NE250 Homework 4 **Mitch Negus**
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Problem 1

a.)

We are attempting to prove

$$\begin{aligned}
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + \frac{\partial(r^2)}{\partial r}\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} + \frac{\partial(1-\mu^2)}{\partial\mu}\psi\right] \\
&= \frac{\mu}{r^2}\left[r^2\frac{\partial\psi}{\partial r} + 2r\psi\right] + \frac{1}{r}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu} - 2\mu\psi\right] \\
&= \mu\frac{\partial\psi}{\partial r} + \frac{2\mu}{r}\psi + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} - \frac{2\mu}{r}\psi \\
\mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu} &= \mu\frac{\partial\psi}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi}{\partial\mu}
\end{aligned}$$

$\square$

b.)

If we now integrate the streaming operator over $-1 \le \mu \le 1$, we have

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\int_{-1}^{1} d\mu \frac{\mu}{r^2}\frac{\partial(r^2\psi)}{\partial r} + \int_{-1}^{1} d\mu \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)\psi}{r}\right]$$

$$\frac{1}{r^2}\int_{-1}^{1} d\mu\, \mu\frac{\partial(r^2\psi)}{\partial r} + \left[\frac{(1-\mu^2)\psi}{r}\right]_{-1}^{1}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\int_{-1}^{1} d\mu\,\mu\,\psi\right).$$

If we note that $\vec{J} = \int_{4\pi} d\hat{\Omega}\,\hat{\Omega}\,\psi = \int_{-1}^{1} d\mu\,\mu\,\psi$ then,

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right)}.$$

c.)

Integrating this expression for the streaming operator over a spherical shell ($r_1 \le r \le r_2$), we have

$$\int_{r_1}^{r_2} 4\pi r^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{J}\right) dr$$

$$4\pi \int_{r_1}^{r_2} \frac{\partial}{\partial r}\left(r^2 \vec{J}\right) dr$$

$$4\pi \left[r^2 \vec{J}\right]_{r_1}^{r_2}$$

$$\boxed{4\pi \left[r_2^2 \vec{J}(r_2) - r_1^2 \vec{J}(r_1)\right]}.$$

Problem 2

a.)

The streaming operator can first be defined as

$$\hat{\Omega} \cdot \nabla \equiv \frac{\partial}{\partial s}.$$

We can then expand this for cylindrical geometries as we did for spherical geometries. In addition to being dependent on position, $\vec{r} = (\rho, \theta, z)$, we note that now the angular flux depends on both angular variables describing the velocity, ξ, the cosine of the polar angle, and ω the azimuthal angle.

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

In the one dimensional case, $\frac{\partial \psi}{\partial \theta} = 0$ and $\frac{\partial \psi}{\partial z} = 0$. Furthermore, because our geometry is cylindrical, $\frac{\partial \xi}{\partial s} = 0$ (the particle's direction with respect to $\hat{z}$ remains constant), so our equation for $\frac{\partial \psi}{\partial s}$ simplifies to

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial s} + \frac{\partial \psi}{\partial \omega}\frac{\partial \omega}{\partial s}.$$

By analyzing the geometry further, we can determine from figure 1-16 in Lewis & Miller figure that

$$\frac{\partial \rho}{\partial s} = \mu \quad \text{and} \quad \frac{\partial \omega}{\partial s} = -\frac{\eta}{\rho}.$$

Finally, our streaming operator becomes

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega}.$$

With manipulation (and specifically noting that $\mu = \frac{\partial \eta}{\partial \omega}$), we can reach the streaming operator in conservation form.

$$\begin{aligned}
\left[\hat{\Omega} \cdot \nabla\right] \psi &= \mu \frac{\partial \psi}{\partial \rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} \\
&= \frac{\mu \rho}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\mu \psi}{\rho} - \frac{\eta}{\rho}\frac{\partial \psi}{\partial \omega} - \frac{\mu \psi}{\rho} \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \mu \psi\right) \\
&= \frac{\mu}{\rho}\left(\rho \frac{\partial \psi}{\partial \rho} + \frac{\partial \rho}{\partial \rho}\psi\right) - \frac{1}{\rho}\left(\eta \frac{\partial \psi}{\partial \omega} + \frac{\partial \eta}{\partial \omega}\psi\right)
\end{aligned}$$

$$\boxed{\left[\hat{\Omega} \cdot \nabla\right] \psi = \frac{\mu}{\rho}\frac{\partial}{\partial \rho}(\rho \psi) - \frac{1}{\rho}\frac{\partial}{\partial \omega}(\eta \psi)}.$$

b.)

