AAE 706: More Expected Utility

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Key Ideas in Expected Utility

Contingent Commodities If either of two events A or B could happen, we define one contingent commodity as consumption if A happens and another if B happens.

State-Specific Budget Constraints In each state we need to define a budget constraint that correctly specifies the set of contingent commodity bundles that a consumer can afford.

von Neumann-Morgenstern utility A consumer's willingness to take various gambles and his willingness to buy insurance will be determined by how he feels about various combinations of contingent commodities.

Monotone Transformation

Any monotone transformations of a utility function does not change underlying preferences. For the purpose of understanding how the degree of relative risk aversion incluces utility it is helpful to add $1 - 1/\rho$ to the utility function, so:

$$U(C) = 1 + \frac{C^{\rho} - 1}{\rho}$$

where ρ is defined by the degree of relative risk aversion, $\sigma : \rho = 1 - 1/\sigma$. Risk-aversion requires $\sigma \geq 0$.

Marginal utility is then defined as:

$$MU(C) = C^{\rho - 1}$$

Calibrated Iso-Elastic Utility

When we have:

$$U(C) = 1 + \frac{C^\rho - 1}{\rho}$$

so

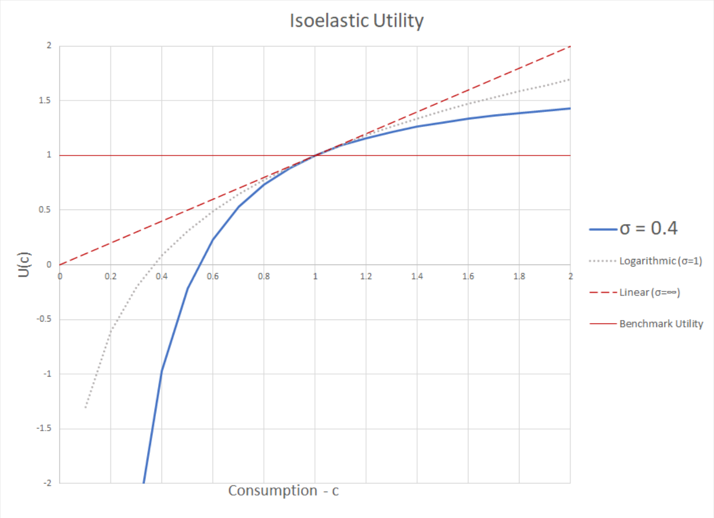
$$MU(C) = C^{\rho-1}$$

This function is *calibrated* so that for any value of σ:

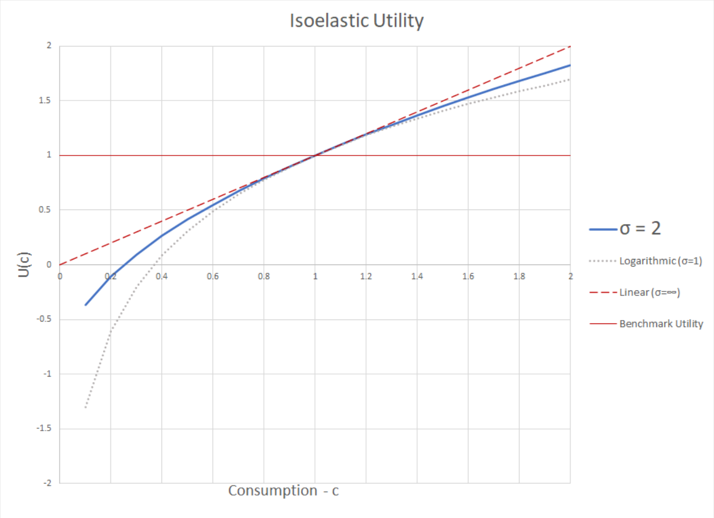
$U(1) = 1$ one unit of consumption provides one unit of utility, and

$MU(1) = 1$ a marginal increase of consumption provides a proportional increase in utility

$\sigma = 0.4$ is highly risk-averse



$\sigma = 2$ is slightly risk-averse



Iso-elastic Expected Utility

Returning to the insurance example from last class, we have:

$$EU(c_a, c_{na}) = \pi U(c_a) + (1-\pi)U(c_n a)$$

in which

$$U(c) = \frac{c^\rho}{\rho}$$

In the absence of insurance expected utility is

$$\overline{EU} = \pi U(M-L) + (1-\pi)U(M)$$

Suppose that we introduce insurance or make another policy change, resulting in expected utility $\tilde{EU} \neq \overline{EU}$. How can we describe this change in expected utility to policy makers? One could be tempted to write:

$$\%\Delta EU = 100 \times \left(\frac{\tilde{EU}}{\overline{EU}} - 1\right)$$

but this can be misleading, as our audience does not know what is meant by utiles. It would be much easier if we could translate the change in expected utility into monetary units.

