cutting plane can be generated from the row of the simplex tableau corresponding to this basic variable.

In this section, we assume that the linear programming problem has the form

$$\begin{array}{llcl} \min & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0, \text{ integer} \end{array} \qquad (IP1)$$

where A is an $m \times n$ matrix and b, c, and x are dimensioned appropriately. In addition, we assume that all the entries of A, b, and c are integer. The results we obtain in this section can be generalized in the case that some of the variables are restricted to be binary. The LP relaxation of this problem is the standard form

$$\begin{array}{llcl} \min & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array} \qquad (LP1)$$

with dual problem

$$\begin{array}{llcccl} \max & b^T y & & & & \\ \text{subject to} & A^T y & + & z & = & c \\ & & & z & \geq & 0 \end{array} \qquad (LD1)$$

3.1 Using the objective function

The Gomory cut from the objective function uses the optimal reduced costs. In an interior point method we have the dual slacks available and these are equivalent to the reduced costs at optimality, so it is possible to use them to generate a cutting plane.

When using the simplex algorithm, the Gomory cut is generated as follows from the optimal solution to the LP relaxation. Let $\bar{c}_i$ be the reduced cost of variable x_i. Let R be the index set of the nonbasic variables. If the optimal value v^{LP} of the LP relaxation is not integer, then the constraint

$$f(v^{LP}) + \sum_{i \in R} f(\bar{c}_i) x_i \geq 1 \tag{1}$$

is valid for the integer programming problem but is violated by the optimal solution to the LP relaxation, so it can be added as a cutting plane.

When using a primal-dual interior point method to solve the LP relaxation, we can use the dual slacks to generate a very similar cut once we are sufficiently close to optimality of the relaxation. We have the following theorem, which explicitly shows how to generate an additional primal constraint and also when this additional constraint is guaranteed to be violated by the current primal iterate.

Theorem 2 *Let $\bar{x}$ and $\bar{y}$ be the current primal and dual solutions, respectively, and let $\bar{z}$ be the current dual slacks. Assume that the primal and dual solutions are feasible. Assume there exists an integer p with $p < b^T\bar{y} \leq c^T\bar{x} < p+1$. Then the constraint*

$$f(b^T\bar{y}) + \sum_{i=1}^{n} f(\bar{z}_i) x_i \geq 1 \tag{2}$$

is valid for $(IP1)$ and is violated by $\bar{x}$. Furthermore, if the LP relaxation $(LP1)$ is nondegenerate, then this constraint is equivalent to the traditional Gomory cut in the limit as $c^T\bar{x} - b^T\bar{y} \to 0$.

Proof: We first show that equation (2) is valid for all feasible solutions to $(IP1)$. For any feasible solution x to $(IP1)$, the primal objective value is integral and can be expressed as

$$\begin{aligned} c^T x &= b^T\bar{y} + \bar{z}^T x \\ &= \lfloor b^T\bar{y} \rfloor + \lfloor \bar{z} \rfloor^T x + f(b^T\bar{y}) + f(\bar{z})^T x \end{aligned}$$

so $f(b^T\bar{y}) + f(\bar{z})^T x$ must be integral. It must take value at least one, since $f(b^T\bar{y}) > 0$.

We now show that the point $\bar{x}$ violates equation (2). For this point,

$$\begin{aligned} 1 &> c^T\bar{x} - p \\ &= c^T\bar{x} - \lfloor b^T\bar{y} \rfloor \\ &= f(b^T\bar{y}) + \bar{z}^T x \\ &\geq f(b^T\bar{y}) + f(\bar{z})^T x. \end{aligned}$$

Finally, if $(LP1)$ is nondegenerate and if $c^T\bar{x} = b^T\bar{y}$ then $\bar{z}_i = \bar{c}_i$, the reduced cost of variable x_i. It follows that the cut defined in equation (2) is the Gomory cut defined from the optimal simplex tableau. □

