Numerical Methods

Math 3338 – Spring 2022

Worksheet 21

Systems of Differential Equations

1 Reading

CP	8.1, 8.2
NMEP	Chapter 7

Table 1: Sections Covered

2 Overview

Believe it or not, not everything depends on only one variable. In these problems it's going to be convenient to consider our independent variable as time, t. Suppose we have functions $x_i(t)$ for $1 \leq i \leq n$ and relations,

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n).$$

This creates a system of differential equations. We want to solve this system.

To solve this, we are going to modify this into a vector equation. Let $\vec{x}(t) = (x_1(t), \ldots, x_n(t))$ and $\vec{f} = (f_1, \ldots, f_n)$. Then, our system becomes

$$\frac{d\vec{x}}{dt} = f(t, \vec{x}).$$

Writing like this allows us to vectorize Runge-Kutta,

$$\begin{aligned}
\vec{k}_1 &= hf(t, \vec{x}) \\
\vec{k}_2 &= hf(t + .5h, \vec{x} + .5\vec{k}_1) \\
\vec{k}_3 &= hf(t + .5h, \vec{x} + .5\vec{k}_2) \\
\vec{k}_4 &= hf(t + h, \vec{x} + \vec{k}_3) \\
\vec{x}_{new} &= \vec{x} + 1/6(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4).
\end{aligned}$$

In other words, multidimensional Runge-Kutta is just Runge-Kutta in each dimension.

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Homework 21 (Due: Tuesday, April 5)

Problem 1 (1 pt) Modify your `runge_kutta` method from last time to handle multidimensional array inputs. In other words, `runge_kutta` should accept a function f that returns an array, and the input y_0 should be an array.

This function should use vectorization *and should be identical (or very similar) to your original function.* For example, when I wrote mine I changed two small things. And that's it.

Problem 2 (1 pt) The Lotka-Volterra equations are a mathematical model of predator-prey interactions between biological species. Let two variables x and y be proportional to the size of populations of two species, traditionally "rabbits" and "foxes".

In the Lotka-Volterra model the rabbits reproduce at a rate proportional to their population, but are eaten by foxes at a rate proportional to their own population and the population of foxes:

$$\frac{dx}{dt} = \alpha x - \beta xy,$$

where α and β are constants. At the same time foxes reproduce at a rate proportional to the rate at which they eat rabbits but also die of old age at a rate proportional to their own population:

$$\frac{dy}{dt} = \gamma xy - \delta y,$$

where γ and δ are constants.

1. Use the Runge-Kutta method for the case $\alpha = 1$, $\beta = \gamma = .5$, and $\delta = 2$, starting with the initial condition $x = y = 2$. Make a graph showing both x and y as a function of time on the same axes from $t = 0$ to $t = 30$. Be sure to label the graph.
2. Describe in words what is going on in the system, in terms of rabbits and foxes.

Problem 3 (1 pt) The Lorenz equations are:

$$\frac{dx}{dt} = \sigma(y - x), \qquad \frac{dy}{dt} = rx - y - xz, \qquad \frac{dz}{dt} = xy - bz$$

where σ, r and b are constants.

These equations were first studied by Edward Lorenz in 1963, who derived them from a simplified model of weather patterns. The reason for their fame is that they wer one of the first good examples of *deterministic chaos*, the occurrence of apparently random motion even though there is no randomness.

Solve the Lorenz equations for the case $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$ in the range $t = 0$ to $t = 50$ with initial conditions $(x, y, z) = (0, 1, 0)$. Use *at minimum* 10,000 steps.

1. Make a plot of y vs t. Note the unpredictable nature of the motion.
2. Make three more plots z vs x, y vs x and z vs y. Ensure the figsize of each is large so you can see detail.

Include all these in a PDF.