Translating ΔEU to Hicksian EV

Consider a thought experiment: in the absence of insurance or any other policy intervention consumer income (= consumption) is multiplied by λ. We can then find the value of λ such that expected utility matches $\tilde{EU}$, i.e.

$$EU(\lambda(M-L), M) = \pi U(\lambda(M-L)) + (1-\pi)U(\lambda M) = \tilde{EU}$$

The calculated value of λ is the *Hicksian equivalent variation* in income – the fractional increase in consumption in a reference equilibrium represented by the outcome of a given scenario.
Substituting for $U(.)$ we find

$$\pi \frac{(\lambda(M-L))^{\rho}}{\rho} + (1-\pi)\frac{(\lambda M)^{\rho}}{\rho} = \tilde{EU}$$

or

$$\lambda = \left(\frac{\tilde{EU}}{\overline{EU}}\right)^{1/\rho} \Rightarrow \mathrm{HEV\%} = 100 \times (\lambda - 1)$$

Money-Metric Utility

Given calculated value of λ we can make a monotone transformation of EU so that expected utility measures equivalent variation directly.
Consider the function:

$$\hat{E}U(c) = \bar{c}\left(\sum_s \pi_s \left(\frac{c_s}{\bar{c}}\right)^{\rho}\right)^{1/\rho}$$

in which s measures states of nature and $\bar{c}$ measures reference consumption. In the insurance example we have $\bar{c} = M$ and use a reference equilibrium in which $L = 0$.
The Hicksian Equivalent change in income comparing consumption vectors $\tilde{c}$ and $\bar{c}$ i then

$$HEV\$ = \bar{c}\left[\left(\sum_s \pi_s \left(\frac{c_s}{\bar{c}}\right)^{\rho}\right)^{1/\rho} - 1\right]$$

EU Application: Imperfect Insurance

A consumer with unit income faces the risk of an accident, $\pi = 0.01$, which will reduce income by $L = 0.5$. The consumer is an expected utility maximizer with iso-elastic preferences and a degree of relative risk aversion, $\sigma = 0.9$, i.e.

$$U(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

States of the World

There are then two states of the world, good (no accident) and bad (accident). States of the world are indexed by $s \in \{G, B\}$, and expected utility is given by:

$$EU(c_G, c_B) = \pi \frac{c_B^\rho}{\rho} + (1 - \pi) \frac{c_G^\rho}{\rho}$$

in which we let $\rho = 1 - 1/\sigma$.

Insurance and Consumption

The consumer has the opportunity to buy insurance with a premium of γ. A payment of γK in each period covers a payment of K on bad days. The budget constraint on good days is then:

$$C_G = 1 - \gamma K$$

and on bad days the budget constraint is

$$C_B = 1 - L + K \times (1 - \gamma)$$

Profitability of Insurance

The profit earned by the insurance company is given by:

$$\Pi = (\gamma - \pi)K$$

in γK is the daily premium revenue and πK is the expected cost of claims on bad days.

Marginal Utility

Let p_s denote the price (marginal utility) of consumption in state s. The consumer demand for consumption in good and bad states of the world depends on expected income:

$$M = p_G + p_B \times (1 - L)$$

N.B. From the consumer's standpoint, insurance breaks even if:

$$\gamma(p_B + p_G) = p_B$$

whereas from the insurance firm's standpoint, the transaction is profitable if:

$$\gamma \geq \pi$$

Define the price index for a unit of consumption as:

$$p_C = \left(\pi \left(\frac{p_B}{\pi} \right)^{1-\sigma} + (1-\pi) \left(\frac{p_G}{1-\pi} \right)^{1-\sigma} \right)^{1/(1-\sigma)},$$

Then consumption in state G is then given by:

$$C_G = \frac{M}{P_C^{1-\sigma}} \left(\frac{p_G}{1-\pi} \right)^{-\sigma}$$

and consumption in the bad state is given by:

$$C_B = \frac{M}{P_C^{1-\sigma}} \left(\frac{p_B}{\pi} \right)^{-\sigma}$$

Upper Bounds on p_B/p_G and γ

In the absence of insurance, the relative price of utility in the bad state relative to the good state is derived from:

$$\frac{C_G}{C_B} = \frac{1}{1-L} = \left(\frac{p_G\ \pi}{p_B(1-\pi)}\right)^{-\sigma}$$

Hence, in the absence of insurance the price of utility in the bad state relative to the good state is maximized:

$$p_B = p_G \frac{1-\pi}{\pi} (1-L)^{-1/\sigma}$$

and this places an upper bound on on the premium if insurance markets are active:

$$\gamma \leq \frac{\pi}{\pi + (1-\pi)(1-L)^{1/\sigma}}$$

The constraint on γ is never binding, but it turns out to be an essential constraint for successfully computing Cournot-Nash equilibria.

Cournot Competition in Provision of Insurance

The insurance market will be imperfectly competitive if firm entry is limited and firms can price above cost. For simplicity assume there are n symmetric Cournot firms. Symmetry assures that the firms each write policies with the same coverage. If we assume Cournot-Nash behavior, the insurance provided by one such firm solves:

$$\max_k \Pi_f(k) = (\gamma(K) - \pi)k$$

where $\frac{n-1}{n}\bar{K}$ designates the output of other firms (taken as given), $k = K - \frac{n-1}{n}\bar{K}$ is the coverage provided by a representative firm, and $\gamma(K)$ indicates the value of γ consistent with insurance demand K.

Caculation of Nash Equilibrium via Diagonalization

Given a fixed number of firms[1], n, we can compute the symmetric Cournot-Nash equilibrium recursively with the following algorithm:

- Initialize $\bar{K} = L$, $\delta = +\infty$
- Repeat until $|\delta| < \mathrm{tolerance}$
    1. Solve $\max_k \Pi_f(k)$
    2. Calculate deviation, $\delta = \bar{K} - K$
    3. Update $\bar{K} = K$

[1] In a more complicated model the number of firms could be determined endogenously.

Profiting from Misinformation

Suppose that a *monopoly insurer* is able to sponsor an advertising campaign which convinces consumers that the risk of a bad state is $\tilde{\pi} > \pi$. We can assume that the cost of the advertising campaign increases with the square of the difference between the values:

$$C(\tilde{\pi}) = 0.002\left(\tilde{\pi}/\pi - 1\right)^2$$

In this week's homework you are asked to evaluate the economic impact of the disinformation strategy by modifying either the Excel or GAMS model accordingly and assessing the impact.

