The Witness Set Constraint

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Abstract

This article is initially concerned with a famous constraint on the class of possible determiners in natural languages: the so-called Conservativity Constraint. We shall briefly illustrate the force of this constraint and informally sketch Keenan and Stavi (1986)'s view according to which the Conservativity Constraint derives from the boolean structure of natural language semantics. We shall proceed to discuss certain well-known linguistic categories that have been argued to be interpreted by non-conservative functions: *only* and the relative proportional determiners *many* and *few*. We shall take the challenge posed by the existence of these categories in order to propose an alternative to the Conservativity Constraint. This alternative will be dubbed the Witness Set Constraint, which is inspired in Barwise and Cooper (1981)'s considerations on the semantic processing of generalized quantifiers. We shall defend that the proposed constraint does not suffer from the empirical shortcomings that have been attributed to the Conservativity Constraint, and indeed, we shall argue in detail that it correctly predicts (a) the existence of conservative determiners, (b) the non-existence of certain non-conservative determiners, such as inner negations, cardinal comparison determiners and the converses of non-trivial proportional determiners, and most importantly, (c) the existence of the non-conservative functions denoted by *only* and the relative proportional determiners *many* and *few*. This line of reasoning suggests that the class of functions from properties to sets of properties denoted in natural languages typically by determiners is constrained by a principle that simplifies the semantic processing of generalized quantifiers.

1 Introduction

As has been clear since Montague (1974), it is possible to characterize quantification in natural languages using the concept of *generalized quantifier*, introduced in logic by Mostowski (1955) and Lindström (1966).

Given a (non-empty) set U –the universe domain of a model–, we shall refer to the subsets of U as **properties** (of elements of U). The set of properties over U is, naturally, the set of all subsets of U, i.e., the power set of U: $\mathcal{P}(U)$. A set of properties over U is called a **generalized quantifier**; accordingly, a generalized quantifier is a subset of $\mathcal{P}(U)$. As a result, the set of generalized quantifiers over U is the set of all subsets of $\mathcal{P}(U)$, namely the power set of $\mathcal{P}(U)$: $\mathcal{P}(\mathcal{P}(U))$. We thus adhere to the following standard definition:

Definition 1. ***Generalized quantifier***
Given a (non-empty) universe domain U, a generalized quantifier over U is a member of $\mathcal{P}(\mathcal{P}(U))$.

For instance, for a universe domain U, if A is a subset of U denoted by a noun, the denotation of the quantified noun phrases $[\![all]\!]$(A), $[\![some]\!]$(A), and $[\![most]\!]$(A) would correspond to the following generalized quantifiers:

$$[\![All]\!](A) = \{X \subseteq U : \emptyset \neq A \subseteq X\}$$
$$[\![some]\!](A) = \{X \subseteq U : X \cap A \neq \emptyset\}$$
$$[\![Most]\!](A) = \left\{X \subseteq U : |A \cap X| > \left|A \cap \overline{X}\right|\right\}$$

Whereas A is the property denoted by a noun, $[\![D]\!]$(A) is the set of properties (i.e., the generalized quantifier) denoted by a syntactic phrase whose head is the determiner *D*.

Consider, for concreteness, the following quantified statements.

(1) a. All men run
b. Some men run
c. Most men run

Given a particular universe domain U that includes the set M of men and the set R of runners, when we calculate the denotation of (1-a) we need to determine whether R belongs to $[\![all]\!](M)$; or in other words, we need to check whether $M \subseteq R$. The proposed definition of *all* reflects the well-known existential presupposition of this determiner in natural languages: the statement *all men run* presupposes that there is at least one man, i.e., that the set M is not empty.[1] Similarly, in order to determine the denotation of (1-b) we must find out whether $R \cap M \neq \emptyset$. Finally, the sentence (1-c) will be true if and only if

[1] Cf., among others, section 6.7 of Heim and Kratzer (1997) for a detailed study of existential presuppositions of quantificational expressions.

the intersection of M and R is larger than the intersection of M and $\overline{R}$ (the complementary set of R, i.e., the set of non-runners).

Within this basic framework, the set $\mathcal{D}_U$ of possible determiner denotations over an arbitrary U is the set of functions from properties over U to sets of properties over U, i.e.,

$$\mathcal{D}_U = \{f : f : \mathcal{P}(U) \to \mathcal{P}(\mathcal{P}(U))\}\,.$$

We emphasize that determiner denotations and generalized quantifiers are different semantic notions (Barwise and Cooper 1981, p. 161-162): a determiner denotation is a function that takes properties as arguments and yields generalized quantifiers as values. Or equivalently, a determiner denotation is a member of $\mathcal{D}_U$ and a generalized quantifier a member of $\mathcal{P}(\mathcal{P}(U))$.

In principle, we could expect that, for any U and any conceivable function belonging to $\mathcal{D}_U$, there would exist a natural language determiner conveying such a function. If this were the case, natural language determiners would realize all semantic possibilities, and consequently, one could argue, the class of natural determiners would have a maximal expressive power: it would be as rich as it would be semantically possible.

However, Barwise and Cooper (1981)'s investigations of how Montague's treatment of quantification can be further developed in order to obtain important implications for a theory of natural languages suggested certain universals that constrain the set of natural determiners.

In this article we shall be initially concerned with perhaps the most popular universal that constrains the set of natural determiners: the so called *Conservativity Constraint* or *Conservativity Universal* (Keenan 1981, Keenan and Stavi 1986).

(2) *Conservativity Constraint* (Keenan and Stavi 1986, p. 260)
Determiners in all languages are interpreted by *conservative* functions

As common, we shall understand that a determiner D is interpreted by a conservative function (or more briefly, that a determiner denotation D is conservative) when the following property is satisfied:

Definition 2. ***The conservativity property***
For all universe domains U, all possible determiner denotations D, and all pairs X, Y of subsets of U, D is conservative iff:

$$Y \in D(X) \Leftrightarrow (X \cap Y) \in D(X).$$

As is well known, the conservativity property has also been studied under the name *intersectivity* by Higginbotham and May (1981) and was originally conceived by Barwise and Cooper (1981) in terms of the "lives on" property, understood as follows:

Definition 3. ***The "lives on" property***
For all universe domains U, all possible determiner denotations D, and all pairs X, Y of subsets of U, the quantifier D(X) lives on X iff:

$$Y \in D(X) \Leftrightarrow (X \cap Y) \in D(X).$$

From Definitions 2 and 3 it is immediately obvious that the conservativity property relative to determiners and the "lives on" property relative to generalized quantifiers are equivalent. We state this in the following basic theorem.

Theorem 1. *For all U and all D, D is conservative iff, for all X, $D(X)$ lives on X.*

The intuition behind the notions of conservativity and "lives on" is that in quantified statements such as those in (1) the argument of the determiner, namely, the set M, has a special status. M provides the *restriction* or the *domain* of individuals relevant for the determiner. When we consider whether the function denoted by the determiner holds between the domain provided by M and the set R, M is *conserved*, i.e., it is *preserved* without being tampered with. It is in this sense that we can say, following Keenan and Stavi (1986)'s terminology, that a determiner is *conservative* with respect to its restriction, and that the denotation of $D([\![NP]\!])$ (a quantifier, in Barwise and Cooper (1981)'s terminology), *lives on* the set $[\![NP]\!]$.

1.1 Basic illustrations

This subsection is mainly illustrative and attempts to be informative for those readers who are not particularly familiarized with our topic of study.[2] We shall focus our attention on the class of cardinal numeral determiners $[\![n]\!]$, such as *one*, *two*, ..., in order to exemplify how the Conservativity Constraint bans certain determiners that would be otherwise expected.

Let us define the quantifier $[\![n]\!](A)$, for any property A over an arbitrary universe domain U.

[2]We refer the interested reader to the following extensive survey works on the topic of generalized quantifiers in linguistics and logic: Szabolcsi (2010), Keenan and Westerståhl (2010), Peters and Westerståhl (2006), Keenan (1996) and Partee et al. (1990).

Definition 4. *For all universe domain U, all $A \subseteq U$ and all natural number $n \geq 1$,*

$$[\![n]\!](A) = \{X \subseteq U : |X \cap A| \geq n\}\,.$$

Therefore the denotation $[\![n]\!](A)$ would be the set of all those subsets of U whose intersection with A has at least n elements. It is easy to see that:

Lemma 1. *$[\![n]\!]$-numerals are conservative functions.*

Proof. Consider an arbitrary numeral $[\![n]\!]$ and let us check whether, for any two subsets A, B of U, $[\![n]\!]$ satisfies the property of being conservative introduced in Definition 2:

(3) $\quad B \in [\![n]\!](A) \Leftrightarrow A \cap B \in [\![n]\!](A).$

If we now apply to (3) the general definition of $[\![n]\!](A)$ provided in Definition 4, we obtain the following biconditional:

(4) $\quad |A \cap B| \geq n \Leftrightarrow |A \cap (A \cap B)| \geq n.$

Note that $A \cap A = A$, since intersection is an idempotent operation, whereby $A \cap B = A \cap (A \cap B)$. As a consequence, the biconditional (4) is necessarily true, and thus the function $[\![n]\!]$ is conservative. $\square$

An instance of $[\![n]\!]$-numeral is *two*; following Definition 4, the quantified noun phrase *two doctors* would denote the following set:

$$[\![\text{two doctors}]\!] = [\![2]\!](D) = \{X \subseteq U : |X \cap D| \geq 2\}\,.$$

Accordingly, a sentence like *two doctors speak French* would be true iff the intersection of the set D of doctors and the set F of French speakers contains at least two members ($|F \cap D| \geq 2$).

We can easily check that *two* is interpreted by a conservative function by following the reasoning developed above for an arbitrary $[\![n]\!]$-numeral. The crucial observation is that $D \cap X = D \cap (D \cap X)$; as a consequence, it has to be the case that

$$|D \cap X| \geq 2 \Leftrightarrow |D \cap (D \cap X)| \geq 2,$$

which implies that $[\![two]\!]$ is conservative.

We can now continue by constructing a quantifier $[\![\widehat{n}]\!](A)$, for any property A over an arbitrary U.

Definition 5. *For all universe domain U, all $A \subseteq U$ and all natural number $n \geq 1$,*

$$[\![\widehat{n}]\!](A) = \left\{X \subseteq U : \left|X \cap \overline{A}\right| \geq n\right\}.$$

For $n = 2$, we obtain the $[\![\widehat{2}]\!]$-numeral –call it *owt*– which denotes the set

$$[\![\widehat{2}]\!](A) = \left\{X \subseteq U : \left|X \cap \overline{A}\right| \geq 2\right\}.$$

In accordance with this definition, the sentence *owt doctors speak French* will be true iff there are at least two individuals in the universe who are not doctors and speak French.

Whereas $[\![n]\!]$-numerals are pervasive in natural languages, $[\![\widehat{n}]\!]$-numerals are plainly unattested. Nonetheless, both type of numerals seem to be logically legitimate; as a matter of fact, the intended meaning of a synthetic $[\![\widehat{n}]\!]$-numeral can be expressed by the combination of an $[\![n]\!]$-numeral and a negative marker *non-* prefixed to the noun.

(5) Two non-doctors speak French

But it is also true that $[\![n]\!]$-numerals are conservative, as we have just shown, whereas $[\![\widehat{n}]\!]$-numerals are non-conservative.[3]

[3] It is common to use not only in informal presentations but also in technical studies a particular type of paraphrase to informally verify whether or not a determiner is conservative. For instance, in order to decide whether *two*, *some* and *all* are conservative, whereas *owt* is not, we can check whether the following biconditionals hold:

(i) a. Two doctors speak French ⇔ Two doctors are doctors that speak French
  b. Some doctors speak French ⇔ Some doctors are doctors that speak French
  c. All doctors speak French ⇔ All doctors are doctors that speak French
  d. Owt doctors speak French ⇔ Owt doctors are doctors that speak French

In examples (i-a), (i-b) and (i-c), if the sentences in the left-hand of the biconditional are true,

Lemma 2. *$[\![\widehat{n}]\!]$-numerals are non-conservative functions.*

Proof. We shall show that, for all A, B, the following biconditional does not hold:

(6) $\quad B \in [\![\widehat{n}]\!](A) \Leftrightarrow A \cap B \in [\![\widehat{n}]\!](A).$

Observe that the statement in the left-hand of the biconditional (6), '$B \in [\![\widehat{n}]\!](A)$', is a contingency: according to Definition 5 of $[\![\widehat{n}]\!]$-determiner, B belongs to $[\![\widehat{n}]\!](A)$ iff the intersection of $\overline{A}$ and B contains at least n elements. The truth of this statement will depend on the particular model and the particular universe. For instance, the statement *owt doctors speak French* is true given a very realistic situation: we would only need to find at least two individuals who are not doctors and speak French.

