

Problem 1**a.)**

We are attempting to prove

$$\begin{aligned}
\mu \frac{\partial \psi}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \psi}{\partial \mu} &= \frac{\mu}{r^2} \frac{\partial(r^2 \psi)}{\partial r} + \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)\psi}{r} \right] \\
&= \frac{\mu}{r^2} \left[r^2 \frac{\partial \psi}{\partial r} + \frac{\partial(r^2)}{\partial r} \psi \right] + \frac{1}{r} \left[(1-\mu^2) \frac{\partial \psi}{\partial \mu} + \frac{\partial(1-\mu^2)}{\partial \mu} \psi \right] \\
&= \frac{\mu}{r^2} \left[r^2 \frac{\partial \psi}{\partial r} + 2r\psi \right] + \frac{1}{r} \left[(1-\mu^2) \frac{\partial \psi}{\partial \mu} - 2\mu\psi \right] \\
&= \mu \frac{\partial \psi}{\partial r} + \frac{2\mu}{r} \psi + \frac{(1-\mu^2)}{r} \frac{\partial \psi}{\partial \mu} - \frac{2\mu}{r} \psi \\
\mu \frac{\partial \psi}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \psi}{\partial \mu} &= \mu \frac{\partial \psi}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \psi}{\partial \mu}
\end{aligned}$$

□

b.)If we now integrate the streaming operator over $-1 \leq \mu \leq 1$, we have

$$\begin{aligned}
&\int_{-1}^1 d\mu \frac{\mu}{r^2} \frac{\partial(r^2 \psi)}{\partial r} + \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)\psi}{r} \right] \\
&\int_{-1}^1 d\mu \frac{\mu}{r^2} \frac{\partial(r^2 \psi)}{\partial r} + \int_{-1}^1 d\mu \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)\psi}{r} \right] \\
&\frac{1}{r^2} \int_{-1}^1 d\mu \mu \frac{\partial(r^2 \psi)}{\partial r} + \left[\frac{(1-\mu^2)\psi}{r} \right]_{-1}^1 \\
&\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \int_{-1}^1 d\mu \mu \psi \right).
\end{aligned}$$

If we note that $\vec{J} = \int_{4\pi} d\hat{\Omega} \hat{\Omega} \psi = \int_{-1}^1 d\mu \mu \psi$ then,

$$\boxed{\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \vec{J} \right)}.$$

c.)Integrating this expression for the streaming operator over a spherical shell ($r_1 \leq r \leq r_2$), we have

$$\int_{r_1}^{r_2} 4\pi r^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \vec{J} \right) dr$$

$$\begin{aligned}
 & 4\pi \int_{r_1}^{r_2} \frac{\partial}{\partial r} (r^2 \vec{J}) dr \\
 & 4\pi \left[r^2 \vec{J} \right]_{r_1}^{r_2} \\
 & \boxed{4\pi \left[r_2^2 \vec{J}(r_2) - r_1^2 \vec{J}(r_1) \right]}.
 \end{aligned}$$

Problem 2

a.)

The streaming operator can first be defined as

$$\hat{\Omega} \cdot \nabla \equiv \frac{\partial}{\partial s}.$$

We can then expand this for cylindrical geometries as we did for spherical geometries. In addition to being dependent on position, $\vec{r} = (\rho, \theta, z)$, we note that now the angular flux depends on both angular variables describing the velocity, ξ , the cosine of the polar angle, and ω the azimuthal angle.

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho} \frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial s} + \frac{\partial \psi}{\partial z} \frac{\partial z}{\partial s} + \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial s} + \frac{\partial \psi}{\partial \omega} \frac{\partial \omega}{\partial s}.$$

In the one dimensional case, $\frac{\partial \psi}{\partial \theta} = 0$ and $\frac{\partial \psi}{\partial z} = 0$. Furthermore, because our geometry is cylindrical, $\frac{\partial \xi}{\partial s} = 0$ (the particle's direction with respect to \hat{z} remains constant), so our equation for $\frac{\partial \psi}{\partial s}$ simplifies to

