

Team Exercise Solutions #1

*Problems 1 & 2 borrowed from Nuclear Engineering 101 homework problem sets, Fall 2016

Problem 1

The radioactive isotope ^{233}Pa can be produced following neutron capture by ^{232}Th when the resulting ^{233}Th decays to ^{233}Pa . In the neutron flux of a typical reactor, neutron capture in 1 g of ^{232}Th produces ^{233}Th at a rate of $2.0 \times 10^{11} \text{ s}^{-1}$.

- What are the activities (in Ci) of ^{233}Th and ^{233}Pa after this sample is irradiated for 1.5 hours?
- The sample is then placed in storage with no further irradiation so that the ^{233}Th can decay away. What are the activities (in Ci) of ^{233}Th and ^{233}Pa after 48 hours of storage?
- The decay of ^{233}Pa results in ^{233}U , which is also radioactive. After the above sample has been stored for 1 year what is the ^{233}U activity in Ci? (Hint: it should not be necessary to set up an additional differential equation to find the ^{233}U activity.)

Nucleus	Half-life
^{233}Th	22.3 min
^{233}Pa	27.0 days
^{233}U	$1.592 \times 10^5 \text{ yr}$

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ s}^{-1}$$

Problem 1 Solution

For all parts of this problem, let R be the rate of neutron capture by 1 g of ^{232}Th , $2.0 \times 10^{11} \text{ s}^{-1}$.

- First, convert all half-lives and irradiation times to seconds for consistency.

$$\lambda_{\text{Th}} = \frac{\ln 2}{13388} = 5.18 \times 10^{-4} \text{ s}^{-1}; \quad \lambda_{\text{Pa}} = \frac{\ln 2}{2332800} = 2.971 \times 10^{-7} \text{ s}^{-1}; \quad 1.5 \text{ hr} = 5400 \text{ s}$$

Thorium-233

The rate of change of the quantity of ^{233}Th , $\frac{dN_{\text{Th}}}{dt}$ is given by the production rate of ^{233}Th , R , minus the decay rate (activity) of ^{233}Th , $\lambda_{\text{Th}}N_{\text{Th}}$.

$$\frac{dN_{\text{Th}}}{dt} = R - \lambda_{\text{Th}}N_{\text{Th}}$$

We solve the differential equation, manipulating the equation so that the left side is only dependent on N_{Th} and the right side is only dependent on dt . Then we integrate:

$$\begin{aligned} \int \frac{dN_{\text{Th}}}{R - \lambda_{\text{Th}}N_{\text{Th}}} &= \int dt \\ \frac{-1}{\lambda_{\text{Th}}} [\ln(R - \lambda_{\text{Th}}N_{\text{Th}})] &= t + C, \quad C = \text{const.} \\ R - \lambda_{\text{Th}}N_{\text{Th}} &= e^{-\lambda_{\text{Th}}t - \lambda_{\text{Th}}C} = e^{-\lambda_{\text{Th}}t}e^{-\lambda_{\text{Th}}C} \end{aligned}$$

We note that since C is an arbitrary constant and λ_{Th} is fixed, we could also say $e^{-\lambda_{\text{Th}}C}$ is an arbitrary constant, and just call it C instead.

$$R - \lambda_{\text{Th}}N_{\text{Th}} = Ce^{-\lambda_{\text{Th}}t}$$

We solve for N_{Th} (explicitly including N_{Th} 's dependence on t), and get

$$N_{\text{Th}}(t) = \frac{R - Ce^{-\lambda_{\text{Th}}t}}{\lambda_{\text{Th}}}.$$

At $t = 0$, $N_{\text{Th}}(0) = \frac{R-C}{\lambda_{\text{Th}}} = 0$, since no ^{233}Th has been formed. We find $C = R$, and use this in the general equation:

$$N_{\text{Th}}(t) = R \frac{1 - e^{-\lambda_{\text{Th}}t}}{\lambda_{\text{Th}}}.$$

