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On QPCCs, QCQPs and Copositive Programs

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Abstract The present paper addresses several topics associated with the three classes of constrained optimization problems: QPCCs for quadratic programs with (linear) complementarity constraints, QCQPs for quadratically constrained quadratic programs, and copositive programs. The subclass of QCQPs, ones that are broader than the QPCCs and fail the Slater constraint qualifications (CQ) can be formulated as QPCCs, therefore a Frank-Wolfe type result holds for this class of QCQPs. We establish a fundamental role of this class of QCQPs in a quadratically constrained non-quadratic program failing the Slater CQ. We also show that such a QCQP, if copositive, can be reformulated as an equivalent completely positive program in the sense of feasibility, boundedness, attainability as well as solvability.

Keywords QCQP · QPCC · Copositive Representation · Local Optimality

1 Introduction

This paper addresses 2 classes of nonconvex constrained optimization problems and their connections as well as their equivalent formulation as a con-

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vex completely positive program via lifting of variables. Playing an analogous role as a quadratic program in the class of nonlinear programs, the class of quadratic programs with (linear) complementarity constraints (QPCC) is a subclass of the class of mathematical programs with complementarity constraints (MPCC) [28,31]. With increasingly many documented applications in diverse engineering fields, the MPCC provides a broad framework for the treatment of such problems as bilevel programs, inverse optimization, Stackelberg-Nash games, and piecewise programming; see [33]. As a subclass of the class of MPCCs, a QPCC [23,13] is almost a quadratic program in the well-known sense except for the constraint that some pairs of variables are required to satisfy the well-known complementary slackness condition; i.e., the 2 members of such a pair of variables are nonnegative and their product is equal to zero. Such a complementarity constraint renders the QPCC a nonconvex disjunctive program even if the objective function is convex. Being a recent entry to the optimization field, the QPCC and its special case of a linear program with (linear) complementarity constraints (LPCC) have recently been studied in [5,22] wherein a logical Benders scheme [14] was proposed for their global resolution.

Unlike the QPCC, the quadratically constrained quadratic program, abbreviated QCQP, has a somewhat longer history of focused study. Specifically, a QCQP is a constrained optimization problem whose objective and constraint functions are all quadratic. In this definition, the QPCC is a special instance of a QCQP. Many of the early study of the QCQP has addressed the issue of existence of optimal solutions [9,29,37] via the well-known Frank-Wolfe theorem originally proved for a quadratic program [19]. Recent works [1,6,7,11, 12,24,30] on the QCQP have developed algorithms of various kinds for solving such a program. Of particular relevance to our work herein are the most recent papers [1,11,12] that cast the QCQP satisfying a boundedness assumption as a completely positive program [10], which is a problem of topical interest.

This paper addresses several topics associated with the QPCC and QCQP: existence of an optimal solution to a QCQP, the formulation of a QCQP as a QPCC, the local optimality conditions of a class of quadratically constrained nonlinear programs failing constraint qualifications, and the formulation of a copositive QCQP as a completely positive program. Proved for a convex polynomial program as early as in the 1977 book [8] and reproved subsequently in [37,29], the existence of an optimal solution to a convex QCQP is fully resolved via the classical Frank-Wolfe theorem, which states that such a program, if feasible, has an optimal solution if and only if the objective function of the program is bounded below on the feasible set. The situation with a nonconvex QCQP is rather different; indeed, while there has been extensive work [2,3,4, 32] published on the existence of optimal solutions to nonconvex programs in general, the sharpest Frank-Wolfe type existence results for a feasible QCQP with a nonconvex (quadratic) objective are obtained in [29] and summarized as follows:

(a) if there is at most one nonlinear (but convex) quadratic constraint, then the QCQP attains its minimum if the objective function is bounded below on the feasible set;
(b) if the objective function is quasiconvex on the feasible region and all the constraints are convex, then the same conclusion as in (a) holds.

The authors in the latter reference also gives an example of a QCQP with two convex quadratic constraints satisfying a Slater condition and a nonconvex objective function bounded below on the feasible set for which the infimum objective value is not attained. Since a Frank-Wolfe type existence result holds for a QPCC, which is a special case of a QCQP albeit with bilinear, thus nonconvex, quadratic constraints, it is natural to ask whether there is a class of QCQPs, broader than the class of QPCCs, for which a Frank-Wolfe type existence result holds. It turns out that the answer to this question is affirmative; interestingly, within this subclass of QCQPs, those which fail the Slater condition but are solvable will have rational optimal solutions if the data are rational numbers to begin with.

In [35], four types of stationary points are defined for MPCCs, among which the so-called Bouligand or B-stationarity [28] yields the strongest conclusions; see also [34] where the concept of stationarity is generalized to nonlinear programs with “structurally nonconvex” feasible sets that include MPCCs. Checking B-stationarity is equivalent to solving a LPCC, therefore it is hard. To date, there is no clear understanding of the stationarity condition for a general nonconvex mathematical program that fails constraint qualifications (CQs). Extending the QCQP, we consider a class of (nonconvex) quadratically constrained optimization problems that fail CQs, and show that checking the local optimality of this class of problems is equivalent to checking the global optimality of a mathematical program with a linearized objective function subject to the same constraints plus an imposed linear constraint. Similar to checking the B-stationarity for MPCCs, checking the latter optimality condition, which serves as a B-stationarity-like condition for the given program is still hard; in the simplest case when the non-quadratic constraints are all linear, the linearized B-stationarity-like condition is equivalent to a LPCC.

Lastly, we show that a QCQP of the type considered in this paper, if copositive, can be recast as an equivalent completely positive program. This extends the recent papers [11,12] that address the copositive representations of binary nonconvex quadratic programs, certain types of quadratically constrained quadratic programs, and a number of other NP-hard problems. An introduction of basic concepts and a summary of recent developments in copositive programming can be found in a survey paper [15] and a book chapter [10]. It follows from our result that a copositive QPCC is completely equivalent to its copositive relaxation in terms of feasibility, boundedness, attainability as well as solvability.

