NE250 Homework 4 **Mitch Negus**
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Problem 1

a.)

We are attempting to prove

$$
\begin{aligned}
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + \frac{\partial(r^2)}{\partial r}\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} + \frac{\partial(1-\mu^2)}{\partial\mu}\psi\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + 2r\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} - 2\mu\psi\right] \\
&= \mu\frac{\partial\psi}{\partial r} + \frac{2\mu}{r}\psi + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} - \frac{2\mu}{r}\psi \\
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu}
\end{aligned}
$$

$\square$

b.)

If we now integrate the streaming operator over $-1 \le \mu \le 1$, we have

$$
\begin{gathered}
\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \int_{-1}^{1} d\mu \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
\frac{1}{r^2}\int_{-1}^{1} d\mu\, \mu\frac{\partial(r^2\psi)}{\partial r} + \left[\frac{(1-\mu^2)\psi}{r}\right]_{-1}^{1} \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\int_{-1}^{1} d\mu\, \mu\, \psi\right).
\end{gathered}
$$

If we note that $\vec{J} = \int_{4\pi} d\hat{\Omega}\, \hat{\Omega}\, \psi = \int_{-1}^{1} d\mu\, \mu\, \psi$ then,

$$
\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right)}.
$$

c.)

Integrating this expression for the streaming operator over a spherical shell ($r_1 \le r \le r_2$), we have

$$
\int_{r_1}^{r_2} 4\pi r^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right) dr
$$

$$4\pi \int_{r_1}^{r_2} \frac{\partial}{\partial r}\left(r^2\vec{J}\right)dr$$

$$4\pi \left[r^2\vec{J}\right]_{r_1}^{r_2}$$

$$\boxed{4\pi \left[r_2^2\vec{J}(r_2) - r_1^2\vec{J}(r_1)\right]}.$$

Problem 2

a.)

The streaming operator can first be defined as

$$\hat{\Omega}\cdot\nabla \equiv \frac{\partial}{\partial s}.$$

We can then expand this for cylindrical geometries as we did for spherical geometries. In addition to being dependent on position, $\vec{r} = (\rho, \theta, z)$, we note that now the angular flux depends on both angular variables describing the velocity, ξ, the cosine of the polar angle, and ω the azimuthal angle.

$$\frac{\partial\psi}{\partial s} = \frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial s} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial s} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial s} + \frac{\partial\psi}{\partial\omega}\frac{\partial\omega}{\partial s}.$$

In the one dimensional case, $\frac{\partial\psi}{\partial\theta} = 0$ and $\frac{\partial\psi}{\partial z} = 0$. Furthermore, because our geometry is cylindrical, $\frac{\partial\xi}{\partial s} = 0$ (the particle's direction with respect to $\hat{z}$ remains constant), so our equation for $\frac{\partial\psi}{\partial s}$ simplifies to

$$\frac{\partial\psi}{\partial s} = \frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial s} + \frac{\partial\psi}{\partial\omega}\frac{\partial\omega}{\partial s}.$$

By analyzing the geometry further, we can determine from figure 1-16 in Lewis & Miller figure that

$$\frac{\partial\rho}{\partial s} = \mu \quad \text{and} \quad \frac{\partial\omega}{\partial s} = -\frac{\eta}{\rho}.$$

Finally, our streaming operator becomes

$$\frac{\partial\psi}{\partial s} = \mu\frac{\partial\psi}{\partial\rho} - \frac{\eta}{\rho}\frac{\partial\psi}{\partial\omega}.$$

With manipulation (and specifically noting that $\mu = \frac{\partial\eta}{\partial\omega}$), we can reach the streaming operator in conservation form.

