A model of Sorting bewteen Workers and Firms (now in Space!)

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Model

- I adopt the framework from (Lise and Robin 2017).
- Introducing a fixed amount of locations and population distribution of workers should be an equilibrium outcome.
- Workers can "partially" direct their search to each location.
- Productivity depends on the quality of match and aggregate state as in the original paper. I want to introduce a third component which is location productivity.
  - Which in turn depends on the skill distribution of workers in each location.

Model Setup

Demographics

- There is a discrete and finite set of locations $\mathcal{J} = \{1, \ldots, |\mathcal{J}|\}$ locations indexed by $j \in \mathcal{J}$.
- Continuum of workers indexed by their ability $x \in \mathcal{X}$.
  - The total measure is normalized to 1.
  - Exogenous distribution $\ell(x)$
  - Endogenous distribution in each location $\ell^j(x)$
  - Denote μ_j the total population in location j.

Demographics

- Continuum of firms indexed by technology $y \in \mathcal{Y}$.
  - Total measure is normalized to 1.
  - Uniformly distributed.

Technology

- There is andexogenous cost of posting v job opportunities in location j is

$$c_j(v) \geq 0$$

assume it is increasing, convex, and independent of the firm type y (but *potentially* dependent on location).
- The aggregate state of the economy is indexed by z_t .
  - Changes from z to z' according to the transition probability $\pi(z, z')$.
- Workers and firms discount the future at the rate β.

Technology

- Workers can move across locations, they *"partially direct"* their search using a mixed strategy:

$$\phi_j^i(x) = \{\phi_j^i(x, j')\}_{j' \in \mathcal{J}}$$

$\phi_j^i(x, j')$ is the probability that x worker from j search in j' and $i \in \{u, s\}$.
- Each strategy has an asociated cost $c_s(\phi_i^j(x))$:

$$c_s(\phi_j^i(x)) = c_1 \left(\sum_{j' \in \mathcal{J}} \phi_j^i(x, j') \log\left(J \phi_j^i(x, j')\right) \right)$$

- When a worker move they must pay a cost $F^{j \to j'} \geq 0$ with $F^{j \to j} = 0$.

Technology

- Unemployed workers instant utility in each location is $b(x, z, j)$.[1]
- Firms have access to a production technology, defined at the match level and depending on the location and the aggregate state of the economy $f(x, y, j, z)$.

Job Search

Timing of the model

The timing is as follows:

[1] In (Lise and Robin 2017) $b(x)$ stands for unemploymet benefits, I want to be more general to be able to include differerences in cost of living across locations

1. At time (t), distributions of employed and unemployed workers are inherited from $t-1$.
   - $u_t^j(x)$ is the measure of type-x Unemployed workers at the location j.
   - $h_t^j(x, y)$ is the measure of type-x workers employed at the firm y at the location j.
   - Note that on each location:

$$u_t^j(x) + \int h_t^j(x, y) dy = \ell^j(x)$$

Timing of the model

2. Aggregate state changes $z_{t-1} \to z_t$.
3. At time $(t+)$ Separations occur:
   - $u_{t+}^j(x)$ is the measure of the type x Unemployed workers in the location j after the shock.
   - $h_{t+}^j(x, y)$ is the measure of type x workers that remain employed at firm y in location j.
4. Unemployed and employed workers draw new offers.

Search Intensity and Vacancies

Search Intensity

Both unemployed and employed workers search. - s is the search intensity of an employed worker - Search intensity of an unemployed worker is normalized to 1. - The total search intensity in location j is:

$$L_t^j = \sum_{j' \in \mathcal{J}} \left[\int \phi_u^{j'}(x, j) u_{t+}^{j'}(x) dx + s \int \int \phi_s^{j'}(x, j) h_{t+}^{j'}(x, y) dx dy \right]$$

Vacancies

- Let $v_t^j(y)$ be the number of job opportunities posted by a firm y at time t in location j.
- The total number of job opportunities posted at the time t in the location j is:

$$V_t^j = \int v_t^j(y) dy$$

Matching

Let $M_t^j = M(L_t^j, V_t^j)$ be the number of job matches in the location j then:

- The probability that an unemployed worker contacts a vacancy in the location j is

$$p_t^j = \frac{M_t^j}{L_t^j}$$

  - sp_t^j is the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker

$$q_t^j = \frac{M_t^j}{V_t^j}$$

- Let $\theta_t^j = V_t^j / L_t^j$ be the market tightness in location j.

