Numerical Methods

Math 3338 – Spring 2022

Worksheet 20

First Order Differential Equations

1 Reading

CP	8.1, 8.2
NMEP	Chapter 7

Table 1: Sections Covered

2 Overview

Suppose y is a function of x. We want to solve equations of the form,

$$\frac{dy}{dx} = f(x, y).$$

3 Euler's Method

This is the simplest numerical method. Here is the process,

1. Start with a point (x_0, y_0) and a step size h
2. $f(x_0, y_0)$ is the slope at the point (x_0, y_0), make the tangent line of the function at this point $y = f(x_0, y_0)(x - x_0) + y_0$.
3. Set $x_1 = x_0 + h$ and $y_1 = f(x_0, y_0)(x_1 - x_0) + y_0$.
4. Continue until you hit the upper limit of x-values.

This is an approximation technique. The smaller the step size, the better the approximation. It's error term is $\mathcal{O}(h^2)$ (this is derived in the book). However, this is a tricky bit here *each step* is accurate to $\mathcal{O}(h)$ and we are potential computing a ton of steps. The error will compound, you'll see this in the homework.

4 Runge-Kutta

We've already learned the first order Runge-Kutta method! It's the same as Euler's method. It's called "first order" because it is accurate to $\mathcal{O}(h)$ (in other words the error term is $\mathcal{O}(h^2)$).

4.1 Second order Runge-Kutta

This is accurate to $\mathcal{O}(h^2)$ with error terms $\mathcal{O}(h^3)$. The book has a full explanation for how this works. Very similar to Euler's method, this is an iterative technique except instead of approximating the slope at x, we approximate the slope at $x + \frac{1}{2}h$.

A single iteration is given by the calculations,

$$\begin{aligned} k_1 &= hf(x, y) \\ k_2 &= hf(x + \frac{1}{2}h, y + \frac{1}{2}k_1) \\ y_{new} &= y + k_2 \end{aligned}$$

where $f(x, y) = \frac{dy}{dx}$.

4.2 Fourth order Runge-Kutta

This is more complicated than the second order, but also far more complicated. It's accurate to $\mathcal{O}(h^4)$, which is amazing.

$$\begin{aligned}
k_1 &= hf(x, y) \\
k_2 &= hf(x + .5h, y + .5k_1) \\
k_3 &= hf(x + .5h, y + .5k_2) \\
k_4 &= hf(x + h, y + k_3) \\
y_{new} &= y + 1/6(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

5 Vector Fields

A vector field is a field of vectors. It tells you which direction of the slope at each point in the plane. These are easy to make in Python,

```
X,Y = np.meshgrid(np.linspace(xmin,xmax,Nx),np.linspace(ymin,ymax,Ny))
plt.quiver(X,Y,1,f(X,Y),color="lightgrey")
```

6 Mean Absolute Error

Suppose you have to vectors x and y. You can determine how "close" they are using the mean absolute error,

$$\frac{\sum_{i=1}^{n} |x_i - y_i|}{n}$$

This is a stats thing, feel free to Google it. It tends to be used quite often in Machine learning to as a measure of how close predictions are to data.

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Homework 20 (Due: Thursday, March 31)

Problem 1 (1 pt) Create three functions, `euler`, `runge_kutta2`, and `runge_kutta`. The inputs to each of these should be (f, x, y_0) where f represents $\frac{dy}{dx}$ so f is a bivariate function, x is an array containing x-values (this will include the step size, h) and y_0 is the initial y-value. They should return an array of y-values.

Problem 2 (1 pt) Consider the differential equation,

$$\frac{dy}{dx} = xy$$

This is separable differential equation, so we can actually solve this. The solution is $y = A \cdot \exp\left(\frac{x^2}{2}\right)$. Plot the following on a single graph,

1. Create a vector field of $\frac{dy}{dx}$ for $-2 \leq x, y \leq 2$. Make the color of the vector field light grey.
2. Plot $f(x) = A \cdot \exp\left(\frac{x^2}{2}\right)$ for $-2 \leq x \leq 2$. You'll need to solve for A. Label this curve "exact"
3. Plot three more curves, one for euler, runge_kutta2 and runga_kutta. Make the x-vector start at -2, end at 2 with 10 equally spaced points between (in other words, a linspace). Label these appropriately.
4. Include a legend.

Include this graph in a PDF along with a visual description of what you see occurring. Note, a visual description uses words. Calculate the mean absolute error of each as well and make a table.

Problem 3 (1 pt) Use the same differential equation from the previous problem. Put the following on a single graph

1. Create a vector field of $\frac{dy}{dx}$ for $-2 \leq x, y \leq 2$. Make the color of the vector field light grey.
2. Plot $f(x) = A \cdot \exp\left(\frac{x^2}{2}\right)$ for $-2 \leq x \leq 2$. You'll need to solve for A. Label this curve "exact"
3. Plot an approximation using Euler's method with 1,000 steps.
4. Plot an approximation using Runge_Kutta2 with 100 steps.
5. Plot an approximation using Runge_Kutta method with 10 steps.
6. Include a legend.

Include this graph in a PDF along with a visual description of what you see occurring. Note, a visual description uses words. Calculate the mean absolute error of each as well and make a table.

Problem 4 (1 pt) Use the same differential equation from the previous problem. Put the following on a single graph

1. Create a vector field of $\frac{dy}{dx}$ for $-2 \leq x, y \leq 2$. Make the color of the vector field light grey.
2. Plot $f(x) = A \cdot \exp\left(\frac{x^2}{2}\right)$ for $-2 \leq x \leq 2$. You'll need to solve for A. Label this curve "exact"
3. Plot an approximation using Euler's method with 10,000 steps.
4. Plot an approximation using Euler's method with 100 steps.
5. Plot an approximation using Euler's method with 10 steps.
6. Include a legend.

Include this graph in a PDF along with a visual description of what you see occurring. Note, a visual description uses words. Calculate the mean absolute error of each as well and make a table.