ECON 736 Presentation
Assortative Matching with Large Firms

Mitchell Valdes-Bobes

December 2, 2021

Roadmap of Talk

Roadmap of Talk

Introduction

Motivation

Research Questions

- **Research Question**
  - Provide a unifying theory of production with a trade-off between hiring more vs better workers.

Research Questions

- **Research Question**
  - Provide a unifying theory of production with a trade-off between hiring more vs better workers.
- **Results**
  - Sorting condition that captures the trade-off between quantity and quality of workers.
  - Characterization of matching in equilibrium.
  - When is matching assortative **(PAM)** or **(NAM)**?
  - Under what conditions more productive firms hire more workers in equilibrium?

Roadmap of Talk

Model Setup: Demographics

- **Workers** indexed by *unidimensional* skill $x \in [\underline{x}, \bar{x}] \subset \mathbb{R}_+$
  - *CDF* $H^w(x)$ and *PDF* h^w
- **Firms** indexed by *unidimensional* productivity $y \in [\underline{y}, \bar{y}] \subset \mathbb{R}_+$
  - *CDF* $H^f(x)$ and *PDF* h^f

Model Setup: Preferences

- **Workers** care about their wage and there is no disutility of work.
- **Firms** maximize their profits.

Model Setup: Production Function

- The output produced by a firm of type y that hires l workers of type x is:

$$F(x, y, l, r)$$

  - r the fraction of y's resources dedicated to x type workers.
  - (x, y) are quality variables and (l, r) are quantity variables.

Model Setup: Production Function

- The output produced by a firm of type y that hires l workers of type x is:

$$F(x, y, l, r)$$

  - r the fraction of y's resources dedicated to x type workers.
  - (x, y) are quality variables and (l, r) are quantity variables.

- F is strictly increasing and strictly concave in (l, r), 0 resources produce 0 output, and standard Inada conditions apply.

- F has constant returns to scale in l and r.

Model Setup: Production Function

- The output produced by a firm of type y that hires l workers of type x is:

$$F(x, y, l, r)$$

  - r the fraction of y's resources dedicated to x type workers.
  - (x, y) are quality variables and (l, r) are quantity variables.

- F is strictly increasing and strictly concave in (l, r), 0 resources produce 0 output, and standard Inada conditions apply.

- F has constant returns to scale in l and r.

- We can write F in terms of **intensity** $\theta = l/r$:

$$f(x, y, \theta) := F(x, y, \theta, 1) \quad \Longrightarrow \quad F(x, y, l, r) = rf(x, y, \theta)$$

Equilibrium

- The equilibrium concept used is the competitive equilibrium.

Equilibrium

- The equilibrium concept used is the competitive equilibrium.
- Firm's problem:
  - Distribution of workers hired by y $\mathcal{L}^y(x) = \int_{\underline{x}}^{x} l^y(\tilde{x})dH^w(\tilde{x})$
  - Distribution of firm y resources $\mathcal{R}^y(x) = \int_{\underline{x}}^{x} r^y(\tilde{x})dH^w(\tilde{x})$

Equilibrium

- The equilibrium concept used is the competitive equilibrium.

- Firm's problem:
  - Distribution of workers hired by y $\mathcal{L}^y(x) = \int_{\underline{x}}^{x} l^y(\tilde{x})dH^w(\tilde{x})$
  - Distribution of firm y resources $\mathcal{R}^y(x) = \int_{\underline{x}}^{x} r^y(\tilde{x})dH^w(\tilde{x})$
  - For any $x \in [\underline{x}, \bar{x}]$ $l^y(x) = \theta^y(x)r^y(x)$ which means

$$\mathcal{L}^y(x) = \int_{\underline{x}}^{x} \theta^y(\tilde{x})d\mathcal{R}^y(\tilde{x}).$$

Equilibrium

- The equilibrium concept used is the competitive equilibrium.