However, if we apply Definition 5 of $[\![\widehat{n}]\!](A)$ to the statement in the right-hand of the biconditional (6), '$A \cap B \in [\![\widehat{n}]\!](A)$', we obtain the formula

(7) $\quad \left|(A \cap B) \cap \overline{A}\right| \geq n,$

which is necessarily false. In order to see this, note that it is always the case that $\left|A \cap \overline{A}\right| = 0$. As a consequence, $\left|(A \cap \overline{A}) \cap B\right| = 0$. And given that intersection is a commutative operation,

$$\left|(A \cap \overline{A}) \cap B\right| = \left|(A \cap B) \cap \overline{A}\right| = 0.$$

Therefore, the statement in (7) must be false, because $[\![\widehat{n}]\!]$-determiners –like $[\![n]\!]$-determiners– are defined for any natural number that is equal to or larger than 1 and $\left|(A \cap B) \cap \overline{A}\right| = 0$. Since the sentences in the left-hand and the right-hand of the biconditional (6) are, respectively, a contingency and a contradiction, $[\![\widehat{n}]\!]$-numerals are non-conservative. □

then the redundant sentences in the right-hand must also be true, and vice versa. Therefore, their respective determiners are conservative. In example (i-d), the biconditional does not hold, for the first sentence (which is equivalent to *two non-doctors speak French*) is a contingency, whereas the second one (which is equivalent to *two non-doctors are doctors that speak French*) is a contradiction. Since the sentences that appear in the left-hand and in the right-hand of the biconditional symbol in examples (i) correspond, respectively, to $D(A, B)$ and $D(A, A \cap B)$, this intuitive procedure is used to verify the conservativity property.

Here we articulate our arguments directly in set-theoretical terms, without resorting to natural language paraphrases, in order to attain the appropriate level of generality; this allows us to study not only particular determiners, such as *some*, *all*, *two* or *owt*, but crucially classes of determiners, such as $[\![n]\!]$-determiners and $[\![\widehat{n}]\!]$-determiners, among others.

Hence, the Conservativity Constraints correctly predicts that there can be no determiner in any natural language that conveys an $[\![\widehat{n}]\!]$-numeral.

1.2 Motivation for the Conservativity Constraint

Our previous illustrations, although far from being comprehensive, clearly suggest that languages instantiate only a portion of all logically possible determiners, and the Conservativity Constraint seems to have an important role in restricting the class of natural determiners. As has been argued by Keenan and Stavi (1986), the Conservativity Constraint is remarkably strong since it bans most ways in which the set $\mathcal{P}(U)$ of properties over U (i.e., of NP denotations) can be mapped into $\mathcal{P}(\mathcal{P}(U))$, the set of generalized quantifiers over U. According to Proposition 4 of Keenan and Stavi (1986, p. 290),

> In a model with n individuals there are $2^{2^{2n}}$ functions from $\mathcal{P}(U)$ into $\mathcal{P}(\mathcal{P}(U))$. Provably only 2^{3^n} of which are conservative.

For instance, as Keenan and Stavi (1986, p. 290) observe, in a model with only two individuals there are $2^2 = 4$ properties, which means that there are $2^4 = 16$ sets of properties. Accordingly, there are $2^{16} = 65,536$ functions from $\mathcal{P}(U)$ into $\mathcal{P}(\mathcal{P}(U))$. However, provably only $2^{3^2} = 2^9 = 512$ of these are conservative.

The Conservativity Constraint, Keenan and Stavi (1986, p. 291) note, "may be interpreted in a way analogous to that in which linguists interpret syntactic constraints: the language learner does not have to seek the meaning of a novel determiner among all logically possible ways in which CNP denotations might be associated with NP denotations. He only has to choose from among those ways which satisfy conservativity". Thus, inasmuch as a syntactic constraint such as the Structure Dependence Principle (Chomsky 1972, Berwick et al. 2011) restricts the class of possible internal merge operations, the Conservativity Constraint would narrow down the set of linguistically possible determiner denotations.

A further important question is why the possible determiner denotations of natural languages would have to be precisely conservative functions. Keenan and Stavi's answer is that the Conservativity Constraint is a byproduct of the boolean structure of language (cf. Keenan and Stavi 1986 as well as van Benthem 1983 for discussion). Following van Benthem's (p. 453-455) simplification of Keenan and Stavi's algebraic account, we can define, for a finite universe U, an initial class of basic conservative determiner functions, say inclusion (*all*) and overlap (*some*). We can now apply to this set of basic determiners boolean operations (the conjunction, the negation and the disjunction of more basic determiners). Then the set of determiner functions that are generated is precisely the set of conservative functions. In other words, "the set of conservative determiners is closed under the boolean operations and contains certain basic simple functions we need to interpret simple determiners (Keenan and Stavi 1986, p. 291)."

At this point we must cast doubt on this result for well-known empirical reasons (see also section 5): in spite of the power of the Conservativity Constraint in banning certain determiner functions, several instances of non-conservative functions have been attested in natural languages. In section 2 we shall present the case of *only* and the case of the relative proportional determiners *many* and *few*, which are instances of linguistic categories that are interpreted by non-conservative functions. In section 3 we shall present the Witness Set Constraint, which is inspired in Barwise and Cooper (1981)'s considerations on the semantic processing of quantified statements. We shall show how this constraint accounts not only for the remarkable absence of certain non-conservative functions, such as $[\![\widehat{n}]\!]$-numerals, but also for the presence of attested non-conservative functions mentioned in section 2. In this sense, the Witness Set Constraint is more general and empirically more accurate than the Conservativity Constraint.

2 Attested non-conservative functions

The best-known instance of non-conservative function denoted by a linguistic item is provided by the linguistic category *only*, contained in the following sentence:

(8) Only Athenians think

We shall consider that the statement (8) is true if the denotation of the predicate *think* is included in the denotation of the NP *Athenians*. In other words, (8) will be true if all the members of a given universe domain that think are also Athenians. In set-theoretical terms, this means that (8) is true if the set T of thinkers is included in the set A of Athenians. Moreover, (8) conveys that effectively there are some individuals in the universe that have the property of thinking, or in other words, that T is non-empty.[4] We recast these two basic

[4] It is debatable whether the requirement that T is not empty is an entailment of *only Athenians think*, or rather a presupposition or an implicature; cf. von Fintel (1997), and references cited therein, for this controversial issue, as well as for a detailed study of the semantics of *only*. Here we shall not be concerned with the pragmatic nature of this requirement and we shall simply express it in Definition 6 as part of the definition of the set $[\![only]\!](A)$. The investigation of how our Witness Set Constraint (see section 3) relates to pragmatic mechanisms requires a much deeper consideration and is left for future research.

observations in the following definition.

Definition 6. *For all U and all $A \subseteq U$,*

$$[\![only]\!](A) = \{X \subseteq U : \emptyset \neq X \subseteq A\}\,.$$

Accordingly the quantifier $[\![only]\!](A)$ is equal to $\mathcal{P}(A) - \emptyset$. As is well known, *only* is not conservative.

Lemma 3. *The function $[\![only]\!]$ is non-conservative.*

Proof. For $[\![only]\!]$ to be conservative it should meet the following requirement for all sets A, B:

(9) $B \in [\![only]\!](A) \Leftrightarrow A \cap B \in [\![only]\!](A)$.

If we apply to (9) the simplified semantics of $[\![only]\!](A)$ described in Definition 6, we obtain the following biconditional:

$$\emptyset \neq B \subseteq A \Leftrightarrow \emptyset \neq (A \cap B) \subseteq A.$$

The sentence in the right-hand of the biconditional is a set-theoretical truth: for any two sets A, B, $A \cap B \subseteq A$. But the truth of this statement does not entail the truth of $\emptyset \neq B \subseteq A$. Therefore, $[\![only]\!]$ is not conservative. $\square$

It is common to consider that the non-conservative nature of *only* is not a problem for the Conservativity Constraint, since the syntactic distribution of *only* suggests that it is not a determiner but rather an adverb "of some type" (cf., among others, van Benthem 1983, von Fintel 1997, von Fintel and Matthewson 2008). For instance, *only* differs from canonical determiners in that it "can combine with pronouns or names", it "can occur on top of other determiners" and it can also "combine with categories other than noun phrases" (von Fintel and Matthewson 2008, p. 163).

(10) a. Only John/they came late
b. Only two guests came late
c. John only stayed for a couple of minutes

Thus, according to this view, the Conservativity Constraint applies solely to determiners; since *only* is not a determiner, it is not a counterexample to the

Constraint. It is fair to note, though, that the paradigm in (10) does not necessarily indicate that *only* is never a determiner: it could be that *only* is a determiner in (8), but not in any of the uses listed in (10). The Conservativity Constraint does not prohibit that an expression that denotes a function from properties to quantifiers can have other uses, but requires these functions to be conservative. Crucially, in the cases where *only* can be interpreted by a function from properties to quantifiers, it should satisfy the Conservativity Constraint, and it does not.

However, in our view, even if the Conservativity Constraint is restricted to determiners and even if it were the case that *only* is never a determiner, the non-conservativity of *only* poses an interesting and genuine problem for the thesis that conservativity plays an important role in natural language semantics. If it is true that there can be no determiners interpreted by non-conservative functions, why is it that an alleged non-determiner can be interpreted by a non-conservative function? If the class of determiners must be semantically constrained in a particular way, why is it that the class of adverbs does not need to meet such a requirement? If it is true that the Conservativity Constraint is a byproduct of the boolean structure of language Keenan and Stavi (1986), should we stipulate that the adverb *only* does not reflect this deep semantic property whereas determiners must adhere to it?

Indeed, conservativity is a property of functions, and a priori functions may be denoted by any type of linguistic categories regardless of their syntactic behavior. Nothing in the notion of conservativity (see Definition 2) entails that it should apply to restrict the set of possible denotations of those categories that belong to the syntactic class of determiners, as the Conservativity Constraint in (2) claims. For this reason, appealing to the syntactic behavior of *only* in order to save the Conservativity Constraint seems far from providing a principled understanding of the availability of a non-conservative function to interpret sentences like (8).

There is a further observation that has forced researchers to question the Conservativity Constraint. It has been known since Westerståhl (1985) that there is a rather uncontroversial determiner, *many*, that casts doubt on the Conservativity Constraint. Consider the so-called relative proportional interpretation of *many* in sentence (11), produced by Westerståhl.

(11) Many Scandinavians have won the Nobel Prize in literature

Indeed, in 1984, by the time Westerståhl considered sentence (11), there were 14 winners of the Nobel Prize in literature from Scandinavia out of a total of 81. This situation makes sentence (11) true under a relative proportional interpretation of *many*.[5]

Westerståhl's account for the relative proportional reading of (11) was based on the idea that the order of arguments of *many* is switched or reversed: although *many* forms a constituent with the NP to the exclusion of the VP in syntax, at the relevant level of semantic interpretation the VP and the NP would be respectively the first and the second argument of *many*. Crucially, *many* would not take its expected restriction, namely the set denoted by the NP *Scandinavians*, but rather the set denoted by the VP *have won the Nobel Prize in literature*. Accordingly, statement (11) would be equivalent to *many of the winners of the Nobel Prize in literature were Scandinavians* (Cohen 2001, p. 42). In this sense, the relative proportional reading of *many* in (11) would be non-conservative on the set denoted by the NP, but rather on the set denoted by the VP (cf. also Keenan 1986, 2002).[6]

This reverse interpretation of the relative proportional reading of *many* is followed as well in Herburger (1997)'s proposal, according to which what triggers the reverse order of arguments of *many* is the focalization of *Scandinavians*. However, here we shall follow Cohen's criticism of the reverse interpretation, and thus we shall assume that in (11) the restriction of *many* is the NP *Scandinavians*.

Let us thus adapt Cohen (2001)'s characterization of the relative proportional interpretation of the determiner *many*. A part from the universe domain U under consideration (say, for instance, the world population in 1984 or the European population in 1984), we need to consider the following two basic sets included in U:

1. the set S of Scandinavians, and
2. the set L of Nobel laureates in literature.

Intuitively, the statement in (11) is true "iff the proportion of Scandinavians who won the Nobel Prize is greater than the proportion of Nobel laureates

[5] The relative proportional interpretation of *many* is different from other interpretations that can be attributed to *many*: the so-called *cardinal interpretation* and *proportional interpretation* (Partee 1988). Note, in this regard, that sentence (i) can mean either that the number of customers that bought the new product is considered large, or that a large proportion of customers bought the new product. In the former case we obtain a cardinal meaning of *many*, whereas in the latter we obtain a proportional reading.

