Exercise 1 (3.15 of Tirole (project riskiness and credit rationing).). Consider the basic, fixed-investment model covered in Section 3.2 of Tirole (2006). In particular, investment is a fixed size I, the entrepreneur borrows $I - A$, the probability of success is either p_H (which yields no private benefit) or p_L (which yields private benefit B), success yields verifiable revenue R while failure yields 0. There are two types, "A" and "B", of the projects, which differ only with respect to "riskiness" defined by $p_H^A R^A = p_H^B R^B$, but $p_H^A > p_H^B$ so that project B is "riskier". The investment cost I is the same for both variants and furthermore, $\Delta p = p_L^A - p_L^A = p_H^B - p_L^B$. Which type of project is less prone to credit rationing?

Answer. From the Second Best financing problem we have that the necessary condition for funding is that

$$\mathcal{P}_0^i \equiv p_H^i \left(R^i - \frac{B}{\Delta p} \right) \geq I - A \qquad i \in \{A, B\}$$

Note that $\mathcal{P}_0^A - \mathcal{P}_0^B = (p_H^A - p_H^B) \left(R^i - \frac{B}{\Delta p} \right) > 0$.
From this, we can see that all other things being equal if a project of type B gets funded then a project of type A will also get funded, but the the opposite is not necessarily true. Thus, the project of type B, the riskier one, is more prone to credit rationing.

Exercise 2 (3.13 of Tirole (lender market power with fixed investment).). The environment is similar to Section 3.2 of Tirole with one exception. An entrepreneur has internal wealth A (which could be negative because of previous debt) and wants to undertake non-negative investment $I > A$ into a fixed-size project. The project yields $R > 0$ with probability p and 0 with probability $1 - p$. The probability of success is p_H if the entrepreneur works and $p_L < p_H$ if he shirks. The entrepreneur obtains private benefit B if she shirks and 0 otherwise. The borrower is protected by limited liability and everyone is risk-neutral. The project is worthwhile only if the entrepreneur behaves.
The exception is that there is a single lender. This lender has access to funds that command an expected rate of return equal to 0 (so the lender would content himself with a 0 rate of return, but will use his market power to obtain a superior rate of return). Assume $V \equiv p_H R - I > 0$ and let $\bar{A}$ and $\widehat{A}$ be defined by

$$\bar{A} \equiv I - p_H \left[R - \frac{B}{\Delta p} \right]$$
$$\widehat{A} \equiv p_H \frac{B}{\Delta p}$$

where $\Delta p = p_H - p_L$. Assume that $\bar{A} > 0$ and that the lender makes a take-it-or-leave-it offer to the borrower (i.e.the lender chooses R_b, the borrower's compensation in the case of success).

2.1. What contract is optimal for the lender? Be sure to state the programming problem explicitly.

Answer. The lender maximizes their profits subject to IC and participation constraints for the borrower holds:

$$
\begin{aligned}
&\max_{R_l, R_b} p_H R_l - I + A \\
\text{s.t. } & p_H R_b \geq p_L R_b + B \\
& p_H R_b \geq A \\
& p_H R_l \geq I - A \\
& R_l + R_b = R
\end{aligned}
$$

Thus, the lender will choose the lowest R_b such that all constraints hold. First let $\hat{A}$ be the net worth where IC_b and PC_b both bind:

$$\frac{B}{\Delta p} = \frac{\hat{A}}{p_H} \Longrightarrow \hat{A} = p_H \frac{B}{\Delta p}$$

Second, observe that the PC_b and PC_l cannot both bind. Suppose not then

$$\frac{A}{p_H} = R - \frac{I - A}{p_H} \Longrightarrow 0 = p_H R - I = V > 0 \Rightarrow\Leftarrow$$

Third, let $\bar{A}$ be the net worth where the IC_b and PC_l both bind:

$$\frac{B}{\Delta p} = R - \frac{I - \bar{A}}{p_H} \Longrightarrow \bar{A} = I - p_H \left(R + \frac{B}{\Delta p} \right)$$

