

Team Exercise Solutions #2

Problem 1

Compute the atomic densities of ^{235}U , ^{238}U , and O in UO_2 when its density is 10.41 g/cm^3 and the uranium is enriched to 5 wt% in ^{235}U .

Problem 1 Solution

First, let's note that we will use the shorthand U5 and U8 in subscripts to denote ^{235}U and ^{238}U respectively.

To begin we note that the densities of ^{235}U and ^{238}U can be found as fractions of the total UO_2 . Since the partial densities are dependent on the mass of the components, the fractions are equivalent to the weight % enrichment.

$$\rho_{\text{UO}_2} = \rho_{\text{U5O}_2} + \rho_{\text{U8O}_2}$$

$$\begin{aligned} \rho_{\text{U5O}_2} &= w\rho_{\text{UO}_2} \\ \rho_{\text{U8O}_2} &= (1-w)\rho_{\text{UO}_2} \end{aligned} \quad \text{where } w = 0.05$$

Now, we can use the fact that the number density of a material, n , is equal to the material's density, ρ , multiplied by Avogadro's number, N_A , and divided by the molar mass, m , of the material.

$$n = \frac{\rho N_A}{m}$$

We find

$$\begin{aligned} n_{\text{U5O}_2} &= \frac{\rho_{\text{U5O}_2} N_A}{m_{\text{U5O}_2}} \\ n_{\text{U8O}_2} &= \frac{\rho_{\text{U8O}_2} N_A}{m_{\text{U8O}_2}} \end{aligned}$$

If we make the simplification that $n_{\text{U5}} = n_{\text{U5O}_2}$ and $n_{\text{U8}} = n_{\text{U8O}_2}$, and also substitute our equations for ρ_{U5O_2} and ρ_{U8O_2} in terms of ρ_{UO_2} from above, then we find

$$\begin{aligned} n_{\text{U5}} &= \frac{w\rho_{\text{UO}_2} N_A}{m_{\text{U5O}_2}} \\ n_{\text{U8}} &= \frac{(1-w)\rho_{\text{UO}_2} N_A}{m_{\text{U8O}_2}} \end{aligned}$$

At this point, we can also separate our molar masses into molar masses of the components, for which we can find data.

$$\begin{aligned} m_{\text{U5O}_2} &= m_{\text{U5}} + 2m_{\text{O}} \\ m_{\text{U8O}_2} &= m_{\text{U8}} + 2m_{\text{O}} \end{aligned}$$

We now have enough information to solve for both of these quantities, and so we plug in values. We also note that $n_{\text{O}} = 2(n_{\text{U5}} + n_{\text{U8}})$ since each uranium atom is bonded to 2 oxygen atoms.

$$\begin{aligned} n_{\text{U5}} &= \frac{w\rho_{\text{UO}_2} N_A}{m_{\text{U5}} + 2m_{\text{O}}} & \boxed{n_{\text{U5}} = \dots} \\ n_{\text{U8}} &= \frac{(1-w)\rho_{\text{UO}_2} N_A}{m_{\text{U8}} + 2m_{\text{O}}} & \boxed{n_{\text{U8}} = \dots} \\ n_{\text{O}} &= 2(n_{\text{U5}} + n_{\text{U8}}) & \boxed{n_{\text{O}} = \dots} \end{aligned}$$

Problem 2

Free neutrons undergo β^- decay with a half-life of 10.4 minutes. Determine the probability that a neutron will decay before being absorbed in an infinite absorbing material (assume no scattering). Estimate this probability for a thermal neutron ($v = 2200$ m/s) in water.

