NE250 Homework 2 **Mitch Negus**
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Problem 1

If the energy distribution for fission neutrons from ^{235}U follows the functional approximation (for energy in MeV)

$$\chi(E) = 0.453e^{-1.036E}\sinh(\sqrt{2.29E}),$$

then the most probable energy of a neutron corresponds to the maximum of the function. A maximum value will be found at a critical point of the function, which can be found via differentiation (specifically when $\frac{d\chi}{dE} = 0$):

$$\begin{aligned}
\frac{d\chi}{dE} = 0 &= \frac{d}{dE}\left(0.453e^{-1.036E_{\text{max}}}\sinh(\sqrt{2.29E_{\text{max}}})\right) \\
0 &= 0.453\frac{d}{dE}\left(e^{-1.036E_{\text{max}}}\sinh(\sqrt{2.29E_{\text{max}}})\right) \\
0 &= 0.453\left[e^{-1.036E}\frac{d}{dE}\sinh(\sqrt{2.29E_{\text{max}}}) + \frac{d}{dE}\left(e^{-1.036E_{\text{max}}}\right)\sinh(\sqrt{2.29E_{\text{max}}})\right] \\
0 &= 0.453\left[e^{-1.036E}\cosh(\sqrt{2.29E})\frac{d}{dE}\left(\sqrt{2.29E_{\text{max}}}\right) - 1.036e^{-1.036E}\sinh(\sqrt{2.29E})\right] \\
0 &= 0.453\left[e^{-1.036E}\cosh(\sqrt{2.29E})\frac{\sqrt{2.29}}{2\sqrt{E_{\text{max}}}} - 1.036e^{-1.036E}\sinh(\sqrt{2.29E})\right] \\
0 &= \frac{\sqrt{2.29}\cosh(\sqrt{2.29E})}{2\sqrt{E_{\text{max}}}} - 1.036\sinh(\sqrt{2.29E}) \\
0 &= 1 - 1.369\sqrt{E_{\text{max}}}\tanh(\sqrt{2.29E_{\text{max}}}) \\
1 &= 1.369\sqrt{E_{\text{max}}}\tanh(\sqrt{2.29E_{\text{max}}})
\end{aligned}$$

$$\boxed{E_{\text{max}} = 0.724 \text{ MeV}}$$

The average energy can be found by finding the expected value of the function on the domain $[0, \infty)$.

$$\begin{aligned}
E_{\text{ave}} &= \int_0^\infty E\,\chi(E)\,dE \\
&= \int_0^\infty E\left(0.453e^{-1.036E}\sinh(\sqrt{2.29E})\right)dE \\
&= 0.453\int_0^\infty E\,e^{-1.036E}\sinh(\sqrt{2.29E})\,dE
\end{aligned}$$

This integral cannot be solved analytically. Solving numerically (with Wolram Alpha),

$$\boxed{E_{\text{ave}} = 1.98 \text{ MeV}}$$

Problem 2

A general reaction rate for process x as a function of energy can be defined as

$$R_x(E) = \Sigma_x(E)\phi(E)$$

where $\Sigma_x(E)$ is the macroscopic cross section for reaction x and ϕ is the neutron flux, both at energy E. The macroscopic cross section can be further decomposed, so that

$$R_x(E) = n_x\sigma_x(E)\phi(E).$$

Comparing neutron-neutron reactions with all neutron-nuclei reactions, the ratio of reaction rates is

$$\frac{R_{nn}(E)}{R_{tot}(E)} = \frac{n_n\sigma_{nn}(E)\phi(E)}{n_{\mathrm{UO_2}}\sigma_{tot}(E)\phi(E)}.$$

