Numerical Methods

Math 3338 – Spring 2022

Worksheet 24

Fourier Transforms

1 Reading

CP	7.1, 7.3
NMEP	-

Table 1: Sections Covered

2 Complex Fourier Transform

There are many ways to represent a function. Perhaps the most useful is a series expansion,

$$f(x) = \sum \gamma_k g_k(x)$$

where the $g_k(x)$'s are "nice" in some way. Naturally this raises the question, what are the γ_k's and how do we find them?

If we choose the g_k's correctly, we'll have

$$\gamma_k = \int_a^b f(x)\bar{g}_k(x)\, dx$$

where $\bar{g}_k(x)$ is the complex conjugate of $g_k(x)$. This is a natural consequence of the $g_k(x)$'s forming an orthonormal basis of the space of functions. That's not really important right now, but it is in life.

Suppose our interval is $[a, b]$, set $L = b - a$ the width of the interval. Then the complex Fourier transform is given by[1],

$$g_k(x) = \frac{1}{\sqrt{L}} e^{k\frac{2\pi i x}{L}}$$

Then we'll be able to write $f(x)$ as,

$$f(x) = \sum_{k=-\infty}^{\infty} \gamma_k g_k(x)$$

Of course, an infinite number of coefficients may not be ideal from a computational point of view. So we restrict to be

$$f(x) = \sum_{k=-N}^{N} \gamma_k g_k(x)$$

Where N is an integer. You do want this to be symmetric if $f(x)$ is a real valued function.

The k's are frequencies and the γ_k is the amplitude at the given frequency.

[1] Typically the $\frac{1}{\sqrt{L}}$ is not included here but there is a $\frac{1}{L}$ later. This is all fine. But MATH wants it there. We'll drop it for good reason later though.

3 Real Fourier Transform

Most functions we deal with are real valued functions. Using the complex Fourier transform is a bit of overkill, so we'd like a real version. This exists. But keeping it real does occasionally lead to issues[2]

As we all know, $e^{ix} = \cos(x) + i\sin(x)$. This is essentially the idea to get the real Fourier transform. In fact, there are two. The cosine transform,

$$c_k(x) = \frac{1}{L}\cos(2k\pi x)$$

and the sine transform,

$$s_k(x) = \frac{1}{L}\sin(2k\pi x)$$

However, since cosine is an even function and function we use the cosine transform must be symmetric about the midpoint of the interval. Similarly, since sine is odd the function must be anti-symmetric about the midpoint. Luckily, these issues are not difficult to overcome, you just double the interval and modify the function. Figure 1 shows $f(x) = x$ on $[0, 2]$. For the cosine transform to work properly, we need to

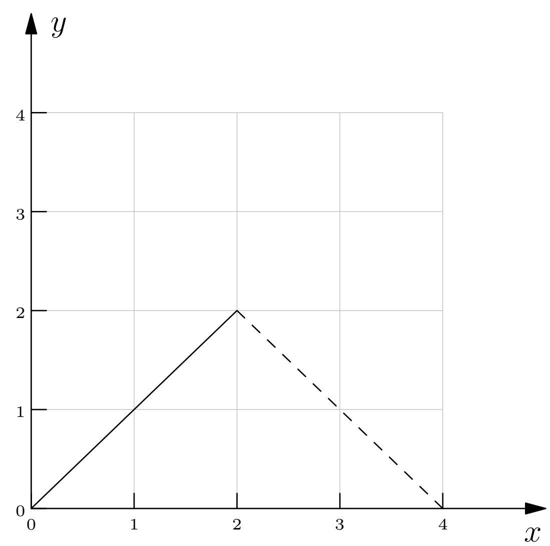

Figure 1: The solid line of $f(x)$, the dashed line gets included for the cosine transform

include the dashed line as well.

[2] `https://www.youtube.com/watch?v=pfz0tDQZhqs`.

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Homework 24 (Due: Thursday, April 14)

Include all graphs in your write up of the problems.

Problem 1 (1 pt) Write two functions `complex_fourier` and `inverse_complex_fourier`. Use the trapezoid method with 1000 subintervals for your integration technique.

For `complex_fourier` the inputs should be a function f, a, b and N. Return a $(2N+1) \times 2$ array where the first column is the frequency k, and the second column is γ_k.

For `inverse_complex_fourier` the inputs should be the array returned by the previous function, x, a, and b. It should return a number.

Problem 2 (1 pt) Let $f(x) = x$ on the interval $[-1, 1]$. Make a plot of $f(x)$ and the inverse fourier transform of $f(x)$ on the same plot for the following situations (make a plot for each situation).

1. $N = 10$
2. $N = 50$
3. $N = 100$

Describe what you see in each plot. Why do you think the graph looks so funky at the ends of the interval?

Problem 3 (1 pt) Let $f(x) = 1 - x^2$ on the interval $[-1, 1]$. Compute the Fourier transform of $f(x)$ using $N = 50$.

1. Make a plot of $f(x)$ and the inverse Fourier transform on the same plot. What happened here?
2. Make a plot of the inverse Fourier transform on the interval $[-5, 5]$. Why did this happen?
3. Make a plot of amplitude vs frequency for the Fourier transform. Be sure to take the absolute value of the amplitudes as the values are complex. This graphs shows you which frequencies are contributing the most to the graph. What is a rough range where the frequencies are "important"?

Problem 4 (1 pt) Write two functions `real_fourier` and `inverse_real_fourier`. Use the cosine transform and the trapezoid method with 1000 subintervals for your integration technique. This transformation will only work for intervals of the form $[a, b]$ with $a, b \geq 0$.

For `real_fourier` the inputs should be a function f, a, b and N. Return a $(N+1) \times 2$ array where the first column is the frequency k, and the second column is γ_k. There is one detail, for the $k = 0$ frequency, you need to divide the amplitude by 2, but only for that frequency[3] [4].

For `inverse_real_fourier` the inputs should be the array returned by the previous function, x, a, and b. It should return a number. For the inverse *you do not need the* $\frac{1}{L}$ *factor.*

Problem 5 (1 pt) Let $f(x) = \cos(6\pi x) + 2\cos(8\pi x) - 3\cos(10\pi x)$ on the interval $[0, 1]$. Let $N = 50$.

1. Make a plot of $f(x)$ and the real inverse Fourier transform on the same plot.
2. Make a plot of amplitude vs frequency for the Fourier transform (you don't need the absolute value). What do you see? Why did this happen?
3. Do the above two again on the interval $[0, 2]$. Does anything change?

[3] There is a reason for this. Try to figure it out.

[4] At the end of the day, it was the real fourier transform that was complex.

Problem 6 (1 pt) Let $f(x) = \cos(6\pi x) + 2\cos(8\pi x) - 3\cos(10\pi x)$ on the interval $[0, b]$. In this problem you'll be making plots of amplitude vs frequency for various values of b. Let $N = 50$.

1. $b = 1$
2. $b = 5$
3. $b = 10$
4. $b = 20$

Explain what you see and why you think it is happening.