1 Trivalent Logic and Presuppositions

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Abstract.
Both sentences *Gloria forgot that John was born in Berlin* and *Gloria did not forget that John was born in Berln* intuitively entail that John was born in Berlin. Such a behavior is problematic from the point of view of classical logic, since in classical logic, only tautologies can be entailed by a sentence S and its negation $\neg S$. Such a sentence is said to *presuppose* (and not just to entail) that John was born in Berlin, both because it seems to be typically used in a situation where this information is taken for granted, and because of the peculiar logical behavior we have just noted. A general question, then, is to predict how presupposed content interacts with various logical operators and syntactic environments, not just negation. That is, we would like to be able to predict the presuppositions of a complex sentence as a function of the presuppositions of its parts – a problem known as the projection problem for presuppositions. A central approach to this problem relies on Trivalent logic: the idea is that a sentence is undefined rather than true or false in a situation where its presuppositions are false. The projection problem then boils down to the problem of specifying an adequate compositional trivalent semantics, which will assign trivalent truth-conditions to every possible sentence, thereby assigning them, among others, definedness conditions.

This chapter presents the core empirical generalizations to be accounted for, and examines the Strong Kleene and Middle Kleene semantics as solutions to the presupposition projection problem, as well as some of the challenges they face. It also includes a discussion of mechanisms such as local and global accommodation. Finally, alternative approaches, including Heim’s dynamic semantics and Schlenker’s Transparency Theory, are briefly reviewed.

1.1 Introduction

In natural languages, the content of declarative sentences tends to be richly structured. In particular, the various inferences that a declarative sentence licenses have different properties, both in terms of how they interact with compositional semantics, and in terms of their role in information exchange. In this chapter, I focus on the notion of *presupposition*, a specific dimension of meaning which arguably cannot be captured within a classical bivalent framework.

One dominant approach to presupposition crucially relies on trivalent logic, and my goal is to provide an introduction to this specific approach.

To illustrate the notion of presupposition, let us consider the following sentence:

(1) Gloria has forgotten that John lives in Paris.

(1) conveys, among others, that John lives in Paris, that Gloria used to know that, and that Gloria currently does not know (anymore) that John lives in Paris.

The first two inferences (John lives in Paris, Gloria used to know that) behave differently from the third inference (Gloria currently does not know that John lives in Paris) in at least two respects. First, when we negate (1), the first two inferences are, somewhat surprisingly from the point of view of classical logic, preserved. That is, (2), just like (1), licenses the inference that John lives in Paris and that Gloria has known this in the past. This is surprising, because in classical logic, if a sentence S entails a sentence T, then if $\neg S$ also entails T, T is a tautology. On the other hand, the third inference (Gloria does not currently know that John lives in Paris) does not survive under negation. That is, (2) does not support the inference that Gloria currently does not know that John lives in Paris.[1]

(2) Gloria has not forgotten that John lives in Paris.

A second property that distinguishes the first two inferences from the third one is that, at an intuitive level, a speaker who uses (1) (as well as (2)) is perceived to *take for granted* that John lives in Paris, and that Gloria used to know this, i.e. is perceived to assume that these facts are already known by their interlocutor, while, on the contrary, the information that Gloria does not now believe that John lives in Paris (in the case of (1)) is not taken for granted. This is the sense in which the author of these sentences is said to *presuppose*, rather than to *assert*, that John lives in Paris and that Gloria used to know that. Inferences that are preserved when a sentence is negated and which are usually perceived as being part of what a speaker using the sentence takes for granted are typically considered to be *presupposed* by the sentence.

It is relevant at this point to distinguish between what a speaker presupposes (i.e. takes for granted) when using a certain sentence in a given context and what a sentence itself can be said to presuppose (Stalnaker 1973, 1998). In general, whether a speaker who uses S is felt to take p as granted depends not just on S (i.e. the specific sentence used), but also on the specific context of utterance. The term *presupposition* is first meant to characterize a certain propositional attitude on the part of the speaker (the verb *presuppose* is a propositional attitude verb). Derivatively, a sentence S can be said to presuppose p if someone who uses S is typically felt to presuppose p. This derivative notion of presupposition seems useful because in many instances, the fact that the author of S is understood to presuppose p is a consequence of the conventional meaning of S. A plausible but nevertheless controversial hypothesis is that certain lexical items are conventionally associated with presuppositions. For instance, it would be part of the lexical entry of the verb *forget* that a sentence of the form *X has forgotten that p* presupposes that p is true and that X used to know that p is true. Under such a view, we can talk of the *semantic presuppositions* of sentences, i.e. those presuppositions which directly follow from the content of lexical items and the way they are combined in a sentence. The assumption that such semantic presuppositions exist (as opposed to the view that only utterances, and not sentences, carry presuppositions) is open to challenge and is in fact debated (cf. Stalnaker 1998; Simons 2013, among many others), but I will nevertheless adopt it in this chapter, for ease of exposition. We should note, furthermore, that in practice such semantic presuppositions are not always 'presupposed' in actual discourse: speakers often use sentences that presuppose certain propositions even when these propositions are not in fact common knowledge (yet). For instance, I could easily tell someone *My sister is visiting me*, thereby using a sentence that presupposes that I have a sister, even if I know that the addressee did not antecedently know that I have a sister. It is standardly assumed that in such cases, the speaker simply counts on the addressee to accept the presupposition that the speaker has a sister ('global accommodation', see section **??**) even though this piece of information is in fact new and cannot be taken for granted. Even in such cases, there is still a feeling that the presupposed content (in this case the fact that the speaker has a sister) is *backgrounded.*

A second caveat is in order. In recent years, linguists have identified various types of 'not-at-issue' content beyond presupposition. The idea is that, in a specific context, a given sentence is used to make a certain point, and that there are good reasons to distinguish between the 'main point' of an utterance, also called its 'at-issue' content, and other aspects of content that are in some sense backgrounded and not 'at-issue' (Tonhauser et al. 2013). Presuppositions are just one type among others of not-at-issue content. Consider for instance:

(3) Gloria, who has just moved to Paris, invited Peter to her home.

(3) conveys, among other things, that Gloria just moved to Paris. This inference, however, is in some intuitive sense backgrounded, and not the main point of the utterance. It also survives when the sentence is negated, as in (4):

(4) It's not the case that Gloria, who has just moved to Paris, invited Peter to her home.

However, the inference that Gloria has just moved to Paris, and generally the contribution of appositive relative clauses and other parentheticals, is typically not viewed as a presupposition. One reason (among others) is that a speaker who uses (3) or (4) is not felt to take for granted that Gloria has just moved to Paris. In fact, it seems that the opposite is true: the author of (4) is perceived to assume that the information that Gloria has just moved to Paris is new information, i.e. is not already common knowledge.[2] This *non-triviality* condition (Potts 2005; Schlenker 2021a), according to which (4) is not felicitous if it is common knowledge that Gloria just moved to Paris, is the *negation* of what standard presuppositions give rise to (namely that their content must be common knowledge).

Presuppositions are therefore just one species among others of not-at-issue content, but the boundaries between the various sorts of not-at-issue content are not clear, and somewhat controversial, to the extent that some researchers doubt that there is a well defined class of phenomena that fall under the rubric of 'presupposition' (Tonhauser, Beaver, and Degen 2018; Degen and Tonhauser 2022). With these various caveats in mind, I will nevertheless proceed as if presuppositions formed a natural class and will assume that at least some of them are lexically triggered.

Presupposition triggers (i.e. lexical items and constructions that trigger presuppositions) are ubiquitous in natural languages. We have just illustrated the case of so-called *factive verbs* such as *forget*, which trigger the presupposition that their complement clause is true. Other factive verbs include *know*, *remember*, *regret*, and many others.[3] For instance, *Gloria knows that it is raining* licenses the inference that it is raining, and so does its negation, *Gloria doesn't know that it is raining*, and in both cases, a user of the sentence is typically perceived to take it for granted that it is raining. Other presupposition triggers include definite descriptions, which carry an existence presupposition (and typically[4] a uniqueness presupposition if they are singular), and so-called change-of-state verbs. Thus, famously, the sentence *The king of France is (not) bald* presupposes that there is a unique king of France, and a sentence such as *Mary has never met Adam's sisters* presupposes that Adam has sisters. As to change-of-state verbs, a sentence such as *Sarah stopped living in Paris* expresses a transition from living in Paris to not living in Paris, and the initial state is presupposed, i.e. this sentence presupposes (and does not merely entail) that Sarah used to live in Paris. The verb *forget* is both a change-of-state and a factive verb. That is, (1) presupposes that John lives in Paris (factive presupposition) but also that Gloria used to know that (the initial state is presupposed). (1) could be paraphrased as *Gloria stopped knowing that John lives in Paris*. Let me also mention cleft constructions, which, like definites, trigger an existence presupposition; thus a sentence such as *It is Gloria and Sue who visited Paul* presupposes that some people visited Paul and asserts that the people who did so are Gloria and Sue. These examples only scratch the surface; in fact, every sentence in natural languages carries multiple presuppositions, starting with proper names, which presuppose that there is a person carrying the proper name in question, pronouns, which presuppose that they have a referent (*John didn't see it* presupposes that something that is not a human exists and asserts John didn't see that thing), or selectional restrictions imposed by verbs and predicates (e.g., both *Mary slept* and *Mary didn't sleep* appear to presuppose that Mary belongs to the class of entities that can sleep). As a result, as Abrusán (2022) points out, it is probably impossible to give a presupposition-free translation of any English sentence, if only because every verb comes with sortal restrictions which are presuppositional in nature.

The main goal of this chapter is to outline how trivalent approaches to presupposition solve the so-called *presupposition projection problem*, i.e. the problem of predicting the presuppositions of any sentence, however complex, in terms of the presuppositions of its parts. It mostly focuses on a propositional fragment. Section 1.2 explains how presuppositions can be captured in a trivalent system. Section 1.3 establishes the core generalizations that any approach to the presupposition projection problem has to account for. Section 1.4 discusses the predictions of the Strong Kleene semantics for presupposition projection. An extended propositional language in which presuppositions are explicitly represented is then introduced in section 1.5. Section 1.6 raises three problems for a theory of presupposition projection based on Strong Kleene, and these problems are discussed in detail in the three subsequent sections. In particular, section 1.7 demonstrates the need to posit a mechanism (known as local accommodation) whereby, under certain circumstances, the third, undefined truth-value is collapsed with the false truth-value; section **??** discusses the need to move from the Strong Kleene truth-tables to asymmetric truth-tables, known as the Middle Kleene truth-tables; section **??** introduces one remaining challenging problem for all current theories of presupposition projection and discusses possible solutions. Section **??** briefly sketches how the trivalent approach to presupposition projection can be extended to a language with quantifiers. Finally, section **??** briefly discusses the two other main approaches to presupposition projection, namely Heim's dynamic semantics approach and Schlenker's Transparency theory.

1.2 Modeling presuppositions in a trivalent framework: definedness conditions and the bridge principle

The fact that a classical bivalent semantics cannot account for the behavior of presuppositions can be shown in the following way (van Fraasen 1968). When a sentence S presupposes a proposition p, p is felt to *follow* from S (e.g, “John lives in Paris” follows from (1)); within a classical semantics approach, the only notion we have to make sense of the idea that a sentence follows from another one is the notion of logical entailment, i.e. in such a case the only option is to say that p is a logical consequence of S. Now, as we have seen, it is a hallmark of presuppositions that they are ‘preserved under negation’, i.e. when S presupposes p, so does S’s negation (e.g.,“John lives in Paris” follows not only from (1), but also from its negation, (2)). But recall that in a classical framework, from the fact that a sentence p is entailed by both S and $\neg S$, it follows that p is a tautology. We are thus led to the absurd conclusion that the sentence “John lives in Paris”, which is felt to follow from both (1) and (2), is a tautology.

The appeal of trivalent semantics comes from the fact that this reasoning is no longer correct for some variants of trivalent semantics and a well-chosen notion of logical consequence. Let us a consider a standard propositional language that includes three propositional constants notated $\top$ (tautology), $\bot$ (contradiction), and U (undefined) and a set of propositional variables (‘propositional atoms’). Let us define a trivalent valuation as a function that maps each propositional atom to a value in $\{0, \#, 1\}$, and maps $\top$ to 1, $\bot$ to 0 and U to #. Let negation be defined as mapping 0 to 1, 1 to 0 and # to #. Suppose further that we define entailment as follows: A entails B if and only if all trivalent valuations that make A true also make B true (this the *ss*-consequence relation in the sense of Cobreros et al. 2012). In this case, for every sentence A, we have both $U \models A$ and $\neg U \models A$, but of course we don’t have, for every A, $\top \models A$.

While this shows in the abstract that the move to trivalent semantics can in principle give us a way out of the problem we noticed, we need to show how one can concretely use trivalent logic to capture presuppositions.