Overall, this paper is organized as follows. Section 2 addresses the formulation of a QCQP, one which fails the Slater condition, as a QPCC. This formulation allows us to conclude that a Frank-Wolfe type result holds for this subclass of QCQPs. Section 3 addresses the more general class of quadratically constrained nonlinear programs failing CQs. The last section shows that a copositive QCQP is totally equivalent to its copositive representation, thereby extending the results in [11,12], which require a boundedness assumption.

2 Two Classes of Quadratic Problems

We begin with the formal definitions of the classes of QPCCs and QCQPs. Specifically, given a symmetric matrix $Q \in \mathbb{R}^{N\times N}$, where $N \triangleq n+2m$, a vector $c \in \mathbb{R}^N$, a matrix $A \in \mathbb{R}^{k\times N}$, and a vector $b \in \mathbb{R}^k$, the QPCC is the minimization problem:

$$\begin{array}{ll} \underset{x \triangleq (x^0,x^1,x^2) \geq 0}{\text{minimize}} & c^T x + \frac{1}{2}\, x^T Q x \\ \text{subject to} & Ax = b \quad \text{and} \quad (\, x^1\,)^T x^2 = 0 \end{array} \tag{1}$$

The special features of this formulation are as follows: (a) the vector x is composed of 3 nonnegative subvectors: $x^0 \in \mathbb{R}^n_+$ and $x^i \in \mathbb{R}^m_+$ for $i = 1,2$; the latter two variables are a complementary pair satisfying the complementarity condition: $0 \leq x^1 \perp x^2 \geq 0$; (b) the other constraints on x are expressed as linear equations; and (c) the matrix Q is *not* necessarily positive semidefinite.

Generalizing the bilinear equation $(x^1)^T x^2 = 0$, which is equivalent to the quadratic inequality $(x^1)^T x^2 \leq 0$ due to the nonnegativity of the variables, to general quadratic constraints, we may state the QCQP as:

$$\begin{array}{ll} \underset{x \geq 0}{\text{minimize}} & f_0(x) \triangleq (\, c^0\,)^T x + \frac{1}{2}\, x^T Q^{\,0} x \\ \text{subject to } Ax = b & \\ \text{and} & f_i(x) \triangleq h_i + (\, c^i\,)^T x + \frac{1}{2}\, x^T Q^{\,i} x \leq 0, \quad i = 1, \cdots, K, \end{array} \tag{2}$$

for some positive integer K, where each $h_i \in \mathbb{R}$, $c^i \in \mathbb{R}^N$, and $Q^{\,i} \in \mathbb{R}^{N\times N}$ for $i = 0, 1, \cdots, N$ are symmetric matrices.

A Frank-Wolfe (FW) result for either (1) and (2) states that the program attains a finite minimum objective value if and only if it is feasible and the quadratic objective function is bounded below on the feasible set. Since the feasible set of the QPCC is the union of finitely many polyhedra and since the FW theorem holds for general quadratic programs, it follows readily that such an attainment result holds for the QPCC. The following results extend this conclusion to a class of QCQPs.

Proposition 1 Assume for each $i = 1, \cdots, K-1$, the constraint $f_i(x) \leq 0$ has no Slater point in the region $\mathcal{P} \triangleq \{x \geq 0 \mid Ax = b\}$, i.e., $x \in \mathcal{P}$ implies $f_i(x) \geq 0$. Then the set

$$\widehat{\mathcal{P}} \triangleq \{\, x \in \mathcal{P} \mid f_i(x) \leq 0, \; i = 1, \cdots, K-1\,\},$$

if nonempty, is the union of finitely many polyhedra.

Proof Indeed, under the constant-sign assumption of the first $K-1$ quadratic functions, it follows that, if $\widehat{\mathcal{P}} \neq \emptyset$, then

$$\widehat{\mathcal{P}} = \left[\underset{x \in \mathcal{P}}{\operatorname{argmin}} \sum_{i=1}^{K-1} f_i(x) \right] = \{ x \in \mathcal{P} \mid f_i(x) = 0, \ i = 1, \cdots, K-1 \}.$$

Write $\sum_{i=1}^{K-1} f_i(x) = \mathbf{h} + \mathbf{q}^T x + \frac{1}{2} x^T \mathbf{Q} x$, where $\mathbf{h} \triangleq \sum_{i=1}^{K-1} h_i$, $\mathbf{q} \triangleq \sum_{i=1}^{K-1} c^i$ and $\mathbf{Q} \triangleq \sum_{i=1}^{K-1} Q^i$. It then follows from [20,21] that $x \in \widehat{\mathcal{P}}$ if and only if $\lambda \in \mathbb{R}^k$ exists such that

$$\begin{aligned} 0 &\leq x \perp \mathbf{q} + \mathbf{Q}x + A^T\lambda \geq 0 \\ 0 &= Ax - b \\ 0 &= \tfrac{1}{2} \left(\mathbf{q}^T x - b^T \lambda \right) + \mathbf{h}. \end{aligned}$$

This is enough to establish the claim that $\widehat{\mathcal{P}}$ is the union of finitely many polyhedra, say $\widehat{\mathcal{P}} = \bigcup_{j=1}^{J} P^j$ for some positive integer J, where each P^j is a polyhedron in $\mathbb{R}^N$. □

Based on the above piecewise representation of the set $\widehat{P}$, we have the following result for the QCQP (2).

Theorem 1 Suppose that the assumptions in Proposition 1 are satisfied and that the remaining quadratic constraint function $f_K(x)$ is convex. Then the FW attainment result holds for the QCQP (2).