GAMS Code: Input Data

```
$title  Nash Equilibrium Insurance

*       Declare some parameters with assigned values (inputs to the analysis)

parameter       pi              Consumer subjective estimate of accident /0.01/,
                L               Loss with a bad outcome /0.5/,
                sigma           Degree of relative risk aversion /0.9/,
                nfirm           Number of firms /1/;

*       GAMS is not case sensitivity, but we following the
*       convention that parameters (exogenous inputs) are
*       in lower case and variables (endogenous outputs) are
*       written in upper case (except for "L").
```

GAMS Code: Declare and Assign Parameters

```
parameter       rho             Primal exponent corresponding to SIGMA ,
                gammamax        Maximum value for GAMMA;

*       The primal-form risk exponent is related
rho = 1 - 1/sigma;

*       The no-insurance outcome determines the maximum amount which can be
*       charged for coverage:

gammamax = pi * (1-L)**(-1/sigma) / ( pi * (1-L)**(-1/sigma) + 1 - pi )
display gammamax;
```

GAMS Code: Declare Variables and Equations

```
variables       P_G     Price index for consumption on a good day,
                P_B     Price index for consumption on a bad day,
                C_G     Consumption in the good day,
                C_B     Consumption in the bad day,
                P_C     Consumption price index,
                M       Income,
                K       Coverage
                GAMMA   Premium for coverage
                PROFIT  Firm profit;

nonnegative variables P_G, P_B, P_C, GAMMA, K;
```

GAMS Code: Define Equations

```
income..        M =E= P_G + P_B*(1-L);

P_Cdef..        P_C =e= ( (1-pi) * (P_G/(1-pi))**(1-sigma) + pi * (P_B/pi)**(1-sigma) )**(1/(1-sigma));

C_Gdef..        C_G =e= (M/P_C * (P_C*(1-pi)/P_G)**sigma);

C_Bdef..        C_B =e= (M/P_C * (P_C*pi/P_B)**sigma);

market_G..      1 =e=  C_G + GAMMA*K;

market_B..      1 - L + K =e= C_B + GAMMA*K ;

demand_K..      GAMMA*(P_G+P_B) =e= P_B;

parameter       kvalue          Lagged value of insurance provision /0/;

profitdef..     PROFIT =e= (GAMMA - pi) * (K - kvalue*(nfirm-1)/nfirm);
```

GAMS Code: Declare MCP and NLP Models

```
*       Declare the model as an equilbrium problem -- this can only be solved with
*       a fixed value of GAMMA:

model insurance /income.M, P_Cdef.P_C,
                market_g.P_G, market_b.P_B,
                C_Gdef, C_Bdef,
                demand_K.K, profitdef.PROFIT/;

*       Declare the model without associating equations and variables so that it can be solved
*       as an optimization problem with an endogenous value of GAMMA:

model nash /all/;
```

GAMS Code: Assign Initial Values and Bounds

```
P_C.L = 1;
P_G.L = 1-pi;
P_B.L = pi;
M.L = 1-pi + pi*(1-L);
P_C.LO = 1e-5;
P_G.LO = 1e-5;
P_B.LO = 1e-5;
P_G.FX = 1-pi;
C_G.L = (1-pi) + pi*(1-L);
C_B.L = (1-pi) + pi*(1-L);
GAMMA.FX = pi;
K.L = (C_B.L-(1-L))/(1-pi);
GAMMA.L = pi;
```

GAMS Code: Replicate the Competive Equilibrium

```
insurance.iterlim = 0;
solve insurance using mcp;
abort$round(insurance.objval,6) "Benchmark replication fails.";
```

GAMS Code: Declare Parameters for Model Output

```
parameter       dev             Deviation from Nash equilibrium,
                iterlog         Iteration log for diagonalization,
                equil           Equilibrium values;

*       Save values from the competitive equilibrium:

equil("Competitive","K") = K.L;
equil("Competitive","K/L") = K.L/L;
equil("Competitive","GAMMA") = GAMMA.L;
equil("Competitive","PROFIT") = PROFIT.L/pi;
equil("Competitive","P_B/pi") = P_B.L/pi;
equil("Competitive","C_G") = C_G;
equil("Competitive","C_B") = C_B;

set     n       Number of symmetric insurance companies /1*15/,
        iter    Nash iterations for diagonalization /1*25/;

GAMMA.LO = 0;
GAMMA.UP = gammamax;
```

GAMS Code: Loop Firm Number of and Iterations

```
loop(n,
        dev = 1;
        nfirm = n.val;
        kvalue = K.L;
        loop(iter$round(dev,5),
          GAMMA.L = gammamax/2;
          solve nash using nlp maximizing PROFIT;

          iterlog(n,iter,"dev") = dev;
          iterlog(n,iter,"K") = K.L;
          iterlog(n,iter,"GAMMA") = GAMMA.L;
          iterlog(n,iter,"PROFIT") = PROFIT.L/pi;
          iterlog(n,iter,"P_B/pi") = P_B.L /pi;
          dev = abs(kvalue - K.L);
          kvalue = K.L;
        );
        equil(n,"dev") = dev;
        equil(n,"K") = K.L;
        equil(n,"K/L") = K.L/L;
        equil(n,"GAMMA") = GAMMA.L;
        equil(n,"PROFIT") = PROFIT.L/pi;
        equil(n,"P_B/pi") = P_B.L/pi;
$ondotl
        equil(n,"C_G") = C_G;
        equil(n,"C_B") = C_B;
);
```

