

Macro-dynamics of Sorting between Workers and Firms (and Locations)

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1 Introduction

Notes:

- I adopt the framework from (Lise and Robin 2017).
- Introducing a fixed amount of locations and population distribution of workers should be an equilibrium outcome.
- Workers can “partially” direct their search to each location.
 - In practice workers randomize over locations paying a cost.
 - Cost of random search is zero.
 - Cost increases unboundedly as the worker directs their search to a particular location.
- Productivity depends on the quality of match and aggregate state as in the original paper. I want to introduce a third component which is location productivity.
 - I intend to use idea exchange as the driver of location productivity.
 - I plan to draw on a framework similar to (davisSpatialKnowledgeEconomy2019a?) to model the value added to production by idea exchange.

TODO:

- ☒ Dynamic programming problem changed to keep track of value of moving across locations.
- ☒ **Important and easy** Change notation so that total number of locations J is not the same letter as value of match $J^j(x, y)$
- ☐ Read (Sims 2003)
- ☒ Add references to literature using similar cost structure for mixed strategies.
- ☐ Compute the distribution of skill in each location.
- ☐ Check the ratios of vacancies of each type of firm in each location when computing the law of motion of the unemployment rate. **Important: I think I have it wrong now.**
- ☐ Check that the unemployment value function is a contraction.
- ☐ Make location productivity a result of the distribution of skill in each location.
- ☐ Add cost of living to the model.
- ☒ **Less Important** Fix Appendix.
- ☐ Think about distribution of firms across and within locations.
- ☐ Solve a modified version of the model where there is no aggregate state.
- ☐ Solve a two location version of the model to gain intuition.
- ☐ Think if I could include the cost of moving to a different location in the cost of the mixed strategy.

2 Model

2.1 Model Setup

2.1.1 Demographics

- There is a discrete and finite set of locations $\mathcal{J} = \{1, \dots, |\mathcal{J}|\}$ locations indexed by $j \in \mathcal{J}$.
- Continuum of workers indexed by their ability $x \in \mathcal{X}$.
 - The total measure is normalized to 1.
 - Exogenous distribution $\ell(x)$
 - Endogenous distribution in each location $\ell^j(x)$
 - Denote μ_j the total population in location j .
- Continuum of firms indexed by technology $y \in \mathcal{Y}$.
 - Total measure is normalized to 1.
 - Uniformly distributed.
- The exogenous cost of posting v job opportunities in location j is

$$c_j(v) \geq 0$$

assume it is increasing, convex, and independent of the firm type y (but *potentially* dependent on location).

- The aggregate state of the economy is indexed by z_t .
 - Changes from z to z' according to the Markov transition probability $\pi(z, z')$.
- Workers and firms discount the future at the rate β .
- Workers can move across locations:
 - Workers choose a mixed strategy to search:

$$\phi_j^i(x) = \{\phi_j^i(x, j')\}_{j' \in \mathcal{J}}$$

where $\phi_j^i(x, j')$ is the probability that a type x worker from location j search in location j' and $i \in \{u, e\}$ refers to the employment status of the worker.

- Each strategy has an associated cost $c_s(\phi_j^i(x))$:¹ works that use this cost structure are (Wu 2020) and (Cheremukhin, Restrepo-Echavarria, and Tutino 2020).
- * (Wu 2020) mentions that this cost structure can be derived from rational inattention as in (Sims 2003) I should read that paper.

$$c_s(\phi_j^i(x)) = c_1 \left(\sum_{j' \in \mathcal{J}} \phi_j^i(x, j') \log (J \phi_j^i(x, j')) \right)$$

- When a worker move they must pay a cost $F^{j \rightarrow j'} \geq 0$ with $F^{j \rightarrow j} = 0$.

2.1.2 Technology

- Unemployed workers instant utility in each location is $b(x, z, j)$.²
- Firms have access to a production technology, defined at the match level and depending on the location and the aggregate state of the economy $f(x, y, j, z)$.

2.1.3 Job Search

The timing is as follows: At time (t) :

1. Distributions of employed and unemployed workers are inherited from $t - 1$.
 - $u_t^j(x)$ is the measure of type- x Unemployed workers at the location j .
 - $h_t^j(x, y)$ is the measure of type- x workers employed at the firm y at the location j .
 - Note that on each location:

$$u_t^j(x) + \int h_t^j(x, y) dy = \ell^j(x)$$

2. Aggregate state changes $z_{t-1} \rightarrow z_t$.
3. At time $(t+)$ Separations occur:
 - $u_{t+}^j(x)$ is the measure of the type x Unemployed workers in the location j after the shock.
 - $h_{t+}^j(x, y)$ is the measure of type x workers that remain employed at firm y in location j .
4. Unemployed and employed workers draw new offers.

