Nuclear Engineering 150 – Discussion Section
Team Exercise Solutions #1

*Problems 1 & 2 borrowed from Nuclear Engineering 101 homework problem sets, Fall 2016

Problem 1

The radioactive isotope ^{233}Pa can be produced following neutron capture by ^{232}Th when the resulting ^{233}Th decays to ^{233}Pa. In the neutron flux of a typical reactor, neutron capture in 1 g of ^{232}Th produces ^{233}Th at of a rate of 2.0×10^{11} s^{-1}.

a) What are the activities (in Ci) of ^{233}Th and ^{233}Pa after this sample is irradiated for 1.5 hours?

b) The sample is then placed in storage with no further irradiation so that the ^{233}Th can decay away. What are the activities (in Ci) of ^{233}Th and ^{233}Pa after 48 hours of storage?

c) The decay of ^{233}Pa results in ^{233}U, which is also radioactive. After the above sample has been stored for 1 year what is the ^{233}U activity in Ci? (Hint: it should not be necessary to set up an additional differential equation to find the ^{233}U activity.)

Nucleus	Half-life
^{233}Th	22.3 min
^{233}Pa	27.0 days
^{233}U	1.592×10^5 yr

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ s}^{-1}$$

Problem 1 Solution

For all parts of this problem, let R be the rate of neutron capture by 1 g of ^{232}Th, 2.0×10^{11} s^{-1}.

a) First, convert all half-lives and irradiation times to seconds for consistency.

$$\lambda_{\text{Th}} = \tfrac{\ln 2}{1338S} = 5.18 \times 10^{-4}\text{s}^{-1}; \qquad \lambda_{\text{Pa}} = \tfrac{\ln 2}{2332800s} = 2.971 \times 10^{-7}\text{s}^{-1}; \qquad 1.5 \text{ hr} = 5400 \text{ s}$$

Thorium-233

The rate of change of the quantity of ^{233}Th, $\frac{dN_{\text{th}}}{dt}$ is given by the production rate of ^{233}Th, R, minus the decay rate (activity) of ^{233}Th, $\lambda_{\text{Th}} N_{\text{Th}}$.

$$\frac{dN_{\text{Th}}}{dt} = R - \lambda_{\text{Th}} N_{\text{Th}}$$

We solve the differential equation, manipulating the equation so that the left side is only dependent on N_{Th} and the right side is only dependent on dt. Then we integrate:

$$\int \frac{dN_{\text{Th}}}{R - \lambda_{\text{Th}} N_{\text{Th}}} = \int dt$$

$$\frac{-1}{\lambda_{\text{Th}}}[\ln(R - \lambda_{\text{Th}} N_{\text{Th}})] = t + C, \quad C = \text{const.}$$

$$R - \lambda_{\text{Th}} N_{\text{Th}} = e^{-\lambda_{\text{Th}} t - \lambda_{\text{Th}} C} = e^{-\lambda_{\text{Th}} t} e^{-\lambda_{\text{Th}} C}$$

We note that since C is an arbitrary constant and λ_{Th} is fixed, we could also say $e^{-\lambda_{\text{Th}} C}$ is an arbitrary constant, and just call it C instead.

$$R - \lambda_{\text{Th}} N_{\text{Th}} = C e^{-\lambda_{\text{Th}} t}$$

We solve for N_{Th} (explicitly including N_{Th}'s dependence on t), and get

$$N_{\mathrm{Th}}(t) = \frac{R - Ce^{-\lambda_{\mathrm{Th}} t}}{\lambda_{\mathrm{Th}}}.$$

At $t = 0$, $N_{\mathrm{Th}}(0) = \frac{R-C}{\lambda_{\mathrm{Th}}} = 0$, since no ^{233}Th has been formed. We find $C = R$, and use this in the general equation:

$$N_{\mathrm{Th}}(t) = R\frac{1 - e^{-\lambda_{\mathrm{Th}} t}}{\lambda_{\mathrm{Th}}}.$$

