AAE 706

Demand, Supply and Market Equilibrium

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Consumer and Producer Demand Models

Uncompensated Demand The consumer demand model translates *preferences*, *income* and *prices* into *utility* and *demand*. Key ideas: the expenditure function, indirect utility, money-metric utility, income- and price elasticities of demand, homotheticity.

Compensated Demand The producer demand model translates *technology* and *factor prices* into *unit cost* and *unit factor demand*. Key ideas: the unit cost function, price elasticities of demand, returns to scale.

A Word Problem

Thomas lives in Ann Arbor where he currently spends 30% of his income on rent. He has an employment offer in Zürich which pays 50% more than he currently earns, but he is hesitant to take the job because rental rates in Zürich are three times higher than in Ann Arbor. Assuming that Thomas has CES preferences with elasticity of substitution σ*; on purely economic grounds, should he move?*

As is the case for all interesting questions in economics, the only good answer to this problem is "It depends.".

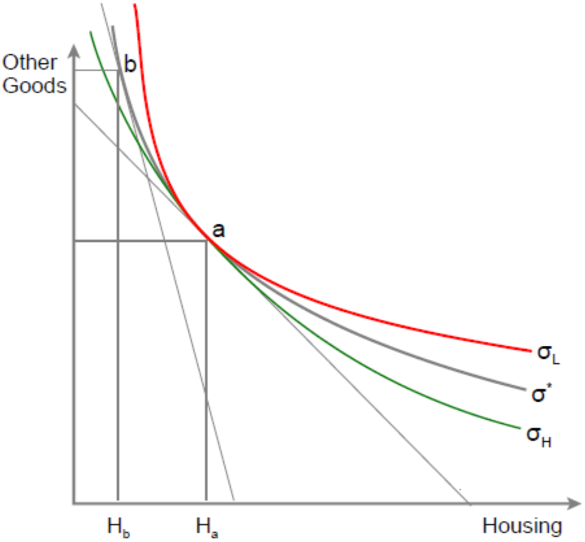
Intuition

Thomas's offer in Zürich does not pay him enough to live exactly the lifestyle that he enjoys in Ann Arbor, as he would need a 60% raise to cover rent and consumption. The elasticity of substitution is key. If it is high, he more willing substitutes consumption of goods and services for housing and thereby lowers his cost of living in Zürich. On the other hand, if the elasticity is low, he is "stuck in his ways", and the move is a bad idea.

Calibration to a Benchmark Equilibrium

We are given information about Thomas's choices in Ann Arbor. This information is essentially an observation of a *benchmark equilibrium*, consisting of the prevailing prices and quantities of goods demand. The benchmark equilibrium data together with assumptions about elasticities are used to evaluate Thomas's choices after a discrete change in the economic environment. The steps involved in solving this little textbook model are identical to those typically employed in applied general equilibrium analysis.

Graphical Representation



CES Preferences

The constant-elasticity-of-substitution utility function can be written:

$$U(C, H) = (\alpha C^\rho + (1-\alpha)H^\rho)^{1/\rho}$$

Exponent ρ is defined by the *elasticity of substitution*, σ, as

$$\rho = 1 - 1/\sigma.$$

The taking the price of other goods as unity, the model of consumer choice is:

$$\max U(C, H) \text{ s.t. } C + p_H H = M$$

Demand

Derivation of demand functions which solve the utility maximization problem involves solving two equations in two unknowns:

$$\frac{\partial U/\partial H}{\partial U/\partial C} = \frac{(1-\alpha)H^{\rho-1}}{\alpha C^{\rho-1}} = p_H;$$

hence

$$\frac{H}{C} = \left(\frac{1-\alpha}{\alpha \ p_H}\right)^{\sigma}$$

Substituting into the budget constraint, we have:

$$H = \frac{M}{p_H + \left(\frac{\alpha \ p_H}{1-\alpha}\right)^{\sigma}} = \frac{(1-\alpha)^{\sigma} M p_H^{-\sigma}}{\alpha^{\sigma} + (1-\alpha)^{\sigma} p_H^{1-\sigma}}$$

and

$$C = \frac{M}{1 + p_H \left(\frac{1-\alpha}{\alpha \ p_H}\right)^{\sigma}} = \frac{\alpha^{\sigma} M}{\alpha^{\sigma} + (1-\alpha)^{\sigma} p_H^{1-\sigma}}$$