GAMS Code: Display Output and Output to Excel

```
option iterlog:3:2:1;
display iterlog, equil;

*       On Windows the data can be written directly to Excel:

execute_unload 'NashInsurance.gdx',iterlog, equil;

*       Exit at this point if we are not on a Windows computer.

$if not %system.filesys%==MSNT $exit

*       If we are on Windows, move the data into an Excel file.

$onecho >gdxxrw.rsp
par=iterlog rng=PivotData!a2 cdim=0 intastext=n
par=equil rng=Equil!a2 cdim=0 intastext=n
$offecho

pexecute 'gdxxrw i=NashInsurance.gdx o=NashInsurance.xlsx @gdxxrw.rsp';
```

The Model in Excel

Parameter	Symbol	Value
Subjective probability of bad outcome	pi	0.01
Loss with a bad outcome	L	0.5
Elasticity	sigma	1.1
Firm accident risk	pi0	0.01
Lagged value of K	k0	0.374853134
Maximum value of GAMMA	gammamax	0.018615202
Number of firms	nfirm	4

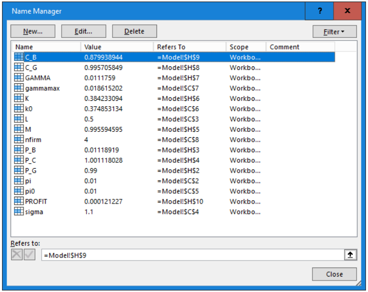
Variable	Symbol	Equilibrium Values	Starting Point
Consumption on a good day	P_G	0.99	0.9
Consumption on a bad day	P_B	0.01118919	0.0
Consumption price index	P_C	1.001118028	
Income	M	0.995594595	0.99
Coverage	K	0.384233094	0.
Premium	GAMMA	0.0111759	0.0
Consumption in the good state	C_G	0.995705849	0.99
Consumption in the bad state	C_B	0.879938944	0.99
Firm profit	PROFIT	0.000121227	=(GAMMA - pi0) * (K - k0*(nfirm-1)/nfirm)

Equations	LHS Value	RHS Value	Balance	Equation
Consumption Market in Good State	1	1	3.2269E-10	1 = C_G + GAMMA*K
Consumption Market in Bad STATe	0.884233094	0.884233094	1.972E-10	1 - L + K = C_B + GAMMA*K
Income	0.995594595	0.995594595	0	M = P_G + P_B*(1-L)
Consumption Price	1.001118028	1.001118028	1.6244E-10	P_C = ((1-pi)*(P_G/(1-pi))^(1-sigma)+pi*(P_B/pi)^(1-sigma))^(1/(1-sigma))
Demand in Good State	0.995705849	0.995705849	-4.219E-14	C_G = (M/P_C * (P_C*(1-pi)/P_G)^sigma)
Demand in Bad State	0.879938944	0.879938974	-3.032E-08	C_B = (M/P_C * (P_C*pi/P_B)^sigma)
Arbitrage Condition for Insurance	0.01118919	0.01118919	3.7551E-12	GAMMA*(P_G+P_B) = P_B
			0	

IterSolve

Deviation: 8.79837E-05

Named Ranges Improve Readability



Name	Value	Refers To	Scope	Comment
C_B	0.879938944	=Model!$H$9	Workbo...	
C_G	0.995705849	=Model!$H$8	Workbo...	
GAMMA	0.0111759	=Model!$H$7	Workbo...	
gammamax	0.018615202	=Model!$C$7	Workbo...	
K	0.384233094	=Model!$H$6	Workbo...	
k0	0.374853134	=Model!$C$6	Workbo...	
L	0.5	=Model!$C$3	Workbo...	
M	0.995594595	=Model!$H$5	Workbo...	
nfirm	4	=Model!$C$8	Workbo...	
P_B	0.01118919	=Model!$H$3	Workbo...	
P_C	1.001118028	=Model!$H$4	Workbo...	
P_G	0.99	=Model!$H$2	Workbo...	
pi	0.01	=Model!$C$2	Workbo...	
pi0	0.01	=Model!$C$5	Workbo...	
PROFIT	0.000121227	=Model!$H$10	Workbo...	
sigma	1.1	=Model!$C$4	Workbo...	

Excel Macro Updates $\bar{K}$ and Solves Model

NashInsurance.xlsm - Module1 (Code)

(General) | IterSolve

```
Sub IterSolve()

'  Update value of k0 and solve the profit maximization model

    Range("k0").Value = Range("K").Value
    SolverReset
    SolverOK SetCell:=Range("PROFIT"), MaxMinVal:=1, ByChange:=Range("$H$3:$H$9")
    SolverAdd cellref:=Range("$D$15:$D$20"), relation:=2, formulaText:=0
    SolverAdd cellref:=Range("GAMMA"), relation:=1, formulaText:=Range("gammamax")
    SolverSolve UserFinish:=False
End Sub
```

Still... Excel is challenging. Bear in mind that this speadsheet solves the model with user-guidance for a given value of n whereas the GAMS code solves the model without user-input for values of n from 1 to 15.

Digression: Discrete Time Markov Chains

A Markov (or Markoff) chain in discrete time descries a process which undergoes transitions from one state s in period t to another state s' in $t+1$. The state-contingent probability distribution of the next state s' depends only on the current state s and not on the sequence of events that preceded it.