3.2 Using the constraints

When using the simplex algorithm, it is possible to generate a cut for any variable which has a fractional value using the optimal simplex tableau. Let $\bar{x}$ be the optimal solution to the LP relaxation $(LP1)$. Let $\bar{a}_{ij}$ denote the (i,j)th entry of the optimal tableau and let R be the index set of the nonbasic variables. Let $\rho(i)$ denote the index of the basic variable corresponding to row i of the optimal tableau. If $\bar{x}_{\rho(i)}$ is fractional then the Gomory cutting plane from this variable is

$$\sum_{j \in R} f(\bar{a}_{ij}) x_j \geq f(\bar{x}_{\rho(i)}) \tag{3}$$

When using an interior point, we could calculate the tableau from the optimal solution to the relaxation and then obtain the Gomory cutting plane. However, this requires a large amount of computational effort. We show in this subsection that the projection matrix calculated at each iteration of an interior point method can be used to generate a cutting plane. This cutting plane is equivalent to the Gomory cutting plane if the relaxation is solved to optimality and is nondegenerate.

When using an interior point method, it is necessary to calculate projections using a matrix of the form

$$P_{AD}^{\perp} = DA^T(AD^2A^T)^{-1}AD$$

for some $n \times n$ diagonal matrix D. Standard choices for D are either X or $X^{1/2}Z^{-1/2}$ or Z^{-1}, where X is a diagonal matrix with $X_{ii} = x_i$ and Z is a diagonal matrix with $Z_{ii} = z_i$. A Cholesky factorization $LL^T = AD^2A^T$ is usually calculated at each iteration, where L is a lower triangular matrix. For calculating cutting planes, we will use the matrix

$$P^{cut} := D^2A^T(AD^2A^T)^{-1}A \tag{4}$$

Let a^i denote the ith column of A. Row i of P^{cut} can be found by solving one system of equations

$$(AD^2A^T)v^i = D_{ii}^2 a^i \tag{5}$$

and then one matrix-vector multiply $v^{i^T}A$. If we have a Cholesky factorization of the matrix AD^2A^T then a row of the matrix P^{cut} can be found using one forward substitution, one backward substitution, and one matrix-vector multiply. Generally, the elements of this diagonal matrix D tend to zero if the corresponding component of x tends to zero. Thus, as the iterates approach optimality, provided the LP relaxation is nondegenerate, we have the following observation, which is similar to a result originally due to Vanderbei *et al.* [17].

Lemma 1 *Assume the problem $(LP1)$ is nondegenerate and that x_i is a basic variable at the optimal solution to $(LP1)$. As the set of iterates tends towards optimality, the ith row of the matrix P^{cut} tends to the row of the optimal simplex tableau corresponding to the basic variable x_i.*

Proof: To simplify the proof, we partition the matrix A into basic and nonbasic columns, and assume without loss of generality that the basic columns constitute the first m columns of A. We write $A = [B \;\; N]$ and define D_B and D_N to be the parts of D corresponding to B and N, respectively. We then have at the optimal solution that $D_N = 0$, and so

$$\begin{aligned} P^{cut} &= \begin{bmatrix} D_B^2 & 0 \\ 0 & D_N^2 \end{bmatrix} \begin{bmatrix} B^T \\ N^T \end{bmatrix} \left([B \;\; N] \begin{bmatrix} D_B^2 & 0 \\ 0 & D_N^2 \end{bmatrix} \begin{bmatrix} B^T \\ N^T \end{bmatrix} \right)^{-1} [B \;\; N] \\ &\rightarrow \begin{bmatrix} D_B^2 B^T \\ 0 \end{bmatrix} (B D_B^2 B^T)^{-1} [B \;\; N] \\ &= \begin{bmatrix} I & B^{-1}N \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The first m rows of the limit of P^{cut} give the optimal simplex tableau. $\square$

Note that the ith row of P^{cut} is given by ${v^i}^T A =: d^i$, where v^i is given by equation (5). Thus, every solution to $Ax = b$ must also satisfy

$$ {d^i}^T x = {v^i}^T A x = {v^i}^T b \tag{6}$$

We can use this equation to generate a valid inequality for $(IP1)$ at any iteration. In the limit, this inequality will be the Gomory cutting plane associated with variable x_i.

Theorem 3 *Let d^i be the ith row of the matrix P^{cut} and let $v^i = D_{ii}^2 (AD^2A^T)^{-1} a^i$. Then the inequality*

$$ f(d^i)^T x \geq f({v^i}^T b) \tag{7}$$

is satisfied by every feasible solution to $(IP1)$. If $(LP1)$ is nondegenerate then this constraint gives the Gomory cutting plane for variable x_i in the limit when the duality gap becomes zero.

