Economic Dynamics Theory and Computation
Excercises Chapter 3

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Problem 1. *Let $\| \cdot \|$ on $\mathbb{R}^k$ be a norm that generates a metric ρ on $\mathbb{R}^k$ via $\rho(x,y) := \|x - y\|$. Show that ρ is indeed a metric, in the sense that it satisfies the three axioms in the definition.*

Answer. Recall the definition of norm:

Definition. 1. *A norm on $\mathbb{R}^k$ is a mapping $\mathbb{R}^k \ni x \mapsto \|x\| \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^k$ and any $\gamma \in \mathbb{R}$*

  1. *$\|x\| = 0$ if and only if $x = 0$*

  2. *$\|\gamma x\| = |\gamma| \|x\|$, and*

  3. *$\|x + y\| \leq \|x\| + \|y\|$.*

and the definition of metric:

Definition. 2. *A metric space is a nonempty set S and a metric or distance $\rho : S \times S \to \mathbb{R}$ such that, for any $x, y, v \in S$,*

  1. *$\rho(x,y) = 0$ if and only if $x = y$,*

  2. *$\rho(x,y) = \rho(y,x)$, and*

  3. *$\rho(x,y) \leq \rho(x,v) + \rho(v,y)$.*

To show that the propoerties of metric hold for ρ consider:

  1. $\rho(x,y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$ for any $x, y \in S$.

  2. $\rho(x,y) = \|x - y\| = \|y - x\| = \rho(y,x)$ for any $x, y \in S$

  3. $\rho(x,y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|y - z\| = \rho(x,z) + \rho(y,z)$ for any $x, y, z \in S$

$\square$

Problem 2. *Let $\| \cdot \|$ be a norm on $\mathbb{R}^k$. Show that for any $x, y \in \mathbb{R}^k$ we have $| \|x\| - \|y\| | \leq \|x - y\|$.*

Answer. Assume w.l.o.g. that $\|x\| \geq \|y\|$ then

$$\left| \|x\| - \|y\| \right| = \|x\| - \|y\|$$

and

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \qquad \implies \qquad \|x\| - \|y\| \leq \|x - y\|$$

$\square$

Problem 3. *Consider the family of norms* $\|\cdot\|_p$ *defined by*

$$\|x\|_p := \left(\sum_{i=1}^{k} |x_i|^p \right)^{1/p} \qquad \left(x \in \mathbb{R}^k \right)$$

where $p \geq 1$, *and*

$$\|x\|_\infty := \max_{1 \leq i \leq k} |x_i|$$

Prove that $\|\cdot\|_p$ *is a norm on* $\mathbb{R}^k$ *for* $p = 1$ *and* $p = \infty$.

Answer. Start with $p = 1$:

1. $\|x\|_1 = 0 \iff \sum_{i=1}^{k} |x_i| = 0 \iff |x_i| = 0 \,\forall i = 1, \ldots, k \iff x = 0$

2. $\|\lambda x\|_1 = \sum_{i=1}^{k} |\lambda x_i| \sum_{i=1}^{k} |\lambda| \, |x_i| = |\lambda| \sum_{i=1}^{k} |x_i| = |\lambda| \|x\|_1$ for all $x \in \mathbb{R}^k$, *lambda* $\in \mathbb{R}$

3. $\|x + y\|_1 = \sum_{i=1}^{k} |x_i + y_i|$ since $|x_i + y_i| \leq |x_i| + |y_i|$ for all $i = 1, \ldots, k$ then $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ for all $x, y \in \mathbb{R}^k$

Then the proof is complete.

