AAE 706: Expected Utility

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Today's Agenda

1. Pervasiveness of uncertainty in economic choice
2. Optimal consumer response to uncertainty
3. State-contingent budget constraints
4. Preferences under uncertainty
5. Choice under uncertainty
6. Insurance
7. Equivalent variation
8. Numerical versions of the model

The Language of Expected Utility

- portfolio
- insurance
- state of nature
- state-contingent contract
- contingencies
- state-specific budget constraints
- endowment bundle
- preferences under uncertainty
- marginal rate of substitution
- marginal utility
- equivalent variation in income
- degree of relative risk aversion
- fair versus unfair insurance premia
- risk-lover
- risk-neutral
- deterministic income
- free entry to the insurance industry
- economic profit from insurance
- slope of the budget constraint in a two state insurance model
- indifference curve
- slope of the expected utility indifference curve

Uncertainty is Pervasive

- What is uncertain in economic systems?
  - tomorrow's prices
  - future wealth
  - future availability of commodities
  - present and future actions of other people.

Uncertainty is Pervasive

- What are rational responses to uncertainty?
    - buying insurance (health, life, auto)
    - a portfolio of contingent consumption goods.

States of Nature

- Possible states of Nature:
  - "car accident" (a)
  - "no car accident" (na).
- Accident occurs with probability π_a and does not with probability π_{na}:

$$\pi_a + \pi_{na} = 1.$$

- Accident causes a loss of \$L.

Contingencies

- A contract implemented only when a particular state of Nature occurs is *state-contingent*.
- E.g. the insurer pays only if there is an accident.

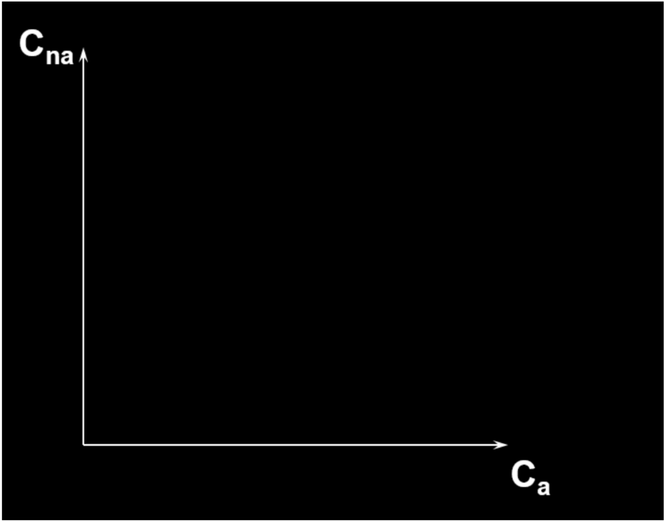
Contingencies

- A state-contingent consumption plan is implemented only when a particular state of Nature occurs.
- E.g. take a vacation only if there is no accident.

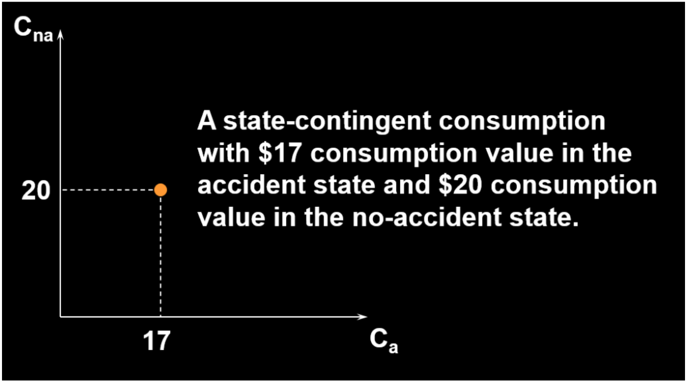
State-Contingent Budget Constraints

- Each \$1 of accident insurance costs γ.
- Consumer has \$m of wealth.
- C_{na} is consumption value in the no-accident state.
- C_a is consumption value in the accident state.