We can know that the neutron flux is $\phi(0.025\text{ eV} = 10^{16}$ neutrons/(cm$^2 \cdot$ s) and they are at thermal energies ($E = 0.025$ eV and traveling at $v = \sqrt{\frac{2(0.025\text{eV})}{m_n}} = 2.190 \times 10^5$ cm/s). The neutrons that are then in a 1 cm^3 volume at any given second is

$$n_n = \frac{\phi(0.025\text{ eV})}{v} = \frac{10^{16}\text{ neutrons}/(\text{cm}^2\cdot\text{s})}{2.190\times10^5\text{ cm/s}} = 4.566\times10^{10}\text{ neutrons/cm}^3$$

For $\mathrm{UO_2}$, $\rho = 10.97$ g/cm^3, $m_{\mathrm{O}} = 16.0$ g/mol, $m_{\mathrm{U8}} = 238.05$ g/mol, and $m_{\mathrm{U5}} = 235.04$ g/mol. If we use an enrichment of 5% (atom percent), then $m_{\mathrm{UO_2}} = 0.95(238.05) + 0.05(235.04) + 2(16.0) = 269.90$ g/mol. For number density, we find

$$n_{\mathrm{UO_2}} = \frac{\rho N_A}{m_{\mathrm{UO_2}}} = \frac{(10.97\text{ g/cm}^3)(6.022\times10^{23})}{269.90\text{ g/mol}} = 2.448\times10^{22}\text{ molecules/cm}^3$$

Additionally, we're given that $\sigma_{nn} = 10$ b, and we can determine the microscopic cross section for $\mathrm{UO_2}$ from tabulated data. (From ENDF/B-VII.1 at 0.025 eV: $\sigma_{tot,\mathrm{U8}} = 11.962\text{ b}, \sigma_{tot,\mathrm{U5}} = 698.856\text{ b},$ and $\sigma_{tot,\mathrm{O}} = 3.852$ b)

$$\sigma_{tot} = 0.95\sigma_{tot,\mathrm{U8}} + 0.05\sigma_{tot,\mathrm{U5}} + 2\sigma_{tot,\mathrm{O}}$$

$$\sigma_{tot} = 0.95(11.962\text{ b}) + 0.05(698.856\text{ b}) + 2(3.852\text{ b})$$

$$\sigma_{tot} = 54.011\text{ b}$$

We can now sove for the ratio of reaction rates (noting that $\phi(E)$ cancels in the numerator and denominator)

$$\frac{R_{nn}(E)}{R_{tot}(E)} = \frac{n_n\sigma_{nn}(E)}{n_{\mathrm{UO_2}}\sigma_{tot}(E)} = \frac{(4.566\times10^{10}\text{ neutrons/cm}^3)(10\text{ b})}{(2.448\times10^{22}\text{ molecules/cm}^3)(54.011\text{ b})}$$

$$\frac{R_{nn}(E)}{R_{tot}(E)} = 3.40\times10^{-13}$$

The rate of neutron-neutron collisions is 13 orders of magnitudes less than the rate of neutron-$\mathrm{UO_2}$ collisions.

Problem 3

Problem 4

First, we define the average scattering cosine $\bar{\mu}_0$ as the average dot product, $\langle \hat{\Omega} \cdot \hat{\Omega}' \rangle$. When normalized by $4\pi\Sigma_s$, the total of cross sections for scattering from any angle $\hat{\Omega}$ to any other angle $\hat{\Omega}'$, this is

$$\bar{\mu}_0 \equiv \langle \hat{\Omega} \cdot \hat{\Omega}' \rangle = \left(\frac{1}{4\pi\Sigma_s} \right) \int_{4\pi} d\hat{\Omega} \int_{4\pi} d\hat{\Omega}' \, \hat{\Omega} \cdot \hat{\Omega}' \Sigma_s(\hat{\Omega} \cdot \hat{\Omega}')$$

In the center of mass system, the probability that a particle scatters in any direction is roughly uniform, $\Sigma_{\mathrm{CM}}(\theta_C) = \frac{\Sigma_s}{4\pi}$,

$$\bar{\mu}_0 = \frac{1}{\Sigma_s} \int_{4\pi} d\hat{\Omega} \int_{4\pi} d\hat{\Omega}' \, \hat{\Omega} \cdot \hat{\Omega}' \Sigma_{\mathrm{CM}}(\theta_C).$$