For the case when $p = \infty$:

1. $\|x\|_\infty = 0 \iff \max_{1 \leq i \leq k} |x_i| = 0 \iff 0 \leq |x_i| \leq \max_{1 \leq i \leq k} |x_i| = 0 \,\forall i = 1, \ldots, k \iff x = 0$

2. $\|\lambda x\|_\infty = \max_{1 \leq i \leq k} |\lambda x_i| = \max_{1 \leq i \leq k} |\lambda| |x_i| = |\lambda| \max_{1 \leq i \leq k} |x_i| = |\lambda| \|x\|_\infty$ for all $x \in \mathbb{R}^k$, *lambda* $\in \mathbb{R}$

3. $\|x + y\|_\infty = \max_{1 \leq i \leq k} |x_i + y_i| = |x_j + y_j|$ for some $j \in 1, \ldots, k$; then $|x_j + y_j| \leq |x_j| + |y_j| \leq \max_{1 \leq i \leq k} |x_i| + \max_{1 \leq i \leq k} |y_i| = \|x\|_\infty + \|y\|_\infty$ for all $x, y \in \mathbb{R}^k$

$\square$

Problem 4. *Let* (x_n) *and* (y_n) *be sequences in* S. *Show that if* $x_n \to x \in S$ *and* $\rho(x_n, y_n) \to 0$, *then* $y_n \to x$

Proof. Recall that:

$$x_n \to x \qquad \iff \qquad \forall \varepsilon_1 > 0 \, \exists N_1 \in \mathbb{N} \mid \rho(x_n, x) < \varepsilon_1$$

and

$$\rho(x_n, y_n) \to 0 \qquad \iff \qquad \forall \varepsilon_1 > 0 \, \exists N_2 \in \mathbb{N} \mid |\rho(x_n, y_n)| < \varepsilon_2$$

Then consider $\rho(y_n, x)$ and note that

$$\rho(y_n, x) < \rho(x_n, y_n) + \rho(x_n, x) \leq \varepsilon = \varepsilon_1 + \varepsilon_2 \qquad \forall \, n > \max\{N_1, N_2\}$$

$\square$

Problem 5. *Let* $(x_n) \subset S$ *and* $x \in S$. *Show that* $x_n \to x$ *if and only if for all* $\varepsilon > 0$, *the ball* $B(\varepsilon; x)$ *contains all but finitely many terms of* (x_n).

Answer. It follows from the definition of convergence that:

$$x_n \to x \qquad \iff \qquad \forall \varepsilon > 0 \, \exists N \in \mathbb{N} \mid \rho(x_n, x) < \varepsilon \qquad \iff \qquad x_n \in B(\varepsilon; x) \forall n \geq N$$

This means that only the N first terms of the sequence are **not** in the ball $B(\varepsilon; x)$. $\square$

Problem 6. *Show that every convergent sequence in S is also bounded.*

Answer. Recall that a subset E of S is called bounded if $E \subset B(n; x)$ for some $x \in S$ and some (suitably large) $n \in \mathbb{N}$. A sequence (x_n) in S is called bounded if its range $\{x_n : n \in \mathbb{N}\}$ is a bounded set.

Assume that (x_n) is a convergent sequence then and select any $\varepsilon > 0$ we know that thre is N such that $x_n \in B(\varepsilon; x)$ for all $n > N$.

For all x_n with $n \leq N$ define $\delta_n = \rho(x_n, x) + 0.1$ then and $\bar{\varepsilon} = \max\{\delta_1, \ldots, \delta_N, \varepsilon\}$. It is clear by construction that $\rho(x_n, x) \leq \bar{\varepsilon}$ thus $\{x_n\}_{n=1}^{\infty} \subset B(\bar{\varepsilon}; x)$, therefore the sequence is bounded. $\qquad\square$

Problem 7. *Show that for $(x_n) \subset S, x_n \to x$ for some $x \in S$ if and only if ever subsequence of (x_n) converges to x.*

Proof. Sufficiency is trivial since the whole sequence is a subsequence of itself.

To show necessity consider $x_n \to x$ and a subsequence $\{x_{k_n}\} \subset \{x_n\}$ since subsequences are defined by increasing functions we know that if the term x_n is "behind" x_m in the original seuqnces then and those terms are in a subsequence then the ordering is preserved. Then for any *varepsilon* > 0 we have that there is a term x_N such that for any term x_n with $n > N$ $\rho(x_n, x_n) < \varepsilon$. Then let consider x_{k_N} the first term in the subsequence that corresponds to x_N in the original sequence this means that any terms that follow x_{k_N} is close enough to x so the definition of convergence holds. $\qquad\square$