# Model Summary and Agenda

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## 1 Agenda

- ☐ Discuss the model with updated bargaining.
- ☐ Discuss aglomerations effects.
  - ☐ Computation of equilibrium.

## 2 Model Summary

### 2.1 Model Setup

#### 2.1.1 Demographics

- There is a discrete and finite set of locations $\mathcal{J} = \{1, \dots, |\mathcal{J}|\}$ locations indexed by $j \in \mathcal{J}$.
- Continuum of workers indexed by their ability $x \in \mathcal{X}$.
  - The total measure is normalized to 1.
  - Exogenous distribution $\ell(x)$
  - Endogenous distribution in each location $\ell^j(x)$
  - Denote $\zeta_j$ the total population in location $j$.
- Continuum of firms indexed by technology $y \in \mathcal{Y}$.
  - Total measure is normalized to 1.
  - Uniformly distributed.

### 2.1.2 Technology

- There is exogenous cost of posting $v$ job opportunities in location $j$ is

$$c_j(v) \geq 0$$

assume it is increasing, convex, and independent of the firm type $y$ (but *potentially* dependent on location).
- Workers and firms discount the future at the rate $\beta$.
- Workers can move across locations:
  - Workers choose a mixed strategy to search:

$$\phi_i^j(x) = \{\phi_i^j(x, j')\}_{j' \in \mathcal{J}}$$

where $\phi_i^j(x, j')$ is the probability that a type $x$ worker from location $j$ search in location $j'$ and $i \in \{u, s\}$ refers to the employment status of the worker.
  - Each strategy has an asociated cost $c_s(\phi_i^j(x))$:

$$c_s(\phi_i^j(x)) = c \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') \log\left(J \phi_j^i(x, j')\right)$$

  - When a worker move they must pay a cost $F^{j \to j'} \geq 0$ (with $F^{j \to j} = 0$).
- Unemployed workers instant utility in each location is $b(x, j)$.
- Firms have access to a production technology, defined at the match level and depending on the location $f(x, y, j)$.

Both unemployed and employed workers search, denote $s$ the search intensity of an employed worker and 1 is the (normalized) search intensity of an unemployed worker. The total search intensity in location $j$ is

$$L^j = \sum_{j' \in \mathcal{J}} \left[ \int \phi_u^{j'}(x, j) u^{j'}(x) dx + s \int \int \phi_s^{j'}(x, j) h^{j'}(x, y) dx dy \right]$$

Let $v^j(y)$ be the number of job opportunities posted by a firm $y$ in location $j$.

- $V^j = \int v^j(y) dy$ is the total number of job opportunities posted location $j$.

Let $M^j = M(L^j, V^j)$ be the number of job matches in the location $j$ then:

- The probability that an unemployed worker contacts a vacancy in the location $j$ is

$$p^j = \frac{M^j}{L^j}$$

  - $sp^j$ is the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker

$$q^j = \frac{M^j}{V^j}$$

- Let $\theta^j = V^j/L^j$ be the market tightness in location $j$.

### 2.1.3 Dynamic Programming Problem

- $U^j(x)$ is the value of being unemployed in location $j$ with type $x$.
- $J^j(x,y)$ is the match value in location $j$ with type $x$ and firm $y$.
- $W_0^{j\to j'}(x,y)$ is the value of a worker in location $j$ when matched with firm $y$ in location $j'$.
- $W_1^{j\to j'}(x,y\to y')$ is the value of a worker in location $j$ when matched with firm $y$ in location $j'$ and then matched with firm $y'$ in location $j'$.

### 2.1.4 Worker

$$U^j(x) = \underbrace{b(x,j)}_{\text{instant utility}} + \beta \max_{\phi_u^j(x)} \Bigg\{ \underbrace{-\, c(\phi_u^j(x))}_{\text{cost of search strategy}} + \sum_{j'\in\mathcal{J}} \underbrace{\phi_u^j(x,j')}_{\text{weight by probability of search in} j'} \Bigg[ \overbrace{(1-p^{j'})U^{j'}(x)}^{\text{no offer, stays unemployed now in} j'} + p^{j'} \underbrace{\int \max\left\{U^{j'}(x), W_0^{j\to j'}(x,y)\right\} \frac{v^{j'}(y)}{V^{j'}} dy}_{\text{if offer, pays cost,moves to } j \text{ and then is matched randomly with some firm}} \Bigg] \Bigg\}$$

