AAE 706: Expected Utility, Risk Aversion and EVPI

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Last class, we began by discussing the Allias paradox which identifies circumstances in which economic decisions seemingly violate the expected utility hypothesis. Recognizing the model's limitations, we carried on in the expected utility setting with Sandmo's model of producer decisions in the face of price uncertainty. At the end of class a model was introduced which considered the interaction of price risk in the "gig economy" in which price risk can be addressed through variation in short-term employee contracts. Today, we begin by review our understanding of the risk premium in a little Excel workbook, and then we see how this can be measured in the gig economy model. Finally we look at a second stochastic decision model: the newsboy problem.

The Simplest Expected Utility Model

Initial wealth is w. There are two earnings outcomes: 1 and 2. The agent earns a_1 with probability $p_1 = p$, or earn $a_2 = 1.5$ with probability $p_2 = 1 - p$. We choose a_2 such the expected earnings equal unity:

$$p \times a_1 + (1 - p) \times 1.5 = 1 \Rightarrow a_1 = \frac{1.5p - 0.5}{p}$$

Ending wealth (x_i in state i) depends on initial wealth and earnings (a_i):

$$x_i = w + a_i$$

CRRA Preferences

In the constant relative risk aversion (CRRA) model we can define utility as a isoelastic function of x:[1]

$$u(x) = \frac{1}{\rho}\left(\left(\frac{x}{\bar{x}}\right)^{\rho} - 1\right) + \bar{x}$$

This function is calibrated so that $u(\bar{x}) = \bar{x}$ and $u'(x)\big|_{x=\bar{x}} = 1$ where $x = w + a$ and $\bar{x} = w + 1$ is the reference value of terminal wealth.

[1] In case $\rho = 0$, the function takes the logarithmic form: $u(x) = \log(x/\bar{x}) + w + 1$

Money Metric Expected Utility

The expected utility model then follows from the state probabilities:

$$EU(x_1, x_2) = p_1 u(x_1) + p_2 u(x_2)$$

When working with applications of this model, it is helpful to apply a montonic transformation of EU so that we have a money-metric form:

$$EU_{mm}(x_1, x_2) = \bar{x}\left(p_1\left(\frac{x_1}{\bar{x}}\right)^{\rho} + p_2\left(\frac{x_2}{\bar{x}}\right)^{\rho}\right)^{1/\rho}$$

Both functions $EU(\bar{x}, \bar{x}) = 1$ and $EU_{mm}(\bar{x}, \bar{x}) = 1$. Furthermore, for any values of x_1 and x_2 the marginal rate of substitution is identical:

$$\begin{aligned} MRS &= \frac{\partial EU/\partial x_1}{\partial EU/\partial x_2} \\ &= \frac{p_1}{p_2}\left(\frac{x_2}{x_1}\right)^{\rho-1} \\ &= \frac{\partial EU_{mm}/\partial x_1}{\partial EU_{mm}/\partial x_2} \end{aligned}$$

The Risk Premium

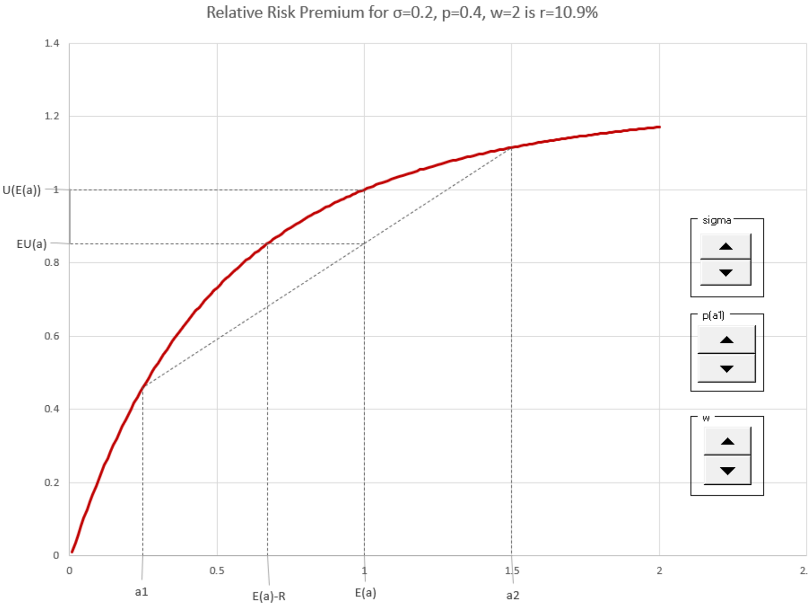
We have constructed the model such that the expected value of terminal wealth is $w + 1$. The risk premium depends on initial wealth, the coefficient of relative risk aversion $(1/\sigma)$, earnings in the low income state (a_1). The risk premium is then:

$$R = \bar{x} - EU_{mm} = \bar{x}\left[1 - \left(p_1\left(\frac{x_1}{\bar{x}}\right)^\rho + p_2\left(\frac{x_2}{\bar{x}}\right)^\rho\right)^{1/\rho}\right]$$

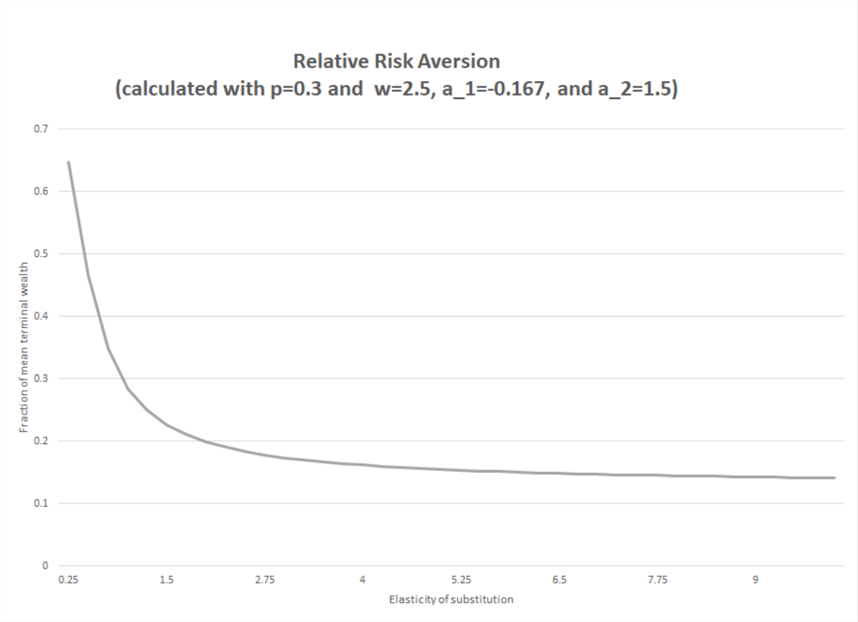
The relative risk premium is

$$r = R/\bar{x} = 1 - \left(p_1\left(\frac{x_1}{\bar{x}}\right)^\rho + p_2\left(\frac{x_2}{\bar{x}}\right)^\rho\right)^{1/\rho}$$

Expected Utility in Excel



Risk Aversion and the CRRA



What Is the *Gig Economy*?

In a gig economy, temporary, flexible jobs are commonplace and companies tend to hire independent contractors and freelancers instead of full-time employees. A gig economy undermines the traditional economy of full-time workers who often focus on their career development.

Understanding the Gig Economy

In a gig economy, large numbers of people work in part-time or temporary positions or as independent contractors. The result of a gig economy is cheaper, more efficient services, such as Uber or Airbnb, for those willing to use them. People who don't use technological services such as the Internet may be left behind by the benefits of the gig economy. Cities tend to have the most highly developed services and are the most entrenched in the gig economy.

Professions in the Gig Economy

A wide variety of positions fall into the category of a gig. The work can range from driving for Lyft or delivering food to writing code or freelance articles. Adjunct and part-time professors, for example, are contracted employees as opposed to tenure-track or tenured professors. Colleges and universities can cut costs and match professors to their academic needs by hiring more adjunct and part-time professors.

Factors Driving the Gig Economy

America is well on its way to establishing a gig economy, and estimates show as much as a third of the working population is already in some gig capacity. Experts expect this working number to rise, as these types of positions facilitate independent contracting work, with many of them not requiring a freelancer to come into an office. Gig workers are much more likely to be part-time workers and to work from home.

