

## Nuclear Reactions

Two types

### 1. Spontaneous (*decay*)

- $\alpha$ :  ${}_Z^AX \rightarrow {}_{Z-2}^{A-4}X + {}_2^4\alpha$
- $\beta$ :  ${}_Z^AX \rightarrow {}_{Z+1}^AX + \beta + \bar{\nu}$
- $\gamma$ :  $X^* \rightarrow X + \gamma$

### Decay Equations

$$\begin{aligned} t &\rightarrow N(t) & dN(t) &= N(t+dt) - N(t) \\ t+dt &\rightarrow N(t+dt) & dN(t) &= -\lambda N(t)dt \text{ where } \lambda \text{ is the decay constant.} \end{aligned}$$

... working through, with B.C.  $N(t=0) = N_0$  ...

$$N(t) = N_0 e^{-\lambda t}$$

### Mean Lifetime:

$\frac{dN(t)}{N_0} = \lambda e^{-\lambda t} dt = p_d(t)$  where  $p_d(t)dt$  is the probability of decay in time  $dt$

$$\bar{t} = \int_0^\infty t p(t) dt = \frac{1}{\lambda}$$

### Half-Life:

$T_{1/2}$  is defined as the time s.t.  $N(T_{1/2}) = \frac{N_0}{2}$

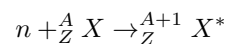
$$T_{1/2} = \frac{\ln 2}{\lambda}$$

### 2. Induced (*projectile/target*)

*This will be the emphasis of NE 250*

#### Neutron-Nucleus

- elastic scattering
- inelastic scattering/compound nuclear reaction



(could result in production of a  $\gamma$  or  $\alpha$ , emission of a  $n$ , or fission)

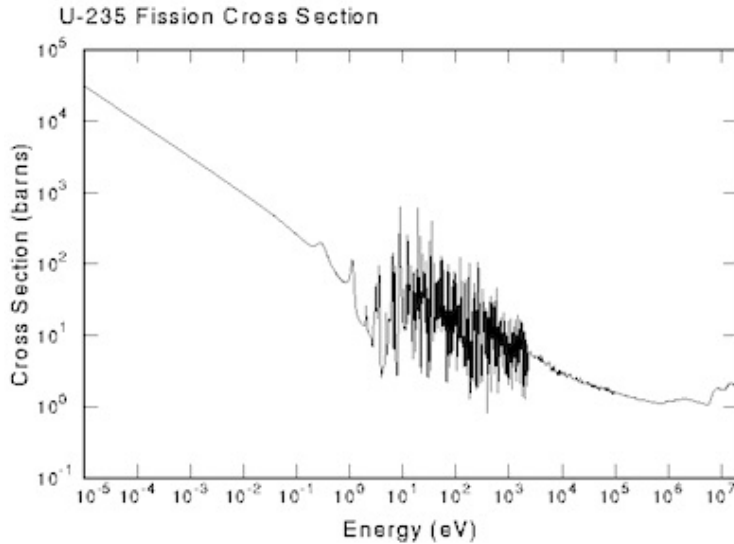
**Capture is a subset of absorption!**

Absorption:  $(n, \alpha), (n, \gamma), (n, f)$

Capture:  $(n, \gamma)$

### Cross Sections ( $\sigma$ )

- A property of the isotope and reaction
- A function of the isotope temperature (vibrational motion) and neutron speed (linear motion)
- Tabulated in XSec libraries (**3 common formats**)
  - ENDF (USA)
  - JEFF (Europe/NEA)
  - JENDL (Japan)



- Resonances in X-Sec plots due to excited energy levels that can be reached; nuclei only all excitation to these levels, and so only neutrons with this energy amount will be absorbed
- Cross sections measured at 300K (room temp); calculated using

$$E = k_B T$$

where  $k_B$  is Boltzmann's constant and  $T = 300 \text{ K} \Rightarrow E = 0.0253 \text{ eV}$

- Higher temperatures cause resonant peak widths to broaden (less time spent near center of vibrational trajectory)  $\rightarrow$  **Doppler Broadening**

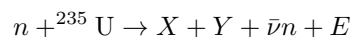
### Units

$$1 \text{ barn} = 10^{-24} \text{ cm}^2$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

### Fission

Can be spontaneous or induced



$\bar{\nu}$  is the average number of neutrons produced in a given fission event.

$E_f \approx 200 \text{ MeV}$  (this is much higher than chemical reactions which are on the order of eV!)

## Fissile Isotopes

$$E_b > E_{\text{threshold}}$$

These neutrons could (almost) be considered as "able to fission from 0 KE neutrons."

Includes  $^{235}\text{U}$ ,  $^{233}\text{U}$ ,  $^{239}\text{Pu}$ ,  $^{241}\text{Pu}$

## Fissionable Isotopes

Fission requires collision with high  $E$  neutrons.

