Macro-dynamics of Sorting between Workers and Firms (and Locations)

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Table of contents

1 Introduction

Notes:

- I adopt the framework from (Lise and Robin 2017).
- Introducing a fixed amount of locations and population distribution of workers should be an equilibrium outcome.
- Workers can "partially" direct their search to each location.
  - In practice workers randomize over locations paying a cost.
  - Cost of random search is zero.
  - Cost increases unboundedly as the worker directs their search to a particular location.
- Productivity depends on the quality of match and aggregate state as in the original paper. I want to introduce a third component which is location productivity.
  - I intend to use idea exchange as the driver of location productivity.
  - I plan to draw on a framework similar to (Davis and Dingel 2019) to model the value added to production by idea exchange.

TODO:

- ☐ Read (Sims 2003)
- ☐ Check that the unemployment value function is a contraction.
- ☐ Think about distribution of firms across and within locations.
- ☐ Solve a modified version of the model where there is no aggregate state.
- ☐ Solve a two location version of the model to gain intuition.
- ☐ Think if I could include the cost of moving to a different location in the cost of the mixed strategy.
- ☐ For now the cost of posting only depends on the number of vacancies, should I introduce a location component? I think It will be important to have only high productive firms posting a lot of vacancies in high productivity locations.

2 Model

2.1 Model Setup

2.1.1 Demographics

- There is a discrete and finite set of locations $\mathcal{J} = \{1, \ldots, |\mathcal{J}|\}$ locations indexed by $j \in \mathcal{J}$.
- Continuum of workers indexed by their ability $x \in \mathcal{X}$.
  - The total measure is normalized to 1.
  - Exogenous distribution $\ell(x)$
  - Endogenous distribution in each location $\ell^j(x)$
  - Denote μ_j the total population in location j.
- Continuum of firms indexed by technology $y \in \mathcal{Y}$.
  - Total measure is normalized to 1.
  - Uniformly distributed.

2.1.2 Technology

- There is exogenous cost of posting v job opportunities in location j is

$$c_j(v) \geq 0$$

  assume it is increasing, convex, and independent of the firm type y (but *potentially* dependent on location).
- The aggregate state of the economy is indexed by z_t .
  - Changes from z to z' according to the transition probability $\pi(z, z')$.
- Workers and firms discount the future at the rate β.
- Workers can move across locations:
  - Workers choose a mixed strategy to search:

$$\phi^i_j(x) = \{\phi^i_j(x, j')\}_{j' \in \mathcal{J}}$$

    where $\phi^i_j(x, j')$ is the probability that a type x worker from location j search in location j' and $i \in \{u, e\}$ refers to the employment status of the worker.

- Each strategy has an asociated cost $c_s(\phi_i^j(x))$: [1]

$$c_s(\phi_j^i(x)) = c_1 \left(\sum_{j' \in \mathcal{J}} \phi_j^i(x, j') \log \left(J \phi_j^i(x, j') \right) \right)$$

- Some works in the literature that use this cost structure are (Wu 2020) and (Cheremukhin, Restrepo-Echavarria, and Tutino 2020).
  - (Wu 2020) mentions that this cost structure can be derived from rational inantention as in (Sims 2003).
- When a worker move they must pay a cost $F^{j \to j'} \geq 0$ (with $F^{j \to j} = 0$).

- Unemployed workers instant utility in each location is $b(x, z, j)$.[2]
- Firms have access to a production technology, defined at the match level and depending on the location and the aggregate state of the economy $f(x, y, j, z)$.

2.1.3 Job Search

The timing is as follows:

1. At time (t), distributions of employed and unemployed workers are inherited from $t-1$.
   - $u_t^j(x)$ is the measure of type-x Unemployed workers at the location j.
   - $h_t^j(x, y)$ is the measure of type-x workers employed at the firm y at the location j.
   - Note that on each location:

$$u_t^j(x) + \int h_t^j(x, y) dy = \ell^j(x)$$

2. Aggregate state changes $z_{t-1} \to z_t$.
3. At time $(t+)$ Separations occur:
   - $u_{t+}^j(x)$ is the measure of the type x Unemployed workers in the location j after the shock.
   - $h_{t+}^j(x, y)$ is the measure of type x workers that remain employed at firm y in location j.
4. Unemployed and employed workers draw new offers.

[1]The cost is proportional to the Kullback–Leibler divergence between the selected distribution and a uniform. Note that if the worker selects random search (i.e. uniform distribution) the associated cost is 0 and the cost grows unboundedly large as the strategy gets closer to perfectly directing search to a particular location. More information on the Kullback–Leibler divergence here

[2]In (Lise and Robin 2017) $b(x)$ stands for unemploymet benefits, I want to be more general to be able to include differerences in cost of living across locations

Both unemployed and employed workers search, denote s the search intensity of an employed worker and 1 is the (normalized) search intensity of an unemployed worker. The total search intensity in location j is

$$L_t^j = \sum_{j' \in \mathcal{J}} \left[\int \phi_u^{j'}(x,j) u_{t+}^{j'}(x) dx + s \int \int \phi_s^{j'}(x,j) h_{t+}^{j'}(x,y) dx dy \right]$$

Let $v_t^j(y)$ be the number of job opportunities posted by a firm y at time t in location j.

