Strong monotonicity and perturbation-proofness of exchange economies

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Abstract

We study the price impact of small perturbations to Walrasian equilibrium in exchange economies, as might be caused by agents' misreports, changes in the supply vector, or changes in the set of participants. A sequence of markets is *perturbation-proof* if the price impact of any perturbation is inversely proportional to the number of agents. Perturbation-proofness implies good large-market incentive properties of Walrasian equilibrium and robustness of prices to small misspecifications. Replica economies are perturbation-proof if and only if the base economy's demand correspondence is *strongly monotone*. When buyers' types are drawn identically and independently from a distribution with a strongly monotone expected demand correspondence, the resulting sequence of economies is perturbation-proof with high probability.

Keywords: Approximate incentive-compatibility, General equilibrium, Market design, Perturbation analysis, Prices, Strong convexity, Strong monotonicity

JEL Codes: C610, D400, D440, D470, D500, D510.

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1 Introduction

Consider a sequence of exchange economies indexed by N, the number of agents in the economy. Suppose we perturb each market slightly by changing the report of an agent, adding or removing some supply, or adding or removing some agents. This paper answers the following question: when does the impact of this perturbation on Walrasian equilibrium prices diminish rapidly in market size, namely at a rate inversely proportional to N?

We call such a sequence of markets *perturbation-proof*. Our motivation for studying perturbation-proofness is its relationship to ex post incentives in Walrasian mechanisms: if a sequence of economies is perturbation-proof, then the benefit of unilateral misreporting in any Walrasian mechanism also diminishes rapidly in market size.

We establish conditions for fast convergence of ex post reporting incentives in general preference domains. This rapid convergence of reporting incentives (at a rate inversely proportional to N) may be desirable for modest-sized markets encountered in applications. Our approach differs from the existing literature on incentives in Walrasian mechanisms, which has focused mostly on incentive compatibility in the limit economy (as in Roberts and Postlewaite (1976) and Jackson (1992)) or on rates of convergence in restricted preference domains (such as the unit-demand double auction of Satterthwaite and Williams (1989) and the linear-quadratic models surveyed by Rostek and Yoon (2020)). Under additional assumptions,[1] we also improve on the conclusions of the "strategy-proofness in the large" result for Walrasian mechanisms due to Azevedo and Budish (2019), which establishes $O\left(1/N^{1/2-\varepsilon}\right)-$interim incentive compatibility for any $\varepsilon > 0$. A corollary of our main result on ex post incentives is a condition under which the Walrasian mechanism is $O(1/N^{1-\varepsilon})-$interim incentive-compatible.

A condition on demand called *strong monotonicity* results in perturbation-proofness. With one consumption good, strong monotonicity simply requires that the slope[2] of the demand curve is bounded away from zero. With multiple goods, strong monotonicity requires that for all prices p, p' such that the demand correspondence D is nonzero and for all $d \in D(p)$ and $d' \in D(p)$, we have that $(d - d') \cdot (p' - p) \geq m\|p' - p\|^2$ for some $m \geq 0$. Strong monotonicity of demand is equivalent to *strong convexity* of the indirect utility function: wherever the indirect utility u is nonzero, all

[1] In fact our assumptions are both stronger and weaker: stronger because we require strong monotonicity of the expected demand correspondence, but weaker because we do not require finiteness of the type space. Note however, that the Azevedo and Budish (2019) result applies to any envy-free mechanism, while we focus on Walrasian mechanisms.

[2] More accurately, all upper and lower derivatives.

second directional derivatives[3] of u are bounded below by m.

If a sequence of markets consists of agents with strongly monotone demand or replica economies of a strongly monotone base economy, that sequence is perturbation-proof. The intuition for this result is as follows: as the number of agents with strongly monotone demand grows, the market demand curve becomes increasingly steep and so movements in the supply curve[4] lead to increasingly small movements in the price coordinate of the intersection of supply and demand. For replica economies, we show that strong monotonicity is also a *necessary* condition for perturbation-proofness.

We then study markets with incomplete information over agents' preferences, in which we relax the assumptions on individual agents' demand. We focus primarily on ex post incentives, in which buyer types are drawn identically and independently according to a known distribution but are common knowledge to all participants in the mechanism. We show that strong monotonicity of expected demand implies that the sequence of markets is perturbation-proof with high probability and in expectation (over market draws). Interim reporting incentives in the Walrasian mechanism—in which each agent's draw is private information—follow as a corollary: strong monotonicity of expected demand implies that the Walrasian mechanism is interim $O\left(1/N^{1-\varepsilon}\right)$-incentive-compatible for any $\varepsilon > 0$.

We then apply our results to economic models with indivisibilities, in which strong monotonicity of expected demand is a condition only on the prices at which demand changes and not the size of these demand changes (each demand change is bounded below by the size of the indivisibility). We provide a simple characterization of strong monotonicity in this setting: expected demand is strongly monotone if the probability that demand changes between any two prices grows at least proportionally to the distance between the two prices. We interpret this as a condition on *variety* in the possible preferences of buyers and *uncertainty* about the reservation prices associated with demand changes (we formalize these notions below).

1.1 Examples

In this section, we contrast two sequences of markets—one in which a buyer has a large influence on the price independently of the market size, and one in which each buyer has a $O\left(1/N\right)$ influence

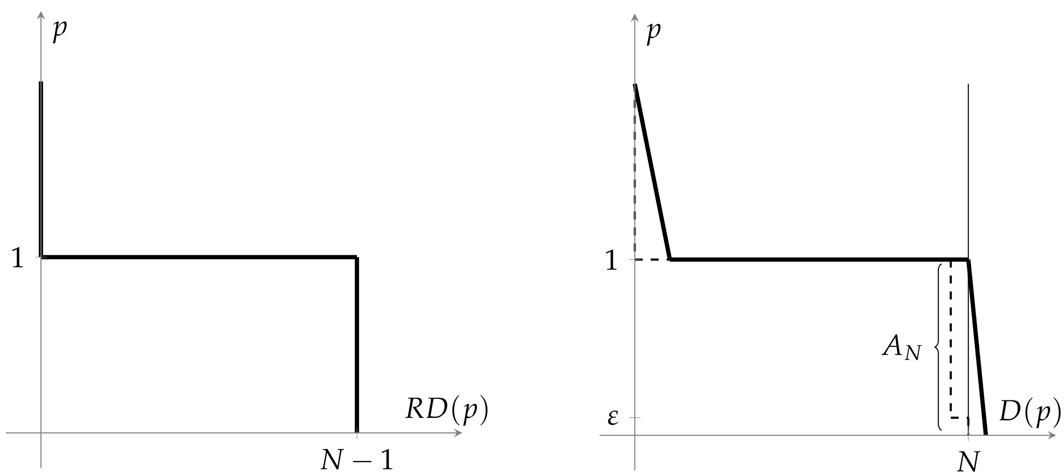
[3]Unless otherwise indicated, we use the term 'directional derivative' to mean the *one-sided* directional derivative as defined in Rockafellar and Wets (2009), which always exists for convex functions.

[4]We show all perturbations described in the first paragraph may be thought of in this way, and therefore define perturbations as such.

on the price in expectation—to illustrate the important role of the demand curve's slope on the price impact of small perturbations.

Example 1.1. Consider an economy with a single consumption good and N buyers. The first $N-1$ buyers have unit demand for the good with value 1, while the N^{th} buyer's demand as a function of price is $D_N(p) = \max\{2-p, 0\}$. The mechanism designer uses a Walrasian mechanism in this market.

Suppose the supply is N and all buyers report their preferences truthfully. In that case, buyer N receives one unit in equilibrium at a price of 1. However, if buyer N misreports and pretends to have unit demand with value ε, the Walrasian equilibria prices are in $[0, \varepsilon]$, so buyer N may effect a price arbitrarily close to zero, regardless of the auctioneer's decision rule in the case of multiple equilibria. The set of Walrasian equilibrium prices attainable to buyer N by some report is $A_N = [0, 1]$. Since the buyer is assigned a single unit of the good under any of these reports, the buyer must be made better off by any report that lowers the price.



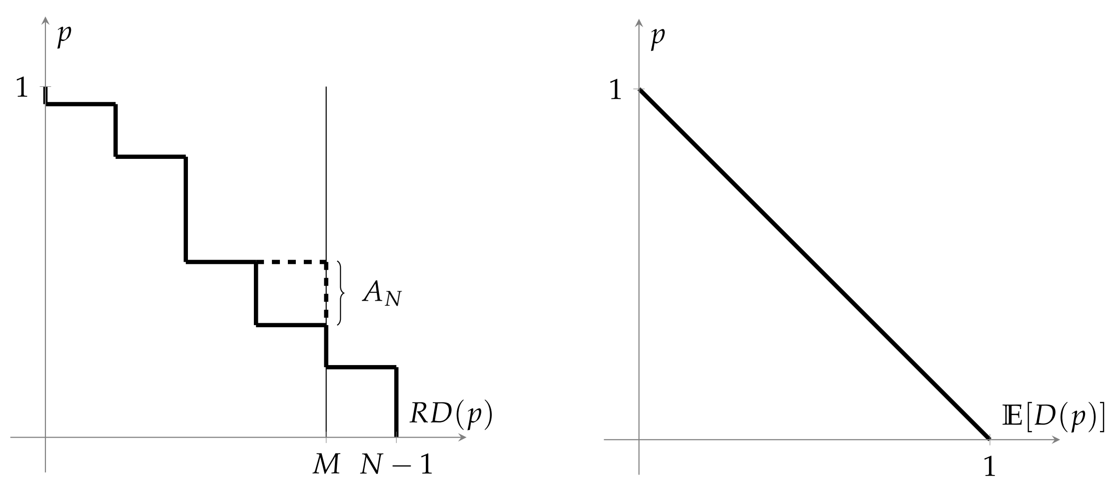
(a) Residual demand (without buyer N) (b) Truthful and perturbed demand with buyer N

Figure 1: Demand functions for Example 1.1

In this case, a small change in report of one agent had a substantial impact on prices, even when that agent is small relative to the size of the economy. We demonstrate below that the main cause is that the residual demand curve, $RD(p)$, the sum of the demand curves of buyers 1 through $N-1$, is flat near the equilibrium price (even in the limit as $N \to \infty$), as illustrated in Figure 1. This allows for a small change in the reported demand function of one agent to move the intersection with the supply curve a relatively large distance in price space.

Example 1.2. Again, consider an economy with N buyers and a single good with supply $M < N$. Buyer $n \in \{1, ..., N\}$ has unit demand for the good with value a_n, where a_n is drawn uniformly on $[0, 1]$. We are interested now in the *expected* influence that any single agent may have on the Walrasian equilibrium price(s).

Consider the problem from the perspective of agent N, supposing that all other agents truthfully report their values to the mechanism designer and the agent is restricted to reporting unit demand. In this case, the set of prices that the agent may, by *some* report,[5] realize is $A_N := [a^{(M-1)}, a^{(M)}]$ where $a^{(i)}$ is the i^{th} order statistic of the $N-1$ *other* draws of the valuation distribution. Because the expected spacing of the uniform order statistics is $O(1/N)$, the expected maximum impact of agent 1 on the equilibrium price is $O(1/N)$ as well. The same logic holds for any valuation distribution which has full support on some interval, with a pdf uniformly bounded away from zero (that is, $f(x) > c > 0$ for all $x \in \text{supp}(f)$).[6]



(a) Residual demand and attainable prices for buyer N

(b) Expected demand curve for an individual agent (equivalently average demand as $N \to \infty$)

Figure 2: Demand functions for Example 1.2

While the residual demand curve has flat segments for finite N, as $N \to \infty$, these flat segments become small when normalized by supply, approaching the negative-sloping demand curve illustrated on the top-right panel of Figure 2. We say that this expected demand curve, which has a slope uniformly bounded away from zero, is *strongly monotone*, introduced formally in Section 3. We show in Section 5 that the strong monotonicity property of the expected demand is what drives the rapid convergence of incentives toward price-taking behavior in this example.