- Firm's problem:
  - Distribution of workers hired by y $\mathcal{L}^y(x) = \int_{\underline{x}}^{x} l^y(\tilde{x}) dH^w(\tilde{x})$
  - Distribution of firm y resources $\mathcal{R}^y(x) = \int_{\underline{x}}^{x} r^y(\tilde{x}) dH^w(\tilde{x})$
  - For any $x \in [\underline{x}, \bar{x}]$ $l^y(x) = \theta^y(x) r^y(x)$ which means

$$\mathcal{L}^y(x) = \int_{\underline{x}}^{x} \theta^y(\tilde{x}) d\mathcal{R}^y(\tilde{x}).$$

  - The total output of the firm can be writen as:

$$\int_{\underline{x}}^{\overline{x}} f\left(x, y, \theta^y(x)\right) d\mathcal{R}^y(x) = \int_{\underline{x}}^{\overline{x}} F\left(x, y, l^y(x), r^y(x)\right) d\mathcal{H}^w(x)$$

  - Firms maximize the difference between output produced and wages paid to workers.

Equilibrium

- Feasible Labor Demand
  - Consider an interval of worker types $(x', x]$
  - The demand of firm y for those workers is $\mathcal{L}^y(x) - \mathcal{L}^y(x')$

Equilibrium

- Feasible Labor Demand
  - Consider an interval of worker types $(x', x]$
  - The demand of firm y for those workers is $\mathcal{L}^y(x) - \mathcal{L}^y(x')$
  - This implies a way to evaluate if a labor demand schedule $\{\mathcal{L}^y\}_{y \in \mathcal{Y}}$ is feasible:

$$\int_y \left[\mathcal{L}^y(x) - \mathcal{L}^y\left(x'\right)\right] dH^f \leq H^w(x) - H^w\left(x'\right) \qquad \forall (x', x] \subseteq \mathcal{X}$$

Equilibrium Definition

- An equilibrium is a tuple of functions $(w, \theta^y, \mathcal{R}^y, \mathcal{L}^y)$ consisting of a non-negative wage schedule $w(x)$ as well as intensity functions $\theta^y(x)$ and resource allocations $\mathcal{R}^y(x)$ with associated feasible labor demands $\mathcal{L}^y(x)$ such that:

Equilibrium Definition

- An equilibrium is a tuple of functions $(w, \theta^y, \mathcal{R}^y, \mathcal{L}^y)$ consisting of a non-negative wage schedule $w(x)$ as well as intensity functions $\theta^y(x)$ and resource allocations $\mathcal{R}^y(x)$ with associated feasible labor demands $\mathcal{L}^y(x)$ such that:
  - **Optimality:** Given the wage schedule $w(x)$, for any firm y, the combination $(\theta^y, \mathcal{R}^y)$ solves:

$$\max_{\theta^y, \mathcal{R}^y} \int \left[f\left(x, y, \theta^y(x)\right) - w(x)\theta^y(x)\right] d\mathcal{R}^y(x)$$

Equilibrium Definition

- An equilibrium is a tuple of functions $(w, \theta^y, \mathcal{R}^y, \mathcal{L}^y)$ consisting of a non-negative wage schedule $w(x)$ as well as intensity functions $\theta^y(x)$ and resource allocations $\mathcal{R}^y(x)$ with associated feasible labor demands $\mathcal{L}^y(x)$ such that:
  - **Optimality:** Given the wage schedule $w(x)$, for any firm y, the combination $(\theta^y, \mathcal{R}^y)$ solves:

$$\max_{\theta^y, \mathcal{R}^y} \int \left[f\left(x, y, \theta^y(x)\right) - w(x)\theta^y(x)\right] d\mathcal{R}^y(x)$$

  - **Market Clearing:** For any $(x', x] \subseteq \mathcal{X}$

$$\text{If } w(x) > 0 \text{ a.e in } (x', x] \implies \int_y \left[\mathcal{L}^y(x) - \mathcal{L}^y\left(x'\right)\right] dH^f = H^w(x) - H^w\left(x'\right)$$

Equilibrium Characterization

- When do better firms hire **better** workers?
- How are wages determined?
- When do better firms employ **more** workers?
- How is that affected by technological change?

Equilibrium Assortativity

Definition (Assortative Matching)

- We say that matching between firms and workers is PAM (NAM) if higher type firms hire higher type workers, i.e., $y > y'$ then, x in the support of $\mathcal{L}^y$ and x' in the support of $\mathcal{L}^{y'}$ only if $x \geq (\leq) x'$.

- Make sure not to forget notes!
- I should use these more
- put your text here!

Equilibrium Characterization

Proposition 1

- If output F is strictly increasing in x and y and the type distributions have nonzero continuous densities, then almost all active firm types y hire exactly one worker type and reach unique size $l(y)$ in an **assortative** equilibrium.
- There is an injective matching function $\mu : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$, between the subset of hired workers and active firms.