¹The cost is proportional to the Kullback–Leibler divergence between the selected distribution and a uniform. Note that if the worker selects random search (i.e. uniform distribution) the associated cost is 0 and the cost grows unboundedly large as the strategy gets closer to perfect directed search to a particular location. More information on the Kullback–Leibler divergence [here](#)

²In (Lise and Robin 2017) $b(x)$ stands for unemployment benefits, I want to be more general to be able to include differences in cost of living across locations

Both unemployed and employed workers search, denote s the search intensity of an employed worker and 1 is the (normalized) search intensity of an unemployed worker. The total search intensity in location j is

$$L_t^j = \sum_{j' \in \mathcal{J}} \left[\int u_{t+}^{j'}(x) dx + s \int \int h_{t+}^{j'}(x, y) dx dy \right]$$

Let $v_t^j(y)$ be the number of job opportunities posted by a firm y at time t in the location j .

- $V_t^j = \int v_t^j(y) dy$ is the total number of job opportunities posted at the time t in the location j .

Let $M_t^j = M(L_t^j, V_t^j)$ The number of job matches in the location j then:

- The probability that an unemployed worker contacts a vacancy in the location j is

$$p_t^j = \frac{M_t^j}{L_t^j}$$

– sp_t^j is the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker

$$q_t^j = \frac{M_t^j}{V_t^j}$$

2.2 Dynamic Programming Problem

Denote:

- $U_t^j(x)$ the value for an unemployed worker of type x at time t at location j .
- The value of getting an offer depends on whether the worker is employed or not:
 - $W_{0,t}^{j' \rightarrow j}(x, y)$ is the value of a type- x unemployed worker at location j' who is hired by a firm of type y at a location j .
 - $W_{1,t}^{j' \rightarrow j}(x, y, y')$ is the value offered at the time t by type y firm at location j to a type x worker employment at a type y' firm in location j' .
- $J_t^j(x, y)$ is the value of a match between a type x worker and a type y firm at time t in location j .

2.2.1 Unemployed Worker

Unemployed workers receive instant utility from living in location j , $b(x, j, z)$, and anticipate next period's aggregate state z_{t+1} , and the probability of getting an offer p_{t+1}^j in each location. They will choose the strategy that maximizes their future expected value knowing that in each location they will receive an offer which can be from any firm with a likelihood proportional to the share of total vacancies posted by each firm in each market. The worker will accept only the offers that promise her a higher value than unemployment:

$$U_t^j(x) = b(x, j, z) + \beta \max_{\phi_u^j(x)} \left\{ -c(\phi_u^j(x)) + \phi_u^j(x, j') \mathbb{E}_t \sum_{j' \in \mathcal{J}} \left[(1 - p_{t+1}^{j'}) U_{t+1}^{j'}(x) + \dots \right. \right. \\ \left. \left. \dots + p_{t+1}^{j'} \int \max \left\{ U_{t+1}^{j'}(x), W_{0,t+1}^{j \rightarrow j'}(x, y) - F^{j \rightarrow j'} \right\} \frac{v_{t+1}^j(y)}{V_{t+1}^j} dy \right] \right\}$$

Assume that workers don't have bargaining power, therefore they are offered their reservation value $U_t^{j'}(x) = W_{0,t+1}^{j \rightarrow j'}(x, y) - F^{j \rightarrow j'}$ by firms:

$$U_t^j(x) = b(x, j, z) + \beta \max_{\phi_u^j(x)} \left\{ \sum_{j' \in \mathcal{J}} \left(\phi_u^j(x, j') \mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] \right) - c(\phi_u^j(x)) \right\}$$

This means that the optimal search policy for each worker is

$$\phi_u^j(x, j') = \frac{e^{\left(\mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] / c_1 \right)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{\left(\mathbb{E}_t \left[U_{t+1}^{\tilde{j}}(x) \right] / c_1 \right)}} \quad (2.1)$$

derivation is in Chapter A.

Substituting the optimal strategy into the Bellman Equation of the unemployed worker we get (derivation in Chapter B):

$$U_t^j(x) = b(x, j, z) + \beta c_1 \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] / c_1 \right) - \log(J) \right)$$

or simply

$$U_t^j(x) = b(x, j, z) + \beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t \left[U_{t+1}^{j'}(x) \right]}{c_1} \right\} \right) - \beta c_1 \log J \quad (2.2)$$

Where $\text{lse}(x \in \mathbb{R}^n)$ is the **log-sum-exp** function.