(i) Many customers bought the new product

[6] That *many* is not conservative (on its first argument) is supported by the observation that sentence (11), under the reverse interpretation, is not equivalent to the sentence *many that are Scandinavian have won the Nobel Prize in literature and are Scandinavian*. Cf. footnote 1 for this type of paraphrase.

in literature in the world (or European) population as a whole (Cohen 2001, p. 54)". Formally, this means that in order to calculate the truth value of (11) we need to check whether

(12)

$$\frac{|S \cap L|}{|S|} > \frac{|U \cap L|}{|U|}.$$

Given that $L \subseteq U$, $U \cap L = L$; this allows us to rewrite (12) in the following terms:

(13)

$$\frac{|S \cap L|}{|S|} > \frac{|L|}{|U|}.$$

We propose, in accordance with these considerations, the following general definition of the quantifier $[\![rpmany]\!](A)$ for all property A:

Definition 7. *For all universe U and all $A \subset U$,*

$$[\![rpmany]\!](A) = \left\{ B \subseteq U : A \not\subseteq B \wedge \frac{|A \cap B|}{|A|} > \frac{|B|}{|U|} \right\}.$$

Observe that, according to this definition, the set A is a proper subset of U and is not a subset of B. We shall motivate these restrictions in subsection 2.1. Now we show that the function denoted by the relative proportional determiner *many* is non-conservative.

Lemma 4. *The determiner denotation $[\![rpmany]\!]$ violates the Conservativity Constraint.*

Proof. Let the universe domain $U = \{a, b, c, d, e\}$, and consider its subsets $A = \{a, b\}$ and $B = \{b, c, d\}$. Then, $|A \cap B| = |\{b\}| = 1$, $|A| = 2$, $|B| = 3$ and $|U| = 5$. On one hand, observe that

$$\frac{|A \cap B|}{|A|} = \frac{1}{2}$$

and that

$$\frac{|B|}{|U|} = \frac{3}{5}.$$

Given that $\frac{1}{2} \not> \frac{3}{5}$, we must conclude that:

$$\frac{|A \cap B|}{|A|} \not> \frac{|B|}{|U|}.$$

Thus, $B \notin [\![rpmany]\!](A)$. But one the other hand note that:

$$\frac{|A \cap (A \cap B)|}{|A|} = \frac{1}{2} > \frac{|A \cap B|}{|U|} = \frac{1}{5},$$

whereby $A \cap B \in [\![rpmany]\!](A)$. As a result, it is not true that

$$B \in [\![rpmany]\!](A) \Leftrightarrow A \cap B \in [\![rpmany]\!](A),$$

which means that $[\![rpmany]\!]$ does not satisfy the Conservativity Constraint. $\square$

The determiner *few* can also have a relative proportional reading. The quantifier $[\![rpfew]\!](A)$ can be described analogously to Definition 7:

Definition 8. *For all universe U and all $A \subset U$,*

$$[\![rpfew]\!](A) = \left\{ B \subseteq U : A \nsubseteq B \wedge \frac{|A \cap B|}{|A|} < \frac{|B|}{|U|} \right\}.$$

Lemma 5. *The determiner denotation $[\![rpfew]\!]$ violates the Conservativity Constraint.*

Proof. Let $U = \{a, b, c, d\}$ and consider $A = \{b, c, d\}$ and $B = \{a, b\}$. Then, $|A \cap B| = 1$, $|A| = 3$, $|B| = 2$ and $|U| = 4$. On one hand, observe that

$$\frac{|A \cap B|}{|A|} = \frac{1}{3}$$

and that

$$\frac{B}{U} = \frac{2}{4}.$$

Given that $\frac{1}{3} < \frac{2}{4}$, we must conclude that

$$\frac{|A \cap B|}{|A|} < \frac{B}{U}.$$

Thus, $B \in [\![few]\!](A)$. But on the other hand note that

$$\frac{|A \cap (A \cap B)|}{|A|} = \frac{1}{3} \not< \frac{|A \cap B|}{|U|} = \frac{1}{4},$$

whereby $A \cap B \notin [\![few]\!](A)$. As a result, it is not true that

$$B \in [\![few]\!](A) \Leftrightarrow A \cap B \in [\![few]\!](A),$$

which means that the determiner denotation $[\![rpfew]\!]$ is a counterexample to the Conservativity Constraint. □

2.1 On the availability of relative proportional readings

The definitions above given for the relative proportional quantifiers incorporate two natural constraints: the restriction A of a relative proportional quantifier cannot be identical to the universe domain and cannot be included in the set B denoted by the predicate. These constraints are introduced in order to account for the unavailability of relative proportional readings in certain cases. We shall discuss them in turn in the following two remarks.

Remark 1. *The relative proportional readings of statements of the form [[many A] B] and [[few A] B] are not available when $A = U$.*

Consider the following reasoning in order to perceive that the relative proportional interpretations are not available when the universe is the restriction. Assume that U is the human population in 2014. Then, the set denoted by the noun *humans* in (14) is identical to the universe we are considering. In this case, the relative proportional readings of the following two sentences are not available.

(14) a. Many humans live in China
b. Few humans live in Greenland

We shall test the unavailability of the relative proportional readings of the sentences in (14). Assume, contrary to what Definitions 7 and 8 state, that U can be the restriction of the determiners $[\![rpmany]\!]$ and $[\![rpfew]\!]$. Then, B would belong to the quantifier $[\![rpmany]\!](U)$ iff:

(15)
$$\frac{|U \cap B|}{|U|} > \frac{|B|}{|U|}.$$

Dually, B would belong to the hypothetical $[\![rpfew]\!](U)$ iff:

(16)
$$\frac{|U \cap B|}{|U|} < \frac{|B|}{|U|}.$$

However, for all subset B of a universe U, $U \cap B = B$, whereby:

(17)
$$\frac{|U \cap B|}{|U|} = \frac{|B|}{|U|}.$$

This means that inequalities (15) and (16) are false, and thus B belongs neither to $[\![rpmany]\!](U)$ nor to $[\![rpmany]\!](U)$.

But these truth conditions do not seem to reflect our semantic intuitions about the sentences in (14): our linguistic competence does not allow us to interpret the sentences (14-a) and (14-b) as contradictions. This clearly indicates that *many* and *few* are not interpreted as relative proportional determiners in (14-a) and (14-b).

Expressions (14-a) and (14-b) admit, for instance, a cardinal reading, according to which they are interpreted as contingencies. Given the present distribution of human populations, statement (14-a) will be true because the cardinality of the intersection of U and the set C of humans living in China may be considered a 'large natural number': more than 1,300,000,000 people live in China out of 7,000,000,000 people, the (approximate) total world population in 2014. Analogously, (14-b) will be true because approximately 55,000 people live in

Greenland, a figure that may be regarded as small.[7]

Remark 2. *The relative proportional readings of statements of the form [many A] B] and [few A] B] are not available when* $A \subseteq B$.

We shall test the unavailability of relative proportional interpretations firstly when $A = B$ and secondly when $A \subset B$.

1. $A = B$. Consider the following sentences, where both the NP and the VP denote the set L of linguists.

   (18) a. Many linguists are linguists
   b. Few linguists are linguists

   Assume, contrary to Definition 7 of $[\![rpmany]\!](A)$, that $A = B$. Recall that, from Remark 1, A is a proper subset of U. Consider the case where $A \neq \emptyset$. Then, A would necessarily belong to $[\![rpmany]\!](A)$, since

   $$\frac{|A \cap A|}{|A|} = 1 > \frac{|A|}{|U|}.$$

   But note that A would not belong to $[\![rpmany]\!](A)$ when $A = \emptyset$. In fact, it is not true that

   $$\frac{0}{0} > \frac{0}{|U|},$$

   because $|U| \cdot 0 = 0 \cdot 0$.

   Accordingly, the sentence *many linguists are linguists* should be true iff the universe domain contains at least one linguist ($L \neq \emptyset$) and at least one non-linguist ($L \subset U$), no matter the exact cardinalities of U and A. If U had billions of individuals, and solely one of them were a linguist, then sentence (18-a) should be a true statement; if U contained just two individuals and one of them were a linguist, then (18-a) should still be true. Given that these truth conditions do not reflect our semantic intuitions about sentence (18-a), we must conclude that *many* is not interpreted as a relative proportional determiner when $A = B$.

   Assume, contrary to Definition 8 of $[\![rpfew]\!](A)$, that $A = B$. Then, statement (18-b) would necessarily be false under a relative proportional reading: A could never belong to $[\![rpfew]\!](A)$, since inequality (19) is always false.

[7] What counts as a small or large number is of course vague and seems to be dependent on context (Westerståhl 1985, Lappin 1988, 1993), but also on the intensions of the NP and the VP denotations (Keenan and Faltz 1985, Fernando and Kamp 1996, Lappin 2000).

(19)

$$\frac{|A \cap A|}{|A|} < \frac{|A|}{|U|}.$$

Let $A \neq \emptyset$. Observe that $\frac{|A \cap A|}{|A|} = 1$ and also that $\frac{|A|}{|U|} < 1$, because $A \subset U$ from Remark 1. In this case, $\frac{|A \cap A|}{|A|} > \frac{|A|}{|U|}$.

Let $A = \emptyset$. Then, (19) becomes the following inequality:

(20)

$$\frac{0}{0} < \frac{0}{|U|},$$

which, as argued above, is false, because $|U| \cdot 0 = 0 \cdot 0$.

Therefore, the inequality (19) will necessarily be false, and thus A would never belong to $[\![rpfew]\!](A)$.

Accordingly, sentence (18-b) should be false no matter the proportion of linguists in the universe domain: it would be false if there are two linguist in a universe with millions of individuals. Since sentence (18-b) is not readily interpreted according to these truth conditions, we must conclude that *few* cannot be interpreted as a relative proportional determiner when $A = B$. This is ensured by the proposed Definition 8 of $[\![rpmany]\!](A)$.

2. $A \subset B$. Consider the following sentences, where the predicate denotation properly includes the NP denotation.

(21) a. Many Italians are European
b. Few Italians are European

We shall first focus our attention on *many*. Assume, contrary to Definition 7 of $[\![rpmany]\!](A)$, that $A \subset B$. Then, B would belong to $[\![rpmany]\!](A)$ iff:

(22)

$$\frac{|A \cap B|}{|A|} > \frac{|B|}{|U|}.$$

Let us inspect when (22) is true. On one hand, given that $A \subset B$, $A \cap B = A$; as a result, $\frac{|A \cap B|}{|A|} = \frac{|A|}{|A|}$, which is equal to 1 iff $A \neq \emptyset$. On the other hand, $\frac{|B|}{|U|} < 1$ iff $B \neq U$. Thus, (22) is true iff $A \neq \emptyset$ and $B \neq U$.

Therefore, sentence (21-a) should be true iff the set of Italians is not empty and the set of Europeans is properly included in the universe domain; or in other words, statement (21-a) should be equivalent to *in the universe* *domain there is at least one Italian and at least one non-European.* These truth conditions, though, are not supported by our semantic intuitions concerning (21-a). This reveals that (21-a) cannot receive a relative proportional interpretation.

Dually, a proper superset B of A would belong to $[\![rpfew]\!](A)$ iff:

(23)
$$\frac{|A \cap B|}{|A|} < \frac{|B|}{|U|}.$$

This inequality, though, is necessarily false. Given that, by assumption, $A \subset B$, if $A \neq \emptyset$, then the left-hand fraction $\frac{|A \cap B|}{|A|}$ is equal to 1. But of course, the right-hand fraction $\frac{|B|}{|U|}$ cannot be larger to 1: it will be smaller than 1 if $B \subset U$ and it will be equal to 1 if $B = U$. If $A = \emptyset$, then $\frac{0}{0} = \frac{|B|}{|U|}$, because $|U| \cdot 0 = 0 \cdot |B|$.

Therefore, statement (21-b) should be a contradiction. But again it seems that the truth conditions we have just described no not correspond to the expression (21-b).

In the remainder of this article we shall argue for the Witness Set Constraint, an alternative to the Conservativity Constraint just reviewed, in order to account not only for the observation that certain non-conservative functions are not expressed by natural language determiners, but also for the non-conservativity of *only* and the relative proportional determiners *many* and *few*.

3 The Witness Set Constraint

In reading Barwise and Cooper (1981) it is clear that these authors were not concerned with deriving the property "lives on" (see above Definition 3) from deeper principles. They defined this property, which had gone unnoticed by semanticists, claimed that they were not aware of any natural language determiner that did not map a common noun denotation A to a quantifier that did not live on A, and assumed the property in order to define certain linguistic universals and demonstrate relevant propositions of their theory. The authors observed as well that the property is used in order to prove quantified statements such as (24), where the determiner *more than half* applies to an infinite set, the set of integers.

(24) More than half of the integers are not prime

The truth of this statement is not dependent "on an a priori logic", Barwise and Cooper note, but rather on "which underlying measure of infinite sets one is using" (Barwise and Cooper 1981, p. 163). This measure may live on different infinite sets: if the chosen metrics lives on the set of prime number, then the statement is false, but "more common measures which do not give special weight to primes will make [(24)] true" (Barwise and Cooper 1981, footnote 3).

What is more relevant to our concerns is that the "lives on" property (or equivalently, the conservativity property) may be related to the semantic processing of quantified statements. Let us focus our attention on this issue.