$$\begin{aligned}
\left[\hat{\Omega}\cdot\nabla\right]\psi &= \mu\frac{\partial\psi}{\partial\rho} - \frac{\eta}{\rho}\frac{\partial\psi}{\partial\omega} \\
&= \frac{\mu\rho}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\mu\psi}{\rho} - \frac{\eta}{\rho}\frac{\partial\psi}{\partial\omega} - \frac{\mu\psi}{\rho} \\
&= \frac{\mu}{\rho}\left(\rho\frac{\partial\psi}{\partial\rho} + \psi\right) - \frac{1}{\rho}\left(\eta\frac{\partial\psi}{\partial\omega} + \mu\psi\right) \\
&= \frac{\mu}{\rho}\left(\rho\frac{\partial\psi}{\partial\rho} + \frac{\partial\rho}{\partial\rho}\psi\right) - \frac{1}{\rho}\left(\eta\frac{\partial\psi}{\partial\omega} + \frac{\partial\eta}{\partial\omega}\psi\right)
\end{aligned}$$

$$\boxed{\left[\hat{\Omega}\cdot\nabla\right]\psi = \frac{\mu}{\rho}\frac{\partial}{\partial\rho}(\rho\psi) - \frac{1}{\rho}\frac{\partial}{\partial\omega}(\eta\psi)}.$$

b.)

We can use this streaming operator in the transport equation to describe one dimensional cylindrical geometries. For simplicity, we will consider that the transport equation is describing a non-multiplying system. In one-dimensional cylindrical coordinate systems, dependence on position, $\vec{r}$, now only depends on ρ, and velocity direction $\hat{\Omega}$, depends on ω and ξ.

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = s(\rho,\omega,\xi,E,t)$$

Integrating this equation over all angles ($0 \leq \omega \leq 2\pi$ and $-1 \leq \xi \leq 1$)

$$\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+ \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, s(\rho,\omega,\xi,E,t)$$

If we let $\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, s(\rho,\omega,\xi,E,t) = S(\rho,E,t)$, then

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

Again, consulting our geometry, we find $\mu = \sqrt{1-\xi^2}\cos\omega$ and $\eta = \sqrt{1-\xi^2}\sin\omega$.

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$- \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \left[\frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\cos\omega \; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi \int_0^{2\pi} d\omega \, \frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega \; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi\sqrt{1-\xi^2}\left[\sin\omega \; \psi(\rho,\omega,\xi,E,t)\right]_0^{2\pi} +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

...

Problem 3

The transport equation in one dimension is

$$\mu \cdot \frac{\partial}{\partial x}\psi(x,\mu,E) + \Sigma_t \psi(x,\mu,E) = q(x,\mu,E)$$

If we look along a characteristic "curve" in one dimension, we can say

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial x}$$

and

$$\frac{\partial}{\partial s}\psi(x_0+\mu s,\mu,E) + \Sigma_t \psi(x_0+\mu s,\mu,E) = q(x_0+\mu s,\mu,E)$$

Since our derivation of the integral form of the transport equation (in class) did not explicitly require the fact that our problem was three-dimensional, we can use the same method for the one-dimensional case. We arrive at the following:

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(x-\rho''\mu,E)\right] q(x-\rho'\mu,\mu,E)$$

where now the exponential represents the attenuation of neutrons as they move from $x-\rho'\mu$ to x, and the source term represents the production of neutrons at $x-\rho'\mu$ into (μ,E).

Since the cross section is uniform in the slab $\Sigma_t(x-\rho''\mu,E) = \Sigma_t$, and

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \int_0^{\rho'} d\rho''\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \rho'\right] q(x-\rho'\mu,\mu,E)$$

The uncollided flux will consist entirely of neutrons with energy E_0 (if the neutron source beam is monoenergetic, neutrons must collide to have energy other than E_0), and direction $\mu = 1$ (if the neutrons are traveling in some other direction, they must have scattered into that direction)

$$\psi_0(x,\mu,E) = \psi_0(x,1,E_0) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \rho'\right] q(x-\rho',1,E_0)$$

Since we are ignoring fission, we note that the source term is simply the beam source, $q = s$.