2.2.2 Firm

- If a match between a worker and a firm in location j is destroyed the firm will get 0 and the worker gets their unemployment value in that location $U_t^j(x)$.
- Matches are destroyed for two reasons:
 - *Exogenous destruction* with probability δ
 - *Endogenous destruction*, if and only if $J_t^j(x, y) < U_t^j(x)$.
 - * Denote $\lambda_t^j(x, y) = \mathbb{1}_{\{J_t^j(x, y) > U_t^j(x, y)\}}$

We can write the Bellman equation of a match value as:

$$\begin{aligned}
 J_t^j(x, y) = & \underbrace{f(x, y, j, z_t)}_{\text{match value added}} + \beta \max_{\phi_s^j(x)} \left\{ \mathbb{E}_t \left[\overbrace{(1 - (1 - \delta)\lambda_{t+1}^j(x, y))}^{\text{match is destroyed}} \underbrace{U_{t+1}^j(x)}_{\text{worker gets unemployment value}} + \dots \right. \right. \\
 & \dots + \underbrace{(1 - \delta)\lambda_t^j(x, y)}_{\text{match survives}} \max_{\phi_s^j(x)} \left\{ -c(\phi_s^j(x)) + \sum_{j' \in \mathcal{J}} \phi_s^j(x, j') \left[\overbrace{(1 - sp_t^{j'})}^{\text{no new offers}} \underbrace{J_{t+1}^j(x, y)}_{\text{stays with same firm}} + \dots \right. \right. \\
 & \left. \left. \left. \dots + sp_t^{j'} \int \underbrace{\max\{J_{t+1}^j(x, y), W_{1,t+1}^{j \rightarrow j'}(x, y', y) - F^{j \rightarrow j'}\}}_{\text{worker only accepts new offers if value is greater than current match}} \frac{v_{t+1}^{j'}(x)}{V_{t+1}^{j'}} dy' \right] \right\} \right\}
 \end{aligned}$$

When a type x worker employed at a type y firm in city j receives an offer from a type y' in city j' then there is a sequential auction like in (Postel-Vinay and Robin 2002). More productive firms can offer higher values. The key difference with (Postel-Vinay and Robin 2002) is that location plays a role: if a firm is located in a different location than the worker's current employer then the poaching firm must cover the cost of moving, this leads to two possible outcomes:

- $J_{t+1}^{j'}(x, y') > J_{t+1}^j(x, y) + F^{j \rightarrow j'}$ the worker moves from $(j, y) \rightarrow (j', y')$ and receives $W_{1,t+1}^{j \rightarrow j'}(x, y', y)$
- $J_{t+1}^j(x, y) > J_{t+1}^{j'}(x, y') - F^{j \rightarrow j'}$ the worker stays at (j, y) and receives $W_{1,t+1}^j(x, y, y')$

to able to poach from different locations the firm must be at least $F^{j \rightarrow j'}$ more productive.

As in (Postel-Vinay and Robin 2002) if the worker is hired by the poaching firm the worker receives the incumbent firm reservation value plus the cost of changing jobs, i.e.

$$J_{t+1}^{j'}(x, y') > J_{t+1}^j(x, y) + F^{j \rightarrow j'} \implies W_{1,t+1}^{j \rightarrow j'}(x, y', y) = J_{t+1}^j(x, y) + F^{j \rightarrow j'}$$

therefore

$$J_t^j(x, y) = f(x, y, j, z_t) + \beta \max_{\phi_s^j(x)} \left\{ \mathbb{E}_t \left[(1 - (1 - \delta)\lambda_{t+1}^j(x, y)) U_{t+1}^j(x) \right. \right. \\ \left. \left. + (1 - \delta)\lambda_t^j(x, y) \sum_{j' \in \mathcal{J}} \phi_s^j(x, j') J_{t+1}^j(x, y) \right] - c(\phi_s^j(x)) \right\}$$

Note that the optimal strategy for employed workers is to engage in random search i.e.:

$$\phi_s^j(x, j') = \frac{1}{|\mathcal{J}|} \quad \forall x \text{ and } j, j' \in \mathcal{J}$$

thus:

$$J_t^j(x, y) = f(x, y, j, z_t) + \beta \mathbb{E}_t \left[(1 - (1 - \delta)\lambda_{t+1}^j(x, y)) U_{t+1}^j(x) + (1 - \delta)\lambda_t^j(x, y) J_{t+1}^j(x, y) \right] \quad (2.3)$$

