

Sorting between Workers and Firms (and Locations)

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This document introduces a spatial sorting model that explores interactions between heterogeneous workers and firms in a frictional labor market. Workers’ “partially” direct their search to specific locations using a mixed strategy. This strategy, derived as an optimal solution under cognitive constraints (known as “rational inattention”), assigns probabilities to locations based on expected surpluses. Agglomeration effects, influenced by skill distribution, play a vital role in this framework. The document outlines the model’s components without delving into equilibrium or solution methods.

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1 Introduction

In this document, I present a spatial sorting model that elucidates the dynamics of interactions between heterogeneous workers and firms. Within this framework, workers exhibit varying skills, while firms exhibit varying levels of productivity. These diverse actors engage in a frictional labor market, where the productivity of workers is contingent upon the skill distribution within their respective locations, a phenomenon commonly referred to as the agglomeration effect.

A fundamental aspect of my research revolves around the exploration of workers' search behavior. Diverging from conventional models, workers can “partially” direct their search to specific locations by employing a mixed strategy. This strategy emerges as the optimal as the solution of the maximization problem of economic agents who weigh the costs associated with acquiring and processing information against their cognitive constraints (this assumption is commonly referred to as *rational inattention* ([Christopher A. Sims 2003](#)) and ([Christopher A. Sims 2006](#))). The optimal strategy assigns a positive probability to all available locations, with the probability increasing in proportion to the worker's anticipated surplus in each locale. The outcomes generated by this approach are consistent with those obtained using a discrete choice framework, as demonstrated in ([Lentz and Moen, n.d.](#)).

Furthermore, I model agglomeration effects as a function of the skill distribution within each location. Borrowing insights from ([Davis and Dingel 2019](#)), I assume that workers derive benefits from interacting with one another, with a greater emphasis on the advantages gained from interacting with skilled workers.

In this document I lay out the ingredients of the model. I do not present the equilibrium, nor do I discuss the solution method.

2 Model

2.1 Model Setup

2.1.1 Demographics

- There exists a discrete and finite set of locations denoted as $\mathcal{J} = \{1, \dots, |\mathcal{J}|\}$, with each location indexed by $j \in \mathcal{J}$.
- A continuum of workers is characterized by their abilities, which are denoted as $x \in \mathcal{X}$.
 - The total measure of workers is normalized to 1.
 - Workers follow an exogenous distribution $\ell(x)$.
 - Within each location, there is an endogenous distribution of workers denoted as $\ell^j(x)$.
 - I represent the total population in location j as μ_j .
- A continuum of firms is characterized by their technology, which is denoted as $y \in \mathcal{Y}$.
 - The total measure of firms is normalized to 1.
 - Firms follow an exogenous distribution Φ .

2.1.2 Technology

- There is an exogenous cost associated with posting v job opportunities in location j , denoted as $c_j(v)$, where $c_j(v) \geq 0$.
 - I assume this cost function is increasing, convex, and potentially dependent on the location.
- Both workers and firms discount the future at the rate β .
- Workers have the ability to move across locations:
 - Workers select a mixed strategy for their job search, denoted as $\phi_i^j(x)$, where $\phi_i^j(x) = \{\phi_i^j(x, j')\}_{j' \in \mathcal{J}}$ represents the probability that a worker of type x from location j searches in location j' , and $i \in \{u, s\}$ refers to the employment status of the worker.
 - Each search strategy incurs an associated cost denoted as $c_s(\phi_i^j(x))$.
 - When a worker moves from one location to another (from j to j'), they must pay a relocation cost denoted as $F^{j \rightarrow j'} \geq 0$, with $F^{j \rightarrow j} = 0$ for movements within the same location.
- Unemployed workers receive instant utility in each location denoted as $b(x, j)$.¹
- Firms have access to a production technology that operates at the match level and depends on the location and the aggregate state of the economy, represented as $f(x, y, j)$.

¹In (Lise and Robin 2017), $b(x)$ represents unemployment benefits. However, I adopt a more general approach to account for differences in the cost of living across locations.

- Worker productivity is influenced by the distribution of skills within each location (see Section 3.1 for more details).

2.1.3 Job Search

Both unemployed and employed workers engage in job search activities. Let's denote s as the search intensity of an employed worker, $s < 1$ is normalized such that an unemployed worker has a search intensity of 1. To compute the total search intensity in location j , we need to consider the following expression:

$$L^j = \sum_{j' \in \mathcal{J}} \left[\int \phi_u^{j'}(x, j) u^{j'}(x) dx + s \int \int \phi_s^{j'}(x, j) h^{j'}(x, y) dx dy \right]$$

Total search intensity (L^j) in location j accounts for the search behaviors of both employed and unemployed workers from all other locations (j') in the model. It takes into account the probabilities of workers from everywhere else searching for jobs in location j , capturing the spatial dynamics of the labor market.

Now, let's introduce some additional definitions:

- Let $v^j(y)$ represent the number of job opportunities posted by a firm y in location j .
- $V^j = \int v^j(y) dy$ represents the total number of job opportunities posted location j .