Problem 5

A critical reactor has a multiplication factor of $k = 1$. The multiplication factor (for an infinite reactor) can be defined as

$$k_\infty \equiv \frac{\#\text{ neutrons produced}}{\#\text{ neutrons absorbed}}$$

Mathematically, the number of neutrons produced is $\int_0^\infty \nu\Sigma_f(E)\phi(E)\,dE$ and the number of neutrons absorbed is $\int_0^\infty \Sigma_a(E)\phi(E)\,dE$. Altogether, we can mathematically describe a critical reactor as

$$1 = \frac{\int_0^\infty \nu\Sigma_f(E)\phi(E)\,dE}{\int_0^\infty \Sigma_a(E)\phi(E)\,dE}$$

or equivalently

$$\int_0^\infty \nu\Sigma_f(E)\phi(E)\,dE = \int_0^\infty \Sigma_a(E)\phi(E)\,dE.$$

Since we are considering only thermal cross sections, we will let $\Sigma_X(E) = \Sigma_X(0.025\text{ eV}) = \Sigma_{X,T}$ and we find

$$\nu\Sigma_{f,T}\int_0^\infty \phi(E)\,dE = \Sigma_{a,T}\int_0^\infty \phi(E)\,dE.$$

The integrals over flux cancel, and so

$$\nu\Sigma_{f,T} = \Sigma_{a,T}.$$

The macroscopic cross sections can be rewritten as $\Sigma_{f,T} = \Sigma_{f,T,\mathrm{f}}$ and $\Sigma_{a,T} = \Sigma_{a,T,\mathrm{f}} + \Sigma_{a,T,\mathrm{m}}$ where subscripts f and m denote fuel and moderator, respectively. Furthermore, each macroscopic cross section for each material can be expressed in terms of the material's number density and microscopic cross section, $\Sigma = n\sigma$. In total

$$\nu n_\mathrm{f}\sigma_{f,T,\mathrm{f}} = n_\mathrm{f}\sigma_{a,T,\mathrm{f}} + n_\mathrm{m}\sigma_{a,T,\mathrm{m}}.$$

The fuel-to-moderator density at criticality can then be expressed as

$$\frac{n_\mathrm{f}}{n_\mathrm{m}} = \frac{\sigma_{a,T,\mathrm{m}}}{\nu\sigma_{f,T,\mathrm{f}} - \sigma_{a,T,\mathrm{f}}}.$$

a.) **Graphite**

$$\frac{n_\mathrm{f}}{n_\mathrm{m}} = 4.55\times10^{-6}$$

b.) **Beryllium**

$$\frac{n_\mathrm{f}}{n_\mathrm{m}} = 1.36\times10^{-5}$$

c.) **Water**

$$\frac{n_\mathrm{f}}{n_\mathrm{m}} = 8.99\times10^{-4}$$

d.) **Graphite**

$$\frac{n_\mathrm{f}}{n_\mathrm{m}} = 1.64\times10^{-6}$$

(see attached Jupyter notebook for full calculations)

Problem 6

a.)

Like in Problem 5,

$$k_\infty = \frac{\int_0^\infty \nu\Sigma_f(E)\phi(E)\,dE}{\int_0^\infty \Sigma_a(E)\phi(E)\,dE}.$$

Again, since we are only considering cross sections averaged over the fast spectrum, we can state that $\Sigma_X(E) = \Sigma_{X,F}$, and then

$$k_\infty = \frac{\nu\Sigma_{f,F}\int_0^\infty \phi(E)\,dE}{\Sigma_{a,F}\int_0^\infty \phi(E)\,dE} = \frac{\nu\Sigma_{f,F}}{\Sigma_{a,F}}.$$

