Nuclear Engineering 150 – Discussion Section
Team Exercise Solutions #1

*Problems 1 & 2 borrowed from Nuclear Engineering 101 homework problem sets, Fall 2016

Problem 1

The radioactive isotope ^{233}Pa can be produced following neutron capture by ^{232}Th when the resulting ^{233}Th decays to ^{233}Pa. In the neutron flux of a typical reactor, neutron capture in 1 g of ^{232}Th produces ^{233}Th at of a rate of 2.0×10^{11} s^{-1}.

a) What are the activities (in Ci) of ^{233}Th and ^{233}Pa after this sample is irradiated for 1.5 hours?

b) The sample is then placed in storage with no further irradiation so that the ^{233}Th can decay away. What are the activities (in Ci) of ^{233}Th and ^{233}Pa after 48 hours of storage?

c) The decay of ^{233}Pa results in ^{233}U, which is also radioactive. After the above sample has been stored for 1 year what is the ^{233}U activity in Ci? (Hint: it should not be necessary to set up an additional differential equation to find the ^{233}U activity.)

Nucleus	Half-life
^{233}Th	22.3 min
^{233}Pa	27.0 days
^{233}U	1.592×10^5 yr

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ s}^{-1}$$

Problem 1 Solution

For all parts of this problem, let R be the rate of neutron capture by 1 g of ^{232}Th, 2.0×10^{11} s^{-1}.

a.)

First, convert all half-lives and irradiation times to seconds for consistency.

$$\lambda_{\text{Th}} = \tfrac{\ln 2}{1338S} = 5.18 \times 10^{-4}\text{s}^{-1}; \qquad \lambda_{\text{Pa}} = \tfrac{\ln 2}{2332800s} = 2.971 \times 10^{-7}\text{s}^{-1}; \qquad 1.5 \text{ hr} = 5400 \text{ s}$$

Thorium-233

The rate of change of the quantity of ^{233}Th, $\frac{dN_{\text{th}}}{dt}$ is given by the production rate of ^{233}Th, R, minus the decay rate (activity) of ^{233}Th, $\lambda_{\text{Th}} N_{\text{Th}}$.

$$\frac{dN_{\text{Th}}}{dt} = R - \lambda_{\text{Th}} N_{\text{Th}}$$

We manipulate the equation so that the left side is only dependent on N_{Th} and the right side is only dependent on dt. Then integrate:

$$\int \frac{dN_{\text{Th}}}{R - \lambda_{\text{Th}} N_{\text{Th}}} = \int dt$$

$$\frac{-1}{\lambda_{\text{Th}}}[\ln(R - \lambda_{\text{Th}} N_{\text{Th}})] = t + C, \quad C = \text{const.}$$

$$R - \lambda_{\text{Th}} N_{\text{Th}} = e^{-\lambda_{\text{Th}} t - \lambda_{\text{Th}} C} = e^{-\lambda_{\text{Th}} t} e^{-\lambda_{\text{Th}} C}$$

We note that since C is an arbitrary constant, we could also say $e^{-\lambda_{\text{Th}}C}$ is an arbitrary constant, and just call it C instead.

$$R - \lambda_{\text{Th}} N_{\text{Th}} = Ce^{-\lambda_{\text{Th}} t}$$

We solve for N_{Th} (explicitly including N_{Th}'s dependence on t), and get

$$N_{\text{Th}}(t) = \frac{R - Ce^{-\lambda_{\text{Th}} t}}{\lambda_{\text{Th}}}.$$

At $t = 0$, $N_{\text{Th}}(0) = \frac{R-C}{\lambda_{\text{Th}}} = 0$, since no ^{233}Th has been formed. We find $C = R$, and use this in the general equation:

$$N_{\text{Th}}(t) = R\frac{1 - e^{-\lambda_{\text{Th}} t}}{\lambda_{\text{Th}}}.$$

With this function of N_{Th}, we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Th}}(t) = \lambda_{\text{Th}} N_{\text{Th}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Th}}(t) = R(1 - e^{-\lambda_{\text{Th}} t}).$$

Using the numerical values for R, λ_{Th}, and t,

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = (2.0 \times 10^{11} s^{-1})(1 - e^{(-5.18 \times 10^{-4} \text{s}^{-1})(5400 s)})$$

$$\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 1.878 \times 10^{11} \text{ Bq}.$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Th}}(1.5 \text{ hr}) = 5.076 \text{ Ci}}.$$

Protactinium-233

We follow a similar procedure for ^{233}Pa, noting that the production rate of ^{233}Pa is just the activity of ^{233}Th as it decays into ^{233}Pa, $\mathcal{A}_{\text{Th}}$.