Proof As a result of Proposition 1, it follows that the feasible region of (2) is equal to $\bigcup_{j=1}^{J} \widehat{P}^j$, where each $\widehat{P}^j \triangleq \left\{ x \in P^j \mid f_K(x) \leq 0 \right\}$ is the zero-level set of the convex quadratic function f_K over the polyhedron P^j. By [29, Theorem 2], the FW attainment result holds for each of the convex singly-quadratically constrained quadratic program $\underset{x \in \widehat{P}^j}{\text{minimize}}\, f_0(x)$. Thus, the same result holds for the QCQP (2). □

While being a nonconvex program, the class of QCQPs satisfying the assumption of Theorem 1 fails the Slater constraint qualification; yet, without the last quadratic constraint $f_K(x) \leq 0$, this class of QCQPs has an interesting property that we highlight in the following result.

Proposition 2 Suppose that for each $i = 1, \cdots, K-1$, the quadratic function $f_i(x)$ does not have a Slater point in the region $\mathcal{P}$. Without the constraint $f_K(x) \leq 0$, if the QCQP (2) has an optimal solution, then it has a rational optimal solution, provided that the input vectors c^i and matrices Q^i for $i = 0, 1, K-1$ and the pair (A, b) have rational components.

Proof By the proof of Theorem 1, it follows that the QCQP (2) has an optimal solution that is an optimal solution of a (possibly nonconvex) quadratic program. Thus, it suffices to show that if the input data of a solvable quadratic program are all rational, then the program has a rational optimal solution. In turn, this claim follows from two facts: (a) the set of global minima of a quadratic program is the union of finitely many polyhedra, a result [18, Exercise 2.9.31] derived from the fact [27] that the objective value of a quadratic program attains finitely many values on the set of stationary points, and (b) defined by finitely many linear inequalities and equations, any nonempty polyhedron must contain at least one rational point, provided that the data of its linear constraints are rational numbers. □

Remark 1 Proposition 2 is false if the QCQP has a quadratic inequality constraint that has a Slater point. The scalar problem: minimize x subject to $x^2 \leq 2$ provides a simple counterexample that illustrates the failure of the proposition under the Slater assumption. □

3 Local Optimality for QC-Problems Failing CQs

Although Theorem 1 has resolved the issue of solvability of the QCQP (2) satisfying the assumptions of this theorem, the question of how to characterize the optimality of a solution to such a problem is not addressed by this theorem, nor is it treated in the current literature, due to the lack of a suitable constraint qualification. In what follows we deal with this issue for the following problem:

$$\begin{array}{ll} \underset{x \in C \cap \mathcal{P}}{\text{minimize}} & \theta(x) \\ \text{subject to} & f_i(x) \triangleq h_i + (c^i)^T x + \frac{1}{2} x^T Q^i x \leq 0, \quad i = 1, \cdots, K-1, \end{array} \tag{3}$$

where C is a closed convex set and $\mathcal{P} \triangleq \{x \geq 0 \mid Ax = b\}$ is a polyhedron in $\mathbb{R}^n$. In (3), the two sets C and $\mathcal{P}$ play distinctive roles. The set C is an arbitrary closed convex set not required to be a polyhedron; e.g., C can be the level-set $\{x \mid f_K(x) \leq 0\}$ of a convex function $f_K(x)$; whereas the set $\mathcal{P}$ is a polyhedron on which each of the quadratic constraints is nonnegative. The feasible region of (3), denoted S, is in general nonconvex. We have the following necessary and sufficient condition of local optimality for (3); in particular, the last assertion of the following theorem extends the conclusion for a "convex" MPCC that is a convex program with additional linear complementarity constraints.

Theorem 2 Suppose that the assumptions in Proposition 1 are satisfied. Let $C \subseteq \mathbb{R}^n$ be closed convex and $\theta : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let $S = \{ x \in C \cap \mathcal{P} \mid q(x) = 0 \}$. Let x^* be a feasible solution of (3). Consider the following statements:

(a) x^* is a locally optimal solution of (3);

(b) x^* is a globally optimal solution of (4).

$$\begin{array}{ll} \underset{x \in S}{\text{minimize}} & (x - x^*)^T \nabla\theta(x^*) \\ \text{subject to} & (x - x^*)^T \displaystyle\sum_{i=1}^{K-1} \nabla f_i(x^*) \leq 0; \end{array} \tag{4}$$

(c) x^* is a locally optimal solution of the following:

$$\underset{x \in S}{\text{minimize}} \ (x - x^*)^T \nabla\theta(x^*). \tag{5}$$

It holds that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If θ is convex on $C \cap \mathcal{P}$, then (b) $\Rightarrow$ (a).

Proof We use the same notation as in Proposition 1. In particular, we write $q(x) \triangleq \sum_{i=1}^{K-1} f_i(x) = \mathbf{h} + \mathbf{q}^T x + \frac{1}{2} x^T \mathbf{Q} x$, By this proposition, we have

$$S = \{ x \in C \cap \mathcal{P} \mid q(x) = 0 \} = C \cap \bigcup_{j=1}^{J} P^j.$$

(a) $\Rightarrow$ (b). Assume by way of contradiction that x^* is not a globally optimal solution to (4). Then there exists $\bar{x}$ feasible to (4) such that $d^T \nabla\theta(x^*) < 0$, where $d \triangleq \bar{x} - x^*$. Note that

$$q(x^* + \lambda d) = q(x^*) + \lambda(\mathbf{q} + \mathbf{Q}x^*)^T d + \frac{\lambda^2}{2} d^T \mathbf{Q} d. \tag{6}$$

Since $q(x) \geq 0$ for all $x \in \mathcal{P}$, we must have $(\mathbf{q} + \mathbf{Q}x^*)^T d \geq 0$; therefore $(\mathbf{q} + \mathbf{Q}x^*)^T d = 0$. Since $q(x^*) = 0$ and $q(\bar{x}) = 0$ we must have $d^T \mathbf{Q} d = 0$. Therefore $x^* + \lambda d$ is feasible in (4) for all $\lambda \in [0,1]$; hence $\theta(x^* + \lambda d) \geq \theta(x^*)$ for all $\lambda \geq 0$ sufficiently small, implying that $d^T \nabla\theta(x^*) \geq 0$. This is a contradiction.