With this function of N_{Th}, we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\mathrm{Th}}(t) = \lambda_{\mathrm{Th}} N_{\mathrm{Th}}(t).$$

Substituting, we find

$$\mathcal{A}_{\mathrm{Th}}(t) = R(1 - e^{-\lambda_{\mathrm{Th}} t}).$$

Using the numerical values for R, λ_{Th}, and t,

$$\mathcal{A}_{\mathrm{Th}}(1.5 \text{ hr}) = (2.0 \times 10^{11} s^{-1})(1 - e^{(-5.18\times10^{-4}\mathrm{s}^{-1})(5400s)})$$

$$\mathcal{A}_{\mathrm{Th}}(1.5 \text{ hr}) = 1.878 \times 10^{11} \text{ Bq}.$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\mathrm{Th}}(1.5 \text{ hr}) = 5.076 \text{ Ci}}.$$

Protactinium-233

We follow a similar procedure for ^{233}Pa, noting that the production rate of ^{233}Pa is just the activity of ^{233}Th as it decays into ^{233}Pa, $\mathcal{A}_{\mathrm{Th}}$.

$$\frac{dN_{\mathrm{Pa}}}{dt} = \mathcal{A}_{\mathrm{Th}} - \lambda_{\mathrm{Pa}} N_{\mathrm{Pa}}$$

From above, we can substitute our function for $\mathcal{A}_{\mathrm{Th}}(t)$,

$$\frac{dN_{\mathrm{Pa}}}{dt} = R(1 - e^{-\lambda_{\mathrm{Th}} t}) - \lambda_{\mathrm{Pa}} N_{\mathrm{Pa}}$$

Since we cannot separate both sides to be dependent only on a single differential, we must try a different method of integration. We will use integrating factors. Still, we start in a similar fashion: collecting the terms dependent on N_{Pa} on the same side.

$$\frac{dN_{\mathrm{Pa}}}{dt} + \lambda_{\mathrm{Pa}} N_{\mathrm{Pa}} = R(1 - e^{-\lambda_{\mathrm{Th}} t})$$

The method of integrating factors suggests that we multiply both sides by an arbitrary exponential. We will use $e^{\lambda_{\mathrm{Pa}} t}$.

$$e^{\lambda_{\mathrm{Pa}} t}\frac{dN_{\mathrm{Pa}}}{dt} + \lambda_{\mathrm{Pa}} e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}} = e^{\lambda_{\mathrm{Pa}} t} R(1 - e^{-\lambda_{\mathrm{Th}} t})$$

We can now observe that the left side of the equation appears to be the result of the product rule when the time derivative of $e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}$ is found. We can then write the equation as

$$\frac{d}{dt}(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = e^{\lambda_{\mathrm{Pa}} t} R(1 - e^{-\lambda_{\mathrm{Th}} t})$$

Moving the dt term to the right side of the equation and using the distributive property, we have

$$d(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = R(e^{\lambda_{\mathrm{Pa}} t} - e^{\lambda_{\mathrm{Pa}} t} e^{-\lambda_{\mathrm{Th}} t})dt$$

or more simply (by exploiting properties of exponents)

$$d(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}) = R(e^{\lambda_{\mathrm{Pa}} t} - e^{\lambda_{\mathrm{Pa}} t - \lambda_{\mathrm{Th}} t})dt.$$

We integrate both sides,

$$\int d\left(e^{\lambda_{\mathrm{Pa}} t} N_{\mathrm{Pa}}\right) = \int R(e^{\lambda_{\mathrm{Pa}} t} - e^{\lambda_{\mathrm{Pa}} t - \lambda_{\mathrm{Th}} t})dt,$$

separate the integral on the right side,

$$\int d\left(e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}}\right) = R\int e^{\lambda_{\mathrm{Pa}}t}dt - R\int e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t}dt$$

and find

$$e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}} = \frac{R}{\lambda_{\mathrm{Pa}}}e^{\lambda_{\mathrm{Pa}}t} - \frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t} + C,\ C = \text{const.}$$