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}} N_{\text{Pa}}$$

From above, we can substitue our function for $\mathcal{A}_{\text{Th}}(t)$,

$$\frac{dN_{\text{Pa}}}{dt} = R(1 - e^{-\lambda_{\text{Th}} t}) - \lambda_{\text{Pa}} N_{\text{Pa}}$$

We note here that we cannot separate both sides to be dependent only on a single differential, so we must try a different method of integration. We will use integrating factors. We do, however start in a similar fashion: collecting the terms dependent on N_{Pa} on the same side.

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} N_{\text{Pa}} = R(1 - e^{-\lambda_{\text{Th}} t})$$

The method of integrating factors suggests that we multiply both sides by an arbitrary exponential. We will use $e^{\lambda_{\text{Pa}} t}$.

$$e^{\lambda_{\text{Pa}} t}\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}} e^{\lambda_{\text{Pa}} t} N_{\text{Pa}} = e^{\lambda_{\text{Pa}} t} R(1 - e^{-\lambda_{\text{Th}} t})$$

We can now observe that the left side of the equation appears to be the result of the product rule when the derivative with respect to time is taken of the expression $e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}$. We can then write the equation as

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}) = e^{\lambda_{\text{Pa}} t} R(1 - e^{-\lambda_{\text{Th}} t})$$

Moving the dt term to the right side of the equation and using the distributive property, we have

$$d(e^{\lambda_{\text{Pa}} t} N_{\text{Pa}}) = R(e^{\lambda_{\text{Pa}} t} - e^{\lambda_{\text{Pa}} t} e^{-\lambda_{\text{Th}} t})dt.$$

We integrate both sides,

$$\int d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = \int R(e^{\lambda_{\text{Pa}}t} - e^{\lambda_{\text{Pa}}t-\lambda_{\text{Th}}t})dt,$$

separate the integral on the right side,

$$e^{\lambda_{\text{Pa}}t}N_{\text{Pa}} = R\int e^{\lambda_{\text{Pa}}t}dt - R\int e^{(\lambda_{\text{Pa}}-\lambda_{\text{Th}})t}dt$$

and find

$$e^{\lambda_{\text{Pa}}t}N_{\text{Pa}} = \frac{R}{\lambda_{\text{Pa}}}e^{\lambda_{\text{Pa}}t} - \frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}}e^{(\lambda_{\text{Pa}}-\lambda_{\text{Th}})t} + C,\ C = \text{const.}$$

Now we factor out the integrating factor back out from both sides and note the explicit time dependence of N_{Pa},

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}}e^{-\lambda_{\text{Th}}t} + Ce^{-\lambda_{\text{Pa}}t}$$

At $t = 0$,

$$N_{\text{Pa}}(0) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}} + C = 0,$$

since no ^{233}Pa has been formed. Solving for C, we find $C = \frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}}$. We plug this back into our equation above, and have the solution for $N_{\text{Pa}}(t)$:

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}} - \frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}}e^{-\lambda_{\text{Th}}t} + \left(\frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}} - \frac{R}{\lambda_{\text{Pa}}}\right)e^{-\lambda_{\text{Pa}}t}$$

and simplifying

$$N_{\text{Pa}}(t) = \frac{R}{\lambda_{\text{Pa}}}(1 - e^{\lambda_{\text{Pa}}t}) + \left(\frac{R}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}}\right)(e^{-\lambda_{\text{Pa}}t} - e^{-\lambda_{\text{Th}}t}).$$

With this function of N_{Pa}, we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\text{Pa}}(t) = \lambda_{\text{Pa}}N_{\text{Pa}}(t).$$

Substituting, we find

$$\mathcal{A}_{\text{Pa}}(t) = R(1 - e^{-\lambda_{\text{Pa}}t}) + \left(\frac{R\,\lambda_{\text{Pa}}}{\lambda_{\text{Pa}}-\lambda_{\text{Th}}}\right)(e^{-\lambda_{\text{Pa}}t} - e^{-\lambda_{\text{Th}}t}).$$