- $V_t^j = \int v_t^j(y) dy$ is the total number of job opportunities posted at the time t in the location j.

Let $M_t^j = M(L_t^j, V_t^j)$ be the number of job matches in the location j then:

- The probability that an unemployed worker contacts a vacancy in the location j is

$$p_t^j = \frac{M_t^j}{L_t^j}$$

  - sp_t^j is the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker

$$q_t^j = \frac{M_t^j}{V_t^j}$$

- Let $\theta_t^j = V_t^j / L_t^j$ be the market tightness in location j.

2.2 Dynamic Programming Problem

Denote:

- $U_t^j(x)$ the value for an unemployed worker of type x at time t at location j.
- The value of getting an offer depends on whether the worker is employed or not:
  - $W_{0,t}^{j' \to j}(x,y)$ is the value of a type-x unemployed worker at location j' who is hired by a firm of type y at a location j.
  - $W_{1,t}^{j' \to j}(x,y,y')$ is the value offered at the time t by type y firm at location j to a type x worker employment at a type y' firm in location j'.
- $J_t^j(x,y)$ is the value of a match between a type x worker and a type y firm at time t in location j.

2.2.1 Unemployed Worker

Unemployed workers receive instant utility from living in location j, $b(x, j, z_t)$, and anticipate next period's aggregate state z_{t+1}, and the probability of getting an offer p^j_{t+1} in each location. They will choose the strategy that maximizes their future expected value knowing that in each location they will receive an offer which can be from any firm with a likelihood proportional to the share of total vacancies posted by each firm in each market. The worker will accept only the offers that promise her a higher value than unemployment:

$$
\begin{aligned}
U^j_t(x) = \underbrace{b(x, j, z_t)}_{\text{instant utility}} + \beta \max_{\phi^j_u(x)} \Bigg\{ & \underbrace{- c(\phi^j_u(x))}_{\text{cost of search strategy}} + \\
& \mathbb{E}_t \sum_{j' \in \mathcal{J}} \underbrace{\phi^j_u(x, j')}_{\text{weight by probability of search in} j'} \Bigg[\overbrace{(1 - p^{j'}_{t+1}) U^{j'}_{t+1}(x)}^{\text{no offer, stays unemployed now in} j'} \\
& + p^{j'}_{t+1} \underbrace{\int \max\left\{ U^{j'}_{t+1}(x), W^{j \to j'}_{0,t+1}(x, y) \right\} \frac{v^{j'}_{t+1}(y)}{V^{j'}_{t+1}} dy - F^{j \to j'}}_{\text{if offer, pays cost,moves to } j \text{ and then is matched randomly with some firm}} \Bigg] \Bigg\}
\end{aligned}
$$

Assume that workers don't have bargaining power, therefore they are offered their reservation value $U^{j'}_t(x) = W^{j \to j'}_{0,t+1}(x, y)$ by firms:

$$
U^j_t(x) = b(x, j, z) + \beta \max_{\phi^j_u(x)} \left\{ \sum_{j' \in \mathcal{J}} \left(\phi^j_u(x, j') \mathbb{E}_t \left[U^{j'}_{t+1}(x) - F^{j \to j'} \right] \right) - c(\phi^j_u(x)) \right\}
$$

This means that the optimal search policy for each worker is

$$
\phi^j_u(x, j') = \frac{e^{\left(\mathbb{E}_t \left[U^{j'}_{t+1}(x) - F^{j \to j'} \right] / c_1 \right)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{\left(\mathbb{E}_t \left[U^{j \to \tilde{j}}_{t+1}(x) \right] / c_1 \right)}} \tag{2.1}
$$

derivation is in Appendix A.

Substituting the optimal strategy into the Bellman Equation of the unemployed worker we get (derivation in Appendix B):

$$
U^j_t(x) = b(x, j, z) + \beta c_1 \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\mathbb{E}_t [U^{j'}_{t+1}(x) - F^{j \to j'}] / c_1 \right) - \log(J) \right)
$$

or simply

$$U_t^j(x) = b(x,j,z) + \beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t[U_{t+1}^{j'}(x) - F^{j\to j'}]}{c_1}\right\}\right) - \beta c_1 \log J \tag{2.2}$$

Where $\text{lse}(x \in \mathbb{R}^n)$ is the **log-sum-exp** function.