[5]Not necessarily an optimal, or even beneficial, report.

[6]See, for example, Satterthwaite and Williams (1989).

1.2 Related literature

The motivating application of our perturbation analysis is to the study of *ex post* incentives in the Walrasian mechanism. Hurwicz (1972) first observed that agents with private information about their preferences may benefit from strategically misreporting demand in order to influence the price vector. This problem is more pervasive than just Walrasian equilibrium: the celebrated theorem of Green and Laffont (1979) implies that there is no mechanism in the quasilinear domain which is strategy-proof, efficient and budget-balanced. Roberts and Postlewaite (1976) studied the question of reporting incentives in large Walrasian economies and showed that the benefits of misreporting for any individual agent must tend to zero, under the condition that the Walrasian equilibrium price correspondence (mapping measures over the function space of possible excess demand functions to prices) is continuous at the limit economy. Jackson (1992) extended this result to show that the L^∞ distance between the true preferences and an optimal report must also tend to zero under the same condition. He, Miralles, Pycia, and Yan (2018) employ a similar condition to establish approximate incentive compatibility in replica economies associated with pseudomarkets à la Hylland and Zeckhauser (1979). However, the *rates* of convergence are not studied in these papers, and so it may be difficult for practical market designers to assess whether to expect good reporting incentives in real-world applications. Furthermore, the regularity and continuity conditions used in these results can be challenging to apply, because they rely on attributes of the Walrasian equilibrium price correspondence rather than underlying properties of the agents' preferences.

The rates of convergence of *ex post* incentives have been studied in several specific models, including the unit-demand double auction of Satterthwaite and Williams (1989) and linear-quadratic finance models surveyed by Rostek and Yoon (2020). Satterthwaite and Williams (1989) show that as long as values and costs are drawn i.i.d. from a full-support distributions with a lower-bounded density, the maximum benefit from misreporting is $O(1/N)$ and the distance between the true and optimal reports is $O(1/N)$. Similar to this paper, the finance literature surveyed by Rostek and Yoon (2020) emphasizes the relationship between the slope of the aggregate demand and the incentives for price-taking behavior in Bayes-Nash equilibrium (again $O(1/N)$ as long as the slope grows with N), specialized to the case of linear-quadratic models.

Al-Najjar and Smorodinsky (2007) take an alternative approach to studying the influence of strategic behavior on market mechanisms. Al-Najjar and Smorodinsky (2007) show that for any level of approximation there exists a sufficiently large $\overline{N}$ such that the outcome associated with any Bayes-

Nash equilibrium (BNE) of a competitive mechanism with at least $\overline{N}$ participants is approximately efficient. Al-Najjar and Smorodinsky (2007) focus on BNE outcomes of competitive mechanisms, whereas we will focus on the influence of deviations of single buyers, which correspond to the usual definitions of incentive compatibility. Also unlike this paper, Al-Najjar and Smorodinsky (2007) require a finite type space and a small probability that agents are not strategic. Moreover, their approach does not characterize the ability of an agent to influence on prices, rather the *number* of agents who can influence prices.

1.3 Organization and notation

The remainder of this paper is organized as follows. In Section 2, we describe the quasilinear exchange economy we study and discuss classical results associated with the model. In Section 3, we discuss strong convexity and strong monotonicity as they pertain to abstract convex functions and monotone correspondences (respectively) and then specialize these notions to the economic setting, applying them to the indirect utility functions and demand correspondences (respectively). In Section 4, we define perturbation-proofness and discuss its desirability for market designers. We then establish perturbation-proofness in large markets in which *all* agents have strongly monotone demand and replica economies associated with markets with strongly monotone demand. In Section 5, we consider the important case of economies with incomplete information, and establish Theorem 7, the key result in this paper: that markets with strongly monotone demand correspondence (in expectation) are perturbation-proof (with high probability and in expectation). We also establish in Corollary 8 the fast convergence of interim incentives for truthful reporting in such markets. Finally, in Section 6, we apply our results to markets with indivisibilities and identify the weak assumptions on models which imply that the expected demand correspondence is strongly monotone and therefore that the associated markets are perturbation-proof (with high probability).

Notation We will model consumption bundles as vectors in Euclidean space $\mathbb{R}^L$, equipped with the standard inner product $x \cdot y = \sum_{\ell=1}^{L} x_\ell y_\ell$, and norm $\|x\| = \sqrt{x \cdot x}$. We use $\geq$ to denote the partial order on $\mathbb{R}^L$ so that $x \geq y$ if and only if $x_\ell \geq y_\ell$ for $\ell = 1, ..., L$. The set $\mathbb{R}^L_+$ is $\{x \in \mathbb{R}^L : x \geq 0\}$, while $\mathbb{R}^L_{++}$ is $\{x \in \mathbb{R}^L : x_\ell > 0 \text{ for } \ell = 1, ..., L\}$. The notation $|\cdot|$ represents either the absolute value (if its argument is a number) or the cardinality (if its argument is a set). The distance between $x \in \mathbb{R}^L$ and a set $S \subseteq \mathbb{R}^L_+$ will be $\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|$. The Hausdorff distance between two sets $S, S' \subseteq \mathbb{R}^L_+$ is $d_H(S, S') = \max\{\sup_{x \in S'} \operatorname{dist}(x, S), \sup_{x \in S} \operatorname{dist}(x, S')\}$.

For a convex function $f : S \to \mathbb{R}$, we say $v \in \mathbb{R}^L_+$ is a subgradient of f at x if for any $x' \in S$, $f(x') - f(x) \geq v \cdot (x' - x)$. The subdifferential $\partial f(x)$ is the nonempty, convex, compact set of subgradients of f at p. Where the gradient of f is well-defined (which, by Rademacher's Theorem, is almost everywhere), we have $\partial f(x) = \{\nabla f(x)\}$.

Finally, we use the asymptotic notation of Knuth (1976), where for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we say $f(x) = O(g(x))$ if $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$; $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$; $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} > 0$; and $f(x) = \Theta(f(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

2 Model and preliminaries

We consider the setting of an exchange economy with L types of consumable good and a numeraire good, money.

There is a finite set of buyers N, which we call a *market*. Each buyer $n \in N$ can consume any bundle of consumable goods $x_n \in X_n$ where X_n is a convex, compact subset of $\mathbb{R}^L_+$, called the *consumption possibility set*. Assume that $0 \in X_n$ for each $n \in N$.

Each buyer has quasilinear preferences over commodity bundles in X_n with a *valuation function* $v_n : X_n \to \mathbb{R}$, so that the agent's *utility* associated with allocation x_n after payment t is $U_n(x_n, t) = v_n(x_n) - t$. We will assume that the valuation functions are drawn from a function space $\mathscr{V}$, such that each $v_n \in \mathscr{V}$ is monotone, concave[7] and satisfies the normalization $v_n(0) = 0$.

There is an exogenous *supply vector* $s \in \mathbb{R}^L_{++}$ for the consumable goods, which is nonnegative in all components. Buyers are unconstrained in their spending of money.

We use the notation $\mathscr{E} = \langle N, s\rangle$ for a market, supply vector pair.

Efficiency, equilibrium and mechanism design An *allocation* $\mathbf{x} = (x_n)_{n\in N}$ is an assignment of consumption bundles $x_n \in X_n$ to each buyer $n \in N$. Allocation $\mathbf{x}$ is *feasible* in $\mathscr{E}$ if $\sum_{n\in N} x_n \leq s$. The set of all feasible allocations for $\mathscr{E}$ is denoted $\mathscr{X}$.

The *surplus* associated with allocation $\mathbf{x}$ is defined by $\mathcal{S}(\mathbf{x}) = \sum_{n\in N} v_n(x_n)$. An *efficient allocation* for $\mathscr{E}$ is a feasible allocation $\mathbf{x}$ that solves the surplus maximization problem

$$\max_{\mathbf{x}\in\mathscr{X}} \mathcal{S}(\mathbf{x}) = \max_{\mathbf{x}\in\mathscr{X}} \sum_{n\in N} v_n(x_n). \tag{OPT}$$

[7]The main function of this assumption is to ensure that Walrasian equilibria exist. For analysis of linear pricing mechanisms, including incentives, without the assumption of concavity, see Milgrom and Watt (2021).

A *Walrasian equilibrium* of $\mathscr{E}$ is a feasible assignment $\mathbf{x} \in \mathscr{X}$ and a price vector $p \in \mathbb{R}^L_+$ such that $\sum_{n \in N} x_n = s$ and the assignment x_n to buyer n maximizes n's utility given price p: that is, x_n is *demanded* by buyer n at price p. Let $D_n : \mathbb{R}^L_+ \rightrightarrows X_n$ be the set of maximizers of $U_n\left(x, p \cdot x\right)$, called the *demand correspondence* of buyer n, so that Walrasian equilibrium requires $x_n \in D_n\left(p\right)$ for each $n \in N$. Throughout we will assume that $D_n\left(p\right) = \{0\}$ for prices p outside of a compact set $\mathscr{P} \subseteq \mathbb{R}^L_+$. The indirect utility function $u_n : \mathbb{R}^L_+ \to \mathbb{R}$ is the negative concave conjugate of v_n, defined by $u_n\left(p\right) = \max_{x \in X_n} U_n\left(x, p \cdot x\right)$. The indirect utility function is convex and related to the demand correspondence by the identity $\partial u_n\left(p\right) = -D_n\left(p\right)$.

Buyers report their preferences to a mechanism designer who determines an outcome and transfers. By the revelation principle, we may restrict our attention to mechanisms in which buyers report their valuation functions v_n (and implicitly the domain X_n) to the market designer. In this analysis, we abstract away from the important question of how bidders communicate these potentially complicated objects to the mechanism designer.[8] In the *Walrasian mechanism*, the mechanism designer (or Walrasian auctioneer) determines[9] and implements the Walrasian equilibrium prices and allocations, with some pre-determined decision rule if the Walrasian equilibrium is not unique. In this paper, we will abstract away from this decision rule by focusing either on economies with unique Walrasian equilibria (in Section 4) or studying changes to the *sets* of equilibrium prices. The motivation of this latter approach is that if the set of equilibrium prices attainable under some misreport is guaranteed to be at a small (Hausdorff) distance from the set of equilibrium prices attained under truthful reporting, then the distance between the prices obtained under any Walrasian mechanism with a specific decision rule must be correspondingly small.

Welfare theorems and problem formulations The fundamental theorems of welfare economics, as formalized by Arrow (1951), imply that the set of allocations associated with Walrasian equilibria coincide with the set of efficient allocations. One way to see this is to consider the Lagrangian $\mathscr{L} : \mathscr{X} \times \mathbb{R}^L_+ \to \mathbb{R}$ associated with (OPT), given by

$$\mathscr{L}\left(\mathbf{x}, p\right) = \sum_{n \in N} v_n\left(x_n\right) + p \cdot \left(s - \sum_{n \in N} x_n\right).$$

[8]The design of bidding languages to report complex preferences has been the subject of substantial study, including by Milgrom (2009), Bichler, Goeree, Mayer, and Shabalin (2014), Bichler, Milgrom, and Schwarz (2020).