With this function of N_{Th} , we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Th}}(t) = \lambda_{\text{Th}} N_{\text{Th}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Th}}(t) = R(1 - e^{-\lambda_{\text{Th}}t}).$$

Using the numerical values for R , λ_{Th} , and t ,

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = (2.0 \times 10^{11} \text{ s}^{-1})(1 - e^{(-5.18 \times 10^{-4} \text{ s}^{-1})(5400 \text{ s})})$$

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 1.878 \times 10^{11} \text{ Bq.}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 5.076 \text{ Ci}.}$$

Protactinium-233

We follow a similar procedure for ^{233}Pa , noting that the production rate of ^{233}Pa is just the activity of ^{233}Th as it decays into ^{233}Pa , \mathcal{A}_{Th} .

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}} N_{\text{Pa}}$$

From above, we can substitute our function for $\mathcal{A}_{\text{Th}}(t)$,

$$\frac{dN_{\text{Pa}}}{dt} = R(1 - e^{-\lambda_{\text{Th}}t}) - \lambda_{\text{Pa}} N_{\text{Pa}}$$

Since we cannot separate both sides to be dependent only on a single differential, we must try a different method of integration. We will use integrating factors. Still, we start in a similar fashion: collecting the terms dependent on N_{Pa} on the same side.

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} N_{\text{Pa}} = R(1 - e^{-\lambda_{\text{Th}}t})$$

The method of integrating factors suggests that we multiply both sides by an arbitrary exponential. We will use $e^{\lambda_{\text{Pa}}t}$.

$$e^{\lambda_{\text{Pa}}t} \frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} e^{\lambda_{\text{Pa}}t} N_{\text{Pa}} = e^{\lambda_{\text{Pa}}t} R(1 - e^{-\lambda_{\text{Th}}t})$$

We can now observe that the left side of the equation appears to be the result of the product rule when the time derivative of $e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}$ is found. We can then write the equation as

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t} R(1 - e^{-\lambda_{\text{Th}}t})$$

Moving the dt term to the right side of the equation and using the distributive property, we have

$$d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t} e^{-\lambda_{\text{Th}}t}) dt$$

or more simply (by exploiting properties of exponents)

$$d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t - \lambda_{\text{Th}}t}) dt.$$

We integrate both sides,

$$\int d(e^{\lambda_{\text{Pa}}t} N_{\text{Pa}}) = \int R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t - \lambda_{\text{Th}}t}) dt,$$

separate the integral on the right side,

$$\int d(e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}) = R \int e^{\lambda_{\text{Pa}} t} dt - R \int e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} dt$$

and find

$$e^{\lambda_{\text{Pa}} t} N_{\text{Pa}} = \frac{R}{\lambda_{\text{Pa}}} e^{\lambda_{\text{Pa}} t} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} + C, \quad C = \text{const.}$$

Now we factor out the integrating factor back out from both sides and note explicitly the time dependence of N_{Pa} ,

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}} t} + C e^{-\lambda_{\text{Pa}} t}$$

At $t = 0$,

$$N_{\text{Pa}}(0) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} + C = 0,$$

since no ^{233}Pa has been formed. Solving for C , we find $C = \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}}$. We plug this back into our equation above, and have the solution for $N_{\text{Pa}}(t)$:

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}} t} + \left(\frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}} \right) e^{-\lambda_{\text{Pa}} t}$$

and simplifying

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} (1 - e^{\lambda_{\text{Pa}} t}) + \left(\frac{R}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) (e^{-\lambda_{\text{Pa}} t} - e^{-\lambda_{\text{Th}} t}).$$

With this function of N_{Pa} , we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Pa}}(t) = \lambda_{\text{Pa}} N_{\text{Pa}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Pa}}(t) = R(1 - e^{-\lambda_{\text{Pa}} t}) + \left(\frac{R \lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) (e^{-\lambda_{\text{Pa}} t} - e^{-\lambda_{\text{Th}} t}).$$

