

Modified Numerals

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Modified numerals are expressions such as *more than three*, *less/fewer than three*, *at least three*, *at most three*, *up to ten*, *between three and ten*, *approximately ten*, *about ten*, *exactly ten*, etc. At first sight, their semantic contribution seems pretty easy to describe. However, this impression is deceptive. Modified numerals raise very serious challenges for formal semantics and pragmatics, many of which have yet to be addressed in a fully satisfactory way. These challenges relate to two broad questions: first, what is the linguistically encoded meaning of modified numerals? Second, how should we divide the work between compositional semantics and pragmatics in order to account for all the inferences they give rise to? These are the two questions we will address in this chapter, focusing on a few striking puzzles.

The difficulties in answering the first question can be illustrated with one example among many others. It seems easy, at first sight, to propose semantic entries for *fewer than three* and *at most two* that capture their meaning contribution in simple sentences such as *Fewer than three students are here* and *At most three students are here*. In standard generalized quantifier theory (cf. Article 124, Quantifiers, Scope, and Pseudo-Scope, section 2), one would typically suggest the following entries:

- (1) a. $\llbracket \text{fewer than three} \rrbracket = \lambda P. \lambda Q. |P \cap Q| < 3$
- b. $\llbracket \text{at most two} \rrbracket = \lambda P. \lambda Q. |P \cap Q| \leq 2$

Whenever these modified numerals combine with count nouns, these entries for *fewer than three* and *at most two* are equivalent, because the conditions ‘ < 3 ’ and ‘ ≤ 2 ’ are equivalent when applied to natural numbers.¹

However, any account where *fewer than three* and *at most two* are assigned equivalent meanings encounters problems when more complex sentences are considered. There are cases where replacing *fewer than three* with *at most two* in a sentence changes its truth-conditions, as in (2).

- (2) a. Fewer than three students managed to lift the piano together.
- b. At most two students managed to lift the piano together.

The point is that (2a) and (2b) are not intuitively equivalent. In a situation where the piano was lifted twice in a row, once by two students and once by

five students, (2a) is intuitively true, but (2b) is not. (2a) seems to just mean that there is a group of fewer than three students who managed to lift the piano together, but (2b), whose meaning is more elusive, rather seems to mean that one cannot find a group of three students or more who managed to lift the piano together. This is one among many different observations that illustrate that the semantic contribution of modified numerals is more complex than it seems at first sight.

The second question can be subdivided into two: 1) Why is it that some modified numerals give rise to epistemic inferences in some syntactic environments but not in others? and 2) Why is it that intuitively synonymous modified numerals differ with respect to the strength of the epistemic inferences they license?

We can illustrate the relevance of these two questions by means of two pairs of examples:

- (3) Context: speaker is assumed to know her age.
 - a. I am allowed to vote: I'm more than 18-years old.
 - b. #I am allowed to vote: I'm at least 18-years old.
- (4) Context: speaker is assumed to know all of her children's age.
 - a. All my children are allowed to vote. They are all more than 18-years old.
 - b. All my children are allowed to vote. They are all at least 18-years old.

The contrast in (3) reflects the fact that (3b), but not (3a), suggests that the speaker does not know how old she is – an implausible state of affairs, and one that the specified context excludes, hence the oddness of (3b). So it seems that *at least 18*, in contrast with *more than 18*, gives rise to a so-called 'ignorance inference', a type of epistemic inference whose content is that the speaker is not knowledgeable about something.²

Now, in (4), neither sentence is odd, and, in particular, they do not suggest that the speaker does not know her children's age, so that they do not conflict with the specified context. That is, even (4b), which features *at least*, does not necessarily trigger a similar ignorance inference. A theory of modified numerals should be able to explain both why *more than* and *at least* differ with respect to ignorance inferences (cf. (3)), and why these ignorance inferences have the distribution they do (cf. the contrast between (3b) and (4b)).

In this paper, we will attempt to provide a survey of existing approaches to the semantics of pragmatics and modified numerals. We will not cover all the relevant empirical and theoretical landscape, but will focus on some salient puzzles and select some of the proposals that have been offered to deal with them. We start with a short typology of modified numerals (section 1). We then discuss the pragmatics of modified numerals and their interactions with modals (section 2). Finally, we discuss a specific puzzle that arises with monotone-decreasing and non-monotonic comparative modified numerals such as *fewer*

than four and between 3 and 6 (section 3).

1 A typology of modified numerals

Modified numerals have different morphological makeups. In English, one can distinguish at least four distinct types.

1. Modified numerals based on superlative morphosyntax: *at least three, at most three, ...*
2. Modified numerals based on comparative morphosyntax: *more than three, fewer than three, no more than three, ...*
3. Modified numerals based on locative, spatial prepositions: *up to ten, over one hundred, between three and seven, around ten, from n to m, ...*
4. Modified numerals in which a bare numeral is modified by a ‘slack-regulator’ adverbial:
exactly/precisely/approximately/about ten, ...

In a seminal paper (Nouwen 2010), Nouwen argued that, in terms of their semantic and pragmatic behavior, modified numerals can be divided into two types, which he calls *Class A* and *Class B* numerals.³ Class A modified numerals include those based on comparative morphology (henceforth ‘comparative modified numerals’) and some based on locative prepositions (*between n and m*), while Class B modified numerals include those based on superlative morphology (henceforth ‘superlative modified numerals’) as well as a few others, including some based on locative prepositions (*minimally/maximally n, up to n, from n to m, ...*), and the classification is not always straightforward. We will focus here on superlative and comparative numerals, which have clearly different behaviors in various dimensions, and provide the main motivation for the distinction between the two classes.⁴

One first observation is that superlative numerals, but not comparative numerals, tend to trigger strong ignorance inferences – as illustrated in (3). Thus Nouwen observed the following contrast:

- (5) a. A hexagon has fewer than 11 sides.
 b. ?A hexagon has at most 10 sides.

(5b) sounds weird, as it suggests either that the speaker does not know how many sides a hexagon has, or that the number of sides can vary from one hexagon to the next. (5a), on the other hand, is not odd in the same way. It expresses a true proposition, if an underinformative one. A completely parallel contrast is obtained if one uses positive instead of negative modified numerals:

- (6) a. A hexagon has more than three sides.
 b. ?A hexagon has at least four sides.

A second puzzling difference between comparative and superlative numerals that was noted by Nouwen pertains to their interpretation in modal environments. Thus consider:

- (7) a. You may invite fewer than 10 people.
- b. You may invite at most 9 people.

While (7a) can be understood as a very weak statement, which in itself does not entail that any specific number of guests is forbidden, (7b) is most naturally understood as forbidding the addressee to invite more than 9 guests - and is thus equivalent to *You must invite fewer than 10 people*⁵. This contrast is surprising, because in plain, simple environments, *fewer than 10* and *at most 9* appear to have the same truth-conditional content. For instance, the following two sentences are truth-conditionally equivalent:

- (8) a. I will invite fewer than 10 people.
- b. I will invite at most 9 people.

Furthermore, there is no such dramatic contrast in meaning when the downward-entailing modified numerals *at most 9* and *fewer than 10* are replaced with their upward-entailing counterparts, *at least 10*, *more than 9*:

- (9) a. You may invite more than 9 people.
- b. You may invite at least 10 people.

While these two sentences may not be felt to be exactly equivalent, they both seem to convey that the upper-bound on the number of guests you are allowed to invite, if there is one, is greater than 9.

2 The pragmatics of modified numerals: ignorance inferences, interactions with modals

In this section, we discuss two issues: the ignorance inference triggered by superlative numerals, and some aspects of their interaction with modals. We will also compare them to comparative numerals.

2.1 Ignorance inferences and lack thereof

Consider the following four sentences in (10) and (11):

- (10) a. John solved at least three problems.
- b. John solved at most three problems.
- (11) a. Every student solved at least three problems.
- b. Every student solved at most three problems.

Both sentences in (10) trigger the inference that the speaker does not know the exact number of problems that John solved. Suppose I have just graded John's

homework, and he turns out to have solved four problems, a fact that I have in mind. Then I would not use either sentence in (10) as a reply to, say, *How did John do at the exam?*. In contrast with this, the sentences in (11) could be used by someone who knows exactly how many problems each student solved, as an answer, to, say, *How did the students do?*. While the kind of judgments we are mentioning here relate such sentences to specific questions, it is in general quite hard to find a context where the sentences in (10) would not trigger an ignorance inference, while this is not similarly hard for the sentences in (11).

