MATP6640/DSES6770 Linear Programming, Homework 3.

Due: Monday, February 25, 2008.
10% penalty for each day late.

The following is a primal-dual pair of linear programs:

$$\begin{array}{llcll} \min & c^T x & & & \\ \text{s.t.} & Ax & = & b & (P) \\ & Hx & = & h & \\ & x & \geq & 0 & \end{array} \qquad \begin{array}{llclcll} \max & b^T y & + & h^T u & & & \\ \text{s.t.} & A^T y & + & H^T u & \leq & c & (D) \end{array}$$

Assume (P) and (D) are both feasible. Further, assume the polyhedron $Q := \{x \geq 0 : Ax = b\}$ is bounded. Let z denote the optimal value of (P) and (D). For any appropriately dimensioned vector π, define

$$\Theta(\pi) := \min_x \{c^T x + \pi^T (h - Hx) : Ax = b, x \geq 0\}$$

The function $\Theta(\pi)$ is concave and provides an underestimate of z for any π. The best lower bound is obtained by maximizing $\Theta(\pi)$.

1. Show that the maximum value of $\Theta(\pi)$ is equal to z.

2. Assume the maximum value of $\Theta(\pi)$ can be overestimated using a piecewise linear function, resulting in a linear programming problem of the form

$$\begin{array}{lcclcll} \max & & & \theta & & & \\ \text{s.t.} & G\pi & + & d\theta & \leq & v & (LD) \end{array}$$

   where G is a matrix, v is a vector, and d is a nonnegative vector. The optimal value of (LD) overestimates the maximum possible value of $\Theta(\pi)$. Assume an optimal solution $\bar{\pi}$, $\bar{\theta}$ for (LD) is known. Let $\bar{x}$ solve $\Theta(\bar{\pi})$, with corresponding dual multipliers $\bar{y}$. Assume $\bar{\theta} > \Theta(\bar{\pi})$. The bound $\bar{\theta}$ can be improved by adding a constraint to (LD). Give a valid linear constraint that must be satisfied by π and $\Theta(\pi)$ for all π.

3. Now assume all of the constraints in (LD) have the same form as the constraint you just found in question 2. What is the relationship between this approach to solving (P) and Dantzig-Wolfe Decomposition?

4. (Based on Chvatal 26.4.) Assume that in some iteration of the Dantzig-Wolfe decomposition algorithm, the subproblem using the dual vector $\hat{\pi}$ has optimal value s^*. Prove that every feasible solution of the original problem has $c^T x \geq s^* + b^T \hat{\pi}$. How can you exploit this?

5. (Chvatal 16.4.) An inequality in a system of inequalities is *redundant* if its removal from the system does not introduce any new solutions. To show that recognizing redundant inequalities is at least as difficult as recognizing solvable systems, prove that $Ax \leq b$ has a solution if and only if the constraint $b^T y \geq 0$ is redundant in the system $A^T y = 0$, $y \geq 0$, $b^T y \geq 0$.

6. The **Project:**
   Along with your solutions to this homework, hand in a brief description of what you would like to do for the project part of this course.

John Mitchell
Amos Eaton 325
x6915.
mitchj at rpi dot edu
Office hours: Thursday February 14th, 2pm – 4pm. Tuesday, February 19th, 1pm – 3pm.
Next week: there will be only one class, on Tuesday February 19.