Proof: We have ${d^i}^T x = {v^i}^T b$ for every x satisfying $Ax = b$, from equation (6). Thus,

$$ f(d^i)^T x + \lfloor d^i \rfloor^T x = f({v^i}^T b) + \lfloor {v^i}^T b \rfloor, $$

so, since $0 \leq f({v^i}^T b) < 1$, equation (7) follows.
In the limit, $d_i^i = 1$ and if the problem is nondegenerate then d^i gives exactly the row of the tableau corresponding to x_i. Further, the value of x_i is ${v^i}^T b$ (see, for example, Vanderbei et al. [17]). $\square$

Since $d_i^i \rightarrow 1$, we may get d_i^i slightly smaller than one, and then taking the fractional part of d^i will only give a weak constraint. Therefore, in practice, it may be advisable to divide the vector d^i by its ith component d_i^i before rounding it down. The fractional part of d^i/d_i^i will then be zero in component i, so the variable x_i does not appear in the constraint.

4 Lift-and-project cutting planes

Balas *et al.* [3] proposed a technique called lift and project for generating cutting planes. This technique requires solving a linear programming problem over a slice through a cone, where the slice (or *normalization*) can be chosen in different ways. As is to be expected, the calculation of the cutting plane usually requires a fair amount of work; however, with one particular normalization the cutting plane can be found purely by examining the optimal tableau for the linear programming relaxation of the integer program. It is the purpose of this section to show that the same cutting plane can also be found when using an interior point method to solve the relaxation.

We follow the notation of Balas *et al.* in this section. Therefore, we define the feasible region for the LP relaxation of the integer problem to be

$$\begin{aligned} K &:= \{x \in \Re^n : Ax \geq b, x \geq 0, x_j \leq 1, j = 1, ..., p\} \\ &:= \{x \in \Re^n : \tilde{A}x \geq \tilde{b}\} \end{aligned}$$

and the feasible region for the integer program is

$$K^0 := \{x \in K : x_j \in \{0,1\}, j = 1, ..., p\}$$

The cutting planes generated have the form

$$\alpha^T x \geq \beta \tag{8}$$

where α and β satisfy

$$\begin{array}{lllllllllll} \alpha & - & \tilde{A}^T u & - & u_0 e_j & & & & & = & 0 \\ \alpha & & & & & - & \tilde{A}^T v & - & v_0 e_j & = & 0 \\ & & \tilde{b}^T u & & & & & & - \ \beta & = & 0 \\ & & & & & & \tilde{b}^T v & + & v_0 \quad - \ \beta & = & 0 \\ & & u, v & \geq & 0 & & & & & & \end{array} \tag{9}$$

for some vectors u and v and scalars u_0 and v_0, where e_j is the jth unit vector. The appropriate component j is chosen using the solution to the LP relaxation. The constraint (8) is valid for the polytope K^0 for any α and β satisfying equation (9). For it to be a cutting plane, it should be violated by the current point $\bar{x}$ in K. Thus, it is desirable to minimize $\alpha^T\bar{x} - \beta$ over all possible choices for α and β. The set of feasible α and β is a cone; therefore, it is necessary to homogenize or normalize the cone before trying to minimize $\alpha^T\bar{x} - \beta$. One possible normalization (*Normalization 4* in the notation of Balas *et al.*) is to insist that

$$v_0 - u_0 = 1.$$

Thus, the linear programming subproblem to be solved in order to obtain a cutting plane is

$$
\begin{array}{llcccccccccccl}
\min & \bar{x}^T\alpha & & & & & & & & & -\ \beta & & & \\
\text{subject to} & \alpha & - & \tilde{A}^T u & - & u_0 e_j & & & & & & = & 0 & \\
& \alpha & & & & & - & \tilde{A}^T v & - & v_0 e_j & & = & 0 & \\
& & & \tilde{b}^T u & & & & & & & -\ \beta & = & 0 & (LP^{L\&P}) \\
& & & & & & & \tilde{b}^T v & + & v_0 & -\ \beta & = & 0 & \\
& & & & - & u_0 & & & + & v_0 & & = & 1 & \\
& & & u, v & \geq & 0 & & & & & & & &
\end{array}
$$