(b) $\Rightarrow$ (c). We show the contrapositive. Assume x^* is not a locally optimal solution to (5). Then a sequence $\{x_k\} \subset S$ converging to x^* exists such that $(x_k - x^*)^T \nabla\theta(x^*) < 0$ for all k. There must exist x_{k_0} such that x_{k_0} and x^* belong to the same piece $P^{j_0} \cap C$. Let $d \triangleq x_{k_0} - x^*$. Since $P^{j_0} \cap C$ is convex, $x^* + \lambda d \in S$ for all $\lambda \in [0,1]$. From the same argument as in the previous part, we again obtain $(\mathbf{q} + \mathbf{Q}x^*)^T d = 0$, or $\nabla q(x^*)^T d = 0$, so $x^* + \lambda d$ is feasible in (4) for all $\lambda \in [0,1]$. Therefore x^* is not a globally optimal solution of (4).

(b) $\Rightarrow$ (a) if θ is convex on $C \cap \mathcal{P}$. Assume by way of contradiction that x^* is not a local minimum to (3). There exists a sequence $\{x_k\}$ feasible to (3) converging to x^* such that $\theta(x_k) < \theta(x^*)$ for all k. As in the proof of (a) implying (b), there exists an x_{k_0} such that $x^* + \lambda d$ is feasible in (3) for all $\lambda \in [0,1]$, where $d \triangleq x_{k_0} - x^*$. Thus $q(x^* + \lambda d) = 0$ for all such λ. The expansion (6) then implies that $(\mathbf{q} + \mathbf{Q}x^*)^T d = 0$. Thus $x^* + \lambda d$ is feasible in (4) for all $\lambda \in [0, 1]$; in particular, so is x_{k_0}. As $\theta(x)$ is convex, we have

$$\theta(x_{k_0}) \geq \theta(x^*) + \nabla\theta(x^*)^T (x_{k_0} - x^*),$$

which implies

$$\nabla\theta(x^*)^T (x_{k_0} - x^*) \leq \theta(x_{k_0}) - \theta(x^*) < 0,$$

which means x^* is not a global optimum for (4). This is a contradiction. $\square$

The example below shows that if x^* is a locally optimal solution to (3), it is not necessarily a *globally* optimal solution of

$$\underset{x \in S}{\text{minimize}} \; (x - x^*)^T \nabla\theta(x^*); \tag{7}$$

thus, to obtain a characterization of x^* as a globally optimal solution of a "linearized problem", it is essential that we add the extra constraint $\nabla q(x^*)^T (x - x^*) \leq 0$, yielding the problem (4). The example also shows that the implication "(c) $\Rightarrow$ (b)" does not hold for the elements of the theorem, even when $\theta(x)$ is convex on $C \cap \mathcal{P}$.

Example 1 Consider the following simple 2-variable QPCC:

$$\begin{aligned}
&\underset{x_1, x_2}{\text{minimize}} && (x_1 - 2)^2 + x_2^2 \\
&\text{subject to} && x_1 + x_2 \geq 3 \\
&\text{and} && 0 \leq x_1 \perp x_2 \geq 0.
\end{aligned}$$

The optimal solution to this "convex" QPCC is $x^* = (3, 0)$. The corresponding problem (7) is

$$\begin{aligned}
&\underset{x_1, x_2}{\text{minimize}} && 2(x_1 - 3) \\
&\text{subject to} && x_1 + x_2 \geq 3 \\
&\text{and} && 0 \leq x_1 \perp x_2 \geq 0,
\end{aligned}$$

whose optimal solutions are all points of the form $(0, x_2)$ with $x_2 \geq 3$, none of these is the point $x^* = (3, 0)$. Adding the linearized complementarity constraint, we obtain the problem:

$$\begin{aligned}
&\underset{x_1, x_2}{\text{minimize}} && 2(x_1 - 3) \\
&\text{subject to} && x_1 + x_2 \geq 3 \\
& && 0 \leq x_1 \perp x_2 \geq 0 \\
&\text{and} && 3x_2 \leq 0,
\end{aligned}$$

whose unique globally optimal solution is precisely x^*. $\square$

The problem (4) generalizes the idea of B-stationarity for a MPCC. Specifically consider the special case of (3) where $x \triangleq (x^0, x^1, x^2)$, the polyhedron $\mathcal{P}$ is a subset of $\mathbb{R}^n \times \mathbb{R}^{2m}_+$, and $q(x) \triangleq (x^1)^T x^2$. For a feasible vector $x^* \triangleq (x^{*,0}, x^{*,1}, x^{*,2})$ of (3), define the 3 index sets pertaining to the complementarity condition: $0 \leq x^1 \perp x^2 \geq 0$:

$$\begin{aligned}
\alpha_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} > 0 = x_i^{*,2} \,\} \\
\beta_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} = 0 = x_i^{*,2} \,\} \\
\gamma_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} = 0 < x_i^{*,2} \,\}.
\end{aligned}$$

These 3 index sets play a key role in the B-stationarity of the MPCC:

$$\begin{array}{ll}
\underset{x \triangleq (x^0, x^1, x^2) \in C \cap \mathcal{P}}{\text{minimize}} & \theta(x) \\
\text{subject to} & 0 \leq x^1 \perp x^2 \geq 0.
\end{array} \tag{8}$$

