NE250 Homework 4 **Mitch Negus**
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Problem 1

a.)

We are attempting to prove

$$\begin{aligned}
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + \frac{\partial(r^2)}{\partial r}\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} + \frac{\partial(1-\mu^2)}{\partial\mu}\psi\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + 2r\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} - 2\mu\psi\right] \\
&= \mu\frac{\partial\psi}{\partial r} + \frac{2\mu}{r}\psi + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} - \frac{2\mu}{r}\psi \\
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu}
\end{aligned}$$

$\square$

b.)

If we now integrate the streaming operator over $-1 \leq \mu \leq 1$, we have

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \int_{-1}^{1} d\mu \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\frac{1}{r^2}\int_{-1}^{1} d\mu\,\mu\frac{\partial(r^2\psi)}{\partial r} + \left[\frac{(1-\mu^2)\psi}{r}\right]_{-1}^{1}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\int_{-1}^{1} d\mu\,\mu\,\psi\right).$$

If we note that $\vec{J} = \int_{4\pi} d\hat{\Omega}\,\hat{\Omega}\,\psi = \int_{-1}^{1} d\mu\,\mu\,\psi$ then,

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right)}.$$

c.)

Integrating this expression for the streaming operator over a spherical shell ($r_1 \leq r \leq r_2$), we have

$$\int_{r_1}^{r_2} 4\pi r^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right) dr$$

$$4\pi \int_{r_1}^{r_2} \frac{\partial}{\partial r}\left(r^2 \vec{J}\right) dr$$

$$4\pi \left[r^2 \vec{J}\right]_{r_1}^{r_2}$$

$$\boxed{4\pi \left[r_2^2 \vec{J}(r_2) - r_1^2 \vec{J}(r_1)\right]}.$$

Problem 2

a.)

The streaming operator can first be defined as

$$\hat{\Omega} \cdot \nabla \equiv \frac{\partial}{\partial s}.$$

We can then expand this for cylindrical geometries as we did for spherical geometries. In addition to being dependent on position, $\vec{r} = (\rho, \theta, z)$, we note that now the angular flux depends on both angular variables describing the velocity, ξ, the cosine of the polar angle, and ω the azimuthal angle.

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

In the one dimensional case, $\frac{\partial \psi}{\partial \theta} = 0$ and $\frac{\partial \psi}{\partial z} = 0$. Furthermore, because our geometry is cylindrical, $\frac{\partial \xi}{\partial s} = 0$ (the particle's direction with respect to $\hat{z}$ remains constant), so our equation for $\frac{\partial \psi}{\partial s}$ simplifies to

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

By analyzing the geometry further, we can determine from figure 1-16 in Lewis & Miller figure that

$$\frac{\partial \rho}{\partial s} = \mu \quad \text{and} \quad \frac{\partial \omega}{\partial s} = -\frac{\eta}{\rho}.$$

Finally, our streaming operator becomes

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega}.$$

With manipulation (and specifically noting that $\mu = \frac{\partial \eta}{\partial \omega}$), we can reach the streaming operator in conservation form.

$$\begin{aligned}
\left[\hat{\Omega} \cdot \nabla\right] \psi &= \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} \\
&= \frac{\mu\rho}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\mu\psi}{\rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} - \frac{\mu\psi}{\rho} \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \mu\psi\right) \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \frac{\partial \rho}{\partial \rho}\psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \frac{\partial \eta}{\partial \omega}\psi\right)
\end{aligned}$$

$$\boxed{\left[\hat{\Omega} \cdot \nabla\right] \psi = \frac{\mu}{\rho}\frac{\partial}{\partial \rho}(\rho\psi) - \frac{1}{\rho}\frac{\partial}{\partial \omega}(\eta\psi)}.$$

b.)

We can use this streaming operator in the transport equation to describe one dimensional cylindrical geometries. For simplicity, we will consider that the transport equation is describing a non-multiplying system. In one-dimensional cylindrical coordinate systems, dependence on position, $\vec{r}$, now only depends on ρ, and velocity direction $\hat{\Omega}$, depends on ω and ξ.

