Model Summary and Agenda

Mitchell Valdes-Bobes

1 Agenda

- ☐ Discuss the model with updated bargaining.
- ☐ Discuss aglomerations effects.
    - ☐ Computation of equilibrium.

2 Model Summary

2.1 Model Setup

2.1.1 Demographics

- There is a discrete and finite set of locations $\mathcal{J} = \{1, \dots, |\mathcal{J}|\}$ locations indexed by $j \in \mathcal{J}$.
- Continuum of workers indexed by their ability $x \in \mathcal{X}$.
    - The total measure is normalized to 1.
    - Exogenous distribution $\ell(x)$
    - Endogenous distribution in each location $\ell^j(x)$
    - Denote μ_j the total population in location j.
- Continuum of firms indexed by technology $y \in \mathcal{Y}$.
    - Total measure is normalized to 1.
    - Uniformly distributed.

2.1.2 Technology

- There is exogenous cost of posting v job opportunities in location j is

$$c_j(v) \geq 0$$

assume it is increasing, convex, and independent of the firm type y (but *potentially* dependent on location).
- Workers and firms discount the future at the rate β.
- Workers can move across locations:
  - Workers choose a mixed strategy to search:

$$\phi_i^j(x) = \{\phi_i^j(x, j')\}_{j' \in \mathcal{J}}$$

    where $\phi_i^j(x, j')$ is the probability that a type x worker from location j search in location j' and $i \in \{u, s\}$ refers to the employment status of the worker.
  - Each strategy has an asociated cost $c_s(\phi_i^j(x))$:

$$c_s(\phi_i^j(x)) = c \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') \log \left(J \phi_j^i(x, j')\right)$$

  - When a worker move they must pay a cost $F^{j \to j'} \geq 0$ (with $F^{j \to j} = 0$).
- Unemployed workers instant utility in each location is $b(x, j)$.
- Firms have access to a production technology, defined at the match level and depending on the location $f(x, y, j)$.

Both unemployed and employed workers search, denote s the search intensity of an employed worker and 1 is the (normalized) search intensity of an unemployed worker. The total search intensity in location j is

$$L^j = \sum_{j' \in \mathcal{J}} \left[\int \phi_u^{j'}(x, j) u^{j'}(x) dx + s \int \int \phi_s^{j'}(x, j) h^{j'}(x, y) dx dy \right]$$

Let $v^j(y)$ be the number of job opportunities posted by a firm y in location j.

- $V^j = \int v^j(y) dy$ is the total number of job opportunities posted location j.

Let $M^j = M(L^j, V^j)$ be the number of job matches in the location j then:

- The probability that an unemployed worker contacts a vacancy in the location j is

$$p^j = \frac{M^j}{L^j}$$

  - sp^j is the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker

$$q^j = \frac{M^j}{V^j}$$

- Let $\theta^j = V^j / L^j$ be the market tightness in location j.

2.1.3 Dynamic Programming Problem

ADD DEFINITION OF OBJECTS HERE

2.1.4 Worker

$$U^j(x) = \underbrace{b(x,j)}_{\text{instant utility}} + \beta \max_{\phi_u^j(x)} \Bigg\{ \underbrace{-c(\phi_u^j(x))}_{\text{cost of search strategy}} + \sum_{j' \in \mathcal{J}} \underbrace{\phi_u^j(x,j')}_{\text{weight by probability of search in} j'} \Bigg[\overbrace{(1-p^{j'})U^{j'}(x)}^{\text{no offer, stays unemployed now in} j'} + p^{j'} \underbrace{\int \max\left\{U^{j'}(x), W_0^{j\to j'}(x,y)\right\} \frac{v^{j'}(y)}{V^{j'}} dy}_{\text{if offer, pays cost,moves to } j \text{ and then is matched randomly with some firm}} \Bigg] \Bigg\}$$

2.2 Value of a match

$$J^j(x,y) = \underbrace{f(x,y,j)}_{\text{match value added}} + \beta \Bigg[\overbrace{(1-\lambda^j(x,y))}^{\text{match is destroyed}} \underbrace{U^j(x)}_{\text{worker gets unemployment value}} + \underbrace{(\lambda^j(x,y)}_{\text{match survives}} \max_{\phi_s^j(x)} \Bigg\{ -c(\phi_s^j(x)) + \sum_{j' \in \mathcal{J}} \phi_s^j(x,j') \Bigg[\overbrace{(1-sp^{j'})}^{\text{no new offers}} \underbrace{J^j(x,y)}_{\text{stays with same firm}} + sp^{j'} \underbrace{\int \max\{J^j(x,y), W_1^{j\to j'}(x,y',y)\} \frac{v^{j'}(x)}{V^{j'}} dy'}_{\text{worker only accepts new offers if value is greater than current match}} \Bigg]$$