It is not difficult to show that with S denoting the feasible region of (8) and $q(x) \triangleq (x^1)^T x^2$, we have

$$\begin{aligned}
&\{\, x \in S \mid \nabla q(x^*)^T (x - x^*) \leq 0 \,\} = \\
&\quad \{\, x \in C \cap \mathcal{P} \mid x_i^2 = 0 \ \forall\, i \in \alpha;\ x_i^1 = 0 \ \forall\, i \in \gamma;\ 0 \leq x_i^1 \perp x_i^2 \geq 0 \}.
\end{aligned}$$

The above expression gives 2 structurally different representations of the same set; the left-hand representation expresses the set as defined by the closed set C intersected with the linearly-quadratically constrained set $\{x \in \mathcal{P} \mid q(x) \leq 0,\ \nabla q(x^*)^T (x - x^*) \leq 0\}$, whereas the right-hand representation reveals the disjunctive structure of the latter set with reference to the given point x^* and shows that it is the union of finitely many closed convex sets. When in addition C is a polyhedron, then the problem (4) is a LPCC with a linear objective function. The upshot of this development is that in this case, the latter LPCC yields the optimality conditions of the quadratically constrained optimization problem (3) that fails the Slater constraint qualification.

4 Copositive Representation of QCQPs

A completely positive program is a linear optimization problem in matrix variables in the form of:

$$\begin{array}{ll}
\underset{X \in \mathcal{S}^n}{\text{minimize}} & < A_0, X > \\
\text{subject to} & < A_i, X > = b_i, \quad i = 1, \cdots, \ell \\
\text{and} & X \in \mathcal{C}_n^*,
\end{array}$$

where $\mathcal{S}^n$ is the space of symmetric $n \times n$ matrices and $\mathcal{C}_n^*$ is the cone of completely positive matrices,

$$\mathcal{C}_n^* \triangleq \operatorname{conv}\left\{\, M \in \mathcal{S}^n \mid M = xx^T,\, x \in \mathbb{R}^n_+ \,\right\},$$

whose dual $\mathcal{C}_n$ is the so-called cone of copositive matrices,

$$\mathcal{C}_n \triangleq \left\{ M \in \mathcal{S}^n \mid x^T M x \geq 0, \; \forall\, x \in \mathbb{R}^n_+ \right\}.$$

The recent paper [11] addresses the copositive representations of a binary or continuous nonconvex quadratic program. Paper [12] formulates a QCQP, one which has a bounded feasible region, as an equivalent completely positive program, given that some other restrictions are satisfied. As a special instance of a QCQP albeit with bilinear quadratic constraints, can a QPCC be cast as a completely positive program? The answer to this question is affirmative if the objective function of the QPCC is convex. This is true even if the feasible set of the QPCC is unbounded. In what follows, we prove a more general result. In particular, the equivalence will be established between a QCQP whose single constraint has no Slater point and whose quadratic term is copositive on the recession cone of the linear constraints and with an objective function that is copositive on a particular subset of this cone, and its completely positive representation. In general, a QPCC with a nonconvex objective function is not guaranteed to have an equivalent copositive representation, unless under limited conditions such as bounded complementarity variables, as we show in the following example.

Example 2

$$\begin{array}{ll} \underset{x \geq 0}{\text{miminize}} & -x_2^2 \\ \text{subject to} & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_4 = 2 \\ \text{and} & x_1 \perp x_2. \end{array}$$

The optimal objective value of the above QPCC is 0 and the only feasible ray is $d \triangleq (1,0,1,1)$. Below is the copositive representation of this QPCC, which is a completely positive program:

$$\begin{array}{ll} \underset{x,\,X}{\text{minimize}} & -X_{2,2} \\ \text{subject to} & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_4 = 2 \\ & X_{1,1} + 2X_{1,2} + X_{2,2} - 2X_{1,3} - 2X_{2,3} + X_{3,3} = 1 \\ & X_{1,1} + X_{4,4} - 2X_{1,4} = 4 \\ & X_{1,2} = 0 \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_5^*. \end{array}$$

Denote $\bar{d} \triangleq (0,1,1,0)$ and it is easy to show that $\begin{pmatrix} 0 & 0 \\ 0 & \bar{d}\,\bar{d}^T \end{pmatrix}$ is a feasible ray of the above completely positive program, therefore this completely positive problem is unbounded below. □

The class of QCQPs that we consider in this section is of the form given in (2), where we concatenate the quadratic constraints by a single quadratic function

$$q(x) \triangleq \mathbf{h} + \mathbf{c}^T x + \tfrac{1}{2}\, x^T \mathbf{Q} x.$$

We assume $\mathbf{Q}$ is copositive on the recession cone of the equality constraints,

$$K \triangleq \{\, d \geqslant 0 \mid A_i\, d = 0,\ i = 1, 2, \cdots, k \,\}, \tag{9}$$

so $d^T \mathbf{Q} d \geq 0$ for all $d \in K$. Further, we assume that any $x \geq 0$ satisfying $Ax = b$ also satisfies $q(x) \geq 0$, so there is no Slater point in the constraint $q(x) \leq 0$. Note that a QPCC satisfies these assumptions. In addition, a binary restriction on a variable x_j represented by the quadratic inequality $x_j - x_j^2 \leq 0$ also satisfies the assumptions, provided the constraints $\{Ax = b,\ x \geq 0\}$ imply $0 \leq x_j \leq 1$. Burer [11] also briefly discussed constraints satisfying the assumption regarding the lack of a Slater point, but the cited reference imposed an additional assumption that is more restrictive than ours.

