AAE 706: Bayesian Logic

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Today's Agenda

- Subjective probability
- Bayesian Inference – Canonical Application
- Climate catastrophe and Bayesian reasoning.
- Who owns the fish?
- Bayesian updating in Clue.

Language of Risk

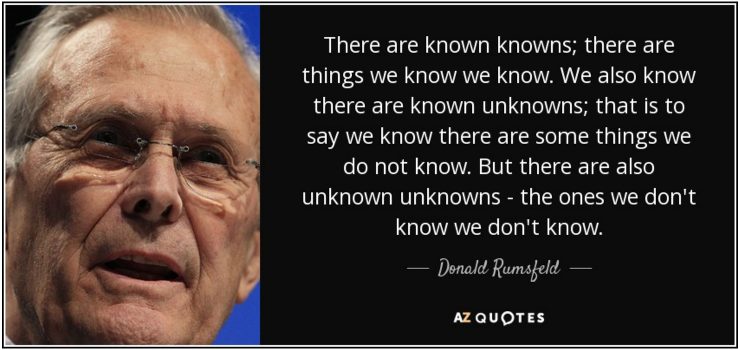
- risky event
- temporal dimension of risk
- chaotic systems
- bounded rationality
- information cost
- probability as relative frequency
- repeatable versus non-repeatable events
- subjective probability
- axioms underlying the existence of probability distributions

- classical probability estimates
- parametric distributions and maximimum likelihood estimation
- fractile method for measuring subjective probability
- prior information and sample information
- deviations from Bayes' rule: "conservatism" and "memory loss"

- learning process
- short- and long-term memory
- costly learning
- information through signals
- framing bias
- nomative applications of Bayesian analysis
- ambiguity

Risk versus Uncertainty

Definition. Risk refers to decision-making situations under which all potential outcomes and their likelihood of occurrences are known to the decision-maker, and uncertainty refers to situations under which either the outcomes and/or their probabilities of occurrences are unknown to the decision-maker.



Risky Events

DEFINITION: A *risky event* is any event that is not known for sure ahead of time. This means:

- uncertainty about the occurrence of events
- temporal dimension of risk

Causality and Risk

According to scientific belief, any event can be explained in a cause-effect framework. This suggests that "risk" does not exist. If so, why are there risky events?

1. because of a limited ability to control and/or measure precisely some of the causing factors (e.g., chaotic systems, flipping a coin, etc.)
2. because of a limited ability to process information (bounded rationality: playing chess, etc.)
3. because of information cost (the economics of information: obtaining and processing information may not be "worth it" if it is too costly)

Probability Theory

Probability theory was proposed by the scientific community as rationalization for the existence of risky events. In that context, a risky event is defined to be any event A for which $Pr(A) < 1$.

Ex: The outcome of flipping a coin is not a random event; it is the outcome of a deterministic process which behaves as if it were a random event.

Thus, a particular event may or may not be risky depending on (1) the quality of measurements; (2) the ability to control it; (3) the ability to obtain and process information; and (4) the cost of information.

As a result, there are a lot of disagreements about what a probability is supposed to be. In general, a probability can be interpreted as measure of anything we don't know. But knowledge can be subjective...

Probability and Relative Frequency

Intuition: A probability is a measure of the "relative frequency" of an event.

But what if

- the event is not repeatable?
- individuals disagree about the magnitude of a probability?

Interpretation of a Probability: A probability is a subjective and personal evaluation of the relative likelihood of an event reflecting the individual's own information and belief.

Elicitation of Probabilities

How to estimate a distribution function or a probability function (or its parameters)?

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How to estimate a distribution function or a probability function (or its parameters)?
In the case of repeatable events:
A probability distribution can be estimated from repeated observations of an event (= the sample information). This is the "classical" approach to statistical analysis.
Examples:

- flipping a coin
- distribution of price changes (if it is stable over time)

Note: This would be similar to a Bayesian approach under "uninformative prior". In this case, the sample information represents all the available information, and Bayesian and classical statistics are fairly similar.

Bayesian Inference: Canonical Application

Suppose, a particular test for whether someone has been using cannabis is 90% sensitive, meaning the true positive rate (TPR) $= 0.90$. Therefore, it leads to 90% true positive results (correct identification of drug use) for cannabis users.

The test is also 80% specific, meaning true negative rate (TNR) $= 0.80$. Therefore, the test correctly identifies 80% of non-use for non-users, but also generates 20% false positives, or false positive rate (FPR) $= 0.20$, for non-users.