In the trivalent approach to presupposition, the key assumption is that a sentence whose presupposition is not true is neither true nor false. That is, presuppositional sentences are undefined, rather than true or false when a certain proposition, namely their presupposition, is false. For instance, *Mary stopped smoking* is undefined if it is not the case that Mary used to smoke. Formally, we can think of certain lexical items as being constrained by *meaning postulates* that together define a set of *admissible models* (In the next sections, a sentence’s truth-value will be relativized to a *world* rather than model, but we can likewise restrict the class of admissible worlds). For instance, when we say that *Mary stopped smoking* presupposes that Mary used to smoke, this means that in every admissible model in which *Mary used to smoke* is false, the sentence *Mary stopped smoking* is itself undefined. From this starting point, the choice of a certain trivalent logic is governed by the following desideratum: correctly predict the presupposition of a complex sentence on the basis of the presuppositions of its parts. This is known as the *projection problem for presuppositions*. Within a trivalent framework, if we keep to a purely extensional fragment, the projection problem for presuppositions is simply a special case of the problem of predicting the truth-value of a complex sentence in terms of the truth-value of its parts, hence of providing an empirically adequate compositional semantics for an extensional fragment of the language.

Before discussing presuppositon projection in a systematic way (in section 1.3), we can start with considering how negation interacts with presuppositions in a trivalent system. We assume, as is the case in most trivalent semantics, that a negative sentence $\neg S$ is true if S is false, false if S is true, undefined if S is undefined. Assume now that a certain sentence S presupposes p, i.e. is undefined if p is false (I will often use lower case letters, esp. the letter p, to refer to propositions that are presupposed, for convenience – in this context, p is not necessarily an atomic sentence). What does $\neg S$ presuppose ? Obviously $\neg S$ is undefined if and only if S is. Hence $\neg S$ is undefined if p is false, hence it also presupposes p. The underlying principle is that for any sentence S, however complex, the presupposition of S is the proposition p such that S is defined if and only if p is true. In other words, the presuppositions of a sentence are equated to its *Definedness Conditions*. Coming back to the problem discussed in the previous section, we can see that if S presupposes p, then whenever S is true, p is true, and whenever $\neg S$ is true, p is true as well, and this is so even though p is not a tautology. That is, if our notion of entailment is truth-preservation from premises to conclusion, we have a case where both S and $\neg S$ entail p, and yet p is not a tautology (both in the sense that it is not entailed by every sentence, and in the sense that it is false under some admissible models).

As discussed above, when a sentence S presupposes p, a speaker who uses S indicates that they take p for granted. This property of presuppositions is typically captured in terms of a *felicity condition*, which states that an utterance of S is not felicitous if S's presuppositions are not *common ground*, i.e. if p is not common knowledge between the participants to the conversation (Stalnaker 2002). Since presuppositions impose a constraint of what is common knowledge, in order to state felicity conditions for presuppositional sentence, we need to model the common assumptions of the participants to a conversation. So we now model the 'common ground' of conversational participants as a set of *possible worlds*, namely the set of worlds that are compatible with what is common knowledge. Given our purposes, we can think of a world simply as an assignment of truth-values to the sentences of the language, i.e. as a model. For instance, if the language is propositional logic, a world is simply a valuation. At a given world, a sentence is true, false or undefined. If we introduce meaning postulates (for instance the postulate that a certain sentence p is necessarily undefined if sentence q is false), it means that worlds in which they do not hold are simply never considered, i.e. are not *admissible* and are never possible worlds. When a sentence is accepted as true, the common ground evolves: the new common ground is the intersection of the initial common ground with the set of worlds at which the sentence is true. [5] This is essentially the framework proposed by Stalnaker in several articles (Stalnaker 1973, 1978, 2002, among others). The relationship between the semantics proper and felicity conditions is governed by the so-called *Stalnaker's bridge principle*, which, within a trivalent system, takes the following form.

(5) **Bridge Principle**
Let a context C be defined as a set of possible worlds. A context represents the common ground between conversation participants at a certain point. A sentence S is felicitous in context C only if, in every world of C, S is not undefined. [Note that we cannot strengthen *only if* into *if and only if*, because there are of courses many other factors that play a role in determining felicity conditions].

Consider, for instance, the sentence *Gloria knows that it is raining*. This sentence presupposes that it is raining. This means that, by virtue of a meaning postulate (which is part of the lexical entry of *know*), this sentence is undefined in every world where *it is raining* is false (a world where *Gloria knows that it is raining* is true but *It is raining* is false is not an admissible world, hence is not a possible world). The bridge principle states that for this sentence to be felicitous in context C, it must be defined in every world of C, hence it must be raining in every world of C. Therefore, in order for this sentence to be felicitous, the context should entail that it is raining, i.e. it should be common ground that it is raining.

1.3 Presupposition Projection and Presupposition Filtering

We present a number of well-known generalizations (since Karttunen's seminal contributions, cf. Karttunen 1973, 1974) about how presuppositions interact with propositional connectives in natural languages (negation, disjunction, conjunction, conditional).

Let S be a sentence that presupposes p. For instance, let S be *Gloria stopped smoking* and p be *Gloria used to smoke*. What happens to the presupposition when S is negated, used in the antecedent or the consequent of a conditional, or in a disjunctive or in a conjunctive sentence? Let us thus consider the following sentences:

(6) a. Gloria didn't stop smoking.
b. If Gloria stopped smoking, she will soon be in better health.
c. If Gloria takes care of her health, she stopped smoking.
d. Gloria takes care of her health, and she stopped smoking.
e. Either Gloria doesn't take care of her health, or she stopped smoking.

It intuitively seems that all of these sentences license the inference that Gloria used to smoke. On this sole basis, a straightforward proposal can be made: if a complex sentence S contains a sentence S' as a subclause and S' presupposes p, then S presupposes p. Within our trivalent framework, this amounts to sayting that if S' receives the undefined truth-value in a certain valuation v, then so does S, if S contains S' as a subclause.

The data are however significantly more complex. In particular, while both sentences in (7) contain the subclause *Gloria stopped smoking*, which in isolation presupposes that Gloria used to smoke, they do not themselves appear to license the conclusion that Gloria used to smoke, hence do not presuppose that Gloria used to smoke.

(7) a. If Gloria used to smoke, she stopped smoking.
b. Either Gloria didn't use to smoke, or she stopped smoking.

Furthermore, even though (8) of course entails that Gloria used to smoke, it does not presuppose it:

(8) Gloria used to smoke and she stopped smoking.

To establish that (8) does not presuppose that Gloria used to smoke, one needs to embed (8) in a construction that tends to 'preserve' presuppositions but not entailments, and observe that the inference that Gloria used to smoke is not preserved, hence does not behave like a presupposition. In principle, as we discussed, prefixing a negation allows us to do precisely this. However, in English, negating a conjunctive sentence made up of two full sentences can only be achieved by means of rather marked constructions like *It is not the case that* … . So instead of using negation, we can embed (8) in the antecedent of a conditional, i.e. an if-clause, since we also know that the antecdent of a conditional typically preserves presuppositions but not entailments (cf. (6b)). And indeed we observe that when one embeds (8) in the antecedent of a conditional, as in (9), the inference that Gloria used to smoke is not preserved, i.e. (9) does not intuitively license the inference that Gloria used to smoke.

(9) If Gloria used to smoke and she stopped smoking, she does not need to be tested.
$\not\rightsquigarrow$ Gloria used to smoke

We can contrast (9) with (10), which, as expected, does license this inference (confirming that the sentence *Gloria stopped smoking*, unlike (8), presupposes that Gloria used to smoke)

(10) If Gloria stopped smoking, she does not need to be tested.

Another test for presupposition consists in turning a declarative sentence into an interrogative one: presuppositions are typically inherited by the interrogative sentence. For instance, someone who asks *Did Gloria stop smoking?* is typically felt to presuppose that Gloria used to smoke. When we turn (8) in to an interrogative, the inference that Mary used to smoke is not preserved (i.e. someone who asks the question is not perceived to take it for granted that Mary used to smoke).[6]

(11) Is it true that Gloria used to smoke and that she stopped smoking?

We can therefore state the following generalization:

(12) Let S be a sentence that presupposes p. When S is embedded in the following frames, the resulting sentence does not inherit p as a presupposition:

a. If p, S (cf. (6b))
b. Either not p, or S (cf. (7b))
c. p and S (cf. our discussion of (8))

In relation with the sentences disussed in (7) and (8), S is *Gloria stopped smoking* and p is *Gloria used to smoke*. Following previous usage (Karttunen 1973), we will say in such cases that the presupposition associated with S if *filtered out*.

In fact, the correct generalization appears to have broader scope: if we replace p, in the frames used in (12), by a sentence p^+ which asymmetrically entails p (i.e. p^+ entails p and p does not entail p^+) , the resulting sentences still do not inherit the presupposition p, even if S presupposes p in isolation. This is illustrated by the following variations on the sentences used in (7) and (8) (with p=*Gloria used to smoke*, p^+ =*Gloria used to smoke a cigarette after each meal*, and S=*Gloria stopped smoking*):

(13) a. If Gloria used to smoke a cigarette after each meal, she stopped smoking.
b. Either Gloria didn't use to smoke a cigarette after each meal, or she stopped smoking.
c. Gloria used to smoke a cigarette after each meal, and she stopped smoking.

None of the above sentences presupposes that Gloria used to smoke (in particular, (13c) entails, but does not presuppose, that Gloria used to smoke, as can be shown in the same way as we did for (8)). We can thus settle for the following, somewhat broader generalization:

(14) **Presupposition Filtering**
Let S be a sentence that presupposes p. Let p^+ be a sentence that entails p. Then when S is embedded in the following frames, the resulting sentence does not inherit p as a presupposition:
a. If p^+, S (cf.(14a))
b. Either not p^+, or S (cf. (14b))
c. p^+ and S (cf. (14c))

1.4 Weak and Strong Kleene and Presupposition Filtering

We now consider a highly idealized and simplistic formalism for presupposition based on a propositional language, whose goal is to characterize how propositional connectives (negation, disjunction, conjunction, conditional) interact with presupposition triggers. We define a trivalent valuation as a function from propositional atoms to $\{0, \#, 1\}$. We assume that presuppositions are semantically attached to certain atomic sentences, by way of of meaning postulates. For instance, a meaning postulate might state that a certain propositional atom p is undefined whenever a certain other atom q is false, i.e. it rules out any valuation where q is true or false but q is false. Such a meaning postulate chracterizes p as presupposing q – for instance p represents the sentence *Gloria stopped smoking* and q the sentence *Gloria used to smoke*.

Now, depending on the choice of truth tables for the connectives, the conditions under which, say, $r \wedge p$ is itself undefined will vary. In the Weak Kleene (WK) Semantics, the undefined truth-value is *contagious*: a sentence that contains a subclause that is undefined under some valuation v is itself undefined under v. That is, sentences of the form $A \vee B$, $A \wedge B$ and $A \rightarrow B$ are all undefined if and only if either A or B is undefined. Otherwise, if neither A nor B are undefined in v, the truth-values of these more complex sentences in v are just those that result from the classical, bivalent semantics for the relevant connectives. Recall that we view presuppositions as definedness conditions: a sentence S presupposes A just in case A has to be true for S to be defined. In the WK semantics, a complex sentence is defined (i.e. is true or false) under a certain valuation v if an only if each of its atom is assigned a standard truth-value by v. So the WK semantics simply predicts that a complex sentence inherits all the presuppositions of its atomic parts. While this seems to be in line with the fact that all the sentences in (6) appear to presuppose that Gloria used to smoke, the WK semantics seems to be falsified by the filtering generalization stated in (14), i.e. the fact that sometimes a sentence's presuppositions are *not* inherited when the sentence is embedded in a larger sentence. For instance, the WK semantics predicts that all sentences in (13) are undefined if Gloria has never smoked, since in that case the second clause (*She stopped smoking*) is undefined, hence it predicts that they presuppose that Gloria has smoked in the past, which is undesirable. Rather, if Gloria has never smoked, we want (13b) to be true and (13c) to be false, and, within the simplistic (and incorrect) translation of natural language conditionals into material conditionals, we would want (13a) to be true (rather than undefined) if Gloria has never smoked.[7]

Let us therefore move to the Strong Kleene (SK) semantics for propositional logic. The core idea of the SK truth tables is that a sentence of the form $A \circ B$, where $\circ$ is a binary connective, can receive a standard truth-value even if, say, B is undefined, provided the truth-value of A is sufficient to determine the truth-value of $A \circ B$, on the basis of the classical, bivalent truth table for $\circ$. Otherwise the sentence is undefined. Consider for instance $p \vee q$, in a situation where p is true and q is undefined. Given the classical truth table for disjunction, the fact that p is true is sufficient to ensure that $p \vee q$ is true, irrespective of q's truth-value. So we count $p \vee q$ as true in this case. On the other hand, if p is false and q is undefined, we do not have enough information to decide whether $p \vee q$ is true or false, so we assign it the undefined truth-value. More formally, to any bivalent boolean function f (a function from $(0, 1)^2$ to $(0, 1)$), we associate its *trivalent SK extension* f' (a function from $(0,1,\#)^2$ to $(0,1,\#)$), defined as follows (George 2008a):