An explanation for this pattern has been offered in Büring (2008), and elaborated upon in Schwarz (2016a).⁶ On this account, the ignorance inferences triggered by the sentences in (10) are Gricean conversational implicatures based on the maxims of quality and quantity, as formalized in Sauerland (2004). The lack of obligatory ignorance inferences in the case of the sentences in (11) is shown to follow from the fact that, in some contexts, the Gricean quantity implicatures that are predicted are themselves compatible with the speaker being knowledgeable. Focusing on *at least*, let us assume that *at least three* evokes the alternatives *exactly three* and *at least four*.⁷ Furthermore, let us assign *at least three* a meaning that makes it equivalent to *three or more than three*. (10a) has the following alternatives:

- (12) Alternatives for (10a) (*John solved at least three problems*).
 - a. John solved exactly three problems.
 - b. John solved more than three problems.

Now, both (12a) and (12b) asymmetrically entail (10a). By the maxim of quantity, one concludes that the author of (10a) does not have the belief that either is true. That is, the following inferences are derived about the speaker's mental state:

- (13) Inferences about the beliefs of the speaker of (10a)
 - a. Speaker believes that John solved three or more than three problems. (by *Quality*)
 - b. Speaker does not have the belief that John solved exactly three. (by *Quantity*)
 - c. Speaker does not have the belief that John solved more than three. (by *Quantity*)

Importantly, none of these weak 'epistemic' implicature of the form *The speaker does not have the belief that S* (which Sauerland 2004 calls *primary implicatures*) can be turned into the stronger statement *The speaker believes that not-S*, on pain of contradiction. For instance, if the speaker believed that John did not solve exactly three, then, since she believes that John solved three or more, she would have to also believe (if consistent) that John solved four or more ... But this contradicts the inference in (13c). Symmetrically, the speaker cannot believe that John did not solve four or more, since if this were the case, given (13a), she would have to believe that John solved exactly three, in contradiction with

(13a). In Fox’s (2007) terms, the sentence (10a) has *symmetric alternatives*, i.e. alternatives that cannot be jointly negated consistently with the sentence itself.

In fact, the inferences in (13) are equivalent to the following, which corresponds to the intuitive understanding of the sentence, and includes the ignorance inferences we wanted to derive:

- (14) The speaker believes that John solved three or more than three problems, and does not know whether John solved just three or more than three problems.

An entirely parallel prediction can be made for (10b), on the assumption that *at most three* competes with *exactly three* and *at most two*, and is synonymous with *three or less than three*. The output of the kind of pragmatic reasoning just sketched is that the speaker does not know whether John solved three or fewer than three problems.⁸

Importantly, the very same pragmatic procedure produces a very different result in cases such as (11). Consider (11a). By assumption, it has two alternatives, given in (15).

- (15) Alternatives for (11a)
- a. Every student solved exactly three problems.
 - b. Every student solved more than three problems.

Both alternatives are logically stronger than the sentence being considered. Hence, we derive the following inferences about the speaker’s mental state:

- (16) Inferences about the beliefs of the author of (11a)
- a. Speaker believes that every student solved three or more than three problems. (by *Quality*)
 - b. Speaker does not have the belief that every student solved exactly three problems. (by *Quantity*)
 - c. Speaker does not have the belief that every student solved more than three problems. (by *Quantity*)

The key difference with the previous case is that these three inferences are by themselves fully compatible with the speaker knowing, for every student, how many problems that student solved. Consider for instance a situation where the speaker believes that half of the students solved exactly three problems, that one forth of them solved exactly four, and all the others more than four. It is easy to see that all three statements in (16) are true in such a situation. This is sufficient to explain the lack of an obligatory ignorance inference for (11a). But we can go even further. Suppose that we are in a context where it is clear that the speaker is knowledgeable - for instance because the speaker is the teacher who gave students problems and has just graded their homework. In such a situation, it is natural to go from the fact that the speaker does not believe a sentence S to the conclusion that she believes not-S (this move is what Sauerland 2004 calls the *epistemic step*, and he called the strong inferences that

result from it *secondary implicatures* – cf. Article 109. Quantity Implicature). In the case of (10a), we saw that such a move resulted into attributing to the speaker contradictory beliefs, and was therefore ruled out. In the case of (11a), making this move results in the following inferences:

- (17) Inferences about the beliefs of the author (11a) in a context where she is known to be knowledgeable.
- a. Speaker believes that every student solved three or more than three problems.
 - b. Speaker believes that not every student solved exactly three problems.
 - c. Speaker believes that not every student solved four or more problems.

To sum up, the prediction is that, in the specified context, the sentence is interpreted as conveying a) that every student solved three or more problems, b) that some solved just three (from the combination of (17a) and (17c)), and c) that some solved more than three (from the combination of (17a) and (17b)). Importantly, these three statements do *not* contradict the statements in (16). They are, in fact, true in the scenario described above, where the speaker believes that half of the students solved just three problems, and that the other half solved four or more. So not only do we not derive ignorance implicatures, we also derive stronger inferences when the context we are in allows it, and these predictions seem correct.⁹

As stressed by Schwarz (2016b), what makes this account particularly plausible is the fact that superlative numerals appear to function very similarly to disjunctions, for which a similar pragmatic account is standard (see, e.g., Sauerland 2004 and Article 109. Quantity Implicature):

- (18) a. John solved the first or the second problem.
 \leadsto Ignorance inference: Speaker does not know whether John solved the first or the second problem.
- b. Every student solved the first or the second problem.
 \leadsto Some students solved the first problem, others solved the second problem.

Now, note that the logic of this explanation carries over to other cases which have a similar ‘logical structure’, so to speak. In particular, given that necessity modals can be viewed as universal quantifiers over possible worlds, it is expected that a sentence such as (19) will not necessarily trigger an ignorance inference and will be able to yield secondary implicatures.

- (19) Mary is required to solve at least three problems.

Consider its alternatives in (20):

- (20) a. Mary is required to solve exactly three problems.
 b. Mary is required to solve at least four problems.

By the maxims of quantity and quality, we know that the author of (19) cannot believe either (20a) or (20b). This, however, does not automatically trigger an ignorance inference, because a speaker who believes that Mary is required to solve at least three problems, and believes that otherwise she can solve any number of problems she wants provided it is more than three, is in a situation to use (19) without violating either quantity and quality; in such a situation, the speaker believes (19) and does not believe either proposition in (20). Furthermore, because the two alternatives can be negated consistently with (19), if the speaker is taken to be knowledgeable, (19) licenses the secondary implicatures according to which Mary is not required to solve exactly three, and is likewise not required either to solve more than three, i.e. that she has a choice between solving just three or more than three. These ‘minimal requirement’ readings (under which the sentence is understood to state that solving three problems is Mary’s minimal requirement), much discussed in the literature, are thus accounted for in a very principled manner, and predicted to arise when the speaker is assumed to be *authoritative*, following Büring’s (2008) terminology.¹⁰

At this point, we still have two important issues to discuss: the contrast between superlative numerals and comparative numerals, and the somewhat puzzling behavior of some modified numerals under possibility modals.

2.2 Unsolved puzzle #1: why are comparative and superlative numerals different?

If the ignorance inferences triggered by *at least* are quantity implicatures, as their distribution suggests, then we would expect them to be modulated by context. Consider the following dialogue, which features a comparative numeral:

- (21) a. Are you allowed to vote?
 b. Yes, I’m more than 18 years old.

The answer in (21b), in this context, does not trigger an inference that the speaker is not knowledgeable about her age, and is not felt to be uncooperative. This is fully expected, as in such a context, it is irrelevant how old the speaker precisely is (I will call such contexts *irrelevance contexts*). In contrast with this, if the question were *How old are you?*, an answer such as *I’m more than 18 years old* would be perceived as quite strange, as it would suggest that the speaker either does not know her age, or is being deliberately uncooperative. That is, comparative numerals might trigger ignorance inferences in some contexts, but only when it is expected that the speaker should provide a precise value (if she has the relevant information).