2.2.2 Joint Value of a Match

- If a match between a worker and a firm in location j is destroyed the firm will get 0 and the worker gets their unemployment value in that location $U_t^j(x)$.
- Matches are destroyed for two reasons:
  - *Exogenous destruction* with probability δ
  - *Endogenous destruction*, if and only if $J_t^j(x,y) < U_t^j(x)$.
    - Denote $\lambda_t^j(x,y) = \mathbb{1}_{\{J_t^j(x,y) > U_t^j(x,y)\}}$

We can write the Bellman equation of a match value as:

$$\begin{aligned} J_t^j(x,y) = \; & \underbrace{f(x,y,j,z_t)}_{\text{match value added}} + \beta \max_{\phi_s^j(x)} \Bigg\{ \mathbb{E}_t \Bigg[\overbrace{(1-(1-\delta)\lambda_{t+1}^j(x,y))}^{\text{match is destroyed}} \underbrace{U_{t+1}^j(x)}_{\text{worker gets unemployment value}} + \\ & + \underbrace{(1-\delta)\lambda_t^j(x,y)}_{\text{match survives}} \max_{\phi_s^j(x)} \Bigg\{ -c(\phi_s^j(x)) + \sum_{j' \in \mathcal{J}} \phi_s^j(x,j') \Bigg[\overbrace{(1-sp_t^{j'})}^{\text{no new offers}} \underbrace{J_{t+1}^j(x,y)}_{\text{stays with same firm}} + \\ & + sp_t^{j'} \underbrace{\int \max\{J_{t+1}^j(x,y), W_{1,t+1}^{j\to j'}(x,y',y) - F^{j\to j'}\} \frac{v_{t+1}^{j'}(x)}{V_{t+1}^{j'}} dy'}_{\text{worker only accepts new offers if value is greater than current match}} \Bigg] \Bigg\} \end{aligned}$$

When a type x worker employed at a type y firm in city j receives an offer from a type y' in city j' then there is a sequential auction like in (Postel-Vinay and Robin 2002). More productive firms can offer higher values. The key difference with (Postel-Vinay and Robin 2002) is that location plays a role: if a firm is located in a different location than the worker's current employer then the poaching firm must cover the cost of moving, this leads to two possible outcomes:

- $J_{t+1}^{j'}(x,y') > J_{t+1}^j(x,y) + F^{j\to j'}$ the worker moves from $(j,y) \to (j',y')$ and receives $W_{1,t+1}^{j\to j'}(x,y',y)$

- $J^j_{t+1}(x,y) > J^{j'}_{t+1}(x,y') - F^{j\to j'}$ the worker stays at (j,y) and receives $W^j_{1,t+1}(x,y,y')$

to able to poach from different locations the firm must be at least $F^{j\to j'}$ more productive.

As in (Postel-Vinay and Robin 2002) if the worker is hired by the poaching firm the worker receives the incumbent firm reservation value plus the cost of changing jobs, i.e.

$$J^{j'}_{t+1}(x,y') > J^j_{t+1}(x,y) + F^{j\to j'} \qquad \Longrightarrow \qquad W^{j\to j'}_{1,t+1}(x,y',y) = J^j_{t+1}(x,y) + F^{j\to j'}$$

therefore

$$\begin{aligned} J^j_t(x,y) = f(x,y,j,z_t) + \beta \max_{\phi^j_s(x)} \Big\{ \mathbb{E}_t \Big[&(1-(1-\delta)\lambda^j_{t+1}(x,y))U^j_{t+1}(x) \\ &+(1-\delta)\lambda^j_t(x,y)\sum_{j'\in\mathcal{J}} \phi^j_s(x,j')J^j_{t+1}(x,y)\Big] - c(\phi^j_s(x))\Big\} \end{aligned}$$

Note that the optimal strategy for employed workers is to engage in random search i.e.:

$$\phi^j_s(x,j') = \frac{1}{\mid \mathcal{J} \mid} \qquad \forall\, x \text{ and } j,j' \in \mathcal{J}$$

thus:

$$J^j_t(x,y) = f(x,y,j,z_t) + \beta\mathbb{E}_t\left[(1-(1-\delta)\lambda^j_{t+1}(x,y))U^j_{t+1}(x) + (1-\delta)\lambda^j_t(x,y)J^j_{t+1}(x,y)\right] \tag{2.3}$$

2.2.3 Match Surplus

Define the surplus of a match between a type x worker in location j and a type y firm in location j' as:

$$S^{j\to j'}_t(x,y) = J^{j'}_t(x,y) - [U^j_t(x) - F^{j\to j'}]$$

After some algebra we obtain the following expression for the surplus of a match:

$$S^{j\to j'}_t(x,y) = s(x,y,j\to j',z_t) - \Lambda^{j'}(x) - F^{j\to j'} + \beta\mathbb{E}_{t+1}\left[\max\left\{0, S^{j'\to j'}_{t+1}(x,y)\right\}\right] \tag{2.4}$$

where $\Lambda^{j'}(x)$ is a function of the expected value of the difference of the instantaneous utility of a type x worker in j' and every other location. For derivations and function definitions see Appendix C.

Note that match surplus encodes all the necessary and sufficient conditions for a firm y' in location j' to poach a worker from a firm y in location j:

$$\begin{aligned} S_t^{j\to j'}(x,y') - S_t^{j\to j}(x,y') &= J_t^{j'}(x,y') - [U_t^j(x) - F^{j\to j'}] - \Big(J_t^j(x,y) - [U_t^j(x) - F^{j\to j}]\Big) \\ &= J_t^{j'}(x,y') - [J_t^j(x,y) + F^{j\to j'}] \end{aligned}$$

therefore worker x is poached by firm y' in location j' from firm y in location j if and only if the surplus obtained from moving to j' and matching with y' is higher than the surplus of staying at j matched with y'.