Audrey Markov

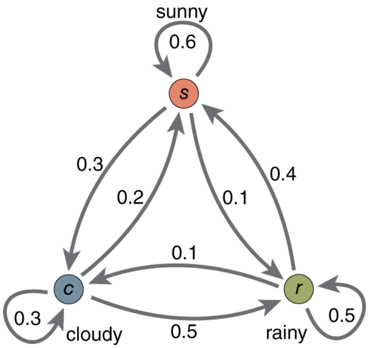
Markov was a Russian mathematician (1856 to 1922) best known for his work on stochastic processes:

Memoryless Property

Loosely speaking, a process satisfies the Markov property if one can make predictions for the future of the process based solely on itst present state just as well as one could knowing the process's full history. i.e., conditional on the present state of the system, its future and past are independent.

An Example

Consider a model in which there are three weather states $s = \{\text{cloudy}, \text{rainy}, \text{sunny}\}$. The likelihood of weather state s in day $t+1$ depends on the weather state in day t, as summarized here:



We can express the *steady-state* equilibrium conditions which relate the probability of weather state s in the subsequent period as a function of the probability of weather state s' in the current period. When the current and future probilities are equivalent, we the Markov chain has achieved a steady-state:

$$
\begin{aligned}
p_S &= 0.4p_R + 0.2p_C + 0.6p_S \\
p_C &= 0.1p_R + 0.3p_C + 0.3p_S \\
p_R &= 0.5p_R + 0.5p_C + 0.1p_S
\end{aligned}
$$

where $\sum_s p_s = 1$

Transition Probabilities

The changes of state of the system are called *transitions*. The probabilities associated with various state changes are called *transition probabilities*. A Markov process is characterized by a state space (s), a transition matrix describing the probabilities of particular transitions ($\pi_{ss'}$), and an initial state (or initial distribution) across the state space (π_s^0). In the previous example we have a tansition probaility matrix:

$$\pi = \begin{pmatrix} 0.4 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.1 \end{pmatrix}$$

By convention, we assume all possible states and transitions have been included in the definition of the process, so there is always a next state, and the process does not terminate. This implies that entries in the transition matrix are nonnegative and the column sums equal unity.

GAMS code: scalar version

```
$title  Solve for Steady-State in a Discrete Time Markov Chain

variables       P_r    Probability of rain
                P_c    Probability of clouds
                P_s    Probability of sun;

equations       p_sum, p_r_def, p_c_def, p_s_def;

*       We look for steady-state probabilitites across
*       states, and these must sum to unity:

p_sum..                P_r + P_c + P_s =e= 1;

*       Declare some parameters to use as "switches" so that
*       we can exclude one equilibrium condition:

parameter       pdef_r  Include steady-state condition for P_r /1/
                pdef_c  Include steady-state condition for P_c /1/
                pdef_s  Include steady-state condition for P_s /1/;

p_r_def$pdef_r..       P_r =e= 0.4 * P_r + 0.2 * P_c + 0.6  * P_s;

p_c_def$pdef_c..       P_c =e= 0.1 * P_r + 0.3 * P_c + 0.3  * P_s;

p_s_def$pdef_s..       P_s =e= 0.5 * P_r + 0.5 * P_c + 0.1  * P_s;

model markov /all/;
```

GAMS code: scalar version (cont.)

We have four equations and three variables. The MCP equation solver in GAMS requires that we provide a balanced system with an equal number of equations and variables. Here we demonstrate that the computed solution is the same regardless of which transition equation is omitted.

```
*       First, solve for steady-state excluding condition p_r_def:

pdef_r = no;
solve markov using mcp;
pdef_r = yes;

parameter       solution        Set of solutions to  the model;
solution("r","Omit_R") = P_r.l;
solution("c","Omit_R") = P_c.l;
solution("s","Omit_R") = P_s.l;

*       Then solve for steady-state excluding condition p_c_def:

pdef_c = no;
solve markov using mcp;
pdef_c = yes;

solution("r","Omit_C") = P_r.l;
solution("c","Omit_C") = P_c.l;
solution("s","Omit_C") = P_s.l;
```

GAMS code: scalar version (cont.)

```
*       Finally solve for steady-state excluding condition p_s_def:

pdef_s = no;
solve markov using mcp;
pdef_s = yes;

solution("r","Omit_S") = P_r.l;
solution("c","Omit_S") = P_c.l;
solution("s","Omit_S") = P_s.l;

display solution;
```

Could be Bergen, Norway:[2]

```
----     48 PARAMETER solution  Set of solutions to  the model

       Omit_R      Omit_C      Omit_S
r       0.429       0.429       0.429
c       0.214       0.214       0.214
s       0.357       0.357       0.357
```

[2] At bus stop in Bergen, it is raining and you ask a kid sitting there "How long has it been raining?" His reply: "I don't know. I'm only 7 years old."

GAMS Code: Vector Syntax

```
$title  Steady-State in a Discrete Time Markov Chain

set     s       States /rain, sun, clouds/;

alias (s,ss);

table   pi(s,ss)        Transition probabilities

                rain    clouds  sun
        rain    0.4     0.2     0.6
        clouds  0.1     0.3     0.3
        sun     0.5     0.5     0.1;

variables       P(s)    Probability of state;

equations       p_sum, p_def;

*       Probabilities sum to unity:

p_sum..                sum(s, P(s)) =e= 1;

*       Include switches to exclude one equilibrium condition:

parameter       pdef(s) Include steady-state condition;

p_def(s)$pdef(s)..     P(s) =e= sum(ss, pi(s,ss)*P(ss));

model markov /all/;
```

