

## Nuclear Engineering 150 – Discussion Section

### Team Exercise Solutions #1

\*Problems 1 & 2 borrowed from Nuclear Engineering 101 homework problem sets, Fall 2016

### Problem 1

The radioactive isotope  $^{233}\text{Pa}$  can be produced following neutron capture by  $^{232}\text{Th}$  when the resulting  $^{233}\text{Th}$  decays to  $^{233}\text{Pa}$ . In the neutron flux of a typical reactor, neutron capture in 1 g of  $^{232}\text{Th}$  produces  $^{233}\text{Th}$  at a rate of  $2.0 \times 10^{11} \text{ s}^{-1}$ .

- What are the activities (in Ci) of  $^{233}\text{Th}$  and  $^{233}\text{Pa}$  after this sample is irradiated for 1.5 hours?
- The sample is then placed in storage with no further irradiation so that the  $^{233}\text{Th}$  can decay away. What are the activities (in Ci) of  $^{233}\text{Th}$  and  $^{233}\text{Pa}$  after 48 hours of storage?
- The decay of  $^{233}\text{Pa}$  results in  $^{233}\text{U}$ , which is also radioactive. After the above sample has been stored for 1 year what is the  $^{233}\text{U}$  activity in Ci? (Hint: it should not be necessary to set up an additional differential equation to find the  $^{233}\text{U}$  activity.)

Nucleus	Half-life
$^{233}\text{Th}$	22.3 min
$^{233}\text{Pa}$	27.0 days
$^{233}\text{U}$	$1.592 \times 10^5 \text{ yr}$

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ s}^{-1}$$

### Problem 1 Solution

For all parts of this problem, let  $R$  be the rate of neutron capture by 1 g of  $^{232}\text{Th}$ ,  $2.0 \times 10^{11} \text{ s}^{-1}$ .

- First, convert all half-lives and irradiation times to seconds for consistency.

$$\lambda_{\text{Th}} = \frac{\ln 2}{13388} = 5.18 \times 10^{-4} \text{ s}^{-1}; \quad \lambda_{\text{Pa}} = \frac{\ln 2}{2332800} = 2.971 \times 10^{-7} \text{ s}^{-1}; \quad 1.5 \text{ hr} = 5400 \text{ s}$$

#### Thorium-233

The rate of change of the quantity of  $^{233}\text{Th}$ ,  $\frac{dN_{\text{Th}}}{dt}$  is given by the production rate of  $^{233}\text{Th}$ ,  $R$ , minus the decay rate (activity) of  $^{233}\text{Th}$ ,  $\lambda_{\text{Th}}N_{\text{Th}}$ .

$$\frac{dN_{\text{Th}}}{dt} = R - \lambda_{\text{Th}}N_{\text{Th}}$$

We solve the differential equation, manipulating the equation so that the left side is only dependent on  $N_{\text{Th}}$  and the right side is only dependent on  $dt$ . Then we integrate:

$$\int \frac{dN_{\text{Th}}}{R - \lambda_{\text{Th}}N_{\text{Th}}} = \int dt$$

$$\frac{-1}{\lambda_{\text{Th}}} [\ln(R - \lambda_{\text{Th}}N_{\text{Th}})] = t + C, \quad C = \text{const.}$$

$$R - \lambda_{\text{Th}}N_{\text{Th}} = e^{-\lambda_{\text{Th}}t - \lambda_{\text{Th}}C} = e^{-\lambda_{\text{Th}}t}e^{-\lambda_{\text{Th}}C}$$

We note that since  $C$  is an arbitrary constant and  $\lambda_{\text{Th}}$  is fixed, we could also say  $e^{-\lambda_{\text{Th}}C}$  is an arbitrary constant, and just call it  $C$  instead.

$$R - \lambda_{\text{Th}}N_{\text{Th}} = Ce^{-\lambda_{\text{Th}}t}$$

We solve for  $N_{\text{Th}}$  (explicitly including  $N_{\text{Th}}$ 's dependence on  $t$ ), and get

$$N_{\text{Th}}(t) = \frac{R - Ce^{-\lambda_{\text{Th}}t}}{\lambda_{\text{Th}}}.$$

