Economic Dynamics Theory and Computation
Excercises Chapter 4

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4 Deterministic Dynamical Systems

4.1 The Basic Model

Problem 4.1. *Show that if (S,h) is a dynamical system, if $x' \in S$ is the limit of some trajectory (i.e., $h^t(x) \to x'$ as $t \to \infty$ for some $x \in S$), and if h is continuous at x', then x' is a fixed point of h.*

Answer. Consider the sequence $\{x_t\} \subset S$ defined as $x_t = h^t(x)$ since h is a continuous function we have:

$$x_t = h^t(x) \to x' \qquad \Longrightarrow \qquad h(x_t) = h^{t+1}(x) \to h(x')$$

Since the limit of a sequence must be unique we have that

$$\boxed{h(x') = x'}$$

$\square$

Problem 4.2. *Prove that if h is continuous on S and $h(A) \subset A$ (i.e., h maps $A \to A$), then $h(\operatorname{cl} A) \subset \operatorname{cl} A$*

Answer. Let $x \in h(\operatorname{cl} A)$ this means that there is $x' \in \operatorname{cl} A$ such that $h(x') = x$. If $x' \in A$ then $h(x') = x \in A$ if $x' \notin A$ then there is $\{x_t\} \subset A$ such that $x_t \to x'$. Note that since h is continuous then $h(x_t) \to h(x') = x$ and since $h(x_t) \in A$ for all t then $x \in \operatorname{cl} A$. $\square$

Problem 4.3. *Prove that x^* is locally stable if and only if there exists an $\epsilon > 0$ such that $B(\epsilon, x^*) \subset \Lambda(x^*)$*

Answer. This problem is straightforward we get sufficiency since $B(\varepsilon; x^*)$ is an oppen set and necessity by definition of an open set that mus include a ball of radious $\varepsilon > 0$ for some ε. $\square$

Problem 4.4. *Prove that if x^* is a fixed point of (S,h) to which every trajectory converges, then x^* is the only fixed point of (S,h).*

Answer. Prove it by contradiction suppose that there are $x^* \neq x^{**}$ fixed points, and that **every** trayectory convertges to x^*, but this must be a contradiction since x^{**} is a fixed point implies that the trayectory $h^t(x^{**}) = x^{**} \to x^{**}$. $\square$

Problem 4.5. *Prove* ***Lemma 4.1.7****:*

If h is a map with continuous derivative h' and x^* is a fixed point of h with $|h'(x^*)| < 1$, then x^* is locally stable.

Answer. Consider x^* is a fixed point of h; by the definition of derivative we have that for any $x_n \to x^*$:

$$|h'(x^*)| = \lim_{n\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_n - x^*)} < 1$$

By the definition of limit we have that for every $\varepsilon > 0$ there are N_1 and N_2 such that $\rho(x_n, x^*) < \varepsilon$ for every $n > N_1$ and

$$\lim_{t\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_t, x^*)} < 1$$

for every $n > N_2$. Define $N = \max\{N_1, N_2\}$ and we have that:

$$\lim_{n\to\infty} \frac{\rho(h(x_n), h(x^*))}{\rho(x_n, x^*)} < 1 \qquad \implies \qquad \rho(h(x_n), h(x^*)) < \rho(x_n, x^*) < \varepsilon$$

therefore $h(x_n) \to h(x^*) = x^*$. We have proved that for any x *"close enough"* to x^* $h(x) \to x^*$. Pick any such point ($x \in B(\varepsilon; x^*)$)

Also note that for any $t > 0$

$$\rho(h^t(x), x^*) = \rho(h(h^{t-1}(x)), x^*) < \rho(h^{t-1}(x), x^*)$$

We can define a sequence ε_t such that $\varepsilon_1 = \varepsilon$, $\varepsilon_t \to 0$ and

$$0 \le \rho(h^t(x), x^*) < \varepsilon_t \qquad \implies \qquad \rho(h^t(x), x^*) \to 0 \qquad \implies \qquad h^t(x) \to x^*$$

Therefore x^* is locally stable.

□

Problem 4.6. *A dynamical system (S, h) is called Lagrange stable if every trajectory is precompact in S. In other words, the set $\{h^n(x) : n \in \mathbb{N}\}$ is precompact for every $x \in S$(i.e every subsequence of the trajectory has a convergent subsubsequence). Show that if S is a closed and bounded subset of $\mathbb{R}^n$, then (S, h) is Lagrange stable for any choice of h.*

Answer. Since S is a bounded and closed subset of $\mathbb{R}^n$ then for any x $\{h^n(x) : n \in \mathbb{N}\} \subset S$ implies that the sequence $x_n = h^n(x)$ is bounded, then by **Bolzano-Weierstrass theorem** it must have a convergent subsequence and since S is closed that subsequence must converge to an element of S. □

Problem 4.7. *Give an example of a dynamical system (S, h) where S is unbounded but (S, h) is Lagrange stable.*

Answer. consider the followign two examples.