Assuming 0.05 prevalence, meaning 5% of people use cannabis, what is the probability that a random person who tests positive is really a cannabis user?

Bayesian Logic

The Positive predictive value (PPV) of a test is the proportion of persons who are actually positive out of all those testing positive, and can be calculated from a sample as:

$$PPV = \text{T}\mathit{ruepositive}/\text{T}\mathit{estedpositive}$$

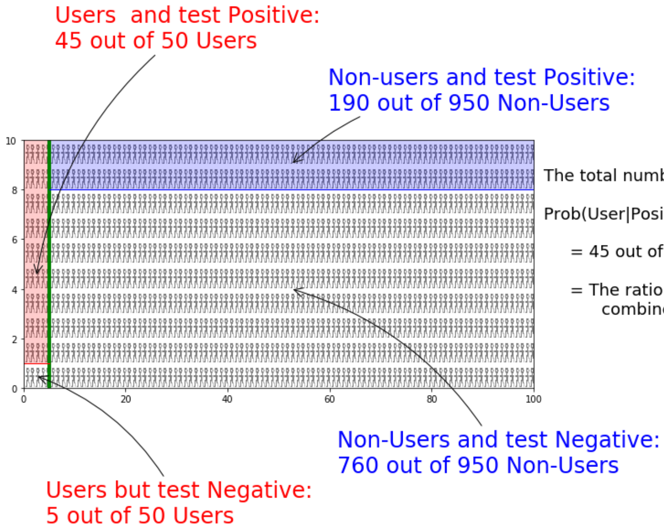
If sensitivity, specificity, and prevalence are known, PPV can be calculated using Bayes theorem. Let

$$P(\text{Positive}) = P(\text{Positive}|\text{User})P(\text{User}) + P(\text{Positive}|\text{Non-user})P(\text{Non-user})$$

is a direct application of the Law of Total Probability.

In this case, it says that the probability that someone tests positive is the probability that a user tests positive, times the probability of being a user, plus the probability that a non-user tests positive, times the probability of being a non-user. This is true because the classifications user and non-user form a partition of a set, namely the set of people who take the drug test. This combined with the definition of conditional probability results in the above statement.

Geometric Intuition



The total number of people tested is 1,000.

Prob(User|Positive) is 19%

= 45 out of 235 people testing Positive

= The ratio of the pink area to the combined pink and blue areas

BayesianLogic

In other words, even if someone tests positive, the probability that they are a cannabis user is only 19% – this is because in this group, only 5% of people are users, and most positives are false positives coming from the remaining 95%.

If 1,000 people were tested:

950 are non-users and 190 of them give false positive (0.20×950) 50 of them are users and 45 of them give true positive (0.90×50) The 1,000 people thus yields 235 positive tests, of which only 45 are genuine drug users, about 19%.

Climate sticker shock: Arctic

www.cnn.com/2013/07/24/world/climate-arctic-methane/index.html?hpt=hp_t3

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Climate sticker shock: Arctic thaw could cost $60 trillion

By **Matt Smith**, CNN

updated 3:31 PM EDT, Wed July 24, 2013

Key Ideas of Catastrophic Climate Change

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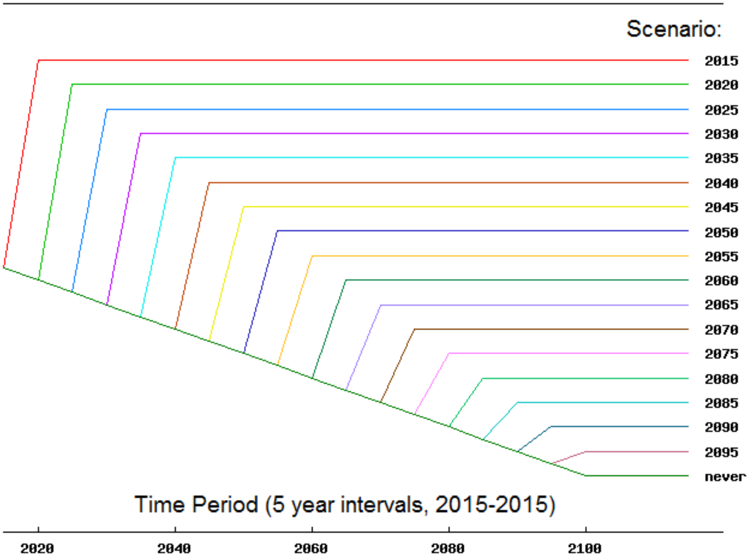
3. Unlike most decision-theory integrated assessment models, the introduction of uncertain catastrophic damages demands a *stochastic control* rather than a *stochastic programming* format.
4. The purpose of this model is to illustrate a minimalist framework for investigating the impact of uncertain catastrophic loss on near term mitigation.