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho} \frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \omega} \frac{\partial \omega}{\partial s}.$$

By analyzing the geometry further, we can determine from figure 1-16 in Lewis & Miller figure that

$$\frac{\partial \rho}{\partial s} = \mu \quad \text{and} \quad \frac{\partial \omega}{\partial s} = -\frac{\eta}{\rho}.$$

Finally, our streaming operator becomes

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho} \frac{\partial \psi}{\partial \omega}.$$

With manipulation (and specifically noting that $\mu = \frac{\partial \eta}{\partial \omega}$), we can reach the streaming operator in conservation form.

$$\begin{aligned}
 \left[\hat{\Omega} \cdot \nabla \right] \psi &= \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho} \frac{\partial \psi}{\partial \omega} \\
 &= \frac{\mu \rho}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\mu \psi}{\rho} - \frac{\eta}{\rho} \frac{\partial \psi}{\partial \omega} - \frac{\mu \psi}{\rho} \\
 &= \frac{\mu}{\rho} \left(\rho \frac{\partial \psi}{\partial \rho} + \psi \right) - \frac{1}{\rho} \left(\eta \frac{\partial \psi}{\partial \omega} + \mu \psi \right) \\
 &= \frac{\mu}{\rho} \left(\rho \frac{\partial \psi}{\partial \rho} + \frac{\partial \rho}{\partial \rho} \psi \right) - \frac{1}{\rho} \left(\eta \frac{\partial \psi}{\partial \omega} + \frac{\partial \eta}{\partial \omega} \psi \right)
 \end{aligned}$$

$$\boxed{\left[\hat{\Omega} \cdot \nabla \right] \psi = \frac{\mu}{\rho} \frac{\partial}{\partial \rho} (\rho \psi) - \frac{1}{\rho} \frac{\partial}{\partial \omega} (\eta \psi)}.$$

b.)

We can use this streaming operator in the transport equation to describe one dimensional cylindrical geometries. For simplicity, we will consider that the transport equation is describing a non-multiplying system. In one-dimensional cylindrical coordinate systems, dependence on position, \vec{r} , now only depends on ρ , and velocity direction $\hat{\Omega}$, depends on ω and ξ .

$$\frac{1}{v} \frac{\partial}{\partial t} \psi(\rho, \omega, \xi, E, t) + \left[\frac{\mu}{\rho} \frac{\partial}{\partial \rho} (\rho \psi(\rho, \omega, \xi, E, t)) - \frac{1}{\rho} \frac{\partial}{\partial \omega} (\eta \psi(\rho, \omega, \xi, E, t)) \right] + \Sigma_t(\rho, E) \psi(\rho, \omega, \xi, E, t) = s(\rho, \omega, \xi, E, t)$$

Integrating this equation over all angles ($0 \leq \omega \leq 2\pi$ and $-1 \leq \xi \leq 1$)

$$\begin{aligned} \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \frac{1}{v} \frac{\partial}{\partial t} \psi(\rho, \omega, \xi, E, t) + \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \left[\frac{\mu}{\rho} \frac{\partial}{\partial \rho} (\rho \psi(\rho, \omega, \xi, E, t)) - \frac{1}{\rho} \frac{\partial}{\partial \omega} (\eta \psi(\rho, \omega, \xi, E, t)) \right] + \\ + \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \Sigma_t(\rho, E) \psi(\rho, \omega, \xi, E, t) = \int_0^{2\pi} d\omega \int_{-1}^1 d\xi s(\rho, \omega, \xi, E, t) \end{aligned}$$

If we let $\int_0^{2\pi} d\omega \int_{-1}^1 d\xi s(\rho, \omega, \xi, E, t) = S(\rho, E, t)$, then

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \left[\frac{\mu}{\rho} \frac{\partial}{\partial \rho} (\rho \psi(\rho, \omega, \xi, E, t)) - \frac{1}{\rho} \frac{\partial}{\partial \omega} (\eta \psi(\rho, \omega, \xi, E, t)) \right] + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t)$$

Again, consulting our geometry, we find $\mu = \sqrt{1 - \xi^2} \cos \omega$ and $\eta = \sqrt{1 - \xi^2} \sin \omega$.