(15) **Strong Kleene Trivalent Extension of a Boolean function** f

a. For any pair (x, y) of elements of $\{0, 1, \#\}$, a bivalent repair of (x, y) is any pair (x', y') of elements of $\{0, 1\}$ such that $x' = x$ if $x \in \{0, 1\}$ and $y' = y$ if $y \in \{0, 1\}$.

b. $f'(x,y) = \begin{cases} 0 & \text{if for every bivalent repair } (x',y') \text{ of } (x,y),\ f(x',y') = 0 \\ 1 & \text{if for every bivalent repair } (x',y') \text{ of } (x,y),\ f(x',y') = 1 \\ \# & \text{otherwise} \end{cases}$

The Strong Kleene truth tables for conjunction, disjunction, and material implication correspond to the trivalent SK extensions of the bivalent Boolean functions they express. They are given in Table 1.1:

Table 1.1
Strong Kleene connectives

A	B	$A \wedge B$
1	1	1
1	#	#
1	0	0
#	1	#
#	#	#
#	0	0
0	1	0
0	#	0
0	0	0

A	B	$A \vee B$
1	1	1
1	#	1
1	0	1
#	1	1
#	#	#
#	0	#
0	1	1
0	#	#
0	0	0

A	B	$A \rightarrow B$
1	1	1
1	#	#
1	0	0
#	1	1
#	#	#
#	0	#
0	1	1
0	#	1
0	0	1

We will now show that the SK truth tables straightforwardly account for the generalization stated in (14), repeated in (16):

(16) Let S be a sentence that presupposes p. Let p^+ be a sentence that entails p. Then when S is embedded in the following frames, the resulting sentence does not inherit p as a presupposition:

a. If p^+, S
b. Either not p^+, or S
c. p^+ and S

Within the model we are developing, this generalization takes the following form.

(17) If p^+ and p are two bivalent sentences (i.e. sentences that are undefined under no valuation) such that $p+$ entails p, and if S presupposes p (i.e. S is undefined if and only if p is false), then the following sentences are themselves presupposition free, i.e. are never undefined (they are either true or false relative to any valuation)

a. $p^+ \to S$
b. $\neg p^+ \lor S$
c. $p^+ \land S$

This generalization is a direct consequence of the SK truth tables. While this can be shown by applying SK truth tables mechanically, as in Table 1.2, it can also be explained with a little bit of reasoning. Assume that p^+ entails p and that p^+ is bivalent (i.e. is either true of false, but cannot be undefined). According to the SK truth tables, if $p+$ is false, (17a) and (17b) are true and (17c) is false (irrespective of the truth-value of S). If p^+ is true, then p is true (since p^+ entails p), and therefore S is defined, hence all sentences in (17) are defined. It follows that no sentence in (17) can ever be undefined.

We can show the same thing using truth tables. The important point is that there will be no line where p^+ is true but p is false (since p^+ entails p). Also, since we assume that p^+ and p are bivalent propositions, there is no line where they receive the undefined truth-value. Finally, S is undefined if and only if p is false. Given these constraints, the truth tables for the sentences in (17) contain exactly 5 lines, defined by the possible truth-values of p^+, p and S, as shown in Table 1.2. As can be read off the table, when S is undefined, p^+ is false, and this is sufficient to ensure that the relevant sentences receive a standard truth-value even though S is undefined.

Table 1.2
truth tables for the sentences in (17), based on SK

p^+	p	S	$p^+ \to S$	$\neg p^+ \lor S$	$p^+ \land S$
1	1	0	0	0	0
1	1	1	1	1	1
0	1	0	1	1	0
0	1	1	1	1	0
0	0	#	1	1	0

1.5 An extended propositional language to represent presuppositions

In order to state generalizations about the predictions of various trivalent logics regarding presupposition projection, it will be useful to explicitly represent presuppositions in our object language. To this end, we move to a formal system in which presuppositions are not introduced by meaning postulates, but by an explicit connective notated '::' – a reversed version of Blamey's (1986) *transplication* operator, which Blamey introduced with the same purpose.[8] We now work with standard valuations which map atomic propositions to either 0 or 1, and never to #. The undefined truth-value (#) is introduced by the connective '::', whose function s to introduce definedness conditions explicitly. That is, a sentence such as $p :: q$, where p and q are atoms, is undefined (#) if q is false, true if p and q are both true, and false if p is false and q is true. The syntax of :: is just that of any binary connective. Therefore, we allow sentences of the form $A :: B$, where A and B are both potentially complex sentences (for instance $(p \vee q) :: (r \wedge s)$), even though we do not assume that English has any way of directly attaching a presupposition to a complex sentence, since we view presuppositions as being lexically triggered. Allowing such formulae in our language of representation will nevertheless prove convenient in order to state the predictions of, say, the SK semantics, in a maximally general way. The truth table for :: is given in Table 1.3. It makes $A :: B$ true if both A and B are true, false if A is false and B is true, undefined in all other cases.

Table 1.3
Presupposition Introducing Connective (Transplication)

A	B	$A :: B$
#	#	#
0	#	#
1	#	#
#	0	#
0	0	#
1	0	#
#	1	#
0	1	0
1	1	1

Within this formalism, a sentence such as *Mary stopped smoking* will be translated, for instance, as $p :: q$, where p translates *Mary doesn't currently smoke* and q translates *Mary used to smoke*. The sentence $p :: q$ is then defined only if

Mary used to smoke (i.e. if q is true), and when defined, it is true if Mary currently doesn't smoke, false otherwise. This system allows for the possibility that a sentence presupposes another sentence which itself has a presupposition, since B in $A :: B$ can be a complex sentence. So the formula $p :: (q :: r)$ represents a sentence which is defined only if $q :: r$ is true, which itself can be the case only if q and r are both true. The kind of sentences that might be translated as $p :: (q :: r)$ are those which have a presupposition which is itself in some intuitive sense presuppositional. An example of such a sentence would be something like *Gloria stopped talking to her brother*, with r standing for *Gloria has exactly one brother*, q for *Gloria used to talk to a brother of hers*, and p for *Gloria currently does not talk to any brother of hers*.

For legibility, I will often use lower-case letters to represent formulae that stand on the right of '::' and capital letters for formulae that stand on the left of '::'. We say that two formulae F and G are equivalent, which is notated $F \equiv G$, if they have the same truth-value under all possible valuations. If we adopt the SK truth tables for '$\wedge$', '$\vee$' and '$\rightarrow$', the following equivalences hold as theorems, for all formulae A, a, B, b, and $\top$, where $\top$ is a special propositional symbol that represents an arbitrary tautological formula, i.e. is an atomic formula that is true under all valuations (cf. Winter 2021 for a similar presentation):

(18) a. $(A :: \top) \equiv A$
b. $\neg(A :: a) \equiv (\neg A) :: a$
c. $(A :: a) \wedge (B :: b) \equiv (A \wedge B) :: ((A \rightarrow b) \wedge (B \rightarrow a))$
d. $(A :: a) \vee (B :: b) \equiv (A \vee B) :: ((\neg A \rightarrow b) \wedge (\neg B \rightarrow a))$
e. $(A :: a) \rightarrow (B :: b) \equiv (A \rightarrow B) :: ((A \rightarrow b) \wedge (\neg B \rightarrow a))$

The equivalence in (18b) expresses the fact that presuppositions are inherited when a sentence is negated. In the special case where a is a tautology, the facts stated in the last three lines reduce to the following (replacing A with $\neg A$ in (18d), and using double negation elimination, which is valid in SK).

(19) a. $A \wedge (B :: b) \equiv (A \wedge B) :: (A \rightarrow b)$
b. $\neg A \vee (B :: b) \equiv (\neg A \vee B) :: (A \rightarrow b)$
c. $A \rightarrow (B :: b) \equiv (A \rightarrow B) :: (A \rightarrow b)$

Note that A and B, a, b stand for arbitrary formulae, which can themselves have occurrences of '::'. Therefore we also have, for instance, replacing A with $A :: a$:

$$(A :: a) \wedge (B :: b) \equiv ((A :: a) \wedge B) :: ((A :: a) \rightarrow b)$$

Let us now define the following notion of entailment, which we call *ts*-entailment (*tolerant-strict* entailment, following Cobreros et al. 2012): F *ts*-entails G if whenever F is not false (i.e. is true or undefined), G is true. Now, $A \rightarrow b$ is a tautology (in the sense of being true in every valuation) if and only if A *ts*-entails b. When this is the case, the three formulae in (19) become equivalent to, respectively, $A \wedge B$, $\neg A \vee B$ and $A \rightarrow B$. In other words, when A *ts*-entails b, the presupposition of B, namely b, is filtered out in the relevant sentences. In the case where A itself is bivalent (can never be assigned the value #), in particular when it does not contain any occurrence of ::, the condition 'A *st*-entails b' boils down to 'A entails b' in the classical sense (whenever A is true, so is b, also called *strict-strict* entailment, *ss*-entailment, cf. Cobreros et al. 2012), and so we recover the generalization in (17).

1.6 Three challenges

It seems that this approach to presupposition projection, based on the SK semantics, meets a number of key desiderata. However, it faces at least three problems.

First, there are cases where the presupposition of a subclause is not inherited by the sentence it occurs in, despite the fact that our system predicts it to be inherited. Consider for instance *Gloria did not stop smoking and she has (in fact) never smoked*. The current system predicts this sentence to presuppose the material conditional *Gloria has never smoked* $\rightarrow$ *Gloria used to smoke*, which is itself equivalent to *Gloria used to smoke*. This presupposition of course is incompatible with the first conjunct, so it is predicted that this sentence should be pereceived as contradictory. Indeed, in our system, this sentence is never true: it is undefined if Gloria has never smoked, and false if Gloria used to smoke. Yet this sentence is perfectly understandable and felicitous in certain contexts, and conveys that Gloria has never smoked. This problem is not only specific to the SK approach, it is also present in a system based on the Middle Kleene truth tables, which we introduce in section **??** to address the problem raised in the next paragraph.

A second problem is that the SK truth tables are defined in a way that is insensitive to linear order. Yet while we seem to get presupposition filtering in a sentence like *Gloria used to smoke a cigarette after dinner and she stopped smoking*, things are much less clear for something like *Gloria stopped smoking and she used to smoke a cigarette after dinner*. Since the SK truth table for $\wedge$ is symmetric with respect to the two clauses it conjoins, these two sentences are predicted to behave exactly the same with respect to filtering.

Finally, a third problem (known as the *Proviso Problem* since Geurts 1996) is that the SK truth tables (but also the Middle Kleene truth tables that we introduce in section **??** in order to solve the second problem) predict apparently too weak presuppositions for sentences of the form $A \circ B$ in cases where there is no natural relationship between, say, A, and the presupposition of B. For instance, a sentence like *If John is well informed, he knows that Mary lives in the countryside* is predicted to presuppose something like *If John is well informed, Mary lives in the countryside*, which is not intuitively correct. Rather, the sentence is perceived to presuppose that Mary lives in the countryside, *unconditionally*.

We now address these three issues in turn, and show how our theory can be modified or enriched so as to solve them.

1.7 Local Accommodation

The sentence *Gloria did not stop smoking and she has never smoked* can be schematized as $\neg(A :: B) \wedge \neg B$, with $A :: B$ standing for *Gloria stopped smoking* and B for *Gloria has been a smoker*. Now, this formula can never be true. It is undefined if B is false (hence it presupposes B), and false if B is true. As mentioned above, this is not desirable, as the corresponding sentence can be understood to simply convey that Gloria has never smoked and that she therefore cannot possibly have stopped smoking – a meaning that can be even more transparently expressed by *Gloria did not stop smoking since she has never smoked* (but we don’t have the tools to translate *since* in our very impoverished toy language). The point is that *Gloria has not stopped smoking* can be understood to mean something like *It is not the case that Gloria has been a smoker and stopped being one*. Under this interpretation, it seems that the presupposition of *Gloria stopped smoking*, namely the proposition that Gloria used to smoke, is interpreted under the scope of negation, instead of ‘escaping’ it, as is normally the case with presuppositions.

It is tempting to analyze such cases as involving a special use of negation where it is used *metalinguistically*, to object to a potential utterance of *Gloria stopped smoking*, on the ground that its presupposition would be false. That negation has such a metalinguistic use, whereby it is used to object to a potential utterance “on any grounds whatever” is uncontroversial (Horn 1985). For instance, in a discourse such as *He didn’t piss, he urinated!*, negation is not used truth-conditionally (we are not negating the proposition ‘he pissed’, as this would contradict the second clause), but as a means to object to the use of a certain word (*piss*). Likewise, the idea would be that a sentence such as *Gloria did not stop smoking* can be interpreted as conveying that it is not proper to say *Gloria stopped smoking*, and one of the possible reasons why this could be so is if the presupposition of *Gloria stopped smoking* is false.