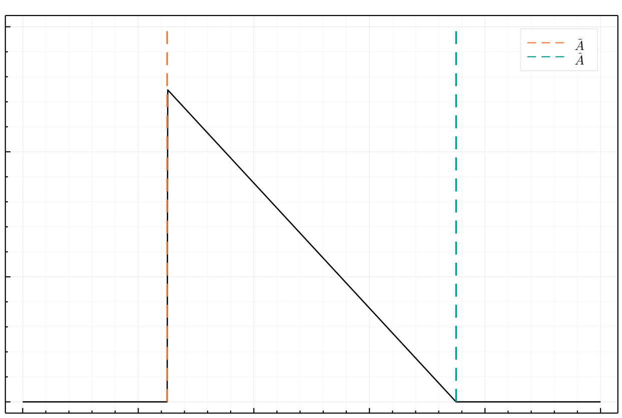
Thus, the optimal lending contract for the lender depends on A. If $A < \bar{A}$, then the lenders participation constraint does not hold, so there's no contract (i.e. credit rationing). If $\bar{A} \leq A < \hat{A}$, the borrowers incentive compatibility constraint binds and the borrowers participation constraint is slack, so $R_b = \frac{B}{\Delta p}$. At $A = \hat{A}$, both the borrowers incentive compatibility and participation constraints bind. If $I > A > \hat{A}$, the borrowers incentive compatibility constraint is slack and the borrowers participation constraint binds, so $R_b = \frac{A}{p_H}$.

2.2. Is the financing decision affected by lender market power (i.e. compared to the case of competitive lenders in Section 3.2)?

Answer. Lender market power does not affect the financing decision.

2.3. Draw the borrower's net utility (i.e. net of A) as a function of A. Note that unlike the monotonic case in Section 3.2, it is nonmonotonic among the regions $(-\infty, \bar{A}), [\bar{A}, \widehat{A}), [\widehat{A}, I)$. Explain.

Answer. The borrower's net utility is:



Exercise 3 (3.5 of Tirole (continuous investment and decreasing returns to scale).)**.** Consider the continuous investment model of Section 3.4 of Tirole (2006) with one modification; investment I yields return $R(I)$ in the case of success and 0 in the case of failure, where $R' > 0$ and $R'' < 0, R'(0) > 1/p_H, R'(\infty) < 1/p_H$. The rest of the model is unchanged. That is, the entrepreneur starts with cash A, the probability of success is either p_H if he behaves or p_L if he misbehaves. The entrepreneur obtains private benefit BI if he misbehaves and 0 otherwise. Only the final outcome is observable. Let I^* denote the level of investment that maximizes total surplus (i.e. $p_H R'(I^*) = 1$).

3.1. How does investment $I(A)$ vary with the level of cash?

Answer. The problem is

$$\begin{aligned}
\max_{I, R_b} \quad & p_H R_b \\
\text{s.t.} \quad & R_b \geq \frac{BI}{\Delta p} \qquad (IC_b) \\
& R_b \leq R(I) - \frac{I - A}{p_H} \qquad (PC_l)
\end{aligned}$$

PC_l binds.if not then R_b can be increased without violating any constraint which increases the objective function.

$$R_b = R(I) - \frac{I - A}{p_H}$$

Differentiating with respect to I, we get:

$$\frac{dR_b}{dI} = R'(I) - \frac{1}{p_H}$$

$R'(I), R_b$ is increasing in I for all $I < I^*$. So the borrower would choose the highest I as possible. Thus, IC_b binds for all $I \leq I^*$. Suppose not, then the borrower would go to a different lender. Taking IC_b and PC_l together:

$$R(I) - \frac{I - A}{p_H} = \frac{BI}{\Delta p}$$

This holds at I^*. By total differentiation and $R'(I^*) - \frac{1}{p_H} = 0$,

$$R'(I^*) dI - \frac{dI - dA}{p_H} = \frac{BdI}{\Delta p} \Longrightarrow \left[R'(I^*) - \frac{1}{p_H}\right] dI + \frac{dA}{p_H} = \frac{BdI}{\Delta p} \Longrightarrow \frac{dI}{dA} = \frac{\Delta p}{Bp_H} > 0$$

So I is increasing in A since all terms in the above expression are positive.

3.2. How does the shadow value v of cash (the derivative of the borrower's gross utility with respect to cash) vary with the level of cash?

Answer.