[9]Here we are also implicitly assuming that the Walrasian equilibrium can be computed efficiently and exactly by the market designer, which is, in general, a non-trivial assumption given that the problem of computing Walrasian equilibrium is PPAD-complete. However, we show in Appendix B that Walrasian equilibrium *can* be approximated efficiently in the strongly convex case using tâtonnement (gradient) methods.

Since Slater's constraint qualification[10] is satisfied in (OPT) (because the zero allocation is in the relative interior of the constraint space), any saddle point of $\mathscr{L}$, that is, any pair $(\mathbf{x}, p)$ that solves

$$\min_{p \in \mathbb{R}^L_+} \max_{\mathbf{x} \in \mathscr{X}} \mathscr{L}(\mathbf{x}, p), \tag{SP}$$

gives rise to a solution $\mathbf{x}$ to the convex program (OPT). Furthermore, the the values of programs (OPT) and (SP) are the same (this is the complementary slackness condition).

From (SP), we see that any saddle point $(\mathbf{x}, p)$ must satisfy $x_n \in \arg\max_{x \in X_n} v_n(x) - p \cdot x$ for each $n \in N$, which is the individual optimality property of Walrasian equilibrium. So the saddle points of (SP)–Walrasian equilibria–correspond to maximizers of (OPT)–efficient allocations–and *vice versa*. This is a statement of the fundamental welfare theorems for quasilinear economies. Moreover, since the objective in (OPT) is bounded and concave and the set $\mathscr{X}$ is compact, (OPT) has a solution and a Walrasian equilibrium exists.

The *dual problem*,

$$\min_{p \in \mathbb{R}^L_+} p \cdot s + \sum_{n \in N} u_n(p), \tag{D}$$

obtained by reorganizing (SP), plays a major role in the analysis of this paper. An advantage of studying the dual problem is that it is an unconstrained convex program. Writing $U(p) = \sum_{n \in N} u_n(p)$ for the total indirect utility function, the first-order (necessary) conditions of (D) are exactly the market-clearing conditions of the Walrasian equilibrium, $s \in -\partial U(p) = \sum_{n \in N} D_n(p)$.

3 Strong monotonicity and strong convexity

We first introduce a stronger notion of convexity that is used in the perturbation analysis of convex programs.[11] In the following definitions, K is a compact, convex subset of $\mathbb{R}^N$ and $f : K \to \mathbb{R}$ is a proper,[12] convex function defined on K.

Definition 3.1 (Strong convexity)**.** The function f is *strongly convex* with constant $m > 0$ if

$$f(y) \geq f(x) + s_x \cdot (y - x) + \frac{m}{2}\|y - x\|^2 \text{ for all } x, y \in K \text{ and } s_x \in \partial f(x).$$

[10] See, for example, Boyd and Vandenberghe (2004).

[11] Strong convexity is used routinely in the analysis of convex optimization problem, see, for example, Boyd and Vandenberghe (2004).

[12] A convex function $f : K \to \mathbb{R} \cup \{\pm\infty\}$ is proper if $f(x) > -\infty$ for all $x \in K$ and $f(x_0) < +\infty$ for some x_0 in K.

Note that by replacing m with zero in Definition 3.1, we recover a definition of convexity of function f so that Definition 3.1 is a stronger condition than convexity. Informally, a function is strongly convex if it is possible to fit a quadratic between the function and all of its tangent planes.

In the same way that the convexity of function f is equivalent to the monotonicity of its subdifferential ∂f, the strong convexity of f is equivalent to *strong monotonicity* of ∂f, as defined below.

Definition 3.2 (Strong monotonicity)**.** Let $s : K \rightrightarrows \mathbb{R}$ be a nonempty-valued correspondence defined on K. For $\gamma > 0$, correspondence s is *strongly monotone* with constant $m > 0$ if

$$(s_y - s_x) \cdot (y - x) \geq m\|y - x\|^2, \text{ for all } x, y \in K \text{ and } s_x \in s(x), s_y \in s(y).$$

Note that by replacing m with zero in Definition 3.2, we obtain the usual definition of a monotone correspondence.

Proposition 1. *Let $f : K \to \mathbb{R}$ be a convex function and $\partial f : K \rightrightarrows \mathbb{R}$ be its subdifferential mapping. Then f is strongly convex with constant m if and only if ∂f is strongly monotone with constant m.*

There are several other well-known characterizations of strong convexity (see Boyd and Vandenberghe (2004)). Function f is strongly convex if and only if:

(a) $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{m}{2}\lambda(1 - \lambda)\|y - x\|^2$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

(b) the function $g : K \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{m}{2}\|x\|^2$ is convex.

(c) for all $x, y \in K$, the second directional derivative of f at x in the direction of $y - x$ is bounded below by m.[13]

Note that in the case that $K \subseteq \mathbb{R}$, this final condition is equivalent to the subdifferential ∂f having directional derivatives bounded away from zero for all $x \in K$.

Strong convexity also has a dual formulation. Recall that the Fenchel dual of a proper convex function $f : K \to \mathbb{R} \cup \{\pm\infty\}$ defined on $K \subseteq \mathbb{R}^N$ is the function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ satisfying $f^*(x^*) = \sup_{x \in K} x^* \cdot x - f(x)$. The following dual characterization of strong convexity is known (see Borwein and Vanderwerff (2010)).

[13]This characterization of strong convexity is typically reported in the literature restricting attention to twice continuously differentiable f, but we provide a proof of this stronger and very useful claim as part of the proof of Theorem 6.

Proposition 2. *A proper convex function $f : K \to \mathbb{R}$ is strongly convex with constant m if and only if the Fenchel dual $f^* : \mathbb{R}^N \to \mathbb{R}$ is* strongly smooth, *that is,*

$$f^*(y^*) \leq f^*(x^*) + s \cdot (y^* - x^*) + \frac{1}{2m}\|y^* - x^*\|^2, \text{ for all } x^*, y^* \in \mathbb{R}^N \text{ and } s \in \partial f^*(x^*).$$

Equivalently, for all $x^*, y^* \in \mathbb{R}^N$, $s_x \in \partial f^*(x^*)$ and $s_y \in \partial f^*(y^*)$, $(s_y - s_x) \cdot (y - x) \leq \frac{1}{m}\|y^* - x^*\|^2$. This latter condition implies the Lipschitz-continuity of ∇f^* wherever it exists.

Strongly monotone demand preferences We will mostly be interested in strong convexity and strong monotonicity as these concepts apply to indirect utility functions and demand correspondences, respectively. A complication is that, in our setup (and in many applications), demand for each good must be nonnegative. If demand at some price p is zero, the law of demand implies that the demand at price $p' = \alpha p$ for $\alpha > 1$ must also be zero. This implies that the inequality in Definition 3.2 cannot be satisfied for prices p, p'. We thus modify the definition of strong monotonicity to apply only at prices where the buyer is *active* in the following sense.

Definition 3.3. Buyer n is *active* at price p if $D_n(p) \neq \{0\}$.

Definition 3.4. Buyer n's demand correspondence $D_n : \mathbb{R}^L_+ \rightrightarrows X_n$ is *strongly monotone* if there exists some $m > 0$ such that for all prices $p, p' \in \mathbb{R}^L_+$ where buyer n is active,

$$(d' - d) \cdot (p - p') \geq m\|p - p'\|^2, \text{ for all } d \in D_n(p) \text{ and } d' \in D_n(p'). \tag{SLD}$$

In this case, we say buyer n has *strongly convex preferences* or *strongly monotone demand*.

For economy $\mathscr{E} = \langle N, s \rangle$, we say the *total* demand correspondence $D = \sum_{n \in N} D_n$ is strongly monotone if inequality (SLD) holds for all p, p' such that *at least* one buyer $n \in N$ is active and $d \in D(p), d' \in D(p')$.

Note the resemblance of (SLD) to the classical law of demand (in which the right-hand side of (SLD) is replaced with a zero). However, while the law of demand is a theorem that applies to *all* demand correspondences, not all demand correspondences are strongly monotone. By Proposition 1, strong monotonicity of demand is equivalent to the strong convexity of the indirect utility function wherever it is nonzero. Proposition 2 implies that it is also equivalent to the strong smoothness of $-v_n$.

4 Perturbation-proofness and complete information exchange economies

In this section, we study the effect of perturbations in the objective function or constraint sets of the dual problem (D) on the set of optimizers, the Walrasian prices. We first define formally what we mean by perturbations and perturbation-proofness.

Definition 4.1. Let $\mathscr{E} = \langle N, s \rangle$ be a market-supply vector pair, which we will call the *original* or *unperturbed economy*. Then for some $\delta s \in \mathbb{R}^L$ such that $s + \delta s \geq 0$, we refer to $\mathscr{E}' = \langle N, s + \delta s \rangle$ as the *perturbed economy*, δs its perturbation and $\|\delta s\|$ the size of the perturbation.

Note here that we are interested in small but *finite* perturbations, distinguishing this analysis from the study of shadow prices, which are relevant only for infinitesimal perturbations.

Definition 4.2. Let $N_1 \subseteq N_2 \subseteq ...$ be a nested sequence of markets. We say that $(N_t)_{t\in\mathbb{N}}$ is $O(\varepsilon(N))$-*perturbation-proof* if for all sequences s_t and for all perturbations δs_t with size $\|\delta s_t\| \leq O(1)$, we have that $d_H(P_t, P_t') \leq O(\varepsilon(|N_t|))$, where P_t is the set of Walrasian prices in the i^{th} original economy $\langle N_t, s_t \rangle$ and P_t' the set of Walrasian prices in the i^{th} perturbed economy $\langle N_t, s_t + \delta s_t \rangle$.

In this paper, we will be almost exclusively interested in $O(1/N)$-perturbation-proofness (or for technical reasons in markets with incomplete information, the very close rate of $O(1/N^{1-\varepsilon})$ for all $\varepsilon > 0$), and so, for expositional purposes (i.e., except in the precise statements of theorems), we omit the big O notation. However, in Appendix A, we discuss conditions that lead to faster or slower rates of convergence.

At first glance, our definition of perturbations may appear very narrow, allowing only for changes in the supply vector. However, we now show that our definition of perturbation-proofness implies robustness to two other important changes to the economy, namely additions (or removals) to the set of agents and misreporting by agents.

Proposition 3. *Suppose that $(N_t)_{t\in\mathbb{N}}$ is $O(\varepsilon(N))$-perturbation proof. For some sequence of supply vectors $(s_t)_{t\in\mathbb{N}}$ let $\mathscr{E}_t = \langle N, s_t \rangle$ and P_t be the associated sequence of Walrasian prices. Consider the following two related economies*

- *(P1)* Misreporting*: For $n_0 \in N_1$, let v_{n_0} be replaced in N_t' by $v' \in \mathscr{V}$, and let $\mathscr{E}' = \langle N_t', s \rangle$ with P_t' its associated sequence of Walrasian prices.*
- *(P2)* Addition of buyers*: Let N_0 be some finite subset of buyers from $\mathscr{V}$, and let $\mathscr{E}_t'' = \langle N_t \cup N_0, s_t \rangle$ with P_t'' its associated sequence of Walrasian prices.*

Then $d_H(P_t, P_t')$ and $d_H(P_t, P_t'')$ are $O(\varepsilon(|N_t))$ as well.