The resulting QCQP is equivalent to the following problem:

$$\begin{aligned} & \underset{x \geq 0}{\text{minimize}} \quad (\,c^0\,)^T x + \tfrac{1}{2}\,\langle\, Q^0,\, X \,\rangle \\ & \text{subject to } A_i\, x \;=\; b_i, \quad i = 1, \cdots, k, \\ & \qquad\qquad q(x) \;\leq\; 0 \quad \text{and} \quad X = x x^T. \end{aligned} \tag{10}$$

By relaxing the rank-1 constraint over the matrix X to the condition that matrix $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ is in the cone of completely positive matrices, we get the following completely positive program (11):

$$\begin{aligned} & \underset{x,\,X}{\text{minimize}} \quad (\,c^0\,)^T x + \tfrac{1}{2}\,\langle\, Q^0,\, X \,\rangle \\ & \text{subject to } A_i\, x \;=\; b_i \quad \text{and} \quad A_i\, X\, A_i^T \;=\; b_i^2, \quad i = 1, \cdots, k, \\ & \qquad\qquad \mathbf{h} + \mathbf{c}^T x + \tfrac{1}{2}\,\langle\, \mathbf{Q}, X \,\rangle \;=\; 0 \\ & \text{and} \qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{1+n}. \end{aligned} \tag{11}$$

Note that we have exploited the assumption regarding the lack of a Slater point in this formulation. The completely positive program (11) is not only a relaxation problem but also an equivalent form of the QCQP. To prove the equivalence, we define the following sets:

$$\begin{aligned}
\Gamma &\triangleq \{\, x \geqslant 0 \mid A_i\, x = b_i,\ i = 1, \cdots, k,\ \text{and } q(x) \leq 0 \,\} \\
L &\triangleq \left\{\, d \geqslant 0 \mid A_i\, d = 0,\ i = 1, 2, \cdots, k,\ \text{and } d^T \mathbf{Q} d = 0 \,\right\} \subseteq K \\
\Gamma^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \mid x \in \Gamma \right\} \\
L^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & dd^T \end{pmatrix} \mid d \in L \right\} \\
\Sigma^+ &\triangleq \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \mid (x, X) \text{ feasible for (11)} \right\}.
\end{aligned} \tag{12}$$

Among the above sets, Γ is the set of feasible solutions of QPCC (1), Γ^+ is the set of feasible solutions of program (10), and Σ^+ is the set of feasible solutions of the completely positive program (11). As a result of the cone constraint $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{1+n}$, it is clear that

1. $\Gamma^+ \subseteq \Sigma^+$,
2. $L^+ \subseteq$ recession cone of Σ^+.

Therefore $\Gamma^+ + L^+ \subset \Sigma^+$. Further, we establish that $\Sigma^+ = \Gamma^+ + L^+$ in the following proposition.

Proposition 3 If $\mathbf{Q}$ is copositive over the cone K, then $\Sigma^+ = \Gamma^+ + L^+$.

Proof Assume (x, X) is feasible to (11). As $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{1+n}$, we have, for some positive integer J,

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{j=1}^{J} \begin{pmatrix} v_j \\ \xi_j \end{pmatrix} \begin{pmatrix} v_j \\ \xi_j \end{pmatrix}^T, \quad \text{with} \quad \begin{pmatrix} v_j \\ \xi_j \end{pmatrix} \in \mathbb{R}^{1+n}_+.$$

Due to the linear constraints in the copositive program (11), for all j, $\begin{pmatrix} v_j \\ \xi_j \end{pmatrix}$ satisfies the following properties:

1. $\sum_{j=1}^{J} v_j^2 = 1$;
2. $A_i \sum_{j=1}^{J} v_j\, \xi_j = \sum_{j=1}^{J} v_j\, (A_i\, \xi_j) = b_i$ for all $i = 1, \cdots, k$;

3. $A_i \left(\sum_{j=1}^{J} \xi_j \, \xi_j^T \right) A_i^T \; = \; \sum_{j=1}^{J} A_i \, \xi_j \, \xi_j^T \, A_i^T \; = \; \sum_{j=1}^{J} (A_i \, \xi_j)^2 \; = \; b_i^2$ for all $i = 1, \cdots, k$;

4. $\mathbf{h} + \sum_{j=1}^{J} v_j \, \mathbf{c}^T \xi_j + \frac{1}{2} \sum_{j=1}^{J} \xi_j^T \mathbf{Q} \xi_j = 0.$

As a result of properties 1, 2 and 3, we have

$$\left(\sum_j v_j \, (A_i \, \xi_j) \right)^2 \; = \; b_i^2 \; = \; \sum_j (A_i \, \xi_j)^2 \, \sum_j v_j^2, \; \forall \, i = 1, \cdots, k.$$

Thus by the Cauchy-Schwartz Inequality, it follows that $A_i \, \xi_j \; = \; v_j \, \delta_i$ for all j and all $i = 1, \cdots k$, for some scalars δ_i. Define the following two index sets:

$$J_+ \; \triangleq \; \{ \, j \; | \; v_j > 0 \, \} \text{ and } J_0 \; \triangleq \; \{ \, j \; | \; v_j = 0 \, \}.$$

Then we have

1. $\sum_{j \in J_+} v_j^2 = 1;$

2. $A_i \, \xi_j = 0$ and $A_i \, \xi_j \xi_j^T A_i^T = 0$ for all $j \in J_0$ and all $i = 1, \cdots, k$;

3. $\sum_{j \in J_+} v_j \, A_i \, \xi_j = b_i$ for all $i = 1, \cdots, k$;

4. $\delta_i = \delta_i \sum_{j \in J_+} v_j^2 = \sum_{j \in J_+} v_j (v_j \, \delta_i) = \sum_{j \in J_+} v_j A_i \xi_j = b_i$ for all $i = 1, \cdots, k$;

5. $A_i \, \xi_j = b_i \, v_j$ for all $j \in J_+$ and all $i = 1, \cdots, k$.