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \left[\frac{\sqrt{1 - \xi^2} \cos \omega}{\rho} \frac{\partial}{\partial \rho} (\rho \psi(\rho, \omega, \xi, E, t)) - \frac{1}{\rho} \frac{\partial}{\partial \omega} (\sqrt{1 - \xi^2} \sin \omega \psi(\rho, \omega, \xi, E, t)) \right] + \\ + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t) \end{aligned}$$

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \left[\frac{\sqrt{1 - \xi^2} \cos \omega}{\rho} \frac{\partial}{\partial \rho} (\rho \psi(\rho, \omega, \xi, E, t)) \right] + \\ - \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \left[\frac{1}{\rho} \frac{\partial}{\partial \omega} (\sqrt{1 - \xi^2} \sin \omega \psi(\rho, \omega, \xi, E, t)) \right] + \\ + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t) \end{aligned}$$

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \int_0^{2\pi} d\omega \int_{-1}^1 d\xi \sqrt{1 - \xi^2} \cos \omega \psi(\rho, \omega, \xi, E, t) \right) + \\ - \frac{1}{\rho} \int_{-1}^1 d\xi \int_0^{2\pi} d\omega \frac{\partial}{\partial \omega} (\sqrt{1 - \xi^2} \sin \omega \psi(\rho, \omega, \xi, E, t)) + \\ + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t) \end{aligned}$$

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \int_0^{2\pi} d\omega \cos \omega \int_{-1}^1 d\xi \sqrt{1 - \xi^2} \psi(\rho, \omega, \xi, E, t) \right) + \\ - \frac{1}{\rho} \int_{-1}^1 d\xi \sqrt{1 - \xi^2} [\sin \omega \psi(\rho, \omega, \xi, E, t)]_0^{2\pi} + \\ + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t) \end{aligned}$$

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(\rho, E, t) + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \int_0^{2\pi} d\omega \cos \omega \int_{-1}^1 d\xi \sqrt{1 - \xi^2} \psi(\rho, \omega, \xi, E, t) \right) + \Sigma_t(\rho, E) \phi(\rho, E, t) = S(\rho, E, t)$$

...

Problem 3

The transport equation in one dimension is

$$\mu \cdot \frac{\partial}{\partial x} \psi(x, \mu, E) + \Sigma_t \psi(x, \mu, E) = q(x, \mu, E)$$

If we look along a characteristic “curve” in one dimension, we can say

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial x}$$

and

$$\frac{\partial}{\partial s} \psi(x_0 + \mu s, \mu, E) + \Sigma_t \psi(x_0 + \mu s, \mu, E) = q(x_0 + \mu s, \mu, E)$$

Since our derivation of the integral form of the transport equation (in class) did not explicitly require the fact that our problem was three-dimensional, we can use the same method for the one-dimensional case. We arrive at the following:

$$\psi(x, \mu, E) = \int_0^\infty d\rho' \exp \left[- \int_0^{\rho'} d\rho'' \Sigma_t(x - \rho''\mu, E) \right] q(x - \rho'\mu, \mu, E)$$

where now the exponential represents the attenuation of neutrons as they move from $x - \rho'\mu$ to x , and the source term represents the production of neutrons at $x - \rho'\mu$ into (μ, E) .

Since the cross section is uniform in the slab $\Sigma_t(x - \rho''\mu, E) = \Sigma_t$, and

$$\psi(x, \mu, E) = \int_0^\infty d\rho' \exp \left[- \int_0^{\rho'} d\rho'' \Sigma_t \right] q(x - \rho'\mu, \mu, E)$$

$$\psi(x, \mu, E) = \int_0^\infty d\rho' \exp \left[- \Sigma_t \int_0^{\rho'} d\rho'' \right] q(x - \rho'\mu, \mu, E)$$

$$\psi(x, \mu, E) = \int_0^\infty d\rho' \exp [-\Sigma_t \rho'] q(x - \rho'\mu, \mu, E)$$

The uncollided flux will consist entirely of neutrons with energy E_0 (if the neutron source beam is monoenergetic, neutrons must collide to have energy other than E_0), and direction $\mu = 1$ (if the neutrons are traveling in some other direction, they must have scattered into that direction)

$$\psi_0(x, \mu, E) = \psi_0(x, 1, E_0) = \int_0^\infty d\rho' \exp [-\Sigma_t \rho'] q(x - \rho', 1, E_0)$$

Since we are ignoring fission, we note that the source term is simply the beam source, $q = s$.