State-Contingent Budget Constraints



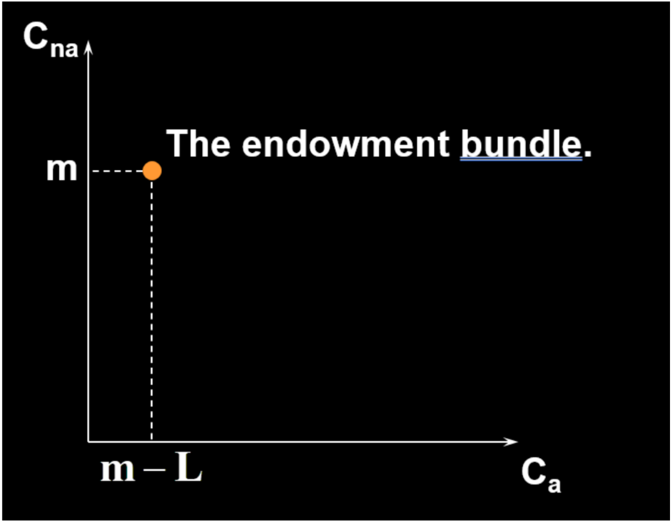
State-Contingent Budget Constraints



State-Contingent Budget Constraints

- Without Insurance
  - $C_a = m - L$
  - $C_{na} = m.$

State-Contingent Budget Constraints



State-Contingent Budget Constraints

- Buy \$ K of accident insurance
- $C_{na} = m - \gamma K$
- $C_a = m - L - \gamma K + K = m - L + (1 - \gamma)K$

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- So

$$K = \frac{C_a - m + L}{1 - \gamma}$$

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- and
$$C_{na} = m - \gamma \frac{C_a - m + L}{1 - \gamma}$$

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- So

$$K = \frac{C_a - m + L}{1 - \gamma}$$

- and

$$C_{na} = m - \gamma \frac{C_a - m + L}{1 - \gamma}$$

- I.e.

$$C_{na} = \frac{m - \gamma L}{1 - \gamma} - \frac{\gamma}{1 - \gamma} C_a$$

State-Contingent Budget Constraints

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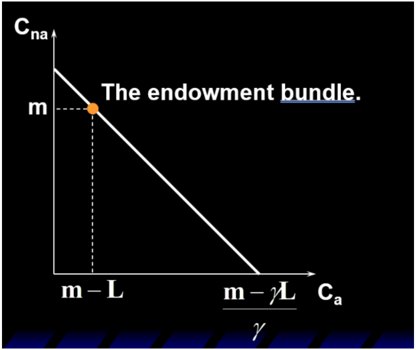

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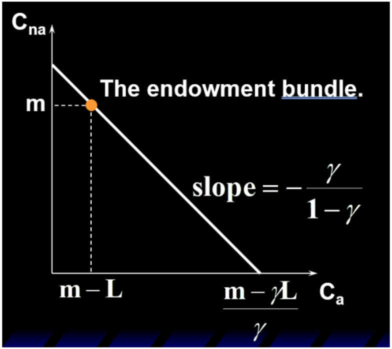

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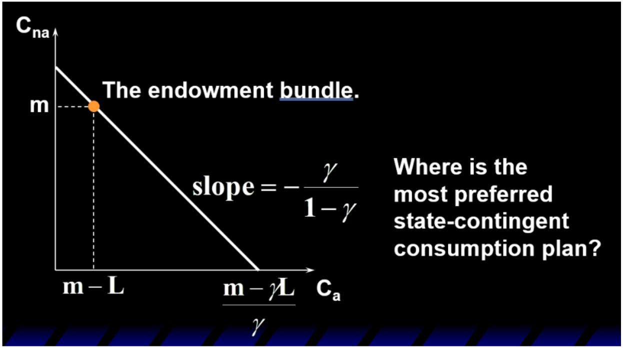

Preferences Under Uncertainty

- Think of a lottery.
- Win \$90 with probability $1/2$ and win \$0 with probability $1/2$.
- $U(\$90) = 12$, $U(\$0) = 2$.
- Expected utility is