We can use this streaming operator in the transport equation to describe one dimensional cylindrical geometries. For simplicity, we will consider that the transport equation is describing a non-multiplying system. In one-dimensional cylindrical coordinate systems, dependence on position, $\vec{r}$, now only depends on ρ, and velocity direction $\hat{\Omega}$, depends on ω and ξ.

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \left[\frac{\mu}{\rho}\frac{\partial}{\partial \rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial \omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = s(\rho,\omega,\xi,E,t)$$

Integrating this equation over all angles ($0 \leq \omega \leq 2\pi$ and $-1 \leq \xi \leq 1$)

$$\begin{aligned}\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \frac{1}{v}\frac{\partial}{\partial t}\psi(\rho,\omega,\xi,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial \rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial \omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \\ + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \Sigma_t(\rho,E)\psi(\rho,\omega,\xi,E,t) = \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t)\end{aligned}$$

If we let $\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, s(\rho,\omega,\xi,E,t) = S(\rho,E,t)$, then

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\mu}{\rho}\frac{\partial}{\partial \rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial \omega}\left(\eta\psi(\rho,\omega,\xi,E,t)\right)\right] + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

Again, consulting our geometry, we find $\mu = \sqrt{1-\xi^2}\cos\omega$ and $\eta = \sqrt{1-\xi^2}\sin\omega$.

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial \rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right) - \frac{1}{\rho}\frac{\partial}{\partial \omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] + \\ +\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)\end{aligned}$$

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{\sqrt{1-\xi^2}\cos\omega}{\rho}\frac{\partial}{\partial \rho}\left(\rho\psi(\rho,\omega,\xi,E,t)\right)\right] + \\ - \int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \left[\frac{1}{\rho}\frac{\partial}{\partial \omega}\left(\sqrt{1-\xi^2}\sin\omega\psi(\rho,\omega,\xi,E,t)\right)\right] + \\ +\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)\end{aligned}$$

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\int_0^{2\pi} d\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\cos\omega\ \psi(\rho,\omega,\xi,E,t)\right) + \\ -\frac{1}{\rho}\int_{-1}^{1} d\xi \int_0^{2\pi} d\omega\, \frac{\partial}{\partial \omega}\left(\sqrt{1-\xi^2}\sin\omega\ \psi(\rho,\omega,\xi,E,t)\right) + \\ +\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)\end{aligned}$$

$$\begin{aligned}\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) + \\ -\frac{1}{\rho}\int_{-1}^{1} d\xi\sqrt{1-\xi^2}\left[\sin\omega\ \psi(\rho,\omega,\xi,E,t)\right]_0^{2\pi} + \\ +\Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)\end{aligned}$$

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\rho,E,t) + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\int_0^{2\pi} d\omega\cos\omega \int_{-1}^{1} d\xi\, \sqrt{1-\xi^2}\psi(\rho,\omega,\xi,E,t)\right) + \Sigma_t(\rho,E)\phi(\rho,E,t) = S(\rho,E,t)$$

...

Problem 3

The transport equation in one dimension is

$$\mu \cdot \frac{\partial}{\partial x}\psi(x,\mu,E) + \Sigma_t \psi(x,\mu,E) = q(x,\mu,E)$$

If we look along a characteristic "curve" in one dimension, we can say

$$\frac{\partial \psi}{\partial s} = \mu \frac{\partial \psi}{\partial x}$$

and

$$\frac{\partial}{\partial s}\psi(x_0+\mu s,\mu,E) + \Sigma_t \psi(x_0+\mu s,\mu,E) = q(x_0+\mu s,\mu,E)$$

Since our derivation of the integral form of the transport equation (in class) did not explicitly require the fact that our problem was three-dimensional, we can use the same method for the one-dimensional case. We arrive at the following:

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(x-\rho''\mu,E)\right] q(x-\rho'\mu,\mu,E)$$

where now the exponential represents the attenuation of neutrons as they move from $x-\rho'\mu$ to x, and the source term represents the production of neutrons at $x-\rho'\mu$ into (μ,E).