When the aggregate state changes from $z_{t-1} \to z_t$ the surplus function determines how does the stock of unemployed and employed workers change:

$$u_{t+}^j(x) = \underbrace{u_t^j(x)}_{\text{inherited from } t} + \overbrace{\int \Big(\underbrace{\mathbb{1}_{S_t^{j\to j}(x,y)<0}}_{\text{endogenous destruction}} + \underbrace{\delta \mathbb{1}_{S_t^{j\to j}(x,y)\geq 0}}_{\text{exogenous destruction}} \Big) h_t^j(x,y)dy}^{\text{new unemployment created by shock}}$$

and

$$h_{t+}^j(x,y) = (1-\delta)\mathbb{1}_{\{S^{j\to j}(x,y)\geq 0\}} h_t^j(x,y)$$

2.2.4 Vacancy Creation

- $B_t^j(y)$ is the expected value of a type y vacancy making contact with a worker in location j. Vacancies are posted in the interim period and meet unemployed and employed type-x workers at a rates

$$\frac{u_{t+}^j(x)}{L_t^j} \qquad \text{and} \qquad s\frac{h_{t+}^j(x,y)}{L_t^j}$$

  The expected value of posting a vacancy is therefore, the surplus that the posting firm expects to add, potential matches with negative surplus are immediately destroyed therefore those add no surplus. In terms of the Bellman equation we can write:

$$B_t^j(y) = \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \phi_u^{j'}(x,j) \underbrace{\frac{u_{t+}^{j'}(x)}{L_t^{j'}}}_{\text{likelihhod of match}} \times \overbrace{S_t^{j' \to j}(x,y)^+}^{\text{match survives}} dx \right)}_{\text{expected value added from hiring unemployed workers}} +$$

$$+ \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \left(\int \underbrace{s\phi_s^{j'}(x,j) \frac{h_{t+}^{j'}(x,y)}{L_t^{j'}}}_{\text{likelihood of match}} \times \overbrace{[S_t^{j' \to j}(x,y) - S_t^{j' \to j'}(x,y')]^+}^{\text{poaching is succesfull}} dx \right) dy \right)}_{\text{expected value added from poaching other firms employees}} \tag{2.5}$$

For simplicity we use the notation $x^+ = \max\{0, x\}$.

Firms will post vacancies such that the marginal cost of the vacancies and the marginal expected benefit B_t^j are equal:

$$c_j'(v_t^j(y)) = q_t^j B_t^j(y) \tag{2.6}$$

Using the value of B_t^j any particular cost and matching function can be used to pin down the number of vacancies posted by each firm in each location. I follow (Lise and Robin 2017) in choosing the following:

$$c_j(v_t^j(y)) = \frac{av^{1+b}}{1+b}$$

and

$$M(L_t, V_t) = \min\{\alpha L_t^\omega V_t^{1-\omega}, L_t, V_t\}$$

as the cost and matching function respectively. Note that this particular matching function implies that

$$q_t^j = \frac{M(L_t^j, V_t^j)}{V_t^j} = \frac{\alpha L_t^\omega V_t^{1-\omega}}{V_t^j} = \alpha(\theta_t^j)^{-\omega}$$

Using the expression in Equation 2.6 we can write:

$$av_t^j(y)^b = q^j B^j(y) \qquad \Longrightarrow \qquad v_t^j(y) = \left(\frac{q_t^j B^j(y)}{a}\right)^{\frac{1}{b}} = \left(\frac{\alpha(\theta_t^j)^{-\omega} B^j(y)}{a}\right)^{\frac{1}{b}}$$

Then integrating over all firms in each location we get the total number of vacancies posted in each location as:

$$V_t^j = \int v_t^j(y) dy = \left((\theta_t^j)^{-\omega}\right)^{\frac{1}{b}} \int \left(\frac{\alpha B^j(y)}{a}\right)^{\frac{1}{b}} dy$$

Multiply by $1/L_t^j$ on both sides to obtain:

$$\theta_t^j = \frac{1}{L_t^j}\left((\theta_t^j)^{-\omega}\right)^{\frac{1}{b}} \int \left(\frac{\alpha B^j(y)}{a}\right)^{\frac{1}{b}} dy$$

solving for θ_t^j we get:

$$\theta_t^j = \left(\frac{1}{L_t^j} \int \left(\frac{\alpha B^j(y)}{a}\right)^{\frac{1}{b}} dy\right)^{\frac{b}{b+\omega}}$$

2.3 Labor Market Flows

Now we characterize the flows of workers in-to and out-of unemployment at each location :

- Let

$$\eta^{j'\to j}(x,y) = \mathbb{1}_{\{S_t^{j'\to j}(x,y)>0\}} \qquad \eta^{j'\to j}(x,y'\to y) = \mathbb{1}_{\{S_t^{j'\to j}(x,y)>S_t^{j'\to j'}(x,y')\}}$$

- And $\widehat{\phi}_u^j(x,j')$ the probability that a type x unemployed worker from location j search in location j' (this the optimal search policy in Equation 2.1).