## 2.2 Value of a match

$$J^j(x,y) = \underbrace{f(x,y,j)}_{\text{match value added}} + \beta \Bigg[ \overbrace{(1-\lambda^j(x,y))}^{\text{match is destroyed}} \underbrace{U^j(x)}_{\text{worker gets unemployment value}} + \underbrace{(\lambda^j(x,y)}_{\text{match survives}} \max_{\phi_s^j(x)} \Bigg\{ -c(\phi_s^j(x)) + \sum_{j'\in\mathcal{J}} \phi_s^j(x,j') \Bigg[ \overbrace{(1-sp^{j'})}^{\text{no new offers}} \underbrace{J^j(x,y)}_{\text{stays with same firm}} + sp^{j'} \underbrace{\int \max\{J^j(x,y), W_1^{j\to j'}(x,y',y)\} \frac{v^{j'}(x)}{V^{j'}} dy'}_{\text{worker only accepts new offers if value is greater than current match}} \Bigg]$$

## 2.3 Surplus of a Match

$$S^{j\to j'}(x,y) = J^{j'}(x,y) - \left[U^j(x) + F^{j\to j'}\right]$$

## 2.4 Bargaining

I use the same bargaining as in (Cahuc, Postel-Vinay, and Robin 2006) this have the following implications for the value that a worker gets from a match:

- Unemployed worker in location $j$ when matched with firm $y$ in location $j'$ gets:

  $$W_0^{j\to j'}(x,y) = (1-\mu)U^{j'}(x)+\mu[J^{j'}(x,y)-F^{j'\to j}] = U^j(x)+\mu\left(J^{j'}(x,y) - [U^j(x) + F^{j'\to j}]\right)$$

- Worker employed in location $j$ by firm $y$ when matched with firm $y'$ in location $j'$ gets:

  $$W_1^{j\to j'}(x,y\to y') = (1-\mu)J^j(x,y)+\mu[J^{j'}(x,y')-F^{j'\to j}] = J^j(x,y)+\mu\left(J^{j'}(x,y') - J^j(x,y) - F^{j'\to j}\right)$$

Plugging this into the Bellman equations we get:

### 2.4.1 Worker

$$\begin{aligned}
U^j(x) &= b(x,j) + \beta \max_{\phi_u^j(x)} \Bigg\{ -c(\phi_u^j(x)) + \sum_{j'\in\mathcal{J}} \phi_u^j(x,j') \Big[(1-p^{j'})U^{j'}(x) \\
&\qquad\qquad + p^{j'} \int \max\left\{U^{j'}(x), U^j(x) + \mu\left(J^{j'}(x,y) - [U^j(x) + F^{j'\to j}]\right)\right\} \frac{v^{j'}(y)}{V^{j'}}dy\Big]\Bigg\} \\
&= b(x,j) + \beta \max_{\phi_u^j(x)} \Bigg\{ -c(\phi_u^j(x)) + \sum_{j'\in\mathcal{J}} \phi_u^j(x,j') \Big[(1-p^{j'})U^{j'}(x) + p^{j'}U^{j'}(x) \\
&\qquad\qquad + p^{j'}\mu \int \max\left\{0, J^{j'}(x,y) - [U^j(x) + F^{j'\to j}]\right\} \frac{v^{j'}(y)}{V^{j'}}dy\Big]\Bigg\} \\
&= b(x,j) + \beta\left[U^j(x)\right] + \beta\Bigg(\sum_{j'\in\mathcal{J}} \phi_u^j(x,j')\left[p^{j'}\mu\int \max\left\{0, J^{j'}(x,y) - [U^j(x) + F^{j'\to j}]\right\} \frac{v^{j'}(y)}{V^{j'}}dy\right] \\
&\qquad\qquad - c\left(\phi_u^j(x)\right)\Bigg)
\end{aligned}$$

Recall that the surplus of a match is defined as

$$S^{j\to j'}(x,y) = J^{j'}(x,y) - [U^j(x) + F^{j\to j'}]$$

We can rewrite the Bellman equation of the unemployed worker as:

$$U^j(x) = b(x,j) + \beta\left[U^j(x)\right] + \beta\left(\sum_{j'\in\mathcal{J}} \phi_u^j(x,j')\left[p^{j'}\mu\int \max\left\{0, S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right] - c\left(\phi_u^j(x)\right)\right)$$

We can derive the optimal strategy for each worker as:

$$\phi_u^j(x,j') = \frac{\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]/c\right)}{\sum_{\tilde{j}\in\mathcal{J}}\exp\left(\left[p^{\tilde{j}}\mu\int\max\left\{0, S^{j\to\tilde{j}}(x,y)\right\}\frac{v^{\tilde{j}}(y)}{V^{\tilde{j}}}dy\right]/c\right)}$$

Plug back into the value function to get:

$$U^j(x) = b(x,j) + \beta\left[\left[U^j(x)\right] + c\log\left(\sum_{\tilde{j}\in\mathcal{J}}\exp\left(\left[p^{j'}\mu\int\max\left\{0, S^{j\to j'}(x,y)\right\}\frac{v^{j'}(y)}{V^{j'}}dy\right]/c\right) + c\log(\mid\mathcal{J}\mid)\right)\right]$$