Now, do we find a similar context-dependency with respect to ignorance inferences in the case of superlative numerals? Most of the theoretical literature assumes that a) ignorance inferences are stronger with superlative numerals than with comparative numerals, b) and are so across contexts. Experimental studies have overall confirmed that ignorance inferences are available with both comparative numerals and superlative numerals, and that they are stronger

with superlative numerals (cf. Cummins and Katsos 2010). However, Westera and Brasoveanu (2014) present results suggesting that the contrast between comparative and superlative numerals is neutralized in irrelevance contexts. It has to be noted, however, that the critical experimental condition in this study involved a superlative modified numeral that was used in a question and then repeated in an answer to the question, as in the following:

- (22) a. Did at least ten students take Experimental Pragmatics?
 b. At least ten students took Experimental Pragmatics.

It might be that the absence of an ignorance inference in such a case is tied to the fact the answer is echoic, as suggested by Ciardelli et al. (2018). Ciardelli et al. (2018) claim that, apart from such echoic uses, superlative modified numerals trigger ignorance inferences even in irrelevance contexts. Furthermore, in recent, so far unpublished work, Cremers et al. (2017) report further experimental results suggesting that the contrast between comparative and superlative numerals, though weakened, is preserved even in irrelevance contexts.

Given this, it is to be expected that the following dialogue, obtained from the one in (21) by replacing the comparative modified numeral with a superlative modified numeral, should trigger the inference that the speaker in (23b) does not know her age - and should in fact sound weird given the default assumption that people know how old they are.

- (23) a. Are you allowed to vote?
 b. Yes, I'm at least 18 years old.

The contrast in the strength of ignorance inferences (in simple sentences with no embedding) between comparative and superlative numerals is a puzzle which, to my knowledge, has not received a satisfactory solution. One possibility would consist in 'semanticizing' at least some of these inferences in the case of superlative numerals. This was the strategy initially pursued in Geurts and Nouwen (2007) and Nouwen (2010), which offered two distinct proposals that both incorporated the relevant ignorance inference in the literal truth-conditions of the relevant sentences. These accounts, however, face some other problems, some of which are discussed in section 2.5.

One may also consider a theory within the framework of the grammatical approach to scalar implicatures (cf. Chierchia et al. 2012 and Article 109. Quantity Implicature), in which even ignorance implicatures are derived 'within the grammar', as is argued on independent grounds by Meyer (2013). In Meyer's (2013) proposal, ignorance inferences arise when the exhaustivity operator, whose meaning is similar to that of *only*, scopes over a silent matrix epistemic operator. Meyer's proposal amounts to analyzing a sentence such as *Mary solved at least eight problems* as having an LF of the form *The speaker **only believes** that Mary solved [eight or more than eight]_F problems*, which is equivalent to: 'The speaker believes Mary solved at least eight, and doesn't believe she solved exactly eight and doesn't believe she solved more than eight'. Under such a perspective, one could speculate that superlative numerals are constrained to

occur under the scope of an exhaustivity operator, which, together with some other assumptions, would predict that they always give rise to an ignorance inference unless secondary implicatures are derived.¹¹ Why they should obey such a constraint, apparently crosslinguistically, would still remain to be explained.

Recently, Ciardelli et al. (2018); Cremers et al. (2017) offered an account within the Inquisitive Semantics framework where ignorance inferences arise through two distinct mechanisms in the case of superlative modified numerals, while only one of these mechanisms (namely, the one posited by Büring 2008 and Schwarz 2016a, which I discussed in the previous section) is operative in the case of comparative numerals – so that ignorance inferences are stronger with superlative modified numerals.

2.3 The alternatives of comparative modified numerals

In order to predict the distribution of ignorance inferences for superlative numerals, one needs to make very specific assumptions about their alternatives. But on what might appear to be the most natural assumption, namely, that numerals form a scale, we would expect that the alternatives for *at least two* would be all expressions of the form *at least n*, and those of *more than two* would be all the expressions of the form *more than three*. On this assumption, as initially noted in Krifka (1999), we would expect sentence such as *Mary has more than two children* and *Mary has at least three children* to trigger the implicature that Mary has exactly three children (by negating the stronger alternatives *John has more than three children* and *Mary has at least four children*, respectively). That this is not so is a puzzle to be explained, on the basis of a general theory of alternatives. In this section, I would like to discuss two lines of work that are related to this issue, in the case of comparative modified numerals, namely Fox and Hackl’s (2006) universal density of measurement hypothesis, and Cummins et al.’s (2012) work on the role of *granularity* in defining alternatives.

Fox and Hackl (2006) proposes that for comparative modified numerals (*more than n*), the alternatives consist of all expressions of the form *more than r*, where *r* can be any real number (rather than just an integer). The key point here is that real numbers are *dense*, i.e. given two distinct real numbers *x* and *y* such that $x < y$, there exists a third real number *z* such that $x < z < y$.¹² Now, for a sentence such as (24) below, negating all stronger alternatives would result in a contradiction. If John is more than 12 years old, then for some real number $x > 12$ he is *x* years old. But then take any number *y* between 12 and *x*. John is then more than *y* years old, and so one cannot negate all the alternatives of the form ‘John is more than *z*-years old’, with $z > 12$, without generating a contradiction.

(24) John is more than 12 years old.

Fox and Hackl propose to extend this account of the lack of ‘exact’ inferences with comparative numerals in unembedded contexts to cases that intuitively involve only integers, as in *John has more than two children*. This is the *universal*

density of measurement scale hypothesis. The idea is that even in such case, as far as implicature computation is concerned, the underlying measurement scale is treated as continuous. So we consider all the alternatives of the form *John has more than x children*, where x can be any real number. Because they cannot be all negated without creating a logical contradiction, no implicature arises, as in (24).

Under a universal modal, though, as in (25), no contradiction arises by negating all stronger alternatives. Specifically, (25) below is predicted to implicate that there is no ϵ such that you are required to be more than $12 + \epsilon$ problems – namely, provided you have passed your 12th birthday, you can access the theater – a welcome result.

- (25) You are required to be more than 12 years old in order to access this theater.

In a case such as (26) below, which intuitively involves discrete quantities, given the universal density of measurement scales, the reasoning proceeds in exactly the same way, and leads to the implicature that there is no ϵ such that we are required to talk to $2 + \epsilon$ witnesses.

- (26) We are required to talk to more than two witnesses.

Now, given the knowledge that if we talk to more than two witnesses, then we talk to three or more, this inference might appear to be a contextual contradiction: if I'm required to talk to more than two witnesses, I am required to talk to at least three. On Fox & Hackl's view, though, what is important is only the fact that, at the purely logical (where such non-purely logical information is ignored), no contradiction arises when stronger alternatives are negated. Given this, an implicature is licensed, and, once contextual assumptions are factored in, the resulting interpretation is that we are not required to talk to more than three witnesses.

Note that this account cannot be extended to superlative modified numerals. For superlative numerals, we can still adopt Büring's (2008) and Schwarz's (2016a) proposal. Consider then the counterpart of (26) where the comparative modified numeral is replaced with its superlative counterpart:

- (27) We are required to talk to at least three witnesses.

For (27), as we saw, Büring's (2008) and Schwarz's (2016a) proposals predict two implicatures: that we are not required to talk to more than three witnesses (due to the alternative based on *more than three*), but also that we are not required to talk to exactly three witnesses (due to the alternative based on *exactly three*). In the case of (26), this second implicature is not predicted to arise on Fox & Hackl's account. So a contrast is expected between (26) and (27), in that the enriched interpretation of (26), but not that of (27), should be interpreted as not excluding the possibility that we are required to talk to exactly three witnesses (i.e. are not allowed to talk to more than three). This subtle prediction is hard to test, given that such obligations typically come with

a minimal requirement, but no maximal requirement.

Another relevant approach to the alternatives of modified numerals consists in assuming that the alternatives of a numeral depend on so-called *granularity levels*. Cummins and Katsos (2010) provide experimental evidence that a sentence such as (28) triggers the implicature that Mary didn't invite 40 more people.

(28) Mary invited more than 30 people.

To account for this, they assume that numerals are associated with different granularity levels which determine the alternatives they are compared with. For instance, multiples of 10 compete with each other, but not with every numeral, so that the relevant alternative for (28) is *Mary invited more than 40 people*. As they note, the logic of this account is that non-round numerals should compete, in some contexts at least, with all other numerals, so that *Mary invited more than 12 people* should implicate *Mary invited exactly 13 people* – contrary to fact. In recent work, Enguehard (2018) observes that the problem is solved if one adopts both the idea of levels of granularity and Buring-types alternatives. On this view, (28) would compete with both sentences in (29):

- (29) a. Mary invited more than 40 people.
b. Mary invited exactly 40 people.