This approach, in terms of a special metalinguistic use of negation, is not tenable, because a very similar point can be made with sentences that do not involve negation. Consider the following discourse:

(20) I am uncertain whether it is raining. If Gloria knows that it is raining, she will tell me to take an umbrella.

In isolation, the second sentence presupposes that it is in fact raining, in line with what our system predicts. However, an interpretation where the second sentence has this presupposition would create a pragmatic clash with the first sentence. If the first sentence is true, then the speaker cannot believe that it is in fact raining, and should not be able to use a subsequent sentence which presupposes that it is raining. In this context, it is quite clear that the second sentence is in fact intepreted as *If it is raining and Gloria knows that it is, she will tell me to take an umbrella*. This is again a case where the presupposition of a subclause is interpreted under the scope of an operator (here *if*), despite the fact that our current system predicts it to ‘escape’ its scope.

Somehow we need a mechanism whereby, in quite limited circumstances, the presupposition of a clause S can be collapsed with its assertive meaning, so that it can then be targeted by operators that take scope over S. In other words, instead of interpreting *Gloria did not stop smoking* as $\neg(A :: B)$, which is itself equivalent to $(\neg A :: B)$ – where B stands for *Gloria used to smoke* and A for *Gloria does not smoke now*, we sometimes want to interpret it as $\neg(A \wedge B)$. Following Heim’s (1983) terminology, we say that in such as case the presupposition of a subclause is ‘locally accommodated’ in the scope of negation (the mechanism of ‘*local* accommodation’ contrasts with another one termed ‘*global* accommodation’, discussed in section **??**).

Within the trivalent approach to presupposition, the standard way to introduce such a local accommodation mechanism is by means of an operator first defined in Bochvar (1939) which we notate $\mathcal{A}$ (Beaver and Krahmer 2001).[9] That is, we add $\mathcal{A}$ to the language, and define $\mathcal{A}(S)$ to be true if S is true, and false if S is either undefined or false:

Table 1.4
Accommodation operator $\mathcal{A}$ (Bochvar’s ‘external assertion’ operator)

S	$\mathcal{A}(S)$
1	1
#	0
0	0

The $\mathcal{A}$ operator collapses presuppositions with assertive meaning. Namely, the following holds, for any sentence S: $\mathcal{A}(B :: C) \equiv \mathcal{A}(B \wedge C)$. In the special case where B and C are themselves bivalent sentences (i.e. do not contain any occurrence of '::'), we have: $\mathcal{A}(B :: C) \equiv (B \wedge C)$.

The sentence *Gloria didn't stop smoking* can then be parsed as $\neg(\mathcal{A}(A :: B))$, where B stands for *Gloria used to smoke*, and A for *Gloria doesn't smoke now*. On this parse, it is equivalent to $\neg(A \wedge B)$, i.e. 'it is not the case that Gloria used to smoke and doesn't smoke now', as desired. As to (20), it can now receive a parse such as $\mathcal{A}(A :: B) \rightarrow C$, with B standing for *it is raining*, A for *Gloria believes that it is raining*, and C for *she will tell me to take an umbrella*. Under such a parse, the sentence is equivalent to $(A \wedge B) \rightarrow C$, i.e. *If it is raining and Gloria knows that it is raining, she will tell me to take an umbrella*.

Of course, unrestricted application of $\mathcal{A}$ would make the overall system self-defeating, since we would no longer predicts that by default sentences of the form $\neg S$ or *If S, T* inherit the presuppositions of S. What is typically assumed is that $\mathcal{A}$-insertion is marked, and that listeners assume a parse with $\mathcal{A}$ only when a parse without it would give rise to pragmatic deviance (as would be the case in the two examples we have considered).

While all current theories of presupposition include a local accommodation mechanism, this specific treatment (based on Bochvar's operator) faces some problems. Chatain and Schlenker (2024) provide evidence against the idea that the local accommodation mechanism is syntactically represented (mostly based on tests involving ellipsis). Chatain and Schlenker (2024) also build on an argument first proposed in Romoli (2011) against the use of Bochvar's operator: when a sentence S presupposes something like $A \wedge B$, sometimes A is locally accommodated but B is not. Without discussing the details, formally, the argument is that a sentence that we would represent (in the absence of $\mathcal{A}$) in our system as $\neg(S :: (A \wedge B))$, where S, A and B are themselves bivalent, can be interpreted as meaning $\neg((A \wedge S) :: B))$. But the accommodation operator, if introduced just under negation $(\neg\mathcal{A}(S :: (A \wedge B)))$, yields as a meaning $\neg(A \wedge B \wedge S)$. That is, either all the presuppositions of a sentence are locally accommodated, or none is, but both Romoli (2011) and Chatain and Schlenker (2024) construct examples that might involve something like 'partial local accommodation', and for this reason argue that local accommodation should not be captured by means of the $\mathcal{A}$-operator. We ignore this complication in the remainder of this chapter.

1.8 Order effects: Strong Kleene and Middle Kleene

As mentioned at the end of section 1.5, the presupposition of a sentence of the form 'A and B' seems to be sensitive to the order of the two conjuncts. Consider for instance:

(21) a. Gloria used to smoke and she stopped smoking.
b. Gloria stopped smoking and she used to smoke.

(21b) feels deviant, in contrast with (21a). Such a contrast has sometimes been used to argue that (21b), unlike (21a), presupposes that Gloria used to smoke. The argument is the following: if (21b) presupposes that Gloria used to smoke, it can only be used in a context where it is already known that Gloria used to smoke, making the second conjunct completely redundant, which would explain the deviance of (21b). This argument, however, is by itself quite weak, given that there are sentences that display a similar sensitivity to order in the absence of presuppositions, as in (22) (Schlenker 2008):

(22) a. Susan lives in France and resides in Paris.
b. #Susan resides in Paris and lives in France.

Schlenker (2008) observes that in both (22a) and (22b), the conjunct *Susan lives in France* is redundant in the sense that the whole sentence is equivalent to the other conjunct, namely *Susan resides in Paris*. However, in (22b), the conjunct *lives in France* is redundant given what precedes it, while in (22a) it is redundant given what follows it. Schlenker (2008) proposes, in a nutshell, that a constituent that is redundant *given preceding material* creates a sense of deviance, but that constituents that are redundant only due material that follows them do not, or at least not to the same extent. Now, based on such a generalization, the contrast between (21a) and (21b) is expected if we accept that *Gloria stopped smoking* entails that Gloria used to smoke. One could say that in (21b), but not in (21a), *she used to smoke* is redundant given preceding material, which would explain why (21b), but not (21a), feels deviant. The key point would be that the contrast would not be due in any way to the presence of a presuppositional trigger in the sentence.

It is however important to note that such an account is not completely straightforward within a trivalent approach to presupposition based on Strong Kleene. This is because on such an approach, neither (21a) nor (21b) are equivalent to *Gloria stopped smoking*, since this last sentence is undefined in a situation where Gloria has never smoked, while the sentences in (21) are false in a such situations. So it is not clear that *she used to smoke* is redundant given preceding material in (21b). It is true that *Gloria stopped smoking* entails *Gloria used to smoke* in the SK approach, provided we define entailment as preservation of truth from premise to conclusion (whenever the premise is true, so is the conclusion), but it is not the case, in the SK semantics, that whenever A entails B in this sense, $A \wedge B$ is equivalent to A, which would be necessary to claim that B is redundant - under a definition of equivalence where two sentences are equivalent if they have the same truth-value at every valuation. Even if A entails B (where entailment is defined as perservation of truth from premise to conclusion), there might still be valuations where A is undefined but $A \wedge B$ is false. This is precisely the case with $A =$ *Gloria stopped smoking* and $B =$ *Gloria used to smoke*, since if Gloria has never smoked, then according to the SK truth table, A is undefined but $A \wedge B$ is false. If we use a weaker notion of equivalence, where two sentences are equivalent is they are true in exactly the same worlds (but may have different truth-values in worlds where they are not true), then if A entails B (in the sense of preservation of truth from premises to conclusions), then A and $A \wedge B$ are equivalent in this weaker sense, and so B could be said to be redundant in a sense.

Be that as it may, we may want to test the role of order in conjunctive sentences involving presuppositional triggers in a case where one could not argue in any way that one of the conjuncts is redundant. Instead of comparing sentences of the form $p \wedge (A :: p)$ and $(A :: p) \wedge p$ (here with $(A :: p) =$ *Gloria stopped smoking* and $p =$ *Gloria used to smoke*), one can move to sentences of the form $p^+ \wedge (A :: p)$, and $(A :: p) \wedge p^+$, where p^+ is a bivalent sentence that asymmetrically entails p, A is a bivalent sentence, and $A \wedge p$ does not entail p^+. In such a case, there is no plausible notion of entailment such that $(A :: p)$ can be said to entail p^+, so neither of the conjunct is redundant in any plausible sense, and so, on the SK view, both orders should behave in the same way – namely both sentences are are never underfined, hence both are expected to have no presupposition. Let us therefore compare the following two sentences (here $p =$ *Gloria used to smoke* and $p^+ =$ *Gloria used to smoke a cigarette after each meal*, and $(A :: p) =$ *Gloria stopped smoking*).

(23) a. Gloria used to smoke a cigarette after each meal, and she stopped smoking.
b. Gloria stopped smoking, and she used to smoke a cigarette after each meal.

The SK approach predicts that neither (23a) nor (23b) presupposes that Gloria used to smoke. Of course, it is still expected that both sentences *entail* that Gloria used to smoke. For this reason, in order to test the prediction that both sentences fail to presuppose that Gloria used to smoke, we need to embed both sentences in a construction which preserves presuppositions, but not entailment. As we discussed above (cf. our discussion of (8)), one way to do this is to embed them in an if-clause, as in (24):[10]

(24) a. If Gloria used to smoke a cigarette after each meal and stopped smoking, she feels better.
b. If Gloria stopped smoking and used to smoke a cigarette after each meal, she feels better.

The empirical question here is whether either of these sentences is felt to license the inference that Gloria used to smoke. If presupposition filtering can happen both from left to right and from right to left, it is expected that in both cases the rate of endorsement of this inference should be lower than in a more simple control case in which the presupposition is expected not to be filtered out. Experimental work (Mandelkern et al. 2020) on precisely this topic has provided evidence that there is no right-to-left filtering in such conjunctive sentences. Specifically, the rate of endorsement of the presupposition when the presuppositional trigger comes first in such a conjunction (as in (24b)) is no different from the one found with more simple control cases which are expected to trigger the same presupposition. Furthermore, things are different when the trigger comes second, as in (24a), in which case the rate of endorsement of the presupposition is significantly lower. In this particular case this means that sentences like (24b) are felt to license the inference that Gloria used to smoke to a larger extent than sentences like (24a). This goes against the SK approach, which is insensitive to order and predicts that the presupposition that Gloria used to smoke should be filtered out in both cases.

It is therefore plausible that presupposition filtering in fact occurs from left to right, i.e. the material that is ‘responsible’ for filtering out the presupposition of a subclause must occur *before* this subclause.[11] There are potentially two ways to modify our theory in order to achieve such a result: we can either change our semantics, i.e. move away from the SK truth tables and adopt asymmetric truth tables, or keep to the SK semantics but modify the bridge principle in (5) in order to make it sensitive to order.

Let us first discuss the second option. The bridge principle as stated in (5) requires that the sentence being used be defined (be true or false) at every world of the context. To make it order-sensitive, we can require instead that at every point in the sentence, the sentence is guaranteed to be defined whatever non-presuppositional expressions follow this point. Consider for instance a sentence $(A :: p) \wedge p^{+}$, where p^{+} entails p, and A, p, and p^{+} are themselves bivalent, i.e. do not themselves contain the connective '::'. In SK, this sentence is defined in every possible world. However, it is not the case that at the point just after the first conjunct is processed, the sentence is guaranteed to be defined whatever material follows which does not itself contain '::'. For instance $(A :: p) \wedge \top$ is not defined in every possible world, since it is undefined when p is false. According to the order-sensitive version of the Bridge Principle, then, this sentence would not be felicitous in a context that does not already entail p, hence would be felt to presuppose p. In contrast, for $p^+ \wedge (A :: p)$, at the point where the clause $(A :: p)$ is encountered, it is already known that the other conjunct is p^+, and that the sentence is going to be defined at every world. The general idea to make a pragmatic principle such as the Bridge Principle sensitive to linear order is inspired by Schlenker (2008), who introduces this technique in order to offer his own, non-trivalent theory of presupposition projection. Here is a modified, order-sensitive version, of the bridge principle, adapted to our extended propositional language, making this idea explicit.

