AAE 706

Linear Programming Duality

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Feasibility Theorems

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness
- Feasibility theorems

The Canonical Linear Programming Model

Linear programming is now used everywhere. While the modeling framework is narrowly defined, the skills involved in LP formulation and analysis carry over to all other model classes.

LP Duality

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness
- Feasibility theorems

Making Beer

The local brewery produces two varieties of beer (lagers and ales) which are marketed in taverns and grocery around town. At the moment, they are planning production for fall. Each beer requires malt, hops and yeast. The lagers return $120 in profit per batch while ales earn only $90 per batch. Lagers are made with German hops, while ales are made with Wisconsin hops. There are currently sufficient German hops in stock for 1000 batches of lager and Wisconsin hops for 1500 batches of ale. Lager requires 4 kg of malt per batch while ale uses only 2 kg. Both beers require one kg of yeast per batch. There are 1,750 kg of yeast and 4800 kg of malt on hand.

What quantities of lager and ale should be produced from these supplies to maximize total profit assuming that all that are made can be sold?

Beer Production Data

Recipe for brewing beer

	kg per batch				
	malt	yeast	German Hops	Wisconsin Hops	Profit ($)
Lager	4	1	1	0	12
Ales	2	1	0	1	9

Current inventory (kg)

	malt	yeast	German Hops	Wisconsin Hops
in stock	4800	1750	1000	1500

Elements of the profit maximization model

1. Decision variables

   x : number of batches of lager produced
   y : number of batches of ales produced

2. Constraints

   $4x + 2y \leq 4800$ (malt budget)
   $x + y \leq 1750$ (yeast budget)
   $x \leq 1000$ (German hops budget)
   $y \leq 1500$ (Wisconsin hops budget)
   $0 \leq x$ (non-negative lager production)
   $0 \leq y$ (non-negative ale production)

3. Objective function

   $\max 12x + 9y$ (profit)

   in which max means *maximize*.

Brewery Profit Model (optimization format)

$$\max_{x,y} 120x + 90y$$

subject to:

$$4x + 2y \leq 4800$$

$$x + y \leq 1750$$

$$0 \leq x \leq 1000$$

$$0 \leq y \leq 1500$$

- Note that this is an instance of a *linear program* (LP), which is a type of optimization model.

Decision Variables and Parameters

$$\max_{x,y} 120x + 90y$$

subject to:

$$4x + 2y \leq 4800$$

$$x + y \leq 1750$$

$$0 \leq x \leq 1000$$

$$0 \leq y \leq 1500$$

- Decision variables are the unknowns (endogenous), and parameters are data (exogenous)

Brewery Profit Model – Generic

$$\max_{x,y} r_x x + r_y y$$

subject to:

$$a_{1x} x + a_{1y} y \leq b_1$$

$$a_{2x} x + a_{2y} y \leq b_2$$

$$\ell_x \leq x \leq u_x$$

$$\ell_y \leq y \leq u_y$$

- By changing the parameters, we create a different *instance* of the same model.
- It is good practice to separate parameters (data) from the algebraic structure of the model.

Estimating Bounds

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness
- Feasibility theorems

Brewery Profit Maximization – The Model

$$\max_{x,y} 12x + 9y$$

subject to:

$$4x + 2y \leq 4800, \quad x + y \leq 1750$$
$$0 \leq x \leq 1000, \quad 0 \leq y \leq 1500$$

Suppose the maximum profit is p^*. How can we *bound* p^*?

- Finding a *lower* bound is easy... pick any feasible point!
  - $\{x = 0, y = 0\}$ is feasible. So $p^* \geq 0$ (we can do better ...)
  - $\{x = 500, y = 1000\}$ is feasible. So $p^* \geq 15000$.
  - $\{x = 1000, y = 400\}$ is feasible. So $p^* \geq 15600$.
- *Each feasible point of the LP yields a lower bound for p.*

Estimating upper bounds

$$\max_{x,y} 12x + 9y$$

subject to:

$$\begin{aligned} 4x + 2y \leq 4800, \quad & x + y \leq 1750 \\ 0 \leq x \leq 1000, \quad & 0 \leq y \leq 1500 \end{aligned}$$

Suppose the maximum profit is p^*. How can we *bound* p^*?