At  $t = 0$ ,  $N_{\text{Th}}(0) = \frac{R-C}{\lambda_{\text{Th}}} = 0$ , since no  $^{233}\text{Th}$  has been formed. We find  $C = R$ , and use this in the general equation:

$$N_{\text{Th}}(t) = R \frac{1 - e^{-\lambda_{\text{Th}}t}}{\lambda_{\text{Th}}}.$$

With this function of  $N_{\text{Th}}$ , we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Th}}(t) = \lambda_{\text{Th}} N_{\text{Th}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Th}}(t) = R(1 - e^{-\lambda_{\text{Th}}t}).$$

Using the numerical values for  $R$ ,  $\lambda_{\text{Th}}$ , and  $t$ ,

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = (2.0 \times 10^{11} \text{ s}^{-1})(1 - e^{(-5.18 \times 10^{-4} \text{ s}^{-1})(5400 \text{ s})})$$

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 1.878 \times 10^{11} \text{ Bq.}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 5.076 \text{ Ci}.}$$

### Protactinium-233

We follow a similar procedure for  $^{233}\text{Pa}$ , noting that the production rate of  $^{233}\text{Pa}$  is just the activity of  $^{233}\text{Th}$  as it decays into  $^{233}\text{Pa}$ ,  $\mathcal{A}_{\text{Th}}$ .

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}} N_{\text{Pa}}$$

From above, we can substitute our function for  $\mathcal{A}_{\text{Th}}(t)$ ,

$$\frac{dN_{\text{Pa}}}{dt} = R(1 - e^{-\lambda_{\text{Th}}t}) - \lambda_{\text{Pa}} N_{\text{Pa}}$$

Since we cannot separate both sides to be dependent only on a single differential, we must try a different method of integration. We will use integrating factors. Still, we start in a similar fashion: collecting the terms dependent on  $N_{\text{Pa}}$  on the same side.

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} N_{\text{Pa}} = R(1 - e^{-\lambda_{\text{Th}}t})$$

The method of integrating factors suggests that we multiply both sides by an arbitrary exponential. We will use  $e^{\lambda_{\text{Pa}}t}$ .

$$e^{\lambda_{\text{Pa}}t} \frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} e^{\lambda_{\text{Pa}}t} N_{\text{Pa}} = e^{\lambda_{\text{Pa}}t} R(1 - e^{-\lambda_{\text{Th}}t})$$

We can now observe that the left side of the equation appears to be the result of the product rule when the time derivative of  $e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}$  is found. We can then write the equation as

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t} R(1 - e^{-\lambda_{\text{Th}}t})$$

Moving the  $dt$  term to the right side of the equation and using the distributive property, we have

$$d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t} e^{-\lambda_{\text{Th}}t}) dt$$

or more simply (by exploiting properties of exponents)

$$d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t - \lambda_{\text{Th}}t}) dt.$$

We integrate both sides,

$$\int d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = \int R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t - \lambda_{\text{Th}}t}) dt,$$

separate the integral on the right side,

$$\int d(e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}) = R \int e^{\lambda_{\text{Pa}} t} dt - R \int e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} dt$$

and find

$$e^{\lambda_{\text{Pa}} t} N_{\text{Pa}} = \frac{R}{\lambda_{\text{Pa}}} e^{\lambda_{\text{Pa}} t} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} + C, \quad C = \text{const.}$$

Now we factor out the integrating factor back out from both sides and note explicitly the time dependence of  $N_{\text{Pa}}$ ,

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}} t} + C e^{-\lambda_{\text{Pa}} t}$$

At  $t = 0$ ,

$$N_{\text{Pa}}(0) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} + C = 0,$$

since no  $^{233}\text{Pa}$  has been formed. Solving for  $C$ , we find  $C = \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}}$ . We plug this back into our equation above, and have the solution for  $N_{\text{Pa}}(t)$ :

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}} t} + \left( \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}} \right) e^{-\lambda_{\text{Pa}} t}$$

and simplifying

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} (1 - e^{\lambda_{\text{Pa}} t}) + \left( \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) (e^{-\lambda_{\text{Pa}} t} - e^{-\lambda_{\text{Th}} t}).$$