For  $^{235}\text{U}$  this is empirically given by

$$\chi(E) = 0.453e^{-1.036E} \sinh(\sqrt{2.29E})$$

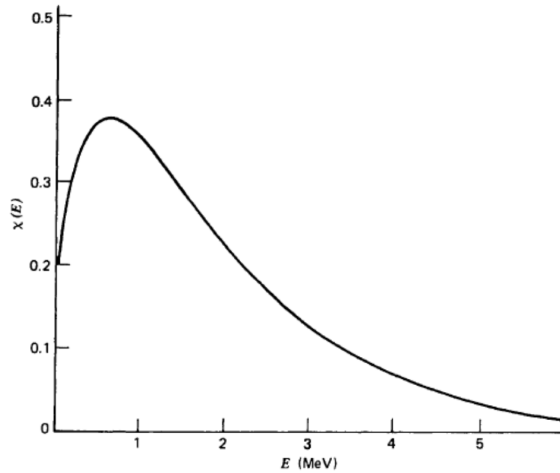


Figure 4: Fission spectrum for thermal neutron induced fission in  $^{235}\text{U}$

Also, note that  $\bar{\nu}$  depends on the isotope. Below is a plot of  $\bar{\nu}$  for  $^{239}\text{Pu}$ ,  $^{233}\text{U}$  and  $^{235}\text{U}$ :

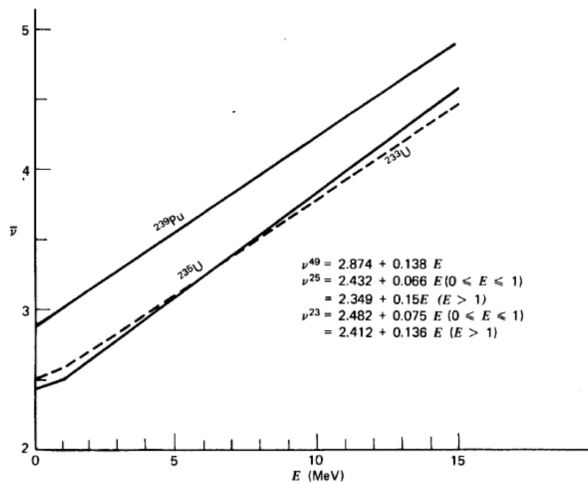


Figure 3: average # of neutrons released per fission as a function of energy

## Fertile Isotopes

Isotopes which either undergo neutron capture (and subsequent decay) to become fissile isotopes.

## Energy breakdown of fission outputs

- 180 MeV in the KE of fission products
- 5 MeV in the kinetic energy of neutrons
- 7 MeV in prompt  $\gamma$ s
- 8 MeV in  $\beta^-$  decay of fission products
- 7 MeV in delayed  $\gamma$ s
- 12 MeV in neutrinos

The energy from all outputs can be captured except for neutrinos.

## Criticality

### Multiplication Factor, $k$

$$k = \frac{\# \text{ neutrons generated}}{\# \text{ neutrons lost}}$$

# neutrons generated = neutrons fission

# neutrons lost = # neutrons absorbed + # neutrons leaked

- $k = 1$ : the reaction is critical; the chain reaction is controlled (reactor)
- $k < 1$ : the reactor is subcritical; boring
- $k > 1$ : the reaction is supercritical; this is a bomb

## Derivation of the Neutron Transport Equation

Solving for the multiplication factor requires that we know:

1.  $n$ : neutron density [ $n/\text{cm}^3$ ]
2.  $N$ : atom/nuclide density [ $\text{nuclei}/\text{cm}^3$ ]
3.  $\sigma$ : microscopic cross section [ $\text{cm}^2$ ]

**Reaction rate:**  $R = nN\sigma$

**Macroscopic cross section** [ $1/\text{cm}$ ]:  $\Sigma = N\sigma$

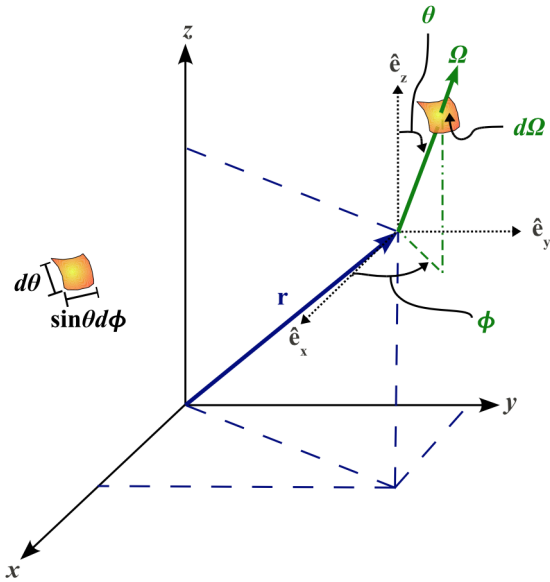
**Angular neutron density** [ $\frac{n}{\text{cm}^3 \cdot \text{sr}}$ ]:  $n(\vec{r}, E, \hat{\Omega}, t)$