Equilibrium Characterization

The proof have two parts:

- First we show that for every hired worker the combination $(x, \theta^y(x))$ solves ▸ Details

$$(x, \theta^y(x)) \in \arg\max \left\{ f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta} w(\tilde{x}) \right\} \qquad \forall x \in \text{supp}\mathcal{R}^y \tag{1}$$

  - An implication is that in equilibirum if a worker is hired then all wokers that are more productive must have strictly possitive wages.

Equilibrium Characterization

- Second, assume that a firm hires two different workers $x' < x$, if the equilibirum is **PAM** then that firm must be the only firm that hire workers in $[x', x]$.

- If there is only one firm active in $[x', x]$ then the aggregate labor demand has zero measure and by market clearing $w(\hat{x}) = 0$ for all $\hat{x} \in (x', x)$.

- This means that there are workers more productive than x' that dont have possitive wages, **Contradiction!**

Conditions for Assortative Equilibrium

- We can restrict our attention to the problem

$$max_{x,\theta(x)}\; f(x,\mu(x),\theta(x)) - \theta(x)w(x)$$

- Taking first and second order conditios ▸ Details we arrive at the expression:

$$\mu'(x)\left[f_{\theta\theta}f_{xy} - f_{y\theta}\left(f_{x\theta} - \frac{f_x}{\theta(x)}\right)\right] < 0$$

Conditions for Assortative Equilibrium

- Note that a **PAM** equilibrium requires $\mu'(x) > 0$, this implies a necessary condition:

$$f_{\theta\theta} f_{xy} - f_{y\theta}\left(f_{x\theta} - \frac{f_x}{\theta(x)}\right) < 0$$

- We can write this condition in terms of F ▸ Details to deal with the potential endogeneity of $\theta(x)$:

$$F_{xy} > \frac{F_{yl} F_{xr}}{F_{lr}}$$

- We have found a necessary condition for the equilibirum matching to be **PAM**, tunrs out that this is also a sufficient condition.

Main Asssortativity Result

Proposition 2

- A necessary and sufficient condition to have equilibria with positive assortative matching is that the following inequality holds:

$$F_{xy} > \frac{F_{yl}F_{xr}}{F_{lr}}$$

for all $(x, y, l, r) \in \mathbb{R}^4_{++}$.
The opposite inequality provides a necessary and sufficient condition for negative assortative matching.

Main Asssortativity Result

- The firm problem is quasi-linear.
- Pareto optimality requires output maximization.
- This is the key idea behind the proof:
  - Assume that the sorting condition holds.
  - Take any matching that is not positive assortative
  - Show that allocation can be strictly improved $\implies$ not an equilibrium.

Main Asssortativity Result

- Consider some matching (x, y, θ) such that a total measure r of resources is deployed in this match, the output generated is

$$F(x, y, \theta r, r) = rf(x, y, \theta)$$

- We can show ▸ Details that the marginal change of shifting an optimal measure of workers of type x from firm y to firm $\hat{y}$:

$$\beta(\hat{y}; x, y, \theta) = f(x, \hat{y}, \hat{\theta}) - \hat{\theta} f_\theta(x, y, \theta) \quad \text{where} \quad f_\theta(x, y, \theta) = f_\theta(x, \hat{y}, \hat{\theta}) \tag{1}$$

Main Asssortativity Result

- Suppose that equilibrium matching is not **PAM**, i.e x_1 is matched to y_1 at intensity θ_1 and x_2 to y_2 at intensity θ_2, but $x_1 > x_2$ while $y_1 < y_2$, for this match to be efficient the following two inequalities mus **simultaneously** hold:

$$\beta(y_1; x_2, y_2, \theta_2) \leq \beta(y_1; x_1, y_1, \theta_1) \tag{1}$$

$$\beta(y_2; x_1, y_1, \theta_1) \leq \beta(y_2; x_2, y_2, \theta_2) \tag{2}$$

- To finalize the proof we show that (1), (2) and the sorting condition cannot simultaneously hold Details.

Equilibrium Assignment

- This model deals with both the intensive and the extensive margin.
- Assortativity is not enough to characterize who matches with whom in equilibrium.
- Firms could hire more or fewer workers.
- We will characterize the matching with a system of differential equations.