When we negate both, we get the inference that Mary invited less than 40 people. But with (30), assuming now that 12, a non-round number, is in the same granularity level as all other numbers, we cannot negate both alternatives without generating a contradiction, which explains the lack of scalar implicature for (30):

- (30) Mary invited more than 12 people.
Alternatives: Mary invited more than 13 people, Mary invited exactly 13 people.

Enguehard (2018) makes a new observation, related to a sentence such as (31), where the underlying quantity being measured is continuous rather than discrete:

(31) Mary walked more than 7 miles to come back home.

(31) can easily trigger the implicature that Mary walked less than 8 miles. 7 being a non-round number, (31) competes with *Mary walked more than 8 miles* and *Mary walked exactly 8 miles*. In this case, they can be both negated without creating any contradiction, thanks to the fact that there is 'space' between 7 and 8 (one can walk a non-integer number of miles).

2.4 Unsolved puzzle #2: superlative numerals in the scope of possibility modals

Consider now the following case:

- (32) Peter is allowed to invite at most 8 people.

This sentence is most naturally understood as stating that Peter is not allowed to invite more than 8 people, and does not necessarily trigger any ignorance implicature. In fact, the speaker might well be the person who decides what is allowed and what is not, and in such a case the sentence is understood as stating that Peter can invite any number of people below 9, and is not allowed to invite 9 or more people. As we shall see, this very natural interpretation is very hard to account for on the basis of the approaches we have discussed so far. And, importantly, all the known attempts to solve the problem either make wrong predictions for other cases, or do not properly generalize to similar cases.

On the basis of our assumptions so far, the sentence's literal meaning is that a state of the world where Peter invites fewer than 9 people is permissible. This is a very weak statement, which does not entail in any way that Peter is not allowed to invite 9 people. Let us look at what pragmatic inferences are predicted if (32) is taken to evoke the following alternatives (paraphrasing 'at most n' by 'n people or less', so that our reasoning is not polluted by the actual meaning of the 'at most' sentences).

- (33) a. Peter is allowed to invite exactly 8 people.
b. Peter is allowed to invite 7 people or less.

Both alternatives in (33) entail the literal reading of (32), so we get the following inferences:

- (34) a. The speaker believes that there is a permissible world where Peter invites 8 people or less.
b. The speaker does not have the belief that there is a permissible world where Peter invites exactly 8 people.
c. The speaker does not have the belief that there is a permissible world where Peter invites 7 people or less.

Importantly, (34b) and (34c) cannot be negated without contradicting (34a). In fact, taken together, these three inferences amount to:

- (35) The speaker believes that there is a permissible world where Peter invites 8 people or less, and is not sure whether there is a permissible world where Peter invites exactly 8 people, and is not sure either whether there is a permissible world where he invites fewer than 8 people.

Equivalently:

- (36) The speaker believes that Peter is allowed to invite 8 people or less, is not sure whether Peter is allowed to invite just 8 people, and is not sure

either whether Peter is allowed to invite fewer than 8 people.

So we only derive ignorance inferences. And the resulting reading still does not entail that Peter is not allowed to invite 9 people. Based on our assumptions so far and an analysis where *at most eight* takes narrow scope, we cannot account for the most natural reading of (32). This may motivate another approach where the relevant inference (Peter is not allowed to invite more than 8 people) is a *semantic entailment*. We discuss such an approach in the next section, and show its shortcomings. Then we will move to another approach in which the relevant reading is viewed as deriving from a so-called ‘free-choice’ inference, and we will see that it, too, encounters problem, before providing an assessment of the current status of the puzzle.

2.5 A semantic approach

Nouwen (2010) proposes that the relevant reading of (32) is one where *at most 8* takes wide scope, and combines with a lambda abstract which represents a property of numbers (degrees), as in (37):

(37) [At most 8]. λn .you are allowed to invite n people.

In Nouwen’s semantics, *At most 8*, when combining with a property of numbers, returns *true* if 8 is the maximum number that has this property. Importantly, Nouwen assumes that the numerical variable n above has an ‘exactly’ reading. Informally speaking λn .you are allowed to invite n people denotes the set of numbers n such that you are allowed to invite exactly n people. Then (37) states that the maximum in this set is 8. This captures the fact that the sentence entails that you are allowed to invite 8 people, and are not allowed to invite more than 8 people. Furthermore, Nouwen introduces the very plausible constraint that the property/set that *at most 8* combines with should be true of more than just one number – the *non-singleton constraint*. Given this constraint, the final reading is that you are allowed to invite 8 people, you are also allowed to invite fewer than 8 people, and you are not allowed to invite more than 8 people. This seems close to what we want to derive.¹³

While this proposal seems appealing, in this simple form it runs into a number of problems. To begin with, without further assumptions, it predicts that a simple sentence such as (10b), repeated below as (38), should always be infelicitous.

(38) John solved at most three problems.

Under the proposed analysis, (38) is parsed as in (39):

(39) [At most three]. λn . John solved n problems

In words: 3 is the maximum number n such that John solved exactly n problems. But obviously, there is only one number such that John solved exactly that number of problems. So the non-singleton constraint cannot be met. Even

ignoring this constraint, the prediction is that (38) should simply mean that John solved exactly three problems, contrary to the observed interpretation. Nouwen’s solution consists in assuming that the LF of sentences may contain a covert epistemic possibility modal, and that this operator is necessarily introduced when needed to satisfy the non-singleton constraint. Under this analysis, the relevant parse is now as in (40), where \Diamond is an epistemic operator standing for *it is possible given the speaker’s beliefs that*

(40) [At most three]. λn . \Diamond (John solved n problems)

In words: 3 the maximal number n such that it is possible that John solved exactly n problems. With the aid of the non-singleton constraint, the desired reading is derived: it is possible that John solved exactly three problems, it is not possible that John solved more than three, and there is at least some other number m such that it is possible that John solved m problems, i.e. the speaker is not sure about the exact number, though 3 is definitely a possibility.

Finally, to capture the ignorance inferences of a sentence such as (10a), repeated in (41), Nouwen offers a completely parallel analysis for *At least*, except that the notion of *maximum* is now replaced with that of *minimum*, giving rise to the parse in (42), where \Diamond is again introduced to satisfy the non-singleton constraint.

(41) John solved at least three problems.

(42) [At least three]. λn . \Diamond (John solved n problems)

In words: 3 is the smallest number n such that it is possible (according to the speaker) that John solved exactly n problems.

This analysis, which seems so far successful, runs however into very serious problems. First, it cannot capture the readings we discussed for the sentences in (11), repeated below in (43):

- (43) a. Every student solved at least three problems.
b. Every student solved at most three problems.

Under a parse where *at least three/at most three* scopes over the universal quantifier, we get a reading amounting to:

(44) 3 is the maximal/minimal number n such that every student solved exactly n problems.

This entails that every student solved the same number of problems, and violates the non-singleton constraint. So \Diamond needs to be introduced, but then ignorance inferences are necessarily derived – which, however, are not obligatory for such sentences, as we have seen. Under a parse where *at least three/at most three* is interpreted under the scope of the universal quantifier, we again have a violation of the non-singleton constraint under the scope of the universal quantifier, unless \Diamond is introduced in an embedded position, resulting in a reading amounting to:

- (45) For every student x , the minimal/maximal number n such that it is possible x solved n problems is 3, and it is possible (according to what the speaker believes), for each student, that she solved fewer than three.

These truth-conditions cannot be satisfied when the speaker is fully knowledgeable about every student – but we saw that there were situations with a fully knowledgeable speaker where the sentence could be felicitously used.

Another, critical problem for this approach is that it makes completely wrong predictions for cases where *at least* n is used in the scope of a possibility modal, as in (46):

- (46) John is allowed to borrow at least three books

Under the proposed analysis, (46) is parsed as in (47), resulting in the truth-conditions in (48):

- (47) [At least three][λn . John is allowed to borrow exactly n books].