Dynamic Programming

- $U_t^j(x)$ the value for an unemployed worker of type x at time t at location j.
- The value of getting an offer depends on whether the worker is employed or not:
  - $W_{0,t}^{j' \to j}(x, y)$ is the value of a type-x unemployed worker at location j' who is hired by a firm of type y at a location j.
  - $W_{1,t}^{j' \to j}(x, y, y')$ is the value offered at the time t by type y firm at location j to a type x worker employment at a type y' firm in location j'.
- $J_t^j(x, y)$ is the value of a match between a type x worker and a type y firm at time t in location j.

Unemployed Worker

Unemployed worker's value function is:

$$\begin{aligned}
U_t^j(x) = b(x, j, z_t) + \beta \max_{\phi_u^j(x)} \Big\{ & -c(\phi_u^j(x)) + \\
& \mathbb{E}_t \sum_{j' \in \mathcal{J}} \phi_u^j(x, j') \Big[(1 - p_{t+1}^{j'}) U_{t+1}^j(x) \\
& + p_{t+1}^{j'} \int \max \Big\{ U_{t+1}^{j'}(x), W_{0,t+1}^{j \to j'}(x, y) \Big\} \frac{v_{t+1}^{j'}(y)}{V_{t+1}^{j'}} dy - F^{j \to j'} \Big] \Big\}
\end{aligned}$$

- Workers don't have bargaining power: they are offered their reservation value $U_t^{j'}(x) - F^{j\to j'} = W_{0,t+1}^{j\to j'}(x,y) - F^{j\to j'}$.
- Define the value of unemployment for a type x worker that moves from j to j' as:

$$U_{t+1}^{j\to j'}(x) = (1-p_{t+1}^{j'})U_{t+1}^{j}(x) + p_{t+1}^{j'}U_{t+1}^{j'}(x) - F^{j\to j'}$$

Unemployed Worker

- We can rewrite the value function as:

$$U_t^j(x) = b(x,j,z) + \beta \max_{\phi_u^j(x)} \left\{ \sum_{j'\in\mathcal{J}} \left(\phi_u^j(x,j')\mathbb{E}_t \left[U_{t+1}^{j\to j'}(x) \right] \right) - c(\phi_u^j(x)) \right\}$$

- And solve for the optimal search strategy and plug into the Bellman:

$$\phi_u^j(x,j') = \frac{e^{\left(\mathbb{E}_t\left[U_{t+1}^{j\to j'}(x)\right]/c_1\right)}}{\sum_{\tilde{j}\in\mathcal{J}} e^{\left(\mathbb{E}_t\left[U_{t+1}^{j\to \tilde{j}}(x)\right]/c_1\right)}}$$

$$U_t^j(x) = b(x,j,z) + \beta c_1 \text{lse}\left(\left\{ \frac{\mathbb{E}_t[U_{t+1}^{j'}(x)]}{c_1} \right\} \right) - \beta c_1 \log J \tag{1}$$

Where $\text{lse}(x\in\mathbb{R}^n)$ is the **log-sum-exp** function.

Joint Value of a Match

- If a match is destroyed the firm will get 0 and the worker gets their unemployment value $U_t^j(x)$.
- Matches are destroyed for two reasons:
  - *Exogenous destruction* with probability δ
  - *Endogenous destruction*, if and only if $J_t^j(x,y) < U_t^j(x)$.
    - Denote $\lambda_t^j(x,y) = \mathbb{1}_{\{J_t^j(x,y) > U_t^j(x,y)\}}$