- Trivial example $S = \mathbb{R}$ and $h(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$.
- Less trivial, same S but $h(x) = x/c$ for some $|c| > 1$. Note that $h^t(x) = x/c^t \to 0$ for any $x \in \mathbb{R}$.

□

Problem 4.8. *Let $S = \mathbb{R}$, and let $h : \mathbb{R} \to \mathbb{R}$ be an increasing function, in the sense that if $x \le y$, then $h(x) \le h(y)$. Show that every trajectory of h is a monotone sequence in $\mathbb{R}$ (either increasing or decreasing).*

Answer. Consider an $x \in \mathbb{R}$ such that $h(x) \le x$ then $h^2(x) \le h(x)$ iterating forward $h^(t+1)(x) \le h^t(x)$ for all $t > 0$ therefore the trayectory is decreasing. Alternatively if $x \le h(x)$ we get an increasing trayectory. □

Problem 4.9. *Now order points in $\mathbb{R}^n$ by setting $x \leq y$ whenever $x_i \leq y_i$ for i in $\{1, \ldots, n\}$ (i.e., each component of x is dominated by the corresponding component of y). Let $S = \mathbb{R}^n$, and let $h : S \to S$ be monotone increasing. (The definition is the same.) Show that the same result no longer holds $-h$ does not necessarily generate monotone trajectories.*

Answer. MISSING!!! □

Problem 4.10. *Let $S = (\mathbb{R}, |\cdot|)$ and $h(x) = ax + b$. Prove that*

$$h^t(x) = a^t x + b \sum_{i=0}^{t-1} a^i \quad (x \in S, t \in \mathbb{N})$$

(Hint: Use induction.) From this expression, prove that (S, h) is globally stable whenever $|a| < 1$, and exhibit the fixed point.

Answer. Since

$$h(x) = h^1(x) = ax + b = a^1 x + b \sum_{i=0}^{0} a^i$$

we have our base case. Now, assume that for $t = k$

$$h^k(x) = a^k x + b \sum_{i=0}^{k-1} a^i$$

We will apply h to h^k to obtain h^{k+1} and check that the formula still holds:

$$\begin{aligned} h^{k+1}(x) = h(h^k(x)) &= a\left(a^k x + b \sum_{i=0}^{k-1} a^i\right) + b \\ &= a^{k+1} x + b \sum_{i=0}^{k-1} a^{i+1} + ba^0 \\ &= a^{k+1} x + b \sum_{i=0}^{k} a^i \end{aligned}$$

Then if $|a| < 1$

$$\lim_{t \to \infty} h^t(x) = b \sum_{i=0}^{\infty} a^i = \frac{b}{1-a} \qquad (\forall x \in S)$$

□

Problem 4.11. *Show that the condition $|a| < 1$ is also necessary, in the sense that if $|a| \geq 1$, then (S, h) is not globally stable. Show, in particular, that $h^t(x_0)$ converges to $x^* := b/(1-a)$ only if $x_0 = x^*$*

Answer. MISSING □

Problem 4.12. *Let (S, h) be as in 4.10 Using Banach's Fixed Point Theorem, prove that (S, h) is globally stable whenever $|a| < 1$.*

Answer. Recall:

Theorem. **1** (Banach)**.** *Let $T : S \to S$, where (S, ρ) is a complete metric space. If T is a uniform contraction on S with modulus λ, then T has a unique fixed point $x^* \in S$. Moreover for every $x \in S$ and $n \in \mathbb{N}$ we have $\rho(T^n x, x^*) \leq \lambda^n \rho(x, x^*)$, and hence $T^n x \to x^*$ as $n \to \infty$*

□

Consider

$$\rho(h(x), h(y)) = \rho(ax + b, ay + b) = \left(a\left(y + \frac{b}{a}\right), a\left(y + \frac{b}{a}\right)\right) = |a||x - y| = |a|\rho(x, y)$$

Then if $|a| < 1$ select $|0 < \lambda < |a| < 1$ and we have

$$\rho(h(x), h(y)) < \lambda\rho(x, y)$$

We have showed that $T : S \to S$ where $Tx = h(x)$ is a uniform contraction on S therefore it has a unique fixed point x^* and $h^n(x) \to x^*$ as $n \to \infty$.