With this function of  $N_{\text{Pa}}$ , we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Pa}}(t) = \lambda_{\text{Pa}} N_{\text{Pa}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Pa}}(t) = R(1 - e^{-\lambda_{\text{Pa}} t}) + \left( \frac{R \lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) (e^{-\lambda_{\text{Pa}} t} - e^{-\lambda_{\text{Th}} t}).$$

Using the numerical values for  $R$ ,  $\lambda_{\text{Th}}$ ,  $\lambda_{\text{Pa}}$ , and  $t$ ,

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) &= (2.0 \times 10^{11} \text{ s}^{-1}) \left( 1 - e^{(-2.971 \times 10^{-7} \text{ s}^{-1})(5400 \text{ s})} \right) \\ &\quad + \left( \frac{2.0 \times 10^{11} \text{ s}^{-1} (2.791 \times 10^{-7} \text{ s}^{-1})}{2.971 \times 10^{-7} \text{ s}^{-1} - 5.18 \times 10^{-4} \text{ s}^{-1}} \right) \left( e^{(-2.971 \times 10^{-7} \text{ s}^{-1})(5400 \text{ s})} - e^{(-5.18 \times 10^{-4} \text{ s}^{-1})(5400 \text{ s})} \right) \end{aligned}$$

$$\mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) = 2.195 \times 10^8 \text{ Bq}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) = 0.006 \text{ Ci}.}$$

b) Let's say that the 1.5 hour mark is now given by  $t = t_0 = 1.5 \text{ hr}$ . We also note that 48 hours = 172,800 seconds.

### Thorium-233

Without irradiation, the rate of change of the quantity of  $^{233}\text{Th}$  is now just the decay rate.

$$\frac{dN_{\text{Th}}}{dt} = -\lambda_{\text{Th}} N_{\text{Th}}$$

We separate the equation and integrate, arriving at the standard exponential decay formula, now including the explicit time dependence.

$$\begin{aligned} \int_{N_{\text{Th}}(t_0)}^{N_{\text{Th}}(t)} \frac{-dN'_{\text{Th}}}{\lambda_{\text{Th}} N'_{\text{Th}}} &= \int_{t_0}^t dt' \\ \frac{-1}{\lambda_{\text{Th}}} [\ln N_{\text{Th}}]_{N_{\text{Th}}(t_0)}^{N_{\text{Th}}(t)} &= [t']_{t_0}^t \end{aligned}$$

$$\ln \frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = -\lambda_{\text{Th}}(t - t_0)$$

$$\frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = e^{-\lambda_{\text{Th}}(t - t_0)}$$

and we have

$$N_{\text{Th}}(t) = N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}$$

Given the definition of activity as  $\mathcal{A} = \lambda N$  and noting  $\lambda_{\text{Th}}N_{\text{Th}}(t_0) = \mathcal{A}_{\text{Th}}(t_0)$ , we can write the activity of  $^{233}\text{Th}$  as

$$\begin{aligned}\mathcal{A}_{\text{Th}}(t) &= \lambda_{\text{Th}}N_{\text{Th}}(t) \\ &= \lambda_{\text{Th}}N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)} \\ &= \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}\end{aligned}$$

Using the numerical values for  $\lambda_{\text{Th}}$ ,  $t$ , and our answer from part (a) for the activity at  $t_0 = 1.5$  hr, we find

$$\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = (5.076 \text{ Ci})e^{-5.18 \times 10^{-4} \text{ s}^{-1}(172800\text{s})}$$

$$\boxed{\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = 6.787 \times 10^{-39} \text{ Ci}}$$

Note: we could also have assumed that since 48 hours is many (more than 100) times longer than the half-life of  $^{233}\text{Th}$ , that the activity would be approximately zero.

### Protactinium-233

We follow the example in part (a) for  $^{233}\text{Pa}$ , again using the activity of  $^{233}\text{Th}$  as the production rate of  $^{233}\text{Pa}$ .