Using the numerical values for R , λ_{Th} , λ_{Pa} , and t ,

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) &= (2.0 \times 10^{11} \text{ s}^{-1}) \left(1 - e^{(-2.971 \times 10^{-7} \text{ s}^{-1})(5400 \text{ s})} \right) \\ &\quad + \left(\frac{2.0 \times 10^{11} \text{ s}^{-1} (2.791 \times 10^{-7} \text{ s}^{-1})}{2.971 \times 10^{-7} \text{ s}^{-1} - 5.18 \times 10^{-4} \text{ s}^{-1}} \right) \left(e^{(-2.971 \times 10^{-7} \text{ s}^{-1})(5400 \text{ s})} - e^{(-5.18 \times 10^{-4} \text{ s}^{-1})(5400 \text{ s})} \right) \end{aligned}$$

$$\mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) = 2.195 \times 10^8 \text{ Bq}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Pa}}(1.5 \text{ hr}) = 0.006 \text{ Ci}.}$$

b) Let's say that the 1.5 hour mark is now given by $t = t_0 = 1.5 \text{ hr}$. We also note that 48 hours = 172,800 seconds.

Thorium-233

Without irradiation, the rate of change of the quantity of ^{233}Th is now just the decay rate.

$$\frac{dN_{\text{Th}}}{dt} = -\lambda_{\text{Th}} N_{\text{Th}}$$

We separate the equation and integrate, arriving at the standard exponential decay formula, now including the explicit time dependence.

$$\begin{aligned} \int_{N_{\text{Th}}(t_0)}^{N_{\text{Th}}(t)} \frac{-dN'_{\text{Th}}}{\lambda_{\text{Th}} N'_{\text{Th}}} &= \int_{t_0}^t dt' \\ \frac{-1}{\lambda_{\text{Th}}} [\ln N_{\text{Th}}]_{N_{\text{Th}}(t_0)}^{N_{\text{Th}}(t)} &= [t']_{t_0}^t \end{aligned}$$

$$\ln \frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = -\lambda_{\text{Th}}(t - t_0)$$

$$\frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = e^{-\lambda_{\text{Th}}(t - t_0)}$$

and we have

$$N_{\text{Th}}(t) = N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\text{Th}}N_{\text{Th}}(t_0) = \mathcal{A}_{\text{Th}}(t_0)$, we can write the activity of ^{233}Th as

$$\begin{aligned}\mathcal{A}_{\text{Th}}(t) &= \lambda_{\text{Th}}N_{\text{Th}}(t) \\ &= \lambda_{\text{Th}}N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)} \\ &= \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}\end{aligned}$$

Using the numerical values for λ_{Th} , t , and our answer from part (a) for the activity at $t_0 = 1.5$ hr, we find

$$\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = (5.076 \text{ Ci})e^{-5.18 \times 10^{-4} \text{ s}^{-1}(172800\text{s})}$$

$$\boxed{\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = 6.787 \times 10^{-39} \text{ Ci}}$$

Note: we could also have assumed that since 48 hours is many (more than 100) times longer than the half-life of ^{233}Th , that the activity would be approximately zero.

Protactinium-233

We follow the example in part (a) for ^{233}Pa , again using the activity of ^{233}Th as the production rate of ^{233}Pa .

