AAE 706

Nonlinear Optimization:

Theory and Application

(See Appendix B in Chavas)

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Optimization in Economics

- Optimization is central to economic analysis and efficiency. Economic rationality implies optimization.
- Each agent maximizes his/her objective function subject to constraints imposed by the economic environment. The objective function is a *utility function* representing the agent's preferences.
- The same model applies to household consumption decisions and firm production and investment decisions.
- Under uncertainty, the utility function reflects risk preferences.
- Key idea: economic decisions are maximization problems subject to feasibility constraints.

Preliminaries

Consider a function $f(x)$, where $x = (x_1, x_2, \dots x_n)$ is an n-vector of real numbers. This means that for each vector x, there exists a unique real number given by $f(x)$.
f is said to be *concave* if, for any x^a and x^b and any α, $0 \leq \alpha \leq 1$:

$$f(\alpha x^a + (1-\alpha)x^b) \geq \alpha f(x^a) + (1-\alpha)f(x^b)$$

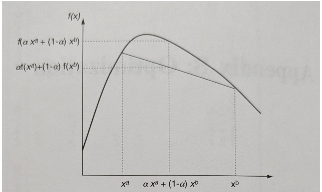


Figure B.1 A concave function

Multivariable Calculus

When the function $f(x)$ is differentiable, let $f'(x) = \partial f / \partial x$ denote the first derivative of f, and let $f''(x) = \partial^2 f / \partial dx^2$ denote the second derivative of f.

The first derivative $f'(x)$ measures the marginal value or local slope of the function at point x.

The second derivative $f''(x)$ reflects the marginal change in the slope at point x.

When f(x) is differentiable and n $= 1$, the function $f(x)$ is *concave* if and only if $f''(x) \leq 0 \quad \forall x$.

Concave and Differentiable

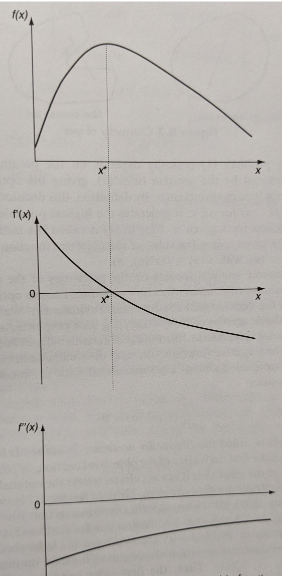


Figure B.2 A concave and differentiable function

Convex Sets

Consider a set of real numbers, denoted by X. X is convex if for every x^a and x^b in X and every number α, $0 \leq \alpha \leq 1$, the point $[\alpha x^a + (1-\alpha)x^b]$ is also in X

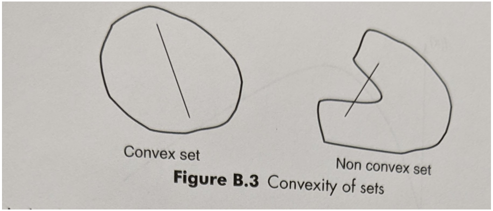


Figure B.3 Convexity of sets

For example, a polyhedron defined by $A \in \mathrm{R}^{m \times n}$ and $b \in \mathrm{R}^m$:

$$X = \{x \in \mathrm{R}^n : Ax \leq b\}$$

Unconstrained Optimization

Consider an economic agent facing the maximization problem:

$$V(\alpha) = \max_{x} f(x, \alpha)$$

where $f(x, \alpha)$ is the objective function, $x = (x_1, x_2, \ldots, x_n)$ is a vector of real numbers representing n decision variables, and $\alpha = (\alpha_1, \ldots, \alpha_m)$ is an m-vector of parameters.

Parameters are real values representing all variables that are not decision variables. They include variables representing the economic environment of the decision-maker.

Decision Rules

The maximization problem typically depends on parameters α. This solution is denoted by the *decisionrule*(α), giving the optimal decision for a given economic environment α.
By definition, this decision rule satisfies

$$f(x^*(\alpha), \alpha) \geq f(x, \alpha) \quad \forall x$$

$V(\alpha) = f(x^*(\alpha), \alpha)$ is the *indirect objective function*.

Economic analysis often focuses on the properties of the decision rule $x^*(\alpha)$ This decision rule summarizes how economic choices optimally adjust to changes in the decision-maker's economic environment.

Optimal Decision Rules – Scalar

The when $N = 1$ and we have a scalar optimal decision $x^*(\alpha)$, this satisfies:

$$\frac{\partial f(x^*(\alpha), \alpha)}{\partial x} = 0$$

This is the *first-order necessary condition*, since it involves the first derivative of the objective function, $\partial f / \partial x$ or f'.