Calibration

It is conventional in applied general equilibrium analysis to employ exogenous elasticities and calibrated value shares. If we follow this approach, σ is then exogenous and α is calibrated.
Choosing units so that the benchmark price of housing ($\bar{p}_H$) is unity, we have:

$$\theta = \bar{p}_H \bar{H} / \bar{M}$$

Substitute into the demand function:

$$1 + \left(\frac{\alpha}{1 - \alpha} \right)^{\sigma} = \frac{\bar{M}}{\bar{H}} = \frac{1}{\theta};$$

and then solve for the preference parameter α:

$$\alpha = \frac{(1 - \theta)^{1/\sigma}}{\theta^{1/\sigma} + (1 - \theta)^{1/\sigma}}.$$

Money Metric Utility

Substitute for α in $U(C,H)$, and denoting the base year expenditure on other goods as $\bar{C} = (1-\theta)\bar{M}$, we have

$$U(C,H) = \kappa \left((1-\theta)^{1/\sigma} C^\rho + \theta^{1/\sigma} H^\rho \right)^{1/\rho}$$

where the κ is a constant which may take on any positive value without altering the preference ordering. It is convenient to assign this value to the benchmark expenditure, so that utility can be measured in money-metric units at benchmark prices.

Noting that $\theta^{1/\sigma} = \theta^{1-\rho}$, we then can write the utility function as:

$$\tilde{U}(C,H) = \bar{M} \left((1-\theta) \left(\frac{C}{\bar{C}} \right)^\rho + \theta \left(\frac{H}{\bar{H}} \right)^\rho \right)^{1/\rho}$$

Indirect Utility

Formally, we have:

$$V(p_H, M) = U(C(p_H, M), M(p_H, M)) = \frac{M}{\left(\alpha^\sigma + (1-\alpha)^\sigma p_H^{1-\sigma}\right)^{1/(1-\sigma)}}$$

In money-metric terms, we can use benchmark income to normalize the utility function:

$$\tilde{V}(p_H, M) = \frac{M}{(1-\theta+\theta p_H^{1-\sigma})^{1/(1-\sigma)}}$$

Demand Functions – Calibrated Share Form

$$H = \bar{H} \; \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{p_H}\right)^{\sigma}$$

$$C = \bar{C} \; \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{1}\right)^{\sigma}$$

where

$$p_U = \left(1 - \theta + \theta p_H^{1-\sigma}\right)^{1/(1-\sigma)}$$

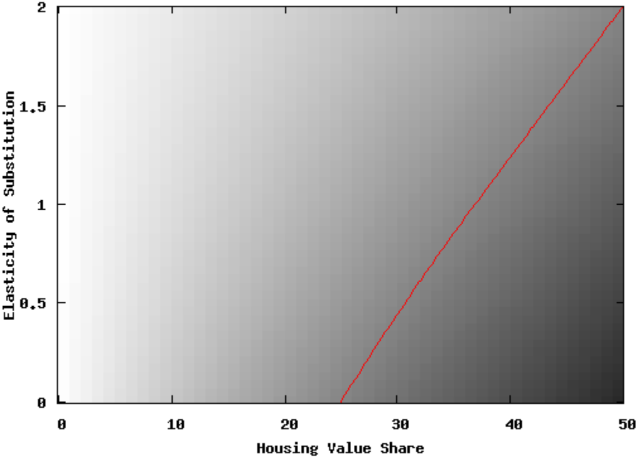
Should Thomas Move?

Thomas's welfare level in Zürich can be easily computed in money-metric terms as:

$$\tilde{V}(p_H = 3, M = 1.5) = \frac{1.5}{(0.7 + 0.3 \times 3^{1-\sigma})^{1/(1-\sigma)}}$$

This expression cannot (to my knowledge) be solved in closed form, however it is easily to solve using Excel. The critical value for σ is that which equates welfare in Zürich with welfare level in Ann Arbor, i.e. $\tilde{V} = 1$. The numerical value is found to be $\sigma^* = 0.441$. The general dependence of welfare on the θ and σ can be illustrated in a contour diagram.

