Economic Dynamics Theory and Computation
Excercises Chapter 3

Mitchell Valdés-Bobes

June 24, 2021

Problem 1. *Let $\|\cdot\|$ on $\mathbb{R}^k$ be a norm that generates a metric ρ on $\mathbb{R}^k$ via $\rho(x, y) := \|x - y\|$. Show that ρ is indeed a metric, in the sense that it satisfies the three axioms in the definition.*

Answer. Recall the definition of norm:

Definition. 1. *A norm on $\mathbb{R}^k$ is a mapping $\mathbb{R}^k \ni x \mapsto \|x\| \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^k$ and any $\gamma \in \mathbb{R}$*

1. $\|x\| = 0$ *if and only if* $x = 0$
2. $\|\gamma x\| = |\gamma|\|x\|$, *and*
3. $\|x + y\| \leq \|x\| + \|y\|$.

and the definition of metric:

Definition. 2. *A metric space is a nonempty set S and a metric or distance $\rho : S \times S \to \mathbb{R}$ such that, for any $x, y, v \in S$,*

1. $\rho(x, y) = 0$ *if and only if* $x = y$,
2. $\rho(x, y) = \rho(y, x)$, *and*
3. $\rho(x, y) \leq \rho(x, v) + \rho(v, y)$.

To show that the propoerties of metric hold for ρ consider:

1. $\rho(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$ for any $x, y \in S$.
2. $\rho(x, y) = \|x - y\| = \|y - x\| = \rho(y, x)$ for any $x, y \in S$
3. $\rho(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|y - z\| = \rho(x, z) + \rho(y, z)$ for any $x, y, z \in S$

□

Problem 2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^k$. Show that for any $x, y \in \mathbb{R}^k$ we have $|\ \|x\| - \ \|y\|\ | \leq \|x - y\|$.*

Answer. Assume w.l.o.g. that $\|x\| \geq \|y\|$ then

$$\big|\|x\| - \|y\|\big| = \|x\| - \|y\|$$

and

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \qquad \implies \qquad \|x\| - \|y\| \leq \|x - y\|$$

□

Problem 3. *Consider the family of norms* $\|\cdot\|_p$ *defined by*

$$\|x\|_p := \left(\sum_{i=1}^{k} |x_i|^p\right)^{1/p} \qquad \left(x \in \mathbb{R}^k\right)$$

where $p \geq 1$*, and*

$$\|x\|_\infty := \max_{1\leq i\leq k} |x_i|$$

Prove that $\|\cdot\|_p$ *is a norm on* $\mathbb{R}^k$ *for* $p = 1$ *and* $p = \infty$*.*

Answer. Start with $p = 1$:

1. $\|x\|_1 = 0 \iff \sum_{i=1}^{k} |x_i| = 0 \iff |x_i| = 0\, \forall i = 1, \ldots, k \iff x = 0$
2. $\|\lambda x\|_1 = \sum_{i=1}^{k} |\lambda x_i| \sum_{i=1}^{k} |\lambda|\, |x_i| = |\lambda| \sum_{i=1}^{k} |x_i| = |\lambda|\|x\|_1$ for all $x \in \mathbb{R}^k$, $lambda \in \mathbb{R}$
3. $\|x + y\|_1 = \sum_{i=1}^{k} |x_i + y_i|$ since $|x_i + y_i| \leq |x_i| + |y_i|$ for all $i = 1, \ldots, k$ then $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ for all $x, y \in \mathbb{R}^k$

Then the proof is complete.

For the case when $p = \infty$:

1. $\|x\|_\infty = 0 \iff \max_{1\leq i\leq k} |x_i| = 0 \iff 0 \leq |x_i| \leq \max_{1\leq i\leq k} |x_i| = 0\, \forall i = 1, \ldots, k \iff x = 0$
2. $\|\lambda x\|_\infty = \max_{1\leq i\leq k} |\lambda x_i| = \max_{1\leq i\leq k} |\lambda||x_i| = |\lambda| \max_{1\leq i\leq k} |x_i| = |\lambda|\|x\|_\infty$ for all $x \in \mathbb{R}^k$, $lambda \in \mathbb{R}$
3. $\|x + y\|_\infty = \max_{1\leq i\leq k} |x_i + y_i| = |x_j + y_j|$ for some $j \in 1, \ldots, k$; then $|x_j + y_j| \leq |x_j| + |y_j| \leq \max_{1\leq i\leq k} |x_i| + \max_{1\leq i\leq k} |y_i| = \|x\|_\infty + \|y\|_\infty$ for all $x, y \in \mathbb{R}^k$

$\square$