2.3 Surplus of a Match

$$S^{j\to j'}(x,y) = J^{j'}(x,y) - \left[U^j(x) + F^{j\to j'}\right]$$

2.4 Bargaining

I use the same bargaining as in (Cahuc, Postel-Vinay, and Robin 2006) this have the following implications for the value that a worker gets from a match:

- Unemployed worker in location j when matched with firm y in location j' gets:

$$W_0^{j\to j'}(x,y) = (1-\mu)U^{j'}(x)+\mu[J^{j'}(x,y)-F^{j'\to j}] = U^j(x)+\mu\left(J^{j'}(x,y)-[U^j(x)+F^{j'\to j}]\right)$$

- Worker employed in location j by firm y when matched with firm y' in location j' gets:

$$W_1^{j\to j'}(x,y) = (1-\mu)J^j(x,y)+\mu[J^{j'}(x,y')-F^{j'\to j}] = J^j(x,y)+\mu\left(J^{j'}(x,y')-J^j(x,y)-F^{j'\to j}\right)$$

Plugging this into the Bellman equations we get:

2.4.1 Worker

$$\begin{aligned}
U^j(x) &= b(x,j)+\beta\max_{\phi_u^j(x)}\Bigg\{-c(\phi_u^j(x))+\sum_{j'\in\mathcal{J}}\phi_u^j(x,j')\Big[(1-p^{j'})U^{j'}(x)\\
&\qquad +p^{j'}\int\max\left\{U^{j'}(x),U^j(x)+\mu\left(J^{j'}(x,y)-[U^j(x)+F^{j'\to j}]\right)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\Big]\Bigg\}\\
&= b(x,j)+\beta\max_{\phi_u^j(x)}\Bigg\{-c(\phi_u^j(x))+\sum_{j'\in\mathcal{J}}\phi_u^j(x,j')\Big[(1-p^{j'})U^{j'}(x)+p^{j'}U^{j'}(x)\\
&\qquad +p^{j'}\mu\int\max\left\{0,J^{j'}(x,y)-[U^j(x)+F^{j'\to j}]\right\}\frac{v^{j'}(y)}{V^{j'}}dy\Big]\Bigg\}\\
&= b(x,j)+\beta\left[U^j(x)\right]+\beta\Bigg(\sum_{j'\in\mathcal{J}}\phi_u^j(x,j')\left[p^{j'}\mu\int\max\left\{0,J^{j'}(x,y)-[U^j(x)+F^{j'\to j}]\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]\\
&\qquad -c\left(\phi_u^j(x)\right)\Bigg)
\end{aligned}$$

Recall that the surplus of a match is defined as

$$S^{j\to j'}(x,y) = J^{j'}(x,y)-[U^j(x)+F^{j\to j'}]$$

We can rewrite the Bellman equation of the unemployed worker as:

$$U^j(x) = b(x,j)+\beta\left[U^j(x)\right]+\beta\left(\sum_{j'\in\mathcal{J}}\phi_u^j(x,j')\left[p^{j'}\mu\int\max\left\{0,S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]-c\left(\phi_u^j(x)\right)\right)$$

We can derive the optimal strategy for each worker as:

$$\phi_u^j(x,j') = \frac{\exp\left(\left[p^{j'}\mu\int\max\left\{0,S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]/c\right)}{\sum_{\tilde{j}\in\mathcal{J}}\exp\left(\left[p^{\tilde{j}}\mu\int\max\left\{0,S^{j\to\tilde{j}}(x,y)\right\}\frac{v^{\tilde{j}}(y)}{V^{\tilde{j}}}dy\right]/c\right)}$$

Plug back into the value function to get:

$$U^j(x) = b(x,j) + \beta\left[[U^j(x)] + c\log\left(\sum_{\tilde{j}\in\mathcal{J}}\exp\left(\left[p^{j'}\mu\int\max\left\{0,S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]/c\right) + c\log(|\mathcal{J}|)\right)\right]$$

2.5 Value of a match

Note tha if the match survives the employed worker chooses their optimal search strategy $\phi_s^j(x)$ search in each of the locations j' with the specified probability $\phi_s^j(x,j')$.

