

# ECON 736 Presentation

## Assortative Matching with Large Firms

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# Roadmap of Talk

## Introduction

## Model

- Model set-up

- Equilibrium

  - Characterization of Equilibrium

  - Assortativity Characterization

  - Equilibrium Assignment

## Simulation

- Simulation Strategy

- Simulation Results

## Appendix

# Introduction

# Motivation

# Research Questions

- **Research Question**

- Provide a unifying theory of production with a trade-off between hiring more vs better workers.

# Research Questions

- **Research Question**

- Provide a unifying theory of production with a trade-off between hiring more vs better workers.

- **Results**

- Sorting condition that captures the trade-off between quantity and quality of workers.
- Characterization of matching in equilibrium.
- When is matching assortative (**PAM**) or (**NAM**)?
- Under what conditions more productive firms hire more workers in equilibrium?

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# Model Setup: Demographics

- **Workers** indexed by *unidimensional* skill  $x \in [\underline{x}, \bar{x}] \subset \mathbb{R}_+$ 
  - CDF  $H^w(x)$  and PDF  $h^w$
- **Firms** indexed by *unidimensional* productivity  $y \in [\underline{y}, \bar{y}] \subset \mathbb{R}_+$ 
  - CDF  $H^f(x)$  and PDF  $h^f$



## Model Setup: Preferences

- Linear utility model.
- **Workers** care about their wage and there is no disutility of work.
- **Firms** maximize their profits.

## Model Setup: Production Function

- The output produced by a firm of type  $y$  that hires  $l$  workers of type  $x$  is:

$$F(x, y, l, r)$$

- $r$  the fraction of  $y$  resources dedicated to  $x$  type workers.
  - $(x, y)$  are quality variables and  $(l, r)$  are quantity variables.
- $F$  is strictly increasing and strictly concave in  $(l, r)$ , 0 resources produce 0 output, and standard Inada conditions apply.
- $F$  has constant returns to scale in  $l$  and  $r$ .

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- $F$  has constant returns to scale in  $l$  and  $r$ .
- We can write  $F$  in terms of **intensity**  $\theta = l/r$  :

$$f(x, y, \theta) := F(x, y, \theta, 1) \quad \implies \quad F(x, y, l, r) = rf(x, y, \theta)$$

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# Equilibrium

- The equilibrium concept used is the competitive equilibrium.

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- Firm's problem:
  - Distribution of workers hired by  $y$   $\mathcal{L}^y(x) = \int_{\underline{x}}^x l^y(\tilde{x}) dH^w(\tilde{x})$
  - Distribution of firm  $y$  resources  $\mathcal{R}^y(x) = \int_{\underline{x}}^x r^y(\tilde{x}) dH^w(\tilde{x})$

# Equilibrium

- The equilibrium concept used is the competitive equilibrium.

- **Firm's problem:**

- Distribution of workers hired by  $y$   $\mathcal{L}^y(x) = \int_{\underline{x}}^x l^y(\tilde{x}) dH^w(\tilde{x})$
- Distribution of firm  $y$  resources  $\mathcal{R}^y(x) = \int_{\underline{x}}^x r^y(\tilde{x}) dH^w(\tilde{x})$
- For any  $x \in [\underline{x}, \bar{x}]$   $l^y(x) = \theta^y(x) r^y(x)$  which means

$$\mathcal{L}^y(x) = \int_{\underline{x}}^x \theta^y(\tilde{x}) d\mathcal{R}^y(\tilde{x}). \quad (1)$$

- The total output of the firm can be written as:

$$\int_{\underline{x}}^{\bar{x}} f(x, y, \theta^y(x)) d\mathcal{R}^y(x) = \int_{\underline{x}}^{\bar{x}} F(x, y, l^y(x), r^y(x)) d\mathcal{H}^w(x)$$

- Firms maximize the difference between output produced and wages paid to workers.

## Feasible Demand

- Consider an interval of worker types  $(x', x]$
- The demand of firm  $y$  for shc workers is  $\mathcal{L}^y(x) - \mathcal{L}^y(x')$
- The total demand for those workers is the integral over all firms.
- This implies a way to evaluate if a labor demand schedule  $\{\mathcal{L}^y\}_{y \in \mathcal{Y}}$  is feasible:

$$\int_{\mathcal{Y}} [\mathcal{L}^y(x) - \mathcal{L}^y(x')] dH^f \leq H^w(x) - H^w(x') \quad \forall (x', x] \subseteq \mathcal{X}$$