Barwise and Cooper mentioned an objection that could be leveled against Montague's treatment of NP's: in order to check the truth of a sentence like *John runs*, "we need to calculate the denotation of $[\text{John}]_{NP}$, namely, the family of all sets X to which John belongs, and then see if the set of runners is one of these sets". This procedure for checking the truth of a simple sentence containing a proper noun seems "well nigh impossible", and "clearly corresponds in no way to the reasoning process actually used by a native speaker of English (Barwise and Cooper 1981, p. 191)". As a consequence, they suggest an intuitive checking procedure for simple NPs, which can be applied in general to quantified expressions, based on the notion of *witness set*. This notion incorporated the property of "lives on" in the following way (Barwise and Cooper 1981, p. 191):[8]

(25) *Witness set* (preliminary definition)
A witness set for a quantifier *D(A)* living on A is any subset w of A such that $w \in D(A)$.

The only witness set for $[\![John]\!]$ is the singleton $\{John\}$. A witness set for $[\![\text{a doctor}]\!]$ and for $[\![\text{most doctors}]\!]$ are, respectively, any non-empty subset of doctors and any subset of doctors that contains most doctors. And the only witness set for $[\![\text{no doctor}]\!]$ is $\emptyset$.

The notion of witness set simplifies the semantic processing of quantified statements by reducing the sets to be considered. In order to perceive this observation, note, for instance, that the set of presidents, the set of men and the set of pianists would all belong to the generalized quantifier $[\![\text{a doctor}]\!]$, for the

[8] Cf. Szabolcsi (2010) and Szabolcsi (1997) for an introduction to the notion of witness set, as well as Beghelli et al. (1997) for an application of this notion to the study of scope ambiguities. It is worth noting that witness sets play a crucial role in some recent investigations on natural language quantification, especially in the study of collective and cumulative readings of quantified sentences in natural languages (cf. Robaldo 2011, Robaldo 2013, Robaldo and Szymanik 2012, Robaldo et al. 2014). Particularly, Robaldo (2013) studies the role of conservativity in relation to these interpretations and argues that it serves to maximize witness sets for quantifiers; conservativity is thus important to ensure correct truth conditions of some interpretations.

intersection of any of those three sets with the set of doctors is non-empty: we may be able to find a president, a man or a pianist who is also a doctor. However, none of those sets would be a witness set for the generalized quantifier $[\![$a doctor$]\!]$, because none of them satisfies the condition of being a subset of the set of doctors, contrary to what the definition of witness set requires; indeed, in a very realistic situation, we may be able to find a president, a man and a pianist who is not a doctor. Consequently, those three sets would be irrelevant when, for instance, the statement *a doctor speaks French* is semantically processed.

We can now decide whether B belongs to the quantifier $D(A)$ living on A by following the procedure in (26), which examines whether there exists a witness set w for $D(A)$ with certain properties (Barwise and Cooper 1981, Szabolcsi 1997).

(26) Let $W[D(A)]$ be the set of witness sets for a quantifier $D(A)$ living on A.

a. If $D(A)$ is monotone increasing, i.e.,

$$(\forall X, Y)\,((X \in D(A) \wedge X \subseteq Y) \Rightarrow Y \in D(A)),$$

then check whether $(\exists w)\,(w \in W[D(A)] \wedge w \subseteq B)$.

b. If $D(A)$ is monotone decreasing, i.e.,

$$(\forall X, Y)\,((X \in D(A) \wedge Y \subseteq X) \Rightarrow Y \in D(A)),$$

then check whether $(\exists w)\,(w \in W[D(A)] \wedge (A \cap B) \subseteq w)$.

c. If $D(A)$ is non-monotonic, i.e., it is neither increasing nor decreasing, then check whether $(\exists w)\,(w \in W[D(A)] \wedge (A \cap B) = w)$.

d. If there is such a w, then we conclude that $B \in D(A)$; otherwise, we conclude that $B \notin D(A)$.

Accordingly, in order to determine whether *a doctor speaks French* is true in a particular model we would need to find firstly a witness set w for $[\![$*a doctor*$]\!]$, i.e., a non-empty subset of the set D of doctors that, according to the model under consideration, belongs to $[\![$*a doctor*$]\!]$. Assume for instance that $w = \{Marie, David\}$, i.e., assume that Marie and David belong to D and that w belongs to the quantifier $[\![$a doctor$]\!]$. Secondly, we should determine whether this quantifier is monotone increasing, monotone decreasing or non-monotonic; as is well known, it is monotone increasing (observe for instance that the set F of French speakers is included in the set of speakers of Romance languages and that the sentence *a doctor speaks French* entails the sentence *a doctor speaks a Romance language*). Thirdly, we would need to follow step (26-a) and check whether w is included in F, i.e., whether Marie and Peter, which belong to D, belong to F as well.

The introduction of the "lives on" property into the definition of witness set was quite natural in Barwise and Cooper (1981), especially because they had previously claimed that they knew "of no counterexamples in the world's languages to the following requirement":

> *U3. Determiner Universal* (Barwise and Cooper 1981, p. 177)
> Every natural language contains basic expressions (called determiners) whose semantic function is to assign to common noun denotations (i.e., sets) A a quantifier that lives on A.[9]

However, as we shall see, it is not necessary to bring the "lives on" property (or equivalently, the conservativity property) into the definition of witness set, and indeed it is desirable to define the concept of witness set without appealing to it, in the following way:

Definition 9. ***Witness set*** *(final definition)*
For all U and all $A \subseteq U$, a set w is a witness set for a generalized quantifier $D(A)$ iff w is a subset of A such that $w \in D(A)$.

Henceforth, $W[D(A)]$ will be the set of witness sets for a given quantifier $D(A)$ defined according to Definition 9.

As we shall argue in detail, this modification of the concept of witness set will allow us to predict:

1. the existence of conservative determiners,
2. the non-existence of certain non-conservative determiners (such as $[\![\widehat{n}]\!]$-numerals, cardinal comparative determiners and the converses of proportionality determiners), and most importantly,
3. the existence of linguistic categories that express non-conservative functions (such as *only* or the determiners *many* and *few* in their relative proportional readings).

[9]Note that, the way it is formulated, Barwise and Coopers's U3 allows for a reading that makes it weaker than Keenan and Stavi's Conservativity Constraint. Under that reading, U3 claims that all natural languages have quantifiers that live on its restriction (or equivalently, conservative determiners), whereas the Conservativity Constraint claims that all determiners of natural languages are conservative. Hence, under this reading, U3 is not falsified by the existence of non-conservative determiners in natural languages. However, it may well be that Barwise and Cooper's intention was that it should be falsified by the existence of non-conservative determiners as suggested by the use of the parentheses around "called determiners". According to this reading, U3 and the Conservativity Constraint would be equivalent.

Thus, if we do not restrict the notion of witness set to quantifiers that live on, we can understand why certain non-conservative functions are attested in natural languages whereas other non-conservative functions are banned.

This will lead us to a new understanding of conservativity. It is not a primitive principle of natural language semantics, but rather a byproduct of a truth calculation constraint that requires appropriate witness sets in order to develop a simpler model for how quantified statements are interpreted following procedure (26). If this is correct, it is not the Conservativity Constraint that dictates what functions from properties to quantifiers can be expressed by natural language determiners, but rather what we shall call the Witness Set Constraint:

(27) *Witness set Constraint*
For all U, if a function $D \in \mathcal{D}_U$ is expressed by a linguistic category, then, for all $A, B \subseteq U$:
a. if $D(A)$ is monotone increasing, then, $B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge w \subseteq B)$
b. if $D(A)$ is monotone decreasing, then $B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge (A \cap B) \subseteq w)$
c. if $D(A)$ is non-monotonic, then $B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge (A \cap B) = w)$

This constraint does not simply claim that a function $D \in \mathcal{D}_U$ can be expressed by a linguistic category if, for any subset A of a universe, $D(A)$ has witness sets. In fact, in section 4 we shall encounter many determiner denotations that yield quantifiers that do have witness sets and nonetheless are not permitted by the Witness Set Constraint. Crucially, the Witness Set Constraint appeals to an *appropriateness requirement*: it requires D functions conveyed by linguistic categories to yield quantifiers that have witness sets that are *appropriate* for the procedure in (26) to calculate the truth of quantified statements. For instance if (i) a quantifier $D(A)$ is monotone increasing, (ii) B does not belong to $D(A)$, and (iii) there is some witness set for $D(A)$ included in B, then the determiner D is not a permissible denotation for a linguistic category according to the Witness Set Constraint. In this case, conditional (27-a) is false, because its antecedent is true ($D(A)$ is monotone increasing) and its subsequent (a biconditional) is false; $W[D(A)]$ is not empty, but none of its elements is an appropriate witness set.

This appropriateness requirement seems a conceptual necessity: if D yields generalized quantifiers that have witness sets that cannot be used to calculate effectively the truth of a quantified statement following (26-a), (26-b) or (26-c), then D should be banned. This requirement, which is crucial to test the adequacy of our proposals, will be discussed in detail in section 4.

The approach we are about to develop provides us with a particularly interesting result: we can now account for the presence of certain categories that express functions that are non-conservative with no further stipulation. It will be immaterial for our concerns whether *only* is a determiner or an adverb from a syntactic point of view, or whether certain determiners are conservative on their first argument, whereas others are conservative on their second argument (Keenan 1996, 2002). A function D from $\mathcal{P}(U)$ to $\mathcal{P}(\mathcal{P}(U))$ can be conveyed by a linguistic category only if all the quantifiers that it yields as output have appropriate witness sets, thereby permitting the procedure (26) to semantically compute quantified statements on the basis of monotonicity properties.

3.1 All conservative functions satisfy the Witness Set Constraint

An initial attractive consequence of our proposal is that it acknowledges a causal relationship between the conservativity property and the appropriateness requirement of the Witness Set Constraint, as expressed in Theorem 2.

Theorem 2. *For all U and all $D \in \mathcal{D}_U$, if D is conservative, then, for all $A \subseteq U$, if $D(A) \neq \emptyset$, it has appropriate witness sets.*

Proof. Let D be conservative. Assume that $A \subseteq U$ and also that $D(A) \neq \emptyset$. This means that there is some set B that belongs to $D(A)$. Given that, by assumption, D is conservative and $B \in D(A)$, then $A \cap B \in D(A)$. But it is a set-theoretical truth that $A \cap B \subseteq A$. Therefore, $D(A)$ has a witness set, namely $A \cap B$.

Thus, all conservative functions have witness sets. We shall now investigate whether all conservative functions have appropriate witness sets, i.e., we shall investigate whether all conservative functions satisfy conditionals (27-a), (27-b) and (27-c) of the Witness Set Constraint.

Let D be conservative and the quantifier $D(A)$ monotone increasing. Then we need to verify whether the following biconditional is true for all sets A, B (cf. (27-a)):

(28)
$$B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge w \subseteq B).$$

Assume the left-hand of (28); then, as we just argued, since D is conservative, $A \cap B$ is a witness set for $D(A)$. This implies the truth of the right-hand of (28), for the witness set $A \cap B$ is included in B. Assume now the right-hand of (28):

there is a witness set w for $D(A)$ –i.e., a subset of A that belongs to $D(A)$– that is included in B. Given that $D(A)$ is monotone increasing, the existence of a set included in B that belongs to $D(A)$ entails the left-hand of (28).

Let D be conservative and the quantifier $D(A)$ monotone decreasing. Then we need to check whether the following biconditional holds for all A, B (cf. (27-b)):

(29)
$$B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge (A \cap B) \subseteq w).$$

Assume the left-hand of (29); then, since D is conservative, the subset $A \cap B$ of A belongs to $D(A)$, in which case $A \cap B$ is a witness set for $D(A)$. Accordingly, the right-hand of (29) is implied. Assume now the right-hand of (29): there is a set w that belongs to $D(A)$ and includes $A \cap B$. The premise that $D(A)$ is monotone decreasing entails that the subset $A \cap B$ of w belongs to $D(A)$: if w belongs to $D(A)$ and $D(A)$ is monotone decreasing, then the subset $A \cap B$ of w also belongs to $D(A)$. And finally, if $A \cap B$ belongs to $D(A)$, as we have just concluded, then B belongs to $D(A)$ as well, because, by assumption, D is conservative.

Let D be conservative and the quantifier $D(A)$ non-monotonic. Then we need to demonstrate that the following biconditional is true for all A, B (cf. (27-c)):

(30)
$$B \in D(A) \Leftrightarrow (\exists w)\,(w \in W[D(A)] \wedge (A \cap B) = w).$$

The left-hand of (30) entails that $A \cap B$ is a witness set for $D(A)$; thus the right-hand of (30) is true as well. Assume now the right-hand of (30): the subset $A \cap B$ of A belongs to $D(A)$; this, along with the premise that $D(A)$ is conservative, implies the left-hand of (30).