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}}N_{\text{Pa}}.$$

We collect terms dependent on  $N_{\text{Pa}}$  on one side,

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}N_{\text{Pa}} = \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

multiply both sides by arbitrary exponential  $e^{\lambda_{\text{Pa}}t}$ ,

$$e^{\lambda_{\text{Pa}}t} \frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}e^{\lambda_{\text{Pa}}t}N_{\text{Pa}} = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

note that the left side is the result of the product rule when  $\frac{d}{dt}$  is taken on  $e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}$

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

multiply by the differential,  $dt$ ,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}dt,$$

and rearrange,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t + \lambda_{\text{Th}}t_0}dt.$$

Then we integrate,

$$\int_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} d(e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0} \int_{t_0}^t e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t'} dt',$$

and find,

$$[e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}]_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} [e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t'}]_{t_0}^t$$

$$e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}(t) - e^{\lambda_{\text{Pa}} t_0} N_{\text{Pa}}(t_0) = \frac{\mathcal{A}_{\text{Th}}(t_0) e^{\lambda_{\text{Th}} t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[ e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} - e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right]$$

Factoring out the integrating factor,

$$N_{\text{Pa}}(t) - e^{-\lambda_{\text{Pa}}(t-t_0)} N_{\text{Pa}}(t_0) = \frac{\mathcal{A}_{\text{Th}}(t_0) e^{\lambda_{\text{Th}} t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[ e^{-\lambda_{\text{Th}} t} - e^{-\lambda_{\text{Pa}} t} e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right]$$

and solve for  $N_{\text{Pa}}(t)$ ,

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0) e^{\lambda_{\text{Th}} t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[ e^{-\lambda_{\text{Th}} t} - e^{-\lambda_{\text{Pa}} t} e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right] + e^{-\lambda_{\text{Pa}}(t-t_0)} N_{\text{Pa}}(t_0).$$

Rearranging,

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[ e^{-\lambda_{\text{Th}}(t-t_0)} - e^{-\lambda_{\text{Pa}}(t-t_0)} \right] + e^{-\lambda_{\text{Pa}}(t-t_0)} N_{\text{Pa}}(t_0).$$

and so

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left( N_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)}$$

Given the definition of activity as  $\mathcal{A} = \lambda N$  and noting  $\lambda_{\text{Pa}} N_{\text{Pa}}(t_0) = \mathcal{A}_{\text{Pa}}(t_0)$ , we can write the activity of  $^{233}\text{Pa}$  as

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(t) &= \lambda_{\text{Pa}} N_{\text{Pa}}(t) \\ &= \lambda_{\text{Pa}} \left( \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left( N_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)} \right) \\ &= \frac{\mathcal{A}_{\text{Th}}(t_0) \lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left( \mathcal{A}_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0) \lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)} \\ &= \mathcal{A}_{\text{Pa}}(t_0) e^{-\lambda_{\text{Pa}}(t-t_0)} + \frac{\mathcal{A}_{\text{Th}}(t_0) \lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left( e^{-\lambda_{\text{Th}}(t-t_0)} - e^{-\lambda_{\text{Pa}}(t-t_0)} \right) \end{aligned}$$

Using the numerical values for  $\lambda_{\text{Th}}$ ,  $\lambda_{\text{Pa}}$ ,  $t$ , and our answer from part (a) for the activities at  $t = t_0 = 1.5$  hr, we find

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(49.5 \text{ hr}) &= (0.006 \text{ Ci}) e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(172800 \text{ s})} \\ &\quad + \frac{(5.076 \text{ Ci})(2.971 \times 10^{-7} \text{ s}^{-1})}{2.971 \times 10^{-7} \text{ s}^{-1} - 5.18 \times 10^{-4} \text{ s}^{-1}} \left( e^{-(5.18 \times 10^{-4} \text{ s}^{-1})(172800 \text{ s})} - e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(172800 \text{ s})} \right) \end{aligned}$$

$$\boxed{\mathcal{A}_{\text{Pa}}(49.5 \text{ hr}) = 0.008 \text{ Ci}}$$

c) Note that  $\lambda_{\text{U}} = \frac{\ln 2}{5.024 \text{ days}} = 1.380 \times 10^{-13} \text{ s}^{-1}$ .

