Homework: Nonlinear Programming Solution

AAE 706: Applied Risk Analysis
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The *optimal lot size problem* of inventory theory is a classical nonlinear programming problem. A firm's inventory of a homogeneous commodity, $I(t)$, is depleted at a constant rate per unit time. The firm reorders an amount x of the commodity, which is delivered immediately, whenever the level of inventory is zero. The annual demand is A, and the firm orders the commodity n times a year where:

$$A = nx.$$

The firm incurs two types of inventory costs: a *holding cost* and an *ordering cost*. The average stock of inventory is $x/2$, and the cost of holding one unit of the commodity is C_h, so $C_h x/2$ is the holding cost.

The firm orders the commodity, as stated above, n times a year, and the cost of placing one order is C_O so $C_O n$ is the ordering cost. The total cost is then:

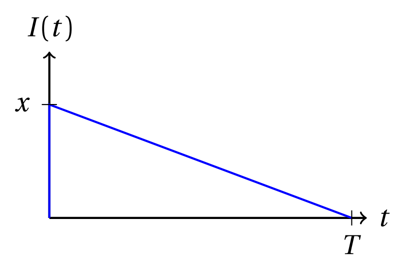
$$C = C_h \frac{x}{2} + C_O n$$

When we have linear demand and orders of magnitude X at time intervals T, the inventory at time t is given by:

$$I(t) = x(1 - \frac{t}{T})$$

Hence, the average inventory holding is give by:

$$\bar{x} = \int_0^T I(t)dt = \frac{x}{2}$$



a. Minimize the cost of inventory, C, by choice of x and n subject to the constraint $A = nx$ using the constraint substitution method. Show that the optimal lot size is given by:

$$x = \sqrt{\frac{2C_o A}{C_h}}$$

We then may specify the cost minimization problem as:

$$\min C = C_o n + C_h \frac{x}{2}$$

s.t.

$$nx = A$$

Substitute for n in the maximand and solve the unconstrained problem:

$$\min \frac{C_o A}{x} + C_h \frac{x}{2}$$

which has a solution:

$$x = \sqrt{\frac{2C_o A}{C_h}}$$

and:

$$C = \sqrt{2C_o A C_h}$$

The Lagrange multiplier on the constraint equals the marginal cost of an additional unit of demand, hence:

$$\lambda = \frac{\partial C}{\partial A} = \sqrt{\frac{C_o C_h}{2A}}$$

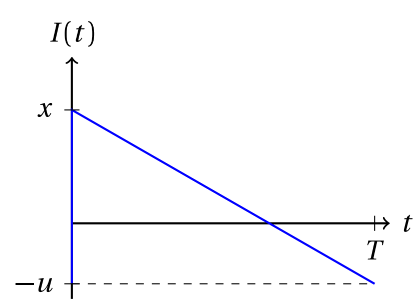
b. Suppose there is a third type of inventory cost: a *penalty cost* for unfilled orders. This cost did not appear above because the firm was never out of inventory. Suppose, however that the firm orders not when inventory is zero but rather when unfilled orders reach a certain level U, at which time all unfilled orders are filled. The cost of one unfilled order is C_p. Show that the optimal levels of x and U are given by:

$$x^* = \frac{C^*}{C_h}$$

and

$$u^* = \frac{C^*}{C_p}$$

where C^* is the minimum cost function which you should obtain explicitly.



Assume that when u orders are unfilled, the inventory is replenished to a level of x. This means that $x+u$ units are ordered n times per year to satisfy a total demand of A. Therefore:

$$A = n(x+u)$$

The average inventory level over an order cycle is given by:

$$\bar{x} = \frac{x}{x+u}\frac{x}{2}$$

and likewise, the average number of backorders stored over an order cycle is given by:

$$\bar{u} = \frac{u}{x+u}\frac{u}{2}.$$

Substituting for n in the objective function, we obtain an equivalent unconstrained optimization problem in two variables:

$$\min C(x,u) = \frac{C_o A}{x+u} + \frac{C_h x^2}{2(x+u)} + \frac{C_p u^2}{2(x+u)}$$

The first order conditions for x and u reduce to

$$x = \frac{C(x,u)}{C_h}$$

and

$$u = \frac{C(x,u)}{C_p}$$

.

Substituting into the objective, we have:

$$C = \sqrt{\frac{2C_o A}{\frac{1}{C_h}+\frac{1}{C_p}}}$$