### 2.4.2 Value of a match

Note tha if the match survives the employed worker chooses their optimal search strategy $\phi_s^j(x)$ search in each of the locations $j'$ with the specified probability $\phi_s^j(x,j')$.

$$\begin{aligned}\max_{\phi_s^j(x)}\Bigg\{\sum_{j'\in\mathcal{J}}\phi_s^j(x,j')\Big[(1-sp^{j'})J^j(x,y)\\ +sp^j\int\max\{J^j(x,y), J^j(x,y)+\mu\left(J^{j'}(x,y')-J^j(x,y)-F^{j'\to j}\right)\}\frac{v^{j'}(x)}{V^{j'}}dy'\Big]\Bigg\}\end{aligned}$$

Wich can be rewritten as:

$$J^j(x,y) + \max_{\phi_s^j(x)}\left\{\sum_{j'\in\mathcal{J}}\phi_s^j(x,j')\left[sp^j\int\max\{0,\mu\left(J^{j'}(x,y')-J^j(x,y)-F^{j'\to j}\right)\}\frac{v^{j'}(x)}{V^{j'}}dy'\right]\right\}$$

Since

$$S^{j\to j'}(x,y') = J^{j'}(x,y') - \left[U^j(x) + F^{j\to j'}\right]$$

then

$$S^{j\to j'}(x,y') - S^{j\to j}(x,y) = J^{j'}(x,y') - J^j(x,y) - F^{j'\to j}$$

Then we can rewrite the previous equation as:

$$J^j(x,y) + \max_{\phi_s^j(x)} \left\{ \sum_{j' \in \mathcal{J}} \phi_s^j(x,j') \left[ sp^j \int \max\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\} \frac{v^{j'}(x)}{V^{j'}} dy' \right] \right\}$$

We can derive the optimal strategy for each worker as:

$$\phi_u^j(x,y,j') = \frac{\exp\left(\left[p^{j'}\mu \int \max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\} \frac{v^{j'}(y')}{V^{j'}} dy'\right]/c\right)}{\sum_{\tilde{j}\in\mathcal{J}} \exp\left(\left[p^{\tilde{j}}\mu \int \max\left\{0, S^{j\to \tilde{j}}(x,y') - S^{j\to j}(x,y)\right\} \frac{v^{\tilde{j}}(y')}{V^{\tilde{j}}} dy'\right]/c\right)}$$

Plug back into the value function:

$$\begin{aligned} J^j(x,y) = f(x,y,j) + \beta\, [&[(1-\lambda^j(x,y))U^j(x) + \lambda^j(x,y) J^j(x,y)] \\ &+ \log\left(\sum_{j'\in\mathcal{J}} \exp\left(\left[p^{j'}\mu \int \max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\} \frac{v^{j'}(y')}{V^{j'}} dy'\right]/c\right)\right) \\ &- c\log(\mid \mathcal{J} \mid)] \end{aligned}$$

### 2.4.3 Surplus Value

Let $s(x,y,j\to j') = f(x,y,j') - b(x,j)$ I satart by writing the expression for the surpluss of staying in the same location:

$$\begin{aligned} S^{j\to j}(x,y) &= J^j(x,y) - U^j(x) \\ &= s(x,y,j\to j) + \beta\lambda^j(x,y) \underbrace{[J^j(x,y) - U^j(x)]}_{S^{j\to j}(x,y)} + \beta c \left[\Lambda_1^j(x,y) - \Lambda_0^j(x)\right] \\ &= s(x,y,j\to j) + \beta \max\{0, S^{j\to j}(x,y)\} + \beta c \left[\Lambda_1^j(x,y) - \Lambda_0^j(x)\right] \end{aligned}$$

The terms $\Lambda_0(x)$ and $\Lambda_1(x,y)$ come from the Bellmans of the worker and the match respectively and are defined as:

$$\Lambda_1(x,y) = \log\left(\sum_{j'\in\mathcal{J}} \exp\left(\left[p^{j'}\mu \int \max\left\{0, S^{j\to j'}(x,y') - S^{j\to j}(x,y)\right\} \frac{v^{j'}(y')}{V^{j'}} dy'\right]/c\right)\right)$$

$$\Lambda_0(x) = \log\left(\sum_{j'\in\mathcal{J}} \exp\left(\left[p^{j'}\mu \int \max\left\{0, S^{j\to j'}(x,y')\right\} \frac{v^{j'}(y')}{V^{j'}} dy'\right]/c\right)\right)$$

Next note that the surplus of moving to a different location is:

$$\begin{aligned} S^{j\to j'}(x,y) &= J^{j'}(x,y) - \left[U^j(x) + F^{j\to j'}\right] \\ &= [J^{j'}(x,y) - U^{j'}(x)] + U^{j'}(x) - \left[U^j(x) + F^{j\to j'}\right] \\ &= S^{j'\to j'}(x,y) - [U^j(x) - U^{j'}(x) + F^{j\to j'}] \end{aligned}$$

# 3 Aglomeration Effects

This section focus on how the distribution of workers (skils) affect both the productivity of matches $f(x,y,j)$ and the instant utility of unemployed workers $b(x,j)$.