$\vec{v} = v\hat{\Omega}$ ,  $|\hat{\Omega}| = 1$  (describes a sphere, formed by  $\theta$  and  $\phi$ )

$dr^3 = dx dy dz$

$dE$

$d\hat{\Omega} = \sin\theta d\theta d\phi$ ;  $d\hat{\Omega}$  is a scalar, about the original position defined by vector  $\hat{\Omega}$ .



Altogether,  $n(\vec{r}, E, \hat{\Omega}, t) d^3r dE d\hat{\Omega}$ , gives the # of neutrons in the small volume about  $\vec{r}$  with energy  $E$ , and moving in direction  $d\hat{\Omega}$  about  $\hat{\Omega}$  at time  $t$ .

**Angular neutron flux (scalar):**  $\phi(\vec{r}, E, \hat{\Omega}, t) = v n(\vec{r}, E, \hat{\Omega}, t)$

**Angular neutron current (vector):**  $\vec{J}(\vec{r}, E, \hat{\Omega}, t) = \hat{\Omega} \phi(\vec{r}, E, \hat{\Omega}, t)$

We can find the number of neutrons in a volume  $V$  using

$$\int_V n(\vec{r}, E, \hat{\Omega}, t) d^3r$$

Change with time is then

$$\frac{\partial}{\partial t} \left[ \int_V n(\vec{r}, E, \hat{\Omega}, t) d^3r \right] dE d\hat{\Omega} = \# \text{ neutrons gained} - \# \text{ neutrons lost}$$

# neutrons gained: source (fission), in-scattering ( $E', \hat{\Omega}' \rightarrow E, \hat{\Omega}$ )

# neutrons lost: absorption, scattering ( $E, \hat{\Omega} \rightarrow E', \hat{\Omega}'$ )

We also add a streaming term, to quantify neutrons leaking out (and in) to the system.

The chance of a collision in the system is given by

$$\left[ \int_V \Sigma_{\text{tot}}(\vec{r}, E) v n(\vec{r}, E, \hat{\Omega}, t) d^3r \right] dE d\hat{\Omega}$$

The chance of fission in system...

(See paper notes...)

## The Transport Equation

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi}{\partial t} + \hat{\Omega} \cdot \nabla \psi + \Sigma_t \psi &= \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) \psi(E', \hat{\Omega}') \\ &+ \frac{\chi(E)}{4\pi} \int_0^\infty dE' \nu(E') \Sigma_f(E') \int_{4\pi} d\hat{\Omega}' \phi(\vec{r}, E', \hat{\Omega}', t) + s(\vec{r}, E, \hat{\Omega}, t) \end{aligned}$$

**Initial condition:**  $\psi(\vec{r}, E, \hat{\Omega}, 0) = \psi_0(\vec{r}, E, \hat{\Omega})$

**Interface condition:** angular flux must be continuous at all points

**Other conditions**

**Fixed condition:** incoming flux is specified  $\psi(\vec{r}_s, E, \hat{\Omega}, t) = \psi_{\text{in}}(\vec{r}, E, \hat{\Omega}, t)$   
(Vacuum or black if  $\psi_{\text{in}}(\vec{r}, E, \hat{\Omega}, t) = 0$ )

**Reflective conditions:** mirror symmetry at some surface,  $\psi(\hat{\Omega}_{\text{in}}, t) = \psi(\hat{\Omega}_{\text{out}}, t)$

**Periodic conditions:**  $\psi(\vec{r}_s) = \psi(\vec{r}_s + \vec{p})$

**Finiteness conditions:** (can't have infinite flux)  $0 \leq \psi(\vec{r}, E, \hat{\Omega}, t) < \infty$

**Source condition:** localized (pt.) sources introduce mathematical singularities  
 $S(\vec{r}, E, \hat{\Omega}, t) = \lim_{\vec{r} \rightarrow \vec{r}_0} \int dS \hat{e} \cdot \hat{\Omega}$

## Approximations to the Transport Equation

### One-speed Transport Equation

Assume all particles are the same speed:  $\vec{v} = v_0 \cdot \hat{\Omega}$

The equation becomes

$$\frac{1}{v} \frac{\partial \psi(\vec{r}, \hat{\Omega}, t)}{\partial t} + \hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t) + \Sigma_t \psi(\vec{r}, \hat{\Omega}, t) = \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t) + \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t) + S(\vec{r}, \hat{\Omega}, t)$$