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = s(\rho,\omega,\xi,E,t)$$

Integrating this equation over all angles ($0 \leq \omega \leq 2\pi$ and $-1 \leq \xi \leq 1$)

$$\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+ \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t)$$

If we let $\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t) = S(\rho,E,t)$, then

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

Again, consulting our geometry, we find $\mu = \sqrt{1-\xi^2}\cos\omega$ and $\eta = \sqrt{1-\xi^2}\sin\omega$.

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$- \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\cos\omega\; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi \int_0^{2\pi} d\omega\, \frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi\sqrt{1-\xi^2}\left[\sin\omega\; \psi(\rho,\omega,\xi,E,t)\right]_0^{2\pi} +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

...

Problem 3

The transport equation in one dimension is

$$\mu \cdot \frac{\partial}{\partial x}\psi(x,\mu,E) + \Sigma_t \psi(x,\mu,E) = q(x,\mu,E)$$

If we look along a characteristic "curve" in one dimension, we can say

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial x}$$

and

$$\frac{\partial}{\partial s}\psi(x_0+\mu s,\mu,E) + \Sigma_t \psi(x_0+\mu s,\mu,E) = q(x_0+\mu s,\mu,E)$$

Since our derivation of the integral form of the transport equation (in class) did not explicitly require the fact that our problem was three-dimensional, we can use the same method for the one-dimensional case. We arrive at the following:

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(x-\rho''\mu,E)\right] q(x-\rho'\mu,\mu,E)$$

where now the exponential represents the attenuation of neutrons as they move from $x-\rho'\mu$ to x, and the source term represents the production of neutrons at $x-\rho'\mu$ into (μ,E).

Since the cross section is uniform in the slab $\Sigma_t(x-\rho''\mu,E) = \Sigma_t$,

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \int_0^{\rho'} d\rho''\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \rho'\right] q(x-\rho'\mu,\mu,E)$$

We can change coordinates to be integrating over x, by noting that $\mu = \cos\theta = x/\rho$. Then $\rho = x/\mu$ and $d\rho = dx$.

$$\psi_0(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] q(x-x',\mu,E)$$

Since we are ignoring fission, we note that the source term is simply the beam source, $q(x-x',\mu,E) = s(x-x',\mu,E)$. If we assume that the beam source can be treated like a plane source at the left boundary, then including the fact that the beam is monoenergetic lets the complete source be written as $s(\mu)\delta(E-E_0)\delta(x-x')$

We can now write the equation for the uncollided flux, using the explicitly defined external source

$$\psi_0(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] s(\mu)\delta(E-E_0)\delta(x-x')$$

$$\psi_0(x,\mu,E) = s(\mu)e^{-\Sigma_t x/\mu}$$

The boundary conditions for this problem are

1. $$J_{0+}(0) = \int_0^1 d\mu \int_0^\infty dE\, \psi_0(0,\mu,E) = I$$

2. $$J_{0-}\left(\frac{a}{\tilde{\mu}}\right) = \int_{-1}^0 \int_0^\infty \psi_0\left(\frac{a}{\tilde{\mu}},\mu,E\right) d\mu\, dE = 0$$

Using our first boundary condition,

$$I = \int_0^1 d\mu \int_0^\infty dE\, \psi_0(0, \mu, E_0)$$

$$I = \int_0^1 d\mu \int_0^\infty dE \left[\int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] s(\mu)\delta(E - E_0)\delta(x - x') \right]_{x=0}$$

The delta function for x picks the value of the integrand at which $x' = x$, and the equation can be simplified to

$$I = \int_0^1 d\mu \int_0^\infty dE \left[\exp\left[-\Sigma_t x/\mu\right] s(\mu)\delta(E - E_0)\right]_{x=0}$$

$$I = \int_0^1 d\mu \int_0^\infty dE\, s(\mu)\delta(E - E_0)$$

The delta function for E now picks the value of the integrand at which $E = E_0$,

$$I = \int_0^1 s(\mu) d\mu$$

If we consider the source beam as monodirectional directly incident on the slab, then for all uncollided neutrons $\mu = 1$.

$$I = \int_0^1 s(1) d\mu$$

$$I = s(1)$$

Then, the flux is

$$\psi_0(x, \mu, E) = s(1) e^{-\Sigma_t x/(1)}$$

$$\boxed{\psi_0(x, \mu, E) = I e^{-\Sigma_t x}}.$$

Problem 4

We know that the once collided flux is given by $\psi_1 = K\psi_0$ where K is the integral operator. To determine K and find the source term for first collisions, we start by looking at the fully expanded form of the integral transport equation.