Now, let's define $M^j = M(L^j, V^j)$ as the number of job matches in location j . With these definitions in place, we can calculate the following probabilities:

- The probability that an unemployed worker contacts a vacancy in location j is given by:

$$p^j = \frac{M^j}{L^j}$$

Here, sp^j represents the probability that an employed worker contacts a vacancy.

- The probability that a firm contacts any searching worker is defined as:

$$q^j = \frac{M^j}{V^j}$$

- Additionally, let $\theta^j = V^j/L^j$, represent the market tightness in location j .

Finally, a job offer is a draw of a firm productivity from the vacancy distribution in each location $\Gamma^j(\cdot)$ with pdf $\gamma^j(\cdot)$, note that

$$\gamma^j(y) = p^j \frac{v^j(y)}{V^j}$$

2.2 Dynamic Programming Problem

Consider the following notation:

- $U^j(x)$ represents the value for an unemployed worker of type x in location j .
- The value of receiving a job offer depends on the employment status of the worker:
 - $W_0^{j \rightarrow j'}(x, y)$ denotes the value of a type- x unemployed worker in location j who is hired by a firm of type y in location j' .
 - $W_1^{j \rightarrow j'}(x, y \rightarrow y')$ represents the value offered by a type y' firm in location j' to a type x worker currently employed at a type y firm in location j .
- $J^j(x, y)$ denotes the value of a match between a type x worker and a type y firm in location j .

Now, let's define the joint surplus of a match between a type x unemployed worker in location j and a type y firm in location j' as:

$$S^{j \rightarrow j'}(x, y) = J^{j'}(x, y) - [U^j(x) + F^{j \rightarrow j'}]$$

$S^{j \rightarrow j'}(x, y)$ represent much better off (or worse off) the worker and the firm are when they are matched compared to when the worker is unemployed in location j and considering the cost of moving from j to j' . If $S^{j \rightarrow j'}(x, y)$ is positive, it indicates a positive gain from the match; if it's negative, it represents a net loss.

2.2.1 Search Strategy

In this section I describe the search strategy of workers. I specify what do I mean by a search strategy and how do workers choose their search strategy. I also describe the cost of search and how it is related to the search strategy. Finally I derive the optimal search strategy for each worker type and location. Throughout this section I abstract from the employment status of the worker.

I assume that workers are *rational inattentive* in the style of (Christopher A. Sims 2003), (Christopher A. Sims 2006) and (Matějka and McKay 2015). Workers have prior knowledge of the value of moving to every location j' $\{u^{j \rightarrow j'}(x)\}_{j' \in \mathcal{J}}$, note that this is dependent on the type of each worker and their location at the moment of decision making. Assume that this prior knowledge is described by a joint distribution $G(\{u^{j \rightarrow j'}(x)\}_{j' \in \mathcal{J}})$.

To refine this knowledge, workers can acquire information about the value of each location by searching. Workers are in essence acquiring information to reduce uncertainty (i.e. reduce entropy) associated with their prior knowledge. I make the simplifying assumption that $\phi_0^j(x)$, the unconditional distribution of choosing each location is uniform, this is:

$$\phi_0^j(x, j') = \frac{1}{|\mathcal{J}|} \quad \forall j' \in \mathcal{J}, x \in \mathcal{X}$$

Thus type x workers in location j are faced with the problem:

$$\begin{aligned}
& \max_{\phi_i^j(x)} \left(\sum_{j' \in \mathcal{J}} \phi_i^j(x, j') u^{j \rightarrow j'}(x) - c(\phi_i^j(x), \phi_0^j(x)) \right) \\
& \text{s.t.} \quad \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') = 1 \\
& \text{and} \quad \phi_i^j(x, j') \geq 0 \quad \forall j' \in \mathcal{J}
\end{aligned}$$

With $c(\phi_i^j(x), \phi_0^j(x))$ being the cost of reducing the entropy of the prior (in this case the assumption is the prior is no information). The cost is proportional to the Kullback–Leibler divergence (also called relative entropy and I -divergence) between the selected distribution and the prior (uniform). More information on the Kullback–Leibler divergence [here](#). For simplicity, denote $c(\phi_i^j(x), \phi_0^j(x)) = c(\phi_i^j(x))$

$$c_s(\phi_i^j(x)) = c \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') \log(J\phi_j^i(x, j')) \quad (2.1)$$

If processing information where costless i.e. $c = 0$ the worker would be able to perfectly direct their search towards the highest value location. With costly information processing ($c > 0$), the worker can still select to search randomly (i.e. uniform distribution) and pay the associated cost of 0, when the worker starts directing their search towards a particular location (the assumption is that this requires, acquiring more information of one of the potential values relative to the others) then the cost grows unboundedly large as the strategy gets closer to perfectly directing search to a particular location.

- Besides ([Matějka and McKay 2015](#)) which formulates the problem in general terms, some works in the literature that use this cost structure are ([Wu 2020](#)) and ([Cheremukhin, Restrepo-Echavarria, and Tutino 2020](#)).