## Equilibrium Definition

An equilibrium is a tuple of functions  $(w, \theta^y, \mathcal{R}^y, \mathcal{L}^y)$  consisting of a non-negative wage schedule  $w(x)$  as well as intensity functions  $\theta^y(x)$  and resource allocations  $\mathcal{R}^y(x)$  with associated feasible labor demands  $\mathcal{L}^y(x)$  (determined as in (1)) such that:

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- 1 **Optimality:** Given the wage schedule  $w(x)$ , for any  $y$ , the combination  $(\theta^y, \mathcal{R}^y)$  solves:

$$\max_{\theta^y, \mathcal{R}^y} \int [f(x, y, \theta^y(x)) - w(x)\theta^y(x)] d\mathcal{R}^y(x)$$

- 2 **Market Clearing:** For any  $(x', x] \subseteq \mathcal{X}$

$$\text{If } w(x) > 0 \text{ a.e in } (x', x] \quad \implies \quad \int_y [\mathcal{L}^y(x) - \mathcal{L}^y(x')] dH^f = H^w(x) - H^w(x')$$

# Equilibrium Characterization

- When do better firms hire **better** workers?
- How are wages determined?
- When do better firms employ **more** workers?
- How is that affected by technological change?

# Equilibrium Assortativity

## Definition (Assortative Matching)

- We say that matching between firms and workers is PAM (NAM) if higher type firms hire higher type workers, i.e.,  $y > y'$  then,  $x$  in the support of  $\mathcal{L}^y$  and  $x'$  in the support of  $\mathcal{L}^{y'}$  only if  $x \geq (\leq) x'$ .

# Equilibrium Characterization

## Proposition 1

- If output  $F$  is strictly increasing in  $x$  and  $y$  and the type distributions have nonzero continuous densities, then almost all active firm types  $y$  hire exactly one worker type and reach unique size  $l(y)$  in an assortative equilibrium.
- There is an injective matching function  $\mu : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ , between the subset of hired workers and active firms.

# Equilibrium Characterization

The proof have two parts:

- First we show that for every hired worker the combination  $(x, \theta^y(x))$  solves [Details](#)

$$(x, \theta^y(x)) \in \arg \max \{ f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta} w(\tilde{x}) \} \quad \forall x \in \text{supp} \mathcal{R}^y \quad (2)$$

- An implication is that in equilibrium if a worker is hired then all workers that are more productive must have strictly positive wages.

# Equilibrium Characterization

- Second, assume that a firm hires two different workers  $x' < x$ , if the equilibrium is **PAM** then that firm must be the only firm that hire workers in  $[x', x]$ .
- If there is only one firm active in  $[x', x]$  then the aggregate labor demand has zero measure and by market clearing  $w(\hat{x}) = 0$  for all  $\hat{x} \in (x', x)$ .
- This contradicts what we showed.

# Conditions for Assortative Equilibrium

- We can restrict our attention to the problem

$$\max_{x, \theta(x)} f(x, \mu(x), \theta(x)) - \theta(x)w(x)$$

- Taking first and second order conditions [▶ Details](#) we arrive at the expression:

$$\mu'(x) \left[ f_{\theta\theta} f_{xy} - f_{y\theta} \left( f_{x\theta} - \frac{f_x}{\theta(x)} \right) \right] < 0$$



# Conditions for Assortative Equilibrium

- Note that a **PAM** equilibrium requires  $\mu'(x) > 0$ , this implies a necessary condition:

$$f_{\theta\theta}f_{xy} - f_{y\theta} \left( f_{x\theta} - \frac{f_x}{\theta(x)} \right) < 0$$

- We can write this condition in terms of  $F$  [► Details](#) to deal with the potential endogeneity of  $\theta(x)$ :

$$F_{xy} > \frac{F_{yl}F_{xr}}{F_{lr}}$$

- We have found a necessary condition for the equilibrium matching to be **PAM**, turns out that this is also a sufficient condition.

# Main Assortativity Result

## Proposition 2

- A necessary and sufficient condition to have equilibria with positive assortative matching is that the following inequality holds:

$$F_{xy} > \frac{F_{yl}F_{xr}}{F_{lr}} \quad (2)$$

for all  $(x, y, l, r) \in \mathbb{R}_{++}^4$ .

The opposite inequality provides a necessary and sufficient condition for negative assortative matching.

# Main Assortativity Result

- Since the firm's problem is quasi-linear then Pareto optimality requires output maximization.
- This is the key idea behind the proof: if (2) holds then the output of any not positive assortative allocation can be strictly improved implying that it is not an equilibrium.