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}}N_{\text{Pa}}.$$

We collect terms dependent on N_{Pa} on one side,

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}N_{\text{Pa}} = \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

multiply both sides by arbitrary exponential $e^{\lambda_{\text{Pa}}t}$,

$$e^{\lambda_{\text{Pa}}t} \frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}e^{\lambda_{\text{Pa}}t}N_{\text{Pa}} = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

note that the left side is the result of the product rule when $\frac{d}{dt}$ is taken on $e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}$

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)},$$

multiply by the differential, dt ,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t - t_0)}dt,$$

and rearrange,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t + \lambda_{\text{Th}}t_0}dt.$$

Then we integrate,

$$\int_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} d(e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0} \int_{t_0}^t e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t'} dt',$$

and find,

$$[e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}]_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} [e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t'}]_{t_0}^t$$

$$e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t) - e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0) = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t} - e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right]$$

Factoring out the integrating factor,

$$N_{\text{Pa}}(t) - e^{-\lambda_{\text{Pa}}(t-t_0)}N_{\text{Pa}}(t_0) = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[e^{-\lambda_{\text{Th}}t} - e^{-\lambda_{\text{Pa}}t}e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right]$$

and solve for $N_{\text{Pa}}(t)$,

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[e^{-\lambda_{\text{Th}}t} - e^{-\lambda_{\text{Pa}}t}e^{(\lambda_{\text{Pa}} - \lambda_{\text{Th}})t_0} \right] + e^{-\lambda_{\text{Pa}}(t-t_0)}N_{\text{Pa}}(t_0).$$

Rearranging,

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left[e^{-\lambda_{\text{Th}}(t-t_0)} - e^{-\lambda_{\text{Pa}}(t-t_0)} \right] + e^{-\lambda_{\text{Pa}}(t-t_0)}N_{\text{Pa}}(t_0).$$

and so

$$N_{\text{Pa}}(t) = \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left(N_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\text{Pa}}N_{\text{Pa}}(t_0) = \mathcal{A}_{\text{Pa}}(t_0)$, we can write the activity of ^{233}Pa as

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(t) &= \lambda_{\text{Pa}}N_{\text{Pa}}(t) \\ &= \lambda_{\text{Pa}} \left(\frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left(N_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0)}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)} \right) \\ &= \frac{\mathcal{A}_{\text{Th}}(t_0)\lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} e^{-\lambda_{\text{Th}}(t-t_0)} + \left(\mathcal{A}_{\text{Pa}}(t_0) - \frac{\mathcal{A}_{\text{Th}}(t_0)\lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \right) e^{-\lambda_{\text{Pa}}(t-t_0)} \\ &= \mathcal{A}_{\text{Pa}}(t_0)e^{-\lambda_{\text{Pa}}(t-t_0)} + \frac{\mathcal{A}_{\text{Th}}(t_0)\lambda_{\text{Pa}}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}} \left(e^{-\lambda_{\text{Th}}(t-t_0)} - e^{-\lambda_{\text{Pa}}(t-t_0)} \right) \end{aligned}$$

Using the numerical values for λ_{Th} , λ_{Pa} , t , and our answer from part (a) for the activities at $t = t_0 = 1.5$ hr, we find

$$\begin{aligned} \mathcal{A}_{\text{Pa}}(49.5 \text{ hr}) &= (0.006 \text{ Ci})e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(172800 \text{ s})} \\ &\quad + \frac{(5.076 \text{ Ci})(2.971 \times 10^{-7} \text{ s}^{-1})}{2.971 \times 10^{-7} \text{ s}^{-1} - 5.18 \times 10^{-4} \text{ s}^{-1}} \left(e^{-(5.18 \times 10^{-4} \text{ s}^{-1})(172800 \text{ s})} - e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(172800 \text{ s})} \right) \end{aligned}$$

$$\boxed{\mathcal{A}_{\text{Pa}}(49.5 \text{ hr}) = 0.008 \text{ Ci}}$$

c) Note that $\lambda_{\text{U}} = \frac{\ln 2}{5.024 \text{ days}} = 1.380 \times 10^{-13} \text{ s}^{-1}$.