Now we factor out the integrating factor back out from both sides and note explicitly the time dependence of N_{Pa},

$$N_{\mathrm{Pa}}(t) = \frac{R}{\lambda_{\mathrm{Pa}}} - \frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}t} + Ce^{-\lambda_{\mathrm{Pa}}t}$$

At $t = 0$,

$$N_{\mathrm{Pa}}(0) = \frac{R}{\lambda_{\mathrm{Pa}}} - \frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}} + C = 0,$$

since no ^{233}Pa has been formed. Solving for C, we find $C = \frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}} - \frac{R}{\lambda_{\mathrm{Pa}}}$. We plug this back into our equation above, and have the solution for $N_{\mathrm{Pa}}(t)$:

$$N_{\mathrm{Pa}}(t) = \frac{R}{\lambda_{\mathrm{Pa}}} - \frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}t} + \left(\frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}} - \frac{R}{\lambda_{\mathrm{Pa}}}\right)e^{-\lambda_{\mathrm{Pa}}t}$$

and simplifying

$$N_{\mathrm{Pa}}(t) = \frac{R}{\lambda_{\mathrm{Pa}}}(1 - e^{\lambda_{\mathrm{Pa}}t}) + \left(\frac{R}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\right)(e^{-\lambda_{\mathrm{Pa}}t} - e^{-\lambda_{\mathrm{Th}}t}).$$

With this function of N_{Pa}, we can determine the activity as a function of time, knowing that

$$\mathcal{A}_{\mathrm{Pa}}(t) = \lambda_{\mathrm{Pa}}N_{\mathrm{Pa}}(t).$$

Substituting, we find

$$\mathcal{A}_{\mathrm{Pa}}(t) = R(1 - e^{-\lambda_{\mathrm{Pa}}t}) + \left(\frac{R\,\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}}}\right)(e^{-\lambda_{\mathrm{Pa}}t} - e^{-\lambda_{\mathrm{Th}}t}).$$

Using the numerical values for R, λ_{Th}, λ_{Pa}, and t,

$$\begin{aligned}\mathcal{A}_{\mathrm{Pa}}(1.5\text{ hr}) = {} & (2.0\times10^{11}\text{ s}^{-1})\left(1 - e^{(-2.971\times10^{-7}\text{s}^{-1})(5400\text{s})}\right) \\ & + \left(\frac{2.0\times10^{11}\text{ s}^{-1}(2.791\times10^{-7}\text{s}^{-1})}{2.971\times10^{-7}\text{s}^{-1} - 5.18\times10^{-4}\text{s}^{-1}}\right)\left(e^{(-2.971\times10^{-7}\text{s}^{-1})(5400\text{s})} - e^{(-5.18\times10^{-4}\text{s}^{-1})(5400\text{s})}\right)\end{aligned}$$

$$\mathcal{A}_{\mathrm{Pa}}(1.5\text{ hr}) = 2.195\times10^{8}\text{ Bq}$$

Finally, we convert this to Curies,

$$\boxed{\mathcal{A}_{\mathrm{Pa}}(1.5\text{ hr}) = 0.006\text{ Ci}}.$$

b) Let's say that the 1.5 hour mark is now given by $t = t_0 = 1.5$ hr. We also note that 48 hours = 172,800 seconds.

Thorium-233

Without irradiation, the rate of change of the quantity of ^{233}Th is now just the decay rate.

$$\frac{dN_{\mathrm{Th}}}{dt} = -\lambda_{\mathrm{Th}}N_{\mathrm{Th}}$$

We separate the equation and integrate, arriving at the standard exponential decay formula, now including the explicit time dependence.