Using the numerical values for R, λ_{Th}, λ_{Pa}, and t,

$$\begin{aligned}\mathcal{A}_{\text{Pa}}(1.5\text{ hr}) = {} & (2.0\times10^{11}\text{ s}^{-1})\left(1 - e^{(-2.971\times10^{-7}\text{s}^{-1})(5400\text{s})}\right) \\ & + \left(\frac{2.0\times10^{11}\text{ s}^{-1}(2.791\times10^{-7}\text{s}^{-1})}{2.971\times10^{-7}\text{s}^{-1} - 5.18\times10^{-4}\text{s}^{-1}}\right)\left(e^{(-2.971\times10^{-7}\text{s}^{-1})(5400\text{s})} - e^{(-5.18\times10^{-4}\text{s}^{-1})(5400\text{s})}\right)\end{aligned}$$

$$\mathcal{A}_{\text{Pa}}(1.5\text{ hr}) = 2.195\times10^{8}\text{ Bq}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\text{Pa}}(1.5\text{ hr}) = 0.006\text{ Ci}}.$$

b.)

Let's say that the 1.5 hour mark is now given by $t = t_0 = 1.5$ hr. We also note that 48 hours = 172,800 seconds.

Thorium-233

Without irradiation, the rate of change of the quantity of ^{233}Th is now just the decay rate.

$$\frac{dN_{\text{Th}}}{dt} = -\lambda_{\text{Th}}N_{\text{Th}}$$

We separate the equation and integrate, arriving at the standard exponential decay formula, now including the explicit time dependence.

$$\int_{N_{\mathrm{Th}}(t_0)}^{N_{\mathrm{Th}}(t)} \frac{-dN'_{\mathrm{Th}}}{\lambda_{\mathrm{Th}} N'_{\mathrm{Th}}} = \int_{t_0}^{t} dt'$$
$$\frac{-1}{\lambda_{\mathrm{Th}}} \left[\ln N_{\mathrm{Th}}\right]_{N_{\mathrm{Th}(t_0)}}^{N_{\mathrm{Th}}(t)} = [t']_{t_0}^{t}$$
$$\ln \frac{N_{\mathrm{Th}}(t)}{N_{\mathrm{Th}}(t_0)} = -\lambda_{\mathrm{Th}}(t - t_0)$$
$$\frac{N_{\mathrm{Th}}(t)}{N_{\mathrm{Th}}(t_0)} = e^{-\lambda_{\mathrm{Th}}(t-t_0)}$$

and we have

$$N_{\mathrm{Th}}(t) = N_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\mathrm{Th}} N_{\mathrm{Th}}(t_0) = \mathcal{A}_{\mathrm{Th}}(t_0)$, we can write the activity of ^{233}Th as

$$\begin{aligned}\mathcal{A}_{\mathrm{Th}}(t) &= \lambda_{\mathrm{Th}} N_{\mathrm{Th}}(t) \\ &= \lambda_{\mathrm{Th}} N_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)} \\ &= \mathcal{A}_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)}\end{aligned}$$

Using the numerical values for λ_{Th}, t, and our answer from part (a) for the activity at $t_0 = 1.5$ hr, we find

$$\mathcal{A}_{\mathrm{Th}}(49.5 \text{ hr}) = (5.076 \text{ Ci}) e^{-5.18\times10^{-4}\mathrm{s}^{-1}(172800\mathrm{s})}$$

$$\boxed{\mathcal{A}_{\mathrm{Th}}(49.5 \text{ hr}) = 6.787 \times 10^{-39} \text{ Ci}}$$

Note: we could also have assumed that since 48 hours is many (more than 100) times longer than the half-life of ^{233}Th, that the activity would be approximately zero.

Protactinium-233

We follow the example in part (a) for ^{233}Pa, again using the activity of ^{233}Th as the production rate of ^{233}Pa.

$$\frac{dN_{\mathrm{Pa}}}{dt} = \mathcal{A}_{\mathrm{Th}} - \lambda_{\mathrm{Pa}} N_{\mathrm{Pa}}.$$