Since the cross section is uniform in the slab $\Sigma_t(x-\rho''\mu,E) = \Sigma_t$,

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t\int_0^{\rho'} d\rho''\right] q(x-\rho'\mu,\mu,E)$$

$$\psi(x,\mu,E) = \int_0^\infty d\rho' \exp\left[-\Sigma_t \rho'\right] q(x-\rho'\mu,\mu,E)$$

We can change coordinates to be integrating over x, by noting that $\mu = \cos\theta = x/\rho$. Then $\rho = x/\mu$ and $d\rho = dx$.

$$\psi_0(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] q(x-x',\mu,E)$$

Since we are ignoring fission, we note that the source term is simply the beam source, $q(x-x',\mu,E) = s(x-x',\mu,E)$. If we assume that the beam source can be treated like a plane source at the left boundary, then including the fact that the beam is monoenergetic lets the complete source be written as $s(\mu)\delta(E-E_0)\delta(x-x')$

We can now write the equation for the uncollided flux, using the explicitly defined external source

$$\psi_0(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] s(\mu)\delta(E-E_0)\delta(x-x')$$

$$\psi_0(x,\mu,E) = s(\mu)e^{-\Sigma_t x/\mu}$$

The boundary conditions for this problem are

1. $$J_{0+}(0) = \int_0^1 d\mu \int_0^\infty dE\, \psi_0(0,\mu,E) = I$$

2. $$J_{0-}\left(\frac{a}{\tilde{\mu}}\right) = \int_{-1}^0 \int_0^\infty \psi_0\left(\frac{a}{\tilde{\mu}},\mu,E\right) d\mu\, dE = 0$$

Using our first boundary condition,

$$I = \int_0^1 d\mu \int_0^\infty dE\, \psi_0(0, \mu, E_0)$$

$$I = \int_0^1 d\mu \int_0^\infty dE\, \left[\int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right] s(\mu)\delta(E - E_0)\delta(x - x') \right]_{x=0}$$

The delta function for x picks the value of the integrand at which $x' = x$, and the equation can be simplified to

$$I = \int_0^1 d\mu \int_0^\infty dE\, \left[\exp\left[-\Sigma_t x/\mu\right] s(\mu)\delta(E - E_0)\right]_{x=0}$$

$$I = \int_0^1 d\mu \int_0^\infty dE\, s(\mu)\delta(E - E_0)$$

The delta function for E now picks the value of the integrand at which $E = E_0$,

$$I = \int_0^1 s(\mu) d\mu$$

If we consider the source beam as monodirectional directly incident on the slab, then for all uncollided neutrons $\mu = 1$.

$$I = \int_0^1 s(1) d\mu$$

$$I = s(1)$$

Then, the flux is

$$\psi_0(x, \mu, E) = s(1)e^{-\Sigma_t x/(1)}$$

$$\boxed{\psi_0(x, \mu, E) = Ie^{-\Sigma_t x}}.$$

Problem 4

We know that the once collided flux is given by $\psi_1 = K\psi_0$ where K is the integral operator. To determine K and find the source term for first collisions, we start by looking at the fully expanded form of the integral transport equation.

$$\psi(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho'' \Sigma_t(\vec{r} - \rho''\hat{\Omega}, E)\right] \left[\frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + \right.$$
$$\left. + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + Q(\vec{r} - \rho'\hat{\Omega}, \hat{\Omega}, E)\right]$$

For the once collided flux, the external source Q is zero, and the angular flux used to calculate the scattering and fission terms is the uncollided flux.

$$\psi_1(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho'' \Sigma_t(\vec{r} - \rho''\hat{\Omega}, E)\right] \left[\frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') + \right.$$
$$\left. + \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E')\right]$$

We can now see that the first collision source is everything within the second set of brackets.

$$q_1(\vec{r} - \rho'\hat{\Omega}, \hat{\Omega}, E) = \frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu\Sigma_f(\vec{r} - \rho'\hat{\Omega}', E')\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E') +$$
$$+ \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}' \to \hat{\Omega}, E' \to E)\psi_0(\vec{r} - \rho'\hat{\Omega}', \hat{\Omega}', E')$$

Since we are still dealing with a one dimensional geometry, this **first collision source** can be simplified to

$$q_1(x-x',\mu,E) = \frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \\ + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E').$$

This source can then be inserted back into the full transport equation (in one dimension).

$$\psi_1(x,\mu,E) = \int_0^\infty dx' \exp\left[-\int_0^{x'/\mu} dx'' \Sigma_t(x-x'',E)\right]\left[\frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \\ + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E')\right]$$