Equilibrium Assignment

- Since the matching $y = \mu(x)$, is singled valued we have $l^y(x) = \theta(x)\mathbb{1}_{\{y=\mu(x)\}}$
- We can show ▸ Details that in equilibrium the demand of labor in any interval $(x, \overline{x}]$ is

$$\int_y \left[\mathcal{L}^y(x) - \mathcal{L}^y(x')\right] dH^f = \int_{\mu(x)}^{\overline{y}} \theta(\mu^{-1}(y)) dH^f = \underbrace{H^w(\overline{x}) - H^w(x)}_{\text{by Labor Market Clearing}}$$

- Differentiating w.r.t x both sides and solving for $\mu'(x)$:

$$\mu'(x) = \frac{\mathcal{H}(x)}{\theta(x)} \qquad \text{with} \qquad \mathcal{H}(x) = \frac{h^w(x)}{h^f(\mu(x))}$$

Equilibrium Assignment

- From the first order condition of the problem we have:

$$w'(x) = \frac{f_x}{\theta(x)}$$

- Using the differenciated version of the FOC and some algebra ▸ Details we get:

$$\theta'(x) = \frac{\mathcal{H}(x) f_{xy}}{\theta(x) f_{x\theta}}$$

Equilibrium Assignment

- The system of differential equations:

$$\begin{cases} \mu'(x) = \dfrac{\mathcal{H}(x)}{\theta(x)} \\ w'(x) = \dfrac{f_x}{\theta(x)} \\ \theta'(x) = \dfrac{\mathcal{H}(x) f_{xy}}{\theta(x) f_{x\theta}} \end{cases}$$

  characterizes the equilibrium.

Roadmap of Talk

Simulation Strategy

- We want to numerically solve a system of ODE's.
- We need an initial condition:

$$\mu(\underline{x}) = \underline{\mu} \qquad \text{and} \qquad \theta(\underline{x}) = \underline{\theta}$$

Simulation Strategy

- We want to numerically solve a system of ODE's.
- We need an initial condition:

$$\mu(\underline{x}) = \underline{\mu} \qquad \text{and} \qquad \theta(\underline{x}) = \underline{\theta}$$

- Positive assortative matching gives us one initial condition:

$$\mu(\underline{x}) = \underline{y}$$

- But we are still unable to pindown $\underline{\theta}$.

Simulation Strategy

- We know a terminal condition for $\mu(x)$:

$$\mu(\overline{x}) = \overline{y}$$

- This turns the initial value problem into a boundary condition problem.

- We can solve this problem using a shooting algorithm.

Simulation Strategy

- We know a terminal condition for $\mu(x)$:

$$\mu(\overline{x}) = \overline{y}$$

- This turns the initial value problem into a boundary condition problem.
- We can solve this problem using a shooting algorithm.
- The idea of the shooting algorithm is to select an initial value for $\underline{\theta}$, solve the system, and compare the obtained value of $\mu(\overline{x})$ with $\overline{y}$ and iteratively update $\underline{\theta}$ until convergence.

Simulation Results

- To simulate the model we will use the following production function:

$$f(x, y, \theta) = \left(\omega_A x^{(1-\sigma_A)/\sigma_A} + (1 - \omega_A)\, y^{(1-\sigma_A)/\sigma_A}\right)^{\sigma_A/(1-\sigma_A)} \theta^{\omega_B}$$

- Parameters:
  - ω_A: captures the importance of the worker type on output.
    - High $\omega_A \implies$ worker type is more determinant.
  - σ_A: captures the degree of complementarity between worker type and firm productivity.
    - High $\sigma_A \implies$ types are less complementarity.
  - ω_B: captures the penalty of for large firm.
    - In the limit $\omega_B \to 1$ eliminates decreasing returns to scale and allocates all labor to the better firms.

Simulation Results

- Computing the sorting condition for this production function we get:

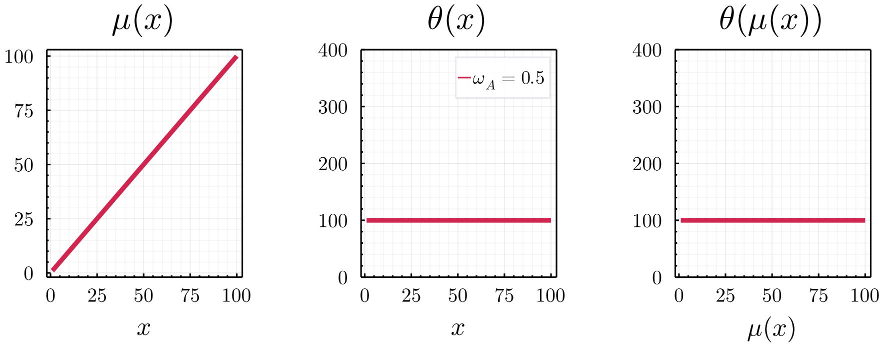
$$-\frac{(1-\sigma_A)(1-\omega_A)\,\omega_A x^{\frac{1}{\sigma_A}} y^{\frac{1}{\sigma_A}} \theta^{\omega_B} \left(\omega_A x^{\frac{1}{\sigma_A}-1} + (1-\omega_A)\, y^{\frac{1}{\sigma_A}-1}\right)^{\frac{\sigma_A}{1-\sigma_A}}}{\sigma_A \left(\omega_A \left(yx^{\frac{1}{\sigma_A}} - xy^{\frac{1}{\sigma_A}}\right) + xy^{\frac{1}{\sigma_A}}\right)^2} > 0$$

- Clearly the condition for **PAM** holds if $\sigma_A < 1$ and we will have **NAM** if $\sigma_A > 1$.

Effect of changing ω_A

- When $\omega_A = 0.5$ workers and firms are equally weighted.

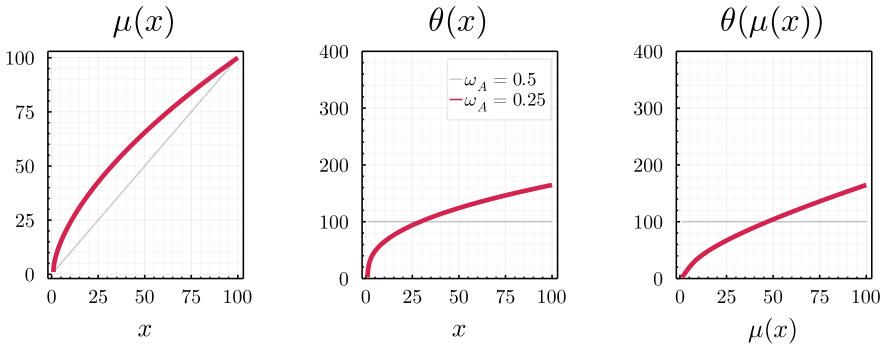
- Fully symmetric model, mathing $\mu(x) = x$, reach constant size



- **Parametrization** $x, y \sim U[0, 1]$, $\omega_B = 0.5$ and $\sigma_A = 0.9$

Effect of changing ω_A

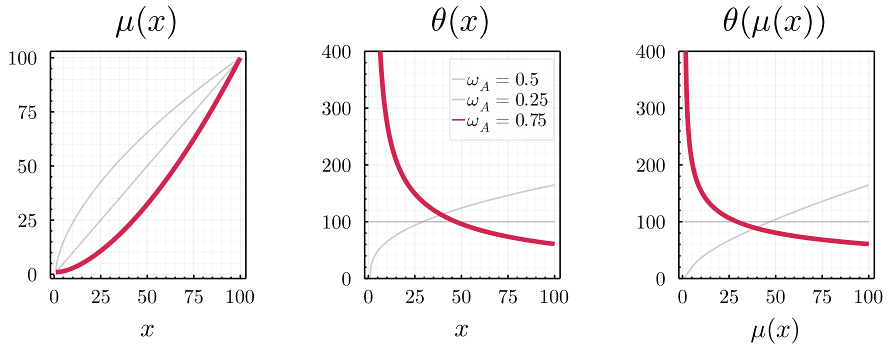
- $\omega_A \in (0.5, 1]$ worker type is more determinant in production.
- The size effect dominates the type effect $\implies$ matching is concave and firm size is increasing.