# Main Assortativity Result

- Consider some matching  $(x, y, \theta)$  such that a total measure  $r$  of resources is deployed in this match, the output generated is

$$F(x, y, \theta r, r) = rf(x, y, \theta)$$

- We can show [▶ Details](#) that the marginal change of shifting an optimal measure of workers from firm  $y$  to firm  $\hat{y}$ :

$$\beta(\hat{y}; x, y, \theta) = f(x, \hat{y}, \hat{\theta}) - \hat{\theta} f_{\theta}(x, y, \theta) \quad \text{where} \quad f_{\theta}(x, y, \theta) = f_{\theta}(x, \hat{y}, \hat{\theta}) \quad (2)$$

# Main Assortativity Result

- Suppose that equilibrium matching is not **PAM**, i.e  $x_1$  is matched to  $y_1$  at intensity  $\theta_1$  and  $x_2$  to  $y_2$  at intensity  $\theta_2$ , but  $x_1 > x_2$  while  $y_1 < y_2$ , for this match to be efficient the following two inequalities must **simultaneously** hold:

$$\beta(y_1; x_2, y_2, \theta_2) \leq \beta(y_1; x_1, y_1, \theta_1) \quad (2)$$

$$\beta(y_2; x_1, y_1, \theta_1) \leq \beta(y_2; x_2, y_2, \theta_2) \quad (3)$$

- To finalize the proof we show that (??), (??) and (2) leads to a contradiction [▶ Details](#).

# Equilibrium Assignment

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- To simulate the model we will use the following production function:

$$f(x, y, \theta) = \left( \omega_A x^{(1-\sigma_A)/\sigma_A} + (1 - \omega_A) y^{(1-\sigma_A)/\sigma_A} \right)^{\sigma_A/(1-\sigma_A)} \theta^{\omega_B}$$

- Computing condition ?? for this production function we get:

$$- \frac{(1 - \sigma_A) (1 - \omega_A) \omega_A x^{\frac{1}{\sigma_A}} y^{\frac{1}{\sigma_A}} \theta^{\omega_B} \left( \omega_A x^{\frac{1}{\sigma_A} - 1} + (1 - \omega_A) y^{\frac{1}{\sigma_A} - 1} \right)^{\frac{\sigma_A}{1 - \sigma_A}}}{\sigma_A \left( \omega_A \left( y x^{\frac{1}{\sigma_A}} - x y^{\frac{1}{\sigma_A}} \right) + x y^{\frac{1}{\sigma_A}} \right)^2} > 0 \quad (4)$$

- Clearly the condition for **PAM** holds if  $\sigma_A < 1$  and we will have **NAM** if  $\sigma_A > 1$ .



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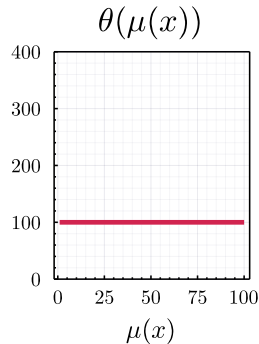
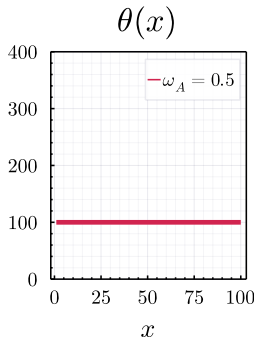
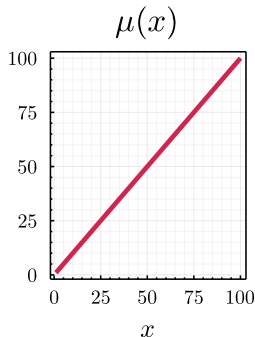
- Simulation Strategy

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## Appendix

## Effect of changing $\omega_A$

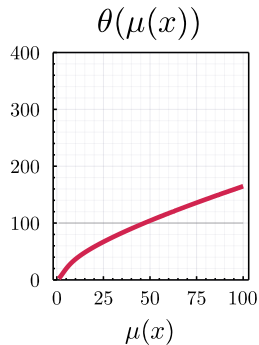
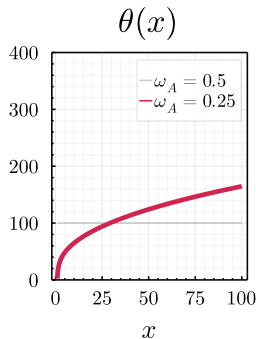
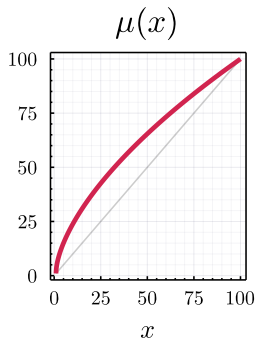
- When  $\omega_A = 0.5$  workers and firms are equally weighted.
- Fully symmetric model, matching  $\mu(x) = x$ , reach constant size