Finally, since Σ_t is constant, the **first-collided flux** can be written as

$$\psi_1(x,\mu,E) = \int_0^\infty dx' \exp\left[-\Sigma_t x'/\mu\right]\left[\frac{\chi(E)}{4\pi}\int_{-1}^{1} d\mu' \int_0^\infty dE' \nu\Sigma_f(x-x',E')\psi_0(x-x',\mu',E') + \\ + \int_{-1}^{1} d\mu' \int_0^\infty dE' \Sigma_s(x-x',\mu'\to\mu,E'\to E)\psi_0(x-x',\mu',E')\right]$$

with boundary conditions

1. $$J_{1+}(0) = \int_0^1 d\mu \int_0^\infty dE\, \psi_1(0,\mu,E) = 0$$

2. $$J_{1-}\left(\frac{a}{\tilde{\mu}}\right) = \int_{-1}^{0}\int_0^\infty \psi_1\left(\frac{a}{\tilde{\mu}},\mu,E\right) d\mu\, dE = 0$$

Problem 5

The integral form of the transport equation is

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t(\vec{r}-\rho''\hat{\Omega},E)\right] q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E)$$

For isotropic sources (scattering and fission are not present in a purely absorbing medium), the source term becomes

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(\vec{r}-\rho'\hat{\Omega},E)}{4\pi}$$

Where the source is a point source,

$$q(\vec{r}-\rho'\hat{\Omega},\hat{\Omega},E) = \frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

Using this in the integral form of the TE, and noting that $\Sigma_t(E) = \Sigma_t$,

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' \exp\left[-\int_0^{\rho'} d\rho''\, \Sigma_t\right]\frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

$$\psi(\vec{r},\hat{\Omega},E) = \int_0^\infty d\rho' e^{-\Sigma_t\rho'}\frac{S(E)}{4\pi}\delta(\vec{r}-\rho'\hat{\Omega})$$

We can integrate this expression over angle to find the scalar flux.

$$\phi(\vec{r}, E) = \int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho'\hat{\Omega})$$

When doing this integration, we can note that

$$d\hat{\Omega} = \frac{dA}{\rho'^{\,2}}$$

$$d\rho'\, dA = dV$$

$$\int_{4\pi} d\hat{\Omega} \int_0^\infty d\rho' = \int_V \frac{dV}{\rho'^{\,2}}$$

and $\rho' = |\vec{r} - \vec{r}\,'|$, so that

$$\phi(\vec{r}, E) = \int_V \frac{dV}{\rho'^{\,2}} e^{-\Sigma_t \rho'} \frac{S(E)}{4\pi} \delta(\vec{r} - \rho'\hat{\Omega})$$

$$\phi(\vec{r}, E) = \int_V \frac{d^3\vec{r}\,'}{4\pi|\vec{r} - \vec{r}\,'|^2} e^{-\Sigma_t |\vec{r} - \vec{r}\,'|} S(E)\delta(\vec{r}\,')$$

The delta function in this equation selects only the value for when $\vec{r}\,' = 0$.

$$\phi(\vec{r}, E) = \frac{1}{4\pi|\vec{r}|^2} e^{-\Sigma_t |\vec{r}|} S(E)$$

Letting $r = |\vec{r}|$,

$$\boxed{\phi(r, E) = \frac{S(E)}{4\pi r^2} e^{-\Sigma_t r}}.$$

Problem 6

In one group diffusion theory, a bare slab reactor is described by the diffusion equation

$$-D\frac{d^2\phi}{dx^2} + \Sigma_a \phi(x) = \Sigma_f(x)\phi(x)$$

or equivalently

$$\frac{d^2\phi}{dx^2} + [B(x)]^2\phi(x) = 0$$

where $[B(x)]^2 = \frac{\Sigma_f(x) - \Sigma_a}{D}$. We also know that the power distribution must be flat, so

$$P(x) = const. = E_f \Sigma_f(x)\phi(x)$$

If we let C be some constant, then

$$C = \Sigma_f(x)\phi(x)$$

$$\Sigma_f(x) = \frac{C}{\phi(x)}$$

Using this in our equation for $B(x)$ and then substituting this back into the diffusion equation to have only an equation in terms of flux gives

$$\frac{d^2\phi}{dx^2} + \left(\frac{C/\phi(x) - \Sigma_a}{D}\right)\phi(x) = 0$$

$$\frac{d^2\phi}{dx^2} + \left(\frac{C - \Sigma_a \phi(x)}{D}\right) = 0$$

$$\frac{d^2\phi}{dx^2} - \frac{\Sigma_a \phi(x)}{D} + \frac{C}{D} = 0$$

Letting $L = \sqrt{D/\Sigma_a}$, the solution to this inhomogeneous ordinary differential equation is