Therefore, all conservative determiners have appropriate witness sets. □

Note that, for any $D \in \mathcal{D}_U$, $D(A)$ is often empty. Consider, for instance, the set P denoted by the noun phrase *planets of the Solar System*; assume $|P| = 8$. The quantifier denoted by the quantifier phrase *exactly ten planets of the Solar System* would be the following, using Barwise and Cooper (1981) notation:

$$[\![10!]\!](P) = \{X \subset U : |P \cap X| = 10\}\,.$$

In this model, there is no property X whose intersection with P has exactly ten elements, because the cardinality of P is eight. In general, for any natural number n, the quantifier $[\![n!]\!](A)$ is empty when $|A| < n$. We must introduce the following lemma concerning empty quantifiers and our Witness Set Constraint.

Lemma 6. *Empty generalized quantifiers lack witness sets and are compatible with the appropriate requirements of the Witness Set Constraint.*

Proof. Let $D(A) = \emptyset$. Then, there is no $B \subseteq U$ such that $B \in D(A)$. Consequently, $D(A)$ lacks witness sets, in which case there is no witness set w for $D(A)$ such that (a) $w \subseteq B$, (b) $(B \cap A) \subseteq w$ or (c) $(B \cap A) = w$. This means that the biconditionals in (27-a), (27-b) and (27-c) of the Witness Set Constraint are true, because their respective left-hand statements and right-hand statements are false. $\square$

We shall continue by illustrating that $[\![n]\!]$-numerals satisfy the Witness Set Constraint. This is of course a consequence of Lemma 1 and Theorem 2: given that $[\![n]\!]$-numerals are conservative and all conservative functions have appropriate witness sets, $[\![n]\!]$-numerals have appropriate witness sets; therefore, they can be conveyed by natural language categories (such as determiners). Secondly, we shall demonstrate that inner negations, which are non-conservative, lack witness sets; they are thus banned as denotations of linguistic categories by the Witness Set Constraint.

Remark 3. *$[\![n]\!]$-numerals satisfy the Witness Set Constraint.*

A witness set for $[\![n]\!](A)$ is, by Definition 9 of witness set, any subset of A that belongs to $[\![n]\!](A)$. In other words, X is a witness set for $[\![n]\!](A)$ iff $X \subseteq$ A and, by Definition (6) of $[\![n]\!](A)$, $|X \cap A| \geq n$. This is the set of witness sets for $[\![n]\!](A)$:

$$W[[\![n]\!](A)] = \{X \subseteq A : |X \cap A| \geq n\}.$$

This means, simply, that any subset of A that contains at least n elements will be a witness set for $[\![n]\!](A)$. But we expect from Theorem 2 that $D(A)$ has appropriate witness sets to calculate the truth of quantified statements on the basis of monotonicity properties, for D is conservative. In this regard, note that, for any $A \subseteq U$, the quantifier $[\![n]\!](A)$ is increasing monotonic, since

$$(\forall X, Y)((X \in [\![n]\!](A) \wedge X \subseteq Y) \Rightarrow Y \in [\![n]\!](A)).$$

Thus, according to the Witness Set Constraint (27), a witness set w for $[\![n]\!](A)$ is appropriate iff the following biconditional holds (cf. (27-a)):

$$B \in [\![n]\!](A) \Leftrightarrow w \subseteq B.$$

Assume that $B \in [\![n]\!](A)$. Then, by the definition of $[\![n]\!](A)$, $|A \cap B| \geq n$. In this case, $A \cap B \in [\![n]\!](A)$ and, moreover, $A \cap B \subseteq A$. Therefore, $A \cap B$ is a witness set for $[\![n]\!](A)$, which, in addition, is included in B. Thus, the truth of the left-hand of the biconditional entails the truth of its right-hand.

Assume now that there is a witness set w for $[\![n]\!](A)$ that is included in B. Given that, by assumption, w is included in A and has at least n elements, and w is also included in B, then $|B \cap A| \geq n$. This entails that $B \in [\![n]\!](A)$. Accordingly, the truth of the right-hand of the biconditional implies the truth of its left-hand. Therefore, $[\![n]\!]$-numerals satisfies the Witness Set Constraint, as expected from Lemma 1 and Theorem 2.

Consider, for concreteness, the $[\![n]\!]$-determiner *two*. In order to find out, using the procedure in (26), whether the statement *two doctors speak French* is true, i.e., whether $F \in [\![two]\!](D)$, we need to follow three steps. Firstly, we must find a witness set w for $[\![two]\!](D)$; assume that $w = \{$Pierre, John, Marie$\}$. Secondly, we must determine whether $[\![two]\!](D)$ is monotone increasing, a monotone decreasing, or non-monotonic; as noted, it is monotone increasing. Finally, given that it is monotone increasing, we must follow the step indicated in (26-a), and check whether $w = \{$Pierre, John, Marie$\}$ is included in F. Only if this is the case can we conclude that $F \in [\![two]\!](F)$.

Consider now $[\![\widehat{n}]\!]$-determiners. We shall see that they are not permitted by the Witness Set Constraint because, for all subsets A of U, the quantifier $[\![\widehat{n}]\!](A)$ does not have a witness set.

Lemma 7. *For all U and all $A \subseteq U$, the generalized quantifier $[\![\widehat{n}]\!](A)$ has no witness set, whereby the function $[\![\widehat{n}]\!]$ cannot be expressed by any linguistic category.*

Proof. Recall that, for an arbitrary set $X \subseteq U$, X belongs to $[\![\widehat{n}]\!](A)$ iff $\left|X \cap \overline{A}\right| \geq n$. Note, though, that for any $[\![\widehat{n}]\!]$-determiner, $X \cap \overline{A}$ must be non-empty, since $[\![\widehat{n}]\!]$-determiners are defined for any natural number n that is equal or larger to 1. Consequently, if X belongs to $[\![\widehat{n}]\!](A)$, then it has some elements that are not elements of A (because they are elements of $\overline{A}$), whereby X cannot be included in A. This leads us to the situation where

$$X \in [\![\widehat{n}]\!](A) \text{ iff } X \nsubseteq A,$$

which implies that there is no witness set for $[\![\widehat{n}]\!](A)$. $\square$

The reasoning just developed for $[\![\widehat{n}]\!]$-determiners, can also be applied to account for the absence in the world's languages of a further non-conservative determiner, the *allnon* determiner discussed by (Chierchia and McConnell-Ginet 2000, p. 426-427):

Definition 10. *For all U and all $A \subseteq U$,*

$$allnon(A) = \{X \subseteq U : (U - A) \subseteq X\}.$$

According to this definition, the following sentence would state that all the individuals of the universe U under consideration that do not belong to the set D of doctors belong to the set F of French speakers.

(31) Allnon doctors speak French

Lemma 8. *For all U and all $A \subseteq U$, the generalized quantifier $[\![allnon]\!](A)$ has no witness set, whereby the function $[\![allnon]\!]$ cannot be expressed by any linguistic category.*

Proof. According to Definition 10, a set X belongs to $[\![allnon]\!]$(A) iff $(U - A) \subseteq X$. Therefore, X belongs to $[\![allnon]\!]$(A) iff X contains all the elements of the universe except those contained in A, in which case X is not included in A. This means that

$$X \in [\![allnon]\!](A) \Leftrightarrow X \nsubseteq A.$$

Therefore, there can be no witness set for $[\![allnon]\!](A)$. □

As for statement (31) particularly, note that, whereas D contains all doctors of U, $[\![allnon]\!](D)$ contains all sets whose members are not doctors. For this reason, there can be no set w that is both included in D and belong to $[\![allnon]\!](D)$, whereby there is no witness set for $[\![allnon]\!](D)$.

Given the Witness Set Constraint, the functions $[\![\widehat{n}]\!]$ and $[\![allnon]\!]$ are not legitimate denotations for linguistic categories. Observe, in this regard, that the quantifiers $[\![\widehat{n}]\!](A)$ and $[\![allnon]\!](A)$ are monotone increasing. For instance, the sentence *owt doctors arrived late* (i.e., "two individuals who are not doctors arrived late") entails the sentence *owt doctors arrived* ("two individuals who are not doctors arrived"), but not the other way around. And similarly, *allnon doctors arrived late* entails the sentence *allnon doctors arrived*, but not the other way around.

However, since these inner negations lack witness sets, we cannot find, for all $A \subseteq U$, a witness set w such that, respectively:

$$w \subseteq A \wedge w \in [\![n]\!](A)$$
$$w \subseteq A \wedge w \in [\![allnon]\!](A).$$

The step described in (26-a) to calculate the truth conditions of this class of quantified statements yields no output: it cannot calculate whether $w \subseteq A$ because there is no witness set w. If witness sets are required, following Barwise and Cooper's insights, in order to provide a more feasible model for the computation of statements that contain generalized quantifiers, then the unavailability of witness sets for inner negations may be the source of the non-existence of this type of determiner.

Importantly, as we shall see immediately in subsection 3.2, the converse to Theorem 2 fails, which is crucial for our proposal: there are certain functions denoted by linguistic categories that satisfy the Witness Set Constraint but violate the Conservativity Constraint.

3.2 Non-conservative denotations of linguistic categories satisfy the Witness Set Constraint

The non-attested inner negations studied above can be banned by both the Conservativity Constraint and the Witness Set Constraint. With the aim of defending that the latter is empirically more adequate than the former we shall show that certain well-attested non-conservative functions expressed by linguistic categories, namely $[\![only]\!]$, $[\![rpmany]\!]$ and $[\![rpfew]\!]$, do satisfy the Witness Set Constraint. This will strongly suggest that the Witness Set Constraint is a more likely candidate for a universal principle that constrains the set of functions from $\mathcal{P}(U)$ to $\mathcal{P}(\mathcal{P}(U))$ that are expressed by linguistic items.

Lemma 9. *The function $[\![only]\!]$ satisfies the Witness Set Constraint.*

Proof. From Definition 6 of $[\![only]\!](A)$, B belongs to $[\![only]\!](A)$ iff it is a non-empty subset of A. Consequently, the set of witness sets for $[\![only]\!](A)$ is equal to the quantifier $[\![only]\!](A)$:

$$W[[\![only]\!](A)] = [\![only]\!](A) = \mathcal{P}(A) - \emptyset.$$

Observe that $[\![only]\!](A)$ is non-monotonic. It is not monotone increasing, because the truth of $B \subseteq A$ does not entail the truth of $C \subseteq A$, for any set C such that $B \subseteq C$. It is neither monotone decreasing because $\emptyset \notin [\![only]\!](A)$.

Thus, $[\![only]\!]$ satisfies the Witness Set Constraint iff the following biconditional holds for all for all $A, B \subseteq U$ (cf. (27-c)):

(32)
$$B \in [\![only]\!](A) \Leftrightarrow (\exists w)\,(w \in W[[\![only]\!](A)] \wedge (A \cap B) = w).$$

If we apply Definition 6 of $[\![only]\!](A)$ to the left-hand of (32) and we take $w = B$, we obtain the following statement:

(33)
$$\emptyset \neq A \subseteq B \Leftrightarrow (B \cap A) = B,$$

which is a set-theoretical truth relative to the properties of intersection and inclusion. Therefore, the function $[\![only]\!]$ satisfies the Witness Set Constraint. □

Let us illustrate the behavior of *only* in relation to the Witness Set Constraint. Consider the three sets $A = \{a, b, c, e\}$, $B = \{b, c\}$ and $C = \{d\}$. Observe that $A \cap B = \{b, c\} = B$, whereas $A \cap C = \emptyset \neq C$. This correctly predicts that B, unlike C, belongs to the quantifier $[\![only]\!](A)$.

More particularly, we semantically process a statement such as *only Athenians think* by checking whether the intersection of the non-empty set T of thinkers and the set A of Athenians is equal to T, following procedure (26). Only if this is the case will the statement under consideration be true.

Let us turn our attention to the non-conservative determiner denotations $[\![rpmany]\!]$ and $[\![rpfew]\!]$. We shall show that they satisfy the Witness Set Constraint.

Lemma 10. *The function $[\![rpmany]\!]$ satisfies the Witness Set Constraint.*

Proof. Let $[\![rpmany]\!](A) \neq \emptyset$. Then, by Definition 7, there is some set B such that $A \nsubseteq B$ and:

(34)
$$\frac{|A \cap B|}{|A|} > \frac{|B|}{|U|}.$$

Note that $A \cap B = A \cap (A \cap B)$. Thus, multiplying out, we see that the inequality (34) is equivalent to the following inequality:

(35)
$$\frac{|A \cap (A \cap B)|}{|A|} > \frac{|B|}{|U|}.$$

Hence, $B \in [\![rpmany]\!](A)$ iff $A \cap B \in [\![rpmany]\!](A)$; furthermore, it is necessary the case that $A \cap B$ is included in A, for the intersection of two arbitrary sets is included in both sets, whereby $A \cap B$ is a witness set for $[\![rpmany]\!](A)$.