The boundary conditions for this problem are

1.

$$J_+(0) = \int_0^1 \int_0^\infty \psi(0, \mu, E) d\mu dE = I$$

2.

$$J_- \left(\frac{a}{\tilde{\mu}} \right) = \int_{-1}^0 \int_0^\infty \psi \left(\frac{a}{\tilde{\mu}}, \mu, E \right) d\mu dE = 0$$

Using our first boundary condition, we can treat the source s as if it is a monodirectional point source appearing at the slab boundary, $x = 0$.

$$\psi_0(x, \mu, E) = \psi_0(x, 1, E_0) = \int_0^\infty d\rho' \exp [-\Sigma_t \rho'] s(x - \rho', 1, E_0) \delta(x - \rho')$$

$$\psi_0(x, \mu, E) = \psi_0(x, 1, E_0) = I e^{-\Sigma_t x}$$

Problem 4

Problem 5

The integral form of the transport equation is

$$\psi(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \exp \left[- \int_0^{\rho'} d\rho'' \Sigma_t(\vec{r} - \rho'' \hat{\Omega}, E) \right] q(\vec{r} - \rho' \hat{\Omega}, \hat{\Omega}, E)$$

For isotropic sources (scattering and fission are not present in a purely absorbing medium), the source term becomes

$$q(\vec{r} - \rho' \hat{\Omega}, \hat{\Omega}, E) = \frac{S(\vec{r} - \rho' \hat{\Omega}, E)}{4\pi}$$

Where the source is a point source,

$$q(\vec{r} - \rho' \hat{\Omega}, \hat{\Omega}, E) = \frac{S(E)}{4\pi} \delta(\vec{r} - \rho' \hat{\Omega})$$

Using this in the integral form of the TE, and noting that $\Sigma_t(E) = \Sigma_t$,

$$\begin{aligned} \psi(\vec{r}, \hat{\Omega}, E) &= \int_0^\infty d\rho' \exp \left[- \int_0^{\rho'} d\rho'' \Sigma_t \right] \frac{S(E)}{4\pi} \delta(\vec{r} - \rho' \hat{\Omega}) \\ \psi(\vec{r}, \hat{\Omega}, E) &= \int_0^\infty d\rho' e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho' \hat{\Omega}) \end{aligned}$$

We can integrate this expression over angle to find the scalar flux.

$$\phi(\vec{r}, E) = \int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho' \hat{\Omega})$$

When doing this integration, we can note that

$$\begin{aligned} d\hat{\Omega} &= \frac{dA}{\rho'^2} \\ d\rho' dA &= dV \\ \int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' &= \int_V \frac{dV}{\rho'^2} \end{aligned}$$

and $\rho' = |\vec{r} - \vec{r}'|$, so that

$$\begin{aligned} \phi(\vec{r}, E) &= \int_V \frac{dV}{\rho'^2} e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho' \hat{\Omega}) \\ \phi(\vec{r}, E) &= \int_V \frac{d^3 \vec{r}'}{4\pi |\vec{r} - \vec{r}'|^2} e^{-\Sigma_t |\vec{r} - \vec{r}'|} S(E) \delta(\vec{r} - \vec{r}') \end{aligned}$$

The delta function in this equation selects only the value for when $\vec{r}' = 0$.