2.2.3 Match Surplus

Define the surplus of a match between a type x worker in location j and a type y firm in location j' as:

$$S_t^{j \rightarrow j'}(x, y) = J_t^j(x, y) - [U_t^j(x) - F^{j \rightarrow j'}]$$

After some algebra we obtain the following expression for the surplus of a match:

$$S_t^{j \rightarrow j'}(x, y) = s(x, y, j \rightarrow j', z_t) - \Lambda^{j'}(x) - F^{j \rightarrow j'} + \beta \mathbb{E}_{t+1} \left[\max \left\{ 0, S_{t+1}^{j' \rightarrow j'}(x, y) \right\} \right] \quad (2.4)$$

where $\Lambda^{j'}(x)$ is a function of the expected value of the difference of the instantaneous utility of a type x worker in j' and every other location. Derivation and function definitions are Chapter C.

Match surplus encodes all the necessary and sufficient conditions for a firm y' in location j' to poach a worker from a firm y in location j . note:

$$S_t^{j \rightarrow j'}(x, y') - S_t^{j \rightarrow j}(x, y') = J_t^{j'}(x, y') - [U_t^j(x) - F^{j \rightarrow j'}] - (J_t^j(x, y) - [U_t^j(x) - F^{j \rightarrow j}]) \\ = J_t^{j'}(x, y') - [J_t^j(x, y) + F^{j \rightarrow j'}]$$

therefore worker x is poached by firm y' in location j' from firm y in location j if and only if the surplus obtained from moving to j' and matching with y' is higher than the surplus of staying at j matched with y' .

When the aggregate state changes from $z_{t-1} \rightarrow z_t$ the surplus function determines how does the stock of unemployed and employed workers change:

$$u_{t+}^j(x) = \underbrace{u_t^j(x)}_{\text{inherited from } t} + \overbrace{\int \left(\underbrace{\mathbb{1}_{S_t^{j \rightarrow j}(x,y) < 0}}_{\text{endogenous destruction}} + \underbrace{\delta \mathbb{1}_{S_t^{j \rightarrow j}(x,y) \geq 0}}_{\text{exogenous destruction}} \right) h_t^j(x,y) dy}_{\text{new unemployment created by shock}}$$

and

$$h_{t+}^j(x,y) = (1 - \delta) \mathbb{1}_{\{S_t^{j \rightarrow j}(x,y) \geq 0\}} h_t^j(x,y)$$

2.2.4 Vacancy Creation

- $V_t^j(y)$ is the expected value of a type y vacancy making contact with a worker in location j . Vacancies are posted in the interim period and meet unemployed and employed type- x workers at a rates

$$\frac{u_{t+}^j(x)}{L_t^j} \quad \text{and} \quad s \frac{h_{t+}^j(x,y)}{L_t^j}$$

The expected value of posting a vacancy is therefore, the surplus that the posting firm expects to add, potential matches with negative surplus are immediately destroyed therefore those add no surplus. In terms of the Bellman equation we can write:

$$V_t^j(y) = \underbrace{\sum_{j' \in \mathcal{J}} \left(\underbrace{\int \frac{u_{t+}^{j'}(x)}{L_t^{j'}}}_{\text{likelihood of match}} \times \overbrace{S_t^{j' \rightarrow j}(x,y)^+ dx}^{\text{match survives}} \right)}_{\text{expected value added from hiring unemployed workers}} + \dots \quad (2.5)$$

(2.6)

$$\dots + \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \left(\underbrace{s \frac{h_{t+}^{j'}(x,y)}{L_t^{j'}}}_{\text{likelihood of match}} \times \overbrace{[S_t^{j' \rightarrow j}(x,y) - S_t^{j' \rightarrow j'}(x,y')]^+ dx}^{\text{poaching is successful}} \right) dy \right)}_{\text{expected value added from poaching other firms employees}} \quad (2.7)$$

For simplicity we use the notation $x^+ = \max\{0, x\}$.

Firms will post vacancies such that the marginal cost of the vacancies and the marginal expected benefit V_t are equal:

$$c_j'(v_t^j(y)) = q_t^j V_t^j(y)$$

2.3 Labor Market Flows

Now we characterize the flows of workers in-to and out-of unemployment at each location :

- Let

$$\eta^{j' \rightarrow j}(x, y) = \mathbb{1}_{\{S_t^{j' \rightarrow j}(x, y) > 0\}}$$

and

$$\eta^{j' \rightarrow j}(x, y' \rightarrow y) = \mathbb{1}_{\{S_t^{j' \rightarrow j}(x, y) > S_t^{j' \rightarrow j'}(x, y')\}}$$

- And $\hat{\phi}_u^j(x, j')$ the probability that a type x unemployed worker from location j search in location j' (this the optimal search policy in Equation 2.1).