Next, we repeat our decomposition of the numerator and denominator, leaving

$$k_\infty = \frac{\nu_{\mathrm{PuO_2}} n_{\mathrm{PuO_2}} \sigma_{F,\mathrm{PuO_2}} + \nu_{\mathrm{UO_2}} n_{\mathrm{UO_2}} \sigma_{F,\mathrm{UO_2}}}{n_{\mathrm{PuO_2}}\sigma_{a,F,\mathrm{PuO_2}} + n_{\mathrm{UO_2}}\sigma_{a,F,\mathrm{UO_2}} + n_{\mathrm{Fe}}\sigma_{a,F,\mathrm{Fe}} + n_{\mathrm{Na}}\sigma_{a,F,\mathrm{Na}}}$$

After solving this equation with the given values, we find

$$\boxed{k_\infty = 1.332}$$

b.)

We can use the densities and volume fractions to determine the fraction of the core mass that is fuel.

$$\boxed{61.6\% \text{ is fuel}}$$

c.)

If $k = 1$ and the non-leakage probability is $P_{NL} = 0.90$, then we can reexpress our earlier equation as

$$k = 1 = P_{NL}\frac{\nu\Sigma_{f,F}\int_0^\infty \phi(E)\,dE}{\Sigma_{a,F}\int_0^\infty \phi(E)\,dE} = P_{NL}\frac{\nu\Sigma_{f,F}}{\Sigma_{a,F}},$$

or

$$P_{NL}\nu\Sigma_{f,F} = \Sigma_{a,F}.$$

When separated into individual materials, this is

$$P_{NL}\sum_i \nu_i n_i \sigma_{f,i} = \sum_j n_j \sigma_{a,j}.$$

or explicitly

$$P_{NL}\left(\nu_{\mathrm{PuO_2}} n_{\mathrm{PuO_2}} \sigma_{F,\mathrm{PuO_2}} + \nu_{\mathrm{UO_2}} n_{\mathrm{UO_2}} \sigma_{F,\mathrm{UO_2}}\right) = n_{\mathrm{PuO_2}}\sigma_{a,F,\mathrm{PuO_2}} + n_{\mathrm{UO_2}}\sigma_{a,F,\mathrm{UO_2}} + n_{\mathrm{Fe}}\sigma_{a,F,\mathrm{Fe}} + n_{\mathrm{Na}}\sigma_{a,F,\mathrm{Na}}$$

We recall that $n_{\mathrm{PuO_2}} = f_{\mathrm{f,PuO_2}}(0.3)\frac{\rho_{\mathrm{PuO_2}}N_A}{m_{\mathrm{PuO_2}}}$ and $n_{\mathrm{UO_2}} = (1 - f_{\mathrm{f,PuO_2}})(0.3)\frac{\rho_{\mathrm{UO_2}}N_A}{m_{\mathrm{UO_2}}}$, while $n_{\mathrm{Na}} = (0.5)\frac{\rho_{\mathrm{Na}}N_A}{m_{\mathrm{Na}}}$ and $n_{\mathrm{Fe}} = (0.2)\frac{\rho_{\mathrm{Fe}}N_A}{m_{\mathrm{Fe}}}$. We can substitute these expressions into the equation above and solve for $f_{\mathrm{f,PuO_2}}$.

$$\boxed{\text{The fuel must be } 10.322\%\ \mathrm{PuO_2} \text{ and } 89.678\%\ \mathrm{UO_2} \text{ for a critical reactor with } P_{NL} = 0.9}$$

(see attached Jupyter notebook for full calculations)

Problem 10

Answer the following questions as true or false, provide a one sentence justification for your answer

1. The integro-differential form of the transport equation expresses a local balance between neutron production and losses.
2. A vacuum boundary condition for the integro-differential transport equation implies a zero outgoing angular flux.
3. In the transport equation in curvilinear coordinates, the redistribution term allows neutrons to migrate between the directions as they move along a straight line.
4. A nuclear system is subcritical if its eigenvalues satisfy max(Re(j)) ¡ 1.
5. The energy spectrum of the fundamental eigenmode of the eigenvalue problem is skewed as though a 1 absorber is present.