Note that we may consider the problem of removing agents by swapping the role of $\mathscr{E}_t$ and $\mathscr{E}_t''$ in Item (P2).

Proof. The analysis is the same for all t, so we ignore the index. The necessary and sufficient conditions for Walrasian equilibrium of the original economy is $D(p) = s$. For problem (P1), suppose that agent n_0 receives allocation x_0 under truthful reporting, while obtains an allocation of $\tilde{x}$ under an announcement that induces Walrasian equilibrium price $\tilde{p}$. Buyer n_0's announcement must satisfy $\sum_{n\in N\setminus\{n_0\}} D_n(\tilde{p}) + \tilde{x} = s$. But this is the same as the necessary and sufficient conditions for equilibrium of the problem $D(p) = s + x_0 - \tilde{x}$, which corresponds to a perturbation of $\mathscr{E}$ by $\delta s = x_0 - \tilde{x}$. Thus the effect of misreporting on prices may be thought of as a perturbation in the sense of Definition 4.1. The same general idea works for the addition of buyers; the equivalent perturbation is $-\sum_{n\in N_0} D_n(\tilde{p})$ where $\tilde{p}$ is the induced price in the perturbed economy. □

Finally, we note the relationship between perturbation-proofness and approximate incentive compatibility of the Walrasian mechanism. We say a mechanism is *ex post $O(\varepsilon(N))$-incentive compatible* if for each agent, holding fixed the truthful reports of the other agents (which are known to all agents), the maximum benefit of misreporting under the mechanism—the utility under the optimal misreport minus the utility received under truthful reporting—is $O(\varepsilon(N))$.

Proposition 4. *If $(N_t)_{t\in I}$ is $O(\varepsilon(N))$-perturbation-proof, then the Walrasian mechanism applied to $\langle N_t, s_t\rangle$ for any (s_t) is $O(\varepsilon(N))$-incentive compatible.*

This follows since indirect utility functions are locally Lipschitz in prices (and therefore globally Lipschitz since utility is bounded), and so small changes in the price vector lead to correspondingly small changes in each agent's indirect utility, which is an upper bound on the realized utility associated with any misreport.

Markets with strongly convex agents We first consider the case where *all* agents have strongly monotone demand functions. This assumption is strong (it is not satisfied, for example, in the unit demand valuations of Example 1.2), but the analysis in this setting provides intuition for other results.

Theorem 5. *Consider a sequence of markets $(N_t)_{t\in\mathbb{N}}$ in which all agents have strongly convex preferences with constant $m > 0$. Let s_t be a sequence of supply vectors and δs_t a sequence of perturbations, and define*

p_t as the equilibrium price in the original economy and p'_t the equilibrium price in the perturbed economy. Then $\|p_t - p'_t\|$ is $O\left(1/N_t^a\right)$ where N_t^a is the number of buyers who are active at prices p_t and p'_t.

We make two remarks on the statement of this theorem: first note that we may speak of 'the' equilibrium price in the original and perturbed economies since the dual objective is strongly convex (and thus strictly convex) so that the equilibrium price is always unique. Second, note that we have not used the 'perturbation-proofness' terminology here, because N_t^a depends on both the set N_t and the sequence of supply vectors s_t and perturbations δs_t. However, this result is similar in spirit to the perturbation-proofness definition.

Before proving Theorem 5, we discuss the intuition for the result in the setting with a single consumable good. The strong monotonicity of each agent's demand implies that the slope of the market demand curve at any price (including the equilibrium price) grows proportionally to the number of active buyers at that price. As the demand curve becomes increasingly steep at the equilibrium price, small perturbations of the supply vector have a progressively smaller effect on the resulting price.[14]

Proof. By assumption, for each active buyer n at prices p, p', we have for any $d_n \in D_n(p)$ and $d'_n \in D_n(p')$

$$(d'_n - d_n) \cdot (p - p') \geq m\|p - p'\|^2, \tag{1}$$

so that

$$\sum_{n \in N_t} (d'_n - d_n) \cdot (p - p') \geq N_t^a m\|p - p'\|^2. \tag{2}$$

That is, the aggregate demand satisfies (SLD) with constant $N_t^a m$. By definition of Walrasian equilibrium, we have $\sum_{n \in N_t} d'_n = s'$ and $\sum_{n \in N_t} d_n = s$, so that

$$(s' - s) \cdot (p - p') \geq N_t^a m\|p - p'\|^2.$$

Since $s' - s = \delta s$, by the Cauchy-Schwarz inequality, we then have

$$\|\delta s\| \left\|p - p'\right\| \geq N_t^a m \left\|p - p'\right\|^2,$$

[14] In Appendix B, we show that this same logic also implies that markets with strongly convex agents are tâtonnement-stable, with a fast (subpolynomial) rate of convergence of tâtonnement.

or on re-arranging,

$$\|p - p'\| \leq \frac{\|\delta s\|}{mN_t^a} \leq O\left(\frac{1}{N_t^a}\right).$$

□

Necessary conditions and replica economies In the remainder of this section and Section 5, we will seek to weaken the assumption in Theorem 5 that all agents have strongly convex preferences. In doing so, we will need to add more structure to the sequence of economies we analyze (as in this section) or accept weaker conclusions (in Section 5 we will settle for probabilistic results).

An alternative (if imprecise) interpretation of the proof of Theorem 5 motivates our approach. Again, we focus on the case with one consumable good. When all agents have strongly monotone demand functions, the *average* (per-capita) demand curve (averaged over the number of active buyers at price p_t) is downward sloping. On the other hand, the per-capita perturbation diminishes at a rate inversely proportional to the number of active buyers at price p_t. Thus, if the average demand curve is not too flat at p_t, the effect of a perturbation on prices diminishes at the same rate as the per-capita size of the perturbation.

This suggests the importance of the *average* demand correspondence for the analysis of perturbations. Given an average demand correspondence, the simplest associated sequence of economies is the replica economies of Debreu and Scarf (1963).

Definition 4.3. Let N be a market and define N_k, its *$k-$fold replica*, as the set of $k|N|$ buyers such that for each $n \in N$, there are k buyers in N_k with the same preferences as n. The $k-$fold replica of a *base economy* $\mathscr{E} = \langle N, s \rangle$ is $\mathscr{E}_k = \langle N_k, ks \rangle$.

In replica economies, the average demand correspondence is constant with respect to the number of replicas. In this setting, Theorem 6 states that strong monotonicity of the average demand correspondence is a necessary and sufficient condition for the conclusions of Theorem 5 to hold for all possible supply vectors s.

Theorem 6. *Let N be a market with total demand correspondence $D = \sum_{n \in N} D_n$, and let N_k be its $k-$fold replica. Then $\mathscr{E}_k = \langle N_k, ks \rangle$ is $O(1/N)$-perturbation-proof for all supply vectors s if and only if D is strongly monotone.*

5 Incomplete information exchange economies

For the remainder of this paper, we study the case where the valuation functions of buyers are not known to the market designer, but are instead random variables drawn identically and independently from a distribution ν over $\mathscr{V}$, now assumed to be a measurable space (equipped with the appropriate Borel σ-algebra).[15] We make one further assumption on the set of buyer types.

Assumption 1. *Given distribution ν on $\mathscr{V}$, there exists a compact set $\mathscr{P} \subseteq \mathbb{R}^L_+$ such that $D_n(p) = \{0\}$ almost surely for $p \notin \mathscr{P}$.*

Assumption 1 is embedded in many auction models and will be required for the main results in this section.[16]

For any price $p \in \mathscr{P}$, define the *expected indirect utility function* by

$$\mathbb{E}_\nu[u(p)] = \int_{\mathscr{V}} u_n(p) d\nu.$$

The *expected demand correspondence* $\mathbb{E}_\nu[D_n(p)]$ is defined using the set-valued integral of Aumann (1965). For any fixed p, the probability measure ν induces a probability measure over the sets $D_n(p)$ associated with valuation function $v_n \in \mathscr{V}$. A selection $\xi : \mathscr{V} \to X$ is a single-valued random vector such that $\xi(v_n)$ ν-almost surely belongs to $D_n(p)$ for each $v_n \in \mathscr{V}$. Then $\mathbb{E}_\nu[D_n(p)]$ is defined as $\mathrm{cl}(\{\mathbb{E}_\nu \xi\})$ over integrable selections ξ. Alternatively, a result of Rockafellar and Wets (1982) implies that $\partial \mathbb{E}_\nu[u(p)] = \mathbb{E}_\nu[D_n(p)]$, so that the expected demand may be defined as the subdifferential of the expected indirect utility. Moreover, a law of large numbers applies to $\mathbb{E}_\nu[D_n(p)]$ so that $d_H\left(\frac{1}{|N_t|}\sum_{n \in N_t} D_n(p), \mathbb{E}_\nu[D_n(p)]\right) \to 0$ as $|N_t| \to \infty$, where N_t is a set of agents drawn i.i.d. from ν (Weil, 1982).

In this section, we assume that the expected demand correspondence is strongly monotone , or, equivalently, that the expected indirect utility function is strongly convex. Note that this does not require that the individual agents' demands are strongly monotone, or even single-valued. For example, in Example 1.2, the individual demand functions are step functions (not strongly monotone), while the expected demand function is strongly monotone.

[15]For example, by a result of Aumann (1963), $\mathscr{V}$ could be taken as the set of bounded, continuous functions on a compact subset of $\mathbb{R}^L_+$, or the set of bounded functions with discontinuities of the first kind, or, more generally, any subset of a Baire class. In particular, we may further restrict to require that the valuation functions are monotone, concave and normalized in accordance with our previous assumptions.

[16]It is simple to modify our results to require only that the equilibrium price belongs to a compact set $\mathscr{P}$ almost surely, but we have chosen the formulation of Assumption 1 because it does not require knowledge of supply vector s.

5.1 Expected perturbation-proofness and *ex post* incentives

For now, despite the uncertainty in buyer types, we maintain the assumption that buyers know each others' types. Our model thus captures uncertainty from the perspective of the modeller or market designer, rather than uncertainty possessed by agents within the model. In this formulation, the quantities of interest, the $d_H(P_i, P_i')$ that arises from the supply perturbations, are random variables, and so we seek to characterize the expectation (over draws of the economy) of this random variable. We will actually prove a stronger result: that the random variable is $O(1/N^{1-\varepsilon})$ with high probability over draws of the economy for any $\varepsilon > 0$. Perturbation-proofness in this setting will thus imply that the *ex post* incentives for truthful reporting diminish rapidly in the market size, both in expectation and with high probability.

Theorem 7. *Suppose that buyer types are drawn i.i.d. from distribution ν on $\mathscr{V}$ such that the expected demand correspondence is strongly monotone. Then for all $\varepsilon > 0$, we have with probability $1 - O(1/|N_i|)$ that $d_H(P_i, P_i') \leq c/|N_i|^{1-\varepsilon}$ for some $c > 0$. In particular, this implies that $\mathbb{E}_\nu[d_H(P_i, P_i')] \leq O\left(1/|N_i|^{1-\varepsilon}\right)$. That is, the sequence of economies is $O(1/N^{1-\varepsilon})$-perturbation proof with high probability. This implies that the Walrasian mechanism is ex post $O(1/N^{1-\varepsilon})-$incentive-compatible with high probability and in expectation for these economies.*

The proof of Theorem 7, presented in Appendix C, relies on Bernstein's Inequality and establishes the concentration of the empirical average demand correspondence around the expected demand correspondence.