We collect terms dependent on N_{Pa} on one side,

$$\frac{dN_{\mathrm{Pa}}}{dt} + \lambda_{\mathrm{Pa}} N_{\mathrm{Pa}} = \mathcal{A}_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)},$$

multiply both sides by arbitrary exponential $e^{\lambda_{\mathrm{Pa}} t}$,

$$e^{\lambda_{\mathrm{Pa}} t} \frac{dN_{\mathrm{Pa}}}{dt} + \lambda_{\mathrm{Pa}} e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}} = e^{\lambda_{\mathrm{Pa}} t} \mathcal{A}_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)},$$

note that the left side is the result of the product rule when $\frac{d}{dt}$ is taken on $e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}$

$$\frac{d}{dt}(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = e^{\lambda_{\mathrm{Pa}} t} \mathcal{A}_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)},$$

multiply by the differential, dt,

$$d(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = e^{\lambda_{\mathrm{Pa}} t} \mathcal{A}_{\mathrm{Th}}(t_0) e^{-\lambda_{\mathrm{Th}}(t-t_0)} dt,$$

and rearrange,

$$d(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = \mathcal{A}_{\mathrm{Th}}(t_0) e^{(\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}})t + \lambda_{\mathrm{Th}} t_0} dt.$$

Then we integrate,

$$\int_{e^{\lambda_{\mathrm{Pa}}t_0}N_{\mathrm{Pa}}(t_0)}^{e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}}(t)} d(e^{\lambda_{\mathrm{Pa}}t}N'_{\mathrm{Pa}}) = \mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}\int_{t_0}^{t} e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t'}dt',$$

and find,

$$\left[e^{\lambda_{\mathrm{Pa}}t}N'_{\mathrm{Pa}}\right]_{e^{\lambda_{\mathrm{Pa}}t_0}N_{\mathrm{Pa}}(t_0)}^{e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}}(t)} = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left[e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t'}\right]_{t_0}^{t}$$

$$e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}}(t) - e^{\lambda_{\mathrm{Pa}}t_0}N_{\mathrm{Pa}}(t_0) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left[e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t} - e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right]$$

Factoring out the integrating factor,

$$N_{\mathrm{Pa}}(t) - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}t} - e^{-\lambda_{\mathrm{Pa}}t}e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right]$$

and solve for $N_{\mathrm{Pa}}(t)$,

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}t} - e^{-\lambda_{\mathrm{Pa}}t}e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right] + e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0).$$

Rearranging,

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}(t-t_0)} - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right] + e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0).$$

and so

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(N_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\mathrm{Pa}}N_{\mathrm{Pa}}(t_0) = \mathcal{A}_{\mathrm{Pa}}(t_0)$, we can write the activity of ^{233}Pa as

$$\begin{aligned}
\mathcal{A}_{\mathrm{Pa}}(t) &= \lambda_{\mathrm{Pa}}N_{\mathrm{Pa}}(t) \\
&= \lambda_{\mathrm{Pa}}\left(\frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(N_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right) \\
&= \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(\mathcal{A}_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)} \\
&= \mathcal{A}_{\mathrm{Pa}}(t_0)e^{-\lambda_{\mathrm{Pa}}(t-t_0)} + \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\left(e^{-\lambda_{\mathrm{Th}}(t-t_0)} - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right)
\end{aligned}$$

Using the numerical values for λ_{Th}, λ_{Pa}, t, and our answer from part (a) for the activities at $t = t_0 = 1.5$ hr, we find

$$\begin{aligned}
\mathcal{A}_{\mathrm{Pa}}(49.5\text{ hr}) = {}&(0.006\text{ Ci})e^{-(2.971\times10^{-7}\mathrm{s}^{-1})(172800s)} \\
&+ \frac{(5.076\text{ Ci})(2.971\times10^{-7}\mathrm{s}^{-1})}{2.971\times10^{-7}\mathrm{s}^{-1} - 5.18\times10^{-4}\mathrm{s}^{-1}}\left(e^{-(5.18\times10^{-4}\mathrm{s}^{-1})(172800s)} - e^{-(2.971\times10^{-7}\mathrm{s}^{-1})(172800s)}\right)
\end{aligned}$$

$$\boxed{\mathcal{A}_{\mathrm{Pa}}(49.5\text{ hr}) = 0.008\text{ Ci}}$$

c.)

Note that $\lambda_{\mathrm{U}} = \frac{\ln 2}{5.024^{12}\mathrm{S}} = 1.380\times10^{-13}\mathrm{s}^{-1}$.