Irrelevance of Location

Employers also have a wider range of applicants to choose from because they don't have to hire someone based on their proximity. Additionally, computers have developed to the point that they can either take the place of the jobs people previously had or allow people to work just as efficiently from home as they could in person.

Economic Motives

Employers Economic reasons also factor into the development of a gig economy. Employers who cannot afford to hire full-time employees to do all the work that needs to be done will often hire part-time or temporary employees to take care of busier times or specific projects.

Employees On the employee's side of the equation, people often find they need to move or take multiple positions to afford the lifestyle they want. It's also common to change careers many times throughout a lifetime, so the gig economy can be viewed as a reflection of this occurring on a large scale.

Impact of the Coronavirus Pandemic

During the coronavirus pandemic of 2020, the gig economy has experienced significant increases as gig workers have delivered necessities to home-bound consumers, and those whose jobs have been eliminated have turned to part-time and contract work for income. Employers will need to plan for changes to the world of work, including the gig economy, when the pandemic has ended.

Criticisms of the Gig Economy

Despite its benefits, there are some downsides to the gig economy. While not all employers are inclined to hire contracted employees, the gig economy trend can make it harder for full-time employees to develop in their careers since temporary employees are often cheaper to hire and more flexible in their availability. Workers who prefer a traditional career path and the stability and security that come with it are being crowded out in some industries.

Firm Decisions in the Gig Economy

Consider the market for a single commodity. Production involves inputs of entrepreneurial effort (R), capital (K), firm employees (E) and gig employee labor (L). The top level product function is Cobb-Douglas withing which we nest a constant elasticity of substitution aggregate of E and L.

$$Y_s = R^{1-\theta_K-\theta_\ell} \left(\frac{K}{\bar{K}}\right)^{\theta_K} \ell^{\theta_\ell}$$

in which

$$\ell = \left(\alpha \left(\frac{E}{\bar{E}}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{\bar{L}}\right)^{\gamma}\right)^{1/\gamma}$$

Entrepreneurial Effort and Returns to Scale

Assuming $R = 1$ commodity outputs are then solely a function of capital and labor. If we were to include entrepreneurial inputs as a decision variable we would write:

$$Y_s = \left(\frac{R}{1}\right)^{1-\theta_K-\theta_\ell} \left(\frac{K}{\bar{K}}\right)^{\theta_K} \ell_s^{\theta_\ell}$$

When we fix $R = 1$, the resource input can be suppressed. Output as a function of capital and labor then exhibits decreasing returns to scale, as evidenced by $\theta_K + \theta_\ell < 1$.

Firm Employees and Gig Employees

When the elasticity of substition in the CES nest $(\mathrm{ESUB} = \frac{1}{1-\gamma})$ is less than one, then E and L_s are gross complements. When $\mathrm{ESUB} > 1$, these labor inputs are gross substitutes. The value of ESUB plays a crucial role in determining whether changes in the cost of finding gig employees increases or decreases the number of permanent employees. Entrepreneurial, capital and employee inputs are the same in all states whereas gig employment varies by state. Calibrating to reference levels of employment $(\bar{E})$ and gig sector labor $(\bar{L})$ then index of labor inputs is given by:

$$\ell_s = \left(\alpha \left(\frac{E}{\bar{E}} \right)^{\gamma} + (1-\alpha) \left(\frac{L_s}{\bar{L}} \right)^{\gamma} \right)^{1/\gamma}$$

State-Contingent Output

When $E = \bar{E}$, $L_s = \bar{L}$ the index of labor inputs equals unity. In state s decisions K and L are fixed, and the only choice variable for the firm is L_s. Output in state s is then:

$$Y_s = \left(\frac{K}{\bar{K}}\right)^{\theta_K} \ell_s^{\theta_\ell}$$

Firm profit in state s depends on the output price, the cost of capital and employees (p_K and p_E) and the price of temporary labor (ω):

$$\pi_s = P_s Y_s - p_K K - p_E E - \omega_s L_s$$

Terminal Wealth

Terminal wealth in state s can be expressed as an index of entering wealth ($w = 1$), current profits (π_s) and long-run average profits ($\bar{\pi}$):