Dependence of Welfare on Benchmark Shares and Elasticity



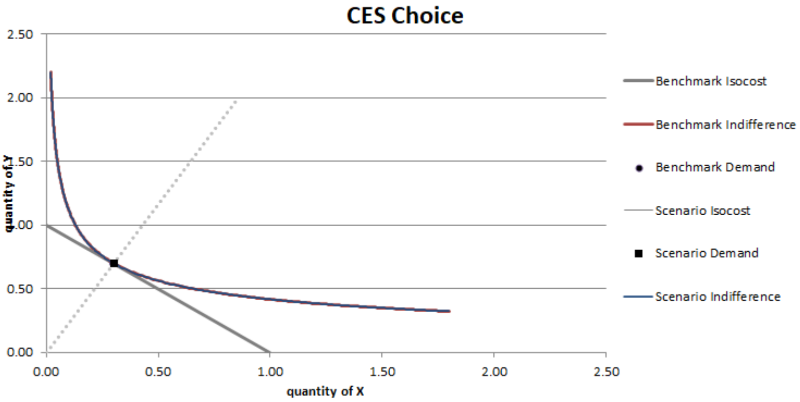
GNUPLOT Script

```
set auto
set style data lines
set xlabel "Housing Value Share"
set ylabel "Elasticity of Substitution"
set view map
set contour base
set xrange [0:50]
set yrange [0:2]
set cntrparam levels discrete 0
set pm3d
set palette gray positive
unset title
unset key
unset colorbox
unset clabel
set isosamples 51,50; set samples 51,50
set xtics
set ytics
unset surface
splot 1.50/(1-x/100+x/100*3**(1-y))**(1/(1-y))-1
```

Homework Assignment Question 1

Nancy currently lives in Madison where she spends a third of her income on housing, half her income on groceries and all the remaining money on opera tickets in Chicago. She loves opera, and she has a lifetime membership at the Madison Overature Center where she is able to (freely) attend every opera which is staged, as many as she chooses to attend in Chicago. Nancy now considers a job offer in London, England. Housing in London is four times as expensive as in Madison, opera tickets are the same price as in Chicago and other goods are twice as expensive as in Madison. What is the minimum salary offer which would induce Nancy to move?

Compensated CES Model in Excel



Calculations

	A	B	C	D	E	F
1						
2	**Exogenous Inputs (from controls)**	Ident				Controls:
3	Benchmark price of X	px0	1			5
4	X benchmark value share	thetax	0.30			3
5	Elasticity of Substitution	sigma	1.01			2
6	Income	m	1.00			5
7	Scenario price of X	px	1.00			8
8	Scenario price of Y	py	1.00			8
9						
10	**Implicit Inputs (calculated):**					
11	Y benchmark value share	thetay	0.70			
12	Benchmark income	m0	1.00			
13	Benchmark demand for x	x0	0.30			
14	Benchmark demand for y	y0	0.70			
15	Primal elasticity exponent	rho	0.01			
16	Cost Index	cost	1.00			
17	Demand Index	d	1.00			
18	Scenario demand for x	x	0.30			
19	Scenario demand for y	y	0.70			
20	X Scenario Value Share	alphax	0.30			
21	Y Scenario Value share	alphay	0.70			
22						
23	**Graph Points and Lines:**					
24	Benchmark Point (Quantities):	0.3	0.7			
25	Benchmark Point (Prices):	1	1			
26	Primal budget line:	1.00	0.00			
27		0.00	1.00			
28	Scenario point	0.30	0.70			
29	Scenario budget line:	1.00	0.00			
30		0.00	1.00			
31	Dual "budget line":	3.33	0.00			
32		0.00	1.43			
33	Benchmark Engle Curve:	0.00	0.00			
34		0.86	2.00			
35	Scenario Engle Curve:	0.00	0.00			
36		0.86	2.00			
37						

Implicit Inputs

Implicit Inputs (calculated):		
Y benchmark value share	thetay	=1-thetax
Benchmark income	m0	=thetay+px0*thetax
Benchmark demand for x	x0	=m0*thetax/px0
Benchmark demand for y	y0	=m0*thetay
Primal elasticity exponent	rho	=1-1/sigma
Cost Index	cost	=(thetax*(px/px0)^(1-sigma)+thetay*py^(1-sigma))^(1/(1-sigma))
Demand Index	d	=m/c_
Scenario demand for x	x	=x0*(c_*px0/px)^sigma*d
Scenario demand for y	y	=y0*(c_/py)^sigma*d
X Scenario Value Share	alphax	=px*x/(px*x+py*y)
Y Scenario Value share	alphay	=py*y/(px*x+py*y)