$$\int_{N_{\mathrm{Th}}(t_0)}^{N_{\mathrm{Th}}(t)} \frac{-dN'_{\mathrm{Th}}}{\lambda_{\mathrm{Th}}N'_{\mathrm{Th}}} = \int_{t_0}^{t} dt'$$

$$\frac{-1}{\lambda_{\mathrm{Th}}}\left[\ln N_{\mathrm{Th}}\right]_{N_{\mathrm{Th}(t_0)}}^{N_{\mathrm{Th}}(t)} = [t']_{t_0}^{t}$$

$$\ln \frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = -\lambda_{\text{Th}}(t - t_0)$$

$$\frac{N_{\text{Th}}(t)}{N_{\text{Th}}(t_0)} = e^{-\lambda_{\text{Th}}(t-t_0)}$$

and we have

$$N_{\text{Th}}(t) = N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\text{Th}}N_{\text{Th}}(t_0) = \mathcal{A}_{\text{Th}}(t_0)$, we can write the activity of ^{233}Th as

$$\begin{aligned}\mathcal{A}_{\text{Th}}(t) &= \lambda_{\text{Th}}N_{\text{Th}}(t)\\ &= \lambda_{\text{Th}}N_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)}\\ &= \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)}\end{aligned}$$

Using the numerical values for λ_{Th}, t, and our answer from part (a) for the activity at $t_0 = 1.5$ hr, we find

$$\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = (5.076 \text{ Ci})e^{-5.18\times10^{-4}\text{s}^{-1}(172800\text{s})}$$

$$\boxed{\mathcal{A}_{\text{Th}}(49.5 \text{ hr}) = 6.787 \times 10^{-39} \text{ Ci}}$$

Note: we could also have assumed that since 48 hours is many (more than 100) times longer than the half-life of ^{233}Th, that the activity would be approximately zero.

Protactinium-233

We follow the example in part (a) for ^{233}Pa, again using the activity of ^{233}Th as the production rate of ^{233}Pa.

$$\frac{dN_{\text{Pa}}}{dt} = \mathcal{A}_{\text{Th}} - \lambda_{\text{Pa}}N_{\text{Pa}}.$$

We collect terms dependent on N_{Pa} on one side,

$$\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}N_{\text{Pa}} = \mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)},$$

multiply both sides by arbitrary exponential $e^{\lambda_{\text{Pa}}t}$,

$$e^{\lambda_{\text{Pa}}t}\frac{dN_{\text{Pa}}}{dt} + \lambda_{\text{Pa}}e^{\lambda_{\text{Pa}}t}N_{\text{Pa}} = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)},$$

note that the left side is the result of the product rule when $\frac{d}{dt}$ is taken on $e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}$

$$\frac{d}{dt}(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)},$$

multiply by the differential, dt,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = e^{\lambda_{\text{Pa}}t}\mathcal{A}_{\text{Th}}(t_0)e^{-\lambda_{\text{Th}}(t-t_0)}dt,$$

and rearrange,

$$d(e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{(\lambda_{\text{Pa}}-\lambda_{\text{Th}})t+\lambda_{\text{Th}}t_0}dt.$$

Then we integrate,

$$\int_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} d(e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}) = \mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}\int_{t_0}^{t} e^{(\lambda_{\text{Pa}}-\lambda_{\text{Th}})t'}dt',$$

and find,

$$\left[e^{\lambda_{\text{Pa}}t}N'_{\text{Pa}}\right]_{e^{\lambda_{\text{Pa}}t_0}N_{\text{Pa}}(t_0)}^{e^{\lambda_{\text{Pa}}t}N_{\text{Pa}}(t)} = \frac{\mathcal{A}_{\text{Th}}(t_0)e^{\lambda_{\text{Th}}t_0}}{\lambda_{\text{Pa}} - \lambda_{\text{Th}}}\left[e^{(\lambda_{\text{Pa}}-\lambda_{\text{Th}})t'}\right]_{t_0}^{t}$$

$$e^{\lambda_{\mathrm{Pa}}t}N_{\mathrm{Pa}}(t) - e^{\lambda_{\mathrm{Pa}}t_0}N_{\mathrm{Pa}}(t_0) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\left[e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t} - e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right]$$