$$\psi(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho'' \Sigma_t(\vec{r} - \rho''\hat{\Omega}, E) \right] \left[\frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + \right.$$
$$\left. + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + Q(\vec{r} - \rho'\hat{\Omega}, \hat{\Omega}, E) \right]$$

For the once collided flux, the external source Q is zero, and the angular flux used to calculate the scattering and fission terms is the uncollided flux.

$$\psi_1(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho'' \Sigma_t(\vec{r} - \rho''\hat{\Omega}, E) \right] \left[\frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + \right.$$
$$\left. + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') \right]$$

We can now see that the first collision source is everything within the second set of brackets.

$$q_1(\vec{r} - \rho'\hat{\Omega}, \hat{\Omega}, E) = \frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') +$$
$$+ \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E')$$

Since we are still dealing with a one dimensional geometry, this **first collision source** can be simplified to

$$q_1(x-x',\mu,E) = \frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \\ + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E').$$

This source can then be inserted back into the full transport equation (in one dimension).

$$\psi_1(x,\mu,E) = \int_0^\infty dx' \exp\left[-\int_0^{x'/\mu} dx''\Sigma_t(x-x'',E)\right]\left[\frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \right. \\ \left. + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E')\right]$$

Finally, since Σ_t is constant, the **first-collided flux** can be written as

$$\psi_1(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right]\left[\frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \right. \\ \left. + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E')\right]$$

with boundary conditions

1. $$J_{1+}(0) = \int_0^1 d\mu \int_0^\infty dE\, \psi_1(0,\mu,E) = 0$$

2. $$J_{1-}\left(\frac{a}{\tilde{\mu}}\right) = \int_{-1}^{0}\int_0^\infty \psi_1\left(\frac{a}{\tilde{\mu}},\mu,E\right) d\mu\, dE = 0$$

Problem 5

The integral form of the transport equation is

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(\vec{r}-\rho''\hat{\Omega},E)\right] q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E)$$

For isotropic sources (scattering and fission are not present in a purely absorbing medium), the source term becomes

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(\vec{r}-\rho'\hat{\Omega},E)}{4\pi}$$

Where the source is a point source,

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

Using this in the integral form of the TE, and noting that $\Sigma_t(E) = \Sigma_t$,

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right] \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' e^{-\Sigma_t\rho'} \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

We can integrate this expression over angle to find the scalar flux.

$$\phi(\vec{r}, E) = \int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho'\hat{\Omega})$$

When doing this integration, we can note that

$$d\hat{\Omega} = \frac{dA}{\rho'^{\,2}}$$

$$d\rho'\, dA = dV$$

$$\int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' = \int_V \frac{dV}{\rho'^{\,2}}$$

and $\rho' = |\vec{r} - \vec{r}\,'|$, so that

$$\phi(\vec{r}, E) = \int_V \frac{dV}{\rho'^{\,2}} e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho'\hat{\Omega})$$

$$\phi(\vec{r}, E) = \int_V \frac{d^3\vec{r}\,'}{4\pi|\vec{r} - \vec{r}\,'|^2} e^{-\Sigma_t|\vec{r} - \vec{r}\,'|} S(E)\delta(\vec{r}\,')$$

The delta function in this equation selects only the value for when $\vec{r}\,' = 0$.

$$\phi(\vec{r}, E) = \frac{1}{4\pi|\vec{r}|^2} e^{-\Sigma_t|\vec{r}|} S(E)$$

Letting $r = |\vec{r}|$,

$$\boxed{\phi(r, E) = \frac{S(E)}{4\pi r^2} e^{-\Sigma_t r}}.$$

Problem 6

In one group diffusion theory, a bare slab reactor is described by the diffusion equation

$$-D\frac{d^2\phi}{dx^2} + \Sigma_a \phi(x) = \Sigma_f(x)\phi(x)$$

or equivalently

$$\frac{d^2\phi}{dx^2} + [B(x)]^2\phi(x) = 0$$

where $[B(x)]^2 = \frac{\nu\Sigma_f(x) - \Sigma_a}{D}$. We also know that the power distribution must be flat, so