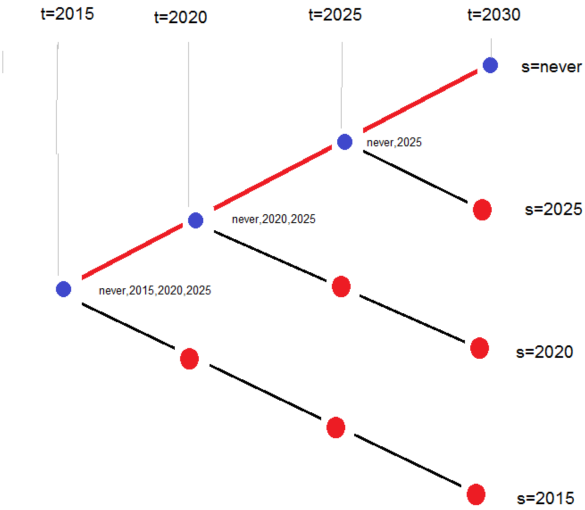
Model Features

1. DICE 2013 (one world, market and non-market damages, costly mitigation, Ramsey growth model, simple climate model driven by carbon emissions)
2. t_E time periods of economic activity (2015,2020,...,2115)
3. $t_C > t_E$ time periods of climate evolution and damages (2015,2020,...,2300)
4. s states of world (*scenarios*) each associated with the year in which catastrophic damage is realized or "*never*"

Scenario Structure



Scenario Indexing



Bayesian Learning

Following Lemoine and Traeger we adopt a simple uniform prior: the probability of a catastrophe *at a temperature* $T > T_{2015}$ is ex-ante assumed to be:

$$p_T = \frac{T - T_{2015}}{\bar{T} - T_{2015}}$$

Let h_t denote the *hazard rate*, the contitional probability of a disaster during period t assuming there has been no catastrophe to that point. If no catastrophe has yet occured in period t when the temperature is T_t, the Bayesian *hazard rate* of catastrophe during period t if the temperature in $t+1$ is T_{t+1} is:

$$h_t = \frac{T_{t+1} - T_t}{\bar{T} - T_t}$$

Probabilities

1. Let ξ_t denote the probability that no catastrophic loss has occured to the start of period t.
2. With a base year of 2015, $\xi_{2015} = 1$
3. Probability that catastrophic loss has not taken place at the start of period $t + 1$ is then:
$$\xi_{t+1} = (1 - h_t)\xi_t$$
4. Probability the scenario s is realized is then:
$$\pi_s = \begin{cases} h_t \xi_t & s = t \\ \\ \xi_{2115} & s = \text{never} \end{cases}$$
5. By definition, $\pi_s \geq 0$ and $\sum_s \pi_s = 1$.

Who owns the fish?

GIVEN

1. In a street there are five houses, painted five different colours.
2. In each house lives a person of different nationality
3. These five homeowners each drink a different kind of beverage, smoke different brand of cigar and keep a different pet.

THE QUESTION: WHO OWNS THE FISH?

Hints

1. The Brit lives in a red house.
2. The Swede keeps dogs as pets.
3. The Dane drinks tea.
4. The Green house is just to the left of the White house.
5. The owner of the Green house drinks coffee.
6. The person who smokes Pall Mall rears birds.
7. The owner of the Yellow house smokes Dunhill.
8. The man living in the centre house drinks milk.
9. The Norwegian lives in the first house.
10. The man who smokes Blends lives next to the one who keeps cats.
11. The man who keeps horses lives next to the man who smokes Dunhill.
12. The man who smokes Blue Master drinks beer.
13. The German smokes Prince.
14. The Norwegian lives next to the blue house.
15. The man who smokes Blends has a neighbour who drinks water.