(25) **Order-sensitive Bridge Principe**

a. Let o be an occurrence of a binary connective, distinct from '::', in a sentence S. We say that S' is a *continuation of S at point o* if S' can be obtained by one or several replacements of subclauses that follow o in S with arbitrary sentences, in such a way that there is no occurrence of '::' after o in S'.[12]

b. For a sentence S to be felicitous, it must be the case that for every occurrence o of a binary connective in S, every continuation S' of S at point o is defined at every world of the context.

Now, a second option consists in keeping to the original bridge principle, which is not sensitive to order, but abandoning the SK truth tables in favor of asymmetric truth tables. The classical proposal in this spirit, originating with Peters (1979) and developed in Beaver and Krahmer (2001), is based on the so-called Middle Kleene truth tables (also called the Peters truth-tables). The logic is similar to that of the SK connectives, but with a left-to-right order asymmetry. Negation is defined exactly as in Strong Kleene. For any binary connectives $\circ$, $A \circ B$ is undefined as soon as A is undefined, but if A is defined, and if A's truth-value is sufficient to determine the truth-value of $A \circ B$ based on the classical semantics for $\circ$, then the whole sentence gets this truth-value, even if B is itself undefined. This gives rise to the following truth tables (the lines that differ from the SK truth tables are in boldface):[13]

Table 1.5
Middle Kleene connectives

A	B	$A \wedge B$	A	B	$A \vee B$	A	B	$A \rightarrow B$
1	1	1	1	1	1	1	1	1
1	#	#	1	#	1	1	#	#
1	0	0	1	0	1	1	0	0
#	1	#	**#**	**1**	**#**	**#**	**1**	**#**
#	#	#	#	#	#	#	#	#
#	**0**	**#**	#	0	#	#	0	#
0	1	0	0	1	1	0	1	1
0	#	0	0	#	#	0	#	1
0	0	0	0	0	0	0	0	1

Similarly to the SK truth tables, and following George (2008a) again, the logic behind these truth tables can be made formally explicit by specifying how binary boolean functions from $\{0, 1\}^2$ to $\{0, 1\}$ are to be extended into functions from $\{0, 1\,\#\}^2$ to $\{0, 1\,\#\}$:

(26) **Middle Kleene Trivalent Extension of a Boolean function f**
Let f be a function from $\{0, 1\}^2$ to $\{0, 1\}$. Then f', the Middle Kleene extension of f, is the function from $\{0, 1, \#\}^2$ to $\{0, 1, \#\}$ defined as follows.

a. For any pair (x, y) of elements of $\{0, 1, \#\}$, a *right-side repair* of (x, y) is a pair (x', y') such that $x' = x$ and $y' \in \{0, 1\}$ and $y' = y$ if $y \in \{0, 1\}$.

b. $$f'(x,y) = \begin{cases} 0 & \text{if } x \neq \# \text{ and for every right-side repair } (x', y') \text{ of } (x, y), f(x', y') = 0 \\ 1 & \text{if } x \neq \# \text{ and for every right-side repair } (x', y') \text{ of } (x, y), f(x', y') = 1 \\ \# & \text{otherwise} \end{cases}$$

On the basis of these truth tables, it is clear that when a presuppositional sentence $A :: p$ occurs on the left side of a connective, the whole coordinated clause inherits p as a presupposition, since whenever $A :: p$ is undefined, the coordinated clause is undefined as well. When a presuppositional sentence $A :: p$ occurs on the right-side of a connective, however, it can be filtered out by the clause that sits on the left side of the connective, in exactly the same way as in SK. For this reason, the generalization stated in (16) is still predicted to hold.

More precisely, the Middle Kleene truth tables, together with the same truth table for the transplication operator (cf. Table 1.3), validate the following equivalence schemas:

1. $(A :: \top) \equiv A$
2. $\neg(A :: a) \equiv (\neg A) :: a$
3. $(A :: a) \wedge (B :: b) \equiv (A \wedge B) :: a \wedge (A \rightarrow b)$
4. $(A :: a) \vee (B :: b) \equiv (A \vee B) :: a \wedge (\neg A \rightarrow b)$
5. $(A :: a) \rightarrow (B :: b) \equiv (A \rightarrow B) :: a \wedge (A \rightarrow b)$

One can see immediately that the presupposition of the left-most clause is always inherited, while the presupposition of the rightmost clause can be filtered out.

If we find left-right asymmetries across the board (i.e. not only with conjunction), then the approach based on Middle Kleene paired with the ‘standard’ bridge principle and the one based on Strong Kleene paired with the order-sensitive bridge principle are equally good at predicting the facts. Suppose however that the order-sensitivity that we found with conjunction is not found with disjunction. In such a case, we could in principle use the Middle Kleene truth table for conjunction but keep to the Strong Kleene truth table for disjunction. The resulting system would not be particularly principled, but it would at least capture the facts. No such move is really possible if we adopt the order-sensitive bridge principle.

The actual empirical situation might in fact be precisely one where we need the symmetric SK truth-table for disjunction but the asymmetric Middle Kleene truth table for conjunction. Before discussing the kinds of data that can help us choose between a symmetric and asymmetric disjunction, let us first consider data that suggest a symmetric behavior of disjunction but which are not in fact dispositive. We can observe that both sentences in (27) intuitively fail to presuppose that that Gloria used to smoke:

(27) a. Either Gloria has never been a smoker, or she stopped smoking.
b. Either Gloria stopped smoking, or she has never been a smoker.

This is of course what the SK truth-table for disjunction predicts: in both cases, there is no way the sentence can be undefined, since when *Gloria stopped smoking* is undefined, *Gloria has never been a smoker* must be true, hence the whole disjunctive sentence is true according to the SK semantics. In contrast, the Middle Kleene truth-table predicts that (27b), but not (27a), presupposes that Gloria has been a smoker. While the lack of a perceived presupposition for (27b) goes in favor of SK, it does not in fact provide strong evidence. This is because even on the Middle Kleene account, one can make sense of this fact, once the possibility of local accommodation (cf. section 1.7) is taken into account. Note indeed that an utterance of a sentence of the form $A \vee B$ typically triggers the inference that the speaker is uncertain about the truth-values of both A and B. Therefore, someone who says (27b) must be uncertain about whether Gloria has ever smoked. The presupposition predicted by the Middle Kleene disjunction, namely the presupposition that Gloria has smoked, clashes with this inference. As discussed in section 1.7, we assume that in order to avoid such a clash, presuppositions can be locally accommodated, i.e. listeners will assume a parse that includes the accommodation operator $\mathcal{A}$. Let $A : a$ be *Gloria stopped smoking* and a be *Gloria has been a smoker*. A possible parse for (27b) is then $\mathcal{A}(A :: a) \vee \neg a$, which is equivalent to $(a \wedge A) \vee \neg a$, which we can paraphrase as *Either Gloria has been a smoker and stopped, or she has never been a smoker*. To recap, under the MK semantics, a sentence of the form $(A :: a) \wedge \neg a$ is undefined if a is false, but since a clashes with the inference that the speaker does not know the truth-value of a, listeners are led to posit an alternative parse, namely $\mathcal{A}(A :: a) \vee \neg a$, and on this parse the sentence neither entails nor presupposes a. So the fact that both orders are felicitous and fail to presuppose that Gloria has been a smoker can in fact be made sense of within the MK semantics, hence does not provide a strong argument in favor of using the SK disjunction.

That being said, Kalomoiros and Schwarz (To appear.) argue in favor of an SK disjunction and and MK conjunction on the basis of experimental data showing that local accommodation with conjunction leads to degraded acceptability and that there is no comparable degradation in acceptability when one move from (27a) to (27b). A more direct way to adjudicate between the MK and SK disjunctions is to consider cases of the form $\neg p^{+} \vee (S :: p)$ and $(S :: p) \vee \neg p^{+}$, where p^{+} s a bivalent proposition that asymmetrically entails p. Indeed, in such a case, the pragmatic inference that the speaker is not opinionated about p^{+} is compatible with a context in which p is presupposed (since one can know that p is true without knowing whether p^{+} is true). Under the MK semantics, $(S :: p) \vee \neg p^{+}$ presupposes p, but not $\neg p^{+} \vee (S :: p)$, and since there is no pressure to remove he presupposition p by means of local accommodation, a difference between these two cases should be detectable. Finding no difference would go in favor of SK. We can thus consider the following two sentences, where p+ is *Gloria used to smoke a cigarette after each meal.* and p is *Gloria use d to smoke.*

(28) a. Either Gloria didn't use to smoke a cigarette after each meal, or she stopped smoking.

b. Either Gloria stopped smoking, or she didn't use to smoke a cigarette after each meal.

To the extent that (28b) is felt to imply *Gloria use to smoke* more clearly than (28a), this would go in favor of adopting the MK disjunction. If on the other hand there is no marked difference between these two cases, then this argues in favor of a symmetric truth table for disjunction, hence in favor of SK. The judgments here are subtle, and seem to require careful experimental work. See Loder, Kuhn, and Spector (2024) for relevant experimental evidence which suggests a symmetric behavior of disjunction in such examples.

If disjunction behaves more symmetrically than conjunction, as existing experimental results suggest, this casts doubt on a unified account of the projective behavior of connectives in terms of either SK or MK. See however, Kalomoiros (2022) for a proposal (not based on trivalence) whose goal is to explain why disjunction apparently behaves more symmetrically than conjunction

1.9 Global Accommodation, conditional presuppositions, and the Proviso Problem

As we have seen, the SK and MK semantics account for the generalization in (14). That is, when, for instance, q entails p, then $q \wedge (A :: p)$ is presupposition-free (if A is itself presupposition free). This is a good result, but what about the more general case where there is no logical relationship between q and p? It follows from the statements in (19) that $q \wedge (A :: p)$, $q \rightarrow (A :: p)$ and $\neg q \vee (A :: p)$ are equivalent, respectively to $(q \wedge A) :: (q \rightarrow p)$, $(q \rightarrow A) :: (q \rightarrow p)$ and $(\neg q \vee A) :: (q \rightarrow p)$. In other words, the presupposition of such sentences is expected to be the conditional statement $q \rightarrow p$.

Let us consider a concrete case, using a conditional sentence.

(29) If Gloria lives in the countryside, she must take good care of her pets.

This sentence is predicted to presuppose the material conditional *Gloria lives in the countryside* $\rightarrow$ *Gloria has pets*, which we allow ourselves to translate into the conditional sentence *If Gloria lives in the countryside, Gloria has pets*. That is, our theory predicts that (29) is felicitous in any context where it is common knowledge that if Gloria lives in the countryside, she has pets. A plausible context with this property is one where it is generally believed that everybody who lives in the countryside has pets. This seems like a good prediction, in the sense that when we situate ourselves in such a context, (29) is not perceived to entail that Gloria in fact has pets in an unconditional way (it does not exclude the possibility that Gloria does not live in the countryside, in which case she may well have no pets). That is, the presupposition of the consequent (*Gloria* *has pets*) is filtered out in such a context, due to the *contextual* entailment from *Gloria lives in the countryside* to *Gloria has pets*.

However, another expectation is that, out of the blue, (29) should be felt to presuppose (and therefore to entail) the conditional statement *If Gloria lives in the countryside, Gloria has pets* but not the unconditional statement *Gloria has pets*. This seems problematic, since out of the blue (29) is perceived to license the inference that Gloria has pets. Likewise, *If it is sunny, Gloria will walk her dog* is predicted by our theory to only presuppose the conditional statement *If it is sunny, Gloria has a dog*, but the sentence intuitively presupposes the unconditional proposition that Gloria has a dog. The very same point can be made with other presupposition triggers. For instance, *If Gloria is reasonable, she will stop smoking* does not intuitively merely presuppose that if Gloria is reasonable, she currently smokes, but rather that Gloria smokes, unconditionally. Yet we do *sometimes* perceive conditional presuppositions out of the blue. Suppose I don't quite know whether the kind of equipment that some sport called *shmub* requires, and then I hear *If Gloria practices shmub, her helmet must be expensive*. I will typically infer that if Gloria practices shmub, she has a helmet, and, in fact, that shmub requires a helmet, but will not infer the stronger proposition that Gloria has a helmet, unconditionally.

To take stock, the trivalent approach to presupposition projection predicts conditional presuppositions across the board for such sentences, but in some cases the perceived presupposition, when the sentence is uttered out of the blue, is unconditional. This problem, known as the *Proviso Problem* (van der Sandt 1992; Geurts 1999), is currently still a challenge for all theories of presupposition: a theory which, unlike the SK and MK accounts, would predict unconditional presuppositions would have difficulty to account for the fact that a sentence such as (29) does not intuitively entail that Gloria has pets when used in a context where it is already believed that everybody who lives in the countryside has pets. Some theories predict a kind of ambiguity between a reading which has a conditional presupposition and one that has a non-conditional presupposition (van der Sandt 1992; Geurts 1999), but such theories still have to explain why some sentences, but not others, out of the blue, do not seem to license at all a conditional presupposition.