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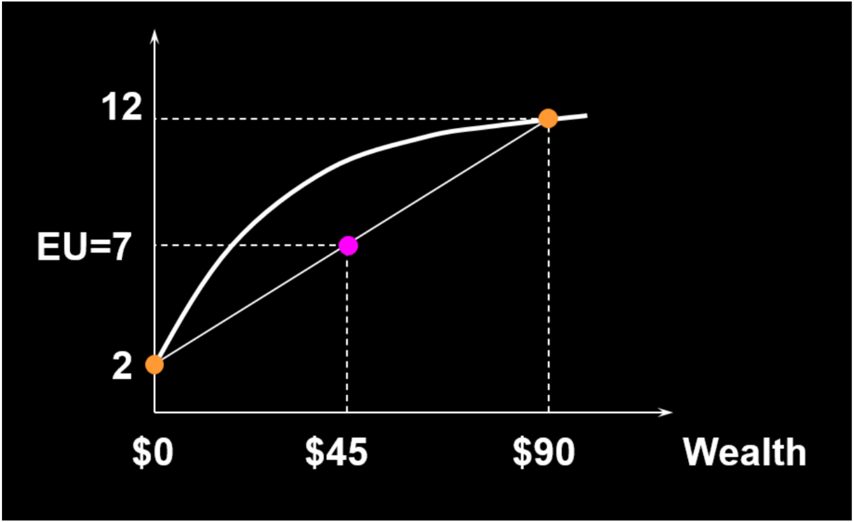
$$\begin{aligned} EU &= \tfrac{1}{2} \times U(\$90) + \tfrac{1}{2} \times U(\$0) \\ &= \tfrac{1}{2} \times 12 + \tfrac{1}{2} \times 2 \end{aligned}$$

Preferences Under Uncertainty

- $EU = 7$ and $EM = \$45$.
- $U(\$45) > 7 \Rightarrow$ \$45 for sure is preferred to the lottery $\Rightarrow$ *risk-aversion*.
- $U(\$45) < 7 \Rightarrow$ the lottery is preferred to \$45 for sure $\Rightarrow$ *risk-loving*.
- $U(\$45) = 7 \Rightarrow$ the lottery is preferred equally to \$45 for sure $\Rightarrow$ *risk-neutral*.

Preferences Under Uncertainty



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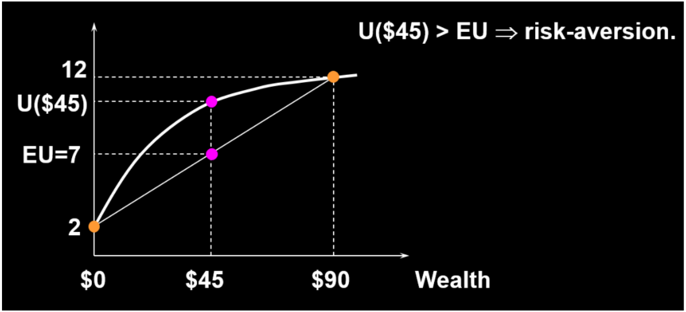

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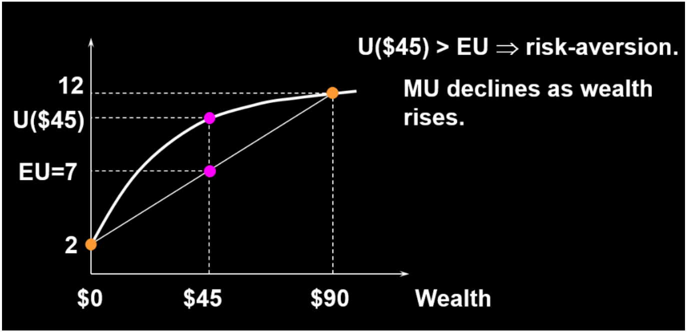

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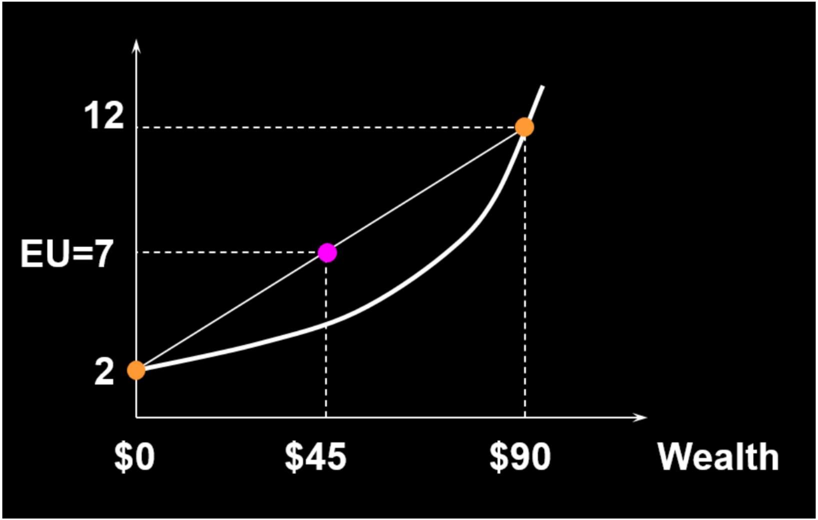