Factoring out the integrating factor,

$$N_{\mathrm{Pa}}(t) - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}t} - e^{-\lambda_{\mathrm{Pa}}t}e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right]$$

and solve for $N_{\mathrm{Pa}}(t)$,

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)e^{\lambda_{\mathrm{Th}}t_0}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}t} - e^{-\lambda_{\mathrm{Pa}}t}e^{(\lambda_{\mathrm{Pa}}-\lambda_{\mathrm{Th}})t_0}\right] + e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0).$$

Rearranging,

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\left[e^{-\lambda_{\mathrm{Th}}(t-t_0)} - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right] + e^{-\lambda_{\mathrm{Pa}}(t-t_0)}N_{\mathrm{Pa}}(t_0).$$

and so

$$N_{\mathrm{Pa}}(t) = \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(N_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)}$$

Given the definition of activity as $\mathcal{A} = \lambda N$ and noting $\lambda_{\mathrm{Pa}}N_{\mathrm{Pa}}(t_0) = \mathcal{A}_{\mathrm{Pa}}(t_0)$, we can write the activity of ^{233}Pa as

$$\begin{aligned}
\mathcal{A}_{\mathrm{Pa}}(t) &= \lambda_{\mathrm{Pa}}N_{\mathrm{Pa}}(t) \\
&= \lambda_{\mathrm{Pa}}\left(\frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(N_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right) \\
&= \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}e^{-\lambda_{\mathrm{Th}}(t-t_0)} + \left(\mathcal{A}_{\mathrm{Pa}}(t_0) - \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\right)e^{-\lambda_{\mathrm{Pa}}(t-t_0)} \\
&= \mathcal{A}_{\mathrm{Pa}}(t_0)e^{-\lambda_{\mathrm{Pa}}(t-t_0)} + \frac{\mathcal{A}_{\mathrm{Th}}(t_0)\lambda_{\mathrm{Pa}}}{\lambda_{\mathrm{Pa}} - \lambda_{\mathrm{Th}}}\left(e^{-\lambda_{\mathrm{Th}}(t-t_0)} - e^{-\lambda_{\mathrm{Pa}}(t-t_0)}\right)
\end{aligned}$$

Using the numerical values for λ_{Th}, λ_{Pa}, t, and our answer from part (a) for the activities at $t = t_0 = 1.5$ hr, we find

$$\begin{aligned}
\mathcal{A}_{\mathrm{Pa}}(49.5\text{ hr}) = {} & (0.006\text{ Ci})e^{-(2.971\times10^{-7}\mathrm{s}^{-1})(172800s)} \\
& + \frac{(5.076\text{ Ci})(2.971\times10^{-7}\mathrm{s}^{-1})}{2.971\times10^{-7}\mathrm{s}^{-1} - 5.18\times10^{-4}\mathrm{s}^{-1}}\left(e^{-(5.18\times10^{-4}\mathrm{s}^{-1})(172800s)} - e^{-(2.971\times10^{-7}\mathrm{s}^{-1})(172800s)}\right)
\end{aligned}$$

$$\boxed{\mathcal{A}_{\mathrm{Pa}}(49.5\text{ hr}) = 0.008\text{ Ci}}$$

c) Note that $\lambda_{\mathrm{U}} = \frac{\ln 2}{5.024^{12}\mathrm{S}} = 1.380\times10^{-13}\mathrm{s}^{-1}$.

In just one day, the probability that any given ^{233}Th nucleus survives is

$$P = e^{-\lambda_{\mathrm{Th}}t_d}$$

$$P = e^{-(5.18\times10^{-4}\ \mathrm{s}^{-1})(86{,}400\ \mathrm{s})} \approx 10^{-20}$$

We can therefore assume that each thorium nucleus produced in the reactor has decayed into ^{233}Pa by the end of the first day of storage. (There were only about 10^{15} thorium nuclei produced in total). The probability that any one of these ^{233}Pa atoms remains after one year—assumed to be 364 more days—is

$$P = e^{-\lambda_{\mathrm{Pa}}(t_y - t_d)}$$

$$P = e^{-(2.971\times10^{-7}\ \mathrm{s}^{-1})(31449600\ \mathrm{s})} = 8.752\times10^{-5}$$

While this probability indicates that some ^{233}Pa nuclei will remain after the year of storage, they hardly make up a substantial fraction of the originally produced set of nuclei. We can assume that in one year, virtually every nucleus of ^{233}Th has decayed at least into ^{233}U.