- Finding an upper bound is harder than finding a lower bound. We need to use the constraints!
  - $12x + 9y \leq 12 \times 1000 + 9 \times 1500 = 25500$. So $p^* \leq 25500$.
  - $12x + 9y \leq 13x + 9y = 2(4x + 2y) + 5(x + y) \leq 2 \times 4800 + 5 \times 1750 = 18350$. So $p^* \leq 18350$.
- Combining the constraints in different ways yields different upper bounds on the optimal profit p^*.

Estimating upper bounds

$$\max_{x,y} 12x + 9y$$

subject to:

$$\begin{aligned} 4x + 2y &\le 4800, & x + y &\le 1750 \\ 0 \le x &\le 1000, & 0 \le y &\le 1500 \end{aligned}$$

Suppose the maximum profit is p^*. How can we *bound* p^*?

What is the **best** upper bound we can find by combining constraints in this manner?

Estimating upper bounds

$$\max_{x,y} 12x + 9y$$

subject to:

$$4x + 2y \le 4800, \quad x + y \le 1750$$
$$0 \le x \le 1000, \quad 0 \le y \le 1500$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$ be the multipliers. If we can choose them such that for any feasible x and y, we have:

$$12x + 9y \le \lambda_1(4x + 2y) + \lambda_2(x + y) + \lambda_3 x + \lambda_4 y$$

Then, using the constraints, we will have the following upper bound on the optimal profit:

$$12x + 9y \le 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Estimating upper bounds

$$\max_{x,y} 12x + 9y$$

subject to:

$$4x + 2y \leq 4800, \quad x + y \leq 1750$$
$$0 \leq x \leq 1000, \quad 0 \leq y \leq 1500$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ be the multipliers. If we can choose them such that for any feasible x and y, we have:

$$12x + 9y \leq \lambda_1(4x + 2y) + \lambda_2(x + y) + \lambda_3 x + \lambda_4 y$$

Rearranging, we get:

$$0 \leq (4\lambda_1 + \lambda_2 + \lambda_3 - 12)x + (2\lambda_1 + \lambda_2 + \lambda_4 - 9)y$$

We can ensure this always holds by choosing $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to make the bracketed terms nonnegative.

Estimating upper bounds

$$\max_{x,y} 12x + 9y$$

subject to:

$$\begin{aligned} 4x + 2y \le 4800, &\quad x + y \le 1750 \\ 0 \le x \le 1000, &\quad 0 \le y \le 1500 \end{aligned}$$

Reiterating: If we choose $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$ such that:

$$4\lambda_1 + \lambda_2 + \lambda_3 \ge 12$$

and

$$2\lambda_1 + \lambda_2 + \lambda_4 \ge 9$$

Then we have an *upper* bound on the optimal profit:

$$p^* \le 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Finding the best (smallest) upper bound to our original is . . . a *different* LP!

Rules for Forming the Dual

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness
- Feasibility theorems

The dual of brewery profit:

$$\max_{x,y} 12x + 9y$$

subject to:

$$\begin{aligned} 4x + 2y \leq 4800, &\quad x + y \leq 1750 \\ 0 \leq x \leq 1000, &\quad 0 \leq y \leq 1500 \end{aligned}$$

To find the best upper bound, solve the **dual** problem:

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

subject to:

$$\begin{aligned} 4\lambda_1 + \lambda_2 + \lambda_3 &\geq 12 \\ 2\lambda_1 + \lambda_2 + \lambda_4 &\geq 9 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0 \end{aligned}$$

The dual of brewery profit:

Primal Problem:

$$\max_{x,y} 12x + 9y$$

subject to:

$$\begin{aligned} 4x + 2y &\leq 4800, & x + y &\leq 1750 \\ 0 \leq x &\leq 1000, & 0 \leq y &\leq 1500 \end{aligned}$$

Solution is p^*

Dual Problem:

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

$$\text{subject to:} \quad \begin{aligned} 4\lambda_1 + \lambda_2 + \lambda_3 &\geq 12 \\ 2\lambda_1 + \lambda_2 + \lambda_4 &\geq 90 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0 \end{aligned}$$