The completely positive matrix $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ can therefore be decomposed as

$$\begin{pmatrix} 1 \; x^T \\ x \; X \end{pmatrix} \; = \; \sum_{j \in J_+} v_j^2 \, \begin{pmatrix} 1 \\ \dfrac{\xi_j}{v_j} \end{pmatrix} \begin{pmatrix} 1 \\ \dfrac{\xi_j}{v_j} \end{pmatrix}^T + \sum_{j \in J_0} \begin{pmatrix} 0 \\ \xi_j \end{pmatrix} \begin{pmatrix} 0 \\ \xi_j \end{pmatrix}^T,$$

where

1. $\sum_{j \in J_+} v_j^2 = 1;$

2. $A_i \, \dfrac{\xi_j}{v_j} \; = \; b_i$ for all $j \in J_+$ and all $i = 1, \cdots, k$;

3. $A_i \, \xi_j \; = 0$ for all $j \in J_0$ and all $i = 1, \cdots k$;

4. $\mathbf{h} + \sum_{j=1}^{J} v_j \, \mathbf{c}^T \xi_j + \frac{1}{2} \sum_{j=1}^{J} \xi_j^T \mathbf{Q} \xi_j = 0.$

It follows that the vector $\dfrac{\xi_j}{v_j}$ must satisfy $q\left(\dfrac{\xi_j}{v_j}\right) \geq 0$, from the assumption regarding the lack of a Slater point, for any $j \in J_+$. In particular, we have

$$0 \leq \mathbf{h} + \frac{1}{v_j}\,\mathbf{c}^T\xi_j + \frac{1}{2v_j^2}\,\xi_j^T\mathbf{Q}\xi_j \quad \forall\, j \in J_+.$$

Multiplying by v_j^2 and adding over J_+, we obtain

$$\begin{aligned} 0 &\leq \mathbf{h}\sum_{j\in J_+} v_j^2 + \sum_{j\in J_+} v_j\,\mathbf{c}^T\xi_j + \tfrac{1}{2}\sum_{j\in J_+}\xi_j^T\mathbf{Q}\xi_j \\ &= \mathbf{h} + \sum_{j} v_j\,\mathbf{c}^T\xi_j + \tfrac{1}{2}\sum_{j\in J_+}\xi_j^T\mathbf{Q}\xi_j \end{aligned}$$

since $\sum_{j\in J_+} v_j^2 = 1$ and $v_j = 0$ for $j \in J_0$. It follows from the earlier point 4, the copositivity of $\mathbf{Q}$ over K, and the nonnegativity of each ξ_j that

$$\mathbf{h} + \frac{1}{v_j}\,\mathbf{c}^T\,\xi_j + \frac{1}{2v_j^2}\,\xi_j^T\mathbf{Q}\xi_j = 0 \quad \forall j \in J_+$$

$$\text{and} \quad \xi_j^T\mathbf{Q}\xi_j = 0 \quad \forall j \in J_0$$

Therefore $\dfrac{\xi_j}{v_j} \in \Gamma$ for all $j \in J_+$. Besides, $\xi_j \in L$ for all $j \in J_0$. Therefore, $\Sigma^+ \subseteq \Gamma^+ + L^+$. On the other hand, $\Gamma^+ + L^+ \subseteq \Sigma^+$ is obvious. $\square$

Based on Proposition 3, we can now establish the claimed equivalence between the QCQP (2) formulated as (10) and the completely positive program (11). This result implies that a QPCC with a convex quadratic objective function is equivalent to its completely positive representation.

Theorem 3 Assume that any point $x \geq 0$ satisfying $Ax = b$ also satisfies $q(x) \geq 0$. Assume further that Q^0 is copositive over L and $\mathbf{Q}$ is copositive over K. The QCQP (2) and the completely positive program (11) are equivalent in the sense that

1. The QCQP (2) is feasible if and only if the completely positive program (11) is feasible.
2. Either the optimal values of the QCQP (2) and the completely positive program (11) are finite and equal, or both of them are unbounded below.
3. Assume both the QCQP (2) and the completely positive program (11) are bounded below, and $(\bar{x}, \bar{X})$ is optimal for the completely positive program, then $\bar{x}$ is in the convex hull of the optimal solutions of the QCQP.
4. The optimal value of the QCQP (2) is attained if and only if the same holds for the completely positive program (11).

Proof 1. Assume (x, X) is a feasible solution of the completely positive program (11). Then

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{j \in J_+} \alpha_j \begin{pmatrix} 1 \\ \xi_j \end{pmatrix} \begin{pmatrix} 1 \\ \xi_j \end{pmatrix}^T + \sum_{j \in J_0} \begin{pmatrix} 0 \\ d_j \end{pmatrix} \begin{pmatrix} 0 \\ d_j \end{pmatrix}^T,$$

where $\alpha_j > 0$ for all $j \in J_+$ and $\sum_{j \in J_+} \alpha_j = 1$. According to the proof of Proposition 3, $\{\xi_j, \forall j \in J_+\}$ are feasible for the QCQP (2). On the other hand, if x is feasible for the QCQP (2), then $\begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ is feasible for the completely positive program (11).

2. Denote the optimal values of the QCQP (2) and the completely positive program (11) as Opt(2) and Opt(11) respectively.

$$\begin{aligned} \text{Opt}(2) &\triangleq \min_{x \in \Gamma} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \right\rangle \\ &= \min_{Y \in \Gamma^+} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle. \end{aligned}$$

Since $\Gamma^+ \subseteq \Sigma^+$,

$$\begin{aligned} &\min_{Y \in \Gamma^+} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle \\ &\quad \geq \text{ Opt}(11) \triangleq \min_{Y \in \Sigma^+ = \Gamma^+ + L^+} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle. \end{aligned}$$

Therefore Opt(2) $\geq$ Opt(11). Since Q^0 is copositive over L,

$$\underset{Y \in \Sigma^+ = \Gamma^+ + L^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle \geq \underset{Y \in \Gamma^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}(c^0)^T \\ \frac{1}{2}c^0 & \frac{1}{2}Q^0 \end{pmatrix}, Y \right\rangle.$$

Therefore, Opt(11) $\geq$ Opt(2). As a result, the two optimum objectives are equal.