For uranium-233 on the other hand, the probability of survival for a nucleus over a year is

$$P = e^{-\lambda_{\mathrm{U}} t_y}$$

$$P = e^{-(1.380\times 10^{-13})(31536000 \text{ s})}$$

$$P = 0.999996$$

We see that while nearly every nucleus decays from ^{233}Th into ^{233}Pa and then into ^{233}U, almost no nuclei seem to decay from ^{233}U in the single year. We can then use our simple formula for the activity of the uranium sample

$$\mathcal{A}_{\mathrm{U}} = \lambda_{\mathrm{U}} N_{\mathrm{U}}$$

Using our rate of production, there are 2.0×10^{11} s^{-1} $*$ (3600 s/hr $\times$ 1.5hr) $\approx 1.08 \times 10^{15}$ produced in the irradiation period. We will assume this is also equal to the number of nuclei of ^{233}U present at the end of the 1 year storage period: $N_U \approx 10^{15}$. We can finally calculate the activity, as

$$\mathcal{A}_{\mathrm{U}} = (1.380 \times 10^{-13} \mathrm{s}^{-1})(1.08 \times 10^{15}).$$

This is 149.04 Bq or

$$\boxed{4.028 \times 10^{-9} \text{ Ci}}$$

Problem 2

Use the following masses for parts (a) and (b):

Nucleus	Atomic Mass
n	1.008665 u
^{1}H	1.007825 u
^{2}H	2.014102 u
^{56}Fe	55.934939 u
^{98}Y	97.922203 u
^{135}I	134.910048 u
^{235}U	235.043924 u

$$1\text{u} \cdot c^2 = 931.502\ \text{MeV}$$

a) Calculate the Q-value of the reaction:

$$^{235}\text{U} + n \rightarrow {}^{135}\text{I} + {}^{98}\text{Y} + 3n$$

b) Calculate the average binding energy per nucleon (in MeV) of ^{2}H, ^{56}Fe, and ^{235}U.

Problem 2 Solution

a) The Q-value of a reaction is given by:

$$Q = [m(\text{x}) + m(\text{X}) - m(\text{y}) - m(\text{Y})]c^2.$$

$^{235}\text{U} + n \rightarrow \ {}^{135}\text{I} + {}^{98}\text{Y} + 3n$

$$Q = [m(^{235}\text{U}) + m(n) - m(^{135}\text{I}) - m(^{98}\text{Y}) - 3m(n)]c^2$$

$$Q = [235.043924\text{u} + 1.008665\text{u} - 134.910048\text{u} - 97.922203\text{u} - 3(1.008665\text{u})]c^2$$

$$Q = 0.194343\text{u} \cdot c^2$$

$$\boxed{Q = 181.031\ \text{MeV}}$$

b) The binding energy $B(Z, A)$ for any nuclei can be found approximately from the equation:

$$M(Z, A) = Zm(^{1}\text{H}) + (A - Z)m_n - B(Z, A)/c^2,$$

where $M(Z, A)$, $Zm(^{1}\text{H})$ and m_n are experimentally calculated values. Per nucleon, the binding energy can be expressed as:

$$B_A(Z, A) = [Zm(^{1}\text{H}) + (A - Z)m_n - M(Z, A)]c^2/A.$$

^{2}H

$$B_A(1, 2) = [m(^{1}\text{H}) + (2 - 1)m_n - M(1, 2)]c^2/2$$

$$B_A(1, 2) = [(1.007825\text{u}) + (1.008665\text{u}) - (2.014102\text{u})]c^2/2$$

$$B_A(1, 2) = 0.001194\text{u} \cdot c^2$$

$\boxed{B_A(1,2) = 1.112 \text{ MeV}}$

^{56}**Fe**

$B_A(26,56) = [26m(^1\text{H}) + (56-26)m_n - M(26,56)]c^2/56$

$B_A(26,56) = [26(1.007825\text{u}) + 30(1.008665\text{u}) - (55.934939\text{u})]c^2/56$