Despite these challenges, I will nevertheless sketch a general perspective on the Proviso Problem which is conceptually very natural, and discuss its limitations. To do so, I first need to discuss the notion of *global accommodation* (Lewis 1979; Heim 1983, with roots in Karttunen 1974). So far, our theory says that if a sentence S presupposes p, then S is infelicitous in a context C in which p is not common knowledge. Viewing C as the set of worlds compatible with what is common knowledge, hence as a proposition, this means that for S to be felicitous, C has to entail p. However, in practice, presuppositional sentences are often felicitously used in a context that does not satisfy this constraint. For instance, I could tell you out of the blue something like *My car broke down, and I brought it to the mechanic*, even if I know that you don't know that I have a car, despite the fact the sentence presupposes that I have a car. This means that the bridge principle must be supplemented with a mechanism explaining what happens in the listener's mind when it is not in fact respected. When we evaluate a sentence out of the blue, this mechanism is involved (since out of the blue we are not in a context that satisfies the sentence's presuppositions). Maybe this mechanism sometimes generates an inference that Gloria has pets when applied to (29), despite the fact that (29) does not presuppose that Gloria has pets.

A very natural approach consists in saying that even when a sentence's presuppositions are not entailed by the context, listeners are still able to make sense of the sentence, and if they accept it as true, the context is intersected with the set of worlds where the sentence is true. So when hearing, for instance, *My car broke down*, the listener, if she accepts the sentence, must now believe that I have a car and that this car broke down. A close variation on such an approach would be that when the context manifestly does not satisfy the presupposition of a sentence S that is being used, the accommodation operator $\mathcal{A}$ is introduced at matrix level to avoid a violation of the bridge principle. Either way (i.e. whether or not $\mathcal{A}$ is involved), the conclusion is that a sentence S that presupposes p is understood as communicating $p \wedge S$. While natural, such an approach offers no solution to the *Proviso Problem*, since the mechanism of global accommodation simply turns the presupposition of a sentence into a standard entailment, but does not ever strengthen a conditional presupposition into an unconditional one. So, for instance, in the case of *If Gloria is reasonable, she will stop smoking*, this approach cannot predict that, out of the blue, this sentence is perceived to entail that Gloria is a smoker.

There is however another possible approach, which can help us make sense of the Proviso Problem. The idea is that when we process a sentence S that presupposes p and p is not entailed by the context, we have to 'adjust' the context before updating it with the sentence, so that the adjusted context entails p (see, e.g., Heim 1983). The phrase 'global accommodation' is often used to describe this adjustment process (sometimes the notion is used to refer to matrix insertion of $\mathcal{A}$, which has the same effect on the final interation).

Now, what does it mean to adjust the context? One possibility consists in saying that when faced with presuppositional failure, to the extent that this revision is not too drastic, we simply adjust the context by removing from the context all the worlds where the presupposition of the sentence is false.

This is equivalent to simply intersecting the context with the presupposition of the sentence. Under such a view, we still don't have a solution to the proviso problem.

However, another perspective is possible regarding the nature of the relevant adjustment process. Beaver (1992, 1999, 2001) proposes that global accommodation is essentially a process whereby the listener *infers* what it is that the speaker pretends to be common knowledge, based on the listener's general notion of what a plausible belief state is. Suppose that someone utters *If Gloria is reasonable, she will stop smoking*. Suppose the listener does not believe the material conditional *Gloria is reasonable* $\rightarrow$ *Gloria is a smoker*. Since this material conditional is the semantic presupposition of the sentence, the listener knows that the speaker must assume, or at least pretend to assume, that the context entails it. The listener now has to come up with a plausible context that the speaker could have in mind and which would entail this material conditional. There are infinitely many possible contexts with this property, so the listener has to reason about what could be a plausible context among all those 'candidate contexts', using her general knowledge of the world. Crucially, a context that entails *Gloria is a smoker* also entails *Gloria is reasonable* $\rightarrow$ *Gloria is a smoker*, so such contexts are among the possible 'candidates'. Now, it is in fact very implausible for someone to believe the material conditional *Gloria is reasonable* $\rightarrow$ *Gloria is a smoker*, without in fact believing the stronger, unconditional proposition that Gloria is a smoker (in this particular case, it would be particularly strange to be unsure whether Gloria smokes but to be sure that if she is reasonable she does, since smoking is not, in fact, a reasonable habit). So the listener, who knows that the speaker is assuming a context (i.e. a belief state) that entails the material conditional *Gloria is reasonable* $\rightarrow$ *Gloria is a smoker*, will conclude that most probably the context the speaker is assuming entails the stronger, unconditional presupposition that Gloria is a smoker.

This proposal predicts that whether a conditional or unconditional presupposition is perceived depends on there being a plausible link between the antecedent and the presupposition of the consequent. If a sentence presupposes $A \rightarrow B$ and there is a plausible link (be it causal or epistemic) between A and B (for instance, there is a plausible link between Gloria's practicing a sport known as 'shmub' and her having a helmet), the listener will conclude that the context that the speaker is assuming entails $A \rightarrow B$ but has no reason to assume that that it also entails B. If, on the contrary, there is no plausible link, then it is difficult to think of a plausible context that would entail $A \rightarrow B$ without entailing B. For instance, one can surmise that if it is common knowledge that if it is sunny, Gloria has a sister, it must in fact be common knowledge that Gloria has a sister, unconditionally, since it is not plausible to believe that Gloria's having a sister depends on the current weather. Another way to put it is that any plausible context that entails the conditional *If it is sunny, Gloria has a sister* happens to entail that Gloria has a sister. In that kind of case, even if the semantic presupposition of a sentence is $A \to B$, the perceived presupposition will be B. For instance, while the sentence *If it is sunny, Gloria will pick up her sister at the airport* semantically only presupposes that if it is sunny, Gloria has a sister, the perceived presupposition is that Gloria has a sister.

Now, while it has often been suggested that the intuitive availability of a conditional presupposition $A \to B$ rather than an unconditional one B depends on the presence of a plausible link between A and B, for instance on B being probabilistically or logically dependent on A (see, e.g., Van Rooij 2007; Lassiter 2012), several observations make it unlikely that such considerations can entirely solve the Proviso Problem. To begin with, as Geurts (1996) points out, there are sentences whose perceived presuppositions are conditional propositions without a plausible link between antecedent and consequent. Consider for instance:

(30) Gloria knows that if it is sunny, I have a sister

(30) is of course a very strange thing to say, precisely because we would typically expect that Gloria could not know such a fact unless she knows that the speaker has a sister, but the important point is that (30) intuitively presupposes only the conditional statement *If it is sunny, the speaker has a sister*, rather than the unconditional statement that the speaker has a sister. So in such a case, despite the implausibility of a common ground that would entail such a proposition without entailing that the speaker has a sister, the perceived presupposition is conditional. The approach just sketched, inspired by Beaver (1992, 1999, 2001), would lead us to expect that (30) should be perceived to presuppose that the speaker has a sister, unconditionally. Potential solutions come to mind. One could for instance think that (30) competes with (31):

(31) Gloria knows that I have a sister.

If (30) is used rather than (31) (which both has a stronger presupposition and stronger assertion), whether I have a sister or not, it must be that Gloria only knows that *if* it is sunny, I have a sister, and that she definitely does not know that I have a sister if I have one (otherwise (31) would have been used). So (30) must be interpreted as attributing to Gloria a strange belief state, in which she does not believe that I have a sister, but she believes that if it is sunny I have one. Given that the speaker is attributing such a weird belief state to Gloria, it becomes more plausible that the speaker is also considering a context in which what is common knowledge is only the material conditional *It is sunny* $\rightarrow$ *I have a sister*. This line of explanation, however, has not been developed, and is only tentative.

A second problem is that sometimes there is in fact a plausible link between the antecedent and the consequent of the predicted conditional presupposition, and yet a stronger, unconditional presupposition is perceived, even in a scenario that is incompatible with the unconditional presupposition. Mandelkern (2016a) offers the following example, among many others:

(32) Context: it is common ground that Smith has gone missing, and we don't know whether he is still alive. A detective enters and says:
If the butler's clothes contain traces of Smith's blood, then we'll soon have Smith's murderer behind bars.

The sentence in (32) is predicted to presuppose the material conditional *The butler's clothes contain traces of Smith's blood* $\rightarrow$ *Smith was murdered*. Mandelkern observes that in (32), the detective is felt to presuppose that Smith was murdered, unconditionally, despite the fact that a context that would entail the conditional presupposition without entailing that Smith was murdered is very plausible (since there is a natural link between finding Smith's blood on the butler's clothes and Smith having been murdered). Mandelkern provides a lot of similar examples, and overall mounts a very serious challenge to the idea that the Proviso Problem could be solved by appealing to a well-motivated theory of accommodation. It is important to note that the Proviso Problem is not specifically a problem for the trivalent approach to presupposition. It is in fact a problem for all theories that predict conditional presuppositions, which include, beyond trivalent approaches, Heim's (1983) extremely influential dynamic approach and Schlenker's Transparency Theory (Schlenker 2007, 2008), which I will briefly discuss in section **??**. Unlike the trivalent approach, van der Sandt's (1992) theory of presupposition projection, based on Discourse Representation Theory (see also Geurts 1999), predicts that the relevant sentences are ambiguous between a reading with the conditional presupposition and a reading with the unconditional presupposition. Such an approach is able to explain why an unconditional presupposition can be perceived even when the context makes the conditional presupposition plausible, but it nevertheless does not explain why, in Mandelkern's examples, the conditional presupposition does not seem to be perceived at all.

Faced with these difficulties, several authors have recently suggested a very different picture. It involves a shift from a view where presuppositions are conditions on the context to one where they are pieces of semantic content that are ‘backgrounded’, but do not necessarily have to be entailed by the context. Consider a sentence *if A, B*, where B presupposes p. Then if we are in a context that entails the material conditional $A \rightarrow p$, then the sentence does not have a presupposition in relation with B (it might have others, depending on what A is). Otherwise, by default, the sentence is perceived to presuppose p (rather than $A \rightarrow p$). This is the picture that is suggested in Winter (2021), with roots in Karttunen (1973, 1974), and Mandelkern (2016b). It is possible to keep our trivalent approach to presupposition if we view it as predicting the conditions under which presuppositions are *filtered out*, but not as determining the perceived presupposition when presuppositions are *not* filtered out. That is, we would distinguish two cases, for any sentence. In the first kind of cases, the context we are in ensures that the sentence receives a classical truth-value (is defined) throughout the worlds of the context; in such a case, no presupposition is perceived. In the second kind of case, the context includes worlds where the sentence is not defined; in such a case, we first identify the lexical items in the sentence that are responsible for the unsatisfied presupposition, and then all the presuppositions triggered by these items in isolation become presuppositions of the sentence. For $A \rightarrow B_p$, then, no presupposition would be perceived in a context that entails $A \rightarrow p$, but in a context where this is not so, the sentence would be felt to presuppose p.

Such an approach may face a challenge that is the reverse of the Proviso Problem: as we saw, sometimes the presupposition that is perceived out of the blue (in a context where it is not already satisfied) *is* conditional, and this would no longer be accounted for. It seems that any adequate theory has to allow for the possibility that we can accommodate either conditional or unconditional presuppositions, without there being, at this point, a fully predictive theory of how this choice is made.

1.10 Interactions between Presuppositions and Quantifiers

So far we have only discussed a propositional logic fragment that can model how propositional operators interact with propositional content. I would like to briefly discuss how to extend the Middle Kleene/Strong Kleene view to quantifiers. I will use first order logic as a means to represent natural language sentences, but I do so for convenience (to avoid introducing a language that includes Generalized Quantifiers, even though this would in fact be much more adequate to model natural languages.)

I will take here the most simple route (in line with Beaver and Krahmer 2001, see also George 2014), which consists in extending the Strong Kleene semantics to the universal and existential quantifiers by viewing them as generalized conjunction and disjunction. In the SK semantics, a conjunctive formula counts as true if all conjuncts are true, false is at least one conjunct is false, and undefined otherwise, i.e. if no conjunct is false and at least one conjunct is undefined (this follows from applying the SK truth-table for conjunction recursively to a conjunctive formula with an airbitrary number of conjuncts). And a disjunctive formula counts as true if at least one disjunct is true, false if all disjuncts are false, and undefined otherwise, i.e. if no disjunct is true and at least one is undefined. This gives rise to the following semantic clauses for the quantifiers:

$$[\![\forall xF]\!]^g = \begin{cases} 1 & \text{iff for all } d,\ [\![F]\!]^{g[x\mapsto d]} = 1 \\ 0 & \text{iff for some } d,\ [\![F]\!]^{g[x\mapsto d]} = 0 \\ \# & \text{otherwise, i.e., if for some } d,\ [\![F]\!]^{g[x\mapsto d]} = \# \text{ and there is no } d \text{ such that } [\![F]\!]^{g[x\mapsto d]} = 0 \end{cases}$$

$$[\![\exists xF]\!]^g = \begin{cases} 1 & \text{iff for some } d,\ [\![F]\!]^{g[x\mapsto d]} = 1 \\ 0 & \text{iff for all } d,\ [\![F]\!]^{g[x\mapsto d]} = 0 \\ \# & \text{otherwise, i.e., if for some } d,\ [\![F]\!]^{g[x\mapsto d]} = \# \text{ and there is no } d \text{ such that } [\![F]\!]^{g[x\mapsto d]} = 1 \end{cases}$$

For negation, disjunction, conjunction and implication let us adopt (for the sake of concreteness) the Middle Kleene connectives.[14] We keep the transplication operator ‘::’ as a propositional connective. This means that when a natural language predicate such as *stopped smoking* triggers a presupposition (namely *used to smoke* in this case), this gets represented as $P(x) :: Q(x)$, where P translated (for instance) *does not smoke* and Q translates *used to smoke*.