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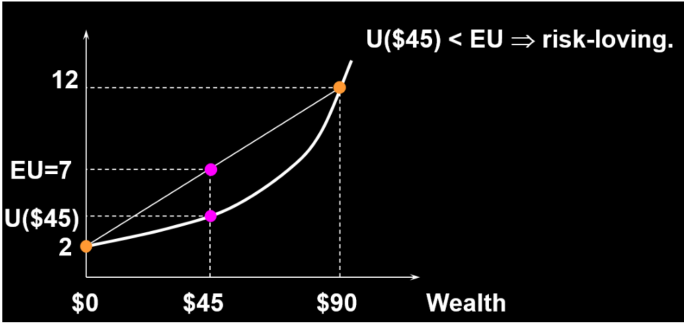

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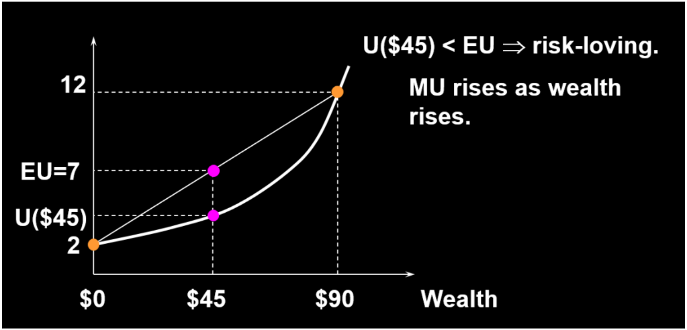

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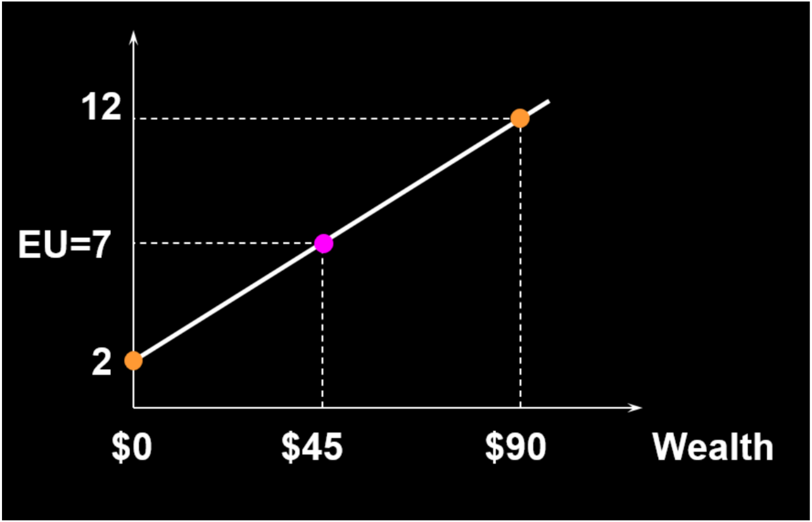

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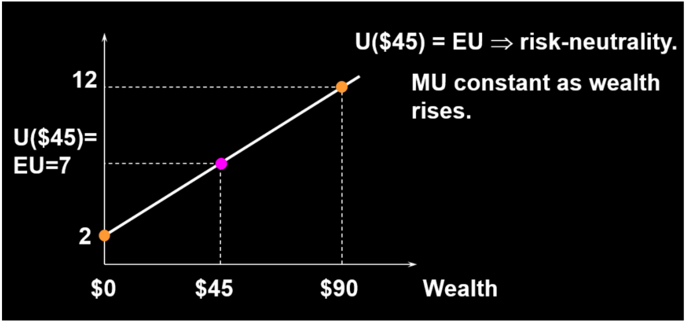
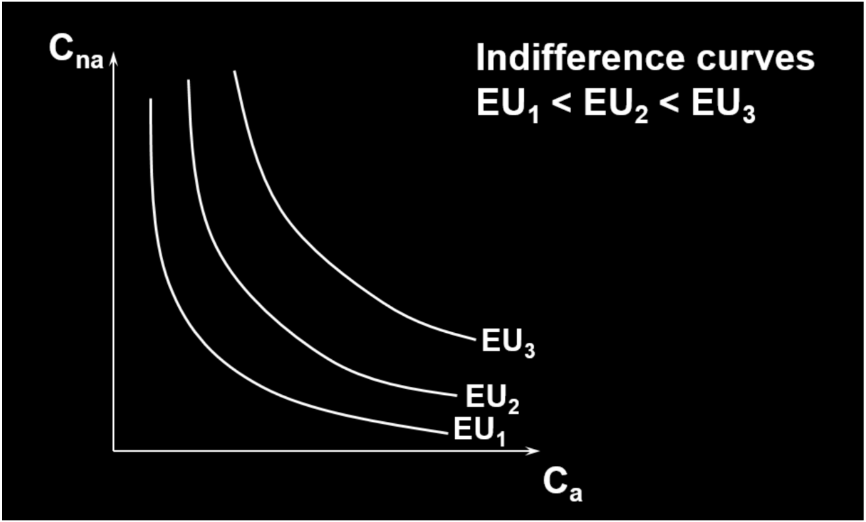


