Numerical Methods

Math 3338 – Spring 2022

Worksheet 13

Solutions to Non-Linear Equations

1 Reading

CP	6.3
NMEP	Chapter 4

Table 1: Sections Covered

2 Solving

Solve the following,

$$2x + 3 = 0$$

Now solve this,

$$x^2 - 3x - 4 = 0$$

How about,

$$x^5 - 4x^3 + 2x - 1 = 0$$

Finally,

$$e^x = x + 3$$

The question you should be asking is *do these actually have solutions?* and *how many solutions are there?*. The first two are easy. We know the third one has at least one real root, it's an odd degree polynomial. The fourth one probably has two roots, based on what we know about e^x. The real way to tell is to draw a graph. Figure 1 shows both of these functions.

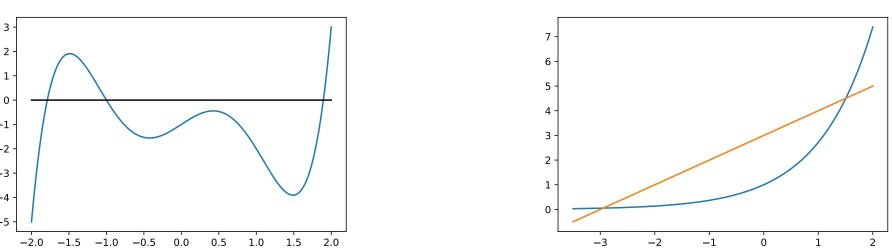

Figure 1: The graphs. Figure out which is which.

The graphs don't actually tell use the solutions. It may look like $x = -3$ is a solution to $e^x = x + 3$, but it clearly isn't.

3 The Bisection Method

Suppose $f(a) < 0$ and $f(b) > 0$. We can get closer to the zero by evaluating $x = \frac{b-a}{2}$ and computing $f(x)$. If $f(x) > 0$ then the zero is in the range $[a, x]$, if $f(x) < 0$ then it's in $[x, b]$ if $f(x) = 0$, then you should probably stop.

Since this method halves the width of the interval at each iteration, it will converge quite quickly.

4 Newton's Method

This is a much, much faster way to approximate the zero of a polynomial. The basic idea to approximate the curve by a line (tangent line) and solve that line. The tangent line should approximate the curve, so the zero should be close to the actual zero. The closer you start to the zero, the closer the approximation should be. You then iterate.

In general, Newton's method follows the recursion,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_0 is a point "close" to the zero.

You may have noticed a liberal use of the word "should". There is are several reasons for that, first we are dividing by $f'(x)$, what if this is zero? Second, if we choose a bad initial point, the tangent line could go far away from the zero. We won't concern ourselves with these issues... At least not today.

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Homework 12 (Due: Tuesday, March 1)

Problem 1 (1 pt) Write a program to compute a zero using the bisection method. Call this function `bisection(f,a,b,tol = 1e-6)` where f is the function, you are searching $a \leq x \leq b$ and *tol* is the tolerance you desire.

Problem 2 (1 pt) Write a function to compute a zero using Newton's method. Call this function `newton(f,fp,a,tol=1e-6)` where f is the function, fp is the derivative, a is the starting point and *tol* is the tolerance you desire.

Problem 3 (1 pt) Write a function to compute a zero using Newton's method, except this time we will numerically calculate the derivative. Call this function `newton_numeric(f,a,tol=1e-6)` where f is the function, a is the starting point and *tol* is the tolerance you desire. Compute the derivative numerically, you could be clever here and use the points x and x' from successive steps of Newton's method (it's not required to do it this way).

Problem 4 (1 pt) Compare computation times for each of your functions. Do do this, use the given table of functions with given inputs. Calculate to an accuracy of $1e-12$. Since these calculations will be very fast, run each one 10000 times and find the total time. In other words, start the timer, run 10000 times, and stop the timer. The total computation should take less than 1 minute (it took approximately 10 seconds on my laptop).

Function	x_0	a	b
$x^5 - 4x^3 + 2x - 1$	-2	-2	-1.5
$x^5 - 4x^3 + 2x - 1$	-1	-1.5	1
$x^5 - 4x^3 + 2x - 1$	2.5	1.5	3
$e^x = x + 3$	-3	-4	-2
$e^x = x + 3$	2	1	2