$$\begin{aligned}\max_{\phi_s^j(x)}\Bigg\{\sum_{j'\in\mathcal{J}}\phi_s^j(x,j')\Big[&(1-sp^{j'})J^j(x,y)\\ &+sp^j\int\max\{J^j(x,y), J^j(x,y)+\mu\left(J^{j'}(x,y')-J^j(x,y)-F^{j'\to j}\right)\}\frac{v^{j'}(x)}{V^{j'}}dy'\Big]\Bigg\}\end{aligned}$$

Wich can be rewritten as:

$$J^j(x,y) + \max_{\phi_s^j(x)}\left\{\sum_{j'\in\mathcal{J}}\phi_s^j(x,j')\left[sp^j\int\max\{0,\mu\left(J^{j'}(x,y')-J^j(x,y)-F^{j'\to j}\right)\}\frac{v^{j'}(x)}{V^{j'}}dy'\right]\right\}$$

Since

$$S^{j\to j'}(x,y') = J^{j'}(x,y') - \left[U^j(x) + F^{j\to j'}\right]$$

then

$$S^{j\to j'}(x,y') - S^{j\to j}(x,y) = J^{j'}(x,y') - J^j(x,y) - F^{j'\to j}$$

Then we can rewrite the previous equation as:

$$J^j(x,y) + \max_{\phi_s^j(x)}\left\{\sum_{j'\in\mathcal{J}}\phi_s^j(x,j')\left[sp^j\int\max\{0,S^{j\to j'}(x,y')-S^{j\to j}(x,y)\}\frac{v^{j'}(x)}{V^{j'}}dy'\right]\right\}$$

We can derive the optimal strategy for each worker as:

$$\phi_u^j(x,y,j') = \frac{\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\}\frac{v^{j'}(y')}{V^{j'}}dy'\right]/c\right)}{\sum_{\tilde{j}\in\mathcal{J}}\exp\left(\left[p^{\tilde{j}}\mu\int\max\left\{0, S^{j\to\tilde{j}}(x,y') - S^{j\to j}(x,y)\right\}\frac{v^{\tilde{j}}(y')}{V^{\tilde{j}}}dy'\right]/c\right)}$$

Plug back into the value function:

$$\begin{aligned}
J^j(x,y) = f(x,y,j) + \beta\,&[[(1-\lambda^j(x,y))U^j(x) + \lambda^j(x,y)J^j(x,y)] \\
&+ \log\left(\sum_{j'\in\mathcal{J}}\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\}\frac{v^{j'}(y')}{V^{j'}}dy'\right]/c\right)\right) \\
&- c\log(\mid\mathcal{J}\mid)]
\end{aligned}$$

2.5.1 Surplus Value

Let $s(x,y,j\to j') = f(x,y,j') - b(x,j)$ I satart by writing the expression for the surpluss of staying in the same location:

$$\begin{aligned}
S^{j\to j}(x,y) &= J^j(x,y) - U^j(x) \\
&= s(x,y,j\to j) + \beta\lambda^j(x,y)\underbrace{[J^j(x,y) - U^j(x)]}_{S^{j\to j}(x,y)} + \beta c\left[\Lambda_1^j(x,y) - \Lambda_0^j(x)\right] \\
&= s(x,y,j\to j) + \beta\max\{0, S^{j\to j}(x,y)\} + \beta c\left[\Lambda_1^j(x,y) - \Lambda_0^j(x)\right]
\end{aligned}$$

The terms $\Lambda_0(x)$ and $\Lambda_1(x,y)$ come from the Bellmans of the worker and the match respectively and are defined as:

$$\begin{aligned}
\Lambda_1(x,y) &= \log\left(\sum_{j'\in\mathcal{J}}\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\}\frac{v^{j'}(y')}{V^{j'}}dy'\right]/c\right)\right) \\
\Lambda_0(x) &= \log\left(\sum_{j'\in\mathcal{J}}\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y')\right\}\frac{v^{j'}(y')}{V^{j'}}dy'\right]/c\right)\right)
\end{aligned}$$

Next note that the surplus of moving to a different location is:

$$
\begin{aligned}
S^{j\to j'}(x,y) &= J^{j'}(x,y) - \left[U^j(x) + F^{j\to j'}\right] \\
&= [J^{j'}(x,y) - U^{j'}(x)] + U^{j'}(x) - \left[U^j(x) + F^{j\to j'}\right] \\
&= S^{j'\to j'}(x,y) - [U^j(x) - U^{j'}(x) + F^{j\to j'}]
\end{aligned}
$$

References

Cahuc, Pierre, Fabien Postel-Vinay, and Jean-Marc Robin. 2006. "Wage Bargaining with On-the-Job Search: Theory and Evidence." *Econometrica* 74 (2): 323–64. https://doi.org/10.1111/j.1468-0262.2006.00665.x.