Joint Value of a Match

We can write the Bellman equation of a match value as:

$$
\begin{aligned}
J_t^j(x,y) = \underbrace{f(x,y,j,z_t)}_{\text{match value added}} + \beta \max_{\phi_s^j(x)} \Bigg\{ \mathbb{E}_t \Bigg[& \overbrace{(1-(1-\delta)\lambda_{t+1}^j(x,y))}^{\text{match is destroyed}} \underbrace{U_{t+1}^j(x)}_{\text{worker gets unemployment value}} + \\
& + \underbrace{(1-\delta)\lambda_t^j(x,y)}_{\text{match survives}} \max_{\phi_s^j(x)} \Bigg\{ -c(\phi_s^j(x)) + \sum_{j' \in \mathcal{J}} \phi_s^j(x,j') \Bigg[\overbrace{(1-sp_t^{j'})}^{\text{no new offers}} \underbrace{J_{t+1}^j(x,y)}_{\text{stays with same firm}} + \\
& + sp_t^{j'} \underbrace{\int \max\{J_{t+1}^j(x,y), W_{1,t+1}^{j\to j'}(x,y',y) - F^{j\to j'}\} \frac{v_{t+1}^{j'}(x)}{V_{t+1}^{j'}} dy'}_{\text{worker only accepts new offers if value is greater than current match}} \Bigg] \Bigg\}
\end{aligned}
$$

Joint Value of a Match

Poaching:

- When worker receives an offer from a type y' in city j' then there is a sequential auction like in (Postel-Vinay and Robin 2002).
- More productive firms can offer higher values.
- The key difference with (Postel-Vinay and Robin 2002) is that location plays a role: the poaching firm must cover the cost of moving, this leads to two possible outcomes:
  - $J_{t+1}^{j'}(x,y') > J_{t+1}^j(x,y) + F^{j\to j'}$ the worker moves from $(j,y) \to (j',y')$ and receives $W_{1,t+1}^{j\to j'}(x,y',y)$
  - $J_{t+1}^j(x,y) > J_{t+1}^{j'}(x,y') - F^{j\to j'}$ the worker stays at (j,y) and receives $W_{1,t+1}^j(x,y,y')$
- To able to poach, the firm must be at least $F^{j\to j'}$ more productive.

Joint Value of a Match

As in (Postel-Vinay and Robin 2002) if the worker is hired by the poaching firm the worker receives the incumbent firm reservation value plus the cost of changing jobs, i.e.

$$
J_{t+1}^{j'}(x,y') > J_{t+1}^j(x,y) + F^{j\to j'} \implies W_{1,t+1}^{j\to j'}(x,y',y) = J_{t+1}^j(x,y) + F^{j\to j'}
$$

therefore

$$J_t^j(x,y) = f(x,y,j,z_t) + \beta \max_{\phi_s^j(x)} \Big\{ \mathbb{E}_t \Big[(1-(1-\delta)\lambda_{t+1}^j(x,y))U_{t+1}^j(x)$$
$$+ (1-\delta)\lambda_t^j(x,y) \sum_{j' \in \mathcal{J}} \phi_s^j(x,j') J_{t+1}^j(x,y) \Big] - c(\phi_s^j(x)) \Big\}$$

Joint Value of a Match

Note that the optimal strategy for employed workers is to engage in random search i.e.:

$$\phi_s^j(x,j') = \frac{1}{\mid \mathcal{J} \mid} \qquad \forall\ x \text{ and } j, j' \in \mathcal{J}$$

Finally we can write the Bellman equation for the joint value of a match as:

$$J_t^j(x,y) = f(x,y,j,z_t) + \beta \mathbb{E}_t \,[\tag{2}$$
$$(1-(1-\delta)\lambda_{t+1}^j(x,y))U_{t+1}^j(x) + (1-\delta)\lambda_t^j(x,y)J_{t+1}^j(x,y)\Big] \tag{3}$$

Match Surplus

Define the surplus of a match between a type x worker in location j and a type y firm in location j' as:

$$S_t^{j \to j'}(x,y) = J_t^{j'}(x,y) - [U_t^j(x) - F^{j \to j'}]$$

After some algebra we obtain the following expression for the surplus of a match:

$$S_t^{j \to j'}(x,y) = s(x,y,j \to j', z_t) - \Lambda^{j'}(x) - F^{j \to j'} \tag{4}$$
$$+ \beta \mathbb{E}_{t+1} \Big[\max \Big\{ 0, S_{t+1}^{j' \to j'}(x,y) \Big\} \Big] \tag{5}$$

- $\Lambda^{j'}(x)$ is a function of the expected value of the difference of the instantaneous utility of a type x worker in j' and every other location.