$$X_s = \frac{w + \pi_s}{w + \bar{\pi}}$$

Expected Utility

We assume that the firm owner is a risk averse expected utility maximizer. If the price of output in each state (P_s) is taken as exogenous, the firm is competitive and the optimal response can be calculated as a non-linear program:

$$\max_{K,E,L_s} EU = \left(\sum_s \frac{X_s^\rho}{|S|} \right)^{1/\rho}$$

GAMS Code for Model Inputs

```
$title    Gig Economy Model

parameter
        vref                Marginal value of profit at reference profit /2/
        piref               Reference profit calibrating sigma /0.25/

        esub                Elasticity of substitution E vs L /4/

        thetak              Value share capital /0.3/,
        thetaE              Value share of employees /0.3/,
        thetaL              Value share of hired labor /0.2/;

set     s                   States of the world /s1*s50/;

parameter
        p(s)                Output price in state s
        wage                Wage of gig workers
        sigma_p             Variance of p /0.5/;
```

Calibration of Risk Attitude

One simple way to characterize risk attitude is through the marginal utility of profit. When $X_s = 1$, the marginal value of an extra dollar of profit can be scaled to one by defining:

$$v(\pi) = \left(\frac{X(\pi)}{\bar{X}}\right)^{\rho-1} = \left(\frac{w+\pi}{w+\bar{\pi}}\right)^{-\sigma}$$

We let initial wealth equal the value of reference output ($\bar{y} = 1$) and assume a value for the marginal utility of provide with half the benchmark profit is given by: $\bar{v} = v(\bar{\pi}/2)$. Given these assumption we can compute

$$\sigma = \frac{\log(1+\bar{\pi}) - \log(1+\bar{\pi}/2)}{\log(\bar{v})}$$

GAMS Model Declarations

```
variables
        EU                Expected utility,
        K(s)              Capital stock,
        E(s)              Full time employees,

        KK                Capital stock (stochastic model)
        EE                Employment (stochastic model)

        L(s)              Gig employees

        X(s)              Ending wealth by state,
        PI(s)             Profit (or social surplus) by state,
        Y(s)              Supply = demand by state,
        EL(s)             Labor-employee nest;

equations       eudef, eldef, ydef, pidef, naK, naE, xdef;

model gigeconomy /all/;
```

GAMS Model Equation Definitions

```
eudef..          EU =e= sum(s, (1/card(s)) * X(s)**rho)**(1/rho);

eldef(s)..       EL(s) =e= (thetaE/(thetaE+thetaL) * (E(s)/e0)**gamma +

                            thetaL/(thetaE+thetaL) * (L(s)/L0)**gamma)**(1/g

ydef(s)..        Y(s) =e= (K(s)/k0)**thetak * EL(s)**(thetaE+thetaL);

pidef(s)..       PI(s) =e= p(s) * Y(s)  - (K(s) + E(s) + wage*L(s));

xdef(s)..        X(s) =e= (1+PI(s))/(1+pi0);

*    Include non-anticipativity constraints in the stochastic model:

naK(s)$stochastic..    K(s) =e= KK;

naE(s)$stochastic..    E(s) =e= EE;
```

Calibrate and Initialize

```
gamma = 1 - 1/esub;
pi0 = 1 - thetak - thetaE - thetaL;
sigma = (log(1+pi0)-log(1+pi0*piref))/log(vref);
rho = 1 - 1/sigma;

*    Assign initial values close to the reference equilibrium:

PI.L(s)   = pi0;
L.L(s)    = thetaL;
K.L(s)    = thetaK;
E.L(s)    = thetaE;
X.L(s) = 1;
Y.L(s) = 1; EL.L(s) = 1;

*    Avoid bad function calls:

X.LO(s)  = 0.001; L.LO(s)=0.01; K.LO(s) = 0.01; E.LO(s) = 0.01;
```

Solve the Stochastic Model

```
*       Prices are log-normally distributed:

p(s) = exp(sigma_p*normal(0,1));

*       Benchmark wage rate:

wage = 1;

stochastic = 1;
solve gigeconomy using nlp maximizing EU;
```

