Economic Dynamics Theory and Computation
Excercises Chapter 3

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Problem 1. *Let $\|\cdot\|$ on $\mathbb{R}^k$ be a norm that generates a metric ρ on $\mathbb{R}^k$ via $\rho(x,y) := \|x-y\|$. Show that ρ is indeed a metric, in the sense that it satisfies the three axioms in the definition.*

Answer. Recall the definition of norm:

Definition. 1. *A norm on $\mathbb{R}^k$ is a mapping $\mathbb{R}^k \ni x \mapsto \|x\| \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^k$ and any $\gamma \in \mathbb{R}$*

1. $\|x\| = 0$ *if and only if* $x = 0$
2. $\|\gamma x\| = |\gamma|\|x\|$, *and*
3. $\|x+y\| \leq \|x\| + \|y\|$.

and the definition of metric:

Definition. 2. *A metric space is a nonempty set S and a metric or distance $\rho : S \times S \to \mathbb{R}$ such that, for any $x, y, v \in S$,*

1. $\rho(x,y) = 0$ *if and only if* $x = y$,
2. $\rho(x,y) = \rho(y,x)$, *and*
3. $\rho(x,y) \leq \rho(x,v) + \rho(v,y)$.

To show that the propoerties of metric hold for ρ consider:

1. $\rho(x,y) = 0 \iff \|x-y\| = 0 \iff x - y = 0 \iff x = y$ for any $x, y \in S$.
2. $\rho(x,y) = \|x-y\| = \|y-x\| = \rho(y,x)$ for any $x, y \in S$
3. $\rho(x,y) = \|x-y\| = \|(x-z)+(z-y)\| \leq \|x-z\| + \|y-z\| = \rho(x,z) + \rho(y,z)$ for any $x, y, z \in S$

□

Problem 2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^k$. Show that for any $x, y \in \mathbb{R}^k$ we have $|\ \|x\| - \|y\|\ | \leq \|x-y\|$.*

Answer. Assume w.l.o.g. that $\|x\| \geq \|y\|$ then

$$\big|\|x\| - \|y\|\big| = \|x\| - \|y\|$$

and

$$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\| \qquad \Longrightarrow \qquad \|x\| - \|y\| \leq \|x-y\|$$

□

Problem 3. *Consider the family of norms $\|\cdot\|_p$ defined by*

$$\|x\|_p := \left(\sum_{i=1}^{k} |x_i|^p\right)^{1/p} \qquad (x \in \mathbb{R}^k)$$

where $p \geq 1$, and

$$\|x\|_\infty := \max_{1\leq i\leq k} |x_i|$$

Prove that $\|\cdot\|_p$ is a norm on $\mathbb{R}^k$ for $p = 1$ and $p = \infty$.

Answer. Start with $p = 1$:

1. $\|x\|_1 = 0 \iff \sum_{i=1}^{k} |x_i| = 0 \iff |x_i| = 0\, \forall i = 1, \ldots, k \iff x = 0$
2. $\|\lambda x\|_1 = \sum_{i=1}^{k} |\lambda x_i| \sum_{i=1}^{k} |\lambda|\, |x_i| = |\lambda| \sum_{i=1}^{k} |x_i| = |\lambda| \|x\|_1$ for all $x \in \mathbb{R}^k$, $lambda \in \mathbb{R}$
3. $\|x + y\|_1 = \sum_{i=1}^{k} |x_i + y_i|$ since $|x_i + y_i| \leq |x_i| + |y_i|$ for all $i = 1, \ldots, k$ then $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ for all $x, y \in \mathbb{R}^k$

Then the proof is complete.

For the case when $p = \infty$:

1. $\|x\|_\infty = 0 \iff \max_{1\leq i\leq k} |x_i| = 0 \iff 0 \leq |x_i| \leq \max_{1\leq i\leq k} |x_i| = 0\, \forall i = 1, \ldots, k \iff x = 0$
2. $\|\lambda x\|_\infty = \max_{1\leq i\leq k} |\lambda x_i| = \max_{1\leq i\leq k} |\lambda||x_i| = |\lambda| \max_{1\leq i\leq k} |x_i| = |\lambda| \|x\|_\infty$ for all $x \in \mathbb{R}^k$, $lambda \in \mathbb{R}$
3. $\|x + y\|_\infty = \max_{1\leq i\leq k} |x_i + y_i| = |x_j + y_j|$ for some $j \in 1, \ldots, k$; then $|x_j + y_j| \leq |x_j| + |y_j| \leq \max_{1\leq i\leq k} |x_i| + \max_{1\leq i\leq k} |y_i| = \|x\|_\infty + \|y\|_\infty$ for all $x, y \in \mathbb{R}^k$

$\square$

Problem 4. *Let (x_n) and (y_n) be sequences in S. Show that if $x_n \to x \in S$ and $\rho(x_n, y_n) \to 0$, then $y_n \to x$*

Proof. Recall that:

$$x_n \to x \qquad \iff \qquad \forall \varepsilon_1 > 0\, \exists N_1 \in \mathbb{N} \mid \rho(x_n, x) < \varepsilon_1$$

and

$$\rho(x_n, y_n) \to 0 \qquad \iff \qquad \forall \varepsilon_1 > 0\, \exists N_2 \in \mathbb{N} \mid |\rho(x_n, y_n)| < \varepsilon_2$$

