NE250 Homework 4 **Mitch Negus**
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Problem 1

a.)

We are attempting to prove

$$
\begin{aligned}
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + \frac{\partial(r^2)}{\partial r}\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} + \frac{\partial(1-\mu^2)}{\partial\mu}\psi\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + 2r\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} - 2\mu\psi\right] \\
&= \mu\frac{\partial\psi}{\partial r} + \frac{2\mu}{r}\psi + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} - \frac{2\mu}{r}\psi \\
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu}
\end{aligned}
$$

$\square$

b.)

If we now integrate the streaming operator over $-1 \le \mu \le 1$, we have

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \int_{-1}^{1} d\mu \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\frac{1}{r^2}\int_{-1}^{1} d\mu\, \mu\frac{\partial(r^2\psi)}{\partial r} + \left[\frac{(1-\mu^2)\psi}{r}\right]_{-1}^{1}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\int_{-1}^{1} d\mu\,\mu\,\psi\right).$$

If we note that $\vec{J} = \int_{4\pi} d\hat{\Omega}\,\hat{\Omega}\,\psi = \int_{-1}^{1} d\mu\,\mu\,\psi$ then,

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right)}.$$

c.)

Integrating this expression for the streaming operator over a spherical shell ($r_1 \le r \le r_2$), we have

$$\int_{r_1}^{r_2} 4\pi r^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right) dr$$

$$4\pi \int_{r_1}^{r_2} \frac{\partial}{\partial r}\left(r^2 \vec{J}\right) dr$$

$$4\pi \left[r^2 \vec{J}\right]_{r_1}^{r_2}$$

$$\boxed{4\pi \left[r_2^2 \vec{J}(r_2) - r_1^2 \vec{J}(r_1)\right]}.$$

Problem 2

a.)

The streaming operator can first be defined as

$$\hat{\Omega} \cdot \nabla \equiv \frac{\partial}{\partial s}.$$

We can then expand this for cylindrical geometries as we did for spherical geometries. In addition to being dependent on position, $\vec{r} = (\rho, \theta, z)$, we note that now the angular flux depends on both angular variables describing the velocity, ξ, the cosine of the polar angle, and ω the azimuthal angle.

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

In the one dimensional case, $\frac{\partial \psi}{\partial \theta} = 0$ and $\frac{\partial \psi}{\partial z} = 0$. Furthermore, because our geometry is cylindrical, $\frac{\partial \xi}{\partial s} = 0$ (the particle's direction with respect to $\hat{z}$ remains constant), so our equation for $\frac{\partial \psi}{\partial s}$ simplifies to

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

By analyzing the geometry further, we can determine from figure 1-16 in Lewis & Miller figure that

$$\frac{\partial \rho}{\partial s} = \mu \quad \text{and} \quad \frac{\partial \omega}{\partial s} = -\frac{\eta}{\rho}.$$

Finally, our streaming operator becomes

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega}.$$

With manipulation (and specifically noting that $\mu = \frac{\partial \eta}{\partial \omega}$), we can reach the streaming operator in conservation form.

$$\begin{aligned}
\left[\hat{\Omega} \cdot \nabla\right] \psi &= \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} \\
&= \frac{\mu \rho}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\mu \psi}{\rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} - \frac{\mu \psi}{\rho} \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \mu \psi\right) \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \frac{\partial \rho}{\partial \rho}\psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \frac{\partial \eta}{\partial \omega}\psi\right)
\end{aligned}$$

$$\boxed{\left[\hat{\Omega} \cdot \nabla\right] \psi = \frac{\mu}{\rho}\frac{\partial}{\partial \rho}(\rho \psi) - \frac{1}{\rho}\frac{\partial}{\partial \omega}(\eta \psi)}.$$

b.)

We can use this streaming operator in the transport equation to describe one dimensional cylindrical geometries. For simplicity, we will consider that the transport equation is describing a non-multiplying system. In one-dimensional cylindrical coordinate systems, dependence on position, $\vec{r}$, now only depends on ρ, and velocity direction $\hat{\Omega}$, depends on ω and ξ.