In practice, we are interested in the presuppositions of complex sentences involving quantifiers such as *every* and *some*, hence in the projection patterns predicted for sentences of the form $\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \wedge Q(x))$. The system we have just set up validates the following equivalences (ower-case letters now are used for variables, capital letters represent predicates on the left of the transplication operator, and Greek lower case letters are used for predicates that appear on the right of the transplication operator, as this makes it easier to distinguish presupposed from asserted material).

(33) a. $\forall x((A(x) : \alpha(x)) \to (B(x) : \beta(x))) \equiv$
$(\forall x(A(x) \to B(x))) :: \exists x(\alpha(x) \wedge A(x) \wedge \beta(x) \wedge \neg B(x)) \vee \forall x(\alpha(x) \wedge (A(x) \to \beta(x)))$

b. $\exists x((A(x) : \alpha(x)) \wedge (B(x) : \beta(x))) \equiv$
$(\exists x(A(x) \wedge B(x))) :: \exists x(\alpha(x) \wedge A(x) \wedge \beta(x) \wedge B(x)) \vee \forall x(\alpha(x) \wedge (A(x) \to \beta(x)))$

To understand what is going on, it will actually be simpler to look at a concrete and more simple case, such as the English sentence in (34a), whose 'translation' is given in (34b), with L standing for *linguist*, S for *being now sober*, and δ for *having been a drinker in the past*

(34) a. Every linguist stopped drinking.
b. $\forall x(L(x) \rightarrow (S(x) : \delta(x)))$

The predicted presupposition is the following (first in Predicate Logic, then 'translated' into English)

(35) a. $\exists x(L(x) \wedge \delta(x) \wedge \neg S(x)) \vee \forall x(L(x) \rightarrow \delta(x))$
b. Either there is a linguist who was a drinker and is not sober now, or every linguist was a drinker at some point.

The predicted presupposition is simply the disjunction of the falsity condition and the truth-condition. The sentence is false if one can find a counterexample to the universal claim, i.e. if there is an individual who did not stop drinking, i.e. who was a drinker and still is. The sentence is true if every linguist was a drinker and is now sober. Why then is the second disjunct in (35) just *every linguist was a drinker* rather than *every linguist was a drinker and is now sober*? This is simply because the two propositions in (36) (using directly English, as this is in this case much easier to see) are logically equivalent ((36b) repeats (35b)).

(36) a. Either there is a linguist who was a drinker and is not sober now, or every linguist was a drinker and is now sober
b. Either there is a linguist who was a drinker and is not sober now, or every linguist was a drinker at some point

That (36a) entails (36b) is obvious. In the other direction: suppose (36b) is true. If the first disjunct of (36b) is true, then (36a) is true. If the first disjunct of (36b) is false, there is no linguist who was a drinker and is not sober, and furthermore the second disjunct must be true. Since the second disjunct is true, every linguist was a drinker, but then every linguit is sober, since there is no linguist who was a drinker and is not sober, and therefore the second disjunct of (36a) is true, hence (36a) is true.

Now, other theories of presupposition projection (such as Heim 1983; Schlenker 2007, 2008), which we briefly discuss in section **??**, make a different prediction for this case. They predict a strictly stronger presupposition, namely *Every linguist was a drinker*.[15] The two theories are hard to tease apart empirically because when the sentence is asserted, it is predicted anyway in both cases to *entail* that every linguist was a drinker. It is somewhat easier to try to tease them apart in principle by using a existential sentence, such as (37):

(37) A linguist stopped drinking

Our theory predicts that (37) presupposes (38):

(38) Either a linguist was a drinker and is now sober or every linguist was a drinker and still is.

The sentence is predicted to entail (assuming that linguistis exist)l that at least one linguist was a drinker, and not that every linguist was a drinker, which is the presupposition predicted by some competing theories, in particular Heim (1983). For this case, our prediction seems intuitively better. On the other hand, when we turn **??** into a polar question, as in (39), we seem to perceive a universal presupposition (i.e. the proposition that every linguist was a drinker in the past), rather than the weaker disjunctive presuppositions in **??**.

(39) Did every linguist stop drinking?

To complicate the matter, intuitive judgments about such sentences do not straightforwardly translate into a clear win for one approach over the other, especially when they seem to lead to conflicting conclusions (as we have just seen). This is because, as we have discussed, on top of the ‘basic’ compositional system that accounts for presupposition projection, we need to posit additional mechanisms that interact with it, such as local accommodation and some kind of pragmatic strengthening mechanism to solve the proviso problem, and these mechanisms can be appealed to when a certain proposal does not appear to be empirically adequate. See Chemla (2009) for a detailed experimental investigation of presuppositions in quantified environments, and a discussion of how these results relate to various theories of presupposition projection.

1.11 Other approaches

This chapter is concerned with the trivalent approach to presupposition projection within the Strong Kleene tradition. There are at least two major alternative approaches, which we briefly present here: the *dynamic approach*, and the *Transparency approach*. The dynamic approach itself makes use of *partial functions*, but the way undefinedness is treated makes it closer to Weak Kleene than to Strong Kleene. The Transparency approach is fully bivalent.

1.11.1 Heim’s dynamic semantics proposal, and it’s relation to the Middle Kleene approach

A major alternative to the trivalent approach that we have just developed is based on *dynamic semantics*, especially the proposal made in Heim (1983). In this framework, instead of viewing the semantic value of a sentence as a function from worlds to truth-values, it is instead viewed as a *function from information states to information states*, which I will henceforth call an *update function* (Heim uses the expression *Context Change Potential*). For our purposes, we can view an information state as a set of worlds, i.e. a proposition. A sentence’s meaning is characterized by how someone’s information state changes as a result of the sentence’s acceptance. For an atomic sentence p that has no presupposition, the idea is quite simple: starting from the ‘classical’ semantic value of p, viewed as the set he set of worlds/valuations in which it is true, and notated $[\![p]\!]$, we define the function $+p$ as that function that maps every information state I to $I \cap [\![p]\!]$:

(40) For any information state I and any atomic sentence p, $I + p = I \cap [\![p]\!]$

So far there does not seem to be any substantial departure from the standard semantics for propositional logic. At this point, we could keep the standard semantics and express a general update rule of the form $I + S = I \cap [\![S]\!]$, where $[\![S]\!]$ is the set of valuations making S true classically. But if we want to compositionally define the update associated with a sentence directly in terms of the updates corresponding to its parts, we can for instance say the following for conjunction, which is essentially a way of restating the standard semantics in this new format (this rule for conjunction is *not* the one the dynamic approach will finally adopt):

(41) For any information state I and any two sentences A and B, $I + (A \wedge B) = (I + A) \cap (I + B)$

Note that, as long as our goal is simply to restate the classical semantics in terms of update rules, we could as well use the following rule, which says that to update a context with $(A \wedge B)$, we first update it with A, and then update the resulting information state with B:

(42) $I + (A \wedge B) = (I + A) + B.$

While this rule seems to create an asymmetry between the two conjuncts, this is in fact not the case in a system where at the end of the day update happens to correspond to intersecting I with the classical semantic value of a sentence (the set of valuations making the sentence true in the standard semantics). This is because intersection is commutative: intersecting I with $[\![A]\!]$ and then with $[\![B]\!]$ is the same as doing it in the opposite order. And both rules would themselves be equivalent to the one in **??**.

For negation, we can state:

(43) $I + \neg A = I - (I + A)$, where $X - Y$ denotes the set of elements that belong to X but not to Y.

The system I have just skteched (with a rule of interpretation for atoms, and these two update rules for conjunction and negation) is exactly equivalent to the standard semantics for the fragment of propositional logic based on $\{\neg, \wedge\}$ in the following sense. For any sentence S of this fragment, let us notate $[\![S]\!]$ the set of valuations that make S true under the standard semantics for propositional logic. Then for any information state I and any sentence S of the relevant fragment, $I + S = I \cap [\![S]\!]$.

Such an update semantics for propositional logic is uninteresting in that it is just a notational variant of the standard classical semantics. However, once we move to a language with presuppositions, things become more interesting. The key new ingredient is that updating with a sentence can *fail* if the input information state does not entail the presupposition of the sentence. Because of this, it is no longer the case that the choice between various classically equivalent ways of stating an update rule for conjunction, is immaterial. In particular, using the rule in **??**, rather than the one in **??**, can make a difference, once the possibility of presupposition failure is incorporated.

So we now assume that some atoms can carry a presuppositions, which in the dynamic setting means that they cannot serve as good update functions for certain input information states that fail to satisfy a certain property. For instance, we might say that an atomic sentence representing *The king of France is bald* cannot serve as a good update for an information state that does not entail that there is a king of France. Let B represent the sentence 'The king of France is bald'. In a certain information state I that does not entail the existence of a king of France, we will say that the update with B *crashes*, which we notate $I + B = \#$. We can then consider again the three potential rules for conjunction discussed above, which are all equivalent in a classical setting (i.e. a language with no presupposition, i.e. where the update rule for atoms is simply intersection), but make different predictions once we consider the possibility that updates might crash.

(44) Three potential update rules for conjunction

a. $I + (A \wedge B) = (I + A) \cap (I + B)$

b. $I + (A \wedge B) = (I + A) + B$

c. $I + (A \wedge B) = (I + B) + A$

Imagine that A and B presuppose, respectively, some propositions a and b, i.e. $I + A = \#$ if I does not entail a, and $I + B = \#$ if I does not entail b. Let us further assume that, when interpreting the statements in (44), if there is a something of the form $I + X$ on the right side such that $I + X = \#$, then the update being defined on the left side crashes. Then according to rule (44a), updating with $A \wedge B$ crashes if, in particular, $I + A$=# or $I + B = \#$, i.e. if I does not entail a or does not entail b. Hence the prediction of (44a) is that a conjunctive sentence inherits the presuppositions of each conjunct, which is similar to what the Weak Kleene approach would predict in a ‘static’, trivalent system. If we use (44b), however, we see that for the update not to crash it is required that a) I can be updated with A without crashing, and b) $(I + A)$ can be updated with B without crashing. So now what is predicted is no longer that I should entail a and b, but rather that I should entail a and $I + A$ should entail b. Let us assume for the time being that A is in fact presupposition-less (i.e. a is a tautology). Saying that $I + A$ entails b, given that A itself is a ‘classical’ sentence (i.e. one with no presupposition) is just the same, it turns out, as saying that I, once intersected with the proposition that A expresses in the classical semantics, entails b. This is in turn equivalent to the requirement that I entail the material conditional $A \rightarrow b$. By using the rule in (44b), then, we obtain the same result as with the Middle Kleene truth table for disjunction in the ‘static’ trivalent approach. It is also clear, of course, that, as with the Middle Kleene approach, $A \wedge B$ is predicted to inherit the presuppositions of A. Finally, using **??** would give rise to a reverse pattern whereby $A \wedge B$ would inherit the presuppositions of B, but the presuppositions of A could be filtered out by B. One can see that, of the three rules, the one that is empirically adequate given the points we made in section 1.3 is the one in **??** (which repeats **??**).

As to negation, the rule in **??** ensures that a sentence of the form $\neg S$ inherits the presuppositions of S, since $I + \neg S = \#$ if and only if $I + S = \#$.

To investigate the relationship between the dynamic approach and the Middle Kleene approach, it will be useful to state the dynamic approach for the very same extended propositional language that we used to present the static trivalent approach. We thus consider the very same language with the operator ‘::’, which will now be reinterpreted. To state the meaning of ‘::’, we need a crucial notion, that of a context entailing a sentence S, where S is now viewed as denoting an update function, rather than a proposition. We will say that an information state I entails S if updating I with S returns I itself - the intuition is that if I is not modified by S, this means that I already contains the information brought by S. In other terms, I entails S if $I + S = I$.