Optimal Decision Rules – Multidimensional

When $N > 1$ under some regularity conditions the *n*-vector of optimal decision rules solve an *n* dimensional *system* of equations:

$$\begin{gathered} \partial f(x,\alpha)/\partial x_1 = 0 \\ \partial f(x,\alpha)/\partial x_2 = 0 \\ \dots \\ \partial f(x,\alpha)/\partial x_N = 0 \end{gathered}$$

These equations implicitly determine the vector of optimal decision rules $x^*(\alpha) = (x_1^*(\alpha), x_2^*(\alpha), \dots, x_n^*(\alpha))$.

Local Sensitivity Analysis

In many applications we wish to identify the *local* dependence of x_i on α, i.e. $\partial x_i/\partial\alpha|_{x=x^*}$. To find this value we can differentiate the optimality conditions:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \quad \frac{\partial f(x,\alpha)}{\partial x} = 0$$

or

$$\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}\alpha} + \frac{\partial^2 f}{\partial x_i \partial \alpha} = 0 \quad \forall i$$

Local Sensitivity Analysis (cont.)

This is a system of n equations in n unknowns:

$$Hdx = -g$$

where H is the $n \times n$ Hessian:

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and g is the gradient vector

$$g_i = \frac{\partial^2 f}{\partial x_i \partial \alpha}$$

which has the solution:

$$\frac{\mathrm{d}x_i}{\mathrm{d}\alpha} = -H^{-1}g = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^{-1} \left(\frac{\partial^2 f}{\partial x_i \partial \alpha}\right)$$

Canonical Example: Optimal Lot Size

A classical nonlinear programming problem in inventory theory is the *optimal lot size problem*. A firm's inventory of a homogeneous commodity, $I(t)$, is depleted at a constant rate per unit time. The firm reorders an amount x of the commodity, which is delivered immediately, whenever the level of inventory is zero.

The annual demand is A, and the firm orders the commodity n times a year where:

$$A = nx.$$

Optimal Lot Size (cont.)

- The firm incurs two types of inventory costs: a *holding cost* and an *ordering cost*. The average stock of inventory is $x/2$, and the cost of holding one unit of the commodity is C_h, so $C_h x/2$ is the holding cost.
- The firm orders the commodity, as stated above, n times a year, and the cost of placing one order is C_O so $C_O n$ is the ordering cost.
- The total cost is then:

$$C = C_h \frac{x}{2} + C_O n$$

Average Inventory Level

When we have linear demand and orders of magnitude X at time intervals T, the inventory at time $t : 0 \leq t \leq T$ is given by:

$$I(t) = x(1 - \frac{t}{T})$$

Hence, the average inventory holding is give by:

$$\bar{x} = \int_0^T I(t)dt = \frac{x}{2}$$

Optimal Lot Size Question 1

Minimize the cost of inventory, C, by choice of x and n subject to the constraint $A = nx$ using constraint substitution. Find the optimal lot size (optimal x) as a function of the parameters C_O, C_h and A.

Homework exercise for Wednesday

Constrained Optimization

A decision maker may face constraints from their economic environment, be they technological, legal, financial or institutional. Consider therefore a generalization of the optimization problem

$$V(\alpha) = \max_x \{f(x, \alpha) : h(x, \alpha) \geq 0\}$$

where $h(x, \alpha) = (h_1(x, \alpha), h_2(x, \alpha), \ldots, h_K(x, \alpha))$ are K functions representing K constraints facing the decision maker: $h_1(x, \alpha) \geq 0, \ldots, h_K(x, \alpha) \geq 0$.

Let $x^*(\alpha)$ denote the optimal decision rule. The value function is the indirect objective function satisfying $V(\alpha) = f(x^*(\alpha), \alpha)$

The Lagrangian: KKT Conditions

$$\mathcal{L}(x, \lambda, \alpha) = f(x, \alpha) + \sum_{k=1}^{K} \lambda_k h_k(x, \alpha)$$

- Primal optimality:

$$\frac{\partial \mathcal{L}(x^*, \lambda^*, \alpha)}{\partial x_i} = 0 \quad \forall i = 1, \ldots, N$$

- Primal feasibility

$$\frac{\partial \mathcal{L}(x^*, \lambda^*, \alpha)}{\partial \lambda} = h_k(x, \alpha) \geq 0 \quad \forall k = 1, \ldots, K$$

- Dual feasibility:

$$\lambda_k^*(\alpha) \geq 0 \quad \forall k = 1, \ldots, K$$

- Complementary slackness

$$\lambda_k^* \frac{\partial \mathcal{L}}{\partial \lambda_k} = \lambda_k^* h_k(x^*, \alpha) = 0 \quad \forall k = 1, \ldots, K$$

Optimal Lot Size Question 2

A third type of inventory cost is the *penalty cost* for unfilled orders. This cost did not appear above because the firm was never out of inventory. Suppose, however that the firm orders not when inventory is zero but rather when unfilled orders reach a certain level U, at which time all unfilled orders are filled. The cost of one unfilled order is C_p. Find the optimal levels of x and U.