The generalized quantifier $[\![rpmany]\!](A)$ is non-monotonic. Consider, in this regard, statement (11). We can see that $[\![rpmany]\!](S)$ is not monotone increasing because *many Scandinavians won the Nobel Prize in Literature* does not entail *Many Scandinavians have won the Nobel Prize*, and it is not monotone decreasing because the latter statement neither entails the former.

Consequently, the function $[\![rpmany]\!]$ satisfies the Witness Set Constraint iff, for all A, B (cf. (27-c)):

$$B \in [\![rpmany]\!](A) \Leftrightarrow (\exists w)\,(w \in W[[\![rpmany]\!](A)] \wedge (A \cap B) = w).$$

But as we have just argued, B belongs to $[\![rpmany]\!](A)$ iff the subset $A \cap B$ of A belongs to $[\![rpmany]\!](A)$, in which case $A \cap B$ is a witness set for $[\![rpmany]\!](A)$. Therefore, $[\![rpmany]\!]$ satisfies the Witness Set Constraint. □

Lemma 11. *The function $[\![rpfew]\!]$ satisfies the Witness Set Constraint.*

Proof. This proof is analogous to the previous one. Note that $[\![rpfew]\!](A)$ is non-monotonic. Accordingly, $[\![rpfew]\!]$ satisfies the Witness Set Constraint iff, for all A, B (cf. (27-c)):

$$B \in [\![rpfew]\!](A) \Leftrightarrow (\exists w)\,(w \in W[[\![rpfew]\!](A)] \wedge (A \cap B) = w).$$

Assume that $B \in [\![rpfew]\!](A)$, i.e., $B \nsubseteq A$ and

$$\frac{|A \cap B|}{|A|} < \frac{|B|}{|U|}.$$

Multiplying out, we see that $A \cap (A \cap B) \in [\![rpfew]\!](A)$, in which case the subset $A \cap B$ of A belongs to $[\![rpfew]\!](A)$. Therefore, $B \in [\![rpfew]\!](A)$ iff $A \cap B$ is a witness set for $[\![rpfew]\!](A)$. $\square$

3.3 Universe Independence

Natural language determiners are mostly universe independent. Roughly, a determiner is universe independent when its behavior does not change when the universe is extended. For instance, the determiner *two* is universe independent because, if *two doctors speak French* is true in a given model, then adding more elements to the universe domain will not modify the truth of this statement. The condition of Universe Independence, formally defined below, is also called *Extension* and *Constancy*.

Definition 11. ***Universe Independence***
A determiner denotation D is universe independent iff:

$$A, B \subseteq U \subseteq U' \Rightarrow (B \in D_U(A) \Leftrightarrow B \in D_{U'}(A)).$$

In Definition 11 we introduce subindexes in order to show that the universe domain with respect to which the quantifier is calculated varies. Throughout, when we suppress the parameter U, the universe is assumed to be constant.

We bring this property into consideration in order to observe that the non-conservative determiners $[\![rpmany]\!]$ and $[\![rpfew]\!]$ are universe dependent.

Lemma 12. *The functions $[\![rpmany]\!]$ and $[\![rpfew]\!]$ are universe dependent.*

Proof. Let $|U| = 10$, $|A| = |B| = 8$ and $|A \cap B| = 4$. Then,

$$\frac{|A \cap B|}{|A|} = \frac{4}{8} < \frac{|B|}{|U|} = \frac{8}{10}.$$

Accordingly, $B \notin [\![rpmany_U]\!](A)$, whereas $B \in [\![rpfew_U]\!](A)$. However, if we add 10 elements to U in order to obtain a new universe domain U', then we obtain the following inequality:

$$\frac{A \cap B}{A} = \frac{4}{8} > \frac{|B|}{|U'|} = \frac{8}{20},$$

in which case $B \in [\![rpmany_{U'}]\!](A)$, whereas $B \notin [\![rpfew_{U'}]\!](A)$. $\square$

This is not an idiosyncratic property of $[\![rpmany]\!]$ and $[\![rpfew]\!]$ (cf. Partee et al. (1990) and Westerståhl (1985)). In this regard, we must observe that non-trivial proportionality determiners such as *more than half* are also universe dependent. For instance, *more than half of students failed* will be true if 25 students failed and the universe is a particular class of 40 students. However, the same statement will be false if 25 students failed and the universe domain is the larger class of 100 students of the entire school. In subsection 4.3 we shall study non-trivial proportionality determiners, as well as their respective converses, which provide an interesting case study for the Witness Set Constraint.

4 On the absence of appropriate witness sets

In this section we shall investigate some mathematical functions that are not conveyed by linguistic categories. The unavailability of these functions as denotations of natural language determiners is commonly accounted for by means of the Conservativity Constraint, since, as we shall indicate, they do not satisfy such a constraint. These case studies are particularly interesting to test our formulation of the Witness Set Constraint: although some of these functions yield quantifiers that have no witness set at all (just like inner negations), there are some cases which do have witness sets; nonetheless, their witness sets are not appropriate, since they cannot be used to calculate the truth of the quantified statements that contain them (unlike relative proportional quantifiers). As a result, we can derive their absence in natural language semantics from the proposed Witness Set Constraint.

4.1 Quantifiers in which the truth of $B \in D(A)$ depends on $B - A$

Recall that the crucial intuition behind the Conservativity Constraint is that we can evaluate the truth of $B \in D(A)$ only on the basis of elements of A –the restriction–, without considering the elements of B or any other sets. It is only the restriction of D that provides the domain of individuals relevant for D. Consider now the following quantifier:

Definition 12. *For any universe domain U and any $A \subseteq U$,*

$$D^{-}(A) = \{X \subseteq U : |X| < |A|\}.$$

In English there is no synthetic determiner, say, *min*, whose denotation would be the function D^{-}, in such a way that a sentence like *min doctors speak French* would mean that the number of French speakers is smaller than the number of doctors. The absence of this determiner can be derived from the Conservativity Constraint and also from the Witness Set Constraint, as we show.

Lemma 13. *The function D^- is non-conservative.*

Proof. Let $B = \{a, b\}$ and $A = \{c\}$. On one hand, it is clear that $B \notin D^-(A)$, since $|B| = 2$, $|A| = 1$, and thus $|B| \not< |A|$. On the other hand, note that $A \cap B \in D^-(A)$, because $A \cap B = \emptyset$ and of course $|\emptyset| < |\{c\}|$. As a consequence, D^- is non-conservative. □

Lemma 14. *For some sets A, B, the quantifier $D^-(A)$ does not have appropriate witness sets to calculate whether $B \in D^-(A)$, whereby the function D^- cannot be expressed by any linguistic category.*

Proof. Note that, for all proper subset A' of A, $|A'| < |A|$. This means that all proper subsets of A belong to $D^-(A)$. Therefore, the set of witness sets for A is

$$W[D^-(A)] = \{A' : A' \subset A\} = \mathcal{P}(A) - A.$$

We shall show that the elements of $W[D^-(A)]$ cannot be used reliably to calculate, for some A, B, whether $B \in D^-(A)$ following monotonicity properties as specified in the procedure proposed in (26) and as required by our Witness Set Constraint.

Observe that the quantifier $D^-(A)$ is monotone decreasing: if $B \in D^-(A)$, because $|B| < |A|$, then, for any subset B' of B, $|B'| < |A|$, in which case $B' \in D^-(A)$. Accordingly, D^- can be expressed by a linguistic category iff, for all $A, B \subseteq U$ (cf. (27-b),

(36)
$$B \in D^-(A) \Leftrightarrow (\exists w)\,(w \in W[D^-(A)] \wedge (A \cap B) \subseteq w).$$

Now consider, for instance, the situation where $A = \{a, b\}$ and $B = \{c, d, e\}$. The set of witness sets for $D^-(A)$ is

$$W[D^-(A)] = \{\{a\}, \{b\}, \emptyset\}.$$

Note that all members of $W[D^-(A)]$ include $A \cap B$, since $\emptyset$ is included in all sets. However $B \notin D^-(A)$, for $|B| \not< |A|$. Indeed, when $A \cap B = \emptyset$, biconditional

(36) is false: its right-hand is necessarily true, because the empty set is included in all sets, and thus in all witness sets for $D^{-}(A)$; however this set-theoretical truth does not entail the truth of $B \in D^{-}(A)$ for all A, B.

Since (36) is false for some A, B, D^{-} is not a legitimate denotation for a linguistic category. □

A further case study is the following non-logical (non-permutation invariant) quantifier:

Definition 13. *For all $i \in U$ and all $A \subseteq U$,*

$$D_i(A) = \{B \subseteq U : i \notin B - A\}.$$

Let $John \in U$; then $B \in D_{John}(A)$ iff $John \notin B - A$. Again, English does present sentences of the type D_{John} *doctors speak French*, whose intended meaning would be that John is a French speaker who is not a doctor.

Lemma 15. *The function D_i is non-conservative.*

Proof. We shall prove that D_i is non-conservative, since the following biconditional does not hold for all sets A, B:

$$B \in D_i(A) \Leftrightarrow (A \cap B) \in D_i(A).$$

If we apply Definition 13 to this biconditional, we obtain the following biconditional:

$$i \notin B - A \Leftrightarrow i \notin (A \cap B) - A.$$

Note that it is necessarily the case that $(A \cap B) - A = \emptyset$; thus, for all A, B, $i \notin (A \cap B) - A$. But the necessary truth of this statement does not entail the truth of $i \notin B - A$. Hence, D_i is non-conservative. □

Lemma 16. *For some sets A, B, the quantifier $D_i(A)$ does not have appropriate witness sets, whereby the function D_i cannot be expressed by any linguistic category.*

Proof. We can see that, for any $A \subseteq U$, if $D_i(A) \neq \emptyset$, it has witness sets. Note that, for any subset A' of A, $A' - A = \emptyset$. Consequently, for any $A' \subseteq A, i \notin A' - A$, which means that the set of witness sets for $D_i(A)$ is

$$W[D_i(A)] = \mathcal{P}(A).$$

Let $B \in D_i(A)$. By Definition 13, this means that $i \notin B - A$. If this is the case, then, for any $B' \subseteq B$, $i \notin B' - A$, whereby $B' \in D_i(A)$. Accordingly, $D_i(A)$ is monotone decreasing.

Assume, for concreteness, that $A = \{a\}$ and $B = \{b\}$. Then, the set of witness sets for $D_b(A)$ is

$$W[D_b(A)] = \{\{a\}, \emptyset\}.$$

Given that $D_b(A)$ is monotone decreasing, we shall conclude that it has appropriate witness sets iff the following biconditional holds (cf. (27-b)):

(37)
$$B \in D_b(A) \Leftrightarrow (\exists w)\,(w \in W[D_b(A)] \wedge (A \cap B) \subseteq w).$$

Observe now that (37) is false. On one hand, $B - A = \{b\}$; thus, $b \in B - A$. Consequently, $B \notin D_b(A)$, which means that the left-hand of the biconditional is false. On the other hand, the right-hand is true, since $A \cap B = \emptyset$, which is included in all witness sets for $D_b(A)$. This violates the appropriateness requirement specified in (27-b) of the Witness Set Constraint: $D_b(A)$ lacks a witness set that is appropriate to calculate the truth of $B \in D_b(A)$ for some sets A, B. □

4.2 Cardinal comparison

We shall now study one-place cardinal comparative determiners constructed on the basis of cardinality relations between two sets. None of these determiners are conveyed by synthetic linguistic categories. They are all correctly banned by the Conservativity Constraint, and also by the Witness Set Constraint, as we shall argue. The investigation of these determiners provides us with a richer picture of the different ways in which the Witness Set Constraint filters out mathematical functions and allows us to corroborate that the proposed constraint, if accurately defined, does not let in certain determiners that map properties into quantifiers that cannot be computed on the basis of monotonicity properties following procedure (26).

Definition 14. *For all U and all $A, B \subseteq U$,*

$$B \in D^{<}(A) \Leftrightarrow |A| < |B|.$$

English does not present sentences such as *blik doctors speak French*, which would claim that the set D of doctors has a smaller cardinality than the set F of French speakers. This gap can be accounted for on the basis of the Conservativity Constraint and the Witness Set Constraint.