5.2 Interim reporting incentives

We now focus on perturbations caused by misreports, under the alternative *interim* informational assumption, that is, buyers choose their reports knowing only their own draw from the type distribution and not those of other buyers. We establish the following result on interim reporting incentives, which follows almost directly from Theorem 7.

Corollary 8. *Suppose that buyer types are drawn i.i.d. from distribution ν on $\mathscr{V}$ such that the expected demand correspondence is strongly monotone. The interim reporting incentives for buyers in this model satisfy the following: the maximum distance between the price induced by an optimal report by buyer n and the price induced by their truthful report tends to zero at $O(1/N^{1-\varepsilon})$. This implies that the Walrasian mechanism is interim $O(1/N^{1-\varepsilon})-$incentive-compatible in expectation for these economies.*

6 Strong monotonicity with indivisible goods

In this section, we specialize our analysis to economies with indivisible goods, so that $X_n \subseteq \mathbb{Z}_+^L$. In models with indivisibilities, strong monotonicity of individual demand cannot be observed since prices are a continuous variable while demand can take on only finitely many values. For this reason, we will focus on models with incomplete information, as in Section 5. The main goal is to establish conditions under which the expected demand correspondence is strongly monotone. In so doing, we will establish two secondary goals: first, we will exhibit applications of our results to settings in which $O(1/N)-$incentive-compatibility has not previously been established, and second, we will argue that the assumptions required to establish expected strong monotonicity are likely to be reasonable assumptions in many economic models.

We study a model with indivisibilities for two key reasons: first, indivisibilities are a natural assumption in many important markets, and second, the change in demand associated with any price change is bounded below by the size of the indivisibility. This allows us to focus, for the purpose of establishing (expected) strong monotonicity, on the prices at which demand changes, rather than also concerning ourselves with the size of these demand changes.

Our analysis will exploit a number of concepts introduced by Baldwin and Klemperer (2019). We will define and illustrate by example the key concepts we use, but the interested reader is referred to Baldwin and Klemperer (2019) for a complete treatment.

Definition 6.1. For buyer n with demand correspondence D_n, the *locus of indifference prices (LIP)* is $\mathscr{L}_n = \left\{ p \in \mathbb{R}_+^L : |D_n(p)| > 1 \right\}$.

The LIP divides price space into *unique demand regions* in which demand is constant, so that demand can only change as prices change through the $(L-1)-$dimensional facets that comprise the LIP.[17] Moreover, Baldwin and Klemperer (2019) show that as prices change between adjacent unique demand regions, demand changes by an integer multiple of the "primitive" normal vector of the associated facet(s) separating the regions. Here, a primitive vector is one in which the greatest common divisor of its entries is 1. This motivates the following definition.

Definition 6.2. The *demand type* $\mathscr{D}_n$ of buyer n is the set of primitive facet normal vectors of the LIP. We refer to an element of $\mathscr{D}_n$ as a *demand subtype*.[18]

[17] Note that the cyclic monotonicity of demand implies that the change in demand as p changes to p' is independent of the path in price space between p and p'. So, unless otherwise specified, when we say a price change from p to p', we will be referring to straight line paths between p and p'.

[18] Note that the "subtype" terminology is not used by Baldwin and Klemperer (2019).

In this section, we will carefully distinguish this "demand type" and the "type" of buyer n, which is the valuation function v_n drawn from ν over $\mathscr{V}$. We illustrate these definitions in the following examples, beginning with a familiar valuation function before describing a more novel example.

Example 6.1 (Unit demand)**.** Consider a unit demand buyer n of a single good (as in Example 1.2), so that $v_n : \{0,1\} \to \mathbb{R}$ with $v_n(x) = a_n x$ for $a_n > 0$. The LIP of buyer n contains only the price $p_x = a_n$, and the demand type of the buyer is $\mathscr{D}_n = \pm\{1\}$.

Note that with one consumable good, this is the only possible demand type (but there are many possible valuation functions associated with this demand type).

Example 6.2 (Additive demand with complementarity)**.** Suppose there are two goods x and y, with $(x, y) \in \{0,1\}^2$. Buyer n has the following valuation function for the goods

$$v_n(x, y) = a_n x + b_n y + c_n xy,$$

where each of a_n, b_n and c_n are strictly positive real numbers. The LIP for buyer n is illustrated in Figure 3. The associated demand type of buyer n is then $\mathscr{D}_n = \pm\{(0,1), (1,0), (1,1)\}$.

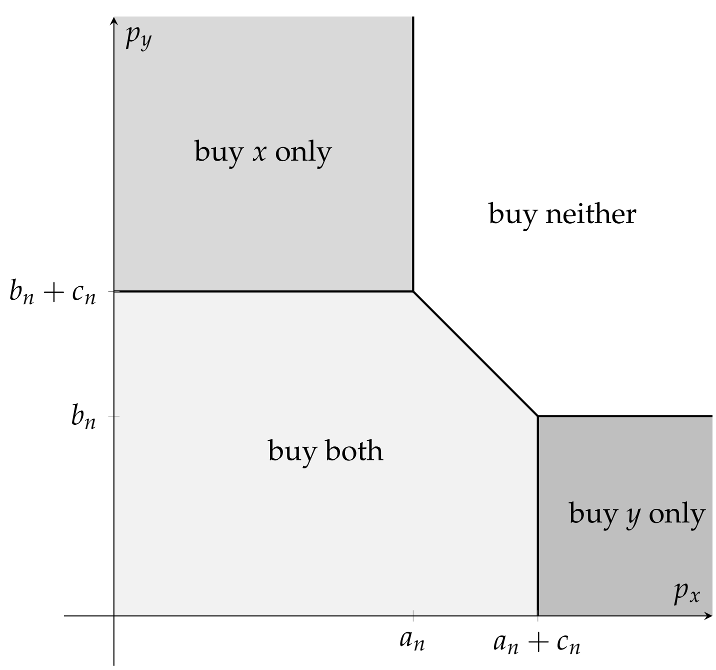


Figure 3: Locus of indifference prices for Example 6.2

6.1 Conditions implying strong monotonicity

Recall that for economies with unit demand buyers and uncertainty in valuations (as in Example 1.2 and Example 6.1), a sufficient condition for the expected maximum influence on price by any single buyer to be $O(1/N)$ is that a_n be drawn from a distribution with full support on an interval in $\mathbb{R}$ with density bounded below by $\alpha > 0$ (Satterthwaite & Williams, 1989). We can see that this condition guarantees strong monotonicity of the expected demand since, for $p' > p$, the change in demand grows at least linearly with $p' - p$, that is

$$\mathbb{E}[d(p') - d(p)] \geq \int_{p}^{p'} \alpha d\tilde{p} = \alpha(p' - p).$$

This implies the required inequality, $\mathbb{E}[(d(p') - d(p))(p' - p)] \geq \alpha(p' - p)^2$.

It should be clear that the unit demand structure is not necessary for this result: all that is required is that, for any price p, there is a positive probability to draw marginal buyers and non-buyers of the good (that is, buyer types who would reduce demand in response to a price increase and types who would increase demand in response to a price decrease) and a condition that corresponds to a lower-bounded density. With more goods, we must also consider the many directions in which price changes can occur at any given price.

We formalize this intuition in the following proposition.

Proposition 9 (Expected strong monotonicity for multiple indivisible goods)**.** *Let $\mathscr{P}$ be a compact, convex subset of $\mathbb{R}^L_+$. Suppose there exists some $\alpha > 0$ such that for all $p, p' \in \mathscr{P}$ with $p \neq p'$ we have $\Pr_\nu[D_v(p) \neq D_v(p')] \geq \min\{\alpha\|p' - p\|, 1\}$ for some $\alpha > 0$. Then the expected demand correspondence associated with ν is strongly monotone.*

This condition may be interpreted in terms of two natural assumptions for economic models with indivisibilities. First, *uncertainty* about where (in price space) a demand change occurs: there must be some probability that demand changes (for a type drawn from ν) associated with *any* price change, and larger price changes must lead to a proportionately larger probability that demand changes. Second, and more subtly, the condition reflects *variety* in the preferences. To see this, fix a price p and consider small price changes in the coordinate directions from p. For each such price change, there must be LIPs of some type passing non-orthogonally through the line segment between the prices (else demand does not change). Taking the limit as the price changes approach zero,[19] we must have that there are LIPs associated with different buyer types passing through p

[19]Assuming a sense of continuity of $\mathscr{V}$: namely that if there are demand correspondences $d_n \in \mathscr{V}$ approaching d in

non-orthogonally. That is, the LIPs passing through any price p (and their normals, the demand types) must span the full space $\mathbb{R}^L$. This implies each price p must be a kind of "marginal price" for different demand changes for various agent types. For example, at price p, some types in the support of ν might be indifferent to buying or not buying good x, while other types are indifferent to buying or not buying good y. This interpretation is reminiscent of Hildenbrand's (1994) argument that the law of demand reflects primarily "heterogeneity" of the population of households.[20]

Finally, we show how Proposition 9 can be used to establish perturbation-proofness (and therefore incentive results) in the environment of Example 6.2. We believe this is suggestive of many new economic environments where approximate incentive-compatibility results may be obtained which have not previously been established.

Example 6.2 (Continued)**.** Let $\mathscr{P} = [0,1]$ and prices $p, p' \in \mathscr{P}$ be given (without loss of generality, suppose $p_x \leq p'_x$). Suppose that a_n and b_n are drawn independently from full support distributions on $[0,1]$ and c_n is drawn from a full support distribution on $[0, 1 - a_n \wedge 1 - b_n]$. There are two possibilities, either

(a) $p_x \neq p'_x$. In this case, we have that buyers with $a_n \leq p_x$, $b_n \leq p'_y$ and $c_n \in [p'_x - a_n, 1 - a_n]$ change demand by a multiple of $(1,0)$ as p changes to p'.

(b) $p_x = p'_x$. In this case, we have that buyers with $b_n < p_y$ and $b_n + c_n < p'_y$ with $a_n > p_x$ change demand by a multiple of $(0,1)$ as p changes to p'.

In each case, we see that the probability of drawing such agents grows in $\|p - p'\|$. Note in both cases that there are other buyers who experience demand changes for this price change, our goal is just to obtain a lower bound on the probability of demand changes associated with the price change. This implies the required condition in Proposition 9 and so markets consisting of additive demand buyers with complementarities are perturbation-proof and the Walrasian mechanism is $O(1/N^{1-\varepsilon})$-incentive compatible.

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the L^∞-norm, then $d \in \mathscr{V}$.

[20]Hildenbrand (1994) argues that the law of demand may be derived at the market level from assumptions on the heterogeneity of household, rather than primarily reflecting the rationality of individual agents, as in the classical approach. While we maintain classical rationality assumptions, the forces driving his law of demand and our strong monotonicity of expected demand are similar.

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A Notions related to strong monotonicity

Here we introduce generalized notions of strong convexity and strong monotonicity, in which the quadratic powers in each definition are replaced by alternative constants.

Definition A.1 (Order γ-strong convexity)**.** For $\gamma > 0$, the function f is *order γ-strongly convex* with constant $m > 0$ if

$$f(y) \geq f(x) + s_x \cdot (y - x) + \frac{m}{2}\|y - x\|^\gamma \text{ for all } x, y \in K \text{ and } s_x \in \partial f(x).$$

Definition A.2 (Order γ-strong monotonicity)**.** Let $s : K \rightrightarrows \mathbb{R}$ be a nonempty-valued correspondence defined on K. For $\gamma > 0$, correspondence s is *order $\gamma-$strongly monotone* with constant $m' > 0$ if

$$(s_y - s_x) \cdot (y - x) \geq m'\|y - x\|^\gamma, \text{ for all } x, y \in K \text{ and } s_x \in s(x), s_y \in s(y).$$

These concepts are related similarly to Proposition 1.