$$\phi(x) = A_1 e^{x/L} + A_2 e^{-x/L} + \frac{C}{\Sigma_a}$$

Since our reactor is a slab, lets say it's width is 2ℓ, and it is centered at the origin. Then, since it is a bare slab, the boundary conditions are

$$\phi(\pm\tilde{\ell}) = 0$$

Using these boundary conditions, we find

$$\phi(\tilde{\ell}) = 0 = A_1 e^{\tilde{\ell}/L} + A_2 e^{-\tilde{\ell}/L} + \frac{C}{\Sigma_a}$$

and

$$\phi(-\tilde{\ell}) = 0 = A_1 e^{-\tilde{\ell}/L} + A_2 e^{\tilde{\ell}/L} + \frac{C}{\Sigma_a}$$

and so $A_1 = A_2$. If we let $A = 2A_1 = 2A_2$, then

$$\phi(x) = A\cosh\left(\frac{x}{L}\right) + \frac{C}{\Sigma_a}$$

Using the boundary conditions again,

$$\phi(\tilde{\ell}) = 0 = A\cosh\left(\frac{\tilde{\ell}}{L}\right) + \frac{C}{\Sigma_a}$$

$$-\frac{C}{\Sigma_a} = A\cosh\left(\frac{\tilde{\ell}}{L}\right)$$

$$C = -A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)$$

The flux solution is finally

$$\phi(x) = A\cosh\left(\frac{x}{L}\right) - \frac{A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{\Sigma_a}$$

$$\phi(x) = A\left(\cosh\left(\frac{x}{L}\right) - \cosh\left(\frac{\tilde{\ell}}{L}\right)\right)$$

Now, we established earlier that $C = \Sigma_f(x)\phi(x)$. Using this fact, we can say

$$\Sigma_f(x) = \frac{C}{\phi(x)}$$

$$\Sigma_f(x) = \frac{C}{A\left(\cosh\left(\frac{x}{L}\right) - \cosh\left(\frac{\tilde{\ell}}{L}\right)\right)}$$

and where $C = -A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)$

$$\Sigma_f(x) = \frac{-A\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{A\left(\cosh\left(\frac{x}{L}\right) - \cosh\left(\frac{\tilde{\ell}}{L}\right)\right)}$$

$$\Sigma_f(x) = \frac{\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{\cosh\left(\frac{\tilde{\ell}}{L}\right) - \cosh\left(\frac{x}{L}\right)}$$

By definition, $\Sigma_f(x) = N_{\text{fuel}}(x)\sigma_f$. Then

$$N_{\text{fuel}}(x)\sigma_f = \frac{\Sigma_a\cosh\left(\frac{\tilde{\ell}}{L}\right)}{\cosh\left(\frac{\tilde{\ell}}{L}\right) - \cosh\left(\frac{x}{L}\right)}$$

$$\boxed{N_{\text{fuel}}(x) = \frac{\Sigma_a \cosh\left(\frac{\tilde{\ell}}{L}\right)}{\sigma_f\left(\cosh\left(\frac{\tilde{\ell}}{L}\right) - \cosh\left(\frac{x}{L}\right)\right)}}.$$

Problem 7

For cylindrical geometries, the diffusion equation takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(\vec{r})}{\partial r}\right) + \frac{\partial^2 \phi(\vec{r})}{\partial z^2} + B^2\phi(\vec{r}) = 0$$

where $B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$ near criticality. Since our cylinder is infinite we lose dependence on z in our equations.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(r)}{\partial r}\right) + B^2\phi(r) = 0$$

Solutions to this equation are are zeroth order Bessel functions of first and second kind, $J_0(r)$ and $Y_0(r)$. Then,

$$\phi_b(r) = A_1 J_0(Br) + A_2 Y_0(Br).$$

Our boundary conditions demand that the flux is finite at $r = 0$, and so $A_2 = 0$. Letting $A_1 = A$, we have

$$\phi_b(r) = AJ_0(Br).$$

For a bare reactor, boundary conditions force the flux to go to zero at $\tilde{R}_b$,

$$\phi(\tilde{R}_b) = 0,$$

and since the flux must be nonnegative, only the first zero of J_0 can satisfy this condition. This occurs at $B\tilde{R}_b = 2.4048$. $B = B_g = \frac{2.4048}{\tilde{R}_b}$. The criticality condition for the bare core is then

$$\left(\frac{\nu\Sigma_f - \Sigma_a}{D}\right)^2 = \left(\frac{2.4048}{\tilde{R}_b}\right)^2$$

and the critical radius is

$$\tilde{R}_b = \frac{2.4048D}{\nu\Sigma_f - \Sigma_a}.$$

For a reflected core, our initial equation and the form of the solutions stay the same within the core region. With a finite flux at $r = 0$, $A_2 = 0$ and if $A_1 = A$, then in the core

$$\phi_r(r) = AJ_0(Br), \qquad r < R.$$

In the reflected region, we still have no z dependence, but our diffusion equation is just