Solving the maximization problem we get the optimal strategy for each worker as:

$$\begin{aligned}
& \max_{\phi_i^j(x)} \left(\sum_{j' \in \mathcal{J}} \phi_i^j(x, j') u^{j \rightarrow j'}(x) - c \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') \log(J\phi_j^i(x, j')) \right) \\
& \text{s.t.} \quad \sum_{j' \in \mathcal{J}} \phi_i^j(x, j') = 1 \\
& \text{and} \quad \phi_i^j(x, j') \geq 0 \quad \forall j' \in \mathcal{J}
\end{aligned}$$

I'll ignore the non-negativity constraints and write the Lagrangean of the problem:

$$\mathcal{L}(\phi^j(x), \lambda) = \sum_{j' \in \mathcal{J}} \phi^j(x, j') u^{j \rightarrow j'}(x) - c \sum_{j' \in \mathcal{J}} \phi^j(x, j') \log(J\phi^j(x, j')) + \lambda \left(\sum_{j' \in \mathcal{J}} \phi^j(x, j') - 1 \right)$$

First order conditions of the problem give us:

$$\begin{aligned}
[\phi^j(x, j')] : \quad & u^{j \rightarrow j'}(x) - c - c \log[J\phi^j(x, j')] = \lambda \\
[\lambda] : \quad & \sum_{j' \in \mathcal{J}} \phi^j(x, j') = 1
\end{aligned}$$

Take any two j_1, j_2 we have that

$$u^{j \rightarrow j_1}(x) - c \log[J\phi^j(x, j_1)] = u^{j \rightarrow j_2}(x) - c \log[J\phi^j(x, j_2)]$$

thus

$$\begin{aligned}
\frac{u^{j \rightarrow j_1}(x) - u^{j \rightarrow j_2}(x)}{c} &= \log \left(\frac{\phi^j(x, j_1)}{\phi^j(x, j_2)} \right) \\
&\Rightarrow \frac{\phi^j(x, j_1)}{\phi^j(x, j_2)} = \frac{e^{u^{j \rightarrow j_1}(x)/c}}{e^{u^{j \rightarrow j_2}(x)/c}}
\end{aligned}$$

Fix any \hat{j} , then we can write any other $j' \in \mathcal{J}$ in terms of \hat{j} and plug into the constraint to get:

$$\begin{aligned}
\sum_{j' \in \mathcal{J}} \phi^j(x, j') &= \sum_{j' \in \mathcal{J}} \frac{e^{u^{j \rightarrow j'}(x)}}{e^{u^{j \rightarrow \hat{j}}(x)}} \phi^j(x, \hat{j}) \\
&= \frac{\phi^j(x, \hat{j})}{e^{u^{j \rightarrow \hat{j}}(x)/c}} \sum_{j' \in \mathcal{J}} e^{u^{j \rightarrow j'}(x)/c} = 1
\end{aligned}$$

Which we can solve to obtain

$$\phi^j(x, \hat{j}) = \frac{e^{u^{j \rightarrow \hat{j}}(x)/c}}{\sum_{j' \in \mathcal{J}} e^{u^{j \rightarrow j'}(x)/c}} \quad (2.2)$$

The non-negativity constraints are satisfied because the exponential function is always positive.

Note: I could have follow (Matějka and McKay 2015) and have more complex prior beliefs, the result of this would be that the optimal strategy would be biased by the prior attractiveness of each location. In terms of closed form solution I would get something like:

$$\phi^j(x, \hat{j}) = \frac{e^{(u^{j \rightarrow \hat{j}}(x) + \zeta^{j \rightarrow \hat{j}}(x))/c}}{\sum_{j' \in \mathcal{J}} e^{(u^{j \rightarrow j'}(x) + \zeta^{j \rightarrow j'}(x))/c}}$$

where $\zeta^{j \rightarrow j'}(x) = c \log(\phi_0^j(x))$ is the prior attractiveness of moving from location j to location \hat{j} . Note that this have the implication: - When an option seems very attractive a priori, then it has a relatively high probability of being selected even if its true value is low. (verbatim quote from (Matějka and McKay 2015)).

I choose to have a uniform prior to simplify the exposition and because I think it is a reasonable assumption. But I also can see how relaxing this assumption could have interesting implications for the model.

We already established that with costless information the limit behavior is that the worker perfectly directs their search to the highest value location. Next, note that $\lim_{c \rightarrow \infty} \exp(u^{j \rightarrow \hat{j}}(x)/c) = 1$ thus the limit behavior as the cost of information rises is disregard new information decide based on the prior

- With uniform prior: $\lim_{c \rightarrow \infty} \phi^j(x, \hat{j}) = \frac{1}{|\mathcal{J}|}$ which is random search.
- With some other prior: $\lim_{c \rightarrow \infty} \phi^j(x, \hat{j}) = \phi_0^j(x)$.

Here the scaling parameter c plays a similar role to the noise parameter in the discrete choice model from (Lentz and Moen, n.d.).