The boundary conditions for this problem are

1. $$J_+(0) = \int_0^1 \int_0^\infty \psi(0,\mu,E) d\mu dE = I$$

2. $$J_-\left(\frac{a}{\tilde{\mu}}\right) = \int_{-1}^0 \int_0^\infty \psi\left(\frac{a}{\tilde{\mu}},\mu,E\right) d\mu\, dE = 0$$

Using our first boundary condition, we can treat the source s as if it is a monodirectional point source appearing at the slab boundary, $x = 0$.

$$\psi_0(x,\mu,E) = \psi_0(x,1,E_0) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \rho'\right] s(x-\rho',1,E_0)\delta(x-\rho')$$

$$\psi_0(x,\mu,E) = \psi_0(x,1,E_0) = Ie^{-\Sigma_t x}$$

Problem 4

Problem 5

The integral form of the transport equation is

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(\vec{r}-\rho''\hat{\Omega},E)\right] q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E)$$

For isotropic sources (scattering and fission are not present in a purely absorbing medium), the source term becomes

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(\vec{r}-\rho'\hat{\Omega},E)}{4\pi}$$

Where the source is a point source,

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

Using this in the integral form of the TE, and noting that $\Sigma_t(E) = \Sigma_t$,

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right] \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' e^{-\Sigma_t\rho'} \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

We can integrate this expression over angle to find the scalar flux.

$$\phi(\vec{r},E) = \int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' e^{-\Sigma_t\rho'} \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

When doing this integration, we can note that

$$d\hat{\Omega} = \frac{dA}{\rho'^{\,2}}$$

$$d\rho'\, dA = dV$$

$$\int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' = \int_V \frac{dV}{\rho'^{\,2}}$$

and $\rho' = |\vec{r}-\vec{r}\,'|$, so that

$$\phi(\vec{r},E) = \int_V \frac{dV}{\rho'^{\,2}} e^{-\Sigma_t\rho'} \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

$$\phi(\vec{r},E) = \int_V \frac{d^3\vec{r}\,'}{4\pi|\vec{r}-\vec{r}\,'|^2} e^{-\Sigma_t|\vec{r}-\vec{r}\,'|} S(E)\delta(\vec{r}\,')$$

The delta function in this equation selects only the value for when $\vec{r}\,' = 0$.

$$\phi(\vec{r},E) = \frac{1}{4\pi|\vec{r}|^2} e^{-\Sigma_t|\vec{r}|} S(E)$$

Letting $r = |\vec{r}|$,

$$\boxed{\phi(r,E) = \frac{S(E)}{4\pi r^2} e^{-\Sigma_t r}}.$$

Problem 6

Problem 7

For cylindrical geometries, the diffusion equation takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(\vec{r})}{\partial r}\right)+\frac{\partial^2\phi(\vec{r})}{\partial z^2}+B^2\phi(\vec{r})=0$$

where $B^2=\frac{\nu\Sigma_f-\Sigma_a}{D}$ near criticality. Since our cylinder is infinite we lose dependence on z in our equations.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r)}{\partial r}\right)+B^2\phi(r)=0$$

Solutions to this equation are are zeroth order Bessel functions of first and second kind, $J_0(r)$ and $Y_0(r)$. Then,

$$\phi_b(r)=A_1J_0(Br)+A_2Y_0(Br).$$

Our boundary conditions demand that the flux is finite at $r=0$, and so $A_2=0$. Letting $A_1=A$, we have

$$\phi_b(r)=AJ_0(Br).$$

For a bare reactor, boundary conditions force the flux to go to zero at $\tilde{R}_b$,

$$\phi(\tilde{R}_b)=0,$$

and since the flux must be nonnegative, only the first zero of J_0 can satisfy this condition. This occurs at $B\tilde{R}_b=2.4048$. $B=B_g=\frac{2.4048}{\tilde{R}_b}$. The criticality condition for the bare core is then

$$\left(\frac{\nu\Sigma_f-\Sigma_a}{D}\right)^2=\left(\frac{2.4048}{\tilde{R}_b}\right)^2$$

and the critical radius is

$$\tilde{R}_b=\frac{2.4048D}{\nu\Sigma_f-\Sigma_a}.$$

For a reflected core, our initial equation and the form of the solutions stay the same within the core region. With a finite flux at $r=0$, $A_2=0$ and if $A_1=A$, then in the core

$$\phi_r(r)=AJ_0(Br),\qquad r<R.$$

In the reflected region, we still have no z dependence, but our diffusion equation is just