$$P(x) = const. = E_f \Sigma_f(x)\phi(x)$$

If we let C be some constant, then

$$C = \Sigma_f(x)\phi(x)$$

$$\Sigma_f(x) = \frac{C}{\phi(x)}$$

Using this in our equation for $B(x)$ and then substituting this back into the diffusion equation to have only an equation in terms of flux gives

$$\frac{d^2\phi}{dx^2} + \left(\frac{C\nu/\phi(x) - \Sigma_a}{D}\right)\phi(x) = 0$$

$$\frac{d^2\phi}{dx^2} + \left(\frac{C\nu - \Sigma_a\phi(x)}{D}\right) = 0$$

$$\frac{d^2\phi}{dx^2} - \frac{\Sigma_a\phi(x)}{D} + \frac{C\nu}{D} = 0$$

Letting $L = \sqrt{D/\Sigma_a}$, the solution to this inhomogeneous ordinary differential equation is

$$\phi(x) = A_1 e^{x/L} + A_2 e^{-x/L} + \frac{C\nu}{\Sigma_a}$$

Since our reactor is a slab, lets say it's width is 2ℓ, and it is centered at the origin. Then, since it is a bare slab, the boundary conditions are

$$\phi(\pm\tilde{\ell}) = 0$$

Using these boundary conditions, we find

$$\phi(\tilde{\ell}) = 0 = A_1 e^{\tilde{\ell}/L} + A_2 e^{-\tilde{\ell}/L} + \frac{C\nu}{\Sigma_a}$$

and

$$\phi(-\tilde{\ell}) = 0 = A_1 e^{-\tilde{\ell}/L} + A_2 e^{\tilde{\ell}/L} + \frac{C\nu}{\Sigma_a}$$

and so $A_1 = A_2$. If we let $A = 2A_1 = 2A_2$, then

$$\phi(x) = A\cosh\left(\frac{x}{L}\right) + \frac{C\nu}{\Sigma_a}$$

Using the boundary conditions again,

$$\phi(\tilde{\ell}) = 0 = A\cosh\left(\frac{\tilde{\ell}}{L}\right) + \frac{C\nu}{\Sigma_a}$$

$$-\frac{C\nu}{\Sigma_a} = A\cosh\left(\frac{\tilde{\ell}}{L}\right)$$

$$C = -\frac{A\Sigma_a}{\nu}\cosh\left(\frac{\tilde{\ell}}{L}\right)$$

The flux solution is finally

$$\phi(x) = A\cosh\left(\frac{x}{L}\right) - \frac{A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{\nu\Sigma_a}$$

$$\phi(x) = A\left(\cosh\left(\frac{x}{L}\right) - \frac{1}{\nu}\cosh\left(\frac{\tilde{\ell}}{L}\right)\right)$$

Now, we established earlier that $C = \Sigma_f(x)\phi(x)$. Using this fact, we can say

$$\Sigma_f(x) = \frac{C}{\phi(x)}$$

$$\Sigma_f(x) = \frac{C}{A\left(\cosh\left(\frac{x}{L}\right) - \frac{1}{\nu}\cosh\left(\frac{\tilde{\ell}}{L}\right)\right)}$$

and where $C = -\frac{A\Sigma_a}{\nu}\cosh\left(\frac{\tilde{\ell}}{L}\right)$

$$\Sigma_f(x) = \frac{-A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{A\nu\left(\cosh\left(\frac{x}{L}\right) - \frac{1}{\nu}\cosh\left(\frac{\tilde{\ell}}{L}\right)\right)}$$

$$\Sigma_f(x) = \frac{\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{\cosh\left(\frac{\tilde{\ell}}{L}\right) - \nu\cosh\left(\frac{x}{L}\right)}$$

By definition, $\Sigma_f(x) = N_{\text{fuel}}(x)\sigma_f$. Then

$$N_{\text{fuel}}(x)\sigma_f = \frac{\Sigma_a \cosh\left(\frac{\tilde{\ell}}{L}\right)}{\cosh\left(\frac{\tilde{\ell}}{L}\right) - \nu\cosh\left(\frac{x}{L}\right)}$$

$$\boxed{N_{\text{fuel}}(x) = \frac{\Sigma_a \cosh\left(\frac{\tilde{\ell}}{L}\right)}{\sigma_f\left(\cosh\left(\frac{\tilde{\ell}}{L}\right) - \nu\cosh\left(\frac{x}{L}\right)\right)}}.$$