Then the law of motion of the unemployment and emplyment are:

$$u_{t+1}^j(x) = \sum_{j'\in\mathcal{J}} \underbrace{\phi_u^{j'}(x,j)u_{t+}^{j'}(x)\left(1 - \int \eta^{j'\to j}(x,y)p^j\frac{v^j(y)}{V^j}dy\right)}_{\text{mass of incoming unemployed workers that are not hired by any firm}} \tag{2.7}$$

To copute the mass of employed workers in the next period define:

- The mass of workers hired from unemployment:

$$h_{u,t+1}^j(x,y) = \sum_{j'\in\mathcal{J}} \phi_u^{j'}(x,j)u_{t+}^{j'}(x)\eta^{j'\to j}(x,y)p^j\frac{v^j(y)}{V^j} \tag{2.8}$$

- The mass of workers that are succesfully poached from other firms:

$$h_{p,t+1}^j = \sum_{j'\in\mathcal{J}} \left(\overbrace{\int \underbrace{h_{t+}^{j'}(x,y')sp_t^j\frac{v_t^j(y)}{V_t^j}\eta^{j'\to j}(x,y'\to y)}_{\text{mass of workers poached from } y' \text{ firms in location } j'} dy'}^{\text{mass of workers that } y \text{ succesfully poach from location} j'}\right) \tag{2.9}$$

- The mass of workers that the firm is able to retain:

$$h^j_{r,t+1} = \underbrace{h_{t+}(x,y)\times}_{\text{employed at interim}} \overbrace{\prod_{j'\in\mathcal{J}} \left[\underbrace{1 - \frac{s}{\mid \mathcal{J} \mid} \int \left(p^{j'}_t \frac{v^{j'}_t(y')}{V^{j'}_t} \eta^{j\to j'}(x, y \to y') \right) dy'}_{\text{no poeached by any firm in location } j'} \right]}^{\text{probability of not being poached}} \tag{2.10}$$

Finally the mass of employed workers in the next period is:

$$h^j(x,y) = h^j_{u,t+1}(x,y) + h^j_{p,t+1}(x,y) + h^j_{r,t+1}(x,y) \tag{2.11}$$

Then I can compute the distribution of skill in each location as:

$$\ell^j_{t+1}(x) = u^j_{t+}(x) + \int h^j_{t+}(x,y)dy$$

and the total population in each location as:

$$\mu^j_{t+1} = \int \ell^j_{t+}(x)dx$$

3 Endogenizing Location Productivity and Cost of Living

This section focus on how the distribution of workers (skils) affect both the productivity of matches $f(x, y, j, z)$ and the instant utility of unemployed workers $b(x, j, z)$.

Location Productivity

I borrow from (Davis and Dingel 2019) in assuming that the productivity of workers in a location j is a result of idea exchange process within each location.

- Assume that a type x worker in location j has a $\Omega(x, \overline{X}^j)$ where $\overline{X}^j$ is the value of idea exchange in location j. Thus:

$$f(x, y, j, z_t) = f\left(\Omega(x, \overline{X}^j), y, z_t\right)$$

- Assume that the value of idea exchange in location j is a function of the distribution of skills in location j:

$$\overline{X}^j = \overline{X}\left(\{\ell^j(x)\}\right)$$

It's natural to make the following assumptions on $\Omega(x, \overline{X}^j)$ and $\overline{X}^j$:

- **Assumption 1**: $\Omega(x, \overline{X}^j)$ should be increasing in x and $\overline{X}^j$ and in the absence of idea exchange, worker productivity is just their type: $\Omega(x, 0) = x$.
- **Assumption 2**: Worker type x and idea exchange environment are complements i.e. $\Omega(x, \overline{X}^j)$ supermodular in $(x\ ,\ \overline{X}^j)$.
- **Assumption 3**: The value of idea exchange in location j is increasing in the number of potential exchange partners μ^j and on the distribution of skills in location j $\ell^j(x)$.
  - With the *FOSD* order of distributions.

I will focus on the following functional forms for $\Omega(x, \overline{X}^j)$ and $\overline{X}^j$:

$$\Omega(x, \overline{X}^j) = x(1 + A\overline{X}^j x)$$

where A is a parameter that captures the scope of gains from idea exchange.

$$\overline{X}\left(\{\ell^j(x)\}\right) = \left(1 - e^{-\nu \int \ell^j(x)dx}\right)\hat{x}^j = \left(1 - e^{-\nu\mu^j}\right)\hat{x}^j$$

The probability of encountering someone during each moment of time seeking is given by $1 - \exp(-\nu\mu^j)$, note that as the number of potential exchange partners (μ^j) increases, the probability of encountering someone also increases, which makes intuitive sense.