- **Parametrization** $x, y \sim U[0, 1]$, $\omega_B = 0.5$ and $\sigma_A = 0.9$

Effect of changing ω_A

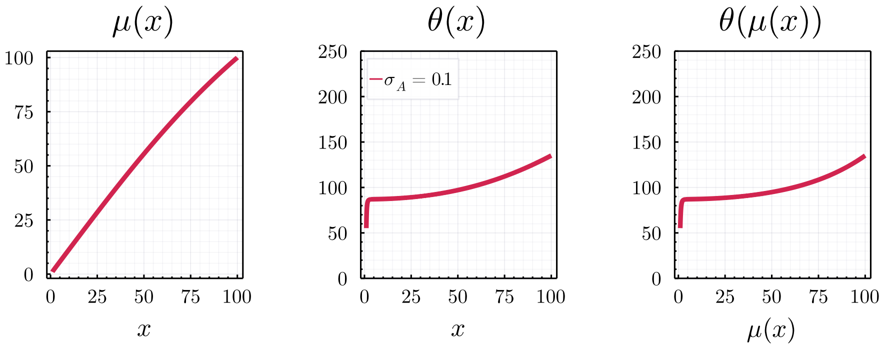
- $\omega_A \in [0, 0.5)$ firm type is more determinant in production.
- The type effect dominates the size effect $\implies$ matching is convex and firm size is decreasing.



- **Parametrization** $x, y \sim U[0,1]$, $\omega_B = 0.5$ and $\sigma_A = 0.9$

Effect of changing σ_A

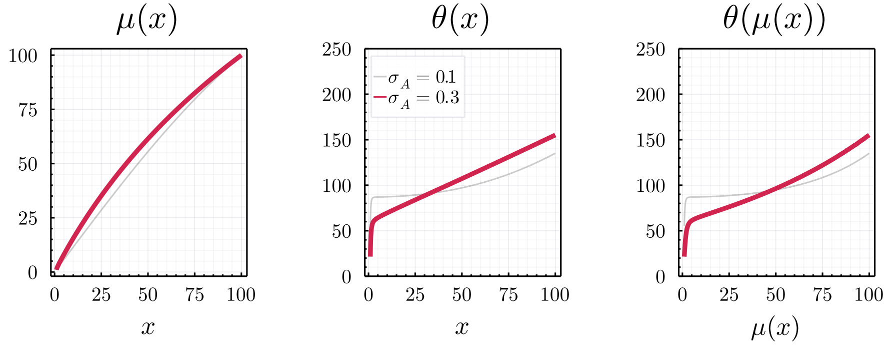
- Higher values of σ_A means that higher type workers are more attractive.
- Since the supply of labor constrained $\implies$ stealing of workers.



- **Parametrization** $x, y \sim U[0, 1]$, $\omega_B = 0.5$ and $\omega_A = 0.75$

Effect of changing σ_A

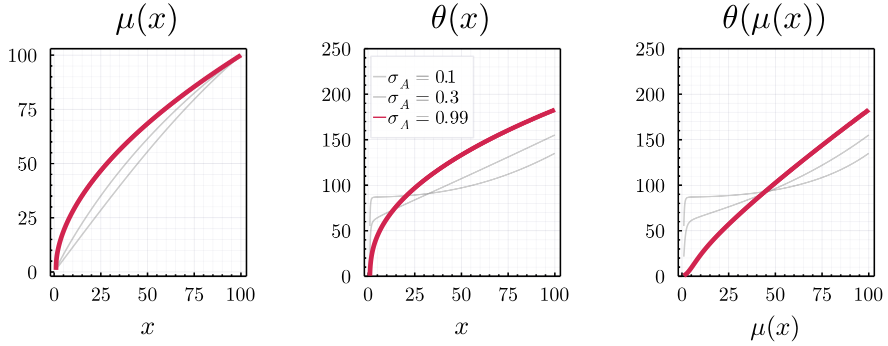
- Higher values of σ_A means that higher type workers are more attractive.
- Since the supply of labor constrained $\implies$ stealing of workers.



- **Parametrization** $x, y \sim U[0,1]$, $\omega_B = 0.5$ and $\omega_A = 0.75$

Effect of changing σ_A

- Higher values of σ_A means that higher type workers are more attractive.
- Since the supply of labor constrained $\implies$ stealing of workers.



- **Parametrization** $x, y \sim U[0,1]$, $\omega_B = 0.5$ and $\omega_A = 0.75$

Thank You!