$$\phi(\vec{r}, E) = \frac{1}{4\pi |\vec{r}|^2} e^{-\Sigma_t |\vec{r}|} S(E)$$

Letting $r = |\vec{r}|$,

$$\boxed{\phi(r, E) = \frac{S(E)}{4\pi r^2} e^{-\Sigma_t r}}$$

Problem 6

Problem 7

For cylindrical geometries, the diffusion equation takes the form:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(\vec{r})}{\partial r} \right) + \frac{\partial^2 \phi(\vec{r})}{\partial z^2} + B^2 \phi(\vec{r}) = 0$$

where $B^2 = \frac{\nu \Sigma_f - \Sigma_a}{D}$ near criticality. Since our cylinder is infinite we lose dependence on z in our equations.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(r)}{\partial r} \right) + B^2 \phi(r) = 0$$

Solutions to this equation are zeroth order Bessel functions of first and second kind, $J_0(r)$ and $Y_0(r)$. Then,

$$\phi_b(r) = A_1 J_0(Br) + A_2 Y_0(Br).$$

Our boundary conditions demand that the flux is finite at $r = 0$, and so $A_2 = 0$. Letting $A_1 = A$, we have

$$\phi_b(r) = A J_0(Br).$$

For a bare reactor, boundary conditions force the flux to go to zero at \tilde{R}_b ,

$$\phi(\tilde{R}_b) = 0,$$

and since the flux must be nonnegative, only the first zero of J_0 can satisfy this condition. This occurs at $B\tilde{R}_b = 2.4048$. $B = B_g = \frac{2.4048}{\tilde{R}_b}$. The criticality condition for the bare core is then

$$\left(\frac{\nu \Sigma_f - \Sigma_a}{D} \right)^2 = \left(\frac{2.4048}{\tilde{R}_b} \right)^2$$

and the critical radius is

$$\tilde{R}_b = \frac{2.4048D}{\nu \Sigma_f - \Sigma_a}.$$

For a reflected core, our initial equation and the form of the solutions stay the same within the core region. With a finite flux at $r = 0$, $A_2 = 0$ and if $A_1 = A$, then in the core

$$\phi_r(r) = A J_0(Br), \quad r < R.$$

In the reflected region, we still have no z dependence, but our diffusion equation is just

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(r)}{\partial r} \right) - \frac{1}{L^2} \phi(r) = 0$$

without the dependence on fission. Solutions to this equation are zeroth order modified bessel functions of the first and second kind, $I_0(r)$ and $K_0(r)$. In the reflector

$$\phi_r(r) = C_1 I_0\left(\frac{r}{L}\right) + C_2 K_0\left(\frac{r}{L}\right), \quad R < r < a$$

We can now impose the boundary condition that the flux must go to zero at the extrapolated distance in the reflector,

$$\phi_r(R + \tilde{a}) = 0.$$

We find

$$0 = C_1 I_0\left(\frac{R + \tilde{a}}{L}\right) + C_2 K_0\left(\frac{R + \tilde{a}}{L}\right).$$

$$C_2 = -C_1 \frac{I_0\left(\frac{R + \tilde{a}}{L}\right)}{K_0\left(\frac{R + \tilde{a}}{L}\right)}$$

If we let $C_1 = C$, then in the reflector

$$\phi_r(r) = CI_0\left(\frac{r}{L}\right) - C \frac{I_0\left(\frac{R+\tilde{a}}{L}\right)}{K_0\left(\frac{R+\tilde{a}}{L}\right)} K_0\left(\frac{r}{L}\right), \quad R < r < a$$

Finally, we use our boundary condition that the flux must be equal at the boundaries.

$$\phi_r(R) = AJ_0(BR) = CI_0\left(\frac{R}{L}\right) - C \frac{I_0\left(\frac{R+\tilde{a}}{L}\right)}{K_0\left(\frac{R+\tilde{a}}{L}\right)} K_0\left(\frac{R}{L}\right)$$

...

Problem 8

The one-dimensional, single energy diffusion equation in each of the material regions are as follows:

Multiplying Core:

$$\frac{d\phi(x)}{dx} - B^2\phi(x) = 0$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Uniform Source Reflector: Multiplying Core:

$$\frac{d\phi(x)}{dx} - B^2\phi(x) = 0$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Problem 9