3. Assume that $(\bar{x}, \bar{X})$ is optimal for the completely positive program (11). Then there exist $\bar{\alpha}_j$, $\bar{\xi}_j$, $\bar{d}_j$, J_+ and J_0 such that

$$\begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} = \sum_{j \in J_+} \bar{\alpha}_j \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix}^T + \sum_{j \in J_0} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix}^T.$$

Here $\bar{\alpha}_j > 0$ for all j and $\sum_{j\in J_+} \bar{\alpha}_j = 1$. In addition, $\{\bar{\xi}_j, \forall\, j \in J_+\} \subset \Gamma$ and $\{\bar{d}_j, \forall\, j \in J_0\} \subset L$. It is obvious that $\left\langle Q^0, \bar{d}_j \bar{d}_j^T \right\rangle = 0$ for all $j \in J_0$. Therefore

$$\left\langle \begin{pmatrix} 0 & \frac{1}{2}\,(c^0)^T \\ \frac{1}{2}\,c^0 & \frac{1}{2}\,Q^0 \end{pmatrix}, \begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} \right\rangle = \sum_{j\in J_+} \bar{\alpha}_j \left[\, (c^0)^T \bar{\xi}_j + \tfrac{1}{2}\, \bar{\xi}_j^T Q^0 \bar{\xi}_j \,\right]$$
$$\geq \sum_{j\in J_+} \bar{\alpha}_j \,\mathrm{Opt}(2) \;=\; \mathrm{Opt}(2).$$

As $\mathrm{Opt}(2) = \mathrm{Opt}(11)$, $(c^0)^T \bar{\xi}_j + \frac{1}{2}\bar{\xi}_j^T Q^0 \bar{\xi}_j = \mathrm{Opt}(2)$ for all $j \in J_+$. In other words, $\{\bar{\xi}_j, \forall\, j \in J_+\}$ are all optimal for QCQP (2). Therefore, $\bar{x} = \sum_{j\in J_+} \bar{\alpha}_j \bar{\xi}_j$ is in the convex hull of the optimal solutions of the QCQP.

4. The QCQP is attainable if and only if it is bounded below, due to the fact that its feasible region is the union of a finite number of polyhedron pieces. If QCQP (2) is attained at $\hat{x}$, the completely positive program (11) is attained at $\begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix}^T$. On the other hand, if the completely positive program (11) is bounded below and attained at

$$\begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{X} \end{pmatrix} = \sum_{j\in J_+} \bar{\alpha}_j \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi}_j \end{pmatrix}^T + \sum_{j\in J_0} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix} \begin{pmatrix} 0 \\ \bar{d}_j \end{pmatrix}^T,$$

then the QCQP is attained at any $\bar{\xi}_j$ such that $j \in J_+$. □

Remark 2 The equivalence in Theorem 3 can be extended to the case where some of the variables in (2) are constrained to be binary, by combining the argument in the paper [11] and our treatment. □

Burer [10] and Eichfelder and Povh [16,17] have shown that the result of [11] can be extended to the case where the requirement that a feasible solution to a QCQP be in the nonnegative orthant is replaced by the requirement that it be in a closed convex cone $\mathcal{K}$. This extension requires that the cone of completely positive matrices $\mathcal{C}_n^*$ be replaced by the cone of completely positive matrices over $\mathcal{K}$, namely

$$\mathcal{C}_n^*(\mathcal{K}) \triangleq \operatorname{conv}\left\{\, M \in \mathcal{S}^n \;\middle|\; M = xx^T,\, x \in \mathcal{K} \,\right\}.$$

It can be readily checked that Proposition 3 and Theorem 3 can be similarly extended. Convex quadratic constraints can be represented as second order cone constraints. Therefore, this extension of Theorem 3 shows the equivalence of a QCQP of the type considered in Theorem 1 with a completely positive program, under only the copositivity assumptions of Theorem 3.

Our results require a copositivity assumption over the recession cone of the linear constraints. In the paper by Sturm and Zhang [36], the authors discuss cones of nonnegative quadratic functions and impose a copositivity restriction on a more general set. More recently, the papers [25, 26] have addressed quadratically constrained quadratic programs by examining cones of nonnegative quadratic functions, based on the programs' KKT conditions that are lifted along with the feasible region of the QCQPs to define a certain cone of symmetric matrices. It would be interesting to investigate the detailed connections of the latter papers with our work; this investigation is left for a future study.

5 Conclusion and Future Work

We have extended the literature to show that certain classes of nonconvex quadratically constrained quadratic programs are guaranteed to satisfy the Frank-Wolfe property. The nonconvex constraints in these problems violate the Slater constraint qualification. In deriving these results, we have exploited a relationship between these problems and quadratic programs with complementarity constraints. Further, we have related local optimality conditions for these QCQPs to stationarity conditions for mathematical programs with equilibrium constraints. If the objective function and constraints for one of these nonconvex QCQPs satisfy some additional constraints then the problem is equivalent to a completely positive program, a convex optimization problem.

By combining Theorems 1 and 3, it follows that (11), which is a special copositive program, has the Frank-Wolfe property, via the equivalence with the QCQP (2) failing the Slater condition with respect to its quadratic inequality constraints. A natural question to ask is whether there is a broader class of copositive programs for which the Frank-Wolfe result holds. At this time, we do not have an answer to this question.

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