Roadmap of Talk

Appendix

Appendix

- Suppose that a firm y that uses strategy $(\theta^y, \mathcal{R}^y)$ to solve the problem

$$\max_{\theta^y, \mathcal{R}^y} \int \left[f\left(x, y, \theta^y(x)\right) - w(x)\theta^y(x) \right] d\mathcal{R}^y(x) \tag{3}$$

- Proceed by contradiction, and suppose that there is a set of hired workers $\tilde{\mathcal{X}}$ for which their assigned resources do not solve

$$(x, \theta^y(x)) \in \arg\max \left\{ f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta} w(\tilde{x}) \right\} \qquad \forall x \in \text{supp}\mathcal{R}^y$$

- Define:

$$\mathcal{X}^* = \left\{ x \in \mathcal{X} \mid (x, \theta^*(x)) \in \arg\max \left\{ f(\tilde{x}, y, \tilde{\theta}) - \tilde{\theta} w(\tilde{x}), \text{for some } \theta^* \right\} \right\}$$

$$\tilde{\mathcal{X}} = \mathcal{X} / \mathcal{X}^*$$

Appendix

- Consider any $x^* \in \mathcal{X}^*$ and a strategy where the firm places or the resources on x^* at intensity θ^* we have:

$$f(x, y, \theta^y(x)) = f(x^*, y, \theta^*) \qquad \forall x \in \mathcal{X}^*$$

$$f(x, y, \theta^y(x)) < f(x^*, y, \theta^*) \qquad \forall x \in \tilde{\mathcal{X}}$$

- Note that the profits pf the firm are:

$$\int_{\mathcal{X}^*} [f(x, y, \theta^y(x)) - w(x)\theta^y(x)]\, d\mathcal{R}^y(x) + \int_{\tilde{\mathcal{X}}} [f(x, y, \theta^y(x)) - w(x)\theta^y(x)]\, d\mathcal{R}^y(x)$$
$$<$$
$$\int_{\mathcal{X}^*} [f(x^*, y, \theta^*) - w(x^*)\theta^*]\, d\mathcal{R}^y(x) + \int_{\tilde{\mathcal{X}}} [f(x^*, y, \theta^*) - w(x^*)\theta^*]\, d\mathcal{R}^y(x)$$

- The firm can strictly increase its profits, therefore the original strategy is not a solution of (3). ▸ Back

Appendix

Back

Appendix

Back

Appendix

- Back Withdraw some optimal measure of workers $\hat{r}\hat{\theta}$ and pair them some firm $\hat{y}$ then the output changes to:

$$rf\left(x,y,\theta-\frac{\hat{r}\hat{\theta}}{r}\right)+\hat{r}f(x,\hat{y},\hat{\theta})$$

- The output variation generated by an infinitesimal change $\hat{r}$ is:

$$\frac{\partial}{\partial\hat{r}}\left(rf\left(x,y,\theta-\frac{\hat{r}\hat{\theta}}{r}\right)+\hat{r}f(x,\hat{y},\hat{\theta})\right)\Bigg|_{\hat{r}=0}=f(x,\hat{y},\hat{\theta})-\hat{\theta}f_{\theta}(x,y,\theta)$$

- The assuption of $\hat{r}$ being optimal implies that the first order condition pins down $\hat{\theta}$:

$$f_{\theta}(x,y,\theta)=f_{\theta}(x,\hat{y},\hat{\theta})=w(x)$$

Appendix

- ▸ Back Define

$$\varphi(y) = \beta\left(y; x_2, y_2, \theta_2\right) - \beta\left(y; x_1, y_1, \theta_1\right)$$

- If the matching is efficient then $\varphi\left(y_1\right) \leq 0 \leq \varphi\left(y_2\right)$, since $\varphi(y)$ is a continuous function of y then there is a value $\tilde{y} \in \left[y_1, y_2\right]$ such that

$$\varphi(\tilde{y}) = 0 \quad \Longrightarrow \quad \beta\left(\tilde{y}; x_2, y_2, \theta_2\right) = \beta\left(\tilde{y}; x_1, y_1, \theta_1\right)$$

- We can define the function $\xi(x)$ such that for all x the following hods:

$$f(x, \tilde{y}, \xi(x)) - \xi(x) f_\theta(x, \tilde{y}, \xi(x)) = \beta\left(\tilde{y}, x_2, y_2, \theta_2\right)$$

  then

$$f(x, \tilde{y}, \xi(x)) - \xi(x) f_\theta(x, \tilde{y}, \xi(x)) = \beta\left(\tilde{y}, x_1, y_1, \theta_1\right)$$

  and $\xi(x_1) = \theta_1$ and $\xi(x_2) = \theta_2$.

- The next step is to implicitly differenciate the above expression with respect to x.