In just one day, the probability that any given  $^{233}\text{Th}$  nucleus survives is

$$\begin{aligned} P &= e^{-\lambda_{\text{Th}} t_d} \\ P &= e^{-(5.18 \times 10^{-4} \text{ s}^{-1})(86,400 \text{ s})} \approx 10^{-20} \end{aligned}$$

We can therefore assume that each thorium nucleus produced in the reactor has decayed into  $^{233}\text{Pa}$  by the end of the first day of storage. (There were only about  $10^{15}$  thorium nuclei produced in total). The probability that any one of these  $^{233}\text{Pa}$  atoms remains after one year—assumed to be 364 more days—is

$$\begin{aligned} P &= e^{-\lambda_{\text{Pa}}(t_y - t_d)} \\ P &= e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(31449600 \text{ s})} = 8.752 \times 10^{-5} \end{aligned}$$

While this probability indicates that some  $^{233}\text{Pa}$  nuclei will remain after the year of storage, they hardly make up a substantial fraction of the originally produced set of nuclei. We can assume that in one year, virtually every nucleus of  $^{233}\text{Th}$  has decayed at least into  $^{233}\text{U}$ .

For uranium-233 on the other hand, the probability of survival for a nucleus over a year is

$$P = e^{-\lambda_U t_y}$$

$$P = e^{-(1.380 \times 10^{-13})(31536000 \text{ s})}$$

$$P = 0.999996$$

We see that while nearly every nucleus decays from  $^{233}\text{Th}$  into  $^{233}\text{Pa}$  and then into  $^{233}\text{U}$ , almost no nuclei seem to decay from  $^{233}\text{U}$  in the single year. We can then use our simple formula for the activity of the uranium sample

$$\mathcal{A}_U = \lambda_U N_U$$

Using our rate of production, there are  $2.0 \times 10^{11} \text{ s}^{-1} * (3600 \text{ s/hr} \times 1.5\text{hr}) \approx 1.08 \times 10^{15}$  produced in the irradiation period. We will assume this is also equal to the number of nuclei of  $^{233}\text{U}$  present at the end of the 1 year storage period:  $N_U \approx 10^{15}$ . We can finally calculate the activity, as

$$\mathcal{A}_U = (1.380 \times 10^{-13} \text{ s}^{-1})(1.08 \times 10^{15}).$$

This is 149.04 Bq or

$$4.028 \times 10^{-9} \text{ Ci}$$

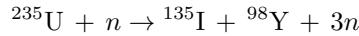
## Problem 2

Use the following masses for parts (a) and (b):

Nucleus	Atomic Mass
n	1.008665 u
<sup>1</sup> H	1.007825 u
<sup>2</sup> H	2.014102 u
<sup>56</sup> Fe	55.934939 u
<sup>98</sup> Y	97.922203 u
<sup>135</sup> I	134.910048 u
<sup>235</sup> U	235.043924 u

$$1\text{u} \cdot c^2 = 931.502 \text{ MeV}$$

a) Calculate the  $Q$ -value of the reaction:

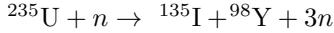


b) Calculate the average binding energy per nucleon (in MeV) of <sup>2</sup>H, <sup>56</sup>Fe, and <sup>235</sup>U.

## Problem 2 Solution

a) The  $Q$ -value of a reaction is given by:

$$Q = [m(x) + m(X) - m(y) - m(Y)]c^2.$$



$$Q = [m(^{235}\text{U}) + m(n) - m(^{135}\text{I}) - m(^{98}\text{Y}) - 3m(n)]c^2$$

$$Q = [235.043924\text{u} + 1.008665\text{u} - 134.910048\text{u} - 97.922203\text{u} - 3(1.008665\text{u})]c^2$$

$$Q = 0.194343\text{u} \cdot c^2$$

$$Q = 181.031 \text{ MeV}$$

b) The binding energy  $B(Z, A)$  for any nuclei can be found approximately from the equation:

$$M(Z, A) = Zm(^1\text{H}) + (A - Z)m_n - B(Z, A)/c^2,$$

where  $M(Z, A)$ ,  $Zm(^1\text{H})$  and  $m_n$  are experimentally calculated values. Per nucleon, the binding energy can be expressed as:

$$B_A(Z, A) = [Zm(^1\text{H}) + (A - Z)m_n - M(Z, A)]c^2/A.$$

### <sup>2</sup>H

$$B_A(1, 2) = [m(^1\text{H}) + (2 - 1)m_n - M(1, 2)]c^2/2$$

$$B_A(1, 2) = [(1.007825\text{u}) + (1.008665\text{u}) - (2.014102\text{u})]c^2/2$$