Proposition 10. *Let $f : K \to \mathbb{R}$ be a convex function and $\partial f : K \rightrightarrows \mathbb{R}$ be its subdifferential mapping.*

(a) If f is order $\gamma-$strongly convex with constant $m > 0$, then ∂f is order $\gamma-$strongly monotone with constant m.

(b) If ∂f is order $\gamma-$strongly monotone with constant $m' > 0$, then f is order $\gamma-$strongly convex with constant $2m'/\gamma$.

The usual notions of strong convexity and strong monotonicity offer several alternative characterizations (as detailed in Section 3), while there are fewer such characterizations for γ-strong convexity when $\gamma \neq 2$. Moreover, where strong convexity relates to the second directional derivatives of a function, γ-strong convexity does not have similar interpretation in terms of higher directional derivatives.

However, we do have the following generalization of Theorem 5 that applies for order γ-strong monotonicity.

Proposition 11. *Consider a sequence of markets $(N_t)_{t\in\mathbb{N}}$ in which all agents have order γ-strongly convex preferences with constant $m > 0$. Let s_t be a sequence of supply vectors and δs_t a sequence of perturbations, and define p_t as the equilibrium price in the original economy and p'_t the equilibrium price in the perturbed economy. Then $\|p_t - p'_t\|$ is $O\left(1/N_t^{a\,\frac{1}{\gamma-1}}\right)$ where N_t^a is the number of buyers who are active at prices p_t and p'_t.*

The proof is almost identical to the proof of Theorem 5. Theorem 7, Corollary 8 and the sufficiency direction of Theorem 6 can also be adapted in obvious ways for the alternative assumption of order γ-strong monotonicity.

Another notion related to strong convexity that may be used when the set of minimizers of a function is not unique is the following.

Definition A.3 (Growth conditions)**.** Let S be the set of minimizers of f on K, which we suppose is non-empty, and let $f_0 = \min_{x\in K} f(x)$. For $\gamma > 0$, the function f satisfies the *order γ-growth condition* if there exists some constant $m > 0$ such that for all $x \in K$,

$$f(x) \geq f_0 + \frac{m}{2}[\mathrm{dist}(x, S)]^{\gamma}. \tag{GC}$$

For $\gamma = 2$, we call this the *quadratic growth condition*. If (GC) is satisfied only in some neighborhood of x, then we refer to it as the *local* order γ-growth condition at x.

Growth conditions were introduced by Shapiro (1992) and thoroughly studied in Bonnans and Shapiro (2013). Because the zero vector is in the subdifferential of f at any minimizer of f, it is clear that the order $\gamma-$growth condition is a weaker concept than order $\gamma-$strong convexity. Proposition 11 is easily modified to apply to $d_H(P_t, P'_t)$ under the assumption that N_t has the order γ-growth condition at P_t for each t.

B Tâtonnement stability of strongly convex economies

Recall the continuous-time tâtonnement process in which prices are adjusted in proportion to the excess demand for the relevant good:

$$\frac{dp}{dt} = \alpha[D(p(t)) - s], \text{ with } p(0) = p_0$$

for some adjustment speed $\alpha > 0$ and starting price $p_0 \in \mathscr{P}$. Here we assume $D(p)$ is single-valued, as is in our case of interest where D is a strongly monotone demand correspondence.

It is well-known that in quasilinear economies (and other economies in which there is a representative consumer) that $\lim_{t\to\infty} p(t)$ is a Walrasian equilibrium price for any starting price p_0.

However, in general, the rate of convergence of prices to equilibrium may be arbitrarily slow. The intuition for this is as follows: there may in general exist prices p at large distance from Walrasian equilibrium price p^* for which the excess demand is very small. This implies that the speed of adjustment of prices is very small, while the distance from equilibrium is very large.

However, under the assumption of strong monotonicity of demand, we have (via the Cauchy-Schwarz inequality) that $\|p - p^*\| \leq \frac{1}{m}\|D(p) - s\|$ so that prices cannot be large when excess demand is small. This will imply that the price adjustment process cannot slow down at prices a long distance from Walrasian equilibrium.

We have the following theorem about the convergence of the continuous-time tâtonnement process to Walrasian equilibrium.

Proposition 12. *Consider the continuous-time tâtonnement process applied to a strongly monotone demand correspondence. The time to convergence to p within an ε-ball*[21] *of a Walrasian equilibrium price p^* is subpolynomial in ε.*

Proof. This involves a simple modification of the classical proof of convergence of tâtonnement for quasilinear economies. Consider the Lyapunov function for the differential equation

$$L(t) = \|p(t) - p^*\|^2.$$

[21] For the analysis of tâtonnement as an algorithm, the ε-neighborhood of p^* is the appropriate subject of study. One reason as to why: p^* may be irrational, and thus we can never expect a computer to converge exactly to p^*.

We have by definition of the tâtonnement process that

$$\frac{dL}{dt} = 2(p(t) - p^*) \cdot \frac{dp}{dt} = 2(p(t) - p^*) \cdot \alpha(D(p(t)) - s).$$

By the definition of strong monotonicity, we then have

$$\frac{dL}{dt} \geq -2\alpha m \|p(t) - p^*\|^2.$$

Solving this differential inequality gives

$$\|p(t) - p^*\| \leq e^{-2\alpha m t}.$$

But then for $t \geq \frac{-1}{2\alpha m} \log(\varepsilon)$, we must have that $\|p(t) - p^*\| \leq \varepsilon$.

$\square$

Despite this, Budish, Cramton, Kyle, Lee, and Malec (2020) find that even under the strong monotonicity assumption, the tâtonnement algorithm may be too slow for practical identification of prices (in their setting, they hope to solve for prices in very large markets once per second). This illustrates the importance of the constant on the practical usefulness of the algorithm. Instead, Budish et al. (2020) find greater success in the use of an interior-point method for the convex program.

C Proofs omitted from the main text

C.1 Proof of Theorem 6

We begin with a helpful lemma.

Lemma 13. *Suppose $d \in D(p)$ and $d' \in D(p')$, with $(d - d') \cdot (p' - p) = 0$. Then $d \in D(p')$ and $d' \in D(p)$.*

Proof. Since $d \in D(p)$, by strong duality, we have from the dual objective associated with supply vector d that

$$\sum_{n \in N} u_n(p) + p \cdot d \leq \sum_{n \in N} u_n(p') + p' \cdot d.$$

Similarly,

$$\sum_{n\in N} u_n(p') + p' \cdot d' \leq \sum_{n\in N} u_n(p) + p \cdot d'.$$

Rearranging, and combining these inequalities, we obtain

$$(p' - p) \cdot d' \leq \sum_{n\in N} u_n(p) - \sum_{n\in N} u_n(p') \leq (p' - p) \cdot d.$$

However, by assumption $(p' - p) \cdot d' = (p' - p) \cdot d$, so that

$$\sum_{n\in N} u_n(p) - \sum_{n\in N} u_n(p') = (p' - p) \cdot d = (p' - p) \cdot d'.$$

Thus, we have that

$$\sum_{n\in N} u_n(p) + p \cdot d = \sum_{n\in N} u_n(p') + p' \cdot d, \text{ and}$$
$$\sum_{n\in N} u_n(p) + p \cdot d' = \sum_{n\in N} u_n(p') + p' \cdot d'.$$

Strong duality then implies that $d \in D(p')$ and $d' \in D(p)$. □

We now proceed to the proof of the Theorem 6.

The sufficiency proof of Theorem 6 follows almost identically to the proof of Theorem 5, except that the demand selections on the left-hand side of (1) are replaced by selections from the total demand correspondence of the base economy, and the number of active buyers N_t^a on the right-hand side of (2) is replaced by the number of replicas k. Since k is $\Theta(|N_k|)$, the conclusion follows.

For necessity, we consider the contrapositive: let $\mathscr{E}$ be a base economy which fails to be strongly monotone and let $D = \sum_{n\in N} D_n(p)$ be its total demand correspondence. Consider any real-valued sequence m_t with $m_t \to 0$, and let sequences of prices p_t, p_t' be such that $p_t \neq p_t'$ and $(d_t - d_t') \cdot (p_t' - p_t) < m_t \|p_t - p_t'\|^2$ for $d_t \in D(p_t)$ and $d_t' \in D(p_t')$ (the existence of such a sequence is assured by the failure of strong monotonicity). By the Bolzano-Weierstrass theorem, it is without loss to assume that $p_t \to p$ and $p_t' \to p'$ for some $p, p' \in \mathscr{P}$. There are two cases:

(1) $p \neq p'$. By Berge's Theorem, D is upper-hemicontinuous so that $d_t \to d \in D(p)$ and $d_t' \to d' \in D(p')$, and we must have $(d - d') \cdot (p' - p) = 0$. Let $s = d$, then p must be a Walrasian equilibrium price in the sequence of economies $\mathscr{E}_k = \langle N_k, kd \rangle$. By Lemma 13, we also have that p' is a Walrasian equilibrium price for $\mathscr{E}_k$.

Without loss of generality,[22] consider any perturbation δs such that $\delta s \cdot (p' - p) > 0$. Note that p' cannot be an equilibrium price of the perturbed economies $\mathscr{E}'_k$ since

$$\sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) - \left(\sum_{n \in N_k} u_n(p') + p' \cdot (kd + \delta s) \right) = \delta s \cdot (p - p') < 0.$$

On the other hand, in the limit as $k \to \infty$, the set of equilibrium prices of $\mathscr{E}'_k$ must approach a (closed, proper) subset of the equilibrium prices of the base economy (also the equilibrium prices of $\mathscr{E}_k$) since

$$\arg\min_p \sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) = \arg\min_p \sum_{n \in N} u_n(p) + p \cdot \left(d + \frac{\delta s}{k} \right),$$

and the objective $\sum_{n \in N} u_n(p) + p \cdot (d + \delta s / k)$ epi-converges (as $k \to \infty$) to the objective of the unperturbed base economy, so that Theorem 7.33 of Rockafellar and Wets (2009) applies. But then p' is a Walrasian equilibrium price of $\mathscr{E}_k$ but not $\mathscr{E}'_k$, and $d_H(P_k, P'_k) \not\to 0$, so cannot be $O(1/|N_k|)$.

(2) $p = p'$. It suffices to consider the case when for all t, $d_t \neq d$, otherwise the argument in the previous case works as well. So, without loss of generality (restricting to a subsequence if necessary), assume that $p_t \to p$ and $p'_t \to p$ in such a way that the angle between $p'_t - p_t$ and $d'_t - d_t$ converges to a constant. For now, let us assume that $\sum_{n \in N} u_n(p) + d \cdot p$ is twice continuously differentiable at p. In this case, we have by assumption that

$$\lim_{t \to \infty} \frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} = 0$$

and this limit is the (negative of the) second directional derivative[23] of $\sum_{n \in N} u_n(p)$ at p in the limiting direction of $p'_t - p_t$. That is, the failure of strong convexity implies a zero second derivative of the objective in some direction at some point.

We now argue that the limiting angle between $d'_t - d_t$ and $p_t - p'_t$ cannot be $90°$ (that is, the demand change cannot approach orthogonality with the price change). To see this, without loss of generality (changing orthonormal coordinates if necessary) suppose that $p_t - p'_t$ approaches unit vector in the direction of the first coordinate (say p_x) and $d_t - d'_t$ approaches

[22] Relabeling p, p' if necessary.