Then consider $\rho(y_n, x)$ and note that

$$\rho(y_n, x) < \rho(x_n, y_n) + \rho(x_n, x) \leq \varepsilon = \varepsilon_1 + \varepsilon_2 \qquad \forall\, n > \max\{N_1, N_2\}$$

$\square$

Problem 5. *Let $(x_n) \subset S$ and $x \in S$. Show that $x_n \to x$ if and only if for all $\varepsilon > 0$, the ball $B(\varepsilon; x)$ contains all but finitely many terms of (x_n).*

Answer. It follows from the definition of convergence that:

$$x_n \to x \qquad \iff \qquad \forall \varepsilon > 0\, \exists N \in \mathbb{N} \mid \rho(x_n, x) < \varepsilon \qquad \iff \qquad x_n \in B(\varepsilon; x) \forall n \geq N$$

This means that only the N first terms of the sequence are **not** in the ball $B(\varepsilon; x)$. $\square$

Problem 6. *Show that every convergent sequence in S is also bounded.*

Answer. Recall that a subset E of S is called bounded if $E \subset B(n; x)$ for some $x \in S$ and some (suitably large) $n \in \mathbb{N}$. A sequence (x_n) in S is called bounded if its range $\{x_n : n \in \mathbb{N}\}$ is a bounded set.

Assume that (x_n) is a convergent sequence then and select any $\varepsilon > 0$ we know that thre is N such that $x_n \in B(\varepsilon; x)$ for all $n > N$.

For all x_n with $n \leq N$ define $\delta_n = \rho(x_n, x) + 0.1$ then and $\bar{\varepsilon} = \max\{\delta_1, \ldots, \delta_N, \varepsilon\}$. It is clear by construction that $\rho(x_n, x) \leq \bar{\varepsilon}$ thus $\{x_n\}_{n=1}^{\infty} \subset B(\bar{\varepsilon; }x)$, therefore the sequence is bounded. $\square$

Problem 7. *Show that for $(x_n) \subset S, x_n \to x$ for some $x \in S$ if and only if ever subsequence of (x_n) converges to x.*

Proof. Sufficiency is trivial since the whole sequence is a subsequence of itself.

To show necessity consider $x_n \to x$ and a subsequence $\{x_{k_n}\} \subset \{x_n\}$ since subsequences are defined by increasing functions we know that if the term x_n is "behind" x_m in the original seuqnces then and those terms are in a subsequence then the ordering is preserved. Then for any $varepsilon > 0$ we have that there is a term x_N such that for any term x_n with $n > N$ $\rho(x, x_n) < \varepsilon$. Then let consider x_{k_N} the first term in the subsequence that corresponds to x_N in the original sequence this means that any terms that follow x_{k_N} is close enough to x so the definition of convergence holds.

$\square$

Problem 8. *Let $f(x, y) = x^2 + y^2$. Show that f is a continuous function from $(\mathbb{R}^2, d_2)$ into $(\mathbb{R}, |\cdot|)$.*

Answer.

$$
\begin{aligned}
(x_n, y_n) \to (x, y) \quad &\iff \quad x_n \to x \quad \textbf{and} \quad y_n \to y \\
&\iff \quad x_n^2 \to x^2 \quad \textbf{and} \quad y_n^2 \to y^2 \\
&\iff \quad \forall \frac{\varepsilon}{2} \exists N \in \mathbb{N} \mid \forall n > N \quad |x_n^2 - x^2| < \frac{\varepsilon}{2} \quad \textbf{and} \quad |y_n^2 - y^2| < \frac{\varepsilon}{2} \\
&\iff \quad \forall \varepsilon > 0 \quad |(x_n^2 + y_n^2) - (x^2 + y^2)| \leq |x_n^2 - x^2| + |y_n^2 - y^2| < \varepsilon \\
&\iff \quad f(x_n, y_n) \to f(x, y)
\end{aligned}
$$

$\square$

Problem 9. *Let f and g be as above, and let S be a metric space. Show that if f and g are continuous, then so are $f + g$ and fg.*

Answer. $\square$

Problem 10. *A function $f : S \to \mathbb{R}$ is called upper-semicontinuous (usc) at $x \in S$ if, for every $x_n \to x$, we have $\limsup_n f(x_n) \leq f(x)$; and lower-semicontinuous (lsc) if, for every $x_n \to x$, we have $\liminf_n f(x_n) \geq f(x)$. Show that f is usc at x if and only if $-f$ is* lsc *at x. Show that f is continuous at x if and only if it is both usc and lsc at x.*

Answer. Recall that

$$
\liminf a_n := \lim_{n \to \infty} \inf_{k \geq n} a_k \qquad \textbf{and} \qquad \limsup a_n := \lim_{n \to \infty} \sup_{k \geq n} a_k
$$

Let f be an uper-semicontinuous function then:

$$
x_n \to x \qquad \Longrightarrow \qquad \limsup_n f(x_n) \leq f(x)
$$

Note that

$$\forall n \qquad \sup_{k \geq n} f(x_k) = -\inf_{k \geq n} -f(x_k)$$

Therefore

$$\limsup_n f(x_n) \leq f(x) \qquad \Longrightarrow \qquad -\limsup_n f(x_n) = \liminf_n f(x_n) \geq -f(x)$$

$\square$