- (48) 3 is the smallest number n such that John is allowed borrow n books

This is actually equivalent to saying that John is *required* to borrow at least three books (if he borrows some books), which is clearly an incorrect prediction. In fact, this prediction is parallel to the one made for (32), which is *correctly* predicted to mean that, if you invite people, you *have* to invite fewer than eight people. But the problem is that in the case of *at least*, this is a wrong prediction. In fact, the lack of parallelism between *at least* and *at most* when they interact with possibility modals is a problem for a number of different approaches.

2.6 Does free-choice solve the puzzle?

Another strategy, suggested for instance in Coppock and Brochhagen (2013) and pursued in Buccola and Haida (2018), is to interpret *at most* numerals in situ, and to capitalize on the fact that possibility modals trigger so-called free-choice inferences. Buccola and Haida’s (2018) proposal is fairly complex and cannot be presented here due to lack of space. I will present the general intuition behind the proposal, without keeping to the letter of their proposal. This will suffice to show the limitations of that sort of account (which are fully discussed and acknowledged by Buccola and Haida 2018).

Very informally, the first step of the proposal is rooted in the observation that, under a standard semantics for *at most* n , where it is synonymous (when applied to discrete objects) with *fewer than* $(n+1)$, the meaning of (32) can be represented as follows (where the symbol \Diamond now represents any possibility modal).

- (49) $\Diamond(\text{You invite exactly 8 people} \vee \text{you will invite fewer than 8 people}).$

As is well known, structures of this form generate a so called free-choice inference whereby they end up being interpreted as in (50):

$$(50) \quad \Diamond(\text{You invite exactly 8 people}) \wedge \Diamond(\text{you invite fewer than 8 people})$$

So one part of the strategy consists in adapting a theory of free-choice inferences such that, with adequate auxiliary assumptions, there is an LF for (32) equivalent to (50).¹⁴ This, however, is not sufficient, since (50) does not by itself entail in any way that you are not allowed to invite more than 8 people. The second ingredient of the strategy consists in assuming that this second entailment arises as a regular quantity implicature. For instance, one could assume that (32), under the relevant LF, competes with all sentences of the form (51a), for $n > 8$, under an LF that makes them equivalent to (51b):¹⁵

- (51) a. You are allowed to invite at most n people.
 b. $\Diamond(\text{You invite exactly } n \text{ people}) \wedge \Diamond(\text{you invite fewer than } n \text{ people})$

Now, because all the LFs of the form of (51b) entail (50) (with $n > 8$), (32), under the LF that makes it equivalent to (50), can be enriched with the negation of all its stronger alternatives (which are infinitely many). The resulting reading (which in Buccola and Haida 2018 is obtained by applying two exhaustivity operators to the sentence, but not quite relative to the alternatives assumed here) is then given by:

$$(52) \quad \begin{aligned} &\Diamond(\text{You invite exactly 8}) \wedge \Diamond(\text{you invite fewer than 8}) \\ &\wedge \neg(\Diamond(\text{You invite exactly 9}) \wedge \Diamond(\text{you invite fewer than 9})) \\ &\wedge \neg(\Diamond(\text{You invite exactly 10}) \wedge \Diamond(\text{you invite fewer than 10})) \\ &\wedge \dots \end{aligned}$$

This is in turn equivalent to:

$$(53) \quad \begin{aligned} &\Diamond(\text{You invite exactly 8 people}) \wedge \Diamond(\text{you invite fewer than 8 people}) \\ &\wedge \neg\Diamond(\text{you invite more than 8 people}) \end{aligned}$$

This is the desired reading.

While this seems a good result, this type of account suffers from the very problem we noticed regarding Nouwen’s account at the end of section 2.5. As noticed in Penka (2014) and discussed in Buccola and Haida (2018), a completely similar reading is predicted for sentences involving a numeral modified by *at least* in the scope of a possibility modal – assuming that parallel assumptions are made regarding the alternatives of *at least* and *at most*. That is, just like Nouwen (2010), this type of proposal predicts that a sentence such as (46) (‘John is allowed to borrow at least three books’) should have a reading that entails that if John borrows some books, he *has* to borrow at least three books – a reading which seems clearly unavailable.

Now, Buccola and Haida (2018) note very interesting facts suggesting that this prediction might be less bad than one could think. For instance, even though that kind of ‘minimal requirement’ reading does not seem available for (46), such a reading becomes available when the *at least*-numeral is itself part of a bigger disjunctive phrase, as in (54):

- (54) In order to pass, students can either do a presentation or solve at least 4 problems.

Surprisingly, (54) has a reading amounting to something like ‘In order to pass, students *must* do a presentation or solve at least 4 problems’. This reading is derivable within Buccola and Haida’s (2018) system. Very informally, (54) can be analyzed as *students can do a presentation, or solve 4 problems, or solve more than 4 problems*. This gives rise to the following free-choice inference: the three options are all possible ways of satisfying the requirements. And a second inference is generated: these three options are the *only* way to satisfy the requirements.

Another important remark made by Buccola and Haida is that even comparative numerals sometimes give rise to similar readings (where, very informally, a possibility modal can be paraphrased by a necessity modal!). Consider for instance (55), used as an answer to *What am I allowed to eat?*. According to Buccola and Haida, (55) suggests that you are not allowed to eat four cookies.

- (55) You are allowed to eat one apple or fewer than than three cookies.

Finally, they observe that *between*-numerals give rise to a similar reading as well (a combination of a free-choice inference and a prohibition regarding both a lower bound and an upper bound):

- (56) For this exam, you are allowed to write between 5 and 10 pages.

(56) feels approximately equivalent to ‘You *must* write between 5 and 10 pages’, and you have a choice as a to the exact number of pages.

In Buccola and Haida’s (2018) perspective, all modified numerals should be able to give rise to such readings, which involve a combination of a free-choice inference and a prohibition expressed as an upper-bound or a lower-bound (or both, as we have just seen). They also note that similar readings arise with other superlative expressions, as in (57):

- (57) You are allowed to leave at 8pm at the latest/earliest.

In both cases, one understands that you *must* leave no later/no earlier than 8pm - strikingly, the asymmetry noted for *at least* and *at most* is not observed with *at the earliest* and *at the latest*.

From this perspective, what is problematic and mysterious is the fact that a reading of this sort is not available with *at least* (and comparative numerals) in more simple structures, e.g. in (46).

2.7 Penka’s decompositional analysis

Penka 2014 takes the opposite view. Following most of the literature, on her view it is *at most*-numerals which are exceptional. She rejects on this basis any account that would generalize from *at most* to *at least*. Her solution involves a specific analysis of *at most*-numerals, in which they are decomposed into a

negative element *ANT* and a part parsed as *at least n*. Crucially, the negative element can take scope independently of *at least*, resulting into a so-called *split scope* readings. On this analysis, the relevant parse for (32) is as follows:

(58) $[ANT - 8][\lambda n. \text{Peter is allowed to invite at least } n \text{ people}]$

The proposed semantics for *ANT* - 8 makes the whole sentence equivalent to ‘for every $n > 8$, it is not the case that Peter is allowed to invite at least n people’. On this analysis, the inference that Peter is not allowed to invite more than eight people thus follows as an entailment, thanks to the split-scope analysis. Penka then shows that under some specific assumptions about how alternatives are computed, and a standard approach to quantity implicatures (of the sort illustrated above), the sentence is not predicted to trigger ignorance inferences, but rather to license the secondary implicature that Peter is allowed to invite 8 people (via the negation of the alternative where 8 is replaced with 7), resulting in the desired reading, according to which Peter is allowed to invite 8 people but is not allowed to invite more. Penka’s analysis explains the difference between *at most* and *at least* by assuming a compositional analysis for *at most* but not for *at least*, and by means of a number of auxiliary assumptions about how alternatives are computed. It is the only proposal on the market that is able to explain the asymmetry between *at most* and *at least*, but at the cost of several stipulations. It also does not straightforwardly generalize to many of the cases discussed in Buccola and Haida (2018).

To conclude this section, it turns out that since Geurts & Nouwen’s seminal paper, no proposal has been able to provide a fully satisfactory account of the interaction of superlative numerals and possibility modals - a failure which is quite extraordinary, as it reveals a surprising amount of complexity behind apparently simple constructions.