Solution is d^*

- Primal is a maximization, dual is a minimization.
- There is a dual variable for each primal constraint.
- There is a dual constraint for each primal variable.
- (any feasible primal point) $\leq p^* \leq d^* \leq$ (any feasible dual point)

The dual of brewery profit:

Primal Problem:

$$\max_{x,y} \begin{bmatrix} 120 \\ 90 \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}$$

$$x, y \geq 0$$

Dual Problem:

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \geq \begin{bmatrix} 120 \\ 90 \end{bmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

Scalar GAMS Code: Primal

```
$TITLE  Brewery Profit Maximization

nonnegative
variables       X       Production of lager,
                Y       Production of ale;

variable        Z       Profit (maximand);

equations       malt, yeast, profit;

malt..          4 * X + 2 * Y =L= 4800;

yeast..         X + Y =L= 1750;

profit..        Z =e= 120 * X + 90 * Y;

*       Include hops constraints as upper bounds:

X.UP = 1000; Y.UP = 1750;

MODEL   PRIMAL /malt, yeast, profit/;

solve PRIMAL using LP maximizing Z;
```

Scalar GAMS Code: Dual

```
VARIABLE        W       Dual ojective;

NONNEGATIVE
VARIABLES       LAMDA1, LAMDA2, LAMDA3, LAMDA4;

EQUATIONS       objdual, dualX, dualY;

objdual..       W =E= 4800*LAMDA1 + 1750*LAMDA2 + 1000*LAMDA3 + 1500*LAMDA4;

dualX.. 4*LAMDA1 + 1*LAMDA2 + 1*LAMDA3 + 0*LAMDA4 =G= 120;

dualY.. 2*LAMDA1 + 1*LAMDA2 + 0*LAMDA3 + 1*LAMDA4 =g= 90;

model DUAL /objdual, dualX, dualY/;

solve DUAL using LP minimizing W;
```

Listing File

```
--------------------------------------------------------------------------------
                           LOWER          LEVEL          UPPER         MARGINAL
---- EQU malt              -INF         4800.0000      4800.0000        15.0000
---- EQU yeast             -INF         1750.0000      1750.0000        60.0000
---- EQU profit              .              .              .             1.0000

                           LOWER          LEVEL          UPPER         MARGINAL
---- VAR X                   .           650.0000      1000.0000           .
---- VAR Y                   .          1100.0000      1750.0000           .
---- VAR Z                 -INF        17700.0000        +INF              .
--------------------------------------------------------------------------------
                           LOWER          LEVEL          UPPER         MARGINAL
---- EQU objdual             .              .              .             1.0000
---- EQU dualX            120.0000       120.0000        +INF          650.0000
---- EQU dualY             90.0000        90.0000        +INF         1100.0000

                           LOWER          LEVEL          UPPER         MARGINAL
---- VAR W                 -INF       177000.0000        +INF              .
---- VAR LAMDA1              .            15.0000        +INF              .
---- VAR LAMDA2              .            60.0000        +INF              .
---- VAR LAMDA3              .              .            +INF          350.0000
---- VAR LAMDA4              .              .            +INF          400.0000
--------------------------------------------------------------------------------
```

Vector GAMS Code: Primal

```
$TITLE  Brewery Profit Maximization

set     j       Products /lager, ale/
        i       Ingredients / malt, yeast, dehops, wihops/;

parameter
     p(j)    Profit by product /lager 120, ale 90/
     s(i)    Supply by ingredient /malt 4800, yeast 1750, dehops 1000, wihops 1750/;

table           a(i,j)  Requirements
                        lager   ale
        malt            4       2
        yeast           1       1
        dehops          1       0
        wihops          0       1;

nonnegative variable   Y(j)    Production levels;

variable               Z       Profit (maximand);

equations       supply(i), objprimal;

supply(i)..     sum(j, a(i,j)*Y(j)) =L= s(i);

objprimal..     Z =e= sum(j, p(j)*Y(j));

MODEL   PRIMAL /supply, objprimal/;

solve PRIMAL using LP maximizing Z;
```

Vector GAMS Code: Dual

```
variable         W       Dual objective;

nonnegative
variables        PI(i)   Shadow price of ingredient i;

equations        profit(j)       Dual constraint,
                 objdual         Dual objectives;

objdual..        W =e= sum(i, PI(i)*s(i));

profit(j)..      sum(i, PI(i)*a(i,j)) =g= p(j);

model DUAL /objdual, profit/;
solve dual using LP minimizing W;
```