### One-dimensional

$\vec{r} = (x, y, z)$

$d\hat{\Omega} = \sin\theta \, d\theta \, d\varphi = d\mu \, d\varphi$ , where  $\mu = \cos\theta$

$$\frac{1}{v} \frac{\partial \psi(x, \hat{\Omega}, t)}{\partial t} + \Omega_x \frac{\partial \psi(x, \hat{\Omega}, t)}{\partial x} + \Sigma_t(x) \psi(x, \hat{\Omega}, t) = \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(x, \hat{\Omega}', t) + \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(x, \hat{\Omega}', t) + S(x, \hat{\Omega}, t)$$

## The Diffusion Equation

Usually the scalar flux is all that is needed to get a fairly accurate picture of our system. Reaction rates usually only depend on the neutron flux, not the direction of neutron motion.

**Assume** that the angular flux depends only weakly on direction.

Recall:

$$\begin{aligned} \phi(\vec{r}, t) &= \int_{4\pi} d\hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) \\ \vec{J}(\vec{r}, t) &= \int_{4\pi} d\hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) \end{aligned}$$

## The Neutron Continuity Equation

### “The Zeroth Moment of the Transport Equation”:

Integrate transport equation over all angles

$$\int_{4\pi} d\hat{\Omega} \left[ \underbrace{\frac{1}{v} \frac{\partial \psi(\vec{r}, \hat{\Omega}, t)}{\partial t}}_1 + \underbrace{\hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t)}_6 + \underbrace{\Sigma_t \psi(\vec{r}, \hat{\Omega}, t)}_2 \right] = \int_{4\pi} d\hat{\Omega} \left[ \underbrace{\int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}, t)}_5 + \underbrace{\frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t)}_3 + \underbrace{s(\vec{r}, \hat{\Omega}, t)}_4 \right]$$

1. **Time:** No approximations in time

$$\frac{1}{v} \frac{\partial}{\partial t} \int_{4\pi} d\hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) = \frac{1}{v} \frac{\partial}{\partial t} \phi(\vec{r}, t)$$

2. **Absorption:** No approximations in absorption

$$\Sigma_t \int_{4\pi} d\hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) = \Sigma_t \phi(\vec{r}, t)$$

3. **Fission:** No approximations in fission

$$\int_{4\pi} d\hat{\Omega} = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi = 4\pi$$

$$\int_{4\pi} d\hat{\Omega} \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t) = \nu \Sigma_f \phi(\vec{r}, t)$$

4. **Source:** No approximations in source

$$\int_{4\pi} d\hat{\Omega} s(\vec{r}, \hat{\Omega}, t) \equiv S(\vec{r}, t)$$

5. **Scattering:** For scattering, interchange the order of integration

$$\int_{4\pi} d\hat{\Omega} \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t) = \int_{4\pi} d\hat{\Omega}' \int_{4\pi} d\hat{\Omega} \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t)$$

Now we assume that scattering is azimuthally symmetric (scattering depends only on cosine). The particle is as likely to scatter at angle  $\theta$  in any direction off  $\hat{\Omega}$ .

$$\int_{4\pi} d\hat{\Omega} \Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) = 2\pi \int_{-1}^1 d\mu \Sigma_s(\mu) = \Sigma_s$$

Then, if we substitute this in above

$$\Sigma_s \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t) = \Sigma_s \phi(\vec{r}, t)$$

6. **Streaming:** To adjust streaming, we first manipulate the order

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t) = \nabla \cdot \int_{4\pi} d\hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) = \nabla \cdot \vec{J}(\vec{r}, t)$$

Put all the above integrations together to get the **neutron continuity equation (NCE)**. Notice that we have three equations (each  $\vec{r}$  has  $x, y, z$  components) and 4 unknown quantities.

$$\frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} + \nabla \cdot \vec{J}(\vec{r}, t) + \Sigma_t \phi(\vec{r}, t) = \Sigma_s \phi(\vec{r}, t) + \nu \Sigma_f \phi(\vec{r}, t) + S(\vec{r}, t)$$

## First angular moment

Multiply the TE by  $\hat{\Omega}$  and integrate. We will drop the fission term (technically we will assume it is part of the source), though the procedure is nearly the same when it is included.