Preferences Under Uncertainty

- State-contingent consumption plans that give equal expected utility are equally preferred.

Preferences Under Uncertainty



Preferences Under Uncertainty

- What is the MRS of an indifference curve?
- Get consumption c_1 with probability π_1 and c_2 with probability π_2 $(\pi_1 + \pi_2 = 1)$
- $EU = \pi_1 U(c_1) + \pi_2 U(c_2)$
- For constant EU, $\mathrm{d}EU = 0$.

Preferences Under Uncertainty

$$EU = \pi_1 U(c_1) + \pi_2 U(c_2)$$

$$\mathrm{d}EU = \pi_1 MU(c_1)\mathrm{d}c_1 + \pi_2 MU(c_2)\mathrm{d}c_2$$

Preferences Under Uncertainty

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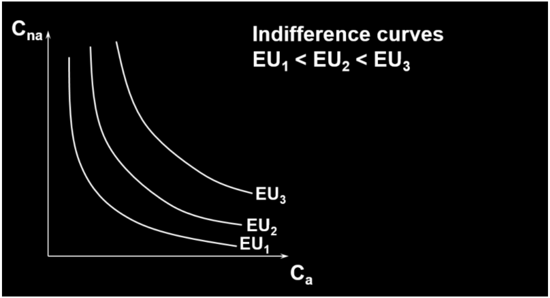
$$\mathrm{d}EU = 0 \Rightarrow \pi_1 MU(c_1)\mathrm{d}c_1 + \pi_2 MU(c_2)\mathrm{d}c_2 = 0$$

$$\Rightarrow \frac{\mathrm{d}c_1}{\mathrm{d}c_2} = -\frac{\pi_2 MU(c_2)}{\pi_1 MU(c_1)}$$

Preferences Under Uncertainty

$$\frac{\mathrm{d}c_{na}}{\mathrm{d}c_a} = -\frac{\pi_a MU(c_a)}{\pi_{na} MU(c_{na})}$$

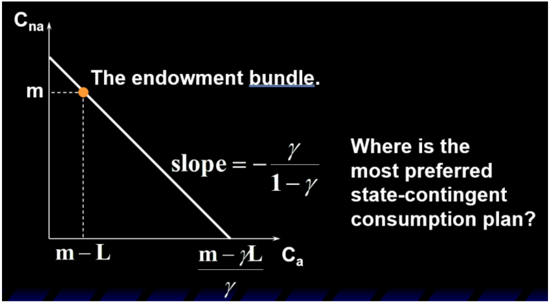


Choice Under Uncertainty

- *Question*: How is a rational choice made under uncertainty?
- *Answer*: Choose the most preferred affordable state-contingent consumption plan.