Assume the relaxation is solved using the simplex algorithm. Let x^t be the optimal solution to the current relaxation, and let B^t be the corresponding basis matrix, so the rows of B^t are the rows of $\tilde{A}$ for which the corresponding slack variables are nonbasic. The optimal solution to $(LP^{L\&P})$ is

$$
\begin{array}{rcl}
u^B & = & ((B^t)_j^{-1})_+ \\
v^B & = & (-(B^t)_j^{-1})_+ \\
\alpha_i & = & ((B^t)_j^{-1})_+^T B_i^t, \quad i = 1, ..., n, \ i \neq j \\
\alpha_j & = & ((B^t)_j^{-1})_+^T B_j^t + ((B^t)_j^{-1})^T d^t - 1 \\
\beta & = & ((B^t)_j^{-1})_+^T d^t
\end{array}
\tag{10}
$$

where $(B^t)_j^{-1}$ denotes the jth row of $(B^t)^{-1}$, B_i^t the ith row of B^t, and d^t is the part of $\tilde{b}$ which corresponds to the rows of B^t. Now, the jth row of $(B^t)^{-1}$ is available from the optimal tableau for the LP relaxation, so α can be calculated straightforwardly.

Now assume the relaxation is solved using an interior point algorithm. At each iteration of the interior point algorithm, a projection matrix of the form

$$
P_{L\&P}^{\perp} = D^{-1}\tilde{A}(\tilde{A}^T D^{-2}\tilde{A})^{-1}\tilde{A}^T D^{-1} \tag{11}
$$

is used, where D is an appropriate diagonal matrix. For example, the entries of D could be the slacks of the constraints $\tilde{A}x \geq \tilde{b}$ as in Adler *et al.* [1]. Generally, an entry of D tends to zero as the corresponding slack variable goes to zero. As in section 3.2, we can approximate the basis matrix B by using an appropriate modification of the projection matrix. Define the vector

$$
w = D^{-2}\tilde{A}(\tilde{A}^T D^{-2}\tilde{A})^{-1} e_j. \tag{12}
$$

It should be noted that w can be found easily using the Cholesky factorization of $\tilde{A}^T D^{-2}\tilde{A}$ formed when calculating projections using $P_{L\&P}^{\perp}$. Then the constraint is defined by

$$
\begin{array}{rcl}
u^w & = & w_+ \\
v^w & = & (-w)_+ \\
\alpha_i & = & \tilde{A}_i^T u^w, i = 1, ..., n, i \neq j \\
\alpha_j & = & \tilde{A}_j^T u^w + w^T \tilde{b} - 1 \\
\beta & = & (u^w)^T \tilde{b}
\end{array}
\tag{13}
$$

where $\tilde{A}_i$ denotes column i of $\tilde{A}$. To verify that this is a valid constraint, it is necessary to show that equation (9) is satisfied by this choice. Notice that

$$\tilde{A}^T u^w - \tilde{A}^T v^w = \tilde{A}^T(u^w - v^w) = \tilde{A}^T w = e_j.$$

Thus, $\tilde{A}_i^T v^w = \tilde{A}_i^T u^w = \alpha_i$ for $i \neq j$. Similarly, $\tilde{b}^T u = \tilde{b}^T v + \tilde{b}^T w$. If we set

$$u_0 = w^T(\tilde{b} - \tilde{A}_j) = w^T \tilde{b} - 1, \qquad v_0 = w^T \tilde{b} \tag{14}$$

then it can be seen that equation (9) holds, and in addition Normalization 4 applies.

As the iterates tend towards optimality, the constraint given in (13) tends towards the constraint given by equation (10).

Theorem 4 *If $\hat{x}^k$ is a sequence of feasible points with strictly positive slacks which tends towards the optimal point x^t, and if the LP relaxation is nondegenerate, then the constraint defined in equation (10) and the limit of the constraints defined in (13) are identical.*

Proof: Assume without loss of generality that the first m constraints of $\tilde{A}x^t \geq \tilde{b}$ are active at the optimal solution x^t. Then B^t is the first m rows of $\tilde{A}$, so for these constraints $\lim_{k\to\infty} D_{ii}^k = 0$ and thus

$$\lim_{k\to\infty} w^k = \begin{bmatrix} ((B^t)^{-1})^T \\ 0 \end{bmatrix} e_j = \begin{bmatrix} (B^t)_j^{-1} \\ 0 \end{bmatrix}$$

The result follows. $\square$

It follows from this theorem and the work of Balas *et al* [3] that in the limit the constraint defined by equation (13) solves the linear programming problem $(LP^{L\&P})$. If the current iterate does not solve the relaxation then the constraint may not solve $(LP^{L\&P})$. We can give an explicit analytic expression for the violation of the constraint defined in (13) by the current iterate.