Attribution

It has been asserted that Albert Einstein wrote this riddle early during the 19th century. He reportedly said that *98% of the world population would not be able to solve it.*

Erwin Kalvalagen pointed me to a web site which comments on Einstein being the author of this riddle (supposedly early on in his career):

> *Finally, you may want to notice that the young Albert Einstein (1879-1955) could not possibly have authored the puzzle in this form: The Pall Mall brand of cigarettes was introduced by Butler & Butler in 1899 (sold to American Tobacco in 1907 and Brown & Williamson in 1994) and Alfred Dunhill was established in 1893 (starting to manufacture pipes in 1907) when Einstein was still a young man. However, the Blue Master brand was introduced by J. L. Tiedemann in 1937, when Einstein was 58!*

GAMS Code: Sets

```
*       Define five sets defining the five characteristics of
*       each individual.  The way the program is designed, the symbols
*       used to define characteristics must be distinct (e.g., you
*       cannot define an element of the smokes set S named "Red").

set
        h       House /h1*h5/
        c       House colors /Red, Green, Yellow, Blue, White/,
        s       Smokes / Pall-Mall, Dunhill, Blends, Prince, Blue-Master/,
        b       beverages / Coffee, Milk, Beer, Water, Tea/,
        p       Pet     /Dogs, Birds, Cats, Horses, Fish/,
        n       Nationality /Brit, Swede, Dane, Norwegian, German/;


*       We need a second reference to the houses in order to express hint 4

alias (h,hh);
```

GAMS Code: Variables and Equation Declarations

```
variable        obj     Objective function (vacuous);

nonnegative
variable
        Z(h,c,s,b,p,n)  Choice variable is 1 if we have h-c-s-b-p-n living

equations  housing, colors, smokes, beverages, pets, nations,
           hint4, hint10, hint11, hint14, hint15,objdef;
```

Equation Definitions

```
$eolcom !

housing(h)..     sum((c,s,b,p,n), Z(h,c,s,b,p,n)) =e= 1; ! One person in eac
colors(c)..      sum((h,s,b,p,n), Z(h,c,s,b,p,n)) =e= 1; ! Each of five colo
smokes(s)..      sum((h,c,b,p,n), Z(h,c,s,b,p,n)) =e= 1; ! Each of five type
beverages(b)..   sum((h,c,s,p,n), Z(h,c,s,b,p,n)) =e= 1; ! Each of five beve
pets(p)..        sum((h,c,s,b,n), Z(h,c,s,b,p,n)) =e= 1; ! Each of five vari
nations(n)..     sum((h,c,s,b,p), Z(h,c,s,b,p,n)) =e= 1; ! Each of five nati
```

Equation Definitions

```
* 4. The Green house is on the left of the White house.

hint4(h)..               sum((s,b,p,n), Z(h,"Green",s,b,p,n)) =L=
                sum((s,b,p,n), Z(h+1,"White",s,b,p,n));

* 10. The man who smokes Blends lives next to the one who keeps Cats.

hint10(h)..     sum((c,b,p,n),Z(h,c,"Blends",b,p,n))
                        =L= sum((c,s,b,n),  Z(h+1,c,s,b,"Cats",n))
                          + sum((c,s,b,n),  Z(h-1,c,s,b,"Cats",n));

* 11. The mann who keeps horses lives next to the man who smokes Dunhill.

hint11(h)..     sum((c,s,b,n),Z(h,c,s,b,"Horses",n))
                 =L= sum((c,b,p,n),  Z(h+1,c,"Dunhill",b,p,n))
                   + sum((c,b,p,n),  Z(h-1,c,"Dunhill",b,p,n));
```

Equation Definitions (cont.)

```
*       14. The Norwegian lives next to the blue house.

hint14(h)..     sum((c,s,b,p),Z(h,c,s,b,p,"Norwegian"))
                 =L= sum((s,b,p,n),  Z(h-1,"Blue",s,b,p,n))
                   + sum((s,b,p,n),  Z(h+1,"Blue",s,b,p,n));

*       15. The man who smokes Blends has a neighbour who drinks Water.

hint15(h)..     sum((c,b,p,n),Z(h,c,"Blends",b,p,n))
                 =L= sum((c,s,p,n),  Z(h-1,c,s,"Water",p,n))
                   + sum((c,s,p,n),  Z(h+1,c,s,"Water",p,n));
```

Objective Function: Irrelevant

```
*       The objective function is not meaningful but required:

objdef..        obj =e= 0;

model einstein /all/;

*       Z=1 makes an assignment:

Z.UP(h,c,s,b,p,n) = 1;
```

The Exclusive OR Function in GAMS

We need to eliminate options based on those hints which link specific pairs of traits. If a hint indicates that two traits are linked, we can drop all permutations in which one or the other but not both characteristics are present. For this purpose the XOR function is useful.
Bruce McCarl's GAMS user's guide provides the following description of the XOR operator:

> *When one wishes to perform an action if and only if one of two or more conditionals apply one can join them with an xor operator. This involves using syntax like*

```
Z.FX(h,c,s,b,p,n)$(logical condition 1 xor logical condition 2) = 0;
```

Simple Bounds

```
*       1. The Brit lives in a red house.

Z.fx(h,c,s,b,p,n)$(SameAs(n,"Brit") xor SameAs(c,"Red")) = 0;

*       2. The Swede keeps Dogs as pets.

Z.fx(h,c,s,b,p,n)$(SameAs(n,"Swede") xor SameAs(p,"Dogs")) = 0;

*       3. The Dane drinks tea.

Z.fx(h,c,s,b,p,n)$(SameAs(n,"Dane") xor SameAs(b,"Tea")) = 0;
```

Simple Bounds (cont.)

```
*       5. The owner of the Green house drinks coffee.

Z.fx(h,c,s,b,p,n)$(SameAs(c,"Green") xor SameAs(b,"Coffee")) = 0;

*       6. The person who smokes Pall Mall rears Birds.

Z.fx(h,c,s,b,p,n)$(SameAs(s,"Pall-Mall") xor SameAs(p,"Birds")) = 0;

*       7. The owner of the Yellow house smokes Dunhill.

Z.fx(h,c,s,b,p,n)$(SameAs(c,"Yellow") xor SameAs(s,"Dunhill")) = 0;

*       8. The man living in the centre house drinks milk.

Z.fx(h,c,s,b,p,n)$(SameAs(b,"Milk") xor SameAs(h,"H3")) = 0;
```

Simple Bounds (cont.)

```
*       9. The Norwegian lives in the first house.

Z.fx(h,c,s,b,p,n)$(SameAs(n,"Norwegian") xor SameAs(h,"H1")) = 0;

*       12. The man who smokes Blue Master drinks Beer.

Z.fx(h,c,s,b,p,n)$(SameAs(b,"Beer") xor SameAs(s,"Blue-Master")) = 0;

*       13. The German smokes Prince.

Z.fx(h,c,s,b,p,n)$(SameAs(n,"German") xor SameAs(s,"Prince")) = 0;
```

For Fun, Count the Options

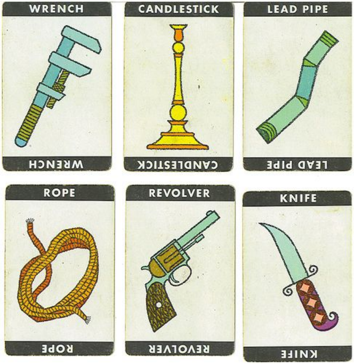
```
parameter npick Number of combinations remaining from which to choose 5;

npick = sum((h,c,s,b,p,n)$Z.up(h,c,s,b,p,n), 1);

*        Display npick with zero decimals:

option npick:0;

display npick;

----    238 PARAMETER npick = 133  Number of combinations ...
```

Solve the Model

```
*       Tell GAMS to omit variables from the model which are fixed.

einstein.holdfixed = yes;

solve einstein using lp minimizing obj;

*        Display with zero decimals as a database list:

option Z:0:0:1;
display Z.L;

----    247 VARIABLE Z.L  Choice variable is 1 if we have h-c-s-b-p-n

h1.Yellow.Dunhill    .Water .Cats  .Norwegian 1
h2.Blue  .Blends     .Tea   .Horses.Dane      1
h3.Red   .Pall-Mall  .Milk  .Birds .Brit      1
h4.Green .Prince     .Coffee.Fish  .German    1
h5.White .Blue-Master.Beer  .Dogs  .Swede     1
```

Details of Clue

- Nine rooms, six weapons and six suspects = 21 cards.
- One room, one weapon, and one suspect are place in the envelope.
- Each of six players is randomly dealt six cards.



Rules of the Game

The player who, by the process of deduction and good plain common sense, first identifies the 3 solution cards hidden in the little envelope, wins the game. This is accomplished by players moving into the rooms and making “suggestions” of what they believe is the room, the person and the weapon for the purpose of gaining information. This information may reveal which cards are in other players’ hands and which cards are missing and must, therefore, be hidden in the little envelope.

A Simplified Game

Forget the board. Instead, deal the cards and take turns making suggestions. After one player makes a suggestion, each of the remaining players must either pass or reveal (to the player making the suggestion) one of the suggested cards (room, weapon or suspect).