Then the law of motion of the unemployment rate is:

$$\begin{aligned} u_{t+1}^j(x) = & \underbrace{u_{t+}^j(x) \left(\sum_{j' \in \mathcal{J}} \hat{\phi}_u^j(x, j') \right)}_{\text{mass of workers leaving location } j} + \dots \\ & \dots + \underbrace{u_{t+}^j(x) \int \left(p_t^j \frac{v_t^j(y)}{V_t^j} \hat{\phi}_u^j(x, j) \eta^{j \rightarrow j}(x, y) \right)}_{\text{staying workers that got a suitable job offer}} + \dots \\ & \dots + \underbrace{\sum_{j \in \mathcal{J}} \left\{ \left(1 - \hat{\phi}_u^{j'}(x, j) \int \left(\frac{v_t^j(y)}{V_t^j} \eta^{j \rightarrow j'}(x, y) dy \right) u_{t+}^{j'}(x) \right) \right\}}_{\text{arriving workers from other locations that dont get a job offer}} \end{aligned} \quad (2.8)$$

and of the distribution of employment

$$\begin{aligned}
h_{t+1}^j(x, y) = & \underbrace{h_{t+}^j(x, y)}_{\text{employed at interim}} \times \prod_{j' \in \mathcal{J}} \overbrace{\left[1 - \frac{s}{|\mathcal{J}|} \int \left(p_t^{j'} \frac{v_t^{j'}(y')}{V_t^{j'}} \eta^{j \rightarrow j'}(x, y \rightarrow y') \right) dy' \right]}^{\text{probability of not being poached}} + \dots \\
& \dots + \sum_{j' \in \mathcal{J}} \left(\overbrace{\int \underbrace{h_{t+}^{j'}(x, y') s p_t^j \frac{v_t^j(y)}{V_t^j} \eta^{j' \rightarrow j}(x, y' \rightarrow y)}_{\text{mass of workers poached from } y' \text{ firms in location } j'} dy'}^{\text{mass of workers that } y \text{ successfully poach from location } j'} \right) \\
& \dots + \underbrace{\sum_{j' \in \mathcal{J}} \left(u_{t+}^{j'}(x) \hat{\phi}_u^{j'}(x, j) p_t \frac{v_t^j(y)}{V_t^j} \eta^{j' \rightarrow j}(x, y) \right)}_{\text{mass of workers hired from unemployment}}
\end{aligned} \tag{2.9}$$

We can compute the distribution of skill in each location as:

$$\ell_{t+1}^j(x) = u_{t+}^j(x) + \int h_{t+}^j(x, y) dy$$

and the total population in each location as:

$$\mu_{t+1}^j = \int \ell_{t+}^j(x) dx$$

3 Endogenizing Location Productivity and Cost of Living

This section focus on how the distribution of workers (skills) affect both the productivity of matches $f(x, y, j, z)$ and the instant utility of workers $b(x, j, z)$.

3.1 Location Productivity

I borrow from (Davis and Dingel 2019) in assuming that the productivity of workers in a location j is a result of idea exchange process within each location.

- Assume that a type x worker in location j has a $B(x, \bar{X}^j)$ where \bar{X}^j is the value of idea exchange in location j . Thus:

$$f(x, y, j, z) = f\left(B(x, \bar{X}^j), y, z\right)$$

- Assume that the value of idea exchange in location j is a function of the distribution of skills in location j : $\bar{X}^j = \bar{X}(\{\ell^j(x)\})$

It's natural to make the following assumptions on $B(x, \bar{X}^j)$ and \bar{X}^j :

- **Assumption 1:** $B(x, \bar{X}^j)$ should be increasing in x and \bar{X}^j and in the absence of idea exchange, worker productivity is just their type: $B(x, 0) = x$.
- **Assumption 2:** Worker type x and idea exchange environment are complements i.e. $B(x, \bar{X}^j)$ supermodular in (x, \bar{X}^j) .
- **Assumption 3:** [TO DO] Idea exchange environment is “increasing” in the distribution of skills in location j . Probable need First Order Stochastic Dominance (FOSD) to define “increasing” in this context.