- **Parametrization**  $x, y \sim U[0, 1]$ ,  $\omega_B = 0.5$  and  $\sigma_A = 0.9$

## Effect of changing $\omega_A$

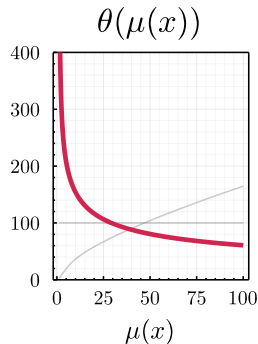
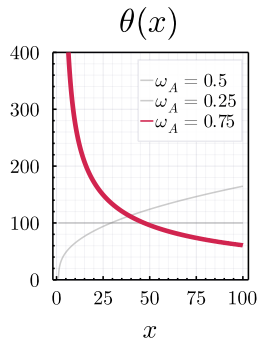
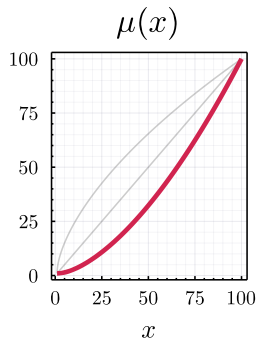
- $\omega_A \in (0.5, 1]$  worker type is more determinant in production.
- The size effect dominates the type effect  $\implies$  matching is concave and firm size is increasing.



- **Parametrization**  $x, y \sim U[0, 1]$ ,  $\omega_B = 0.5$  and  $\sigma_A = 0.9$

## Effect of changing $\omega_A$

- $\omega_A \in [0, 0.5)$  firm type is more determinant in production.
- The type effect dominates the size effect  $\implies$  matching is convex and firm size is decreasing.



- **Parametrization**  $x, y \sim U[0, 1]$ ,  $\omega_B = 0.5$  and  $\sigma_A = 0.9$

- [Effect in wages](#)

- Suppose that a firm  $y$  that uses strategy  $(\theta^y, \mathcal{R}^y)$  to solve the problem

$$\max_{\theta^y, \mathcal{R}^y} \int [f(x, y, \theta^y(x)) - w(x)\theta^y(x)] d\mathcal{R}^y(x) \quad (5)$$

- Proceed by contradiction, and suppose that there is a set of hired workers  $\tilde{\mathcal{X}}$  for which their assigned resources do not solve

$$(x, \theta^y(x)) \in \arg \max \{f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta}w(\tilde{x})\} \quad \forall x \in \text{supp}\mathcal{R}^y$$

- Define:

$$\mathcal{X}^* = \{x \in \mathcal{X} \mid (x, \theta^*(x)) \in \arg \max \{f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta}w(\tilde{x}), \text{ for some } \theta^*\}\}$$

$$\tilde{\mathcal{X}} = \mathcal{X} / \mathcal{X}^*$$

- Consider any  $x^* \in \mathcal{X}^*$  and a strategy where the firm places or the resources on  $x^*$  at intensity  $\theta^*$  we have:

$$f(x, y, \theta^y(x)) = f(x^*, y, \theta^*) \quad \forall x \in \mathcal{X}^*$$

$$f(x, y, \theta^y(x)) < f(x^*, y, \theta^*) \quad \forall x \in \tilde{\mathcal{X}}$$

- Note that the profits pf the firm are:

$$\begin{aligned} & \int_{\mathcal{X}^*} [f(x, y, \theta^y(x)) - w(x)\theta^y(x)] d\mathcal{R}^y(x) + \int_{\tilde{\mathcal{X}}} [f(x, y, \theta^y(x)) - w(x)\theta^y(x)] d\mathcal{R}^y(x) \\ & \qquad \qquad \qquad < \\ & \int_{\mathcal{X}^*} [f(x^*, y, \theta^*) - w(x^*)\theta^*] d\mathcal{R}^y(x) + \int_{\tilde{\mathcal{X}}} [f(x^*, y, \theta^*) - w(x^*)\theta^*] d\mathcal{R}^y(x) \end{aligned}$$

- The firm can strictly increase its profits, therefore the original strategy is not a solution of (5). [▶ Back](#)

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# slide 3

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# slide 3

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# Use it to intimidate audiences!

Now you can make it clear you've done a shitload of work  
without having to show everything! [▶ Back](#)

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You can make an object a link using the `\hyperlink{label}{object}` command