The average ability of the individuals you encounter in these exchanges is denoted as $\hat{x}^j$. This is a weighted average of the abilities of the people you meet

$$\hat{x}^j = \frac{1}{\mu^j}\int x\ell^j(x)dx = \frac{\mathbb{E}^j[x]}{\mu^j}$$

where $\mathbb{E}^j[x]$ is the average ability of workers in location j.

Finally I assume that the production function is a Cobb-Duglas with the state of the economy z_t being a Hicks neutral modifier

$$f(x, y, j, z_t) = z_t\Omega(x, \overline{X}^j)^\alpha y^{1-\alpha} \tag{3.1}$$

Let $y^*(x, 1, j)$ be the firm type that maximizes production for a worker of type x in location j when the aggregate shock is equal to 1 and $\hat{b}$ an adjustment factor, then, home production is characterized as:

$$b(x, j) = \hat{b}f(x, y^*(x, 1), 1, j)$$

Note that with the functional form I'm considering $y^*(x, 1, j) = \max \mathcal{Y} = y^*$ for all x and j, thus:

$$b(x, j) = \hat{b}f(x, y^*, 1, j)$$

Cost of living

- As in (Davis and Dingel 2019) and (Behrens, Duranton, and Robert-Nicoud 2014) every individual in location j pays the congestion cost :

$$\theta\mu_j^\gamma$$

4 Computation of the Equilibrium

4.1 Parametrization

- $\{x_1, \ldots, x_{N_x}\} \subset [0]$ is a discretization of the continuum of worker types.
  - N_x=21 as in the paper.
- Distribution of x-types is Beta$(2.15, 12.0)$.
  - Parameters for the distribution are estimated by the authors.
- $\{y_1, \ldots, y_{N_y}\} \subset [0]$ is a discretization of the continuum of firm types.
  - N_y=21 as in the paper. <!—
- Assume the productivity shock follows the following AR(1) process:

$$\log z_t = \rho \log z_{t+1} + \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma^2(1-\rho^2))$$

  - I discretize this process into a Markov with values $\{z_1, \ldots, z_{N_z}\}$ and transition matrix $\pi(z \mid z')$ using Tauchen's method.
  - $N_z = 51$ as in the paper.
  - $\rho = 0.999$ and $\sigma = 0.071$ as in the paper.
- I set $\beta = 0.95$ consistent with a 5% discount rate as in the paper. —>
- For now I'm assuming that the aggregate state of the world is fixed and equal to 1.
- As in (Lise and Robin 2017) value added at the match level takes the functional form:

$$f(x, y, z_t) = z_t \times (p_1 + p_2 x + p_3 y + p_4 x^2 + p_5 y^2 + p_6 xy)$$

  - With $(p_1, p_2, p_3, p_4, p_5, p_6) =$
- Denote $y^*(x, z)$ the firm type that maximizes the surplus of worker x When the aggregate state is z.
- Home production is characterized as

$$b(x) = b_{share} f(x, y^*(x, 1), 1)$$

- $b_{share} = 0.7$ the same that the authors use (they take it from (Hall 2005))

- The matching function is

$$M(L_t, V_t) = \min\{\alpha L_t^{\omega} V_t^{1-\omega}, L_t, V_t\}$$

  - The authors estimate $\alpha = 0.497$ and $\omega = 0.5$.

- The cost of creating a v vacancies are

$$c(v) = \frac{c_0 v^{1-c_1}}{1 - c_1}$$

  - The authors estimate $c_0 = 0.028$ and $c_1 = 0.084$.

A Optimal Search Strategy Unemployed Workers

Unemployed agents solve the following maximization problem:

$$\max_{\phi_u^j(x)} \left\{ \sum_{j' \in \mathcal{J}} \phi_u^j(x,j') \mathbb{E}_t \Big[U_{t+1}^{j'}(x) - F^{j \to j'} \Big] - c(\phi_u^j(x)) \right\}$$

$$\text{subject to: } \sum_{j' \in \mathcal{J}} \phi_u^j(x,j') = 1$$

$$\phi_u^j(x,j') \geq 0 \qquad \forall j' \in \mathcal{J}$$

I'll ignore the non-negativity constraints and write the Lagrangean of the problem:

$$\mathcal{L}(\phi_u^j(x), \lambda) = \sum_{j' \in \mathcal{J}} \phi_u^j(x,j') \mathbb{E}_t \Big[U_{t+1}^{j'}(x) - F^{j \to j'} \Big] - c(\phi_u^j(x)) - \lambda \left(\sum_{j' \in \mathcal{J}} \phi_u^j(x,j') - 1 \right)$$