In just one day, the probability that any given ^{233}Th nucleus survives is

$$P = e^{-\lambda_{\mathrm{Th}}t_d}$$

$$P = e^{-(5.18\times10^{-4}\ \mathrm{s}^{-1})(86{,}400\ \mathrm{s})} \approx 10^{-20}$$

We can therefore assume that each thorium nucleus produced in the reactor has decayed into ^{233}Pa by the end of the first day of storage. (There were only about 10^{15} thorium nuclei produced in total). The probability that any one of these ^{233}Pa atoms remains after one year—assumed to be 364 more days—is

$$P = e^{-\lambda_{\mathrm{Pa}}(t_y - t_d)}$$

$$P = e^{-(2.971\times10^{-7}\ \mathrm{s}^{-1})(31449600\ \mathrm{s})} = 8.752 \times 10^{-5}$$

While this probability indicates that some ^{233}Pa nuclei will remain after the year of storage, they hardly make up a substantial fraction of the originally produced set of nuclei. We can assume that in one year, virtually every nucleus of ^{233}Th has decayed at least into ^{233}U.

For uranium-233 on the other hand, the probability of survival for a nucleus over a year is

$$P = e^{-\lambda_{\mathrm{U}} t_y}$$

$$P = e^{-(1.380\times10^{-13})(31536000\ \mathrm{s})}$$

$$P = 0.999996$$

We see that while nearly every nucleus decays from ^{233}Th into ^{233}Pa and then into ^{233}U, almost no nuclei seem to decay from ^{233}U in the single year. We can then use our simple formula for the activity of the uranium sample

$$\mathcal{A}_{\mathrm{U}} = \lambda_{\mathrm{U}} N_{\mathrm{U}}$$

Using our rate of production, there are $2.0 \times 10^{11}\ \mathrm{s}^{-1} * (3600\ \mathrm{s/hr} \times 1.5\mathrm{hr}) \approx 1.08 \times 10^{15}$ produced in the irradiation period. We will assume this is also equal to the number of nuclei of ^{233}U present at the end of the 1 year storage period: $N_U \approx 10^{15}$. We can finally calculate the activity, as

$$\mathcal{A}_{\mathrm{U}} = (1.380 \times 10^{-13}\mathrm{s}^{-1})(1.08 \times 10^{15}).$$

This is 149.04 Bq or

$$\boxed{4.028 \times 10^{-9}\ \mathrm{Ci}}$$

Problem 2

Use the following masses for parts (a) and (b):

Nucleus	Atomic Mass
n	1.008665 u
^{1}H	1.007825 u
^{2}H	2.014102 u
^{56}Fe	55.934939 u
^{98}Y	97.922203 u
^{135}I	134.910048 u
^{235}U	235.043924 u

$$1\text{u} \cdot c^2 = 931.502 \text{ MeV}$$

a) Calculate the Q-value of the reaction:

$$^{235}\text{U} + n \rightarrow {}^{135}\text{I} + {}^{98}\text{Y} + 3n$$

b) Calculate the average binding energy per nucleon (in MeV) of ^{2}H, ^{56}Fe, and ^{235}U.

Problem 2 Solution

a) The Q-value of a reaction is given by:

$$Q = [m(\text{x}) + m(\text{X}) - m(\text{y}) - m(\text{Y})]c^2.$$

$^{235}\text{U} + n \rightarrow \ {}^{135}\text{I} + {}^{98}\text{Y} + 3n$

$$Q = [m(^{235}\text{U}) + m(n) - m(^{135}\text{I}) - m(^{98}\text{Y}) - 3m(n)]c^2$$

$$Q = [235.043924\text{u} + 1.008665\text{u} - 134.910048\text{u} - 97.922203\text{u} - 3(1.008665\text{u})]c^2$$

$$Q = 0.194343\text{u} \cdot c^2$$

$$\boxed{Q = 181.031 \text{ MeV}}$$

b) The binding energy $B(Z, A)$ for any nuclei can be found approximately from the equation:

$$M(Z, A) = Zm(^{1}\text{H}) + (A - Z)m_n - B(Z, A)/c^2,$$

where $M(Z, A)$, $Zm(^{1}\text{H})$ and m_n are experimentally calculated values. Per nucleon, the binding energy can be expressed as:

$$B_A(Z, A) = [Zm(^{1}\text{H}) + (A - Z)m_n - M(Z, A)]c^2/A.$$

^{2}H

$B_A(1, 2) = [m(^{1}\text{H}) + (2 - 1)m_n - M(1, 2)]c^2/2$

$B_A(1, 2) = [(1.007825\text{u}) + (1.008665\text{u}) - (2.014102\text{u})]c^2/2$