$B_A(26,56) = 0.009437\text{u} \cdot c^2$

$\boxed{B_Z(26,56) = 8.791 \text{ MeV}}$

^{235}**U**

$B_A(92,235) = [92m(^1\text{H}) + (235-92)m_n - M(92,235)]c^2/235$

$B_A(92,235) = [92(1.007825\text{u}) + 143(1.008665\text{u}) - (235.043924\text{u})]c^2/235$

$B_A(92,235) = 0.00814924u\text{u} \cdot c^2$

$\boxed{B_Z(92,235) = 7.591 \text{ MeV}}$

Problem 3

a) Solve the first order differential equation

$$\frac{dy}{dx} + 3y = 0$$

b) Solve the second order differential equation (A and B are constants)

$$\frac{d^2y}{dx^2} - A^2y = B$$

The boundary condition is $y(\pm\frac{1}{A}) = 0$.

Problem 3 Solution

a)

$$\frac{dy}{dx} = -3y$$
$$\frac{dy}{y} = -3\,dx$$
$$\int \frac{dy}{y} = -3\int dx$$
$$\ln y = -3x + C$$
$$y = e^{-3x+C}$$

$$\boxed{y = Ce^{-3x}}$$

b) Solve the second order differential equation

$$\frac{d^2y}{dx^2} - A^2y = B$$

For the homogeneous equation, $\frac{d^2y}{dx^2} - A^2y = 0$, try $y_c = X_1e^{Ax} + X_2e^{-Ax}$ as the complementary solution (X_1 and X_2 are constants). Then

$$\frac{d^2y}{dx^2} = X_1A^2e^{Ax} + X_2A^2e^{-Ax}$$

and

$$(X_1A^2e^{Ax} + X_2A^2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) = 0$$
$$A^2(X_1e^{Ax} + X_2e^{-Ax}) - A^2(X_1e^{Ax} + X_2e^{-Ax}) = 0$$

This is <u>true</u>.

For the inhomogeneous equation, $\frac{d^2y}{dx^2} - A^2y = B$, $y_P = -B/A^2$ is the only particular solution satisfying the equation. The general solution is the sum of the complementary and particular solutions, $y = y_c + y_p$.

$$y = X_1e^{Ax} + X_2e^{-Ax} - \frac{B}{A^2}$$

Now we can solve for X_1 and X_2.

$$y(\frac{1}{A}) = X_1e^{A(\frac{1}{A})} + X_2e^{-A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

and

$$y(-\frac{1}{A}) = X_1 e^{A(-\frac{1}{A})} + X_2 e^{A(\frac{1}{A})} - \frac{B}{A^2} = 0$$

We can note that in these two equations, X_1 and X_2 can be interchanged freely, and so must be equal. We will say, $X_1 = X_2 = X$. Then, we have

$$0 = Xe^{A(-\frac{1}{A})} + Xe^{A(\frac{1}{A})} - \frac{B}{A^2}$$

$$0 = X(e^{-1} + e^{1}) - \frac{B}{A^2}$$

$$X = \frac{B}{A^2(\frac{1}{e} + e)}$$

The we plug this X into the final solution, which gives

$$y = \frac{B}{A^2(\frac{1}{e} + e)}(e^{Ax} + e^{-Ax}) - \frac{B}{A^2}$$

Note that $e^{Ax} + e^{-Ax}$ is similar in form to $\cosh(Ax)$ which we could have also used to solve this problem.