GAMS Code: Alternative Normalizations

```
parameter       solution        Comparison of solution across alternative normalizations;


alias (s,sd);
loop(sd,
        pdef(s) = yes$(not sameas(s,sd));
        solve markov using mcp;

        solution(s,sd) = P.L(s);
);

display solution;
```

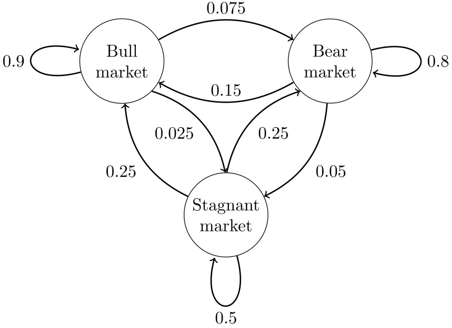
```
----     44 PARAMETER solution  Comparison of solution across alternative normalizations

              rain         sun      clouds

rain         0.429       0.429       0.429
sun          0.357       0.357       0.357
clouds       0.214       0.214       0.214
```

Another Example

Consider a model in which there are three market states $s = \{\text{bull}, \text{bear}, \text{stagnant}\}$. The likelihood of market state s in month $t+1$ depends on the market state in month t, as summarized here:



Steady-State Equilibrium

N.B. We have four equations in three unknowns! One of the first three equations is redundant. We need to drop an equation in order to have a valid model.

$$
\begin{array}{lcrcrcr}
P_{Bull} & = & 0.900P_{Bull} & + & 0.15P_{Bear} & + & 0.25P_{Stagnant} \\
P_{Bear} & = & 0.075P_{Bull} & + & 0.80P_{Bear} & + & 0.25P_{Stagnant} \\
P_{Stagnant} & = & 0.025P_{Bull} & + & 0.05P_{Bear} & + & 0.50P_{Stagnant} \\
1 & = & P_{Bull} & + & P_{Bear} & + & P_{Stagnant}
\end{array}
$$

GAMS Code: Business Cycle Example

```
$title  Steady-State in a Discrete Time Markov Chain

set     s       States /bull, bear, stagnant/;

alias (s,ss);

table   pi(s,ss)        Transition probabilities
                        bull    bear    stagnant
        bull            0.900   0.15    0.25
        bear            0.075   0.80    0.25
        stagnant        0.025   0.05    0.50;

variables       P(s)    Probability of state;

equations       p_sum, p_def;

p_sum..                sum(s, P(s)) =e= 1;

parameter       sseq(s) Include steady-state condition;

p_def(s)$sseq(s)..     P(s) =e= sum(ss, pi(s,ss)*P(ss));

model markov /all/;
```

Optimal Betting

γ	Pay out per dollar
M	Aggregate budget for betting and consumption
π	Subjective probability
x	Size of the wager ($x < M$)
$U()$	Utility function

Choice problem:

$$\max_{x} EU(x) = \pi U(M + \gamma x) + (1 - \pi) U(M - x)$$

With isoelastic utility:

$$U(c) = \frac{c^{\rho}}{\rho}$$

where $\rho = 1 - 1/\sigma$.

First order condition for x

$$\pi MU(M + \gamma x) = (1 - \pi) MU(M - x)$$

or

$$M + \gamma x = \left(\frac{1 - \pi}{\pi \gamma}\right)^{1/(\rho - 1)} (M - x)$$

Betting Fraction

Letting $\xi = \frac{x}{M} \quad (0 \leq x < 1)$, and noting that $\rho - 1 = -1/\sigma$, we obtain:

$$\xi = \begin{cases} \frac{\left(\frac{\pi\gamma}{1-\pi}\right)^{1/\sigma} - 1}{\left(\frac{\pi\gamma}{1-\pi}\right)^{1/\sigma} + \gamma} & \pi\gamma > 1 - \pi \\ \\ 0 & \pi\gamma \leq 1 - \pi \end{cases}$$

Calibration

Under certain circumstances observed choices reveal attitudes toward risk. In order to uncover the value of σ we need to know M, x, γ and π. It is not so difficult to guess or estimate values of the first three of these, but the subjective probability is somewhat problematic. If $x > 0$ the model requires that:

$$\pi \geq \frac{1}{1+\gamma},$$

i.e. the expected return has to be positive.
If, however, we are given values of γ, $\xi > 0$, π, we can find the degree of relative risk aversion by solving for σ using the demand function for ξ on the previous slide.

$$\sigma = \frac{\log(\pi\gamma) - \log(1-\gamma)}{\log(1+\gamma\xi)}$$

Sample Application: Betting

You are thinking of betting on whether the Chiefs or the Buccaneers will with the Super Bowl this year. A local gambler will bet with you at odds of 10 to 1 against the Chiefs. You think the probability that the Chiefs will win is $p = 0.2$. If you don't bet, you are certain to have \$1,000 to spend on consumption goods. Your behavior satisfies the expected utility hypothesis and your von Neumann-Morgenstern utility function.

Contingent Commodities

The contingent commodities are dollars if the Chiefs with the game and dollars if the Buccaneers win. Let c_C be your consumption contingent on the Chiefs winning and c_B be your consumption contingent on a Buccaneers win. Betting on the Chiefs at odds of 10 to 1 means that if you bet \$$x$ on the Chiefs, then if the Chiefs win, you make a net gain of \$$10x$, but if they don't, you have a net loss of \$$x$.

The Budget Constraint

Since you had \$1,000 before betting, if you bet \$$x$ on the Chiefs and they won you would have $c_C = 1,000 + 10x$ to spend on consumption. If you bet \$$x$ on the Chiefs and they didn't win, you would lose \$$x$, and you would have $c_B = 1,000 - x$.