2.2.2 Unemployed Worker

Unemployed workers receive instant utility from living in location j , $b(x, j)$, and anticipate the probability of getting an offer p^j in each location. They will choose the strategy that maximizes their future expected value knowing that in each location they will receive an offer which can be from any firm with a likelihood proportional to the share of total vacancies posted by each firm in each market. The worker will accept only the offers that promise her a higher value than unemployment:

$$U^j(x) = \underbrace{b(x, j)}_{\text{instant utility}} + \beta \max_{\phi_u^j(x)} \left\{ \underbrace{-c(\phi_u^j(x))}_{\text{cost of search strategy}} + \sum_{j' \in \mathcal{J}} \underbrace{\phi_u^j(x, j')}_{\text{weight by probability of search in } j'} \left[\begin{array}{l} \text{no offer, stays unemployed now in } j' \\ \frac{(1 - p^{j'}) U^j(x)}{\quad} \end{array} \right. \right. \\ \left. \left. + p^{j'} \underbrace{\int \max \{U^j(x), W_0^{j \rightarrow j'}(x, y)\} \frac{v^{j'}(y)}{V^{j'}} dy}_{\text{if offer, pays cost, moves to } j \text{ and then is matched randomly with some firm}} \right] \right\}$$

I use the same bargaining as in (Cahuc, Postel-Vinay, and Robin 2006) and (Bagger and Lentz 2019). The Nash bargaining solution implies that the worker receives a constant share ξ of the match rent, where ξ is the bargaining power of the worker:

$$W_0^{j \rightarrow j'}(x, y) = U^j(x) + \xi S^{j \rightarrow j'}(x, y) = (1 - \xi) U^j(x) + \xi [J^{j'}(x, y) - F^{j \rightarrow j'}]$$

From (2.2) we know that the optimal strategy for each unemployed worker is:

$$\phi_u^j(x, j') = \frac{\exp \left(\left[\xi \int \max \{0, S^{j \rightarrow j'}(x, y)\} d\Gamma^{j'}(y) \right] / c \right)}{\sum_{\tilde{j} \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow \tilde{j}}(x, y)\} d\Gamma^{\tilde{j}}(y) \right] / c \right)} \quad (2.3)$$

Substituting the optimal strategy in the Bellman equation we get:

$$U^j(x) = b(x, j) + \beta \left[U^j(x) + c \log \left(\frac{1}{|\mathcal{J}|} \sum_{\tilde{j} \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow \tilde{j}}(x, y)\} d\Gamma^{\tilde{j}}(y) \right] / c \right) \right) \right] \quad (2.4)$$

2.2.3 Joint Value of a Match

- If a match between a worker and a firm in location j is destroyed the firm will get 0 and the worker gets their unemployment value in that location $U^j(x)$.
- Matches are destroyed for two reasons:
 - *Exogenous destruction* with probability δ
 - *Endogenous destruction*, if and only if $J^j(x, y) < U^j(x)$.
 - Denote $\lambda^j(x, y) = (1 - \delta) \mathbb{1}_{\{J^j(x, y) > U^j(x, y)\}}$ the probability that a match survives accounting for both exogenous and endogenous destruction.

We can write the Bellman equation of a match value as:

$$J^j(x, y) = \underbrace{f(x, y, j)}_{\text{match value added}} + \beta \left[\underbrace{\frac{1}{(1 - \lambda^j(x, y))}}_{\text{match is destroyed}} \underbrace{U^j(x)}_{\text{worker gets unemployment value}} + \underbrace{\left(\lambda^j(x, y) \max_{\phi_s^j(x, y)} \left\{ -c(\phi_s^j(x, y)) + \sum_{j' \in \mathcal{J}} \phi_s^j(x, y, j') \left[\underbrace{\frac{1}{(1 - sp^{j'})}}_{\text{no new offers}} \underbrace{J^j(x, y)}_{\text{stays with same firm}} + \right. \right. \right. \right. \right. \\ \left. \left. \left. \left. \underbrace{sp^{j'} \int \max \{J^j(x, y), W_1^{j \rightarrow j'}(x, y', y)\} \frac{v^{j'}(x)}{V^{j'}} dy'}_{\text{worker only accepts new offers if value is greater than current match}} \right\} \right) \right]}_{\text{match survives}} \right]$$

When a type x worker employed at a type y firm in location j meet potential type y' poaching firm at location j' (assume that $y' > y$ is productive enough to poach the worker, I'll show later what the poaching condition is), competition between the two employers over the worker's services occurs as a second price auction. No employer will offer more than the entire match value $J^j(x, y)$, thus this becomes the reservation value for the worker. Nash bargaining solution implies that the worker will obtain their outside option plus a share ξ of the match surplus.