Match Surplus

Match surplus encodes all the necessary and sufficient conditions for a firm y' in location j' to poach a worker from a firm y in location j:

$$\begin{aligned} S_t^{j\to j'}(x,y') - S_t^{j\to j}(x,y') &= J_t^{j'}(x,y') - [U_t^j(x) - F^{j\to j'}] - \left(J_t^j(x,y) - [U_t^j(x) - F^{j\to j}]\right) \\ &= J_t^{j'}(x,y') - [J_t^j(x,y) + F^{j\to j'}] \end{aligned}$$

- Worker x is poached by firm y' in location j' from firm y in location j if and only if the surplus obtained from moving to j' and matching with y' is higher than the surplus of staying at j matched with y'.

Interim Distributions

- The surplus function determines how does the stock of unemployed and employed workers change when $z_{t-1} \to z_t$:

$$u_{t+}^j(x) = \underbrace{u_t^j(x)}_{\text{inherited from } t} + \overbrace{\int \Big(\underbrace{\mathbb{1}_{S_t^{j\to j}(x,y)<0}}_{\text{endogenous destruction}} + \underbrace{\delta \mathbb{1}_{S_t^{j\to j}(x,y)\geq 0}}_{\text{exogenous destruction}} \Big) h_t^j(x,y)dy}^{\text{new unemployment created by shock}}$$

  and

$$h_{t+}^j(x,y) = (1-\delta)\mathbb{1}_{\{S^{j\to j}(x,y)\geq 0\}} h_t^j(x,y)$$

Vacancy Creation

- $B_t^j(y)$ is the expected value of a type y vacancy making contact with a worker in location j.
- Vacancies are posted in the interim period and meet unemployed and employed type-x workers at a rates

$$\frac{u_{t+}^j(x)}{L_t^j} \quad \text{and} \quad s\frac{h_{t+}^j(x,y)}{L_t^j}$$

- The expected value of posting a vacancy is therefore, the surplus that the posting firm expects to add, potential matches with negative surplus are immediately destroyed therefore those add no surplus.

Vacancy Creation

In terms of the Bellman equation we can write:

$$B_t^j(y) = \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \phi_u^{j'}(x,j) \underbrace{\frac{u_{t+}^{j'}(x)}{L_t^{j'}}}_{\text{likelihhod of match}} \times \overbrace{S_t^{j' \to j}(x,y)^+}^{\text{match survives}} \, dx \right)}_{\text{expected value added from hiring unemployed workers}} + \tag{6}$$

$$+ \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \left(\int \underbrace{s\phi_s^{j'}(x,j) \frac{h_{t+}^{j'}(x,y)}{L_t^{j'}}}_{\text{likelihood of match}} \times \overbrace{[S_t^{j' \to j}(x,y) - S_t^{j' \to j'}(x,y')]^+}^{\text{poaching is succesfull}} \, dx \right) dy \right)}_{\text{expected value added from poaching other firms employees}} \tag{7}$$

For simplicity we use the notation $x^+ = \max\{0, x\}$.

Vacancy Creation

Firms will post vacancies such that the marginal cost of the vacancies and the marginal expected benefit B_t^j are equal:

$$c_j'(v_t^j(y)) = q_t^j B_t^j(y)$$

Using the value of B_t^j any particular cost and matching function can be used to pin down the number of vacancies posted by each firm in each location.

Labor Market Flows

Now we characterize the flows of workers in-to and out-of unemployment at each location:

- Let

  $$\eta^{j' \to j}(x,y) = \mathbb{1}_{\{S_t^{j' \to j}(x,y) > 0\}}$$

  and

  $$\eta^{j' \to j}(x, y' \to y) = \mathbb{1}_{\{S_t^{j' \to j}(x,y) > S_t^{j' \to j'}(x,y')\}}$$

Labor Market Flows

- The law of motion of the unemployment rate is:

$$u^j_{t+1}(x) = \sum_{j'\in\mathcal{J}} \underbrace{\phi^{j'}_u(x,j)u^{j'}_{t+}(x)\left(1-\int \eta^{j'\to j}(x,y)p^j\frac{v^j(y)}{V^j}dy\right)}_{\text{mass of incoming unemployed workers that are not hired by any firm}} \tag{8}$$