Note:

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} = 0$$

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} = \frac{4\pi}{3} \bar{\bar{I}}, \quad \bar{\bar{I}} \text{ is the identity tensor:}$$

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega}_i \hat{\Omega}_j = \begin{cases} 0, & i \neq j \\ \frac{4\pi}{3}, & i = j \end{cases}$$

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \hat{\Omega} = 0$$

When multiplied out, we have

$$\begin{aligned} & \int_{4\pi} d\hat{\Omega} \hat{\Omega} [\text{TE}] \\ &= \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \frac{1}{v} \frac{\partial \psi(\vec{r}, \hat{\Omega}, t)}{\partial t}}_1 + \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \nabla \psi(\vec{r}, \hat{\Omega}, t)}_5 + \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \Sigma_t \psi(\vec{r}, \hat{\Omega}, t)}_2 \\ &= \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \int_{4\pi} d\hat{\Omega}' X s_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t)}_4 + \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t)}_3 + \int_{4\pi} d\hat{\Omega} \hat{\Omega} S(\vec{r}, \hat{\Omega}, t) \end{aligned}$$

Like we did to derive the continuity equation, we can break down each term.

1. **Time:**

$$\frac{1}{v} \frac{\partial}{\partial t} \int_{4\pi} d\hat{\Omega}' \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) = \frac{1}{v} \frac{\partial \vec{J}}{\partial t}$$

2. **Absorption:**

$$\Sigma_t \int_{4\pi} d\hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) = \Sigma_t \vec{J}(\vec{r}, t)$$

3. **Source (fission now absent):**

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} S(\vec{r}, \hat{\Omega}, t) \equiv S(\vec{r}, t)$$

4. **Scattering:** Expand scattering cross section in Legendre Polynomials (a sequence of orthogonal polynomials)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

Expand  $\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega})$  in Legendre polynomials

$$\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) = \sum_0^{\infty} \frac{2\ell + 1}{4\pi} \Sigma_{s\ell} P_\ell(\hat{\Omega}') P_\ell(\hat{\Omega})$$

•  $\ell = 0$  is isotropic

$$P_0(\hat{\Omega}) = 1 \quad \Rightarrow \quad \Sigma_s(\hat{\Omega}', \hat{\Omega}) \approx \frac{1}{4\pi} \Sigma_{s0}$$

•  $\ell = 1$  is linearly anisotropic

$$P_1(\hat{\Omega}) = \hat{\Omega} \quad \Rightarrow \quad \Sigma_s(\hat{\Omega}', \hat{\Omega}) \approx \frac{1}{4\pi} (\Sigma_{s0} + 3\hat{\Omega}' \cdot \hat{\Omega} \Sigma_{s1})$$

Assume scattering is at most linearly anisotropic (if its not, there's some "weird" stuff going on")



Substitute the linearly anisotropic approximation into the expression for streaming. We note the previously defined identities and definition of neutron current:

$$\begin{aligned} \frac{1}{4\pi} \int_{4\pi} d\hat{\Omega} \hat{\Omega} \int_{4\pi} d\hat{\Omega} (\Sigma_{s0} + 3\hat{\Omega}' \cdot \hat{\Omega} \Sigma_{s1}) \psi(\vec{r}, \hat{\Omega}, t) &= \frac{1}{4\pi} \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega}}_0 \int_{4\pi} \hat{\Omega}' \Sigma_s \psi(\vec{r}, \hat{\Omega}', t) \\ &+ \frac{1}{4\pi} \underbrace{\int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega}}_{4\pi/3\bar{I}} \underbrace{\int_{4\pi} d\hat{\Omega}' \hat{\Omega}' 3\Sigma_{s1}}_{3\Sigma_{s1}\bar{J}(\vec{r}, t)} \psi(\vec{r}, \hat{\Omega}', t) \end{aligned}$$

Substituting those identities and simplifying, we get

$$\left(\frac{1}{4\pi}\right) \left(\frac{4\pi}{3\bar{I}}\right) (3\Sigma_{s1}\bar{J}(\vec{r}, t)) = \Sigma_{s1} J(\vec{r}, t)$$

#### 5. Streaming:

$$\int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t) = \nabla \cdot \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t)$$

Putting each of these 5 components back together the **current continuity equation (CCE)** is

$$\frac{1}{v} \frac{\partial \bar{J}}{\partial t} + \nabla \cdot \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) + \Sigma_t \bar{J}(\vec{r}, t) = \Sigma_{s1} \bar{J}(\vec{r}, t) + S_1(\vec{r}, t)$$

Now we have 2 moment equations (zeroth and first) so a total of 4 equations (neutron continuity and 3 tensor equations). There are now 10 unknowns  $\phi$  (1),  $\bar{J}$  (3), and the new tensor term (6). There is little point to continuing with this technique any longer.