Problem 2 Solution

Since we are assuming no scattering, at any given time t there are three possible “fates” of our neutron in the next infinitesimal timestep, dt . These are (1) the particle continues with no decay or interaction, (2) the neutron decays, or (3) the neutron is absorbed. We want to find the probability that, given a destructive event has taken place (situation 2 or 3), the event was decay, rather than absorption. Formally, we write this as

$$p(E = d | E \in \{d, a\})$$

which we read as “the instantaneous probability that an event is decay, given the event was decay or absorption.” Statistically this is equivalent to

$$\frac{p(E = d)}{p(E \in \{d, a\})} \text{ or } \frac{p(d)}{p(d) + p(a)}. \quad (1)$$

(If you’re not comfortable with the statistics here, you can reason through this intuitively. We have some event occurring, being either a decay or an absorption collision. The probability of that event depends on the individual probabilities of the independent events, decay and absorption. The probability of that event being a decay is just the fraction of the event’s total probability that is attributable to the decay.)

From here on, we will write the argument of the probability function as a subscript, which will allow us to note the probability’s dependence on either time or space. (For example, $p(d) \rightarrow p_d$.)

Now, we know that the exponential decay law is $N(t) = N_0 e^{-\lambda t}$. The probability that a neutron decays in some time t is then

$$P_d(t) = 1 - \frac{N(t)}{N_0} = 1 - e^{-\lambda t}$$

We can calculate that the instantaneous probability that a neutron decays between times t and $t + dt$ is

$$\frac{dP_d(t)}{dt} = \lambda e^{-\lambda t}$$

From this equation, we note that $dP_d(t) = p_d(t)dt$. Remember, $P_d(t)$ is the total probability of decay, and so $p_d(t)$ is the instantaneous probability of decay per time. When multiplied by dt , this is $p_d(t)$.

$$p_d(t) = p_d(t) dt = \lambda e^{-\lambda t} dt$$

We rewrite this equation in terms of $N(t)$, and

$$p_d(t) = \frac{\lambda N(t)}{N_0} dt$$

Additionally, we know that the attenuation of neutrons moving through a material also follows an exponential function. The number of neutrons remaining after traveling through a material with macroscopic absorption cross section Σ_a , is

$$N(x) = N_0 e^{-\Sigma_a x}$$

(note: $\Sigma_a = \sigma_a n$, where σ_a is the microscopic absorption cross section and n is the number density of the material in question)

We repeat the procedure from above, but now in space rather than time. The probability for a collision is

$$P_a(x) = 1 - \frac{N(x)}{N_0} = 1 - e^{-\Sigma_a x}$$

and the instantaneous probability is

$$\frac{dP_a(x)}{dt} = \Sigma_a e^{\Sigma_a x}$$

or

$$p_a(x) = p_a(x) dx = \Sigma_a e^{\Sigma_a x} dx.$$

Again, eliminating dx and writing in terms of $N(x)$, we have

$$p_a(x) = \frac{\Sigma_a N(x)}{N_0} dx$$

Now, we observe that using $p_d(t)$ and $p_a(x)$ in equation (1) causes all terms to be dependent on N , the number of neutrons. Unfortunately, we have p_d reliant on $N(t)$, the number of neutrons surviving until time t , and p_a reliant on $N(x)$, the number of neutrons reaching distance x . We are told, however, that these neutrons are thermal, and so are moving at $v = 2200$ m/s. If we say $dx = v dt$, then we can rewrite $N(x)dx$ as a function of time!

$$N(x) dx = N(t) v dt$$

and

$$p_a(t) = \frac{\Sigma_a N(t) v}{N_0} dt$$

Finally, we do substitute this into equation (1), and get

$$\begin{aligned} p(E = d | E \in \{d, a\}) &= \frac{\frac{\lambda N(t)}{N_0} dt}{\frac{\lambda N(t)}{N_0} dt + \frac{\Sigma_a N(t) v}{N_0} dt} \\ &= \frac{\lambda}{\lambda + \Sigma_a v} \\ &= \left(1 + \frac{\Sigma_a v}{\lambda}\right)^{-1} \end{aligned}$$

We can look up Σ_a to find it is approximately 0.022 cm^{-1} (or 2.2 m^{-1} , and can calculate that 10.2 minutes is 612 seconds. Using this, along with $v = 2200 \text{ m/s}$, and $\lambda = \frac{\ln 2}{612} = 1.13 \times 10^{-3} \text{ s}^{-1}$, we can find the probability.