Problem 4.13. *Let $S := (0, \infty)$ with $\rho(x, y) := |\ln x - \ln y|$. Prove that ρ is a metric on S and that (S, ρ) is a complete metric space. Consider the growth model $k_{t+1} = h(k_t) = sAk_t^\alpha$, where $s \in (0, 1], A > 0$ and $\alpha \in (0, 1)$. Convert this into a dynamical system on (S, ρ), and prove global stability using theorem Banach's Fixed Point Theorem.*

Answer. We start by proving the ρ is a metric:

(i) $\rho(x, y) = 0 \iff |\ln x - \ln y| = 0 \iff \ln x = \ln y \iff \exp(\ln x) = \exp(\ln y) \iff x = y$

(ii) $\rho(x, y) = |\ln x - \ln y| = |\ln y - \ln x| = \rho(y, x) \qquad \forall x, y \in S$

(iii) $\rho(x, y) = |\ln x - \ln y| = |(\ln x - \ln z) - (\ln y - \ln z)| \leq |\ln x - \ln z| + |\ln y - \ln z| = \rho(x, z) + \rho(y, z) \qquad \forall x, y, z \in S$

Next we show that (S, ρ) is a Complete Space. Consider a function $f : (S, \rho) \to (\mathbb{R}, |\cdot|)$ such that $f(x) = \ln x$ and a Cauchy sequence $\{x_n\}$:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \mid \rho(x_n, x_m) = |\ln x - \ln y| = |f(x) - f(y)| < \varepsilon \quad \forall n, m > N$$

then $\{f(x_n)\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ which is a compelte space thus $f(x_n) \to y = \ln x$ for some $x \in (0, \infty)$. Using the definition of limit we have:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \mid |f(x_n) - y| = |\ln x_n - \ln x| = \rho(x_n, x) < \varepsilon \quad \forall n > N$$

We have showed that $x_n \to x$ in (S, ρ), therefore (S, ρ) is complete.

Let $h : S \to S$ where $h(x) = sAx^\alpha$ consider $x, y \in S$:

$$\rho(h(x), h(y)) = |\ln(sAx^\alpha) - \ln(sAx^\alpha)| = |\cancel{\ln s} + \cancel{\ln A} + \alpha \ln x - \cancel{\ln s} - \cancel{\ln A} - \alpha \ln y| = |\alpha(\ln x - \ln y)| = \alpha\rho(x, y)$$

Since $\alpha \in (0, 1)$ we can select $0 < \lambda < \alpha < 1$ to get

$$\rho(h(x), h(y)) < \lambda\rho(x, y)$$

Since we have proved that h is a uniform contraction with modulus λ then we can apply Banach's Fixed Point Theorem to prove that $h$has only one fixed point x^* and that $\lim_{t\to\infty} h^t(x) = x^*$ thus concluding the prove that the dynamical system (S, h) is globally stable.

□

Problem 4.14. *Consider the mapping $h(x) = Ex + b$ where E is an $n \times n$ matrix and $b \in \mathbb{R}^n$, let $\|\cdot\|$ be any norm on $\mathbb{R}^n$, and define*

$$\lambda := \max\{\|Ex\| : x \in \mathbb{R}^n, \|x\| = 1\}$$

If you can, prove that the maximum exists. Using the properties of norms and linearity of E, show that $\|Ex\| \leq \lambda\|x\|$ for all $x \in \mathbb{R}^n$. Show in addition that if $\lambda < 1$, then $(\mathbb{R}^n, h)$ is globally stable.

Answer. We start by showing that the maximum exists. First define the set $A = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, clearly A is a bounded and closed, therefore compact (by Heine-Borel); next consider the continuous function $f : A \to \mathbb{R}$ defined as $f(x) = \|Ex\|$, since f is a continous function on a compact set it attains it's supremum.