### Angular Flux Approximation

Assume now that the flux is at most linearly anisotropic (for the current continuity equation we assumed that *scattering* was linearly anisotropic).

$$\psi(\hat{\Omega}) \approx \frac{1}{4\pi} (\psi_0 + 3\hat{\Omega} \cdot \vec{\psi}_1)$$

Substitute expansion into the streaming term of the current continuity equation.

$$\begin{aligned} \nabla \cdot \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \psi(\vec{r}, \hat{\Omega}, t) &= \nabla \cdot \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \frac{1}{4\pi} (\psi_0 + 3\hat{\Omega} \cdot \vec{\psi}_1) \\ \nabla \cdot \frac{1}{4\pi} \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} (\psi_0 + 3\hat{\Omega} \cdot \vec{\psi}_1) &= \nabla \cdot \frac{1}{4\pi} \left[ \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \psi_0 + 3 \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} \hat{\Omega} \cdot \vec{\psi}_1 \right] \end{aligned}$$

Using the same identities as before, we have

$$\nabla \cdot \frac{1}{4\pi} \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} (\psi_0 + 3\hat{\Omega} \cdot \vec{\psi}_1) = \nabla \cdot \frac{1}{4\pi} \frac{4\pi}{3} \bar{I} \phi(\vec{r}, t)$$

(Note, that this assumes  $\psi_0 = \phi \leftarrow$  figure out where this comes from) and

$$\nabla \cdot \frac{1}{4\pi} \int_{4\pi} d\hat{\Omega} \hat{\Omega} \hat{\Omega} (\psi_0 + 3\hat{\Omega} \cdot \vec{\psi}_1) = \frac{1}{3} \nabla \phi(\vec{r}, t)$$

Now the current continuity equation is

$$\frac{1}{v} \frac{\partial \bar{J}}{\partial t} + \frac{1}{3} \nabla \phi(\vec{r}, t) + \Sigma_t \bar{J}(\vec{r}, t) = \Sigma_{s1} \bar{J}(\vec{r}, t) + S_1(\vec{r}, t)$$

Define absorption and transport cross sections:

$$\Sigma_a \equiv \Sigma_t - \Sigma_{s0}$$

$$\Sigma_{tr} \equiv \Sigma_t - \Sigma_{s1}$$

Using these cross sections we can reform both the neutron and current continuity equations.

**NCE:**

$$\frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} + \nabla \cdot \vec{J}(\vec{r}, t) + \Sigma_a \phi(\vec{r}, t) = S(\vec{r}, t)$$

**CCE:**

$$\frac{1}{v} \frac{\partial \vec{J}}{\partial t} + \frac{1}{3} \nabla \phi(\vec{r}, t) + \Sigma_{tr} \vec{J}(\vec{r}, t) = S_1(\vec{r}, t)$$

**Fick's Law**

With the following conditions

- Steady state ( $\frac{\partial}{\partial t}(X) = 0$ )
- Isotropic source ( $S_1 = 0$ )

The current continuity equation becomes

$$\frac{1}{3} \nabla \phi(\vec{r}) + \Sigma_{tr} \vec{J}(\vec{r}) = 0$$

which we can solve for  $\vec{J}$ :

$$\vec{J}(\vec{r}) = -\frac{1}{3\Sigma_{tr}} \nabla \phi(\vec{r}).$$

If we let  $D = \frac{1}{3\Sigma_{tr}} = \frac{1}{3(\Sigma_t - \Sigma_{s1})}$  be the diffusion coefficient, then this simplifies further to

$$\vec{J}(\vec{r}) = -D \nabla \phi(\vec{r})$$

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Recall:

$$\Sigma_{s1} = \int d\hat{\Omega} \hat{\Omega} \Sigma_s$$

Include azimuthally symmetric assumptions and  $\Sigma_{s1} = \bar{\mu}_0 \Sigma_s$ , where  $\bar{\mu}_0$  = average scattering cosine =  $\frac{2}{3A}$ , where  $A$  is the atomic mass number.

Then  $\Sigma_{tr} = \Sigma_t - \bar{\mu}_0 \Sigma_s$ .

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From this implementation of Fick's law, we can write the diffusion equation as only a function of  $\phi$ :

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(\vec{r}, t) - \nabla \cdot D \nabla \phi(\vec{r}, t) + \Sigma_a \phi(\vec{r}, t) = \nu \Sigma_f \phi(\vec{r}, t) + S(\vec{r}, t)$$

Due to our assumptions, however, the diffusion equation is not valid at

1. Boundaries/interfaces
2. Sources
3. Strong absorbers
4. Voids

The angular flux expansion can then be written as

$$\psi(\vec{r}, \hat{\Omega}, t) \approx \frac{1}{4\pi} \left( \phi(\vec{r}, t) - \frac{1}{\Sigma_{tr}} \nabla \phi(\vec{r}, t) \right)$$

At equilibrium,  $\vec{J} = 0$  (net migration is opposed to gradient, hence negative into Fick's Law)

**Mean Free Path:** the median distance from the last collision

$$\lambda_t = \frac{1}{\Sigma_t} \text{ or } \lambda_{tr} = \frac{1}{\Sigma_{tr}}$$