$B_A(1, 2) = 0.001194\text{u} \cdot c^2$

$\boxed{B_A(1,2) = 1.112 \text{ MeV}}$

^{56}**Fe**

$B_A(26,56) = [26m(^1\text{H}) + (56-26)m_n - M(26,56)]c^2/56$

$B_A(26,56) = [26(1.007825\text{u}) + 30(1.008665\text{u}) - (55.934939\text{u})]c^2/56$

$B_A(26,56) = 0.009437\text{u} \cdot c^2$

$\boxed{B_Z(26,56) = 8.791 \text{ MeV}}$

^{235}**U**

$B_A(92,235) = [92m(^1\text{H}) + (235-92)m_n - M(92,235)]c^2/235$

$B_A(92,235) = [92(1.007825\text{u}) + 143(1.008665\text{u}) - (235.043924\text{u})]c^2/235$

$B_A(92,235) = 0.00814924u\text{u} \cdot c^2$

$\boxed{B_Z(92,235) = 7.591 \text{ MeV}}$

Problem 3

a) Solve the first order differential equation

$$\frac{dy}{dx} + 3y = 0$$

b) Solve the second order differential equation (A and B are constants)

$$\frac{d^2y}{dx^2} - A^2y = B$$

The boundary condition is $y(\pm\frac{1}{A}) = 0$.

Problem 3 Solution

a)

$$\frac{dy}{dx} = -3y \tag{1}$$
$$\frac{dy}{y} = -3\,dx \tag{2}$$
$$\int \frac{dy}{y} = -3\int dx \tag{3}$$
$$\ln y = -3x + C \tag{4}$$
$$y = e^{-3x+C} \tag{5}$$
$$\tag{6}$$

$$\boxed{y = Ce^{-3x}}$$

Solve the second order differential equation

$$\frac{d^2y}{dx^2} - A^2y = B$$

For the homogeneous equation, $\frac{d^2y}{dx^2} - A^2y = 0$, try $y_c = X_1e^{Ax} + X_2e^{-Ax}$ as the complementary solution (X_1 and X_2 are constants). Then

$$\frac{d^2y}{dx^2} = X_1A^2e^{Ax} + X_2A^2e^{-Ax}$$

and

$$(X_1A^2e^{Ax} + X_2A^2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) = 0$$
$$A^2(X_1e^{Ax} + X_2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) = 0$$

This is true.

For the inhomogeneous equation, $\frac{d^2y}{dx^2} - A^2y = B$, $y_P = -B/A^2$ is the only particular solution satisfying the equation. The general solution is the sum of the complementary and particular solutions, $y = y_c + y_p$.

$$y = X_1e^{Ax} + X_2e^{-Ax} - \frac{B}{A^2}$$

Now we can solve for X_1 and X_2.

$$y(\frac{1}{A}) = X_1e^{A(\frac{1}{A})} + X_2e^{-A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

and

$$y(-\frac{1}{A}) = X_1e^{A(-\frac{1}{A})} + X_2e^{A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

We can note that in these two equations, X_1 and X_2 can be interchanged freely, and so must be equal. We will say, $X_1 = X_2 = X$. Then, we have

$$0 = Xe^{A(-\frac{1}{A})} + Xe^{A(\frac{1}{A})} - \frac{B}{A^2}$$

$$0 = X(e^{-1} + e^{1}) - \frac{B}{A^2}$$

$$X = \frac{B}{A^2(\frac{1}{e} + e)}$$

The we plug this X into the final solution, which gives

$$y = \frac{B}{A^2(\frac{1}{e} + e)}(e^{Ax} + e^{-Ax}) - \frac{B}{A^2}$$

Note that $e^{Ax} + e^{-Ax}$ is similar in form to $\cosh(Ax)$ which we could have also used to solve this problem.

Problem 4

Classify the following cross section plots. They are, in no particular order:

(1) ^{155}Gd absorption

(2) ^{235}U fission

(3) ^{238}U absorption

(4) ^{238}U fission

(5) ^{239}Pu fission

Problem 4 Solution

(4) ^{238}U fission (high fast region)
(5) ^{239}Pu fission (resonance peak)
(1) ^{155}Gd absorption (consistently high absorption)
(2) ^{235}U fission (high thermal fission cross section)
(3) ^{238}U absorption (large absorptive resonances)