Create a Reporting “Subroutine”

```
parameter    results        Summary of results;

*       Generate a reporting "subroutine" which takes a single
*       argument (the scenario identifier):

$onechov    >%gams.scrdir%report.gms
results("%1",s,"Y") = Y.L(s);
results("%1",s,"EL") = EL.L(s);
results("%1",s,"L") = L.L(s)/thetaL;
results("%1",s,"K") = K.L(s)/thetaK;
results("%1",s,"E") = E.L(s)/thetaE;
results("%1",s,"P") = p(s);
results("%1",s,"PI") = PI.L(s);
results("%1",s,"wage") = wage;
$offecho

*       Call the reporting routine:

$batinclude %gams.scrdir%report ref_stochastic
```

Declare Some Additional Reporting Parameters

```
parameter       rp      Risk premium (%),
                EU_s    Expected utility in stochastic model,
                EU_d    Expected utility in the deterministic model,
                evpi    Expected value of perfect information;


rp = 100 * (sum(s, X.L(s))/card(s) / EU.L - 1);
EU_s = EU.L;
```

Solve the Deterministic Model and Report the EVPI

```
stochastic = 0;
solve gigeconomy using nlp maximizing EU;

$batinclude %gams.scrdir%report ref_deterministic

EU_d = sum(s,X.L(s))/card(s);
display EU_d, EU_s;

evpi = 100 * (EU_d / EU_s - 1);
display rp, evpi;
```

```
----     EU_d = 3.143  Expected utility in the deterministic model
         EU_s = 0.942  Expected utility in stochastic model

----     rp   = 14.517  Risk premium (%)
         evpi =233.711  Expected value of perfect information
```

Newsvendor Problem

The *Newsvendor Problem* is a classical example of optimization under uncertainty. It used to be called the Newsboy Problem but now has a gender-neutral name!

A person sells newspapers in the streets. Each morning she purchases a number of papers from the distributor, at a price of c_p. Each day, there is a demand, y, for papers, which is unknown[1] in the morning. Throughout the day, she sells as many papers as she can at a price of c_s. Of course, she can't sell any more papers than she purchased in the morning, but if she sells fewer, the remaing ones can be sold as fish wrapping paper at a price of c_f. In the morning, she has money for at most N papers.

Her main decision is: *How many papers, x, should I purchase?* We assume that each day looks the same in terms of the *probability distribution* of demand for papers, so that the probability that i papers are demanded on any given day is p_i, $i = 0, 1, 2, ..., \infty$.

What is the optimal number of papers, $\hat{x}$, to buy in order to

Maximize Profits!

Of course, there is no *fixed* number of papers that will maximize profits *every day*. Some days will be better than others. So she decides to

Maximize Expected Profits.

This is a problem she can formulate mathematically. First, she determines the connection between x, the number of papers purchased, and y, the number of papers demanded, and represents this as the stochastic profit function

$$F(x,y) = \begin{cases} c_s x - c_p x & \text{if} \quad x \le y \\ c_s y + c_f(x-y) - c_p x & \text{if} \quad x > y \end{cases}$$

With this setup-up, one possible formulation is as follows:

$$\begin{array}{ll} \underset{x \in \Re}{\text{Max}} & E_y F(x,y) \\ \text{S.t.} & 0 \le x \le N, \end{array}$$

where E_y is the expections operator with respect to the stochastic variable y.

Solution by straight calculation

The problem can easily be solved by a straigt-forward calculation: For each x on $0, ..., N$, calculate the expected profit

$$G(x) = E_y F(x, y) = \sum_{y=0}^{x} p_y F(x, y) \tag{1}$$

and pick the value $\hat{x}$ for which G is maximized.
There are also more "elegant" methods. But we're interested in formulating the problem as an optimization problem!