Bargaining implies that a worker employed in location j by firm y when matched with firm y' in location j' gets:

$$W_1^{j \rightarrow j'}(x, y \rightarrow y') = J^j(x, y) + \xi [J^{j'}(x, y') - J^j(x, y) - F^{j' \rightarrow j}]$$

Note that this means that to be able to poach from different locations the firm must be at least $F^{j \rightarrow j'}$ more productive. In terms of joint surplus we can rewrite

$$W_1^{j \rightarrow j'}(x, y \rightarrow y') = J^{j'}(x, y') + \xi [S^{j \rightarrow j'}(x, y') - S^{j \rightarrow j}(x, y)]$$

Focussing on the maximization problem in the Bellman equation we can write:

$$J^j(x, y) + \max_{\phi_s^j(x, y)} \left\{ \sum_{j' \in \mathcal{J}} \phi_s^j(x, y, j') \left[sp^j \int \max\{0, \xi [S^{j \rightarrow j'}(x, y') - S^{j \rightarrow j}(x, y)]\} d\Gamma^{j'}(y') \right] \right\}$$

From (2.2) we know that the optimal strategy for each employed worker is:

$$\phi_s^j(x, y, j') = \frac{\exp \left([p^{j'} \xi \int \max\{0, S^{j \rightarrow j'}(x, y') - S^{j \rightarrow j}(x, y)] d\Gamma^{j'}(y') \right] / c}{\sum_{\tilde{j} \in \mathcal{J}} \exp \left([p^{\tilde{j}} \xi \int \max\{0, S^{j \rightarrow \tilde{j}}(x, y') - S^{j \rightarrow j}(x, y)] d\Gamma^{\tilde{j}}(y') \right] / c}$$

Plug back into the value function:

$$J^j(x, y) = f(x, y, j) + \beta \left[(1 - \lambda^j(x, y)) U^j(x) + \lambda^j(x, y) J^j(x, y) \right] + \log \left(\frac{1}{|\mathcal{J}|} \sum_{j' \in \mathcal{J}} \exp \left([p^{j'} \xi \int \max\{0, S^{j \rightarrow j'}(x, y') - S^{j \rightarrow j}(x, y)] d\Gamma^{j'}(y') \right] / c \right) \right]$$

2.2.4 Match Surplus

Next I characterize the surplus function. First, define the instant surplus of a match between a type x worker in location j and a type y firm in location j' as:

$$s(x, y, j \rightarrow j') = f(x, y, j, z) - b(x, j')$$

We can compute the surplus of matches in the same location as:

$$\begin{aligned} S^{j \rightarrow j}(x, y) &= J^j(x, y) - U^j(x) \\ &= s(x, y, j \rightarrow j) + \beta \lambda^j(x, y) \underbrace{[J^j(x, y) - U^j(x)]}_{S^{j \rightarrow j}(x, y)} + \beta c [\Lambda_1^j(x, y) - \Lambda_0^j(x)] \\ &= s(x, y, j \rightarrow j) + \beta(1 - \delta) \max\{0, S^{j \rightarrow j}(x, y)\} + \beta c [\Lambda_1^j(x, y) - \Lambda_0^j(x)] \end{aligned} \quad (2.5)$$

Notice that $\lambda^j(x, y) > 0$ if and only if $S^{j \rightarrow j}(x, y) > 0$, thus

$$(1 - \delta) \max\{0, S^{j \rightarrow j}(x, y)\} = \lambda^j(x, y) S^{j \rightarrow j}(x, y)$$

The terms $\Lambda_0(x)$ and $\Lambda_1(x, y)$ come from the Bellmans of the worker and the match respectively and are defined as:

$$\Lambda_1(x, y) = \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow j'}(x, y') - S^{j \rightarrow j}(x, y)\} \Gamma^{j'}(y') \right] / c \right) \right)$$

$$\Lambda_0(x) = \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow j'}(x, y')\} d\Gamma^{j'}(y') \right] / c \right) \right)$$

Finally we can use the expression in (2.5) to write the surplus of a match between a type x worker in location j and a type y firm in location j' as:

$$\begin{aligned} S^{j \rightarrow j'}(x, y) &= J^{j'}(x, y) - [U^j(x) + F^{j \rightarrow j'}] \\ &= [J^{j'}(x, y) - U^{j'}(x)] + U^{j'}(x) - [U^j(x) + F^{j \rightarrow j'}] \\ &= S^{j' \rightarrow j'}(x, y) - [U^j(x) - U^{j'}(x) + F^{j \rightarrow j'}] \end{aligned} \quad (2.6)$$

From the expressions in (2.5) and (2.6) the following is evident:

- A worker in location j can be hired from a firm in location j' by firm y if and only if the surplus of the match is positive:

$$S^{j \rightarrow j'}(x, y) \geq 0 \quad \Longleftrightarrow \quad J^{j'}(x, y) - U^{j \rightarrow j'}(x) \geq 0$$

- A worker employed at location j by firm y can be poached by a firm y' in location j' if and only if the surplus of the match is higher than the surplus of staying at the same firm in the same location:

$$\begin{aligned} S^{j \rightarrow j'}(x, y') > S^{j \rightarrow j}(x, y) &\Longleftrightarrow J^{j'}(x, y') - U^{j \rightarrow j'}(x) > J^j(x, y) - U^{j \rightarrow j}(x) \\ &\Longleftrightarrow J^{j'}(x, y') - [U^j(x) - F^{j \rightarrow j'}] > J^j(x, y) - U^j(x) \\ &\Longleftrightarrow J^{j'}(x, y') > J^j(x, y) + F^{j \rightarrow j'} \end{aligned}$$