Next we will show that h is a uniform contraction of modulus λ.

$$\|Ex\| = \frac{\|x\|}{\|x\|}\|Ex\| = \left\| E\left(\frac{x}{\|x\|}\right)\right\| \|x\| \leq \lambda\|x\|$$

Showing that $(\mathbb{R}, h)$ is globally stable is just an aplicaction of Banach's Fixed Point Theorem. □

Problem 4.15. *Prove that Long and Plosser's system is stable in the following way: Let $A = (a_{ij})$ be an $n \times n$ matrix where the sum of any of the rows of A is strictly less than 1 (i.e., $\max_i \alpha_i < 1$, where $\alpha_i := \sum_j |a_{ij}|$). Using the norm $\|\cdot\|_\infty$ in 4.14, show that for A we have $\lambda < 1$. Now argue that in Long and Plosser's model, (y_t) converges to a limit y^*, which is independent of initial output y_0, and, moreover, is the unique solution to the equation $y^* = Ay^* + b$.*

Answer. Consider $x \in S$ such that $\|x\|_\infty = \max_{i=1\dots n} |x_i| = 1$ and $\|Ax\|_\infty$ since $x = (x_1, \dots, x_n) = \sum_{i=1}^n x_i e_i$ where e_i is the canonical base of $\mathbb{R}^n$ then

$$\|Ax\|_\infty = \left\| A \sum_{i=1}^n x_i e_i \right\| \leq \sum_{i=1}^n |x_i| \|Ae_i\|_\infty \leq \|x\|_\infty \sum_{i=1}^n \|Ae_i\|_\infty$$

Notice that

$$Ae_i = \begin{pmatrix} a_{i1} \\ \dots \\ a_{in} \end{pmatrix} = \sum_{j=1}^n a_{ij} e_j$$

then

$$\|Ae_i\|_\infty \leq \sum_{j=1}^n |a_{ij}| \cancelto{1}{\|e_j\|} \leq \max_i \alpha_i < 1$$

Thus we have that

$$\lambda := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\} < 1$$

Now consider

$$h(y) = Ay + b$$

we know that

$$h^t(y_0) = A^t y_0 + b \sum_{i=0}^{t-1} A^i$$

First note that

$$0 \leq \lim_{t\to\infty} \|A^t y_0\| \leq \lim_{t\to\infty} \|y_0\| \left\| A^t \left(\frac{y}{\|y\|} \right) \right\| \leq \lim_{t\to\infty} \lambda^t \|y_0\| = 0$$

therefore we have showed that the limit does not depend on the initial output y_0.

Next consider $y^* = Ay^* + b$ and

$$\begin{aligned}
y^* - b\sum_{i=0}^{t-1} A^i &= Ay^* + b - b\sum_{i=0}^{t-1} A^i = Ay^* - b\sum_{i=1}^{t-1} A^i = A\left(y^* - b\sum_{i=1}^{t-1} A^{i-1}\right) = A\left(y^* - b\sum_{i=0}^{t-2} A^i\right) \\
&= A\left(Ay^* + b - b\sum_{i=0}^{t-1} A^i\right) = A\left(Ay^* - b\sum_{i=1}^{t-2} A^i\right) = A^2\left(y^* - b\sum_{i=1}^{t-2} A^{i-1}\right) = A\left(y^* - b\sum_{i=0}^{t-3} A^i\right) \\
&\dots \\
&= A^{t-1}(y^* - b) = A(Ay^* + b - b) = A^t y^*
\end{aligned}$$

Finally

$$\lim_{t\to\infty} \|h^t(y_0) - y^*\|_\infty = \|\lim_{t\to\infty} h^t(y_0) - y^*\|_\infty = \left\| \lim_{t\to\infty} y^* - b\sum_{i=0}^{t-1} A^i \right\|_\infty = \|\lim_{t\to\infty} A^t y^*\|_\infty = \lim_{t\to\infty} \|A^t y^*\|_\infty = 0$$

Therefore

$$h^t(y_0) \to y^* \qquad \forall y_0 \in \mathbb{R}^n$$

□

Problem 4.16. *Let $B = (b_{ij})$ be an $n \times n$ matrix where the sum of any of the columns of B is strictly less than 1 (i.e., $\max_j \beta_j < 1$, where $\beta_j := \sum_i |b_{ij}|$). Using the norm $\|\cdot\|_1$ in 4.14, show that for B we have $\lambda < 1$. Conclude that if $h(x) = Bx + b$, then $(\mathbb{R}^n, h)$ is globally stable.*

Answer. Follows for Problem 4.15 □

Problem 4.17. *Suppose that h is uniformly contracting on complete space S, so (S, h) is globally stable. Prove that if $A \subset S$ is nonempty, closed and invariant under h (i.e., $h(A) \subset A$), then the fixed point of h lies in A.*

Answer. From Banach's Fixed Point Theorem h has a unique fixed point x^* and $h^n(x) \to x^*$ for any $x \in S$ in particular select $x \in A$ and define $x_n = h^n(x)$, since A is invariant under h then $\{x_n\} \subset A$ and since $x_n \to x^*$ and A is closed then $x^* \in A$. □