Homework exercise for Wednesday.

Application to Consumer Choice

The problem of preference maximization can be written as

$$\max u(\mathbf{x})$$

such that

$$\mathbf{px} \leq m$$

$$x \geq 0$$

Relating to the previous discussion *parameters* of the consumer problem are prices and income: $\alpha = (\mathbf{p}, m)$ The **demand function** corresponding to $x^*(\alpha)$ is thus $\mathbf{x}(\mathbf{p}, m)$.

When preferences are *strictly convex*, there is a *unique* bundle which maximizes utility.

Indirect Utility

Provided that preferences are well behaved, the consumer's value-function can be written as:

$$v(\mathbf{p}, m) = \max u(\mathbf{x})$$

such that

$$\mathbf{px} = m$$

We refer to $v()$ as the *indirect utility function*.

The Expenditure Function

Provided that welfare is monotonically increasing in income, we can invert the *indirect utility* to obtain the *expenditure function*:

$$e(\mathbf{p}, u) = \min \mathbf{px}$$

such that

$$u(\mathbf{x}) \geq \bar{u}$$

The expenditure function relates the minimum cost of achieving a fixed level of utility ($\bar{u}$).

Money-Metric Utility

We define $m(\mathbf{p}, \mathbf{x})$ as follows:

$$m(\mathbf{p}, \mathbf{x}) \equiv e(\mathbf{p}, u(x))$$

This can be called a *money metric utility function*. $m(\mathbf{p}, \mathbf{x})$ is the minimum expenditure required when prices are p to achive the utility consistent with consumption bundle $\mathbf{x}$.
A dual form of this function is the *money metric indirect utility function*:

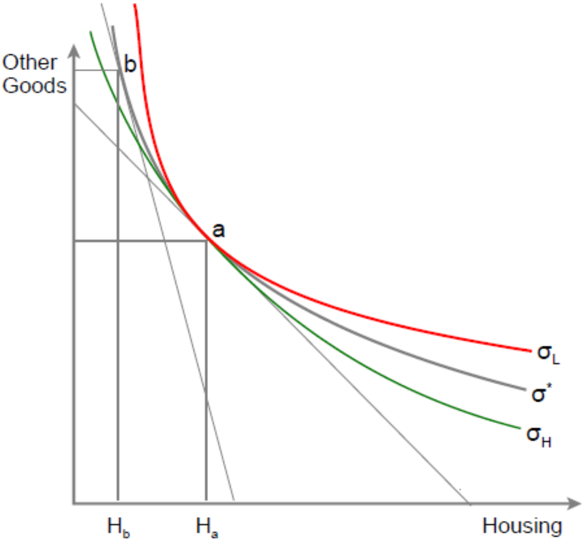
$$\mu(\mathbf{p}; \mathbf{q}, m) \equiv e(\mathbf{p}, v(\mathbf{q}, m)).$$

A Word Problem

> *Thomas lives in Ann Arbor where he currently spends 30% of his income on rent. He has an employment offer in Zürich which pays 50% more than he currently earns, but he is hesitant to take the job because rental rates in Zürich are three times higher than in Ann Arbor. Assuming that Thomas has CES preferences with elasticity of substitution* σ*; on purely economic grounds, should he move?*

As is the case for all interesting questions in economics, the only good answer to this problem is "It depends.".

Graphical Representation



Intuition

Thomas's offer in Zürich does not pay him enough to live exactly the lifestyle that he enjoys in Ann Arbor, as he would need a 60% raise to cover rent and consumption. The elasticity of substitution is key. If it is high, he more willing substitutes consumption of goods and services for housing and thereby lowers his cost of living in Zürich. On the other hand, if the elasticity is low, he is "stuck in his ways", and the move is a bad idea.

Preferences

The CES utility function:

$$U(C,H) = (\alpha C^{\rho} + (1-\alpha)H^{\rho})^{1/\rho}$$

Exponent ρ is defined by the elasticity of substitution, σ, as

$$\rho = 1 - 1/\sigma.$$

The model of consumer choice is:

$$\max U(C,H) \text{ s.t. } C + p_H H = M$$

Calibration to a Benchmark Equilibrium

We are given information about Thomas's choices in Ann Arbor. This information is essentially an observation of a *benchmark equilibrium*, consisting of the prevailing prices and quantities of goods demand. The benchmark equilibrium data together with assumptions about elasticities are used to evaluate Thomas's choices after a discrete change in the economic environment. The steps involved in solving this little textbook model are identical to those typically employed in applied general equilibrium analysis.

Hint: Assume that in Ann Arbor $p_H = 1$, $M = 1$, $\bar{C} = 0.7$ and $\bar{H} = 0.3$. If σ is given, what value of α is consistent with these choices?