Lemma 17. *The function $D^{<}$ is non-conservative.*

Proof. Let $B \in D^{<}(A)$; then, $|A| < |B|$. However, $|A| \geq |A \cap B|$, because, for all sets A, B, $A \cap B \subseteq A$; thus $A \cap B \notin D^{<}(A)$. □

Lemma 18. *For all U and all $A \subseteq U$, the non-empty quantifier $D^{<}(A)$ has no witness set, whereby the function $D^{<}$ cannot be expressed by any linguistic category.*

Proof. For all $A' \subseteq A$, $|A| \not< |A'|$, whereby there is no subset of A that belongs to $D^{<}(A)$, i.e., no witness set for $D^{<}(A)$. This does not entail, though, that $D^{<}(A) = \emptyset$. □

Definition 15. *For any U and any $A, B \subseteq U$,*

$$B \in D^{\neq}(A) \Leftrightarrow |A| \neq |B| \,.$$

Lemma 19. *The function $D^{\neq}$ is non-conservative.*

Proof. Let $A = \{a\}$ and $B = \{a, b\}$; then, $B \in D^{\neq}(A)$ because $|A| = 1$ and $|B| = 2$. However, $A \cap B \notin D^{\neq}(A)$, since $A \cap B = A$. □

Lemma 20. *For all U, if $A = \emptyset$, then the non-empty quantifier $D^{\neq}(A)$ has no witness set, whereby the function $D^{\neq}$ violates the Witness Set Constraint.*

Proof. Choose $A = \emptyset$ and $B \neq \emptyset$. Since $|\emptyset| \neq |B|$, we can conclude that $B \in D^{\neq}(A)$. In this case $D^{\neq}(A) \neq \emptyset$, and nonetheless $D^{\neq}(A)$ has no witness set: A has a single subset A', namely $\emptyset$, which does not belong to $D^{\neq}(A)$, because $|A| = |A'|$. Since it is possible that $D^{\neq}(A)$ is non-empty and lacks witness sets, the function $D^{\neq}$ is not a permissible denotation of a linguistic category, according to the Witness Set Constraint. □

Definition 16. *For any U and any $A, B \subseteq U$,*

$$B \in D^{>}(A) \Leftrightarrow |A| > |B| \,.$$

Lemma 21. *The function $D^{>}$ is non-conservative.*

Proof. Let $A = \{a, b\}$ and $B = \{b, c, d\}$. Observe that $A \not> B$; thus, $B \notin D^{>}(A)$. However, $A \cap B = \{b\}$, in which case $(A \cap B) \in D^{>}(A)$, because $|A| > |A \cap B|$. □

Lemma 22. *For some sets A, B, the quantifier $D^{>}(A)$ does not have appropriate witness sets to calculate whether $B \in D^{>}(A)$, whereby the function $D^{>}$ cannot be expressed by any linguistic category.*

Proof. Note that, for any proper subset A' of A, $|A| > |A'|$. Consequently, the set of witness for $D^{>}(A)$ is

$$W[D^{>}(A)] = \mathcal{P}(A) - A.$$

Assume that $|A| > |B|$; then, for any subset B' of B, $|A| > |B'|$. In other words,

$$(\forall A, B)\,((B \in D^{>}(A) \wedge B' \subseteq B) \Rightarrow B' \in D^{>}(A)).$$

Thus, $D^{>}(A)$ is monotone decreasing.

Given that $D^{>}(A)$ is monotone decreasing, we check whether $D^{>}(A)$ has appropriate witness sets by verifying the following biconditional ((27-b)):

(38)
$$B \in D^{>}(A) \Leftrightarrow (\exists w)\,(w \in W[D^{>}(A)] \wedge (A \cap B) \subseteq w).$$

Let $A = \{a, b\}$ and $B = \{c, d\}$. The set of witness sets for $D^{>}(A)$ is

$$W[D^{>}(A)] = \{\{a\}, \{b\}, \emptyset\}.$$

Observe that $A \cap B = \emptyset$, which is included in all elements of $W[D^{>}(A)]$; consequently, the right-hand of (38) is true. Nonetheless, the left-hand of (38) is false, since

$$|\{a, b\}| \not> |\{c, d\}|.$$

Therefore, there are some sets A, B for which the non-empty quantifier $D^{>}(A)$ lacks appropriate witness sets. □

Definition 17. *For all U and all $A, B \subseteq U$,*

$$B \in D^{\geq}(A) \Leftrightarrow |A| \geq |B|.$$

Lemma 23. *The function $D^{\geq}$ is non-conservative.*

Proof. Let $A = \{a\}$ and $B = \{a, b\}$. Then, $B \notin D^{\geq}(A)$, because $|A| \not\geq |B|$. However, $(A \cap B) \in D^{\geq}(A)$, since $A = A \cap B$.

Since B does not belong to $D^{\geq}(A)$ and $A \cap B$ belongs to $D^{\geq}(A)$, the determiner $D^{\geq}$ is not conservative. $\square$

Lemma 24. *For some sets A, B, the quantifier $D^{\geq}(A)$ lacks appropriate witness sets to calculate whether $B \in D^{\geq}(A)$, whereby the function $D^{\geq}$ cannot be expressed by any linguistic category.*

Proof. The set of witness sets for $D^{\geq}(A)$ is

$$W[D^{\geq}(A)] = \{A' \subseteq A : |A| \geq |A'|\} = \mathcal{P}(A).$$

$D^{\geq}(A)$ is monotone decreasing: for all $B' \subseteq B$, $|A| \geq |B| \Rightarrow |A| \geq |B'|$. Thus, with the aim of deciding whether $D^{\geq}(A)$ is permitted by the Witness Set Constraint we must verify whether the following biconditional holds for all A, B (cf. (27-b)):

(39)
$$B \in D^{\geq}(A) \Leftrightarrow (\exists w)\,(w \in W[D^{\geq}(A)] \wedge (B \cap A) \subseteq w).$$

Let $A = \{d\}$ and $B = \{a, b, c\}$. Then, the left-hand of (39) is false, because $A \not\geq B$. However, the right-hand of (39) is true; observe that the set of witness sets is $W[D^{\geq}(A)] = \{\{d\}, \emptyset\}$, whose two elements include $B \cap A = \emptyset$.

Since there are sets A, B for which biconditional (39) is false, the function $D^{\geq}$ is not a possible denotation for a linguistic category, according to the Witness Set Constraint. $\square$

Definition 18. *For any U and any $A, B \subseteq U$,*

$$B \in D^{\leq}(A) \Leftrightarrow |A| \leq |B|.$$

Lemma 25. *The function $D^{\leq}$ is non-conservative.*

Proof. Let $A = \{a\}$ and $B = \{b, c\}$; then $|A| = 1, |B| = 2$ and $|A \cap B| = 0$. Accordingly, $B \in D^{\leq}(A)$, because $1 \leq 2$, and $A \cap B \notin D^{\leq}(A)$, because $1 \not\leq 0$. $\square$

Lemma 26. *For some sets A, B, the quantifier $D^{\leq}(A)$ does not have appropriate witness sets to calculate whether $A \in D^{\leq}(A)$, whereby the function $D^{\leq}$ cannot be the denotation of any linguistic category.*

Proof. There is solely one subset A' such that $|A| \leq |A'|$, namely A. Then, $W[D^{\leq}(A)] = \{A\}$.

$D^{\leq}(A)$ is monotone increasing: if $|A| \leq |B|$ then, for any C such that $B \subseteq C$, $A \leq C$. Thus, according to the Witness Set Constraint, $D^{\leq}(A)$ is a legitimate denotation for a linguistic category iff, for all A, B (cf. (27-a)):

(40)
$$B \in D^{\leq}(A) \Leftrightarrow (\exists w)\,(w \in W[D^{\leq}(A)] \wedge w \subseteq B).$$

Let $A = \{d\}$ and $B = \{a, b, c\}$. In this case, we know that $B \in D^{\leq}(A)$, because $|A| \leq |B|$. However, A, the unique witness set for $D^{\leq}(A)$, is not included in B, whereby there in no witness set for $D^{\leq}(A)$ that is included in B. Since there are sets A, B for which (40) does not hold, the function $D^{\leq}$ is banned by the Witness Set Constraint. □

Definition 19. *For any U and any $A, B \subseteq U$,*

$$B \in D^{=}(A) \Leftrightarrow |A| = |B|\,.$$

Lemma 27. *The function $D^{=}$ is non-conservative.*

Proof. Let $A = \{a\}$ and $B = \{b\}$. Then, $B \in D^{=}(A)$ and $A \cap B \notin D^{=}(A)$. □

Lemma 28. *For some sets A, B, the quantifier $D^{=}(A)$ does not have appropriate witness sets to calculate whether $B \in D^{=}(A)$, whereby the function $D^{=}$ cannot be expressed by any linguistic category.*

Proof. For any set A, there is a single subset of A that belongs to $D^{=}(A)$, namely A itself. Thus,

$$W[D^{=}(A)] = \{A\}\,.$$

The quantifier $D^{=}(A)$ is clearly non-monotonic. As a consequence, it is permitted by the Witness Set Constraint iff, for all A, B (cf. (27-c)):

(41)
$$B \in D^{=}(A) \Leftrightarrow (\exists w)\,(w \in W[D^{=}(A)] \wedge (A \cap B) = w).$$

Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$. In this case, the left-hand of the biconditional (41) is true, since $|A| = |B|$, whereas its second-hand is false, because

$A \cap B = \emptyset$ and $\emptyset \notin W[D^{=}(A)]$. Since (41) is false for some sets A, B, the function $D^{=}$ is filter out by the Witness Set Constraint. $\square$

4.3 Converses of non-trivial proportionality quantifiers

For any possible determiner denotation D we can define its converse D^c as follows:

$$D^c(B, A) = 1 \Leftrightarrow D(A, B) = 1.$$

We bring this issue into consideration because *only*, when it is interpreted by a non-conservative function, is the converse of *all*. Recall that $[\![only]\!](B, A) = 1$ iff $A \neq \emptyset$ and $A \subseteq B$, and that $[\![all]\!](A, B) = 1$ iff $A \neq \emptyset$ and $A \subseteq B$. In this subsection we shall show that the converses of relative proportional functions are filtered out as denotations of linguistic categories by the Witness Set Constraint.

Let us begin by defining non-trivial proportionality quantifiers. Consider the quantificational expression *more than half (of)*, $\textsc{more}\frac{1}{2}$, whose denotation would be as follows: for any U and any $A, B \subseteq U$,

$$\textsc{more}\frac{1}{2}(A, B) = 1 \Leftrightarrow 2 \cdot |A \cap B| > |A|.$$

This quantifier is just a particular instance of a larger class of proportional quantifiers, which provide denotations for an infinity of quantificational expressions such as *more than one third (of)*, *more than one forth (of)*, etc. We can characterize these proportional quantifiers in the following way:

Definition 20. *Let $m \neq 0$ and $\frac{n}{m}$ such that $0 < \frac{n}{m} < 1$ (i.e., $m > n > 0$). Then,*

$$\textsc{more}\frac{n}{m}(A, B) = 1 \Leftrightarrow m \cdot |A \cap B| > n \cdot |A|.$$

Lemma 29. *The function $\textsc{more}\frac{n}{m}$ is conservative.*

Proof. We shall demonstrate that for any U and any $A, B \subseteq U$, the following biconditional holds:

(42)
$$\textsc{more}\frac{n}{m}(A, B) = 1 \Leftrightarrow \textsc{more}\frac{n}{m}(A, A \cap B) = 1.$$

Following Definition 20 of proportional quantifier, we rewrite (42) in terms of (43):

(43)
$$m \cdot |A \cap B| > n \cdot |A| \Leftrightarrow m \cdot |A \cap (A \cap B)| > n \cdot |A| \,.$$

Given that $A \cap B = A \cap A \cap B$, (43) is a tautology. Therefore, we conclude that $\text{MORE}\frac{n}{m}$ is conservative. □

Lemma 30. *By Theorem 2 and Lemma 29, the determiner* $\text{MORE}\frac{n}{m}$ *satisfies the Witness Set Constraint.*

We now construct, for any determiner $\text{MORE}\frac{n}{m}$, its converse $[\text{More}\frac{n}{m}]^c$.

Definition 21. *For any U and any $A, B \subseteq U$,*

$$[\text{MORE}\frac{n}{m}]^c(A,B) = \text{MORE}\frac{n}{m}(B,A) = 1 \Leftrightarrow m \cdot |B \cap A| > n \cdot |B| \,.$$

The absence of the converses of non-trivial proportional determiners as linguistic denotations is predicted by the Conservativity Constraint and the Witness Set Constraint.