Problem 7

For cylindrical geometries, the diffusion equation takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(\vec{r})}{\partial r}\right) + \frac{\partial^2\phi(\vec{r})}{\partial z^2} + B^2\phi(\vec{r}) = 0$$

where $B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$ near criticality. Since our cylinder is infinite we lose dependence on z in our equations.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r)}{\partial r}\right) + B^2\phi(r) = 0$$

Solutions to this equation are are zeroth order Bessel functions of first and second kind, $J_0(r)$ and $Y_0(r)$. Then,

$$\phi_b(r) = A_1 J_0(Br) + A_2 Y_0(Br).$$

Our boundary conditions demand that the flux is finite at $r = 0$, and so $A_2 = 0$. Letting $A_1 = A$, we have

$$\phi_b(r) = AJ_0(Br).$$

For a bare reactor, boundary conditions force the flux to go to zero at $\tilde{R}_b$,

$$\phi(\tilde{R}_b) = 0,$$

and since the flux must be nonnegative, only the first zero of J_0 can satisfy this condition. This occurs at $B\tilde{R}_b = 2.4048$. $B = B_g = \frac{2.4048}{\tilde{R}_b}$. The criticality condition for the bare core is then

$$\left(\frac{\nu\Sigma_f - \Sigma_a}{D}\right)^2 = \left(\frac{2.4048}{\tilde{R}_b}\right)^2$$

and the critical radius is (to within the extrapolated distance)

$$R_b = \frac{2.4048D}{\nu\Sigma_f - \Sigma_a}.$$

For a reflected core, our initial equation and the form of the solutions stay the same within the core region. With a finite flux at $r = 0$, $A_2 = 0$ and if $A_1 = A$, then in the core

$$\phi_r(r) = AJ_0(Br), \qquad r < R.$$

In the reflected region, we still have no z dependence, but our diffusion equation is just

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r)}{\partial r}\right) - \frac{1}{L^2}\phi(r) = 0$$

without the dependence on fission. Solutions to this equation are zeroth order modified bessel functions of the first and second kind, $I_0(r)$ and $K_0(r)$. In the reflector

$$\phi_r(r) = C_1 I_0\left(\frac{r}{L}\right) + C_2 K_0\left(\frac{r}{L}\right), \qquad R < r < a$$

We can now impose the boundary condition that the flux must go to zero at the extrapolated distance from the reflector,

$$\phi_r(R+\tilde{a}) = 0.$$

We find

$$0 = C_1 I_0(\frac{R+\tilde{a}}{L}) + C_2 K_0(\frac{R+\tilde{a}}{L}).$$

$$C_2 = -C_1 \frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}$$

If we let $C_1 = C$, then in the reflector

$$\phi_r(r) = C I_0(\frac{r}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})} K_0(\frac{r}{L}), \qquad R < r < a$$

Together, the flux equations describing our reflected reactor are now

$$\phi_r(r) = A J_0(Br), \qquad r < R$$

$$\phi_r(r) = C I_0(\frac{r}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})} K_0(\frac{r}{L}), \qquad R < r < a$$

Finally, we use our boundary condition that the flux and currents must be equal at the interface.

$$A J_0(BR) = C I_0(\frac{R}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})} K_0(\frac{R}{L})$$

and

$$-D_c \frac{d}{dx}\left(A J_0(Bx)\right)\Big|_{x=R} = -D_r \frac{d}{dx}\left(C I_0\left(\frac{x}{L}\right) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})} K_0\left(\frac{x}{L}\right)\right)\Big|_{x=R}$$

$$-ABD_c J_1(BR) = \frac{CD_r}{L}\left(I_1\left(\frac{R}{L}\right) + \frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})} K_1\left(\frac{R}{L}\right)\right)$$

From these two equations for the flux in the core and the reflector, as well as the 4 boundary conditions (finite flux at $r=0$, zero flux at $r = R+\tilde{a}$, continuous flux at $r=R$, and continuous current at $r=R$), it is *possible* to solve for R and B. (haha, good luck)

Assuming this value is R', the reflector savings $\mathcal{S}$ is

$$\mathcal{S} = R' - \frac{2.4048D}{\nu\Sigma_f - \Sigma_a},$$

and the criticality condition will be the expression for B.