$$B_A(1, 2) = 0.001194\text{u} \cdot c^2$$

$$B_A(1, 2) = 1.112 \text{ MeV}$$

$^{56}\text{Fe}$

$$B_A(26, 56) = [26m(^1\text{H}) + (56 - 26)m_n - M(26, 56)]c^2/56$$

$$B_A(26, 56) = [26(1.007825\text{u}) + 30(1.008665\text{u}) - (55.934939\text{u})]c^2/56$$

$$B_A(26, 56) = 0.009437\text{u} \cdot c^2$$

$$B_Z(26, 56) = 8.791 \text{ MeV}$$

$^{235}\text{U}$

$$B_A(92, 235) = [92m(^1\text{H}) + (235 - 92)m_n - M(92, 235)]c^2/235$$

$$B_A(92, 235) = [92(1.007825\text{u}) + 143(1.008665\text{u}) - (235.043924\text{u})]c^2/235$$

$$B_A(92, 235) = 0.00814924\text{u} \cdot c^2$$

$$B_Z(92, 235) = 7.591 \text{ MeV}$$

## Problem 3

a) Solve the first order differential equation

$$\frac{dy}{dx} + 3y = 0$$

b) Solve the second order differential equation ( $A$  and  $B$  are constants)

$$\frac{d^2y}{dx^2} - A^2y = B$$

The boundary condition is  $y(\pm\frac{1}{A}) = 0$ .

## Problem 3 Solution

a)

$$\begin{aligned}\frac{dy}{dx} &= -3y \\ \frac{dy}{y} &= -3dx \\ \int \frac{dy}{y} &= -3 \int dx \\ \ln y &= -3x + C \\ y &= e^{-3x+C}\end{aligned}$$

$$y = Ce^{-3x}$$

b) Solve the second order differential equation

$$\frac{d^2y}{dx^2} - A^2y = B$$

For the homogeneous equation,  $\frac{d^2y}{dx^2} - A^2y = 0$ , try  $y_c = X_1e^{Ax} + X_2e^{-Ax}$  as the complementary solution ( $X_1$  and  $X_2$  are constants). Then

$$\frac{d^2y}{dx^2} = X_1A^2e^{Ax} + X_2A^2e^{-Ax}$$

and

$$\begin{aligned}(X_1A^2e^{Ax} + X_2A^2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) &= 0 \\ A^2(X_1e^{Ax} + X_2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) &= 0\end{aligned}$$

This is true.

For the inhomogeneous equation,  $\frac{d^2y}{dx^2} - A^2y = B$ ,  $y_p = -B/A^2$  is the only particular solution satisfying the equation. The general solution is the sum of the complementary and particular solutions,  $y = y_c + y_p$ .

$$y = X_1e^{Ax} + X_2e^{-Ax} - \frac{B}{A^2}$$

Now we can solve for  $X_1$  and  $X_2$ .

$$y(\frac{1}{A}) = X_1e^{A(\frac{1}{A})} + X_2e^{-A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

and

$$y\left(-\frac{1}{A}\right) = X_1 e^{A\left(-\frac{1}{A}\right)} + X_2 e^{A\left(\frac{1}{A}\right)} - \frac{B}{A^2} = 0$$

We can note that in these two equations,  $X_1$  and  $X_2$  can be interchanged freely, and so must be equal. We will say,  $X_1 = X_2 = X$ . Then, we have

$$\begin{aligned} 0 &= X e^{A\left(-\frac{1}{A}\right)} + X e^{A\left(\frac{1}{A}\right)} - \frac{B}{A^2} \\ 0 &= X(e^{-1} + e^1) - \frac{B}{A^2} \\ X &= \frac{B}{A^2\left(\frac{1}{e} + e\right)} \end{aligned}$$

Then we plug this  $X$  into the final solution, which gives

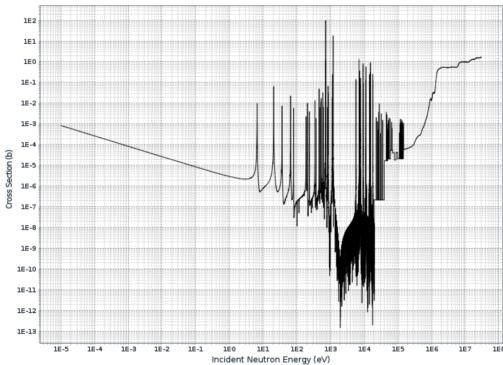
$$y = \frac{B}{A^2\left(\frac{1}{e} + e\right)}(e^{Ax} + e^{-Ax}) - \frac{B}{A^2}$$

Note that  $e^{Ax} + e^{-Ax}$  is similar in form to  $\cosh(Ax)$  which we could have also used to solve this problem.