In just one day, the probability that any given ^{233}Th nucleus survives is

$$\begin{aligned} P &= e^{-\lambda_{\text{Th}}t_d} \\ P &= e^{-(5.18 \times 10^{-4} \text{ s}^{-1})(86,400 \text{ s})} \approx 10^{-20} \end{aligned}$$

We can therefore assume that each thorium nucleus produced in the reactor has decayed into ^{233}Pa by the end of the first day of storage. (There were only about 10^{15} thorium nuclei produced in total). The probability that any one of these ^{233}Pa atoms remains after one year—assumed to be 364 more days—is

$$\begin{aligned} P &= e^{-\lambda_{\text{Pa}}(t_y - t_d)} \\ P &= e^{-(2.971 \times 10^{-7} \text{ s}^{-1})(31449600 \text{ s})} = 8.752 \times 10^{-5} \end{aligned}$$

While this probability indicates that some ^{233}Pa nuclei will remain after the year of storage, they hardly make up a substantial fraction of the originally produced set of nuclei. We can assume that in one year, virtually every nucleus of ^{233}Th has decayed at least into ^{233}U .

For uranium-233 on the other hand, the probability of survival for a nucleus over a year is

$$P = e^{-\lambda_{\text{U}} t_y}$$

$$P = e^{-(1.380 \times 10^{-13})(31536000 \text{ s})}$$

$$P = 0.999996$$

We see that while nearly every nucleus decays from ^{233}Th into ^{233}Pa and then into ^{233}U , almost no nuclei seem to decay from ^{233}U in the single year. We can then use our simple formula for the activity of the uranium sample

$$\mathcal{A}_{\text{U}} = \lambda_{\text{U}} N_{\text{U}}$$

Using our rate of production, there are $2.0 \times 10^{11} \text{ s}^{-1} * (3600 \text{ s/hr} \times 1.5\text{hr}) \approx 1.08 \times 10^{15}$ produced in the irradiation period. We will assume this is also equal to the number of nuclei of ^{233}U present at the end of the 1 year storage period: $N_{\text{U}} \approx 10^{15}$. We can finally calculate the activity, as

$$\mathcal{A}_{\text{U}} = (1.380 \times 10^{-13} \text{ s}^{-1})(1.08 \times 10^{15}).$$

This is 149.04 Bq or

$$4.028 \times 10^{-9} \text{ Ci}$$

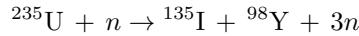
Problem 2

Use the following masses for parts (a) and (b):

Nucleus	Atomic Mass
n	1.008665 u
¹ H	1.007825 u
² H	2.014102 u
⁵⁶ Fe	55.934939 u
⁹⁸ Y	97.922203 u
¹³⁵ I	134.910048 u
²³⁵ U	235.043924 u

$$1\text{u} \cdot c^2 = 931.502 \text{ MeV}$$

a) Calculate the Q -value of the reaction:

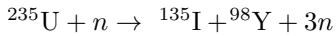


b) Calculate the average binding energy per nucleon (in MeV) of ²H, ⁵⁶Fe, and ²³⁵U.

Problem 2 Solution

a) The Q -value of a reaction is given by:

$$Q = [m(x) + m(X) - m(y) - m(Y)]c^2.$$



$$Q = [m(\text{²³⁵U}) + m(n) - m(\text{¹³⁵I}) - m(\text{⁹⁸Y}) - 3m(n)]c^2$$

$$Q = [235.043924\text{u} + 1.008665\text{u} - 134.910048\text{u} - 97.922203\text{u} - 3(1.008665\text{u})]c^2$$

$$Q = 0.194343\text{u} \cdot c^2$$

$$Q = 181.031 \text{ MeV}$$

b) The binding energy $B(Z, A)$ for any nuclei can be found approximately from the equation:

$$M(Z, A) = Zm(\text{H}) + (A - Z)m_n - B(Z, A)/c^2,$$

where $M(Z, A)$, $Zm(\text{H})$ and m_n are experimentally calculated values. Per nucleon, the binding energy can be expressed as:

$$B_A(Z, A) = [Zm(\text{H}) + (A - Z)m_n - M(Z, A)]c^2/A.$$

²H

$$B_A(1, 2) = [m(\text{H}) + (2 - 1)m_n - M(1, 2)]c^2/2$$

$$B_A(1, 2) = [(1.007825\text{u}) + (1.008665\text{u}) - (2.014102\text{u})]c^2/2$$

$$B_A(1, 2) = 0.001194 \text{u} \cdot c^2$$

$$B_A(1, 2) = 1.112 \text{ MeV}$$

^{56}Fe

$$B_A(26, 56) = [26m(^1\text{H}) + (56 - 26)m_n - M(26, 56)]c^2/56$$

$$B_A(26, 56) = [26(1.007825\text{u}) + 30(1.008665\text{u}) - (55.934939\text{u})]c^2/56$$

$$B_A(26, 56) = 0.009437 \text{u} \cdot c^2$$

$$B_Z(26, 56) = 8.791 \text{ MeV}$$

^{235}U

$$B_A(92, 235) = [92m(^1\text{H}) + (235 - 92)m_n - M(92, 235)]c^2/235$$

$$B_A(92, 235) = [92(1.007825\text{u}) + 143(1.008665\text{u}) - (235.043924\text{u})]c^2/235$$

$$B_A(92, 235) = 0.00814924 \text{u} \cdot c^2$$

$$B_Z(92, 235) = 7.591 \text{ MeV}$$

Problem 3

a) Solve the first order differential equation

$$\frac{dy}{dx} + 3y = 0$$

b) Solve the second order differential equation (A and B are constants)

$$\frac{d^2y}{dx^2} - A^2y = B$$

The boundary condition is $y(\pm\frac{1}{A}) = 0$.

Problem 3 Solution

a)

$$\begin{aligned}\frac{dy}{dx} &= -3y \\ \frac{dy}{y} &= -3 dx \\ \int \frac{dy}{y} &= -3 \int dx \\ \ln y &= -3x + C \\ y &= e^{-3x+C}\end{aligned}$$

$$y = Ce^{-3x}$$

b) Solve the second order differential equation

$$\frac{d^2y}{dx^2} - A^2y = B$$

For the homogeneous equation, $\frac{d^2y}{dx^2} - A^2y = 0$, try $y_c = X_1e^{Ax} + X_2e^{-Ax}$ as the complementary solution (X_1 and X_2 are constants). Then

$$\frac{d^2y}{dx^2} = X_1A^2e^{Ax} + X_2A^2e^{-Ax}$$

and

$$\begin{aligned}(X_1A^2e^{Ax} + X_2A^2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) &= 0 \\ A^2(X_1e^{Ax} + X_2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) &= 0\end{aligned}$$

This is true.

For the inhomogeneous equation, $\frac{d^2y}{dx^2} - A^2y = B$, $y_p = -B/A^2$ is the only particular solution satisfying the equation. The general solution is the sum of the complementary and particular solutions, $y = y_c + y_p$.

$$y = X_1e^{Ax} + X_2e^{-Ax} - \frac{B}{A^2}$$

Now we can solve for X_1 and X_2 .

$$y\left(\frac{1}{A}\right) = X_1e^{A(\frac{1}{A})} + X_2e^{-A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

and

$$y(-\frac{1}{A}) = X_1 e^{A(-\frac{1}{A})} + X_2 e^{A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