State-Contingent Budget Constraints

$$C_{na} = \frac{m - \gamma L}{1 - \gamma} - \frac{\gamma}{1 - \gamma} C_a$$



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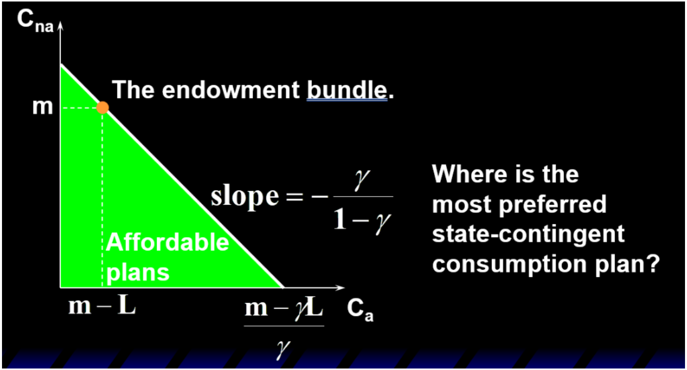
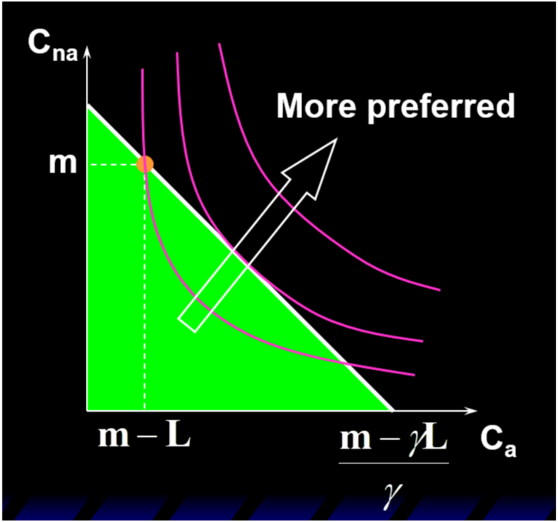


State-Contingent Budget Constraints

What is the most preferred state-contingent consumption plan?



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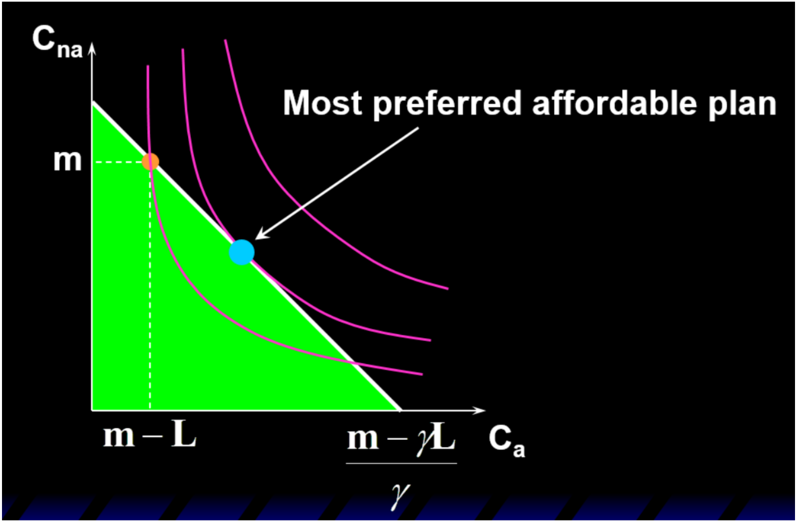

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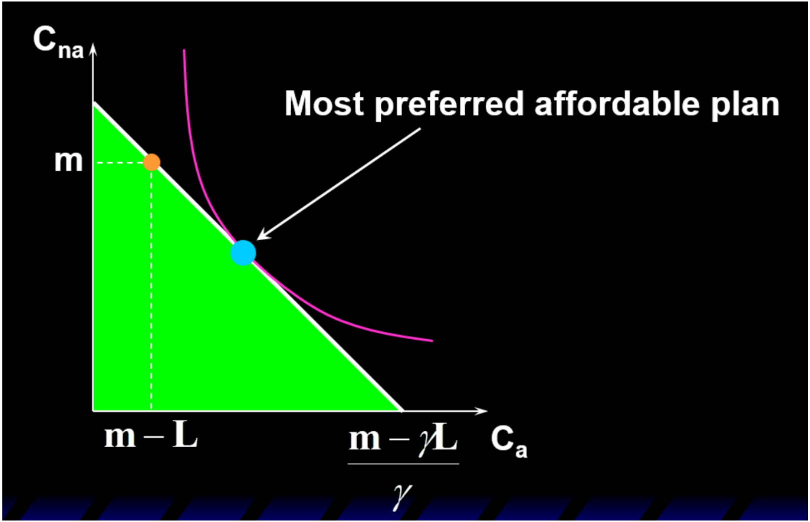
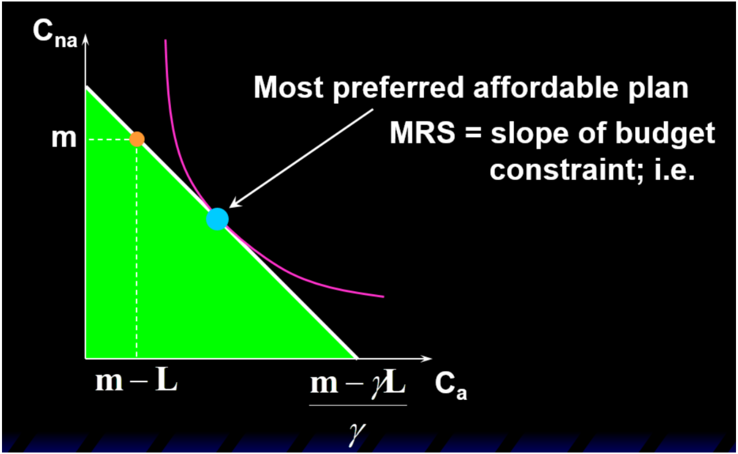