Appendix

- Back To obtain the derivate $\xi'(x)$

$$\xi'(x) = \frac{f_x}{\xi(x) f_{\theta\theta}} - \frac{f_{x\theta}}{f_{\theta\theta}}$$

- And:

$$\frac{\partial}{\partial x}\left(f_y(x, \tilde{y}, \xi(x))\right) = f_{xy} + f_{y\theta}\xi'(x)$$
$$= \overbrace{\frac{1}{f_{\theta\theta}}}^{<0} \underbrace{\left(f_{\theta\theta} f_{xy} + f_{y\theta}\frac{f_x}{\xi(x)} - f_{y\theta} f_{x\theta}\right)}_{<0 \text{ Ths is the sorting comndition}}) > 0$$

Appendix

- Back TThis means that $f_y(x, \tilde{y}, \xi(x))$ is decreasing in x and since $x_1 > x_2$ and $\xi(x_1) = \theta_1$ and $\xi(x_2) = \theta_2$ we have

$$f_y(x_2, \tilde{y}, \theta_2) < f_y(x_1, \tilde{y}, \theta_1)$$

- Since

$$\beta_1(\hat{y}; x, y, \theta) = f_y(x, \hat{y}, \hat{\theta})$$

then

$$\beta_1(\tilde{y}; x_2, y_2, \theta_2) < \beta_1(\tilde{y}; x_1, y_1, \theta_1)$$

- $\beta(\tilde{y}; x_1, y_1, \theta_1)$ grows strictly faster than $\beta(\tilde{y}; x_2, y_2, \theta_2)$

- Plug $\tilde{y} = y_2$ and we have found our contradiction.

Appendix

- Back By market-clearing of the labor market, it must be true in equilibrium that:

$$\int_y \left[\mathcal{L}^y(x) - \mathcal{L}^y(x')\right] dH^f = H^w(x) - H^w(x')$$

- Re-write the LHS of the above expression in terms of θ as:

$$\begin{aligned}\int_y \left[\mathcal{L}^y(x) - \mathcal{L}^y(x')\right] dH^f &= \int_y \left[\int_{\underline{x}}^{\bar{x}} \theta(s) d\mathcal{R}^y(x) - \int_{\underline{x}}^{x} \theta(s) d\mathcal{R}^y(s)\right] dH^f \\ &= \int_y \left[\int_x^{\bar{x}} \theta(s) d\mathcal{R}^y(s)\right] dH^f = \int_y \left[\int_x^{\bar{x}} \theta(s) \mathbb{1}_{\{y=\mu(s)\}} dH^w\right]\end{aligned}$$

- Note $y = \mu(x')$ with $x' \notin [x, \bar{x}]$ then $\mathbb{1}_{\{y=\mu(s)\}} = 0$ integrate over $[\mu(x), \bar{y}]$:

$$\int_{\mu(x)}^{\bar{y}} \left[\int_x^{\bar{x}} \theta(x) I_{\{s=\mu^{-1}(y)\}} dH^w\right] dH^f = \int_{\mu(x)}^{\bar{y}} \theta\left(\mu^{-1}(y)\right) dH^f$$

$$\implies \int_{\mu(x)}^{\bar{y}} \theta\left(\mu^{-1}(y)\right) dH^f = H^w(x) - H^w(x')$$

Appendix

- Back The differentiated FOC is:

$$f_{xx} - \theta(x)w''(x) = -\mu'(x)f_{xy} - \theta'(x)\left[f_{x\theta} - w'(x)\right]$$

- Manipulating this expression we get:

$$\theta(x)w''(x) + \theta'(x)w'(x) = \mu'(x)t + f_{xy} + \theta'(x)f_{x\theta} + f_{xx}$$
$$\implies \frac{\partial}{\partial x}\left(\theta(x)w'(x)\right) = \mu'(x)t + f_{xy} + \theta'(x)f_{x\theta} + f_{xx}$$

- From the expression for $w'(x)$:

$$w'(x)\theta(x) = f_x \quad \implies \quad \frac{\partial}{\partial x}\left(\theta(x)w'(x)\right) = f_{xx}$$

- Combining both equations:

$$\cancel{f_{xx}} = \mu'(x)t + f_{xy} + \theta'(x)f_{x\theta} + \cancel{f_{xx}} \quad \implies \quad \theta'(x) = \frac{\mathcal{H}(x)f_{xy}}{\theta(x)f_{x\theta}}$$