I will focus on the following functional forms for $B(x, \bar{X}^j)$ and \bar{X}^j :

$$B(x, \bar{X}^j) = x(1 + A\bar{X}^j x) \tag{3.1}$$

where A is a parameter that captures the scope of gains from idea exchange.

$$\overline{X}(\{\ell^j(x)\}) = (1 - e^{-\nu \int \ell^j(x) dx}) \hat{x}^j = (1 - e^{-\nu \mu^j}) \hat{x}^j \quad (3.2)$$

The probability of encountering someone during each moment of time seeking is given by $1 - \exp(-\nu \mu^j)$, note that as the number of potential exchange partners (μ^j) increases, the probability of encountering someone also increases, which makes intuitive sense.

The average ability of the individuals you encounter in these exchanges is denoted as \hat{x}^j . This is a weighted average of the abilities of the people you meet

$$\hat{x}^j = \frac{1}{\mu^j} \int x \ell^j(x) dx = \frac{\mathbb{E}^j[x]}{\mu^j}$$

where $\mathbb{E}^j[x]$ is the average ability of workers in location j .

4 Computation of the Equilibrium

4.1 Parametrization

- $\{x_1, \dots, x_{N_x}\} \subset [0, 1]$ is a discretization of the continuum of worker types.
 - $N_x = 21$ as in the paper.
- Distribution of x -types is Beta(2.15, 12.0).
 - Parameters for the distribution are estimated by the authors.
- $\{y_1, \dots, y_{N_y}\} \subset [0, 1]$ is a discretization of the continuum of firm types.
 - $N_y = 21$ as in the paper.

- Assume the productivity shock follows the following AR(1) process:

$$\log z_t = \rho \log z_{t+1} + \varepsilon_t \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2(1 - \rho^2))$$

- I discretize this process into a Markov with values $\{z_1, \dots, z_{N_z}\}$ and transition matrix $\pi(z | z')$ using Tauchen's method.
- $N_z = 51$ as in the paper.
- $\rho = 0.999$ and $\sigma = 0.071$ as in the paper.
- I set $\beta = 0.95$ consistent with a 5% discount rate as in the paper.
- Value added at the match level takes the functional form

$$f(x, y, z_t) = z_t \times (p_1 + p_2x + p_3y + p_4x^2 + p_5y^2 + p_6xy)$$

- With $(p_1, p_2, p_3, p_4, p_5, p_6) = (0.03, 2.053, -0.140, 8.035, -1.907, 6.596)$ as estimated by the authors.
- Denote $y^*(x, z)$ the firm type that maximizes the surplus of worker x When the aggregate state is z .
- Home production is characterized as

$$b(x) = b_{share} f(x, y^*(x, 1), 1)$$

- $b_{share} = 0.7$ the same that the authors use (they take it from [\(Hall 2005\)](#))

- The matching function is

$$M(L_t, V_t) = \min\{\alpha L_t^\omega V_t^{1-\omega}, L_t, V_t\}$$

- The authors estimate $\alpha = 0.497$ and $\omega = 0.5$.

- The cost of creating a v vacancies are

$$c(v) = \frac{c_0 v^{1-c_1}}{1 - c_1}$$

- The authors estimate $c_0 = 0.028$ and $c_1 = 0.084$.

A Optimal Search Strategy Unemployed Workers

Unemployed agents solve the following maximization problem:

$$\begin{aligned} \max_{\phi_u^j(x)} & \left\{ \sum_{j' \in \mathcal{J}} \phi_u^j(x, j') \mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] - c(\phi_u^j(x)) \right\} \\ \text{subject to: } & \sum_{j' \in \mathcal{J}} \phi_u^j(x, j') = 1 \\ & \phi_u^j(x, j') \geq 0 \quad \forall j' \in \mathcal{J} \end{aligned}$$

I'll ignore the non-negativity constraints and write the Lagrangean of the problem:

$$\mathcal{L}(\phi_u^j(x), \lambda) = \sum_{j' \in \mathcal{J}} \phi_u^j(x, j') \mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] - c(\phi_u^j(x)) - \lambda \left(\sum_{j' \in \mathcal{J}} \phi_u^j(x, j') - 1 \right)$$

First order conditions of the problem give us:

$$\begin{aligned} [\phi_u^j(x, j')] : & \quad \mathbb{E}_t \left[U_{t+1}^{j'}(x) \right] - c_1 - c_1 \log[J\phi_u^j(x, j')] = \lambda \\ [\lambda] : & \quad \sum_{j' \in \mathcal{J}} \phi_u^j(x, j') = 1 \end{aligned}$$