Graph Coordinates

Graph Points and Lines:		
Benchmark Point (Quantities):	=x0	=y0
Benchmark Point (Prices):	=px0	=1
Primal budget line:	=m0/px0	0
	0	=m0
Scenario point	=x	=y
Scenario budget line:	=x+py*y/px	0
	0	=px*x/py+y
Dual "budget line":	=m0/(x0)	0
	0	=m0/(y0)
Benchmark Engle Curve:	0	0
	=MIN(2,x0*2/y0)	=MIN(2,y0*2/x0)
Scenario Engle Curve:	0	0
	=MIN(2,2*x/y)	=MIN(2,2*y/x)

The Canonical Producer Demand Model

- Compensated unit demand
- Technology: $f(k, \ell)$
- Factor prices: p_k, p_ℓ
- Unit cost function:

$$c(p_k, p_\ell) = \min p_k k + p_\ell \ell \quad \text{s.t.} \quad f(k, \ell) = 1$$

- Unit factor demand functions: $k(p_k, p_\ell), \ell(p_k, p_\ell)$
- Price elasticities of demand:

$$\epsilon_L = \frac{\partial k(p_k, p_\ell)}{\partial p_k} \frac{p_k}{k} \quad y = 1$$

- Constant returns to scale: $f(\lambda k, \lambda \ell) = \lambda f(k, \ell)$

CES Factor Demand Model

- Calibrated output:

$$y = f(k, \ell) = \bar{y}\left(\theta_k \left(\frac{k}{\bar{k}}\right)^{\rho} + \theta_\ell \left(\frac{\ell}{\bar{\ell}}\right)^{\rho}\right)^{1/\rho}$$

- Unit cost function:

$$c(p_k, p_\ell) = \left(\theta_k \left(\frac{p_k}{\bar{p}_k}\right)^{1-\sigma} + \theta_\ell \left(\frac{p_\ell}{\bar{p}_\ell}\right)^{1-\sigma}\right)^{1/(1-\sigma)}$$

Calibrated CES Demand

- Demand functions are then:

$$k = \bar{k} \left(\frac{c(p_k, p_\ell)\bar{p}_k}{p_\ell} \right)^{\sigma} \quad \left(\frac{y}{\bar{y}} \right)$$

and

$$\ell = \bar{\ell} \left(\frac{c(p_k, p_\ell)\bar{p}_\ell}{p_\ell} \right)^{\sigma} \quad \left(\frac{y}{\bar{y}} \right)$$

Isoelastic CES Demand Model

We can replace output (y) by a downward sloping demand function which is a function of the unit cost of k and ℓ. Factor demand functions are then:

$$k = \bar{k} c^{-\epsilon} \left(\frac{c \bar{p}_k}{p_k} \right)^{\sigma}, \quad \ell = \bar{\ell} c^{-\epsilon} \left(\frac{c \bar{p}_\ell}{p_\ell} \right)^{\sigma}$$

in which the unit c is a function of the factor prices:

$$c = \left(\theta_k \left(\frac{p_k}{\bar{p}_k} \right)^{1-\sigma} + \theta_\ell \left(\frac{p_\ell}{\bar{p}_\ell} \right)^{1-\sigma} \right)^{1/(1-\sigma)}$$

The Factor Demand Model

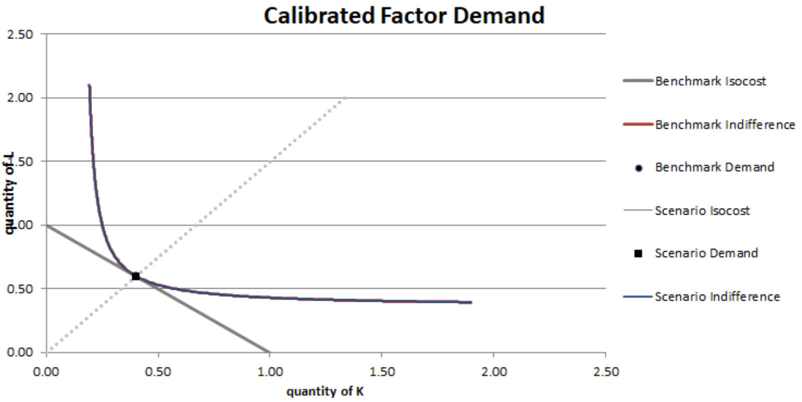
When $\epsilon = 0$ in the isoelastic model we have the conventional uncompensated factor demand, solutions to the producer's problem:

$$\min p_k k + p_\ell \ell$$

s.t.