Notation:

Let:

c	Cards
p, q	Players
$\mathcal{E}$	Envelope
$\mathcal{W}$	Weapons
$\mathcal{R}$	Rooms
$\mathcal{S}$	Suspects
π_c	Prior probability that card c is the envelope
μ_{cp}	Prior probability that card c is held by player
$\tilde{\pi}, \tilde{\mu}$	Updated priors

Probabilities with Random Deal

Assuming that the cards are randomly dealt, what is your prior about the locations of the cards?

Probabilities with Random Deal

Assuming that the cards are randomly dealt, what is your prior about the locations of the cards?

Assuming that the cards are randomly shuffled the probability that a card is in the envelope depends on whether it is a suspect, weapon or room:

$$\pi_c = Pr(c \in \mathcal{E}) = \begin{cases} 1/6 & c \in \mathcal{W} \cup \mathcal{S} \\ 1/9 & c \in \mathcal{R} \end{cases}$$

Likewise, we can calculate the probability that card c is held by player p:

$$\mu_{cp} = Pr(c \in p) = (1 - \pi_c) \times \frac{1}{6}$$

Permutations

What is the size of the sample space defined in terms of elementary equi-probable events?

Permutations

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Number of equi-probable combinations of cards in the envelope is equal to $6 \times 6 \times 9$. If we enumerate all possible permutations of the remaining cards distributed among the 6 players, we have a space of elementary events which is very large:

$$N = 6 \times 6 \times 9 \times \binom{18}{3} \times \binom{15}{3} \times \binom{12}{3} \times \binom{9}{3} \times \binom{6}{3} = 21,794,572,800$$

Priors after Deal

After examining the cards you have been dealt, how do you update your prior? What do you regard as the most likely contents of the envelope?

Priors after Deal

After examining the cards you have been dealt, how do you update your prior? What do you regard as the most likely contents of the envelope?

Let $\mathcal{D}$ denote the cards dealt to me. A Bayesian update of my prior on card in the envelope is:

$$\tilde{\pi}_c = \begin{cases} \frac{1}{6-|\mathcal{D}\cap\mathcal{W}|} & c \in \mathcal{W} \\ \frac{1}{6-|\mathcal{D}\cap\mathcal{S}|} & c \in \mathcal{S} \\ \frac{1}{9-|\mathcal{D}\cap\mathcal{R}|} & c \in \mathcal{R} \end{cases}$$

And my prior on whether card c is held by another player p is:

$$\tilde{\mu}_{cp} = \begin{cases} \frac{1-\pi_c}{5} & c \notin \mathcal{D} \\ 0 & c \in \mathcal{D} \end{cases}$$

Update after your suggestion

You make a suggestion and then observe responses of the other players. How do you update your priors?

Update after your suggestion

You make a suggestion and then observe responses of the other players. How do you update your priors?

Initialize revised priors $\tilde{\pi}_c = \pi_c$ and $\tilde{\mu}_{cp} = \mu_{cp}$.

- If player p is *unable to refute* the suggestion *wrs* she reveals that

$$\tilde{\mu}_{wp} = \tilde{\mu}_{rp} = \tilde{\mu}_{sp} = 0$$

.

- If player p refutes with card c ($c \in \{w, r, s\}$, we can conclude that

$$\tilde{\mu}_{cp} = 1$$

$$\tilde{\mu}_{cq} = 0 \quad \forall q \neq p,$$

and

$$\tilde{\pi}_c = 0.$$

Update after another player's suggestion

A different player makes a suggestion and I observe responses. How do I update my priors?

Update after another player's suggestion

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- If player p is *unable to refute* the suggestion *wrs* she reveals that

$$\tilde{\mu}_{wp} = \tilde{\mu}_{rp} = \tilde{\mu}_{sp} = 0$$

  .

  This is no different that in the case I had made the suggestion.

- The difference when the suggestion is made by another player is that we observe a refutation, but we do not observe the card whch has been used to make the refutation. Here we need to apply a Bayesian update.

A Bayesian Logic Homework Problem

Three prisoners, A, B, and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the two who are going to be executed:

> *If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, secretly flip a coin to decide whether to name B or C.*

The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $1/3$ to $1/2$, as it is now between him and C. Prisoner A secretly tells C the news, who reasons that A's chance of being pardoned is unchanged at $1/3$, but he is pleased because his own chance has gone up to $2/3$. Which prisoner is correct?