[23] Recall that the second directional derivative of a C^2 function at p in the direction of u is $u' D^2 f(p) u$. This is easily seen to equal the limit above.

the unit vector in the direction of the second coordinate (say p_y). In this case, we must have that $\frac{\partial d_x}{\partial p_x}(p) = 0$ and $\frac{\partial d_y}{\partial p_x}(p) \neq 0$. But then, by symmetry of the Slutsky matrix (equivalently, recognizing that these are mixed partials in the same coordinates and by Schwarz's Theorem), we must have $\frac{\partial d_x}{\partial p_y}(p) \neq 0$. But this would imply that the Hessian of the objective at p is not positive semidefinite, which contradicts the convexity of the objective.

Thus, we have that $(d_t - d'_t) \cdot (p'_t - p_t) \geq c\|d_t - d'_t\|\|p'_t - p_t\|$ for some $c > 0$ and for all t. By restricting to a subsequence if necessary, we may take $\|d_t - d'_t\| = O(1/N)$ and then the only way that

$$\frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} \geq \frac{c\|d_t - d'_t\|}{\|p_t - p'_t\|}$$

can tend to zero is if $\|p_t - p'_t\| \geq \Omega(1/N)$.

This argument is easily adapted to the case that the objective is not twice continuously differentiable replacing the second-order directional derivative above with the generalized second-order directional derivative of Yang and Jeyakumar (1992) and replacing the use of Schwartz's Theorem above by noting that the generalized Hessian of the objective in the direction of the limiting price change must be non-symmetric, contradicting the symmetry established by Hiriart-Urruty, Strodiot, and Nguyen (1984).

C.2 Proof of Theorem 7

Notation We write $V(p) = \sum_{n \in N} u_n(p)$ for the realized total indirect utility and $D(p) = \partial V(p)$ for the realized demand correspondence. Recall that P is the set of minimizers of (D) which has the objective $V(p) + s \cdot p$, while P' is the set of minimizers of the objective $V(p) + (s + \delta s) \cdot p$. We will abuse notation to write inequalities like $\|D(p) - D(p')\| \geq \|\delta s\|$ as shorthand for $\|d - d'\| \geq \|\delta s\|$ for all $d \in D(p)$ and $d' \in D(p')$.

Proof approach Consider economy $\mathscr{E} = \langle N, s \rangle$ obtained by drawing $N := |N|$ buyers from distribution ν over $\mathscr{V}$ which satisfies the conditions of Theorem 7. We will prove the slightly stronger claim that $d_H(P, P')$ is $O_p\left(1/N^{1-\varepsilon}\right)$ for all $\varepsilon > 0$. [24] Then the fact that $d_H(P, P')$ is bounded (and therefore uniformly integrable) will imply the claim that $\mathbb{E}_{\mathscr{E} \sim \nu^N}[d_H(P, P')]$ is $O\left(1/N^{1-\varepsilon}\right)$.

[24]Recall the 'big O in probability' notation: a random variable $X(t)$ is $O_p(f(t))$ if there exists an $c > 0$ such that $\lim_{t \to \infty} \Pr\left[\left|\frac{X(t)}{f(t)}\right| > c\right] = 0$.

Our approach will be to show that with high probability (henceforth, w.h.p.) [25] over draws of the economy $\mathscr{E}$, that for *all* price vectors p with $\text{dist}(p, P) > c/N^{1-\varepsilon}$ (for a constant c to be chosen later), we have that $\|D(p) - s\| > \|\delta s\|$. That is, w.h.p., the demand at prices p outside a neighborhood of size $c/N^{1-\varepsilon}$ from P must differ (in magnitude) from the supply vector s by more than the size of the perturbation $\|\delta s\|$. This will imply on that measure of economies that any price in P' must be within distance $c/N^{1-\varepsilon}$ of P, so that w.h.p. $d_H(P, P')$ will be less than $\frac{c}{N^{1-\varepsilon}}$.

Before completing the proof, we offer some high-level intuition for our approach and divide the proof into a number of steps.

1. **Concentration**: For any fixed p, p' at a distance of $c/N^{1-\varepsilon}$, we show using the Bernstein Inequality that w.h.p. $M(p, p') := \min_{d \in D(p), d' \in D(p')}(d - d') \cdot (p' - p)$ is at least $\frac{mN}{2}\|p - p'\|^2$. That is, w.h.p., the definition of strong monotonicity with constant $m/2$ holds for fixed prices p, p'. This will imply via the Cauchy-Schwarz Inequality logic used in Theorem 5 that for large enough c, w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for $k > 1$.

2. **Extension to discretized sphere**: Fixing p, we then extend the result that $\|D(p) - D(p')\| > k\|\delta s\|$ to *all* prices p' at distance of at least $c/N^{1-\varepsilon}$. To do so, we first discretize the $c/N^{1-\varepsilon}$ unit sphere and employ a union bound, which critically relies on the subexponential tail bound obtained from the Bernstein Inequality in Step 1.[26]

3. **Extension to sphere via regularization**: We then extend the result to the full $c/N^{1-\varepsilon}-$sphere centered at p under the assumption that the *realized* correspondence is Lipschitz continuous. At the end of the proof (in the paragraph titled *Regularization*), we show that this additional assumption is without loss of generality because in economies with non-Lipschitz demand correspondences, it is possible to analyze a *regularized* version of the economy with Lipschitz demand for which $d_H(P, P')$ is approximately (up to $o\left(1/N^{1-\varepsilon}\right)$) equal to the original economy.

4. **Extension to exterior of sphere**: Using convexity, we then show that this implies $\|D(p) - D(p')\| > k\|\delta s\|$ for all p' with distance at least $c/N^{1-\varepsilon}$ from p.

5. **Uniformization over** p: Finally, we extend the result of Step 4 to a fine grid of prices over $\mathscr{P}$,

[25] Throughout, we will use the term 'high probability' to refer to a probability that tends to 1 as $N \to \infty$. Then, X_t is $O_p(f(t))$ if $\left|\frac{X(t)}{f(t)}\right| < c$ w.h.p..

[26] This is why the Bernstein Inequality is used rather than the simpler Chebyshev's Inequality, which is sufficient to obtain result in Step 1.

and use the Lipschitzian property of demand in the regularized economy to establish that w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for all p, p' at distance of at least $c/N^{1-\varepsilon}$. This concludes the proof.

We now fill in the details in these steps to complete the proof.

Step 1: Concentration Consider any fixed $p, p' \in \mathscr{P}$ with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. Define for each $n \in N$, $M_n(p, p') = \min_{d \in D_n(p), d' \in D_n(p')} (d - d') \cdot (p' - p)$ and let $M(p, p') = \sum_{n \in N} M_n(p, p')$. By the strong convexity assumption, $M_n(p, p')$ is a real-valued random variable satisfying $\mathbb{E}_\nu \left[M_n(p, p')\right] \geq m\|p - p'\|^2$. It will help us to write $\mu_{p,p'} := \mathbb{E}_\nu \left[M_n(p, p')\right]$.

We will apply the Bernstein Inequality[27]: given independent real-valued random variables $X_1, X_2, ..., X_N$ with $|X_i| \leq B$, we have

$$\Pr\left[\left|\sum_i X_i - \sum_i \mathbb{E}[X_i]\right| \geq t\right] \leq 2\exp\left(\frac{-\frac{1}{2}t^2}{\sum_i \mathbb{E}[X_i^2] + \frac{1}{3}Bt}\right).$$

To apply the Bernstein Inequality to $M_n(p, p')$, we require an estimate of the second moment of $M_n(p, p')$. We use the Bhatia and Davis (2000) inequality to obtain an upper bound: for any real-valued random variable X with mean μ and $m \leq X \leq M$ a.s., $\text{Var}[X] \leq (M - \mu)(\mu - m)$. Since $M_n(p, p')$ is bounded below by zero (by the monotonicity of d_n) and $M_n(p, p')$ is bounded a.s. above by $2X_{max}\|p - p'\|$ (using the Cauchy-Schwarz inequality), we have that

$$\mathbb{E}_\nu[M_n(p, p')^2] \leq 2X_{max}\|p - p'\|\mu_{p,p'}.$$

Thus, applying the Bernstein Inequality to $M_n(p, p')$, we obtain

$$\begin{aligned}
\Pr\left[M(p, p') \geq \frac{1}{2}N\mu_{p,p'}\right] &\geq 1 - 2\exp\left(\frac{-\frac{1}{8}N^2\mu_{p,p'}^2}{2NX_{max}\|p - p'\|\mu_{p,p'} + \frac{1}{3}NX_{max}\|p - p'\|\mu_{p,p'}}\right) \\
&= 1 - 2\exp\left(\frac{-3N\mu_{p,p'}}{56X_{max}\|p - p'\|}\right).
\end{aligned}$$

[27] See, for example Boucheron, Lugosi, and Massart (2013).

Since $\mu_{p,p'} \geq m\|p-p'\|^2$ and $\|p-p'\| \geq c/N^{1-\varepsilon}$, we have

$$\Pr\left[M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2\right] \geq 1-2\exp\left(\frac{-3Nm\|p-p'\|^2}{56X_{max}\|p-p'\|}\right)$$
$$\geq 1-2\exp\left(\frac{-3cN^{\varepsilon}m}{56X_{max}}\right)$$

The above probability tends to 1 as $N \to \infty$. Note that the event $M(p,p') \geq \frac{mN}{2}\|p-p'\|^2$ for $\|p-p'\| = \frac{2k\|\delta s\|}{mN^{1-\varepsilon}}$ (that is, $c = 2k\|\delta s\|/m$, in our previous notation) is equivalent to the event that $(D(p)-D(p'))\cdot(p-p') \geq k\|\delta s\|N^{\varepsilon}\|p-p'\|$. By the Cauchy-Schwarz Inequality, this implies $\|d-d'\| \geq k\|\delta s\|N^{\varepsilon}$. For $k>1$ and sufficiently large N, if $p \in P$, this implies the event that p' could not be in P'. In later arguments, it will help to choose k larger than 1 to leave room for other approximations.

Step 2: Extension to discretized sphere With the same fixed p as in Step 1, we now consider $\mathbb{S}_c(p)$, the $c/N^{1-\varepsilon}$-sphere around p. By standard covering arguments, it is possible to identify $O(N^{(3+\varepsilon)L})$ points on the sphere of radius $O(1/N^{1-\varepsilon})$ such that the distance between each pair is at most $O(N^{-4})$. Let such a discretization be $\mathbb{D}_c(p)$.

Note that the number of pairs p, p' with $p' \in \mathbb{D}_c(p)$ is polynomial in N. A union bound over the events $M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2$ over $p' \in \mathbb{D}_c(p)$ thus implies

$$\Pr\left[M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2 \textbf{ for all } p' \in \mathbb{D}_c(p)\right] \geq 1-2O(N^{(3+\varepsilon)L})\exp\left(\frac{-3cN^{\varepsilon}m}{56X_{max}}\right),$$

which also tends to 1 as $N \to \infty$. Thus $\|D(p)-D(p')\| > k\|\delta s\|$ for all p' in $\mathbb{D}_c(p)$ w.h.p. for large enough N (where again, we have set $c = 2k\|\delta s\|/m$ in the above).

Assumption: In Steps 3 and 5, we assume that the realized demand correspondence is $O(N^2)$-Lipschitzian. We justify this assumption in our discussion on regularization below.