State-Contingent Budget Constraints

$$\frac{\gamma}{1-\gamma} = \frac{\pi_a MU(C_a)}{\pi_{na} MU(C_{na})}$$



Competitive Insurance

- Suppose entry to the insurance industry is free.
- Expected economic profit $= 0$.
- I.e. $\gamma K - \pi_a K - (1 - \pi_a)0 = (\gamma - \pi_a)K = 0$.
- I.e. free entry $\Rightarrow \gamma = \pi_a$.
- If the price of \$1 insurance = accident probability, the insurance is *fair*.

Competitive Insurance

- When insurance is fair, rational insurance choices satisfy

$$\frac{\gamma}{1-\gamma} = \frac{\pi_a}{1-\pi_a} = \frac{\pi_a MU(C_a)}{\pi_{na} MU(C_{na})}$$

Competitive Insurance

- When insurance is fair, rational insurance choices satisfy

$$\frac{\gamma}{1-\gamma} = \frac{\pi_a}{1-\pi_a} = \frac{\pi_a MU(C_a)}{\pi_{na} MU(C_{na})}$$

- I.e., $MU(C_a) = MU(C_{na})$
- Marginal utility of income must be the same in both states.

Competitive Insurance

- How much fair insurance does a risk-averse consumer buy?

$$MU(C_a) = MU(C_{na})$$

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$$MU(C_a) = MU(C_{na})$$

- Risk-aversion $\Rightarrow MU(c) \downarrow$ as $C \uparrow$
- Hence $C_a = C_{na}$
- I.e., full-insurance.

“Unfair” Insurance

- Suppose insurers make positive expected economic profit.
- I.e., $\gamma K - \pi_a K - (1 - \pi_a)0 = (\gamma - \pi_a)K > 0$.

“Unfair” Insurance

- Suppose insurers make positive expected economic profit.
- I.e., $\gamma K - \pi_a K - (1 - \pi_a)0 = (\gamma - \pi_a)K > 0$.
- Then $\Rightarrow \gamma > \pi_a \Rightarrow \frac{\gamma}{1-\gamma} > \frac{\pi_a}{1-\pi_a}$

"Unfair" Insurance

- Rational choice requires

$$\frac{\gamma}{1-\gamma} = \frac{\pi_a MU(c_a)}{\pi_{na} MU(c_{na})}$$

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- Hence $C_a < C_{na}$ for a risk-averter.

"Unfair" Insurance

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- Since $\frac{\gamma}{1-\gamma} > \frac{\pi_a}{1-\pi_a}$,

$$MU(C_a) > MU(C_{na})$$

- Hence $C_a < C_{na}$ for a risk-averter.
- I.e., a risk-averter buys less than full "unfair" insurance.

Quantification of Economic Cost

- Consider the optimal choice when the loss in the no-accident state is zero (i.e., $L = 0$). In this case $C_a = C_{na} = m$.
- Given expected utility of $\tilde{EU}$ we can describe this as a *equivalent change in deterministic income*. I.e.

$$U(\tilde{C}) = U(\tilde{m}) = \tilde{EU}$$

- The economic cost of a change in the stochastic environment (L, π or γ) can then be characterized as an *equivalent variation in deterministic income*, i.e.

$$EV = 100 \times \left(1 - \frac{\tilde{m}}{m}\right)$$

where $\tilde{m}$ solves

$$U(\tilde{m}) = \tilde{EU}$$

Iso-Elastic Utility

$$U(C) = \frac{C^{\rho}}{\rho}$$

where ρ is defined by the degree of relative risk aversion:

$$\rho = 1 - 1/\sigma$$

Risk-netrality assumes $\sigma \geq 0$.