3 Comparative modified numerals, plural semantics and van Benthem’s problem

In this section, we focus on a puzzle that concerns specifically monotone-decreasing comparative numerals (e.g., *fewer than three*) and non-monotonic ones (e.g., *between n and m*), following the discussion in Buccola and Spector (2016) (but see also Winter 2001; Ben-Avi and Winter 2003, as well as Article 124 [Quantification, Scope, and Pseudo-Scope] for monotonicity in quantifiers in general). Consider the three following sentences:

- (59) a. Fewer than three students have blue eyes.
 b. Fewer than three students lifted the piano together.
 c. Fewer than five guests drank over twenty beers between them.

(59a) triggers the inference that the maximal number n such that it may be the case that n students have blue eyes is 2 (we will call this inference a *maximality*

inference), and does not rule out the possibility that no student has blue eyes. Unlike (59a), (59b) triggers an *existential inference* according to which there are students who lifted the piano together, and does not exclude the possibility that the relevant piano was lifted once by fewer than three students, and possibly another time by five students. Finally, (59c), under the intended cumulative interpretation (forced here by ‘between them’), also triggers an existential inference (the sentence is false if no guest drank any beer), and does not either trigger a maximality inference: (59c) does not exclude the possibility that five or more guests drank over twenty beers together, and it even strongly suggests that this might well be the case. Consider for instance a situation where four guests each had six beers, and a fifth guest had two beers. Then it is both true that fewer than five guests drank over twenty beers between them (the four first guests drank 24 beers together) and that five guests drank over twenty beers between them (since the five guests had 26 beers between them). The three sentences in (59) illustrate the following generalization:

- (60) A sentence of the form *fewer than n NP VP* means:
- a. If VP is distributive, that no plurality consisting of n NPs or more has the property denoted by VP.
 - b. If VP is collective or cumulative¹⁶, that there exists a plurality *X* consisting of fewer than n NPs such that *P* has the property denoted by VP.

(60a) predicts the absence of an existential inference and the presence of a maximality inference for (59a), and (60b) predicts the presence of an existential inference and the absence of a maximality inference for (59b) and (59c).

Now, there is a straightforward way of analyzing modified numerals which would correctly explain (60b). We start with the idea that *fewer than n* denotes a predicate of pluralities - when applied to a plurality *X*, it returns *true* if *X* has fewer than *n* atomic parts. When it combines with a noun, the resulting meaning is obtained by predicate modification (i.e. intersection). So *fewer than five boys* is a predicate which is true of any plurality *X* if *X* is the sum of fewer than five atomic individuals that are boys. Then we add a mechanism of existential closure to turn this predicate into an existential quantifier. For concreteness, let us assume that there is a silent existential determiner, noted \emptyset , which can apply to a predicate to turn it into a generalized quantifier. All this is cashed out in (61):

- (61) a. $\llbracket \text{fewer than three} \rrbracket = \lambda x. |x| < 3$ [where $|x|$ is the cardinality of the set of atomic members of *x*]
b. $\llbracket \emptyset \rrbracket = \lambda P. \lambda Q. \exists x (P(x) \wedge Q(x))$
c. $\llbracket \text{Fewer than three boys} \rrbracket = \lambda x. |x| < 3 \wedge \llbracket \text{boys} \rrbracket (x)$
d. $\llbracket [\emptyset [\text{Fewer than three boys}]] \rrbracket = \lambda Q. \exists x (\llbracket \text{Fewer than three boys} \rrbracket (x) \wedge Q(x))$
 $= \lambda Q. \exists x (|x| < 3 \wedge \llbracket \text{boys} \rrbracket (x) \wedge Q(x))$

Now, while this approach fares well when the modified numeral combines with a non-distributive predicate, it derives disastrous predictions when it combines with a distributive predicate. This issue was first raised in van Benthem (1986: 52f), and is now known as *van Benthem's problem*. For a sentence such as (59a), the resulting meaning is 'there is a plurality S consisting of fewer than three students such S has blue eyes'. Now, because the predicate *has blue eyes* is *distributive*, this comes out equivalent to: 'There is a plurality S consisting of fewer than three students such that every member of S has blue eyes'. The critical observation, here, is that this last statement is made true even if one hundred students have blue eyes: if this were the case, there would indeed be a plurality consisting of fewer than three students who have blue eyes, and so the predicted meaning does not entail that there aren't more than two students with blue eyes - clearly a terrible result. In fact, this reading is just equivalent to 'At least one student has blue eyes': in every situation where at least one student has blue eyes, there is a plurality of students (maybe reduced to a 'plurality' of just one student) containing fewer than three members such that every member of this plurality has blue eyes.

A completely parallel issue arises with non-monotonic modified numerals, such as *between four and six*. Thus consider:

- (62) a. Between four and six students have blue eyes.
 b. Between four and six students lifted the piano together.
 c. Between four and six students drank over twenty beers between them.

Both (62b) and (62c) can be analyzed as involving existential quantification over pluralities whose cardinalities are between four and six: *there exists a plurality S consisting of four to six students such that P lifted the piano/drank over twenty beers*. But a similar construal for (62a), namely *there exists a plurality S of four to six students such that every member of S has blue eyes* is true even if 100 students have blue eyes, and comes out equivalent to 'at least four students have blue eyes'.¹⁷ This is again van Benthem's problem - namely, existential quantification makes the upper-bound of a numerical quantifier irrelevant in distributive contexts.

Buccola and Spector (2016) (henceforth B&S) discuss in details this puzzle and several approaches to a solution. I will sketch here only one of the strategies discussed in this paper, which consists in using a somewhat non-standard semantics for modified numerals, based on a notion of *maximal informativity*.

3.1 Maximal informativity vs. standard maximality

The key idea consists in redefining the notion of maximality that underlies the semantics of modified numerals. Let us start with a quite standard approach to modified numerals (again, presented informally):

- (63) *[Fewer than five NPs] [VP]* is true if the *maximal* number n such that there is a plurality of n *NPs* in the extension of *VP* is smaller than five.

When the VP is distributive, the maximal number n such that n NPs VP is true is simply the number of atomic individuals which are both in the denotation of NP and VP. Suppose exactly 5 students have blue eyes. Then for any n in $[1, 5]$, there is a plurality consisting of n people who have blue eyes, and the maximal such n is therefore 5. So in this situation, the sentence *Fewer than five students have blue eyes* is false. And a sentence of the form *Fewer than n students have blue eyes* is true for any n greater than 5, false for any n equal to or smaller than 5, which is the desired result. Because we want such a sentence to be true if no student has blue eyes, we need to stipulate that the maximal number in question is 0 when the intersection of NP and VP is empty.

Let us now consider what happens when the VP is not distributive:

- (64) Fewer than five students lifted the piano together.

The predicted interpretation for (64) is simply that no group of five students or more lifted the piano together. This is not the reading we are after, as it has an upper-bound implication and no existential entailment.

B&S argues that by modifying the underlying notion of *maximality*, it is possible to obtain the same result as in the standard story for distributive predicates, without the undesirable consequences noted for a case such as (64). The key insight is to use a notion that we can term *logical maximality*, which I now introduce.¹⁸ Consider first all the propositions of the form *n students have blue eyes*. They are linearly ordered in terms of logical strength: in (65), each proposition on the left entails the proposition to its right.¹⁹

- (65) 5 students have blue eyes \Rightarrow 4 students have blue eyes \Rightarrow ... \Rightarrow 1 student has blue eyes.

Consider a world where exactly 4 students have blue eyes. In such a world, there is exactly one proposition in the above sequence which is true and such that any strictly stronger proposition (i.e. any proposition to its left) is false, namely the proposition expressed by *4 students have blue eyes*. We will say that in such a world, the number 4 is *logically maximal* relative to the property of numbers $\lambda n.n$ students have blue eyes. In general, a number m is logically maximal with respect to this property just in case it is the greatest number which has this property. We will say that (64) is true if there is a number m , smaller than 5, such that m is logically maximal with respect to the property $\lambda n.n$ students have blue eyes, which boils down to saying that there is a number smaller than 5 which is the maximal number m such that *m students have blue eyes* is true. Another way to express the same idea consists in saying this sentence is true just in case the following holds:

- (66) There is a number m , smaller than 5, such that *m students have blue eyes* is true, and no other true proposition of this form entails that *m students have blue eyes*.

So we now can replace our rule in (63) with the following one:

- (67) *[Fewer than five NPs] [VP]* is true in a world w if there is a number m , smaller than 5, such that m is logically maximal, in w , relative to the property $\lambda n.n$ *n NPs VPs*.