General LP Duality:

Primal Problem (P):

$$\max_{x} c^T x$$

$$\text{s.t.} Ax \leq b$$

$$x \geq 0$$

Dual Problem (D):

$$\min_{x} b^T \lambda$$

$$\text{s.t.} A^T \lambda \geq c$$

$$\lambda \geq 0$$

If x and y are feasible points of (P) and (D), respectively:

$$c^T x \leq p^* \leq d^* \leq b^T \lambda$$

Powerful fact: if p^* and d^* exist and are finite, then $p^* = d^*$. This property is known as strong duality.

General LP Duality:

Primal Problem (P):

$$\max_x c^T x$$

$$\text{s.t.} Ax \leq b$$

$$x \geq 0$$

1. optimal p^* is attained
2. unbounded: $p^* = +\infty$
3. infeasible: $p^* = -\infty$

Dual Problem (D):

$$\min_x b^T \lambda$$

$$\text{s.t.} A^T \lambda \geq c$$

$$\lambda \geq 0$$

1. optimal d^* is attained
2. unbounded: $d^* = -\infty$
3. infeasible: $d^* = +\infty$

Which combinations are possible?

General LP Duality:

Primal Problem (P):

$$\max_x c^T x$$
$$\text{s.t.} Ax \leq b$$
$$x \geq 0$$

Dual Problem (D):

$$\min_x b^T \lambda$$
$$\text{s.t.} A^T \lambda \geq c$$
$$\lambda \geq 0$$

There are **exactly four** possibilities:

1. (P) and (D) are both feasible and bounded, and $p^* = d^*$.
2. $p^* = +\infty$ (unbounded primal) and $d^* = +\infty$ (infeasible dual)
3. $p^* = -\infty$ (infeasible primal) and $d^* = -\infty$ (unbounded dual)
4. $p^* = -\infty$ (infeasible primal) and $d^* = +\infty$ (infeasible dual)

More properties of the dual

To find the dual of an LP that is not in standard form:

1. convert the LP to standard form
2. write the dual
3. make simplifications

Example What is the dual of the dual? . . . The primal!

$$\begin{aligned} \min_{\lambda} \quad & b^T\lambda \\ \text{s.t.} \quad & A^T\lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} -\max_{\lambda} \quad & (-b)^T\lambda \\ \text{s.t.} \quad & (-A^T)\lambda \leq (-c) \\ & \lambda \geq 0 \end{aligned}$$

dual

$$\begin{aligned} -\min_{z} \quad & (-c)^Tz \\ \text{s.t.} \quad & (-A)z \geq (-b) \\ & z \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} \max_{z} \quad & c^Tz \\ \text{s.t.} \quad & Az \leq b \\ & z \geq 0 \end{aligned}$$

Different Duals

Standard Form

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_\lambda \quad & b^\mathsf{T} \lambda \\ \text{s.t.} \quad & \lambda \ge 0 \\ & A^\mathsf{T} \lambda \ge c \end{aligned}$$

Free Form

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_\lambda \quad & b^\mathsf{T} \lambda \\ \text{s.t.} \quad & \lambda \ge 0 \\ & A^\mathsf{T} \lambda = c \end{aligned}$$

Mixed Constraints

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & Fx = g \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda,\mu} \quad & b^\mathsf{T} \lambda + g^\mathsf{T} \mu \\ \text{s.t.} \quad & \lambda \ge 0 \\ & \mu \text{ free} \\ & A^\mathsf{T} \lambda + F^\mathsf{T} \mu = c \end{aligned}$$

Equivalences between Primal and Dual

Minimization	Maximization
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

Sensitivity Analysis

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness
- Feasibility theorems

Sensitivity: The General Case

Primal Problem (P):

$$\max_x c^T x$$
$$\text{s.t.} Ax \leq b + e$$
$$x \geq 0$$

Dual Problem (D):

$$\min_x (b + e)^T \lambda$$
$$\text{s.t.} A^T \lambda \geq c$$
$$\lambda \geq 0$$

Suppose we add a small e to the constraint vector b.