First order conditions of the problem give us:

$$[\phi_u^j(x,j')] : \quad \mathbb{E}_t \Big[U_{t+1}^{j'}(x) - F^{j \to j'} \Big] - c_1 - c_1 \log[J \phi_u^j(x,j')] = \lambda$$

$$[\lambda] : \quad \sum_{j' \in \mathcal{J}} \phi_u^j(x,j') = 1$$

Take any two j_1, j_2 we have that

$$\mathbb{E}_t \left[U_{t+1}^{j_1}(x) - F^{j \to j_1} \right] - c_1 \log[J \phi_u^j(x,j_1)] = \mathbb{E}_t \left[U_{t+1}^{j_2}(x) - F^{j \to j_2} \right] - c_1 \log[J \phi_u^j(x,j_2)]$$

thus

$$\frac{\mathbb{E}_t\left[U_{t+1}^{j_1}(x) - F^{j\to j_1}\right] - \mathbb{E}_t\left[U_{t+1}^{j_2}(x) - F^{j\to j_2}\right]}{c_1} = \log\left(\frac{\phi_u^j(x,j_1)}{\phi_u^j(x,j_2)}\right)$$
$$\implies \quad \frac{\phi_u^j(x,j_1)}{\phi_u^j(x,j_2)} = \frac{e^{\left(\mathbb{E}_t\left[U_{t+1}^{j_1}(x)-F^{j\to j_1}\right]/c_1\right)}}{e^{\left(\mathbb{E}_t\left[U_{t+1}^{j_2}(x)-F^{j\to j_2}\right]/c_1\right)}}$$

Fix any $\hat{j}$, then we can write any other $j' \in \mathcal{J}$ in therms of $\hat{j}$ and plug into the constraint to get:

$$\begin{aligned}
\sum_{j'\in\mathcal{J}} \phi_u^j(x,j') &= \sum_{j'\in\mathcal{J}} \frac{e^{\left(\mathbb{E}_t\left[U_{t+1}^{j'}(x)-F^{j\to j'}\right]/c_1\right)}}{e^{\left(\mathbb{E}_t\left[U_{t+1}^{\hat{j}}(x)-F^{j\to \hat{j}}\right]/c_1\right)}}\phi_u^j(x,\hat{j}) \\
&= \frac{\phi_u^j(x,\hat{j})}{e^{\left(\mathbb{E}_t\left[U_{t+1}^{\hat{j}}(x)-F^{j\to \hat{j}}\right]/c_1\right)}} \sum_{j\in\mathcal{J}} e^{\left(\mathbb{E}_t\left[U_{t+1}^{j'}(x)-F^{j\to j'}\right]/c_1\right)} = 1
\end{aligned}$$

Which we can solve to obtain

$$\phi_u^j(x,j') = \frac{e^{\left(\mathbb{E}_t\left[U_{t+1}^{j'}(x)-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}} e^{\left(\mathbb{E}_t\left[U_{t+1}^{j\to\tilde{j}}(x)\right]/c_1\right)}} \tag{A.1}$$

Note that the non-negativity constraints are satisfied because the exponential function is always positive.

B Derivation of Unemployed Bellman Equation

The Bellman of a unemployed worker is (omitting time and worker types):

$$U^j = b + \beta \max_{\phi} \left\{ \sum_{j' \in \mathcal{J}} \left(\phi(j') \mathbb{E}_t \left[U^{j'} - F^{j \to j'} \right] \right) - c(\phi) \right\}$$

Where the optimal strategy we obtained in Appendix A

$$\phi(j') = \frac{e^{\left(\mathbb{E}_t \left[U^{j'} - F^{j \to j'} \right] / c_1 \right)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{\left(\mathbb{E}_t \left[U^{\tilde{j}} - F^{j \to \tilde{j}} \right] / c_1 \right)}} \tag{B.1}$$

and the cost of a search policy (strategy) is:

$$c(\phi) = c_1 \sum_{j \in \mathcal{J}} \phi(j') \log(J \phi(j'))$$

We want to plug the best policy and obtain the Bellman. The continuation value takes the form:

$$\sum_{j'\in\mathcal{J}}\left[\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]-c_1\left(\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\right)\right.$$

$$\left.\log\left(J\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\right)\right]$$

$$\implies\quad\sum_{j'\in\mathcal{J}}\left\{\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\left[\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]-c_1\log\left(J\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\right)\right]\right\}$$

$$\implies\quad\sum_{j'\in\mathcal{J}}\left\{\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\left[\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]-c_1\left(\log(J)+\log\left(e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}\right)-\right.\right.\right.$$

$$\left.\left.\left.\log\left(\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}\right)\right)\right]\right\}$$

$$\implies\quad c_1\sum_{j'\in\mathcal{J}}\left\{\frac{e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\left[\log\left(\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}\right)-\log(J)\right]\right\}$$

$$\implies\quad c_1\left[\log\left(\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}\right)-\log(J)\right]\frac{1}{\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}}\sum_{j'\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{j'}-F^{j\to j'}\right]/c_1\right)}$$

Thus we obtain:

$$U^j=b+\beta c_1\left[\log\left(\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{\tilde{j}}-F^{j\to\tilde{j}}\right]/c_1\right)}\right)-\log(J)\right]$$

Denote $U^{j\to j'}=U^j-F^{j\to j'}$ then

$$U^{j\to j'}=b+\beta c_1\left[\log\left(\sum_{\tilde{j}\in\mathcal{J}}e^{\left(\mathbb{E}_t\left[U^{j\to\tilde{j}}\right]/c_1\right)}\right)-\log(J)\right]-F^{j\to j'}$$

C Derivation of Surplus Bellman Equation

First note that the differerence in unemployement value between two locations is:

$$U_t^j(x) - U_t^{j'}(x) = b(x, j, z_t) - b(x, j', z_t)$$

we will use this fact later.