Lemma 2 *The value of the slack in the constraint $\alpha^T x - \beta \geq 0$ is*

$$(1 - \bar{x}_j)(\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j(\tilde{A}\bar{x} - \tilde{b})^T v^w + \bar{x}_j^2 - \bar{x}_j$$

at the current point $\bar{x}$.

Proof: We have

$$\begin{aligned}
\alpha^T \bar{x} - \beta &= (u^w)^T \tilde{A}\bar{x} + (w^T\tilde{b} - 1)\bar{x}_j - (u^w)^T\tilde{b} \text{ from equation (13)} \\
&= (\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j \tilde{b}^T w - \bar{x}_j \\
&= (\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j(\tilde{b} - \tilde{A}\bar{x} + \tilde{A}\bar{x})^T w - \bar{x}_j \\
&= (\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j(\tilde{b} - \tilde{A}\bar{x})^T w + \bar{x}_j^2 - \bar{x}_j \\
&\qquad\qquad \text{since } \tilde{A}^T w = e_j \text{ from equation (12)} \\
&= (\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j(\tilde{b} - \tilde{A}\bar{x})^T(u^w - v^w) + \bar{x}_j^2 - \bar{x}_j \\
&\qquad\qquad \text{since } w = u^w - v^w \text{ from equation (13)} \\
&= (1 - \bar{x}_j)(\tilde{A}\bar{x} - \tilde{b})^T u^w + \bar{x}_j(\tilde{A}\bar{x} - \tilde{b})^T v^w + \bar{x}_j^2 - \bar{x}_j.
\end{aligned}$$

This is the desired result. $\square$

The constraint defines a cutting plane if the quantity given in the theorem is negative. In the limit as the solution sequence tends to optimality, the vector w is only nonzero at active constraints, and the vector $\tilde{A}^T\bar{x} - \tilde{b}$ is only nonzero at constraints which are not active. Thus, the quantities $(\tilde{A}\bar{x} - \tilde{b})^T u^w$ and $(\tilde{A}\bar{x} - \tilde{b})^T v^w$ tend to zero as the solution sequence tends to optimality. The quantity $(\bar{x}_j)^2 - \bar{x}_j$ is negative if the variable $\bar{x}_j$ is fractional. Thus, the constraint may give a valid cutting plane before the relaxation is solved to optimality.

5 Conclusions

We have shown how variables can be fixed and how cutting planes can be generated when using an interior point algorithm in a branch and cut method for integer programming problems. All of the methods developed in this paper can be applied before the current relaxation has been solved completely to optimality. The method for fixing variables uses the current dual feasible solution and requires a simple, cheap test. Similarly, the method for generating a Gomory-like cutting plane from the objective function uses the current dual solution, and once it is obvious from the primal and dual values that the optimal value of the relaxation must be fractional, the constraint generated is guaranteed to cut off the current primal iterate. It is very cheap to find this cutting plane: we just round down a vector.

The methods for generating a Gomory-like cutting plane from a fractional variable, and for generating a lift-and-project type cutting plane both require more computational work: it is necessary to solve a system of equations, using the Cholesky factors of part of the projection matrix, where these factors have already been calculated. This work is actually comparable to the amount of work required when using the revised simplex method, because the tableau is not then stored explicitly and must be calculated using a factorization of the basis matrix. When using an interior point method, various indicators are available for determining whether a variable is likely to be basic or nonbasic at the optimal solution — see El-Bakry *et al.* [7] for a survey. Because of the overhead involved in calculating potential cutting planes, they should only be calculated when the indicators suggest that a variable is very likely to be fractional at the optimal solution to the relaxation. Once the constraint has been determined, it should only be added if it does indeed cut off the current primal iterate.

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