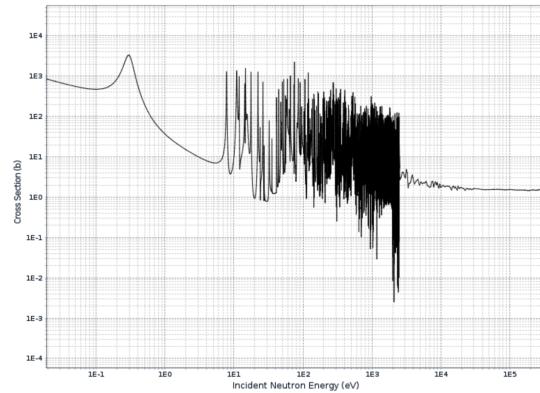
## Problem 4

Classify the following cross section plots. They are, in no particular order:

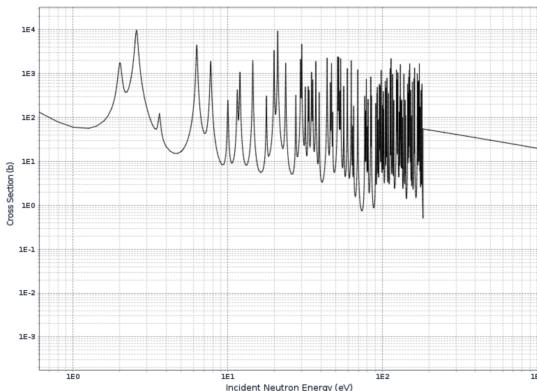
- (1)  $^{155}\text{Gd}$  absorption
- (2)  $^{235}\text{U}$  fission
- (3)  $^{238}\text{U}$  absorption
- (4)  $^{238}\text{U}$  fission
- (5)  $^{239}\text{Pu}$  fission



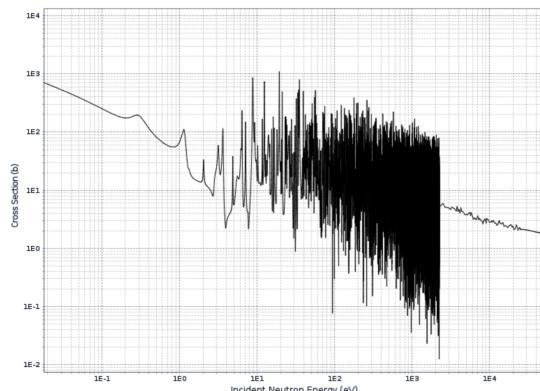
(a)



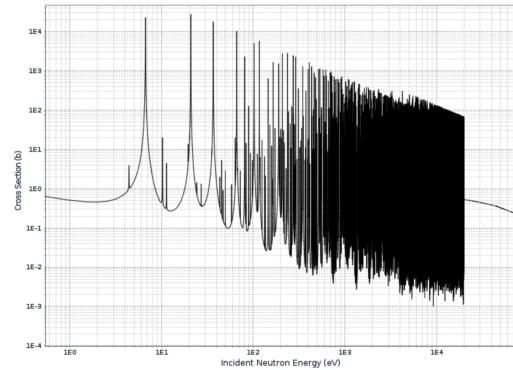
(b)



(c)



(d)



(e)

## Problem 4 Solution

- (a) 4  $^{238}\text{U}$  fission (high fast region)
- (b) 5  $^{239}\text{Pu}$  fission (resonance peak)
- (c) 1  $^{155}\text{Gd}$  absorption (consistently high absorption)
- (d) 2  $^{235}\text{U}$  fission (high thermal fission cross section)
- (e) 3  $^{238}\text{U}$  absorption (large absorptive resonances)