Step 3: Extension to sphere via regularization Consider $p'' \in \mathbb{S}_c(p) \setminus \mathbb{D}_c(p)$. Since p'' is at a distance of at most $O(N^{-4})$ from p' in $\mathbb{D}_c(p)$ and $(D(p)-D(p'))\cdot(p-p') \geq k\|\delta s\|N^{\varepsilon}\|p-p'\|$

w.h.p. for all p' in $\mathbb{D}_c(p)$, using the Cauchy Schwarz inequality, we obtain

$$
\begin{aligned}
&(D(p) - D(p'')) \cdot (p - p'') \\
&\geq (D(p) - D(p')) \cdot (p - p') - \|D(p') - D(p'')\| \|p - p'\| - \|D(p) - D(p')\| \|p' - p''\| \\
&\geq k\|\delta s\| N^{\varepsilon} \|p - p'\| - O(N^2) \cdot O(N^{-4}) \|p - p'\| - \|D(p) - D(p')\| O(N^{-4}) \\
&\geq k' \|\delta s\| N^{\varepsilon} \|p - p'\|
\end{aligned}
$$

for any $k' < k$ and sufficiently large N, where the second line uses the $O(N^2)$-Lipschitz property of demand. This implies that all $p'' \in \mathbb{S}_c(p)$ have $\|D(p'') - D(p)\| \geq k'\|\delta s\|$ w.h.p. for sufficiently large N.

Step 4: Extension to exterior of sphere Now let p'' be a point outside of $\mathbb{S}_c(p)$ and let p' be the point on $\mathbb{S}_c(p)$ which is on the line between p and p''. By convexity, we have for all $d' \in D(p')$ and $d'' \in D(p'')$ that $(d'' - d') \cdot (p' - p'') \geq 0$. Since $p' - p'' = \frac{\|p' - p''\|}{\|p - p'\|}(p - p')$, we also have $(d'' - d') \cdot (p - p') \geq 0$. But then since for all $d' \in D(p')$, we have $(d' - d) \cdot (p - p') \geq k'\|\delta s\| N^{\varepsilon} \|p - p'\|$ w.h.p., by adding the previous expression, we obtain $(d'' - d) \cdot (p - p') \geq k'\|\delta s\| N^{\varepsilon}\|p - p'\|$ with the same probability. But this implies that $\|d'' - d\| \geq k'\|\delta s\| N^{\varepsilon}$, as required. Thus, we have for all p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$ that $\|d - d'\| \geq k'\|\delta s\| N^{\varepsilon}$ with high probability.

Step 5: Uniformization over p Up until now, we have held p fixed, but we now wish to extend the conclusion of Step 4 above to any realized $p \in P$. To do so, we apply another discretization of $\mathscr{P}$ with points at distance $\Theta(N^{-4})$. Again, by standard covering arguments (since $\mathscr{P}$ is compact), $O(N^{4L})$ points are required for such a covering of $\mathscr{P}$. We may again apply a union bound to obtain the conclusions of Step 4 for *all* p in the discretization. Because the realized demand is $O(N^2)$-Lipschitz, this implies (via the same logic as in Step 3) the same result for $p \in P$ not in the covering for sufficiently large N.

This implies that with probability approaching 1 as $N \to \infty$, for any $p \in P$ and p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$, that $\|s - d'\| \geq k'\|\delta s\| N^{\varepsilon}$ for all $d' \in D(p')$. By choosing c large enough for $k' > 1$, we then have $\|s - d'\| > \delta s$ for all $d' \in D(p')$, which means p' cannot be in P'. This implies that $d_H(P, P') < \frac{c}{N^{1-\varepsilon}}$ with probability approaching 1 for sufficiently large c, that is the random variable $d_H(P, P')$ is $O_p\left(\frac{1}{N^{1-\varepsilon}}\right)$. Finally, since the distance $d_H(P, P')$ is a bounded random variable, this implies that $\mathbb{E}[d_H(P, P')]$ is $O\left(\frac{1}{N^{1-\varepsilon}}\right)$.

Regularization In Steps 3 and 5 above, we assumed that the realized demand correspondence is an $O(N^2)-$Lipschitz. Here, we show that this assumption is without loss of generality by exploiting the Moreau-Yosida regularization of convex functions. The explicit construction of the Moreau-Yosida approximation will not be important for our argument (although it is not complicated—see, for example, Rockafellar and Wets (2009)), so instead we state the result as an existence theorem.

Proposition 14 (Moreau-Yosida)**.** *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semi-continuous function defined on a convex, compact subset X of a Hilbert space. Then for all $\gamma > 0$, there exists a function $\tilde{f} : X \to \mathbb{R} \cup \{\infty\}$, the γ-*Moreau envelope *of f, with the following properties:*

- *$\tilde{f}$ is convex, $\frac{1}{\gamma}$-Lipschitz-continuous and Fréchet-differentiable with gradient $\nabla \tilde{f}$ which is $\frac{1}{\gamma}$-Lipschitz continuous*
- *f and $\tilde{f}$ have the same minimizers.*

Furthermore, if f is L-Lipschitz continuous, then $\tilde{f}$ is also L-Lipschitz, and for all $x \in X$,

$$\tilde{f}(x) \leq f(x) \leq \tilde{f}(x) + \frac{\gamma L^2}{2}.$$

The inverse mapping of the gradient of $\tilde{f}$ and the inverse mapping of the subdifferential of f are related by

$$(\nabla \tilde{f})^{-1}(x^*) = \gamma x^* + (\partial f)^{-1}(x^*).$$

Note that V is proper, convex and $X_{max}-$Lipschitz, where X_{max} is defined as the maximum magnitude demand vector, $\max_{v_n \in \mathrm{supp}(\nu)} \max_{x \in \mathrm{dom}(v_n)} \|x\|$ (which exists by the assumption of compactness of the consumption possibility sets). The $\frac{1}{N^2}-$Moreau envelope of V, $\tilde{V}$, is thus convex, $\max\{X_{max}, N^2\}$-Lipschitz continuous and Fréchet differentiable with gradient (i.e. demand function) which is N^2-Lipschitz.

We now show that it suffices for us to analyze the $1/N^2$ regularized dual objective. Let $\tilde{P}$ and $\tilde{P}'$ be the unperturbed and perturbed Walrasian prices (respectively) for the regularized demand.

First, we note that $d_H(P, P') = d_H(\tilde{P}, \tilde{P}') + O(1/N^2)$, which implies that if $d_H(\tilde{P}, \tilde{P}')$ is $O(1/N^{1-\varepsilon})$, so is $d_H(P, P')$. To see this, note that $\tilde{P} = (\nabla \tilde{V})^{-1}(s)$, $\tilde{P}' = (\nabla \tilde{V})^{-1}(s + \delta s)$, $P = (\partial V)^{-1}(s)$ and $P' = (\partial V)^{-1}(s + \delta s)$ so that by the last identity in Proposition 14, $\tilde{P} = P + \frac{1}{N^2}s$ and $\tilde{P}' = P' + \frac{1}{N^2}(s + \delta s)$ which, since $\|s\| < X_{max}$ and $\|s + \delta s\| < X_{max}$, implies the first claim.

Second we claim that $\mathbb{E}[\nabla \tilde{V}]$ is m'-strongly monotone for all $m' < m$ and sufficiently large N and $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. To see this, consider the expression $\mathbb{E}[(p - p') \cdot (\tilde{d}' - \tilde{d})]$ for $\tilde{d} = \nabla V(p)$

and $\tilde{d}' = \nabla V(p')$. Note $p \in (\nabla\tilde{V})^{-1}(\tilde{d})$ and $p' \in (\nabla\tilde{V})^{-1}(\tilde{d}')$ so that $p - \frac{1}{N^2}\tilde{d} \in (\partial V)^{-1}(\tilde{d})$ and $p' - \frac{1}{N^2}\tilde{d}' \in (\partial V)^{-1}(\tilde{d}')$. But then

$$\begin{aligned}
\mathbb{E}[(p-p')\cdot(\tilde{d}'-\tilde{d})] &= \mathbb{E}[(p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}')\cdot(\tilde{d}'-\tilde{d}) + \frac{1}{N^2}(\tilde{d}-\tilde{d}')\cdot(\tilde{d}'-\tilde{d})] \\
&\geq m\|p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{m}{N^2}\|\tilde{d}-\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{1+m}{N^2}X_{max}^2,
\end{aligned}$$

where the second line above follow by the strong monotonicity property of ∂V. So, for $\|p-p'\| \geq \frac{c}{N^{1-\varepsilon}}$ the second term is asymptotically dominated by the first, and the claim follows. Together, these two claims imply that we may replicate the arguments in Steps 1-2 above for the regularized demand $\nabla\tilde{V}$ (which is Lipschitz) and Steps 3-5 imply the required result.

C.3 Proof of Corollary 8

In the proof of Theorem 7, we showed that with subexponential probability—in fact, with probability $1 - O(1/N)$, we have that the maximum distance between the price associated with the truthful report of an agent and any alternative report of that agent is $O(1/N^{1-\varepsilon})$. In the complementary $O(1/N)$ measure of draws of economies, we have that the maximum influence on price is $O(1)$, since by assumption the set of possible prices $\mathscr{P}$ is compact. Thus, we have that the expected maximum influence of any report, including the *interim* optimal report, on price is $(1-O(1/N))O(1/N^{1-\varepsilon}) + O(1/N)O(1) = O(1/N^{1-\varepsilon})$

C.4 Proof of Proposition 9

Consider any price change $p \mapsto p'$. For all the demand subtypes δ associated with buyers in $\mathscr{V}$ (note there are finitely many possible subtypes for L goods with bounded demand), we have either that $\delta\cdot(p'-p) = 0$ or $\delta\cdot(p'-p) > 0$. In the first case, p and p' must both lie on the same facet of the LIP, so that demand does not change along p to p'. In the other case, since the number of possible demand subtypes is finite, there is a least $\delta\cdot(p'-p)$ among them: let δ' be that subtype and let $\delta'\cdot(p'-p) = k\|p'-p\|$ for some $k > 0$. (In other words, k is the least product of $\|\delta\|$ among the demand subtypes and the cosine of the angle between δ and $p'-p$. This is why our expression maintains $\|p'-p\|$ as a constant of proportionality.)

For now, suppose that $\alpha\|p - p'\| \leq 1$. In this case, we have that a lower bound on $\mathbb{E}[(D(p) - D(p')) \cdot (p' - p)]$ is given by the $\Pr_\nu[D(p) \neq D(p')]$ multiplied by the least value of $(D(p) - D(p')) \cdot (p' - p)$ conditional on a demand change. Since by assumption $\Pr_\nu[D(p) \neq D(p')] \geq \alpha\|p - p'\|$ and the least value of the projected demand change is $k\|p' - p\|$, we obtain the lower bound on $\mathbb{E}[(D(p) - D(p')) \cdot (p' - p)]$ of $\alpha k\|p' - p\|^2$, which is the required inequality for strong monotonicity.

If $\alpha\|p - p'\| \geq 1$, we can divide the line segment up into pieces $p, p_1, p_2, ..., p_N, p'$ where between p and p_1, p_1 and p_2, p_2 and p_3 etc., demand changes occur with probability 1, and between p_N and p', demand changes occur with at least $\alpha\|p_N - p'\|$. In this case, since the demand changes are lower bounded by the size of an indivisibility, it is still clear that the size of the demand change is proportional to the distance $\|p - p'\|$.