**Location Productivity**

I borrow from (Davis and Dingel 2019) in assuming that the productivity of workers in a location $j$ is a result of idea exchange process within each location.

- Assume that a type $x$ worker in location $j$ has a $\Omega(x,\overline{X}^j)$ where $\overline{X}^j$ is the value of idea exchange in location $j$. Thus:

$$f(x,y,j) = f\left(\Omega(x,\overline{X}^j), y\right)$$

- Assume that the value of idea exchange in location $j$ is a function of the distribution of skills in location $j$:

$$\overline{X}^j = \overline{X}\left(\{\ell^j(x)\}\right)$$

It's natural to make the following assumptions on $\Omega(x,\overline{X}^j)$ and $\overline{X}^j$:

- **Assumption 1**: $\Omega(x,\overline{X}^j)$ should be increasing in $x$ and $\overline{X}^j$ and in the absence of idea exchange, worker productivity is just their type: $\Omega(x,0) = x$.

- **Assumption 2**: Worker type $x$ and idea exchange environment are complements i.e. $\Omega(x, \overline{X}^j)$ supermodular in $(x \ , \overline{X}^j)$.
- **Assumption 3**: The value of idea exchange in location $j$ is increasing in the number of potential exchange partners $\mu^j$ and on the distribution of skills in location $j$ $\ell^j(x)$.
  - With the *FOSD* order of distributions.

I will focus on the following functional forms for $\Omega(x, \overline{X}^j)$ and $\overline{X}^j$:

$$\Omega(x, \overline{X}^j) = x(1 + A\overline{X}^j x)$$

where $A$ is a parameter that captures the scope of gains from idea exchange.

$$\overline{X}\left(\{\ell^j(x)\}\right) = \left(1 - e^{-\nu \int \ell^j(x)dx}\right)\hat{x}^j = \left(1 - e^{-\nu\mu^j}\right)\hat{x}^j$$

The probability of encountering someone during each moment of time seeking is given by $1 - \exp(-\nu\mu^j)$, note that as the number of potential exchange partners $(\mu^j)$ increases, the probability of encountering someone also increases, which makes intuitive sense.

The average ability of the individuals you encounter in these exchanges is denoted as $\hat{x}^j$. This is a weighted average of the abilities of the people you meet

$$\hat{x}^j = \frac{1}{\mu^j}\int x\ell^j(x)dx = \frac{\mathbb{E}^j[x]}{\mu^j}$$

where $\mathbb{E}^j[x]$ is the average ability of workers in location $j$.

$$f(x, y, j, z_t) = \Omega(x, \overline{X}^j)^\alpha y^{1-\alpha} \tag{1}$$

**Cost of living**

- As in (Davis and Dingel 2019) and (Behrens, Duranton, and Robert-Nicoud 2014) every individual in location $j$ pays the congestion cost :

$$c_j = c(\zeta_j) = \theta\zeta_j^\gamma$$

## 4 Computation of Equilibrium

Right now I'm brute forcing the computation of the equilibrium, I'm using the following algorithm:

Take $\ell(x)$ as a primitive. Make initial guesses for $\ell^j(x)$. Initialize $S^{j\to j'}(x,y) \equiv 0$ for all $j$, $j'$, $x$ and $y$.

1. Using $S^{j\to j'}(x,y)$ update the distribution of employed and unemployed workers in each location $j$.
2. Compute the value $B^j(y)$ of vacancy creation.
3. Compute:
   - $v^j(y)$ the number of vacancies posted by a firm $y$ in location $j$.
   - $V^j = \int v^j(y)dy$ the total number of job opportunities posted location $j$.
   - $L^j$ the total search intensity in location $j$.
   - $\theta^j = V^j/L^j$ the market tightness in location $j$.
4. Compute $S^{j\to j'}(x,y)$, and $U^j$.
   - Obtain wokers optimal strategies $\phi_u^j(x)$ and $\phi_s^j(x)$.
5. Update the distribution of workers in each location $j$.
   - If the distribution of workers in each location $j$ has converged stop, otherwise update worker productivity in each location and cost of living and go back to Step 1.

## References


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