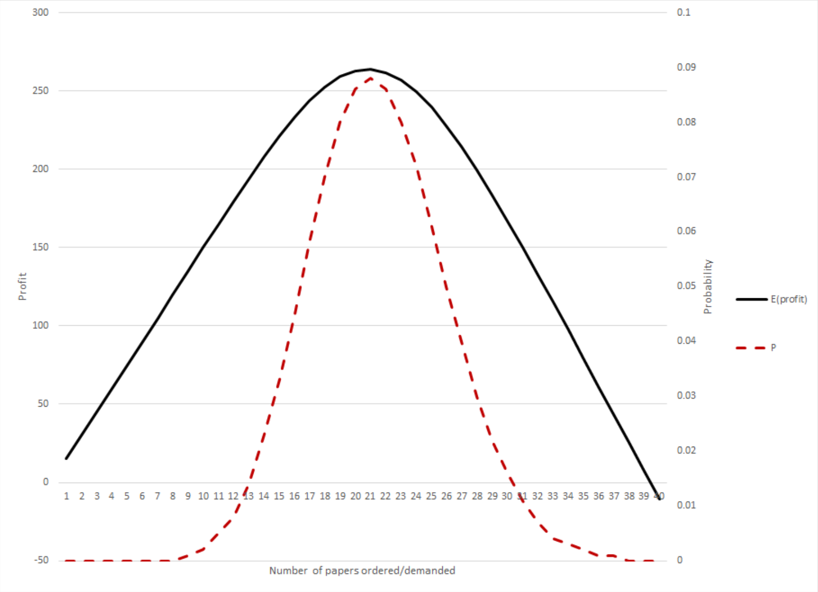
The GAMS Code

```
$title  Soren Nielsen's Newsvendor Model

set     i       Papers to be ordered /1*100/;

alias( i, j);

parameter p(i) "Probility that i papers will be demanded";

parameter lambda        Parameter of the truncated Poisson distribution / 2

p("1") = exp(-lambda) * lambda;

*       Make an assignment to the next value.  When i reaches 100, the
*       reference off the end of the set is ignored:

loop(i, p(i+1) = p(i) * lambda / i.val;);

*       Normalize so the p(i)'s sum to 1:

p(i) = p(i) / sum(j, p(j));
display p;
```

Brute Force Solution

```
parameter f(i,j) "Profit if i papers are purchased and j are demanded";

F(i,j) = c_s * min(i.val, j.val) + c_f * max(0, i.val-j.val) - c_p * i.val;
display F;

*       Brute force calculation of the optimal value:

parameter g(i) "Expected profit if i papers are purchased";
g(i)$(i.val<=N) = sum(j, p(j)*F(i,j));
display g;

*       Find the highest possible expected profit:

set             i_hat(i)        Papers orders for which profit is maximized
parameter       max_prof        The maximum profit;

max_prof = smax(i, g(i)); i_hat(i) = yes$(g(i)=max_prof);
display max_prof, i_hat;

---- max_prof = 263.765  (the maximum profit)
---- i_hat = 21          (Paper order for which profit is maximized)
```

Profit (left) and Demand Probability (right)



Optimization Model

Set up the newsvendor problem as a two-stage, stochastic program. *

The first-stage decision is: How many papers to purchase, represented by x.

The second-stage decision is: How many papers to sell, represented by z_s.

The index s is the "scenario", that is, how many papers are demanded.

Notice the two-stage decision process (two decision points):

1. Purchases in the morning, then wait and see how many are demanded/sold,
2. Decide how many papers to sell as fish wrapping, in this case a trivial decision.

Declare Variables and Constraints

```
alias  (i,s);

nonnegative variables
       X      "How many papers to purchase",
       Z(s)   "How many papers are sold";

*       First stage constraint is a simple upper bound -
*       the maximum number we can purchase is n:
X.UP = n;

*       Constraints on the second-stage variables.

EQUATION purchased(s)  Can't sell more than we have purchased;

purchased(s)..  Z(s) =L= X;

*       The number sold in state s cannot exceed the number
*       demanded in state s:

Z.UP(s) = s.val;
```

Declare Objective Function

```
*       Formulate the objective: expected profit.  Note that
*       we declare profit(s) as a free variable so that we may
*       have scenarios in which profits are negative.

*       The NLP objective (EXP_PROFIT) must be a free variable.

variables
        PROFIT(s)      "Profit under each scenario",
        EXP_PROFIT     "Expected profit of the whole day";

equations profit_def(s), exp_profit_def;

profit_def(s)..
        PROFIT(s) =e= c_s * Z(s) + c_f * (X-Z(s)) - c_p * X;

exp_profit_def..
        EXP_PROFIT =E= sum(s, p(s) * PROFIT(s) );

model two_stage /all/;

solve two_stage maximizing EXP_PROFIT using lp;
```