We can note that in these two equations, X_1 and X_2 can be interchanged freely, and so must be equal. We will say, $X_1 = X_2 = X$. Then, we have

$$0 = X e^{A(-\frac{1}{A})} + X e^{A(\frac{1}{A})} - \frac{B}{A^2}$$

$$0 = X(e^{-1} + e^1) - \frac{B}{A^2}$$

$$X = \frac{B}{A^2(\frac{1}{e} + e)}$$

The we plug this X into the final solution, which gives

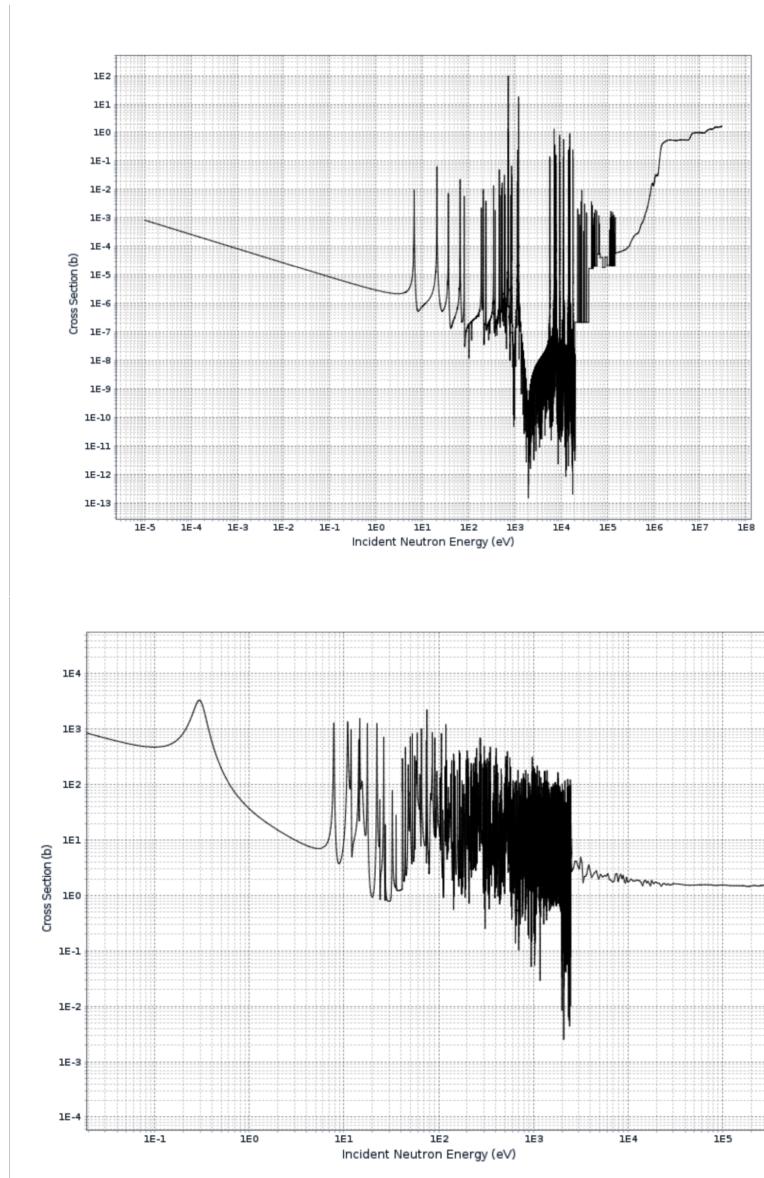
$$y = \frac{B}{A^2(\frac{1}{e} + e)}(e^{Ax} + e^{-Ax}) - \frac{B}{A^2}$$