- The optimal x^* (and therefore p^*) may change, since we are changing the feasible set of (P). Call new values $\hat{x}^*$ and $\hat{p}^*$.
- As long as e is sufficiently small, the optimal λ will not change, since the feasible set (D) is the same.
- Before $p^* = b^T \lambda^*$. After $\hat{p}^* = (b + e)^T \lambda^*$
- Therefore $(\hat{p}^* - p^*) = e^T \lambda^*$. I.e., $\nabla_b (p^*) = \lambda^*$.

Another Example

A factory makes n products from m resources. Each unit of product j requires a_{ij} units of resource i and makes a profit of c_j dollars. Each day, the factory has b_j units of resource i available.

Mathematical Model

$$\text{maximize} \sum_{j=1}^{n} c_j x_j$$

$$\begin{aligned} \text{subject to} \quad \textstyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad & (i = 1, 2, \ldots, m) \\ x_j \geq 0 \quad & (j = 1, 2, \ldots, n) \end{aligned}$$

Suppose the resource constraints are not hard; i.e., we can either buy or sell each resources at certain fixed price.

Question: What is the *fair price* for each resource?

Fair price?

A fair price is one for which there is no advantage to either buy or sell small amount the resource.

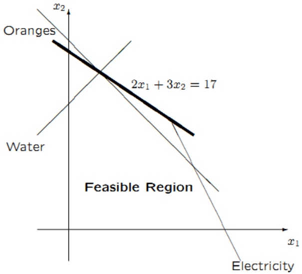
Fair prices are also called *shadow prices*.

```
$TITLE  Linear Programming Duality -- An Illustration

nonnegative
variables        X1       Fresh juice,
                 X2       Frozen juice;

variable         R        Profit (revenue);

equations        revenue, electricity, oranges, water;

revenue..        R =E= 2*X1 + 3*X2;

electricity..    2*X1 + X2 =L= 10;

oranges..        X1 + X2 =L= 6;

water..          -X1 + X2 =L= 4;

model OJ /all/;
```

Geometric Intuition

$$\text{maximize } 2x_1 + 3x_2$$

$$\begin{aligned}
\text{subject to} \quad 2x_1 + x_2 &\le 10 \quad \text{(Electricity)} \\
x_1 + x_2 &\le 6 \quad \text{(Oranges)} \\
-x_1 + x_2 &\le 4 \quad \text{(Water)} \\
x_1, x_2 &\ge 0
\end{aligned}$$



Question: What is a fair price for water?

Answer: $\frac{1}{2}$

GAMS Listing

```
Objective :          17.000000

                    LOWER     LEVEL     UPPER    MARGINAL
---- EQU revenue      .         .         .        1.0000
---- EQU electrici~ -INF       7.0000   10.0000     .
---- EQU oranges    -INF       6.0000    6.0000    2.5000
---- EQU water      -INF       4.0000    4.0000    0.5000

                   LOWER     LEVEL     UPPER    MARGINAL
---- VAR X1          .         1.0000   +INF       .
---- VAR X2          .         5.0000   +INF       .
---- VAR R         -INF       17.0000   +INF       .
```

The Explicit Dual

```
nonnegative
variables         PELEC   Price of electricity,
                  PORNG   Price of oranges,
                  PWATR   Price of water;

equations         profitjuice, profitconcentrate, resourcerent;

profitjuice..             2*PELEC + PORNG - PWATR =G= 2;

profitconcentrate..       PELEC + PORNG + PWATR =G= 3;

resourcerent..            R =e= 10*PELEC + 6*PORNG + 4*PWATR;

model OJdual /profitjuice, profitconcentrate, resourcerent/;

solve OJdual using LP minimizing R;
```

Solution Listing – Dual

```
Optimal solution found.
Objective :          17.000000

                   LOWER     LEVEL     UPPER     MARGINAL
---- EQU profitjui~  2.0000    2.0000    +INF        1.0000
---- EQU profitcon~  3.0000    3.0000    +INF        5.0000
---- EQU resourcer~   .         .         .          1.0000

                  LOWER     LEVEL     UPPER     MARGINAL
---- VAR R        -INF      17.0000    +INF         .
---- VAR PELEC      .         .        +INF       3.0000
---- VAR PORNG      .        2.5000    +INF         .
---- VAR PWATR      .        0.5000    +INF         .
```

Dual Variables: Fair Prices

- The dual variables associated with a linear program problem can be interpreted as “fair prices”.
- When confronted with the possibility of buying or selling resources at the optimal dual prices, there is no incentive to buy and there is also no incentive to sell.