Marginal utility is then defined as:

$$MU(C) = C^{\rho-1}$$

and if we normalize $m = 1$ the equivalent variation in income is given by:

$$EV(C) = 100 \times \left[\left(\sum_s \pi_s C_s^{\rho} \right)^{1/\rho} - 1 \right]$$

EV Monotonic Transformation of EU

$EV(\vec{C})$ is a *monotonic transformation* of $EU(\vec{C})$, hence maximization of $EV(\vec{C})$ produces the same state-contingent consumption choices as maximization of $EU(\vec{C})$.

Uncertainty is Pervasive

- What are rational responses to uncertainty?
    - ✓ buying insurance (health, life, auto)

Uncertainty is Pervasive

- What are rational responses to uncertainty?
  - ✓ buying insurance (health, life, auto)
  - **?** a portfolio of contingent consumption goods.

GAMS Code: Input Data

```
$title  Optimal Insurance

*       Declare some parameters with assigned values.

parameter       pi      True probability of bad outcome /0.01/
                L       Loss with a bad outcome /0.5/,
                gamma   Premium for coverage /0.02/,
                sigma   Elasticity /0.5/;

*       GAMS is not case sensitivity, but we following the
*       convention that parameters (exogenous inputs) are
*       in lower case and variables (endogenous outputs) are
*       written in upper case (except for "L").

*       Declare a parameter whose value will be assigned:

parameter       rho     Risk exponent;
rho = 1 - 1/sigma;
```

GAMS Code: Model Declaration

```
*         Declare and solve the model:

variables          EU       Expected utility,
                   C_G      Consumption on a good day,
                   C_B      Consumption on a bad day,
                   K        Coverage;

*         Declare some equations:

equations          eudef, budget_g, budget_b;

*         Represent the utility function as a macro:

$macro U(C)        (C**rho/rho)

eudef..            EU =e= (1-pi) * U(C_G) + pi * U(C_B);

budget_G..         C_G =e= 1 - gamma * K;

budget_B..         C_B =e= 1 - L + (1-gamma) * K;
```

GAMS Code: Initialize and Solve

```
C_G.L = 1; C_B.L = 1; K.L = 1;

solve insurance using nlp maximizing EU;

parameter       solution        Report of model solution for comparison across models;

solution("C_G","Max_EU") = C_G.L;
solution("C_B","Max_EU") = C_B.L;
solution("K","Max_EU") = K.L;
```

GAMS Code: Formulation with Money-Metric Expected Uti

```
variable        EV      Equivalent variation;

equation        evdef;

evdef..         EV =e= 100 * (( (1-pi) * C_G**rho + pi * C_B**rho )**(1/rho) - 1);

model insurance_ev /budget_G, budget_B, evdef/;

solve insurance_ev using nlp maximizing EV;

solution("C_G","Max_EV") = C_G.L;
solution("C_B","Max_EV") = C_B.L;
solution("K","Max_EV") = K.L;
```

GAMS Code: Representation as an Equilibrium Problem

```
*       Declare a macro to compute marginal utility:

$macro MU(c)     (C**(rho-1))

*       This is a complementarity constraint -- if the marginal cost
*       exceeds the marginal benefit, then K must be zero.

*               Marginal cost of the insurance          =g= Marginal benefit

equation coverage;

*       Cost of insurance is the premium (gamma) which must be paid
*       in each state of the world.  Benefit is the expected value of the
*       payment made in the bad state.

coverage..      gamma * ((1-PI) * MU(c_g) + PI*MU(C_B)) =g= PI * MU(C_B);

*       Declare the model as an equilibrium problem corresponding to the first
*       order conditions of the nonlinear programming model:

model equilibrium /eudef.EU, evdef.EV, budget_g.C_G, budget_B.C_B, coverage.K/;

solve equilibrium using mcp;

solution("C_G","Equilibrium") = C_G.L;
solution("C_B","Equilibrium") = C_B.L;
solution("K","Equilibrium") = K.L;
```

GAMS Code: Comparison of Results

```
----     100 PARAMETER solution  Report of model solution for comparison across models

          Max_EU      Max_EV  Equilibri~

C_G        0.996       0.996       0.996
C_B        0.701       0.701       0.701
K          0.205       0.205       0.205
```