$$p(E = d | E \in \{d, a\}) = \left(1 + \frac{(2.2 \text{ m}^{-1})(2200 \text{ m/s})}{1.13 \times 10^{-3} \text{ s}^{-1}}\right)^{-1}$$

$$\boxed{p(E = d | E \in \{d, a\}) = 2.33 \times 10^{-7}}$$

Problem 3

Consider a 1000 MWE reactor with a 33% efficiency conversion from MWT to MWE. What is the minimum volume of UO_2 , enriched to 3 (atom) % ^{235}U that could theoretically supply the yearly energy production of this reactor. Treat energy contributions as coming only from the fission of ^{235}U . These fission events release about 200 MeV with 95% of that energy staying in the reactor.

Problem 3 Solution

We need to find the minimum required volume of UO_2 , V , so let's begin by expressing that volume of UO_2 in terms of the number of UO_2 molecules it contains.

First, we use M_{UO_2} as the total mass of UO_2 and ρ_{UO_2} as the density of UO_2 (about 10.4 g/cm^3 for our purposes)

$$V = \frac{M_{\text{UO}_2}}{\rho_{\text{UO}_2}}$$

Then, we can write the mass of UO_2 as the number of UO_2 molecules, N_{UO_2} , times the mass of a UO_2 molecule, m_{UO_2} . Since we know that the UO_2 is enriched such that 3% of the molecules contain ^{235}U , we can say $m_{\text{UO}_2} = 0.03m_{\text{U}5} + 0.97m_{\text{U}8} + 2m_{\text{O}}$. Then

$$M_{\text{UO}_2} = N_{\text{UO}_2}(0.03m_{\text{U}5} + 0.97m_{\text{U}8} + 2m_{\text{O}})$$

We can use this equality in our total volume equation to find

$$V = \frac{N_{\text{UO}_2}(0.03m_{\text{U}5} + 0.97m_{\text{U}8} + 2m_{\text{O}})}{\rho_{\text{UO}_2}} \quad (2)$$

Now we need to relate this volume, in terms of number of UO_2 molecules, to the electric power produced by the reactor. Say P_E is the electric power, P_T is thermal power, and ε is the efficiency.

$$P_E = \varepsilon P_T$$

The total energy produced by the reactor in time t can be found by simply multiplying the power by the time of production.

$$E = P_T t$$

or equivalently

$$P_T = \frac{E}{t}.$$

The energy from one fission is E_f and the energy harnessed from one fission is $0.95E_f$. The total energy harnessed can then be expressed in terms of the number of fissions, N_f and the energy captured per event,

$$E = 0.95E_f N_f$$

If we consider the extreme and highly unrealistic case that *all* ^{235}U atoms fission, then the number of ^{235}U atoms required is $N_{\text{U}5} = N_f$. Since ^{235}U is 3% of the total uranium by atom, then $N_{\text{U}5} = 0.03N_{\text{U}}$. Taking this one step further, there is a one-to-one ratio of uranium atoms to UO_2 molecules, so $N_{\text{U}} = N_{\text{UO}_2}$.

We combine these facts together to relate the electric power produced to the number of uranium dioxide molecules required.

$$\begin{aligned} P_E &= \varepsilon P_T \\ &= \varepsilon \frac{E}{t} \\ &= \frac{0.95\varepsilon E_f N_f}{t} \\ &= \frac{(0.95)(0.03)\varepsilon E_f N_{\text{UO}_2}}{t} \end{aligned}$$

We solve for N_{UO_2} to get

$$N_{\text{UO}_2} = \frac{P_E t}{(0.95)(0.03)\varepsilon E_f} \quad (3)$$

Now, we use equation (3) for the number of UO_2 molecules required for the given power in equation (2) for the volume of UO_2 required. We find

$$V = \frac{P_E t (0.03 m_{\text{U5}} + 0.97 m_{\text{U8}} + 2 m_{\text{O}})}{(0.95)(0.03)\varepsilon E_f \rho_{\text{UO}_2}}$$