Take any two j_1, j_2 we have that

$$\mathbb{E}_t \left[U_{t+1}^{j_1}(x) \right] - c_1 \log[J\phi_u^j(x, j_1)] = \mathbb{E}_t \left[U_{t+1}^{j_2}(x) \right] - c_1 \log[J\phi_u^j(x, j_2)]$$

thus

$$\frac{\mathbb{E}_t[U_{t+1}^{j_1}(x)] - \mathbb{E}_t[U_{t+1}^{j_2}(x)]}{c_1} = \log\left(\frac{\phi_u^j(x, j_1)}{\phi_u^j(x, j_2)}\right) \implies \frac{\phi_u^j(x, j_1)}{\phi_u^j(x, j_2)} = \frac{e^{(\mathbb{E}_t[U_{t+1}^{j_1}(x)]/c_1)}}{e^{(\mathbb{E}_t[U_{t+1}^{j_2}(x)]/c_1)}}$$

Fix any \hat{j} , then we can write any other $j' \in \mathcal{J}$ in terms of \hat{j} and plug into the constraint to get:

$$\sum_{j' \in \mathcal{J}} \phi_u^j(x, j') = \sum_{j' \in \mathcal{J}} \frac{e^{(\mathbb{E}_t[U_{t+1}^{j'}(x)]/c_1)}}{e^{(\mathbb{E}_t[U_{t+1}^{\hat{j}}(x)]/c_1)}} \phi_u^j(x, \hat{j}) = \frac{\phi_u^j(x, \hat{j})}{e^{(\mathbb{E}_t[U_{t+1}^{\hat{j}}(x)]/c_1)}} \sum_{j' \in \mathcal{J}} e^{(\mathbb{E}_t[U_{t+1}^{j'}(x)]/c_1)} = 1$$

Which we can solve to obtain

$$\boxed{\phi_u^j(x, j') = \frac{e^{(\mathbb{E}_t[U_{t+1}^{j'}(x)]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t[U_{t+1}^{\tilde{j}}(x)]/c_1)}}$$

Note that the non-negativity constraints are satisfied because the exponential function is always positive.

B Derivation of Unemployed Bellman Equation

The Bellman of a unemployed worker is (omitting time and worker types):

$$U^j = b + \beta \max_{\phi} \left\{ \sum_{j' \in \mathcal{J}} (\phi(j') \mathbb{E}_t [U^{j'}]) - c(\phi) \right\}$$

Where the optimal strategy we obtained in Chapter A

$$\phi(j') = \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \quad (\text{B.1})$$

and the cost of a search policy (strategy) is:

$$c(\phi) = c_1 \sum_{j \in \mathcal{J}} \phi(j') \log(J \phi(j'))$$

We want to plug the best policy and obtain the Bellman. The continuation value takes the form:

$$\begin{aligned} & \sum_{j' \in \mathcal{J}} \left[\frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \mathbb{E}_t [U^{j'}] - c_1 \left(\frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \right) \log \left(J \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \right) \right] \\ \Rightarrow & \sum_{j' \in \mathcal{J}} \left\{ \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \left[\mathbb{E}_t [U^{j'}] - c_1 \log \left(J \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \right) \right] \right\} \\ \Rightarrow & \sum_{j' \in \mathcal{J}} \left\{ \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \left[\mathbb{E}_t [U^{j'}] - c_1 \left(\log(J) + \log \left(e^{(\mathbb{E}_t [U^{j'}]/c_1)} \right) - \log \left(\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)} \right) \right) \right] \right\} \\ \Rightarrow & c_1 \sum_{j' \in \mathcal{J}} \left\{ \frac{e^{(\mathbb{E}_t [U^{j'}]/c_1)}}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \left[\log \left(\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)} \right) - \log(J) \right] \right\} \\ \Rightarrow & c_1 \left[\log \left(\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)} \right) - \log(J) \right] \frac{1}{\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t [U^{\tilde{j}}]/c_1)}} \sum_{j' \in \mathcal{J}} e^{(\mathbb{E}_t [U^{j'}]/c_1)} \end{aligned}$$

Thus we obtain:

$$U^j = b + \beta c_1 \left[\log \left(\sum_{\tilde{j} \in \mathcal{J}} e^{(\mathbb{E}_t[U^{\tilde{j}}]/c_1)} \right) - \log(J) \right]$$

C Derivation of Surplus Bellman Equation

First note that the difference in unemployment value between two locations is:

$$U_t^j(x) - U_t^{j'}(x) = b(x, j, z_t) - b(x, j', z_t)$$

we will use this fact later.