Equivalently:

- (68) *[Fewer than five NPs] [VP]* is true in a world w if there is a number m , smaller than 5, such that m *NPs VP* is true and no other true proposition of this form entails m *NPs VP*.

Now, the crucial observation is that with collective predicates, the linear ordering in terms of logical strength given in (65) disappears. That is, the following propositions are logically independent:

- (69) A plurality of 5 students lifted the piano (together)
 A plurality of 4 students lifted the piano (together)
 ...
 A plurality of 2 students lifted the piano (together)
 1 student lifted the piano.

Precisely because *lifted the piano together* is not a distributive predicate, there is simply no entailment relation from $(n+1)$ to n in the above sequence, nor the other way around. For this reason, if a proposition of the form *A plurality of m students lifted the piano* is true in a world w , it is necessarily the case that m is logically maximal in w relative to the property $\lambda n.n$ *students lifted the piano*, since there cannot be any proposition of the same form that entails it. In other words, if the piano was lifted twice, once by 3 students and once by 6 students, both 3 and 6 count as logically maximal. Let us now apply the rule in (68) to (64):

- (70) *Fewer than five students lifted the piano together* is true in w if:
 There is a number m smaller than 5, such that m *students lifted the piano together* is true and no other true proposition of the same form entails m *students lifted the piano together*.

The underlined part is always vacuously true: as we have just seen, there is simply no entailment relation between propositions of the form n *students lifted the piano*. So (70) reduces to (71):

- (71) *Fewer than five students lifted the piano together* is true in w if:
 There is a number m smaller than 5, such that m *students lifted the piano together* is true.

And this is basically the reading we are after, which does not imply any upper bound. It is, in particular, true in a situation where the piano was lifted twice, once by 3 students and once by 6 students.

At this point, we should notice that there is a non-trivial complication in the case where the VP in a structure of the form *Fewer than n NP VP* has an empty extension. We might consider that the number 0 has the property

$\lambda n.n$ *NP VP* just in case the intersection of the denotations of *NP* and *VP* is empty. This will make a sentence such as (59) true if no students have blue eyes. This is a good result. But we also predict that the sentence (64) is true if no group of students lifted the piano together. We would therefore fail to predict that (64) has an existential entailment. Alternatively, if we don't make such a move, we will correctly predict the existential entailment of (64) but we will also generate an unwanted existential entailment for (59a). One possible way out is to modify our ontology and to add to our domain of pluralities a *null individual* (which contains no atomic member) which would be in the extension of all distributive predicates but not in that of non-distributive predicates, and to adjust our semantics for quantifiers accordingly. See B&S, section 8.3, for a discussion of such an approach.

3.2 *At most* vs. *Fewer than* in collective contexts

While, at first sight, *At most 4* and *Fewer than 3* yield the same truth-conditions when they combine with distributive predicates, this is not clearly the case when they combine with non-distributive predicates. Thus consider:

- (72) a. Fewer than 3 students managed to lift the piano together.
 b. At most 4 students managed to lift the piano together.

(72a), as we have discussed at length, has an existential entailment and does not imply an upper bound. The reverse is true for (72b), which seems true if no students managed to lift the piano together, and false if a group of 5 students lifted the piano together. This reading would be the one that would result if *At most n* were governed by the rule in (63), which we rejected in the case of *fewer than n*. Such an approach is defended, for instance, in Kennedy (2015).²⁰

4 Conclusion

Despite its technical nature, the rich formal semantics literature on modified numerals has important implications for our understanding of the human language faculty, for at least two reasons. First, as I hope to have convinced the reader, developing an explicit theory of the semantics and the pragmatics of such expressions leads to extremely difficult problems, which are not yet fully resolved. This is a fascinating situation, suggesting that this domain can teach us something deep about how humans compute the meaning of natural language sentences. Second, semantic and pragmatic accounts of modified numerals appear to interact with quite fundamental issues in semantics and pragmatics, such as which concept of maximality is relevant to the semantics of natural language, scalar implicatures and related exhaustivity effects, and the division of labor between semantics and pragmatics. It is thus a fertile testing ground for theories that address these issues.

References

- Alexandropoulou, Stavroula, Jakub Dotlačil, and Rick Nouwen (2016). At least ignorance inferences come at a processing cost: Support from eye movements. In *Semantics and Linguistic Theory*, Volume 26, pp. 795–813.
- Beck, Sigrid and Hotze Rullmann (1999). A Flexible Approach to Exhaustivity in Questions. *Natural Language Semantics* 7, 249–298.
- Ben-Avi, Gilad and Yoad Winter (2003). Monotonicity and collective quantification. *Journal of Logic, Language and Information* 12(2), 127–151.
- van Benthem, Johan (1986). *Essays in Logical Semantics*. Dordrecht: D. Reidel.
- Breheny, Richard (2008). A new look at the semantics and pragmatics of numerically quantified noun phrases. *Journal of Semantics* 25(2), 93–139.
- Buccola, Brian and Andreas Haida (2018). A surface-scope analysis of authoritative readings of modified numerals. In Robert Truswell, Chris Cummins, Caroline Heycock, Brian Rabern, and Hannah Rohde (Ed.), *Proceedings of Sinn und Bedeutung* 21, pp. 233–248.
- Buccola, Brian and Benjamin Spector (2016). Modified numerals and maximality. *Linguistics and Philosophy* 39(3), 151–199.
- Büring, Daniel (2008). The least *at least* can do. *West Coast Conference on Formal Linguistics (WCCFL)* 26, 114–120.
- Chierchia, Gennaro, Danny Fox, and Benjamin Spector (2012). Scalar implicature as a grammatical phenomenon. In Claudia Maienborn, Paul Portner, and Klaus von Heusinger (Eds.), *Semantics: An International Handbook of Natural Language Meaning*, Volume 3, pp. 2297–2331. Berlin/Boston: de Gruyter.
- Ciardelli, Ivano, Elizabeth Coppock, and Floris Roelofsen (2018). Implicatures of modified numerals: quality or quantity. In Robert Truswell, Chris Cummins, Caroline Heycock, Brian Rabern, and Hannah Rohde (Ed.), *Proceedings of Sinn und Bedeutung* 21, pp. 283–300.
- Cohen, Ariel and Manfred Krifka (2014). Superlative quantifiers and meta-speech acts. *Linguistics and Philosophy* 37(1), 41.
- Coppock, Elizabeth and Thomas Brochhagen (2013). Raising and resolving issues with scalar modifiers. *Semantics and Pragmatics* 6(3), 1–57.
- Cremers, Alexandre, Elizabeth Coppock, Jakub Dotlacil, and Floris Roelofsen (2017). Modified numerals: Two routes to ignorance. Manuscript, ILLC, University of Amsterdam.
- Cummins, Chris and Napoleon Katsos (2010). Comparative and superlative quantifiers: Pragmatic effects of comparison type. *Journal of Semantics* 27(3), 271–305.

- Cummins, Chris, Uli Sauerland, and Stephanie Solt (2012). Granularity and scalar implicature in numerical expressions. *Linguistics and Philosophy* 35(2), 135–169.
- Dayal, Vaneeta (1996). *Locality in WH Quantification: Questions and Relative Clauses in Hindi*. Studies in Linguistics and Philosophy. Kluwer Academic Publishers.
- Enguehard, Emile (2018). Modified numerals revisited. In *Proceedings of Semantics and Linguistic Theory (SALT)* 28.
- von Fintel, Kai, Danny Fox and Sabine Iatridou (2014). Definiteness as maximal informativeness. In Luka Crnić and Uli Sauerland (Eds.), *The Art and Craft of Semantics: A Festschrift for Irene Heim*, Volume 1, pp. 165–174. MITWPL 70.
- Fox, Danny (2007). Too many alternatives: density, symmetry, and other predicaments. *Semantics and Linguistic Theory (SALT)* 17, 89–111.
- Fox, Danny and Martin Hackl (2006). The universal density of measurement. *Linguistics and Philosophy* 29(5), 537–586.
- Geurts, Bart (2006). Take ‘five’: the meaning and use of a number word. In Svetlana Vogeleer and Liliane Tasmowski (Eds.), *Non-definiteness and plurality*, pp. 311–329. Amsterdam/Philadelphia: Benjamins.
- Geurts, Bart and Rick Nouwen (2007). ‘At Least’ et al.: The Semantics of Scalar Modifiers. *Language* 83(3), 533–559.
- Kennedy, Chris (2015). A “de-Fregean” semantics (and neo-Gricean pragmatics) for modified and unmodified numerals. *Semantics and Pragmatics* 8(10), 1–44.
- Krifka, Manfred (1999). At least some determiners aren’t determiners. In Ken Turner (Ed.), *The Semantics/Pragmatics Interface from Different Points of View*, Volume 1, pp. 257–291. Elsevier.
- Marty, Paul, Emmanuel Chemla, and Benjamin Spector (2015). Phantom readings: the case of modified numerals. *Language, Cognition and Neuroscience* 30(4), 462–477.
- Mayr, Clemens (2013). Implicatures of modified numerals. In Caponigro, Ivano and Carlo Cecchetto (Ed.), *From grammar to meaning: The spontaneous logicity of language*, pp. 139–171. Cambridge University Press Cambridge.
- Meyer, Marie-Christine (2013). *Ignorance and grammar*. Ph. D. thesis, Massachusetts Institute of Technology.
- Nouwen, Rick (2010). Two kinds of modified numerals. *Semantics and Pragmatics* 3(3), 1–41.