$$f(k, \ell) = \left(\theta_k \left(\frac{k}{\bar{k}} \right)^\rho + \theta_\ell \left(\frac{\ell}{\bar{\ell}} \right)^\rho \right)^{1/\rho} = 1$$

Isoelastic Factor Demand Model in Excel



Homework Assignment Question 2

Consider the production function which combines labor (ℓ) and capital (k) with a constant elasticity of substitution equal to σ to produce output (y):

$$y = f(k, \ell) = \bar{y}\left(\theta\left(\frac{k}{\bar{k}}\right)^{\rho} + (1-\theta)\left(\frac{\ell}{\bar{\ell}}\right)^{\rho}\right)^{1/\rho}$$

in which $\rho = 1 - 1/\sigma$ and $\theta = \bar{k}/(\bar{k} + \bar{\ell})$. Suppose that the market wage and the capital input are fixed (i.e., $p_\ell = 1$ and $k = \bar{k}$).
Defind the price elasticity of supply as:

$$\eta = \left.\frac{\partial y}{\partial p}\frac{p}{\bar{y}}\right|_{p=1}$$

Show that when the benchmark value shares of labor and capital are equal the price elasticity of supply equals the elasticity of substitution, i.e. $\eta = \sigma$.

Mathiesen's Equilibrium Programming Framework

Consider an economy or a sector with production. Suppose that it has m commodities and n activities with constant returns to scale production. We base our presentation of the modeling format and the algorithm on the assumption that production is characterized by a linear technology matrix with fixed input-output coefficients.

Model Primitives

For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let

$b = (b_i)$ denote the vector of endowments,

$d(p) = (d_i(p))$ denote the market demand functions, which we assume to be point-to-point and continuously differentiable,

$c = (c_j)$ denote the vector of unit costs of operating the activities, and finally, let

$A = (a_{ij})$ denote the technology matrix of input-output coefficients consistent with unit production, where $a_{ij} > 0 (a_{ij} < 0)$ denotes an output (input).

Equilibrium Variables

$p = (p_i)$ denote the vector of prices,

$y = (y_j)$ denote the vector of activity levels,

Equations which Characterize an Equilibrum

Because of the generality of the theory of economic equilibrium, there are several ways to characterize an equilibrium. We shall use the following definition (Scarf 1973):

A price vector p^* and a vector of activity levels y^* constitute a competitive equilibrium if:

i. No activity earns a positive profit;

$$c_j - \sum_i a_{ij} p_i^* \geq 0 \quad \forall j$$

ii. No commodity is in excess demand;

$$b_i + \sum_j a_{ij} y_j^* \geq d_i(p^*) \quad \forall i$$

iii. Prices and activity levels are non-negative;

$$p_i^* \geq 0 \quad \forall i, y_j^* \geq 0 \quad \forall j$$

Characterizing an Equilibrum (cont.)

i. An activity earning a deficit is not used and an operated activity has no loss;

$$(c_j - \sum_i a_{ij} p_i^*) y_j^* = 0 \quad \forall j$$

ii. A commodity in excess supply has zero price, and a positive price implies market clearance;

$$p_i^* \left(b_i + \sum_j a_{ij} y_j^* - d_i(p^*) \right) = 0 \quad \forall i$$

General Versus Partial Equilibrium

- The vector c of operating costs represents factors of production that are exogenous to the economy or sector under consideration. It typically applies in a partial equilibrium setting.
- When we describe a general equilibrium problem (of a closed economy) the cost vector $c = 0$ because all prices will be determined simultaneously and no single price will be exogeneously given.
- In this case, demands $d_i(p)$ are functions of all prices in the economy, i.e., both product and factor prices. Furthermore, these demand functions will usually be specified in a manner consistent with individual household utility maximization.