If scattering is

- isotropic  $\rightarrow \bar{\mu}_0 = 0, \lambda_t = \lambda_{tr}$
- forward peaked  $\rightarrow \bar{\mu}_0 > 0, \lambda_t > \lambda_{tr}$

**Initial Conditions:**  $\phi(\vec{r}, 0) = \phi(\vec{r}) \forall \vec{r} \in V$

Basic requirements:

- real and nonnegative ( $\phi \geq 0$ )
- bounded ( $\phi < \infty$ )

**Interface conditions:**

-Zeroth and First moments must be continuous

$$\begin{aligned} \phi_1(\vec{r}, t) &= \phi_2(\vec{r}, t) \\ \vec{J}_1(\vec{r}, t) &= \vec{J}_2(\vec{r}, t) \end{aligned}$$

**Vacuum BCs:**

In the DE use partial current

$$\begin{aligned} J_-(\vec{r}, t) &= \int_{2\pi^-} d\hat{\Omega} \hat{\Omega} \cdot \hat{e}_s \psi(\vec{r}, \hat{\Omega}, t) = 0 \\ J_+(\vec{r}, t) &= \int_{2\pi^+} d\hat{\Omega} \hat{\Omega} \cdot \hat{e}_s \psi(\vec{r}, \hat{\Omega}, t) = 0 \end{aligned}$$

$$J_{\mp} = \int_{2\pi^-} d\hat{\Omega} \hat{\Omega} \hat{e}_s \left( \frac{1}{4\pi} \left( \phi \pm \frac{1}{\Sigma_{tr}} \nabla \phi \right) \right) \approx \frac{1}{4} \phi \pm \frac{D}{2} \hat{e}_s \cdot \nabla \phi = 0$$

In 1D problems  $J_-(\vec{r}_s, t) = J_-(x_s, t) = \frac{1}{4} \phi(x_s, t) + \frac{D}{2} \frac{d\phi}{dx} \Big|_{x_s} = 0$

The extrapolation distance, where  $\phi = 0$  is

$$\tilde{x}_s = x_s + 2D = x_s + \frac{2}{3} \lambda_{tr}$$

and so replace  $J_-(x_s) = 0$  with  $\phi(\tilde{x}_s) = 0$ .

## Helmholtz Form of the Diffusion Equation

$$\nabla^2 \phi - \frac{1}{L^2} \phi = \frac{-Q}{D}$$

where  $L \equiv \sqrt{\frac{D}{\Sigma_a}}$

The solution to this equation is  $\phi = \phi_H(\vec{r}) + \phi_P(\vec{r})$ , where the homogeneous solution is  $\phi_H = Ae^{\frac{-|\vec{r}|}{L}} + Be^{\frac{|\vec{r}|}{L}}$

## Criticality

For the equation

$$\nabla \cdot D \nabla \phi + \Sigma_a \phi = \nu \Sigma_f \phi, \phi(\pm \tilde{x}_s) = 0.$$

There is no general solution unless we get it exactly right.

Introduce  $k$ . For any  $k$ , there is always a solution. We use an iterative process to find  $k$ . You might consider  $k$  as tuning the neutrons produced per fission.

$$\nabla \cdot D \nabla \phi + \Sigma_a \phi = \frac{1}{k} \nu \Sigma_f \phi$$

This causes our problem to become an eigenvalue problem. At long times, the non-negative largest, real eigenvalue is the dominant mode (eigenvalue  $k$ ; eigenvector  $\phi$ ).

## Prompt and Delayed Neutrons

Fission produces

- Prompt  $n$ 's (within  $10^{-10}$  s of fission)
- Fission Products (FPs)

*Delayed neutrons* come from the decay of FPs. For example  $^{87}\text{Br}$  has a half-life of 55.9 s.

Use  $T_{1/2}$  to bin delayed neutrons into either 6 or 8 groups (generally). Define the group index as  $j$ .

**Decay constant of group  $j$ :**  $\lambda_j^d$

**Delayed neutron fraction:**  $\beta_j$

**Reactivity:**  $\rho = \frac{k-1}{k}$

$0 < \rho < \beta$  : delayed supercritical

$\rho = 0$  : critical

$\rho > \beta$  : prompt supercritical

$\rho = \beta$  : 1\$

The total delayed neutron fraction ( $\beta$ ) from thermal fission in  $^{235}\text{U}$  is 0.0065, and the average neutron lifetime is 0.1 s.