Problem 8

The one-dimensional, single energy diffusion equation in each of the material regions are as follows:

Multiplying Core:

$$\frac{d^2\phi(x)}{dx^2} + B^2\phi(x) = 0, \qquad 0 < x < a$$

where $B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$.

Uniform Source Reflector:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = \frac{-S_0}{D}, \qquad x > a$$

where $L^2 = \frac{D}{\Sigma_a}$.

Boundary Conditions:

1. $J_+(0) = 0$
2. $\phi_c(a) = \phi_r(a)$
3. $\vec{J}_c(a) = \vec{J}_r(a)$
4. $\phi(x) < \infty$

In the multiplying core, solutions to the differential equation are

$$\phi(x) = A_1 \sin(Bx) + A_2 \cos(Bx)$$

We can relate J_+ to the flux using the relation

$$J_+(0) = 0 = \frac{\phi(0)}{4} - \frac{D}{2}\frac{d\phi}{dx}\bigg|_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{D}{2}\left[A_1 B\cos(Bx) - A_2 B\sin(Bx)\right]_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{A_1 BD}{2}$$

$$A_2 = 2A_1 BD$$

Letting $A_1 = A$,

$$\phi_c(x) = A\sin(Bx) + 2ABD\cos(Bx)$$

In the uniform reflector, the homogeneous solution will be

$$\phi_h(x) = C_1 e^{x/L} + C_2 e^{-x/L},$$

the particular solution will be

$$\phi_p(x) = \frac{S_0 L^2}{D},$$

and thus the general solution in the reflector will be

$$\phi_r(x) = C_1 e^{x/L} + C_2 e^{-x/L} + \frac{S_0 L^2}{D}$$

Noting that $\phi(x) < \infty$ for all x, we see that this is violated in the above equation as $x \to \infty$ if $C_1 \neq 0$. Letting $C_2 = C$,

$$\phi_r(x) = Ce^{-x/L} + \frac{S_0 L^2}{D}$$

Using our interface condition, we can see that

$$\phi_c(a) = \phi_r(a)$$

$$A\sin(Ba) + 2ABD\cos(Ba) = Ce^{-a/L} + \frac{S_0 L^2}{D}$$

$$C = AD\frac{\sin(Ba) + 2BD\cos(Ba) - S_0 L^2}{De^{-a/L}}$$

In total, the flux in this system is

$$\boxed{\phi(x) = \begin{cases} A\left(\sin(Bx) + 2BD\cos(Bx)\right), & 0 < x < a \\ AD\left(\frac{\sin(Ba)+2BD\cos(Ba)-S_0L^2}{De^{-a/L}}\right)e^{-x/L} + \frac{S_0L^2}{D}, & x > a \end{cases}}$$

Problem 9

The diffusion equation for a bare spherical reactor is

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + B^2\phi(r) = 0$$

where $B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$. We have neglected the angular components of the Laplacian and used only one dimension due to the symmetry of our system. It can be found that the solution to this differential equation is

$$\phi(r) = \frac{A_1}{r}\sin(Br) + \frac{A_2}{r}\cos(Br).$$

Using our boundary condition that the flux must be finite everywhere ($\phi(r) < \infty$) we see that $A_2 = 0$ or else $\frac{\cos(Br)}{r} \to \infty$ as $x \to 0$. If we let $A = A_1$, then we can use our vacuum boundary condition for bare reactors, that $\phi(\tilde{R}) = 0$ to find B.

$$\phi(r) = \frac{A}{r}\sin(Br)$$

$$\phi(\tilde{R}) = 0 = \frac{A}{\tilde{R}}\sin(B\tilde{R})$$

and so $B\tilde{R} = n\pi$ where n is an integer. Since the flux can never be negative, $n = 1$, and

$$B^2 = \left(\frac{\pi}{\tilde{R}}\right)^2$$

Using our previous equation for B^2 and solving for $\tilde{R}$ as the critical radius of the reactor

$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

$$\tilde{R} = \pi\sqrt{\frac{D}{\nu\Sigma_f - \Sigma_a}}$$

We know that

$$\Sigma_a = \Sigma_a^{D2O} + \Sigma_a^{U235}$$

and we're given $\Sigma_a^{D2O} = 3.3 \times 10^{-5}$ cm^{-1}. We can use this, the provided microscopic cross section of D2O, $\sigma_a^{D2O} = 0.001$ b to find the number density of D2O.

