Economic Dynamics Theory and Computation
Excercises Chapter 4

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4 Deterministic Dynamical Systems

4.1 The Basic Model

Problem 4.1. *Show that if (S, h) is a dynamical system, if $x' \in S$ is the limit of some trajectory (i.e., $h^t(x) \to x'$ as $t \to \infty$ for some $x \in S$), and if h is continuous at x', then x' is a fixed point of h.*

Answer. Consider the sequence $\{x_t\} \subset S$ defined as $x_t = h^t(x)$ since h is a continuous function we have:

$$x_t = h^t(x) \to x' \qquad \Longrightarrow \qquad h(x_t) = h^{t+1}(x) \to h(x')$$

Since the limit of a sequence must be unique we have that

$$\boxed{h(x') = x'}$$

□

Problem 4.2. *Prove that if h is continuous on S and $h(A) \subset A$ (i.e., h maps $A \to A$), then $h(\operatorname{cl} A) \subset \operatorname{cl} A$*

Answer. Let $x \in h(\operatorname{cl} A)$ this means that there is $x' \in \operatorname{cl} A$ such that $h(x') = x$. If $x' \in A$ then $h(x') = x \in A$ if $x' \notin A$ then there is $\{x_t\} \subset A$ such that $x_t \to x'$. Note that since h is continuous then $h(x_t) \to h(x') = x$ and since $h(x_t) \in A$ for all t then $x \in \operatorname{cl} A$. □

Problem 4.3. *Prove that x^* is locally stable if and only if there exists an $\epsilon > 0$ such that $B(\epsilon, x^*) \subset \Lambda(x^*)$*

Answer. This problem is straightforward we get sufficiency since $B(\varepsilon; x^*)$ is an oppen set and necessity by definition of an open set that mus include a ball of radious $\varepsilon > 0$ for some ε. □

Problem 4.4. *Prove that if x^* is a fixed point of (S, h) to which every trajectory converges, then x^* is the only fixed point of (S, h).*

Answer. Prove it by contradiction suppose that there are $x^* \neq x^{**}$ fixed points, and that **every** trayectory convertges to x^*, but this must be a contradiction since x^{**} is a fixed point implies that the trayectory $h^t(x^{**}) = x^{**} \to x^{**}$. □

Problem 4.5. *Prove **Lemma 4.1.7**:*

If h is a map with continuous derivative h' and x^* is a fixed point of h with $|h'(x^*)| < 1$, then x^* is locally stable.

Answer. Consider x^* is a fixed point of h; by the definition of derivative we have that for any $x_n \to x^*$:

$$|h'(x^*)| = \lim_{n\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_n - x^*)} < 1$$

By the definition of limit we have that for every $\varepsilon > 0$ there are N_1 and N_2 such that $\rho(x_n, x^*) < \varepsilon$ for every $n > N_1$ and

$$\lim_{t\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_t, x^*)} < 1$$

for every $n > N_2$. Define $N = \max\{N_1, N_2\}$ and we have that:

$$\lim_{n\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_n, x^*)} < 1 \qquad \Longrightarrow \qquad \rho(h(x_n), h(x^*)) < \rho(x_n, x^*) < \varepsilon$$

therefore $h(x_n) \to h(x^*) = x^*$. We have proved that for any x *"close enough"* to x^* $h(x) \to x^*$. Pick any such point ($x \in B(\varepsilon; x^*)$)

Also note that for any $t > 0$

$$\rho(h^t(x), x^*) = \rho(h(h^{t-1}(x)), x^*) < \rho(h^{t-1}(x), x^*)$$

We can define a sequence ε_t such that $\varepsilon_1 = \varepsilon$, $\varepsilon_t \to 0$ and

$$0 \le \rho(h^t(x), x^*) < \varepsilon_t \qquad \Longrightarrow \qquad \rho(h^t(x), x^*) \to 0 \qquad \Longrightarrow \qquad h^t(x) \to x^*$$

Therefore x^* is locally stable.

□

Problem 4.6.

Answer. □

Problem 4.7.

Answer. □

Problem 4.8.

Answer. □

Problem 4.9.

Answer. □

Problem 4.10.

Answer. □

Problem 4.11.

Answer. □

Problem 4.12.

Answer. □

Problem 4.13.

Answer. □

Problem 4.14.

Answer. □

Problem 4.15.

Answer. □