We start by deriving the expression for the surplus created by a match within a location $S^{j\to j}(x, y)$. Re-write Equation 2.3 as:

$$J_t^j(x, y) = f(x, y, j, z_t) + \beta\mathbb{E}_t\left[(1-\delta)\lambda_{t+1}^j(x, y)\left(J_{t+1}^j(x, y) - U_{t+1}^j(x)\right)\right] + \beta\mathbb{E}_t\left[U_{t+1}^j(x)\right] \tag{C.1}$$

Note that

$$\mathbb{E}_t\left[U_{t+1}^j(x)\right] - U_t^j(x) = \mathbb{E}_t\left[U_{t+1}^j(x)\right] - b(x, j, z) - \beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1}\right\}\right) - \beta c_1 \log J$$

Focus on the term:

$$\beta\mathbb{E}_t\left[U_{t+1}^j(x)\right] - \beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1}\right\}\right)$$

we take advantaje of the following translation identity property of the lse function:

$$\text{lse}\left(\{x_i\}\right) = \text{lse}\left(\{x_i - a\}\right) + a \qquad \Longrightarrow \qquad a - \text{lse}\left(\{x_i\}\right) = -\text{lse}\left(\{x_i - a\}\right)$$

thus we can write:

$$\begin{aligned}
\beta\mathbb{E}_t\left[U_{t+1}^j(x)\right] - \beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1}\right\}\right) &= -\beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t[U_{t+1}^{j'}(x)] - \mathbb{E}_t\left[U_{t+1}^j(x)\right]}{c_1}\right\}\right) \\
&= -\beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t\left[U_{t+1}^{j'}(x) - U_{t+1}^j(x)\right]}{c_1}\right\}\right) \\
&= -\beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t\left[b(x, j', z_{t+1}) - b(x, j, z_{t+1})\right]}{c_1}\right\}\right)
\end{aligned}$$

Subtracting $U_t^j(x)$ to both sides of Equation C.1 and using the previous result we get:

$$J_t^j(x,y) - U_t^j(x) = f(x,y,j,z_t) - b(x,j,z_t) - \beta c_1 \log(J) - \beta c_1 \text{lse}\left(\left\{\frac{\mathbb{E}_t\left[b(x,j',z_{t+1}) - b(x,j,z_{t+1})\right]}{c_1}\right\}\right) + ...$$
$$... + \beta \mathbb{E}_t\left[(1-\delta)\lambda_{t+1}^j(x,y)\left(J_{t+1}^j(x,\right.\right.$$

Define $s(x,y,j \to j',z_t) = f(x,y,j,z_t) - b(x,j',z_t)$ and

$$\Lambda^j(x) = \beta c_1 \left(\log(J) + \text{lse}\left(\left\{\frac{\mathbb{E}_t\left[b(x,j',z_{t+1}) - b(x,j,z_{t+1})\right]}{c_1}\right\}\right)\right)$$

and the Bellman equation becomes:

$$S_t^{j\to j}(x,y) = s(x,y,j\to j,z_t) - \Lambda^j(x) + \beta \mathbb{E}_{t+1}\left[\max\left\{0, S_{t+1}^{j\to j}(x,y)\right\}\right]$$

Next consider the the surplus created by a match between a worker x in location j_1 and a firm y in location j_2:[1].

$$\begin{aligned}
S_t^{j_1\to j_2} &= J_t^{j_2}(x,y) - [U_t^{j_1}(x) + F^{j_1\to j_2}] \\
&= \left(J_t^{j_2}(x,y) - U_t^{j_2}(x)\right) - \left(U_t^{j_1}(x) - U_t^{j_2}(x)\right) - F^{j_1\to j_2} \\
&= \left(J_t^{j_2}(x,y) - U_t^{j_2}(x)\right) - (b(x,j_1,z_t) - b(x,j_2,z_t)) - F^{j_1\to j_2} \\
&= S_t^{j_2\to j_2}(x,y) - (b(x,j_1,z_t) - b(x,j_2,z_t)) - F^{j_1\to j_2} \\
&= s(x,y,j_2\to j_2,z_t) - \Lambda^{j_2}(x) + \beta\mathbb{E}_{t+1}\left[\max\left\{0, S_{t+1}^{j_2\to j_2}(x,y)\right\}\right] - (b(x,j_1,z_t) - b(x,j_2,z_t)) - F^{j_1\to j_2} \\
&= s(x,y,j_1\to j_2,z_t) - \Lambda^{j_2}(x) + \beta\mathbb{E}_{t+1}\left[\max\left\{0, S_{t+1}^{j_2\to j_2}(x,y)\right\}\right] - F^{j_1\to j_2}
\end{aligned}$$

[1] Note that $F^{j\to j} = 0$ therefore we are generalizing the case where $j_1 = j_2$

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