Finally, we can include our values. We note the following

- $P_E = 1000 \text{ MWE} = 1000 \text{ J/s}$
- $t = 1 \text{ yr} = 31,557,600 \text{ s}$
- $E_f = 200 \text{ MeV} = 3.204 \times 10^{-11} \text{ J}$
- $m_{\text{U5}} = 235.044 \text{ amu} = 3.9028 \times 10^{-22} \text{ g}$
- $m_{\text{U8}} = 238.051 \text{ amu} = 3.9528 \times 10^{-22} \text{ g}$
- $m_{\text{O}} = 15.995 \text{ amu} = 2.656 \times 10^{-23} \text{ g}$

$$V = \frac{(1000 \text{ J/s})(31557600 \text{ s})(0.03(3.9028 \times 10^{-22} \text{ g}) + 0.97(3.9528 \times 10^{-22} \text{ g}) + 2(2.656 \times 10^{-23} \text{ g}))}{(0.95)(0.03)(0.33)(3.204 \times 10^{-11} \text{ J})(10.41 \text{ g/cm}^3)}$$

$V = 4.51 \text{ cm}^3$

Problem 4

Given UO_2 with 2.5×10^{21} atoms/cm³ of ^{235}U and 2.0×10^{22} atoms/cm³ of ^{238}U , find the partial densities of ^{235}U , ^{238}U , and O, and determine the enrichment.

Problem 4 Solution

The number density of material i can be found using the formula

$$n_i = \frac{\rho_i N_A}{M_i}$$

where ρ_i is the partial density of material i , M_i is the molar mass of i , and N_A is Avogadro's number. We can reverse this formula to find the partial density of a material from the atomic density.

$$\rho_i = \frac{N_i M_i}{N_A}$$

We can use this formula for the three isotopes we are considering. We also note the following:

- $n_{\text{O}} = 2(n_{\text{U5}} + n_{\text{U8}})$
- $M_{\text{U5}} = 235.04$ g/mol
- $M_{\text{U8}} = 238.05$ g/mol
- $M_{\text{O}} = 16.00$ g/mol

$$\rho_{\text{U5}} = \frac{n_{\text{U5}} M_{\text{U5}}}{N_A}$$
$$\rho_{\text{U5}} = \frac{(2.5 \times 10^{21} \text{ atoms/cm}^3)(235.04 \text{ g/mol})}{6.022 \times 10^{23} \text{ atoms/mol}}$$

$$\boxed{\rho_{\text{U5}} = 0.976 \text{ g/cm}^3}$$

$$\rho_{\text{U8}} = \frac{n_{\text{U8}} M_{\text{U8}}}{N_A}$$
$$\rho_{\text{U8}} = \frac{(2.0 \times 10^{22} \text{ atoms/cm}^3)(238.05 \text{ g/mol})}{6.022 \times 10^{23} \text{ atoms/mol}}$$

$$\boxed{\rho_{\text{U8}} = 7.91 \text{ g/cm}^3}$$

$$\rho_{\text{O}} = \frac{n_{\text{O}} M_{\text{O}}}{N_A}$$
$$\rho_{\text{O}} = \frac{2(n_{\text{U5}} + n_{\text{U8}}) M_{\text{O}}}{N_A}$$
$$\rho_{\text{O}} = \frac{2(2.25 \times 10^{22} \text{ atoms/cm}^3)(16.00 \text{ g/mol})}{6.022 \times 10^{23} \text{ atoms/mol}}$$

$$\boxed{\rho_{\text{O}} = 1.20 \text{ g/cm}^3}$$

The enrichment of ^{235}U is therefore $\frac{\rho_{\text{U5}}}{\rho_{\text{U5}} + \rho_{\text{U8}}} = 0.1098 \Rightarrow \boxed{10.98\%}$