We start by deriving the expression for the surplus created by a match within a location $S^{j \rightarrow j}(x, y)$. Re-write Equation 2.3 as:

$$J_t^j(x, y) = f(x, y, j, z_t) + \beta \mathbb{E}_t \left[(1 - \delta) \lambda_{t+1}^j(x, y) \left(J_{t+1}^j(x, y) - U_{t+1}^j(x) \right) \right] + \beta \mathbb{E}_t \left[U_{t+1}^j(x) \right] \quad (\text{C.1})$$

Note that

$$\mathbb{E}_t \left[U_{t+1}^j(x) \right] - U_t^j(x) = \mathbb{E}_t \left[U_{t+1}^j(x) \right] - b(x, j, z) - \beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1} \right\} \right) - \beta c_1 \log J$$

Focus on the term:

$$\beta \mathbb{E}_t \left[U_{t+1}^j(x) \right] - \beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1} \right\} \right)$$

we take advantage of the following translation identity property of the lse function:

$$\text{lse}(\{x_i\}) = \text{lse}(\{x_i - a\}) + a \quad \implies \quad a - \text{lse}(\{x_i\}) = -\text{lse}(\{x_i - a\})$$

thus we can write:

$$\begin{aligned}
\beta \mathbb{E}_t [U_{t+1}^j(x)] - \beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1} \right\} \right) &= -\beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x)] - \mathbb{E}_t[U_{t+1}^j(x)]}{c_1} \right\} \right) \\
&= -\beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x) - U_{t+1}^j(x)]}{c_1} \right\} \right) \\
&= -\beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[b(x, j', z_{t+1}) - b(x, j, z_{t+1})]}{c_1} \right\} \right)
\end{aligned}$$

Subtracting $U_t^j(x)$ to both sides of Equation C.1 and using the previous result we get:

$$\begin{aligned}
J_t^j(x, y) - U_t^j(x) &= f(x, y, j, z_t) - b(x, j, z_t) - \beta c_1 \log(J) - \beta c_1 \text{lse} \left(\left\{ \frac{\mathbb{E}_t[b(x, j', z_{t+1}) - b(x, j, z_{t+1})]}{c_1} \right\} \right) + \dots \\
&\quad \dots + \beta \mathbb{E}_t [(1 - \delta) \lambda_{t+1}^j(x, y) (J_{t+1}^j(x, y) - U_{t+1}^j(x))]
\end{aligned}$$

Define $s(x, y, j \rightarrow j', z_t) = f(x, y, j, z_t) - b(x, j', z_t)$ and

$$\Lambda^j(x) = \beta c_1 \left(\log(J) + \text{lse} \left(\left\{ \frac{\mathbb{E}_t[b(x, j', z_{t+1}) - b(x, j, z_{t+1})]}{c_1} \right\} \right) \right)$$

and the Bellman equation becomes:

$$S_t^{j \rightarrow j}(x, y) = s(x, y, j \rightarrow j, z_t) - \Lambda^j(x) + \beta \mathbb{E}_{t+1} [\max \{0, S_{t+1}^{j \rightarrow j}(x, y)\}]$$

Next consider the the surplus created by a match between a worker x in location j_1 and a firm y in location j_2 :¹.

$$\begin{aligned}
S_t^{j_1 \rightarrow j_2} &= J_t^{j_2}(x, y) - [U_t^{j_1}(x) + F^{j_1 \rightarrow j_2}] \\
&= (J_t^{j_2}(x, y) - U_t^{j_2}(x)) - (U_t^{j_1}(x) - U_t^{j_2}(x)) - F^{j_1 \rightarrow j_2} \\
&= (J_t^{j_2}(x, y) - U_t^{j_2}(x)) - (b(x, j_1, z_t) - b(x, j_2, z_t)) - F^{j_1 \rightarrow j_2} \\
&= S_t^{j_2 \rightarrow j_2}(x, y) - (b(x, j_1, z_t) - b(x, j_2, z_t)) - F^{j_1 \rightarrow j_2} \\
&= s(x, y, j_2 \rightarrow j_2, z_t) - \Lambda^{j_2}(x) + \beta \mathbb{E}_{t+1} [\max \{0, S_{t+1}^{j_2 \rightarrow j_2}(x, y)\}] - (b(x, j_1, z_t) - b(x, j_2, z_t)) - F^{j_1 \rightarrow j_2} \\
&= s(x, y, j_1 \rightarrow j_2, z_t) - \Lambda^{j_2}(x) + \beta \mathbb{E}_{t+1} [\max \{0, S_{t+1}^{j_2 \rightarrow j_2}(x, y)\}] - F^{j_1 \rightarrow j_2}
\end{aligned}$$

¹Note that $F^{j \rightarrow j} = 0$ therefore we are generalizing the case where $j_1 = j_2$

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