- Penka, Doris (2014). ‘At most’ at last. In *Sinn und Bedeutung* 19, pp. 463–480.
- Sauerland, Uli (2004). Scalar implicatures in complex sentences. *Linguistics and Philosophy* 27(3), 367–391.
- Schlenker, Philippe (2012). Informativity-based maximality conditions. *Snippets* 26, 18–19.
- Schwarz, Bernhard (2016a). Consistency preservation in quantity implicature: The case of *at least*. *Semantics and Pragmatics* 9(1), 1–47.
- Schwarz, Bernhard (2016b). *At least* and ignorance: A reply to Coppock & Brochhagen 2013. *Semantics and Pragmatics* 9(10), 1–17.
- Schwarz, Bernhard, Brian Buccola, and Michael Hamilton (2012). Two types of class B numeral modifiers: A reply to Nouwen 2010. *Semantics and Pragmatics* 5(1), 1–25.
- Spector, Benjamin (2013). Bare numerals and scalar implicatures. *Language and Linguistics Compass* 7(5), 273–294.
- Spector, Benjamin (2014). Global positive polarity items and obligatory exhaustivity. *Semantics and Pragmatics* 7(11), 1–61.
- Westera, Matthijs and Adrian Brasoveanu (2014). Ignorance in context: The interaction of modified numerals and quds. In *Semantics and Linguistic Theory*, Volume 24, pp. 414–431.
- Winter, Yoad (2001). *Flexibility Principles in Boolean Semantics: The Interpretation of Coordination, Plurality, and Scope in Natural Language*. The MIT Press.

Notes

¹Within an ontology with plural individuals, the most straightforward treatment would be the following:

- (73) a. $\llbracket \text{fewer than three} \rrbracket = \lambda P. \lambda Q. \neg \exists X (|X| \geq 3 \wedge P(X) \wedge Q(X))$
 b. $\llbracket \text{at most two} \rrbracket = \lambda P. \lambda Q. \neg \exists X (|X| > 2 \wedge P(X) \wedge Q(X))$ where $|X|$
 denotes the number of atomic parts of X

The point we make here would apply as well if we used such entries.

²The claim that there is such a contrast has typically been made on the basis of informal introspective judgments. Section 2.2 discusses some relevant experimental results.

³Nouwen was building on observations made in Geurts and Nouwen 2007

⁴See Schwarz et al. (2012) for arguments that the Class A/Class B distinction needs to be further refined.

⁵This reading is presumably available also for (73a) as well, under a certain intonation but is certainly not the only reading, and probably isn't the most salient one, in contrast with (73b).

⁶Büring's paper was not concerned with sentences such as those in (11), but rather sentences like (19), but his account carries over to the examples in (11).

⁷This is what Büring (2008) assumes. See Schwarz (2016a) for discussion, and Mayr (2013) for a related but different proposal.

⁸Things become significantly more complex if we consider all the alternatives of the form *at least/at most n* and *exactly n*. In that case, one needs to adopt a more complex implicature computation procedure, but one that is well motivated on independent grounds. See Schwarz (2016a) for a detailed discussion of how different theories of quantity implicatures can deal with such cases.

⁹Of course, in some contexts, a sentence such as (11a) *can* trigger an ignorance inference, i.e. the hearer can *learn* from an utterance of (11a) that the speaker is partly ignorant about how many problems each student solved. A possible way to address this issue would be to assume that *at least three* in (11a) can take scope over *every*, which, together with some other assumptions about degree semantics, would ensure that the *exactly n* alternatives and the *at least n* alternatives would now be 'symmetric' in Fox's (2007) sense.

¹⁰Alexandropoulou et al. (2016) present eye-tracking data suggesting that the ignorance inferences triggered by *at least* come at a processing cost, which is

consistent with a pragmatic account of such inferences.

¹¹This might also partly explain why *at least*-numerals are positive polarity items (cf. Cohen and Krifka 2014; Spector 2014), given the assumption that exhaustivity operators are not licensed if they are vacuous and tend to be disfavored in a downward-entailing environment.

¹²The density property also holds for rational numbers, so the account would work as well if the alternatives was based on rational numbers rather than on all real numbers.

¹³On such an analysis, *at most three* covertly scopes out of the modal. There could be a variant of this analysis where *at most* alone would scope out. What is not clear is whether a structure in which the surface position *at most* is above the modal has the reading we are after:

(74) You are at most allowed to invite 8 people

(74) clearly has a reading without an ignorance inference ('I don't know exactly how many people you are allowed to invite, but it's at most eight). An informal survey suggests that the reading we are after ('8 the maximal permitted number') is accessible for such a sentence for some speakers but not others. If this reading is accessible, it might suggest that the best analysis of this reading involves an LF where *at most* scopes out without the numeral, but I don't know of any actual implementation of this idea. Penka (2014), which I briefly discuss below, is an interesting proposal where the relevant readings arise through scoping out *at most eight*.

¹⁴Importantly, (74) itself is not, on this account, the LF of (32), but is equivalent to it once free-choice is taken into consideration.

¹⁵This is actually *not* the way Buccola and Haida's (2018) proposal works, but again I care more here about the general logic of the account than about its technical details.

¹⁶Cf. Article 21, "Distributivity, Collectivity, and Cumulativity"

¹⁷Interestingly, there is experimental evidence that some speakers can access this pathological reading with non-monotonic quantifiers, cf. Marty et al. (2015).

¹⁸This idea has roots in von Stechow and Iatridou (2014) (see also Dayal 1996; Beck and Rullmann 1999; Fox and Hackl 2006; Schlenker 2012 for related discussions).

¹⁹Actual sentences of the form *n students have blue eyes* are preferably interpreted under an exact reading, in which case the entailments in (65) do not hold,

but this is immaterial to our discussion here: the actual meaning of bare numerals is not what matters, we just use such sentences just for ease of exposition, as shortcuts for *there is a plurality of n students with blue eyes*. Regarding the interpretation of bare numerals, see, a.o., Geurts 2006; Breheny 2008; Spector 2013; Kennedy 2015.

²⁰*At most* and *at least* are focus-sensitive adverbials that can associate not only with numerals but with all sorts of scalar items, quantifiers, and even referring expression (e.g., *At most Peter and Sue came*) – cf. Krifka (1999). I am not aware of any full-fledged semantics for *At most* that treats it as a focus-sensitive adverbial that can associate with expressions of different types and at the same time specifically accounts for the interpretation of *At most n* when it combines with a non-distributive predicate. In the case of (72b), it seems that the maximality component of *At most* must be defined in terms of the standard ordering of numbers. In other cases, however, it seems that what is relevant is logical strength. Coppock and Brochhagen (2013) proposes an interesting semantics for *At most* in the framework of inquisitive semantics, in which its meaning is defined relative to a ranking of alternatives in terms of *pragmatic strength*, a notion that includes logical strength as a special case, but may be related to other types of ordering as well. While such an approach might give us enough flexibility to use different types of ranking (logical strength vs. say, the natural ordering of natural numbers), it does not fully explain why different rankings are relevant depending on the linguistic environment in which *at most* occurs. Furthermore, Schwarz (2016b) shows that Coppock and Brochhagen’s (2013) proposal does not fully capture the ignorance inferences triggered by *at least*.