By increasing the amount \$$x$ that you bet, you can make c_C larger and c_B smaller. We can use the above two equations to solve for a budget equation. From the second equation, we have

$$x = 1,000 - c_B \Rightarrow c_C + 10c_B = 11,000,$$

or equivalently,

$$0.1c_C + c_B = 1,100$$

von Neumann-Morgenstern Utility

Given the subjective probability of a Chiefs victory equal to $\pi = 0.2$ and a utliity function $U(c) = \sqrt{c}$, you will choose your contingent consumption bundle (c_C, c_B) to

$$\max U(c_C, c_B) = 0.2\sqrt{c_C} + .8\sqrt{c_B}$$

subject to the budget constraint,

$$0.1c_C + c_B = 1,100$$

Find a point on the budget line where the marginal rate of substitution equals the slope of the budget constraint.

$$MRS = \frac{0.2\sqrt{c_B}}{0.8\sqrt{c_C}} = 0.1 \Rightarrow c_B = 0.16c_C$$

This equation, together with the budget equation implies that $c_C = \$4,230.77$ and $c_B = \$676.92$. You achieve this bundle by betting \$323.08 on the Chieves. If the Chiefs win you will have $\$1,000 + 10 \times 323.08 = \$4,230.80$. If not, you will have \$676.92 (rounded to the nearest penny.)

The St. Petersburg Paradox

Consider the game: flip a coin repeatedly until a head is obtained for the first time; receive the reward $\$(2^n)$ if the first head is obtained on the n^{th} toss. What is the maximum willingness-to-pay (WTP) of individuals to play this game?

Empirically, it is typically found that individuals have a WTP for the game that is positive but <u>finite</u>. Yet, the expected value of the reward is:

$$E(\text{reward}) = \sum_{n\geq 1} 2^n (1/2)^n = 1 + 1 + \ldots = \infty.$$

Thus, if individuals behaved in a way consistent with the maximization of expected rewards, their WTP for the game should be <u>infinite</u>. The fact that the WTP is finite indicates that people <u>typically do not behave as if they maximized the expected value of rewards</u> (Bernoulli, 18th century).

In general, most people are "risk averse" as they see exposure to risk as having a negative impact on their welfare.

The Expected Utility Hypothesis

People make decisions on the basis of the expected utility of rewards.

Notation: Consider a choice among alternative risky rewards: a_1, a_2, a_3, ... (e.g. alternative monetary rewards). Each a_i is a random variable with a given subjective probability distribution.

Let $a_1 \sim^* a_2$ denote indifference between a_1 and a_2,
$a_1 \geq^* a_2$ denote that a_2 is not preferred to a_1,
$a_1 >^* a_2$ denote that a_1 is preferred to a_2.

Under the expected utility hypothesis, choices among the a_i's are made so as to maximize EU(a), where E is the expectation operator and U(.) is a utility function representing preferences.

But is the Expected Utility Hypothesis reasonable? And how do we know that U(a) exists?

Existence of the Utility Function $U(a)$

Assumptions:

A1- (ordering and transitivity)

. For any a_1 and a_2, exactly one of the following must hold:

$$a_1 >^* a_2,\ a_2 >^* a_1, \text{ or } a_1 \sim^* a_2.$$

. If $a_1 \geq^* a_2$ and $a_2 \geq^* a_3$, then $a_1 \geq^* a_3$. (transitivity).

A2- (independence)

For any a_1, a_2, a_3 and any α $(0 < \alpha < 1)$, then $a_1 \leq^* a_2$ if and only if

$$[\alpha\, a_1 + (1-\alpha)a_3] \leq^* [\alpha\, a_2 + (1-\alpha)\, a_3].$$

(the preferences between a_1 and a_2 are independent of a_3).

1

Existence of the Utility Function (cont.)

A3- (continuity)

For any a_1, a_2, a_3 where $a_1 <^* a_3 <^* a_2$, then there exists numbers (α, β), $0 < \alpha < 1$, $0 < \beta < 1$, such that

$$a_3 <^* [\alpha\, a_2 + (1-\alpha)\, a_1] \text{ and } a_3 >^* [\beta\, a_2 + (1-\beta)\, a_1].$$

A4- For any a_1, a_2 satisfying $Pr[a_1\!: a_1 \leq^* r] = Pr[a_2\!: a_2 \geq^* r] = 1$ for some r, then $a_2 \geq^* a_1$.

A5- . If there exists an integer m_0 such that $\{a_1 \mid a_1 \geq^* s_m\} \geq^* a_2$, $s_m \geq^* s_{m+1} \geq^* s_{m+2} \ldots$, for every $m \geq m_0$, then $a_1 \geq^* a_2$.

. If there exists an integer n_0 such that $\{a_1 \mid a_1 \leq^* t_n\} \leq^* a_2$, $t_n \leq^* t_{n+1} \leq^* t_{n+2} \ldots$, for every $n \geq n_0$, then $a_1 \leq^* a_2$.

The Expected Utility Theorem

Theorem: Under assumptions A1-A5, for any a_1, a_2, there exists a utility function U(a) such that

$a_1 \geq^* a_2$ if and only if $EU(a_1) \geq EU(a_2)$,

where U(a) is a utility function defined up to a positive linear transformation.

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Note: $EU(a_i) = \Sigma_j U(a_{ij}) Pr(a_{ij})$ (in the discrete case)

Thus, under the EUH, the objective function of the decision maker is linear in the probabilities. This follows from the independence assumption (A2).