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi(r)}{\partial r}\right) - \frac{1}{L^2}\phi(r) = 0$$

without the dependence on fission. Solutions to this equation are zeroth order modified bessel functions of the first and second kind, $I_0(r)$ and $K_0(r)$. In the reflector

$$\phi_r(r) = C_1 I_0(\frac{r}{L}) + C_2 K_0(\frac{r}{L}), \qquad R < r < a$$

We can now impose the boundary condition that the flux must go to zero at the extrapolated distance in the reflector,

$$\phi_r(R + \tilde{a}) = 0.$$

We find

$$0 = C_1 I_0(\frac{R + \tilde{a}}{L}) + C_2 K_0(\frac{R + \tilde{a}}{L}).$$

$$C_2 = -C_1 \frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}$$

If we let $C_1 = C$, then in the reflector

$$\phi_r(r) = CI_0(\frac{r}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{r}{L}), \qquad R < r < a$$

Finally, we use our boundary condition that the flux must be equal at the boundaries.

$$\phi_r(R) = AJ_0(BR) = CI_0(\frac{R}{L}) - C\frac{I_0(\frac{R+\tilde{a}}{L})}{K_0(\frac{R+\tilde{a}}{L})}K_0(\frac{R}{L})$$

...

Problem 8

The one-dimensional, single energy diffusion equation in each of the material regions are as follows:

Multiplying Core:

$$\frac{d^2\phi(x)}{dx^2} + B^2\phi(x) = 0, \qquad 0 < x < a$$

where $B^2 = \frac{\Sigma_f - \Sigma_a}{D}$.

Uniform Source Reflector:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = \frac{-S_0}{D}, \qquad x > a$$

where $L^2 = \frac{D}{\Sigma_a}$.

Boundary Conditions:

1. $J_+(0) = 0$
2. $\phi_c(a) = \phi_r(a)$
3. $\vec{J}_c(a) = \vec{J}_r(a)$
4. $\phi(x) < \infty$

In the multiplying core, solutions to the differential equation are

$$\phi(x) = A_1\sin(Bx) + A_2\cos(Bx)$$

We can relate J_+ to the flux using the relation

$$J_+(0) = 0 = \frac{\phi(0)}{4} - \frac{D}{2}\frac{d\phi}{dx}\bigg|_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{D}{2}\left[A_1B\cos(Bx) - A_2B\sin(Bx)\right]_{x=0}$$

$$0 = \frac{A_2}{4} - \frac{A_1BD}{2}$$

$$A_2 = 2A_1BD$$

Letting $A_1 = A$,

$$\phi_c(x) = A\sin(Bx) + 2ABD\cos(Bx)$$

In the uniform reflector, the homogeneous solution will be

$$\phi_h(x) = C_1e^{x/L} + C_2e^{-x/L},$$

the particular solution will be

$$\phi_p(x) = \frac{S_0 L^2}{D},$$

and thus the general solution in the reflector will be

$$\phi_r(x) = C_1 e^{x/L} + C_2 e^{-x/L} + \frac{S_0 L^2}{D}$$

Noting that $\phi(x) < \infty$ for all x, we see that this is violated in the above equation as $x \to \infty$ if $C_1 \neq 0$. Letting $C_2 = C$,

$$\phi_r(x) = C e^{-x/L} + \frac{S_0 L^2}{D}$$

Using our interface condition, we can see that

$$\phi_c(a) = \phi_r(a)$$

$$A\sin(Ba) + 2ABD\cos(Ba) = Ce^{-a/L} + \frac{S_0 L^2}{D}$$

$$C = AD\frac{\sin(Ba) + 2BD\cos(Ba) - S_0 L^2}{De^{-a/L}}$$

In total, the flux in this system is

$$\boxed{\phi(x) = \begin{cases} A\left(\sin(Bx) + 2BD\cos(Bx)\right), & 0 < x < a \\ AD\left(\frac{\sin(Ba)+2BD\cos(Ba)-S_0L^2}{De^{-a/L}}\right)e^{-x/L} + \frac{S_0 L^2}{D}, & x > a \end{cases}}$$

Problem 9