2.2.5 Vacancy Creation

- $B^j(y)$ is the expected value of a type y vacancy making contact with a worker in location j . Vacancies meet unemployed and employed type- x workers at a rates

$$\frac{u_+^j(x)}{L^j} \quad \text{and} \quad s \frac{h_+^j(x, y)}{L^j}$$

The expected value of posting a vacancy is therefore, the surplus that the posting firm expects to add, potential matches with negative surplus are immediately destroyed therefore those add no surplus. In terms of the Bellman equation we can write:

$$\begin{aligned}
B^j(y) = & \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \underbrace{\phi_u^{j'}(x, j)}_{\text{likelihood of match}} \underbrace{\frac{u_+^{j'}(x)}{L^j}}_{\text{match survives}} \times \max\{0, S^{j' \rightarrow j}(x, y)\} dx \right)}_{\text{expected value added from hiring unemployed workers}} \\
& + \underbrace{\sum_{j' \in \mathcal{J}} \left(\int \left(\int \underbrace{s\phi_s^{j'}(x, y', j)}_{\text{likelihood of match}} \underbrace{\frac{h_+^{j'}(x, y)}{L^j}}_{\text{poaching is succesfull}} \times \max\{0, S^{j' \rightarrow j}(x, y) - S^{j' \rightarrow j'}(x, y')\} dx \right) dy' \right)}_{\text{expected value added from poaching other firms employees}} \quad (2.7)
\end{aligned}$$

Firms will post vacancies such that the marginal cost of the vacancies and the marginal expected benefit B^j are equal:

$$c'_j(v^j(y)) = q^j B^j(y) \quad (2.8)$$

Since the cost function is increasing and concave there is a unique vacancy posting level that clears the market.

2.3 Labor Market Flows

Now we characterize the flows of workers in-to and out-of unemployment at each location :

- I start by denoting the following indicator functions

$$\eta^{j' \rightarrow j}(x, y) = \mathbb{1}_{\{S^{j' \rightarrow j}(x, y) > 0\}} \quad \eta^{j' \rightarrow j}(x, y' \rightarrow y) = \mathbb{1}_{\{S^{j' \rightarrow j}(x, y) > S^{j' \rightarrow j'}(x, y')\}}$$

note that these characterize (if and only if) when a worker can hired from unemployment or to be poached from another firm.

- Interactign the indicators with workers strategies we can obtain the actual probability that a woker moves across locations and firms:

$$\hat{\phi}_u^{j \rightarrow j'}(x, y) = \phi_u^j(x, j') \eta^{j' \rightarrow j}(x, y) \quad \hat{\phi}_s^{j \rightarrow j'}(x, y \rightarrow y') = \phi_s^j(x, y, j') \eta^{j' \rightarrow j}(x, y \rightarrow y') \quad (2.9)$$

We can write the *interim* distributions in terms of theese indicators as:

$$u_+^j(x) = u^j(x) + \int (1 - (1 - \delta) \eta^{j \rightarrow j}(x, y)) h^j(x, y) dy \quad (2.10)$$

$$h_+^j(x, y) = (1 - \delta) \eta^{j \rightarrow j}(x, y) h^j(x, y) \quad (2.11)$$

I split the mass of employed workers into three components:

$$h^j(x, y) = h_U^j(x, y) + h_P^j(x, y) - h_L^j(x, y) \quad \implies \quad h^j(x, y) + h_L^j(x, y) = h_U^j(x, y) + h_P^j(x, y) \quad (2.12)$$

those workers that are hired from unemployment, those that are poached from other firms and those that are poached by other firms.

- The mass of workers hired from unemployment:

$$h_U^j(x, y) = \gamma^j(y) \sum_{j' \in \mathcal{J}} u_+^{j'}(x) \hat{\phi}_u^{j' \rightarrow j}(x, y) \quad (2.13)$$

- The mass of workers that are succesfully poached from other firms:

$$h_P^j(x, y) = s\gamma^j(y) \sum_{j' \in \mathcal{J}} \left[\int h_+^{j'}(x, y') \hat{\phi}_s^{j' \rightarrow j}(x, y' \rightarrow y) dy' \right] \quad (2.14)$$

- The mass of workers that the firm loses to other firms:

$$h_L^j(x, y) = sh_+^j(x, y) \sum_{j' \in \mathcal{J}} \left[\int \hat{\phi}_s^{j \rightarrow j'}(x, y \rightarrow y') d\Gamma^{j'}(y') \right] \quad (2.15)$$

Take the the LHS in (2.12) and substitute the expression for $h_+^j(x, y)$ from (2.11):

$$h(x, y) \left(1 + s(1 - \delta)\eta^{j \rightarrow j}(x, y) \sum_{j' \in \mathcal{J}} \left[\int \hat{\phi}_s^{j \rightarrow j'}(x, y \rightarrow y') d\Gamma^{j'}(y') \right] \right) \quad (2.16)$$