$$N^{D2O} = \Sigma_a^{D2O}/\sigma_a^{D2O} = 3.3 \times 10^{-2}\ \text{cm}^{-3}$$

We are also told that $N^{D20} = 2000N^{U235}$, so

$$N^{U235} = 1.65 \times 10^{-5}\ \text{cm}^{-3}$$

Given the microscopic cross section of U235, $\sigma_a^{U235} = 678$ b, we can find the macroscopic absorption cross section of U235,

$$\Sigma_a^{U235} = N^{U235}\sigma_a^{U235} = (1.65 \times 10^{-5}\ \text{cm}^{-3})(674\ \text{b}) = 1.12 \times 10^{-2}\ \text{cm}^{-1}$$

Summing the two macroscopic cross sections together,

$$\Sigma_a = \Sigma_a^{D2O} + \Sigma_a^{U235}$$

$$\Sigma_a = 3.3 \times 10^{-5}\ \text{cm}^{-1} + 1.12 \times 10^{-2}\ \text{cm}^{-1}$$

$$\Sigma_a \approx 1.12 \times 10^{-2}\ \text{cm}^{-1}$$

Since we are given η_{U235}, the number of neutrons produced by U235 to those absorbed by U235, we can say

$$\eta_{U235} = \frac{\nu\Sigma_f}{\Sigma_a^{U235}}$$

where we note that fission can only occur in U235 so the fission cross section does not explicitly note U235. Using the approximation $\nu = 2.4$,

$$2.06 = \frac{2.4(\Sigma_f)}{1.12 \times 10^{-2} \text{ cm}^{-1}}$$

$$\Sigma_f = 9.613 \times 10^{-3} \text{ cm}^{-1}$$

The only additional piece of information to find is the diffusion coefficient, D. By definition

$$D \equiv \frac{1}{3\Sigma_{tr}}$$

and

$$\Sigma_{tr} \equiv \Sigma_t - \Sigma_a$$

under the assumption that scattering is isotropic. Breaking the transport cross section into its components

$$\Sigma_{tr} = \Sigma_{tr}^{U235} + \Sigma_{tr}^{D2O}$$

$$\Sigma_{tr} = \Sigma_t^{U235} - \Sigma_a^{U235} + \Sigma_{tr}^{D2O}$$

We can find Σ_{tr}^{D2O} from the provided diffusion coefficient,

$$D_{D2O} = \frac{1}{3\Sigma_{tr}^{D2O}}$$

$$\Sigma_{tr}^{D2O} = \frac{1}{3D_{D2O}}$$

If we use the total cross section for U235 at thermal energies, $\sigma_t^{U235} = 699$ b, then

$$\Sigma_{tr} = N^{U235}\sigma_t^{U235} - \Sigma_a^{U235} + \frac{1}{3D_{D2O}}$$

$$\Sigma_{tr} = (1.65 \times 10^{-5} \text{ cm}^{-3})(699 \text{ b}) - 1.12 \times 10^{-2} \text{ cm}^{-1} + \frac{1}{3(0.87 \text{ cm})}$$

$$\Sigma_{tr} = 1.15 \times 10^{-2} \text{ cm}^{-1} - 1.12 \times 10^{-2} \text{ cm}^{-1} + 0.383 \text{ cm}^{-1}$$

$$\Sigma_{tr} \approx 0.383 \text{ cm}^{-1}$$

(we could have also found this by simply setting our total transport cross section equal to the transport cross section of D2O by noting that at thermal energies the total cross section of U235 is approximately equal to the absorption cross section, and so the transport cross section of U235 is almost zero) From this, we find that $D \approx D_{D2O}$, and so the critical radius is

$$\tilde{R} = \pi\sqrt{\frac{D_{D2O}}{\nu\Sigma_f - \Sigma_a}}$$

$$\tilde{R} = \pi\sqrt{\frac{0.87 \text{ cm}}{(2.4)(9.613 \times 10^{-3} \text{ cm}^{-1}) - (1.12 \times 10^{-2} \text{ cm}^{-1})}}$$

$$\tilde{R} = \pi\sqrt{0.73 \times 10^2 \text{ cm}^2}$$

$$\boxed{\tilde{R} = 26.9 \text{ cm}}$$