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = s(\rho,\omega,\xi,E,t)$$

Integrating this equation over all angles ($0 \le \omega \le 2\pi$ and $-1 \le \xi \le 1$)

$$\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+ \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t)$$

If we let $\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t) = S(\rho,E,t)$, then

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

Again, consulting our geometry, we find $\mu = \sqrt{1-\xi^2}\cos\omega$ and $\eta = \sqrt{1-\xi^2}\sin\omega$.

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial\rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$- \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{1}{\rho}\frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\cos\omega\; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi \int_0^{2\pi} d\omega\, \frac{\partial}{\partial\omega}\left(\sqrt{1-\xi^2}\sin\omega\; \psi(\rho,\omega,\xi,E,t)\right) +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) +$$
$$-\frac{1}{\rho}\int_{-1}^{1} d\xi\sqrt{1-\xi^2}\left[\sin\omega\; \psi(\rho,\omega,\xi,E,t)\right]_0^{2\pi} +$$
$$+\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

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Problem 3

Problem 4

Problem 5

Problem 6

Problem 7

For cylindrical geometries, the diffusion equation takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(\vec{r})}{\partial r}\right) + \frac{\partial^2 \phi(\vec{r})}{\partial z^2} + B^2\phi(\vec{r}) = 0$$

where $B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$ near criticality. Since our cylinder is infinite we lose dependence on z in our equations.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(r)}{\partial r}\right) + B^2\phi(r) = 0$$

Solutions to this equation are are zeroth order Bessel functions of first and second kind, $J_0(r)$ and $Y_0(r)$. Then,

$$\phi_b(r) = A_1 J_0(Br) + A_2 Y_0(Br).$$

Our boundary conditions demand that the flux is finite at $r = 0$, and so $A_2 = 0$. Letting $A_1 = A$, we have

$$\phi_b(r) = AJ_0(Br).$$

For a bare reactor, boundary conditions force the flux to go to zero at $\tilde{R}_b$,

$$\phi(\tilde{R}_b) = 0,$$

and since the flux must be nonnegative, only the first zero of J_0 can satisfy this condition. This occurs at $B\tilde{R}_b = 2.4048$. $B = B_g = \frac{2.4048}{\tilde{R}_b}$. The criticality condition for the bare core is then

$$\left(\frac{\nu\Sigma_f - \Sigma_a}{D}\right)^2 = \left(\frac{2.4048}{\tilde{R}_b}\right)^2$$

and the critical radius is

$$\tilde{R}_b = \frac{2.4048D}{\nu\Sigma_f - \Sigma_a}.$$

For a reflected core, our initial equation and the form of the solutions stay the same within the core region. With a finite flux at $r = 0$, $A_2 = 0$ and if $A_1 = A$, then in the core

$$\phi_r(r) = AJ_0(Br), \qquad r < R.$$

In the reflected region, we still have no z dependence, but our diffusion equation is just

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(r)}{\partial r}\right) - \frac{1}{L^2}\phi(r) = 0$$

without the dependence on fission. Solutions to this equation are zeroth order modified bessel functions of the first and second kind, $I_0(r)$ and $K_0(r)$. In the reflector

$$\phi_r(r) = C_1 I_0(\frac{r}{L}) + C_2 K_0(\frac{r}{L}), \qquad R < r < a$$

We can now impose the boundary condition that the flux must go to zero at the extrapolated distance in the reflector,

$$\phi_r(R+\tilde{a}) = 0.$$

We find

$$0 = C_1 I_0(\frac{R+\tilde{a}}{L}) + C_2 K_0(\frac{R+\tilde{a}}{L}).$$

$$C_2 = -C_1 \frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}$$

If we let $C_1 = C$, then in the reflector

$$\phi_r(r) = CI_0(\frac{r}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{r}{L}), \qquad R < r < a$$

Finally, we use our boundary condition that the flux must be equal at the boundaries.

$$\phi_r(R) = AJ_0(BR) = CI_0(\frac{R}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{R}{L})$$

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Problem 8

The one-dimensional, single energy diffusion equation in each of the material regions are as follows:

Multiplying Core:

$$\frac{d\phi(x)}{dx} - B^2\phi(x) = 0$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Uniform Source Reflector: Multiplying Core:

$$\frac{d\phi(x)}{dx} - B^2\phi(x) = 0$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Problem 9