Problem 4

Classify the following cross section plots. They are, in no particular order:

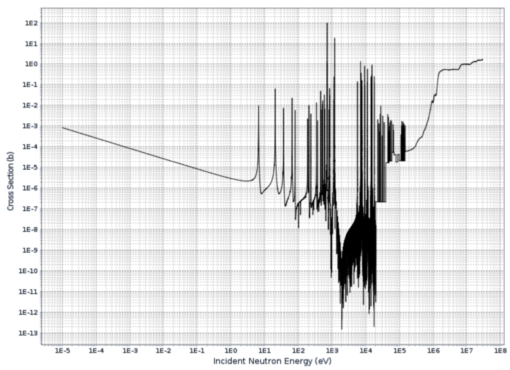
(1) ^{155}Gd absorption

(2) ^{235}U fission

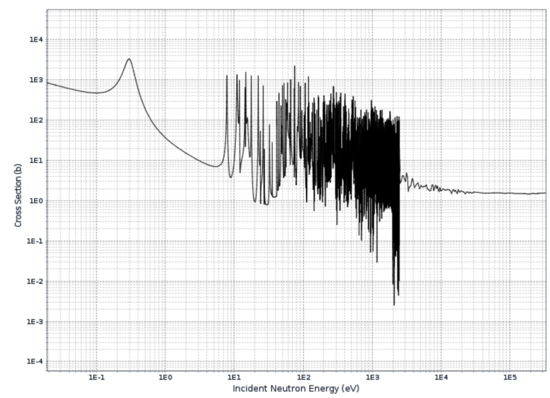
(3) ^{238}U absorption

(4) ^{238}U fission

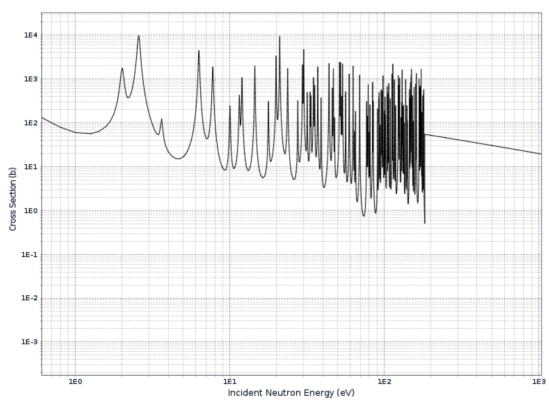
(5) ^{239}Pu fission



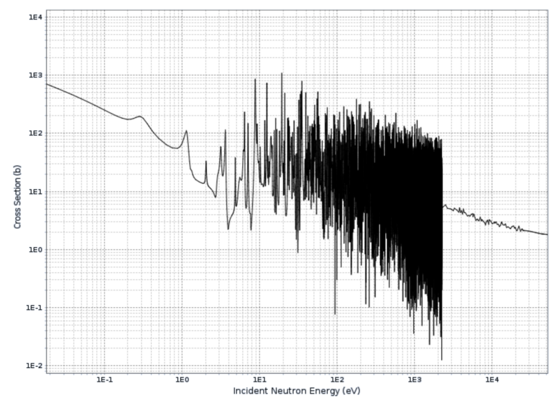
(a)



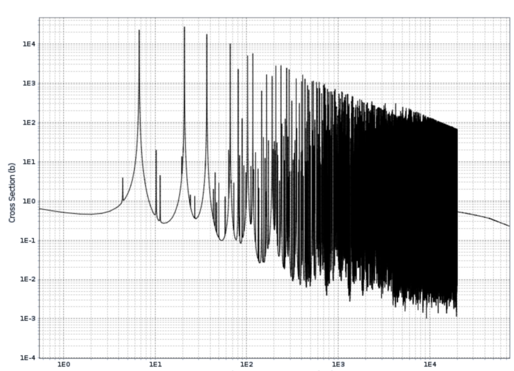
(b)



(c)



(d)



(e)

Problem 4 Solution

(a)	4	^{238}U fission (high fast region)
(b)	5	^{239}Pu fission (resonance peak)
(c)	1	^{155}Gd absorption (consistently high absorption)
(d)	2	^{235}U fission (high thermal fission cross section)
(e)	3	^{238}U absorption (large absorptive resonances)