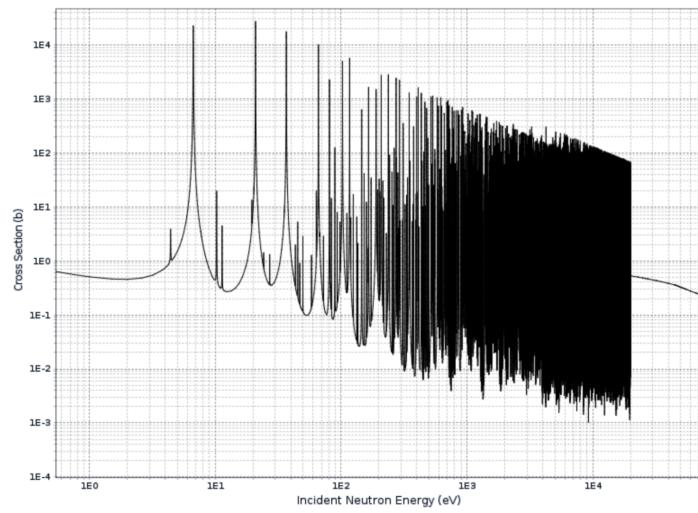
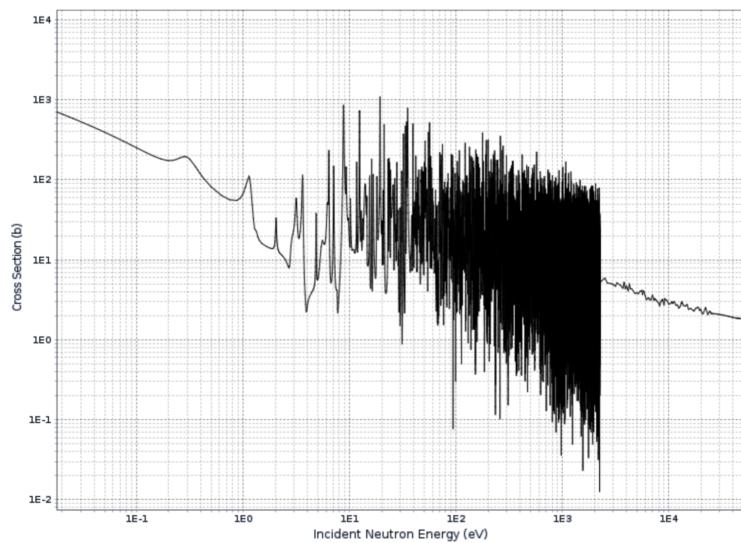
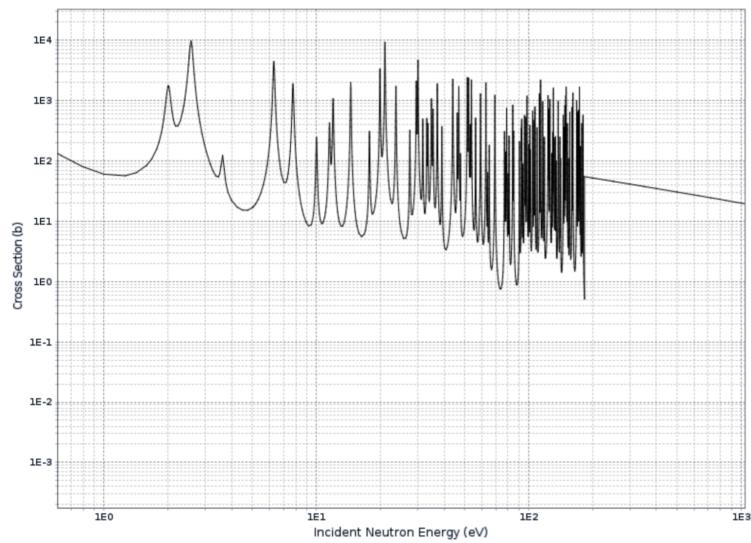
Note that $e^{Ax} + e^{-Ax}$ is similar in form to $\cosh(Ax)$ which we could have also used to solve this problem.

Problem 4

Classify the following cross section plots. They are, in no particular order:

- (1) ^{155}Gd absorption
- (2) ^{235}U fission
- (3) ^{238}U absorption
- (4) ^{238}U fission
- (5) ^{239}Pu fission





Problem 4 Solution

- (a) 4 ^{238}U fission (high fast region)
- (b) 5 ^{239}Pu fission (resonance peak)
- (c) 1 ^{155}Gd absorption (consistently high absorption)
- (d) 2 ^{235}U fission (high thermal fission cross section)
- (e) 3 ^{238}U absorption (large absorptive resonances)