And the RHS in (2.12):

$$\gamma^j(y) \left(\sum_{j' \in \mathcal{J}} \left(u_+^{j'}(x) \hat{\phi}_u^{j' \rightarrow j}(x, y) + s \int h_+^{j'}(x, y') \hat{\phi}_s^{j' \rightarrow j}(x, y' \rightarrow y) dy' \right) \right) \quad (2.17)$$

This give us the following expression for the law of motion of the employment distribution:

$$h^j(x, y) = \frac{\gamma^j(y) \left(\sum_{j' \in \mathcal{J}} \left(u_+^{j'}(x) \hat{\phi}_u^{j' \rightarrow j}(x, y) + s \int h_+^{j'}(x, y') \hat{\phi}_s^{j' \rightarrow j}(x, y' \rightarrow y) dy' \right) \right)}{1 + s(1 - \delta)\eta^{j \rightarrow j}(x, y) \sum_{j' \in \mathcal{J}} \left[\int \hat{\phi}_s^{j \rightarrow j'}(x, y \rightarrow y') d\Gamma^{j'}(y') \right]} \quad (2.18)$$

The distribution of unemployed workers is determined by the existing distribution of unemployed at “interim” an the probability that they are not hired by any firm in any location:

$$\begin{aligned}
u^j(x) &= u_+(x) \left(1 - \prod_{j \in \mathcal{J}} \int \hat{\phi}_u^{j \rightarrow j'}(x, y) d\Gamma(y') \right) \\
&= \left[u^j(x) + \int (1 - (1 - \delta)\eta^{j \rightarrow j}(x, y)) h^j(x, y) dy \right] \left(1 - \prod_{j \in \mathcal{J}} \int \hat{\phi}_u^{j \rightarrow j'}(x, y) d\Gamma(y') \right) \\
\Rightarrow u^j(x) &= \left[\int (1 - (1 - \delta)\eta^{j \rightarrow j}(x, y)) h^j(x, y) dy \right] \frac{\left(1 - \prod_{j \in \mathcal{J}} \int \hat{\phi}_u^{j \rightarrow j'}(x, y) d\Gamma(y') \right)}{\prod_{j \in \mathcal{J}} \int \hat{\phi}_u^{j \rightarrow j'}(x, y) d\Gamma(y')} \quad (2.19)
\end{aligned}$$

Note that (2.19) is a standard steady-state condition for unemployment, or Beveridge curve. Here the flow out of unemployment equals the flow into unemployment in every location at every skill level.

Then I can compute the distribution of skill in each location as:

$$\ell^j(x) = u^j(x) + \int h^j(x, y) dy$$

and the total population in each location as:

$$\mu^j = \int \ell^j(x) dx$$

3 Endogenizing Location Productivity and Cost of Living

This section focus on how the distribution of workers (skills) affect both the productivity of matches $f(x, y, j)$ and the instant utility of unemployed workers $b(x, j)$.

3.1 Location Productivity

I borrow from (Davis and Dingel 2019) in assuming that the productivity of workers in a location j is a result of idea exchange process within each location.

- Assume that a type x worker in location j has a $\Omega(x, \bar{X}^j)$ where \bar{X}^j is the value of idea exchange in location j . Thus:

$$f(x, y, j, z_t) = f\left(\Omega(x, \bar{X}^j), y, z_t\right)$$

- Assume that the value of idea exchange in location j is a function of the distribution of skills in location j :

$$\bar{X}^j = \bar{X}(\{\ell^j(x)\})$$

It's natural to make the following assumptions on $\Omega(x, \bar{X}^j)$ and \bar{X}^j :

- **Assumption 1:** $\Omega(x, \bar{X}^j)$ should be increasing in x and \bar{X}^j and in the absence of idea exchange, worker productivity is just their type: $\Omega(x, 0) = x$.
- **Assumption 2:** Worker type x and idea exchange environment are complements i.e. $\Omega(x, \bar{X}^j)$ supermodular in (x, \bar{X}^j) .
- **Assumption 3:** The value of idea exchange in location j is increasing in the number of potential exchange partners μ^j and on the distribution of skills in location j $\ell^j(x)$.
 - With the *FOSD* order of distributions.

I will focus on the following functional forms for $\Omega(x, \bar{X}^j)$ and \bar{X}^j :

$$\Omega(x, \bar{X}^j) = x(1 + A\bar{X}^j x)$$

where A is a parameter that captures the scope of gains from idea exchange.

$$\bar{X}(\{\ell^j(x)\}) = \left(1 - e^{-\nu \int \ell^j(x) dx}\right) \hat{x}^j = \left(1 - e^{-\nu \mu^j}\right) \hat{x}^j$$

The probability of encountering someone during each moment of time seeking is given by $1 - \exp(-\nu\mu^j)$, note that as the number of potential exchange partners (μ^j) increases, the probability of encountering someone also increases, which makes intuitive sense.