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Note: Assumptions A4-A5 are rather technical: they are made to guarantee that EU(·) is "measurable" when distributions are unbounded... Assumptions A1-A3 are the more crucial ones: they are the ones that tend to be questioned if the predictions from the EUH are found to be inconsistent with actual behavior under risk.

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Note: If U(a) is a utility function, then the positive linear transformation $W(a) = \alpha + \beta U(a)$ is a utility function exhibiting the same preferences as U(a) for any $\beta > 0$.

Proof: $a_1 \geq^* a_2$ iff $EU(a_1) \geq EU(a_2)$ iff $\alpha + \beta EU(a_1) \geq \alpha + \beta EU(a_2)$ iff $EW(a_1) \geq EW(a_2)$.

Note: The above discussion assumes that EU(a) exists (i.e., that $EU(a) < \infty$).

Direct Elicitation of Preferences

1- Case of monetary rewards where a = scalar: (a = $)

Assume: $a_L \leq a \leq a_U$,

$U(a)$ = a strictly increasing function.

a/ Questionnaire Design: Find the answer to the following questions:

1. Find the reward a_1 obtained with certainty which is regarded by the person as equivalent to the lottery:

   $\{a_L$ with probability 1/2; a_U with probability 1/2$\}$.
2. Find the reward a_2 obtained with certainty which is regarded as being equivalent to the lottery:

   $\{a_1$ with probability 1/2; a_U with probability 1/2$\}$.
3. Find the reward a_3 obtained with certainty which is regarded as being equivalent to the lottery:

   $\{a_1$ with probability 1/2; a_L with probability 1/2$\}$.
4. etc...

Direct Elicitation (cont.)

b/ Finding U(a) from the questionnaire results:

. Choose $U(a_L) = 0$ and $U(a_U) = 1$ (because U(a) is defined up to a positive linear transformation).

. From "1.", $a_1 \sim^* [a_L$ with probability 1/2; a_U with probability 1/2]. Under the EUH, this implies that $U(a_1) = 1/2\ U(a_L) + 1/2\ U(a_U) = .5$.

. From "2.", $a_2 \sim^* [a_1$ with probability 1/2; a_U with probability 1/2]. Under the EUH, this implies that $U(a_2) = 1/2\ U(a_1) + 1/2\ U(a_U) = .75$.

. From "3.", $a_3 \sim^* [a_1$ with probability 1/2; a_L with probability 1/2]. Under the EUH, this implies that $U(a_3) = 1/2\ U(a_1) + 1/2\ U(a_L) = .25$.

. etc...

. Plot U(a) and draw a line through the points to obtain an estimate of the utility function of the individual.

This shows that, under the expected utility model, the utility function of an individual is empirically tractable.

Direct Elicitation (cont.)

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This shows that, under the expected utility model, the utility function of an individual is empirically tractable.

Note: The method can be modified using lotteries with probabilities other that 1/2, or lotteries comparing risky outcomes.

Note: Empirical validation of the method can be done by asking additional questions. For example, find the reward A obtained with certainty and regarded as being equivalent to the lottery {a_2 with probability 1/2; a_3 with probability 1/2}. Under the EUH, it follows that $U(A) = 1/2\ U(a_2) + 1/2\ U(a_3) = 1/2$. Thus, $U(A) = U(a_1) = 1/2$. U(a) being strictly increasing, this implies $A = a_1$. If $A \neq a_1$, then:

. either there is need to iterate the questions until $A = a_1$ (assuming that the EUH holds);

. or the behavior of the individual is not consistent with the EUH.

Multidimensional Utility

2- Multidimensional Case: U(x) where $x = (x_1, x_2, ...)$ = multidimensional vector (e.g. x_1 = money, x_2 = leisure, etc...).

The procedure discussed in "1-" can be used changing one dimension at a time, the other dimensions being held constant. But this gets complicated for dimensions greater than two.

Addititve Utility

2- Multidimensional Case: $U(x)$ where $x = (x_1, x_2, ...)$ = multidimensional vector (e.g. x_1 = money, x_2 = leisure, etc...).

The procedure discussed in "1-" can be used changing one dimension at a time, the other dimensions being held constant. But this gets complicated for dimensions greater than two.

The Case of Additive Utility: $U(x) = \Sigma_i k_i U_i(x_i)$, $0 \leq U_i(x_i) \leq 1$, $0 \leq k_i \leq 1$, $\Sigma_i k_i = 1$.

a/ Use the procedure in "1-" to estimate each $U_i(x_i)$, $i = 1, 2, ...$

b/ Estimate k_i:

Let x_i^+ = most preferred level of x_i with $U_i(x_i^+) = 1$,

x_i^- = least preferred level of x_i with $U_i(x_i^-) = 0$,

for all $i = 1, 2, ...$

Using a questionnaire, find the probability p_1 such that the person is indifferent between $\{(x_1^+, x_2^-, x_3^-, ...)$ with certainty$\}$ and $\{(x_1^+, x_2^+, x_3^+, ...)$ with probability p_1; $(x_1^-, x_2^-, x_3^-, ...)$ with probability $(1-p_1)\}$. Under the EUH, this implies

$U(x_1^+,x_2^-,x_3^-,..) = p_1\, U(x_1^+,x_2^+,x_3^+,..) + (1-p_1)\, U(x_1^-,x_2^-,x_3^-,..)$

or

$k_1 = p_1[k_1 + k_2 + ...] + (1-p_1)\,[0]$

or

$k_1 = p_1$.

Repeat the procedure with $p_2, p_3, ...$ to estimate $k_2, k_3, ...$