**Critical:** Prompt + Delayed holds criticality

**Prompt Critical:** Critical only on prompt  $n$ 's

**Prompt Supercritical:**  $k > 1$  from only prompt  $n$ 's

### Thought experiment:

$\ell$  = mean  $n$  generation time

$$n(t + \ell) = n(t) + \ell \frac{dn}{dt} = kn(t)$$

We can solve for the rate of neutron change

$$\frac{dn}{dt} = \left( \frac{k-1}{\ell} \right) n(t),$$

and then solve the differential equation

$$n(t) = n_0 e^{\frac{(k-1)t}{\ell}}$$

The reactor period is given by  $T \equiv \frac{\ell}{k-1}$ .

If a reactor has  $k = 1.005$ , at  $t = 1$  s, what is the ratio of  $n(t)$  to  $n_0$  for

a.)  $\ell = 0.1$  s

$$n(1\text{s})/n_0 = e^{(1.0005-1)(1\text{s})/0.1\text{s}}$$

$$\boxed{n(1\text{s})/n_0 = 1.005}$$

b.)  $\ell = 1 \times 10^{-4}$  s

$$n(1\text{s})/n_0 = e^{(1.0005-1)(1\text{s})/(1 \times 10^{-4})\text{s}}$$

$$\boxed{n(1\text{s})/n_0 = 148.4}$$

## Diffusion Equation and the $P_N$ equations

We assumed that the angular flux is linearly anisotropic ( $P_1$  expansion), and it is valid away from boundaries, sources, sinks, and voids. Other assumptions included

- one speed
- isotropic source
- azimuthally symmetric, linearly anisotropic scattering
- $n$  current changes slowly compared to the mean collision time ( $\frac{1}{|\vec{J}(\vec{r}, t)|} \frac{\partial \vec{J}}{\partial t} \ll v \Sigma_t$ )

Fick's Law and the 1-speed Diffusion Equation gave us

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = S(\vec{r}, t) = \Sigma_a(\vec{r}) \phi(\vec{r}, t) + \nabla \cdot D(\vec{r}) \nabla \phi(\vec{r}, t)$$

Start with the energy depend

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = S(\vec{r}, E, t) + \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E) \phi - \Sigma_t(\vec{r}) \phi(\vec{r}, E) - \nabla \cdot \vec{J}(\vec{r}, E, t)$$

We apply assumptions

1. Slow current change
2. Isotropic source,  $S = 0$
3. Isotropic scattering,  $\Sigma_{s1}(E' \rightarrow E) = 0$

Condition #3 is usually too strong, so we introduce a different diffusion coefficient.

$$D(\vec{r}, E) = \frac{1}{3} \left[ \Sigma_t(\vec{r}, E) - \frac{\int_0^\infty \Sigma_{s1}(\vec{r}, E' \rightarrow E) J_i}{J_i(\vec{r}, E, t)} \right]^{-1}$$

where  $J_i$  denotes the current in  $i = x, y, z$ . This gives

...

This isn't great, so we neglect anisotropic contribution to energy transfer in scattering.

..

$$\int_0^\infty dE' \Sigma_{s1}(\vec{r}, E) \delta(E' \rightarrow E) J_i(\vec{r}, E, t) = \bar{\mu}_0 \Sigma_s(\vec{r}, E) J_i(\vec{r}, E, t)$$

..

which, when plugged into the neutron continuity equation gives

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = S(\vec{r}, E, t) + \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E) \phi(\vec{r}, E, t) - \Sigma_t(\vec{r}, E) \phi(\vec{r}, E, t) + \nabla \cdot D(\vec{r}, E) \nabla \phi(\vec{r}, E, t)$$

↑ check that

Before, we had just 1 speed. Now we have 1 group (a set of different speeds in 1 bin). To get the group, we integrate the equation over energy, and we weight based on the flux.

\*Note this requires we solve for the flux, based on the flux. We often make estimations of the flux shape.

The group constants:

The effective group cross section,  $\Sigma_{t,1}(\vec{r}) = \frac{\int_0^\infty dE \Sigma_t(\vec{r}, E) \phi(\vec{r}, E, t)}{\int_0^\infty dE \phi(\vec{r}, E, t)}$   $\phi_1(\vec{r}, t) = \int_0^\infty dE \phi(\vec{r}, E, t)$

...

Integrate the source:  $\int_0^\infty dE S(\vec{r}, E, t) = S(\vec{r}, t)$

$$\int_0^\infty dE \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E) \phi(\vec{r}, E', t) = \int_0^\infty dE' \Sigma(\vec{r}) \phi(\vec{r}, E', t) = \Sigma_{s,1} \phi_1$$

...

The one group diffusion equation is then

$$\frac{1}{v} \frac{\partial \phi_1}{\partial t} = S_1(\vec{r}, t) - \Sigma_a(\vec{r}) \phi_1(\vec{r}, t) + \nabla \cdot D_1(\vec{r}) \nabla \phi_1(\vec{r}, t)$$