Lemma 31. *The function* $[\text{More}\frac{n}{m}]^c$ *is non-conservative.*

Proof. The determiner $[\text{More}\frac{n}{m}]^c$ is conservative iff, for all sets A, B:

(44) $\quad B \in [\text{More}\frac{n}{m}]^c(A) \Leftrightarrow A \cap B \in [\text{More}\frac{n}{m}]^c(A).$

If we apply Definition 21 of $[\text{More}\frac{n}{m}]^c$ to (44), we obtain the following biconditional:

(45) $\quad m \cdot |A \cap B| > n \cdot |B| \Leftrightarrow m \cdot |(A \cap B) \cap A| > n \cdot |A \cap B|.$

Observe that the $|A \cap B|$ must be equal to $|(A \cap B) \cap A|$, because $A \cap B$ and $(A \cap B) \cap A$ are the same set; thus the first term of the left-hand inequality and the first term of right-hand inequality of (45) are identical. However, it is not necessary the case that the second terms $n \cdot |B|$ and $n \cdot |A \cap B|$ are identical, because $|B|$ can be different from $|A \cap B|$. Accordingly, biconditional (45) is not necessarily true for all sets A, B, whereby $[\text{More}\frac{n}{m}]^c$ is not a conservative determiner. □

Let us illustrate the non-conservativity of this class of determiner. Choose $m = 6$, $n = 4$, $A = \{a, b\}$ and $B = \{a, b, d, e\}$; then, $|A \cap B| = 2$ and $|B| = 4$. Observe that $B \notin [\text{MORE}\frac{4}{6}]^c(A)$, because

$$6 \cdot |A \cap B| = 12 \ngtr 4 \cdot |B| = 16.$$

However, $A \cap B \in [\text{MORE}\frac{4}{6}]^c(A)$, since

$$6 \cdot |(A \cap B) \cap A| = 12 > 4 \cdot |A \cap B| = 8.$$

Since $B \notin [\text{MORE}\frac{4}{6}]^c(A)$ and $A \cap B \in [\text{MORE}\frac{4}{6}]^c(A)$, we must conclude that $[\text{MORE}\frac{4}{6}]^c$ is not conservative.

Lemma 32. *For some sets A, B, the quantifier $[\text{MORE}\frac{n}{m}]^c(A)$ does not have appropriate witness sets to calculate whether $B \in [\text{MORE}\frac{n}{m}]^c(A)$, whereby $[\text{MORE}\frac{n}{m}]^c$ cannot be expressed by a linguistic category.*

Proof. A subset A' of A is a witness for $[\text{MORE}\frac{n}{m}]^c(A)$ iff it belongs to $[\text{MORE}\frac{n}{m}]^c(A)$, i.e, iff it satisfies the following condition:

(46) $\quad m \cdot |A' \cap A| > n \cdot |A'|.$

Observe that, for any $A' \subseteq A$, $A' \cap A = A'$, and that $m > n$; this implies that condition (46) is satisfied when $|A'| \neq 0$. Consequently, the set of witness sets for $[\text{MORE}\frac{n}{m}]^c(A)$ is

$$W[[\text{MORE}\frac{n}{m}]^c(A)] = \mathcal{P} - \emptyset.$$

The quantifier $[\text{MORE}\frac{n}{m}]^c(A)$ is not monotone increasing. Let $m = 2$ and $n = 1$. Choose $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, b, c, d, e\}$. Observe that $|A \cap B| = 2$ and $|B| = 3$. Thus,

$$2 \cdot |A \cap B| = 4 > 1 \cdot |B| = 3.$$

Consequently, $B \in [\text{MORE}\frac{1}{2}]^c(A)$. Note that $B \subseteq C$ and $C \notin \text{MORE}^c\frac{1}{2}(A)$, since $|A \cap C| = 2$, $|C| = 5$ and

$$2 \cdot |A \cap C| = 4 \not> 1 \cdot |C| = 5.$$

The quantifier $[\text{MORE}\frac{n}{m}]^c(A)$ is neither monotone decreasing. Assume that $m \cdot |A \cap B| > n \cdot |B|$. Observe now that $\emptyset$ is a subset B' of B for which $m \cdot |A \cap B'| \not> n \cdot |B'|$, since $|A \cap \emptyset| = |\emptyset| = 0$.

Therefore, $[\text{MORE}\frac{n}{m}]^c(A)$ is non-monotonic. In order to determine whether $W[[\text{MORE}\frac{n}{m}]^c(A)]$ contains appropriate witness sets for $[\text{MORE}\frac{n}{m}]^c(A)$ we must check whether the following biconditional holds for all A, B:

(47)

$$B \in [\text{MORE}\frac{n}{m}]^c(A) \Leftrightarrow (\exists w)\,(w \in W[[\text{MORE}\frac{n}{m}]^c(A)])(A \cap B = w).$$

Let $m = 2$ and $n = 1$. Consider the two sets $A = \{a, b\}$ and $B = \{a, b, c, d\}$. Then, B does not belong to $[\text{MORE}\frac{1}{2}]^c(A)$, because

$$2 \cdot |A \cap B| = 4 \not> 1 \cdot |B| = 4,$$

in which case the left-hand statement of (47) is false. However, the subset $A \cap B$ of A does belong to $[\text{MORE}\frac{1}{2}]^c(A)$, because

$$2 \cdot |(A \cap B) \cap A| = 4 > 1 \cdot |A \cap B| = 2.$$

Consequently, we must conclude that $A \cap B$ is a witness set for $\text{MORE}\frac{n}{m}^c(A)$, in which case the right-hand of (47) is true.

Since (47) is false, the function $[\text{MORE}\frac{n}{m}]^c$ is filtered out by the Witness Set Constraint. $\Box$

5 Discussion

In this article we have argued that the Witness Set Constraint determines what functions from properties to generalized quantifiers are denoted by linguistic categories. Our proposal has several interesting consequences for the understanding of what universal principles determine the set of possible determiner denotations in natural languages.

Firstly, the existence of conservative determiner denotations is expected if the Witness Set Constraint is the semantic principle that determines the determiner denotations that are linguistically legitimate: provably, all conservative functions satisfy the Witness Set Constraint (Theorem 2).

Secondly, we can understand why inner negations (such as the *allnon* determiner or the $[\![\widehat{n}]\!]$-determiners) are banned; these non-conservative functions are not permitted because none of the generalized quantifiers associated to them have witness sets. If witness sets are necessary to provide a feasible account for the semantic processing of sentences with generalized quantifiers, as suggested by Barwise and Cooper (1981), then we can understand the absence of these non-conservative functions as a result of a simplicity condition that is required in order to calculate the truth conditions of quantified statements in a simpler and more realistic way.

Thirdly, certain familiar linguistic categories that have been argued to denote non-conservative functions, namely *only* and the relative proportional determiners *many* and *few*, map properties into generalized quantifiers that have appropriate witness set when they are not empty sets. Therefore, their existence is anticipated by the Witness Set Constraint. Indeed, the inverse of Theorem 2 does not hold, since there are functions that have appropriate witness sets and are non-conservative. We do not need to consider whether or not *only* is a determiner from a syntactic viewpoint, whether the relative proportional determiners are conservative on the second argument, or whether any other operations, such as focalization, are involved in order to allow determiners that are non-conservative on their first argument.

Fourthly, we have shown that certain mathematical quantifiers, such as cardinal comparative quantifiers or the converses of non-trivial proportional quantifiers, do have witness sets. However, these witness sets are not appropriate, in the technical sense we have specified, to calculate the truth of some quantified statements. The Witness Set Constraint correctly predicts that the functions that yield these quantifiers are not possible denotations of linguistic categories.

In sum, not only the absence of certain non-conservative categories but most interestingly the existence of certain non-conservative categories can be derived from a presumable constraint involved in the semantic processing of generalized quantifiers. This indicates the empirical superiority of the Witness Set Constraint compared to the Conservativity Constraint.

Finally, we would like to remark that the approach developed in these pages, according to which it is a constraint on semantic processability that restricts the set of generalized quantifiers permitted in natural languages, has a more restrictive and accurate power of definability than the argument developed by Keenan and Stavi (1986). According to this argument (informally sketched in section 1.2), natural language determiners would need to be conservative as a result of the boolean structure of natural language semantics. However, as noted by van Benthem (1983), every binary relation between subsets of a universe can be defined using boolean operations, "for every such relation may be viewed as a (finite) disjunction of singleton cases" (cf. van Benthem 1983, p. 455 for a detailed discussion). This "simple amendment" introduced by van Benthem allows us to generate inner negations, such as *allnon* –as van Benthem himself notes– and $[\![\widehat{n}]\!]$-determiners). This leads us "to excessive power of definability", since finite disjunctions of singletons generate non-attested determiners that indeed are non-conservative, such as inner negations.

Therefore, the Witness Set Constraint is not only empirically but also theoretically superior to the Conservativity Constraint, to the extent that the Witness Set Constraint can be naturally motivated in terms of processing strategies, whereas the above-mentioned attempt to motivate the Conservativity Constraint is formally problematic.

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References

Barwise, J. & R. Cooper (1981), Generalized quantifiers and natural language. *Linguistics and philosophy* 4(2):159-219.

Beghelli, F., D. Ben-Shalom & A. Szabolcsi (1997), Variation, distributivity, and the illusion of branching. In A. Szabolcsi (ed.), *Ways of scope taking*. Springer. 29-69.

Berwick, R. C., P. Pietroski, B. Yankama & N. Chomsky (2011), Poverty of the stimulus revisited. *Cognitive Science* 35(7):1207-1242.

Chierchia, G. & S. McConnell-Ginet (2000), *Meaning and grammar: an introduction to semantics*. The MIT press.

Chomsky, N. (1972), *Problems of knowledge and freedom: the Russell lectures*. Vintage Books.

Cohen, A. (2001), Relative readings of 'many', 'often', and generics. *Natural Language Semantics* 9(1):41-67.

Fernando, T. & H. Kamp (1996), Expecting 'many'. *Proceedings of SALT*, vol. 6, 53-68.

Heim, I. & A. Kratzer (1997), *Semantics in generative grammar*. Blackwell Publishers.

Herburger, E. (1997), Focus and weak noun phrases. *Natural Language Semantics* 5(1):53-78.

Higginbotham, J. & R. May (1981), Questions, quantifiers and crossing. *The Linguistic Review* 1(1):41-80.

Keenan, E. L. (1981), A boolean approach to semantics. In J. Groenendijk, T. Janssen, & M. Stockhof (eds.), *Truth, Interpretation and Information*. Foris Publications. 65-98.

Keenan, E. L. (2002), Some properties of natural language quantifiers: generalized quantifier theory. *Linguistics and Philosophy* 25(5):627-654.

Keenan, E. L. (1996), The semantics of determiners. In S. Lappin (ed.), *The handbook of contemporary semantic theory*. Blackwell Publishers. 41-65.

Keenan, E. L. & L. M. Faltz (1985), *Boolean semantics for natural language*. Springer Science & Business Media.

Keenan, E. L. & J. Stavi (1986), A semantic characterization of natural language determiners. *Linguistics and philosophy* 9(3):253-326.

Keenan, E. L. & D. Westerståhl (2010), Generalized quantifiers in linguistics and logic. In J. van Benthem & A. Ter Meulen (eds.), *Handbook of logic and language*. Elsevier. 837-893.

Lappin, S. (1988), The semantics of 'many' as a weak determiner. *Linguistics* 26(6):977-1020.

Lappin, S. (1993), 'many' as a two-place determiner function. *SOAS working papers in linguistics and phonetics* 3:337-358.

Lappin, S. (2000), An intensional parametric semantics for vague quantifiers. *Linguistics and philosophy* 23(6):599-620.

Lindström, P. (1966), First order predicate logic with generalized quantifiers. *Theoria* 32(3):186-195.

Montague, R. (1974), *Formal philosophy: selected papers of Richard Montague*, edited and with an introduction by Richmond Thomason. Yale University Press.

Mostowski, A. (1955), On a generalization of quantifiers. *Fundamenta Mathematicae* 44:12-36.

Partee, B. (1988), Many quantifiers. *Proceedings of the Fifth Eastern States Conference on Linguistics*. The Ohio State University. 383-402.

Partee, B., A. Ter Meulen & R. Wall (1990), *Mathematical methods in linguistics*. Springer Science & Business Media.

Peters, S. & D. Westerståhl (2006), *Quantifiers in language and logic*. Oxford University Press.

Robaldo, L. (2011), Distributivity, collectivity, and cumulativity in terms of (in)dependence and maximality. *Journal of Logic, Language and Information* 20(2):233-271.

Robaldo, L. (2013), Conservativity: a necessary property for the maximization of witness sets. *Logic Journal of IGPL*. jzt018.

Robaldo, L. & J. Szymanik (2012), Pragmatic identification of the witness sets. *Proceedings of the 8th International Conference on Language Resoures and Evaluation (LREC)*.

Robaldo, L., J. Szymanik & B. Meijering (2014), On the identification of quantifiers' witness sets: a study of multi-quantifier sentences. *Journal of Logic, Language and Information* 23(1):53-81.

Szabolcsi, A. (1997), Background notions in lattice theory and generalized quantifiers. In A. Szabolcsi (ed.), *Ways of scope taking*. Springer. 1-27.

Szabolcsi, A. (2010), *Quantification*. Cambridge University Press.

van Benthem, J. (1983), Determiners and logic. *Linguistics and Philosophy* 6(4):447-478.

von Fintel, K. (1997), Bare plurals, bare conditionals, and 'only'. *Journal of Semantics* 14(1):1-56.

von Fintel, K. & L. Matthewson (2008), Universals in semantics. *The Linguistic Review* 25(1-2):139-201.

Westerståhl, D. (1985), Logical constants in quantifier languages. *Linguistics and Philosophy* 8(4):387-413.