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r)}{\partial r}\right)-\frac{1}{L^2}\phi(r)=0$$

without the dependence on fission. Solutions to this equation are zeroth order modified bessel functions of the first and second kind, $I_0(r)$ and $K_0(r)$. In the reflector

$$\phi_r(r)=C_1I_0(\frac{r}{L})+C_2K_0(\frac{r}{L}),\qquad R<r<a$$

We can now impose the boundary condition that the flux must go to zero at the extrapolated distance in the reflector,

$$\phi_r(R+\tilde{a})=0.$$

We find

$$0=C_1I_0(\frac{R+\tilde{a}}{L})+C_2K_0(\frac{R+\tilde{a}}{L}).$$

$$C_2=-C_1\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}$$

If we let $C_1 = C$, then in the reflector

$$\phi_r(r) = CI_0(\frac{r}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{r}{L}), \qquad R < r < a$$

Finally, we use our boundary condition that the flux must be equal at the boundaries.

$$\phi_r(R) = AJ_0(BR) = CI_0(\frac{R}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{R}{L})$$

...

Problem 8

The one-dimensional, single energy diffusion equation in each of the material regions are as follows:

Multiplying Core:

$$\frac{d^2\phi(x)}{dx^2} + B^2\phi(x) = 0, \qquad 0 < x < a$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Uniform Source Reflector:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = \frac{-S_0}{D}, \qquad x > a$$

where $L^2 = \frac{D}{\Sigma_a}$.

Boundary Conditions:

1. $J_+(0) = 0$
2. $\phi_c(a) = \phi_r(a)$
3. $\vec{J}_c(a) = \vec{J}_r(a)$
4. $\phi(x) < \infty$

In the multiplying core, solutions to the differential equation are

$$\phi(x) = A_1\sin(Bx) + A_2\cos(Bx)$$

We can relate J_+ to the flux using the relation

$$J_+(0) = 0 = \frac{\phi(0)}{4} - \frac{D}{2}\frac{d\phi}{dx}\bigg|_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{D}{2}\left[A_1B\cos(Bx) - A_2B\sin(Bx)\right]_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{A_1BD}{2}$$

$$A_2 = 2A_1BD$$

Letting $A_1 = A$,

$$\phi_c(x) = A\sin(Bx) + 2ABD\cos(Bx)$$

In the uniform reflector, the homogeneous solution will be

$$\phi_h(x) = C_1e^{x/L} + C_2e^{-x/L},$$

the particular solution will be

$$\phi_p(x) = \frac{S_0L^2}{D},$$

and thus the general solution in the reflector will be

$$\phi_r(x) = C_1 e^{x/L} + C_2 e^{-x/L} + \frac{S_0 L^2}{D}$$

Noting that $\phi(x) < \infty$ for all x, we see that this is violated in the above equation as $x \to \infty$ if $C_1 \neq 0$. Letting $C_2 = C$,

$$\phi_r(x) = C e^{-x/L} + \frac{S_0 L^2}{D}$$

Using our interface condition, we can see that

$$\phi_c(a) = \phi_r(a)$$

$$A \sin(Ba) + 2ABD \cos(Ba) = C e^{-a/L} + \frac{S_0 L^2}{D}$$

$$C = AD \frac{\sin(Ba) + 2BD \cos(Ba) - S_0 L^2}{D e^{-a/L}}$$

In total, the flux in this system is

$$\boxed{\phi(x) = \begin{cases} A\left(\sin(Bx) + 2BD\cos(Bx)\right), & 0 < x < a \\ AD\left(\frac{\sin(Ba)+2BD\cos(Ba)-S_0L^2}{De^{-a/L}}\right) e^{-x/L} + \frac{S_0 L^2}{D}, & x > a \end{cases}}$$

Problem 9