Calculating the Fair Price

Let:

$$x_j = \text{the } \textit{optimal} \text{ amount of product } j \text{ produced.}$$
$$y_i = \text{the fair price of resource } i.$$

1. $y_i < 0 \implies$ we would buy unlimited amounts of resource i (assuming "free disposal") **Contradiction.** Hence,

$$y_i \geq 0$$

.

2. One unit of product j costs $\sum_{i=1}^{m} a_{ij} y_i$. Hence, $\sum_{i=1}^{m} a_{ij} y_i < c_j \implies$ we would buy unlimited amounts of these resources to make and sell product j. **Contradiction**. Hence,

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

Complementary Slackness

- Brewery optimization: a canonical LP model
- LP duality: estimating bounds
- LP duality: the rules
- Sensitivity and shadow prices
- Complementary slackness

Additional Properties of a “Fair” Price

3. $\sum_j a_{ij} x_j < b_i$ and $y_i > 0 \implies$. We would sell the surplus resource i and increase profit. **Contradiction.** Hence,

$$\text{if } \sum_j a_{ij} x_j < b_i \quad \text{then} \quad \mathrm{y_i} = 0.$$

4. $x_j > 0$ and $\sum_i a_{ij} y_i > c_j \implies$ stop making product j and sell the resources to increase profits. **Contradiction.** Hence,

$$\text{if } \sum_i a_{ij} y_i > c_j \quad \text{then} \quad \mathrm{x_j} = 0.$$

Complementary slackness

- Given two numbers a and b, if $a \times b = 0$, then one of these numbers must equal zero. Either $a = 0$ and/or $b = 0$
- Given two *nonnegative* n-vectors, $\mathbf{a}$ and $\mathbf{b}$. That is, $a_i \geq 0$ and $b_i \geq 0$ for all components i, then

$$\mathbf{a}^T\mathbf{b} = 0 \implies a_i b_i = 0 \quad \forall i$$

- Complementary slackness can be expressed using the "perp" symbol, $\perp$. That is:

$$\mathbf{a} \geq 0, \mathbf{b} \geq 0, \mathbf{a} \perp \mathbf{b} \quad \implies \quad a_i = 0 \text{ and/or } \mathrm{b_i} = 0 \quad \forall \mathrm{i}$$

Complementary Slackness – LP Optimality Conditions

- Primal feasiblity (constraints on x)

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\ x_j & \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

- Dual feasiblity (constraints on y)

$$\begin{array}{rcll} \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, 2, \ldots, n) \\ y_j & \geq & 0 & (i = 1, 2, \ldots, m) \end{array}$$

- Complementary slackness (relating x and y)

$$\begin{array}{rcll} y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) & = & 0 & (i = 1, 2, \ldots, m) \\ x_j \left(c_j - \sum_{i=1}^{m} a_{ij} y_i \right) & = & 0 & (j = 1, 2, \ldots, n) \end{array}$$

Complementary Slackness (CS) Conditions

The following relates the optimal primal and dual solutions, x^* and y^*:

$$x_j^* = 0 \quad \text{or} \quad \sum_i a_{ij} y_i^* = c_j \quad \text{(or both)} \quad \forall j$$

A more useful statement of this condition is:

$$x_j^* > 0 \quad \Rightarrow \quad \sum_i a_{ij} y_i^* = c_j \quad \forall j$$

and likewise

$$y_i^* > 0 \quad \Rightarrow \quad \sum_j a_{ij} x_j^* = b_j \quad \forall i$$

Complementary Slackness (CS) Theorem

Theorem

Suppose that x^* is feasible for (P) and y^* is feasible for (D). Optimality of x^* and y^* implies that the CS condition holds.

Theorem

Suppose that x^* is optimal for (P). Then there exists a y^* which is optimal for (D), and the CS condition holds for this pair of vectors.