(45) Dynamic semantics for our extended propositional language – Preliminary notions

a. A valuation is a function from atoms to $\{0, 1\}$

b. For any atom p, we define $[\![p]\!]$ as the set of valuations that maps p to 1

c. A *standard* information state is a set of valuations. The *degenerate* information state is a distinguished entity which is not a set of valuations, and is notated #.[16]

d. If I and J are standard information states, $I - J$ is the set of valuations that are in I but not in J

e. $\# - I = I - \# = \#$

(46) Dynamic Semantics for our extended propositional language – Recursive Semantics.
Let I be a non-degenerate information state.

a. If p is an atom, $I + p = I \cap [\![p]\!]$

b. If A and B are sentences, then:

(i) $I + (A :: B) = \#$ if $I + B \neq I$; otherwise, $I + (A :: B) = I + A$

(ii) $I + \neg A = I - (I + A)$

(iii) $I + (A \wedge B) = (I + A) + B$

(iv) $I + (A \vee B) = I + \neg(\neg A \wedge \neg B)$

(v) $I + (A \rightarrow B) = I + \neg(A \wedge \neg B)$

c. For any sentence S, $\# + S = \#$.

This system is provably equivalent to the trivalent approach based on the Middle Kleene truth-tables, in the following sense:

(47) For any sentence S of our extended propositional language, let us notate $[\![S]\!]^{mk}$ the set of valuations that make S true according the MK truth tables (augmented with the transplication operator ‘::’). Then:

a. For any sentence S and any non-degenerate information state I, $I + S = \#$ just in case there is a valuation in I that makes S undefined according the MK truth tables, i.e. just in case Stalnaker’s bridge principle is not met for $[\![S]\!]^{mk}$ if I represents what is common knowledge in the context of utterance.

b. For any sentence S, any non-degenerate information state I, if $I + S \neq \#$, then $I + S = I \cap [\![S]\!]^{mk}$

While these two systems are equivalent, things change when we extend the semantics to quantifiers. The dynamic semantics approach is relatively unconstrained, and so in principle could make different choices regarding the lexical entries of quantifiers. Heim (1983) predicts a pattern of *universal projection* for all quantifiers (cf. footnote **??**), which, as discussed in the previous section, is not what the most natural way of extending the MK approach to quantifiers delivers.

For a more systematic discussion of what makes a dynamic system properly distinct from some other systems that could be said to be static, see Rothschild and Yalcin (2017). See also Rothschild (2011), which discusses how dynamic semantics lexical entries could be derived in a principled manner.

1.11.2 Schlenker's Transparency Theory

Another influential, and more recent account of presupposition has been proposed in Schlenker (2007, 2008), the *Transparency account*. The account is couched in purely bivalent and classical semantics. In this account, presupposed content is a part of content that comes with a specific pragmatic constraint: it should be *locally redundant*, in a sense that I will now explain. Let me first illustrate with a specific example:

(48) Mary lives in the countryside, and Mary's garden is beautiful.

The second clause presupposes that Mary has a garden. The Transparency Condition says, informally, that inserting the proposition that Mary has a garden in *in the position of the second conjunct* should provide no more information than inserting a tautological proposition given the context of utterance and what precedes this position. This is explained in (49).

(49) Transparency condition for (49): the two following sentences should be contextually equivalent, for any possible clause *S*:
   a. Mary lives in the countryside, and <u>Mary has a garden</u> and *S*.
   b. Mary lives in the countryside and *S*

In other words, for (48) to be felicitous in a given context *C*, *C* should entail '(49a) $\Leftrightarrow$ (49b)', for every possible *S*. Will a little bit of thinking, one can see that this condition is satisfied just in case *C* entails the material conditional 'Mary lives in the countryside $\rightarrow$ Mary has a garden'.

We now make the Transparency condition more explicit, in the context of the the extended propositional language introduced in section 1.5. The operator '::' is now interpreted as equivalent to conjunction, but it signals that the second conjunct is subject to a specific pragmatic condition: it should be redundant given what linearly precedes it, and the context of utterance.

(50) **Transparency Condition**
Let S be a sentence that contains an occurrence of a subclause of the form $(A :: B)$, which we notate $[X\ (A :: B)\ Y]$, where X and Y are the strings that precede and follow $(A :: B)$. For S to be felicitous in a context C, the following should be the case: for every sentence A' and every string Y' that could replace Y in S without causing ill-formedness, the sentences $[X\ (A' \wedge B)\ Y']$ and $[X\ A'\ Y']$ are equivalent in context C (have the same truth-value in every world of C).

This condition, coupled with the classical truth-table for propositional logic, gives rise to predictions which are exactly equivalent to those based on Middle Kleene and Heim's dynamic system. As Schlenker discusses, it is possible to modify the Transparency Condition in such a way that the linear order of constituents does not matter, so as to make exactly the same predictions as one based on Strong Kleene.

(51) **Symmetric Transparency Condition**.
Let S be a sentence that contains an occurrence of a subclause of the form $(A :: B)$, which we notate $[X\ (A :: B)\ Y]$, where X and Y are the strings that precede and follow $(A :: B)$. For S to be felicitous in a context C, the following should be the case: for every sentence A' the sentences $[X\ (A' \wedge B)\ Y]$ and $[X\ A'\ Y]$ are equivalent in context C (have the same truth-value in every world of C).

Schlenker shows how to extend the Transparency Condition to a language with generalized quantifiers, and with some minimal assumptions derives predictions which are exaclyl identical to Heim's system (and different from those based on Middle Kleene and Strong Kleene, cf. section **??**). Furthermore, in a related paper (Schlenker 2009), Schlenker establishes a link between the Transparency Theory and Dynamic Semantics. In Heim's system, as presented in the previous section, we can define the *local context* of a constituent (a clause in the case of the propositional fragment) as the information state that is being updated by this clause given the recursive definition of updates represented by a complex sentence. For instance, in the sentence $A \wedge B$, interpreted in a context C (viewed as an information state), the local context of B is $C + A$ (given the rule $I + (A \wedge B) = (I + A) + B$). Schlenker proposes to *define* the local context of a clause, in a given 'global' context C, as the strongest proposition (smallest set of worlds) p which is 'redundant' in the position of the clause in question:

(52) **Local Contexts for propositional logic** Let S be a sentence containing a certain occurrence of a clause A, which we notate $[X\ A\ Y]$, where

> X and Y are the strings that precede and follow A. The local context of A in a given context C (a set of worlds) is the smallest set of worlds p such that, for any clause A' and any string Y' that could replace Y in S, the sentences $[X\ (p \wedge A')\ Y]$ and $[X\ A'\ Y]$ are equivalent in C.

Schlenker shows that this definition predicts the local context of a clause to be exactly the one that Heim (1983) predicts; hence, in the case of quantified sentences, it makes predictions that are different from the trivalent approach to quantification discussed in section **??**.

1.12 Conclusion

This chapter has illustrated how trivalent logic can be used to model the *projection* of presuppositions in natural language, i.e. the way complex sentences inherit the presuppositions of their parts. It is important to bear in mind that accounting for projection patterns leaves a lot of key questions unaddressed, the first of which is certainly why presuppositions exist to begin with. The *triggering* problem, i.e. the problem of understanding why certain expressions or constructions give to rise to presuppositions, is in no way solved by a solution to the *projection* problem. But one might hope that a solution to the triggering problem would at the same time provide a theory of projection. It is to be expected that once we understand what it is that makes certain inferences presuppositional, we should also understand their projective behavior. Existing pragmatic accounts of presupposition triggering (see in particular Simons 2013; Abusch 2002; Abrusán 2011; Schlenker 2021b), however, remain either incomplete or speculative, and do not in practice give rise to a genuinely predictive theory of projection that would be itself purely based on pragmatic principles (but see Schlenker 2021b for an interesting proposal).

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Notes

Chapter 1

1. In fact, (2) appears to support the inference that the content of the third inference is false, i.e. that Gloria in fact knows that John lives in Paris. This is also surprising from the standpoint of classical logic, since in general the fact that S entails T does not let us expect that $\neg S$ entails $\neg T$.
2. Another reason why appositive clauses are not viewed as contributing presuppositions is that their detailed projection profile, i.e. the way they interact with compositional semantics, is typically different from that of standard presuppositions, which we discuss at length in the next sections.
3. Degen and Tonhauser (2022), however, challenges the idea that there is a clear-cut distinction between factive and non-factive attitude verbs.
4. Definite descriptions, when used anaphorically, do not come with a uniqueness presupposition, cf. Elbourne (2005) and Schwarz (2009), a.o.
5. See however Goldstein and Kirk-Giannini 2022 for an argument that such a view is logically incompatible with some quite minimal and hard-to-reject assumptions regarding how individuals update their beliefs about the world and each other
6. Compare (11) with *Is it true that Gloria stopped smoking?*, which typically suggests that Gloria used to smoke.
7. The consensus view is that natural language conditionals, be they indicative or subjunctive, are no equivalent to their material conditional counterparts in propositional logic. Given that we focus on a propositional fragment in this chapter, we are nevertheless forced to do as if natural language conditionals were material conditionals. Heim (1992) discusses how her own dynamic approach to presupposition projection, which we present in section **??**, can be extended to indicative conditionals analyzed as variably strict conditionals (Stalnaker 1981; Lewis 1973). As she discusses, the presupposition she predicts for indicative conditionals (which can itself be characterized as a material conditional statement) is the same as the one is generated for material conditionals in her initial system (Heim 1983) as well as in the trivalent approach we develop in this paper. From this point of view,

our simplistic treatment of conditionals as material conditionals is an acceptable idealization.

8. Blamey used the symbol '/' to represent the transplication operator, and our own version reverses the order of the two arguments; that is, we translate what Blamey would write as B/A as $A :: B$. Note that (as pointed out to me by Paul Egré, p.c.) $A :: B$ is in fact equivalent to the De Finetti conditional $B \rightarrow A$, which is undefined if B is false or undefined, and, when defined, true if A is true and false if A is false (De Finetti 1936).
9. Bochvar called this operator the 'external assertion' operator, and viewed it as corresponding to the natural language truth predicate. In his view, a sentence such as *S is true* is false, not meaningless, if S is itself meaningless, and he took the third truth-value to correspond to cases where a sentence is meaningless
10. The second conjunct does not include the pronoun *she*, because otherwise the sentences would feel slightly odd, so these sentences actually test presupposition projection in coordinated predicates rather than in coordinated sentences, but we ignore this complication here.
11. In Mandelkern et al.'s (2020) experiments, even sentences like **??** are felt by some participants to license to some degree the inference that Gloria used to smoke (the task involved graded judgments), though much less than sentences like **??**. This is surprising from the point of view of most current theories of presupposition projection. van der Sandt's (1992) theory of presupposition projection, based on Discourse Representation Theory (see also Geurts 1999), can however accomodate this fact, as it predicts that **??** is ambiguous between a reading where the presupposition is filtered out and one where it's not. Mandelkern et al. discusses how several aspects of their experimental task might explain this finding. It remains that the contrast that was found between cases like **??** and cases like **??** suggests that presupposition filtering occurs from left to right.
12. That is, in S', everything that follows S is bivalent: sentences used to replace subclauses are all bivalent sentences (i.e. without '::'), and all subclauses of the form $A :: B$ that follow o in S have been eliminated in S'.
13. As George (2008b) points out, Middle Kleene connectives are exactly equivalent to what results from the so-called *short circuit* evaluation of boolean operators in a number of programming languages: a Boolean formula is evaluated from left to right and its value is returned as soon as it can be determined. The connection between short circuit evaluation and presupposition projection has been made explicitly in Kracht (1994).
14. There is no real discrepancy between the treatment for quantifiers and the one for binary connectives, despite the fact that we use Strong Kleene for the quantifiers but Middle Kleene for the connectives. In actual conjunctive and disjunctive formulae, disjuncts and conjuncts need to be linearitzed, and the Middle Kleene semantics is sensitive to order; but even if we view the universal quantifier (resp. the existential quantifier) as corresponding to generalized conjunction (resp. disjunction), this generalized conjunction is of course not linearized, and so there is no meaningful way of creating any asymmetry between the 'conjuncts'. In other

words, a statement of the form $\forall x P(x)$ can be viewed (informally) as a kind of shortcut for a possibly infinite conjunctive statement $P(a_1) \wedge \ldots \wedge P(a_n) \wedge \ldots$, but there is no principled way to decide the order in which the conjuncts should appear in this 'translation', hence no principled way to create any asymmetry between the possible values of x when evaluate the sentence's truth-value.

15. In general, Heim's approach to quantification predicts *universal presuppositions* for all English sentences of the form *DET P Q*: if *Q* presupposes *R* (in the same sense as *stop smoking* presupposes *used to smoke*), then Heim's approach predicts *DET P Q* to presuppose that every entity that has property *P* has property *R*.
16. One should think of the degenerate information state as what results from updating a normal information state with a sentence whose presupposition is not entailed by it – it is not to be conflated with the empty set, which is rather the information state that results from updating an information state with a sentence that contradicts it.

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