The average ability of the individuals you encounter in these exchanges is denoted as \hat{x}^j . This is a weighted average of the abilities of the people you meet

$$\hat{x}^j = \frac{1}{\mu^j} \int x \ell^j(x) dx = \frac{\mathbb{E}^j[x]}{\mu^j}$$

where $\mathbb{E}^j[x]$ is the average ability of workers in location j .

Cost of living

- As in (Davis and Dingel 2019) and (Behrens, Duranton, and Robert-Nicoud 2014) every individual in location j pays the congestion cost of $\theta\mu_j^\gamma$. Denote $b^j(x)$ the home production of type x workers in location j then the instant utility of unemployed workers in location j is given by:

$$b(x, y) = b^j(x) - \theta\mu_j^\gamma$$

4 Equilibrium

4.1 Equilibrium Definition

Concept:

- Steady state equilibrium.

Elements:

- Set of parameters $\Xi = \{\dots\}$.
- Distribution of unemployed workers and matches $\{u(x), h(x, y)\}_{j \in \mathcal{J}}$.
- Set of workers search strategies $\{\phi_u^j(x, \phi_s^j(x, y))\}_{j \in \mathcal{J}}$.
- A distribution of vacancies $\{\Gamma^j(y)\}_{j \in \mathcal{J}}$.
- Surplus function $S^{j \rightarrow j'}(x, y)$.
- Unemployment value function $U^j(x)$.
- Value of vacancy posting $B^j(y)$.

Such that:

Conditions:

1. Search strategies solve the maximization problem of employed and unemployed workers.
2. $S^{j \rightarrow j'}(x, y)$ and $U^j(x)$ solve (2.4), (2.5) and (2.6).
3. Value of vacancy posting $B^j(y)$ is determined by (2.7).
4. Distribution of unemployed workers and matches satisfy (2.19) and (2.18).

4.2 Existence of Value Functions and Distribution of Vacancies

In this section I will show that the value functions $S^{j \rightarrow j'}$ and $U^j(x)$ and the distribution of vacancies $\Gamma^j(y)$ exist and are unique for a given set of parameters Ξ and distribution of unemployed workers and matches $\{u(x), h(x, y)\}_{j \in \mathcal{J}}$.

4.2.1 Defininig the Space of Functions

Let

$$\mathcal{S} = \{S : \mathcal{J}^2 \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}\}$$

and

$$\mathcal{G} = \{\Gamma : \mathcal{J} \times \mathcal{Y} \rightarrow \mathbb{R}\}$$

Note a distributions of unemployed workers u and matched h there is a one to one mapping from \mathcal{S} to \mathcal{G} given by:

$$\Gamma^j(y | S) = \frac{M \left(L(u, h), \int_0^1 v^j(y' | S) d\Phi(y') \right)}{L(u, h)} \frac{\int_0^y v^j(y' | S) d\Gamma^j(y')}{\int_0^1 v^j(y' | S) d\Phi(y')}$$

where $v^j(y | S)$ is the total number of vacancies posted by firms of type y at location j given the surplus function S .

Define the operator $T : \mathcal{S} \rightarrow \mathcal{S}$ as:

$$TS = \begin{cases} s(x, y, j \rightarrow j) + \beta(1 - \delta) \max\{0, S^{j \rightarrow j}(x, y)\} + \beta c [\Lambda_1^j(x, y) - \Lambda_0^j(x)] & \text{if } j = j' \\ TS^{j' \rightarrow j'}(x, y) - [U^j(x) - U^{j'}(x) + F^{j \rightarrow j'}] & \text{if } j \neq j' \end{cases} \quad (4.1)$$

with

$$U^j(x) = \frac{1}{1 - \beta} [b(x, j) + \beta c \Lambda_0^j(x) - \beta c \log(|\mathcal{J}|)]$$

and

$$\begin{aligned} \Lambda_1^j(x, y) &= \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow j'}(x, y') - S^{j' \rightarrow j}(x, y)\} \Gamma^{j'}(y' | S) \right] / c \right) \right) \\ \Lambda_0^j(x) &= \log \left(\sum_{j' \in \mathcal{J}} \exp \left(\left[\xi \int \max \{0, S^{j \rightarrow j'}(x, y')\} d\Gamma^{j'}(y' | S) \right] / c \right) \right) \end{aligned}$$

Since the

Let $S_1, S_2 \in \mathcal{S}$ such that $S_1^{j \rightarrow j'}(x, y) > S_2^{j \rightarrow j'}(x, y)$ for all $j, j' \in \mathcal{J}$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then for $j = j'$ we have:

$$\begin{aligned} TS_1^{j \rightarrow j'}(x, y) - TS_2^{j \rightarrow j'}(x, y) &\leq \beta c [\Lambda_{1,1}^j(x, y) - \Lambda_{0,1}^j(x) - \Lambda_{1,2}^j(x, y) + \Lambda_{0,1}^j(x)] \\ &\leq \beta c \left[\frac{\xi}{c} S_1^{j \rightarrow j'}(x, y) - \Lambda_{1,2}^j(x, y) + \Lambda_{0,1}^j(x) \right] \end{aligned}$$

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