- For ease of exposition we write the mass of employed workers as the sum of the following three terms:

$$h^j(x,y) = h^j_{u,t+1}(x,y) + h^j_{p,t+1}(x,y) + h^j_{r,t+1}(x,y) \tag{9}$$

Labor Market Flows

- The mass of workers hired from unemployment:

$$h^j_{u,t+1}(x,y) = \sum_{j'\in\mathcal{J}} \phi^{j'}_u(x,j)u^{j'}_{t+}(x)\eta^{j'\to j}(x,y)p^j\frac{v^j(y)}{V^j} \tag{10}$$

Labor Market Flows

- The mass of workers that are succesfully poached from other firms:

$$h^j_{p,t+1} = \sum_{j'\in\mathcal{J}} \left(\overbrace{\int \underbrace{h^{j'}_{t+}(x,y')sp^j_t\frac{v^j_t(y)}{V^j_t}\eta^{j'\to j}(x,y'\to y)}_{\text{mass of workers poached from } y' \text{ firms in location } j'} dy'}^{\text{mass of workers that } y \text{ succesfully poach from location} j'}\right) \tag{11}$$

Labor Market Flows

- The mass of workers that the firm is able to retain:

$$h^j_{r,t+1} = \underbrace{h_{t+}(x,y)\times}_{\text{employed at interim}} \overbrace{\prod_{j'\in\mathcal{J}} \left[\underbrace{1 - \frac{s}{\mid \mathcal{J} \mid} \int \left(p^{j'}_t \frac{v^{j'}_t(y')}{V^{j'}_t} \eta^{j\to j'}(x, y \to y') \right) dy'}_{\text{no poeached by any firm in location } j'} \right]}^{\text{probability of not being poached}} \tag{12}$$

Endgogenizing Location Productivity

- This section focuses on the endogenous determination of location productivity.
- The key idea is that the productivity of a location depends on the skill distribution of workers in that location.
- (Davis and Dingel 2019) in assuming that the productivity of workers in a location j is a result of idea exchange process within each location.

Worker Productivity

- Assume that a type x worker in location j has a $\Omega(x, \overline{X}^j)$ where $\overline{X}^j$ is the value of idea exchange in location j. Thus:

$$f(x, y, j, z_t) = f\left(\Omega(x, \overline{X}^j), y, z_t\right)$$

- Assume that the value of idea exchange in location j is a function of the distribution of skills in location j:

$$\overline{X}^j = \overline{X}\left(\{\ell^j(x)\}\right)$$

Asumptions

- **Assumption 1**: $\Omega(x, \overline{X}^j)$ should be increasing in x and $\overline{X}^j$ and in the absence of idea exchange, worker productivity is just their type: $\Omega(x, 0) = x$.
- **Assumption 2**: Worker type x and idea exchange environment are complements i.e. $\Omega(x, \overline{X}^j)$ supermodular in (x , $\overline{X}^j$).
- **Assumption 3**: The value of idea exchange in location j is increasing in the number of potential exchange partners μ^j and on the distribution of skills in location j $\ell^j(x)$.
  - With the *FOSD* order of distributions.

Functional Forms

- I will focus on the following functional forms for $\Omega(x, \overline{X}^j)$ and $\overline{X}^j$:

$$\Omega(x, \overline{X}^j) = x(1 + A\overline{X}^j x)$$

where A is a parameter that captures the scope of gains from idea exchange.

$$\overline{X}\left(\{\ell^j(x)\}\right) = \left(1 - e^{-\nu \int \ell^j(x)dx}\right)\hat{x}^j = \left(1 - e^{-\nu\mu^j}\right)\hat{x}^j$$

- The average ability of the individuals you encounter in these exchanges is denoted as $\hat{x}^j$:

$$\hat{x}^j = \frac{1}{\mu^j}\int x\ell^j(x)dx$$

Cost of Living

- As in (Davis and Dingel 2019) and (Behrens, Duranton, and Robert-Nicoud 2014) every individual in location j pays the congestion cost :

$$\theta\mu_j^\gamma$$

References

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