

# LINEAR PRICING MECHANISMS FOR MARKETS WITHOUT CONVEXITY

*Now: Walrasian Mechanisms for Non-Convex Economies and the Bound-Form First Welfare Theorem*

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2. Stanford University. Supported by Koret Fellowship and Ric Weiland Graduate Fellowship.

# quasilinear two-sided exchange economy

$L$  commodities (+ money)

$n = 1, 2, \dots, N$  **buyers** with quasilinear preferences  $v_n(x) - t$ .

**Valuations**  $v_n(x)$  are bounded, nondecreasing and  $v_n(0) = 0$ .

$f = 1, 2, \dots, F$  **sellers** with profit  $t - c_f(y)$ .

**Costs**  $c_f(y)$  are bounded, nondecreasing and  $c_f(0) = 0$ .

**Valuations may be non-concave and costs may be non-convex**

**$\Rightarrow$  Walrasian equilibria (market-clearing prices) may not exist**

Goal: to offer extensions of the **Walrasian mechanism**, sharing its desirable properties, without the assumption of convexity

# non-convexities are pervasive



startup & switching costs, economies of scale, indivisibilities, complementarities, externalities, exposure concerns

# our new mechanisms

## **Markup Mechanisms** $(\alpha, p, \omega)$

Sellers' prices  $p$

Buyers' prices  $(1 + \alpha)p$

Allocation  $\omega$  in supply and demand sets at respective prices, enforcing **physical feasibility**  
production  $\geq$  consumption

and **budget feasibility**

buyer payments  $\geq$  seller payments

*for markets with two-sided non-convexity*

*(there may be no feasible allocations that exactly clear the market)*



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*for markets with two-sided non-convexity*

*(there may be no feasible allocations that exactly clear the market)*

## Rationing Mechanisms $(p, \omega)$

Buyers & sellers both face price  $p$

Allocation  $\omega$  may involve **rationing**: assignments that are not preferred given the price vector  $p$ .

$(p, \omega)$  chosen to satisfy **physical feasibility** and **exact budget balance**

*for markets with one-sided convexity*

*(exact market-clearing may be guaranteed)*

# bound-form first welfare theorem

If no equilibrium exists, any (price, allocation) pair must feature either:

- $\text{supply} > \text{demand} \Rightarrow$  **budget deficit**
- $\text{demand} > \text{supply} \Rightarrow$  **rationing**

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- demand > supply  $\Rightarrow$  **rationing**

Given **any** feasible allocation  $\omega = (x, y)$ , **any** non-negative price vector  $p \in \mathbb{R}_+^L$ , and the surplus function  $S: \Omega \rightarrow \mathbb{R}$ , and an efficient  $\omega^* \in \Omega$ ,

$$S(\omega^*) - S(\omega) \leq p \cdot \left( \sum_f y_f - \sum_n x_n \right) + \left( \sum_f \mathcal{R}_f(p, y_f) + \sum_n \mathcal{R}_n(p, x_n) \right).$$

Welfare loss from  $\omega$     $\leq$    Revenue deficit    $+$    Rationing losses for sellers & buyers

where  $S(\omega) = \sum_n v_n(x_n) - \sum_f c_f(y_f)$  and **rationing losses** are defined as

- $\mathcal{R}_f(p, y_f) =$  maximum profit at price  $p$  minus profit at  $y_f$  given  $p$
- $\mathcal{R}_n(p, x_n) =$  maximum utility at price  $p$  minus utility at  $x_n$  given  $p$

# markup mechanisms: results

## Markup Mechanisms $(\alpha, p, \omega)$

Sellers' prices  $p$

Buyers' prices  $(1 + \alpha)p$

$\omega$  in supply and demand sets  
at respective prices

Enforcing **physical feasibility**  
and **budget feasibility**

## Large market assumptions:

- Number of buyers  $N$  and sellers  $\rightarrow \infty$  with  $\frac{\# \text{ buyers}}{\# \text{ sellers}} \rightarrow \phi$
- Non-convexities are bounded by  $\delta \sim O(1)$ 
  - Radius of (non-convex) demand/supply set
- Growing gains from trade: efficient surplus  $\Omega(N)$
- Set of approximately clearing prices is bounded



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**Theorem:** There exists a markup mechanism  $(\alpha, p, \omega)$  with  $\alpha \sim O(1/N)$ , which is:

1. Nearly efficient: proportion of surplus lost cf. efficient allocation is  $O(\delta/N)$ .
2. Nearly incentive-compatible\*: maximum gain to a misreport is approx.  $O(1/N)$
3. Easy to compute\* (using convex optimization & bisection search for  $\alpha$ ).

\* Under additional assumption of strong monotonicity (as in Walrasian mechanism!)

# proof idea: finding $(\alpha, p, \omega)$

1. For a **fixed**  $\alpha$

- Convexify economy
  - Scale down buyer values to  $\frac{1}{1+\alpha} \text{cav}(u(x))$
  - Add  $\delta L$  units of ‘auctioneer demand’ for each good
- Find Walrasian equilibrium  $(p, \omega^*)$ .

Worst case allowance to be  
adjusted in practice

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Distance from  $\sum S_n$  to  $\sum \text{co}(S_n)$  is no more than  $\delta L$

2. ‘**Round**’: use Shapley-Folkman Theorem to find feasible allocation in demand and supply sets at  $p$  with  $||\hat{\omega} - \omega^*|| \leq \delta L$ .

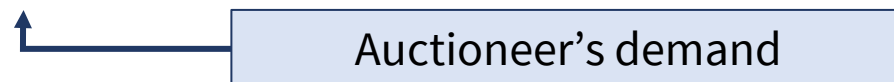
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3. Search to find the smallest  $\alpha \geq 0$  such that budgets balance.  
 $\alpha$  needs to be large enough to cover excess supply  $\sim 1$  buyer’s demand  $\Rightarrow O(1/N)$



# proof idea: approximate efficiency

Use **bound-form first welfare theorem** with  $(p, \omega)$ .

$$\text{loss of } \omega \leq \text{rationing losses} + \text{budget deficits}$$

Each seller's rationing loss at  $p$  is 0.

Each buyer's rationing loss at  $p$  is  $o(\alpha) = o(1/N)$  via envelope theorem.

- $N$  buyers implies  $o(1)$  total loss due to rationing.

Budget deficit from excess supply is  $O(1)$ .

Leads to  $O(1)$  total loss

# incentives: strong monotonicity (Watt, 2022)

For simplicity, suppose just 1 good. Demand is **strongly monotone** if there exists  $m > 0$  such that  $\frac{\partial D}{\partial p} \leq -m < 0$  for all  $p$  such that  $D(p) \neq \{0\}$ .\*

Strongly monotone supply is defined analogously.

\* With  $L$  goods:  $(d - d') \cdot (p' - p) \geq \mu \|p - p'\|^2$  for all  $d \in D(p), d' \in D(p')$



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Walrasian and markup mechanisms are **ex post  $O(1/N)$  –incentive compatible** if *each* agent has strongly monotone preferences.

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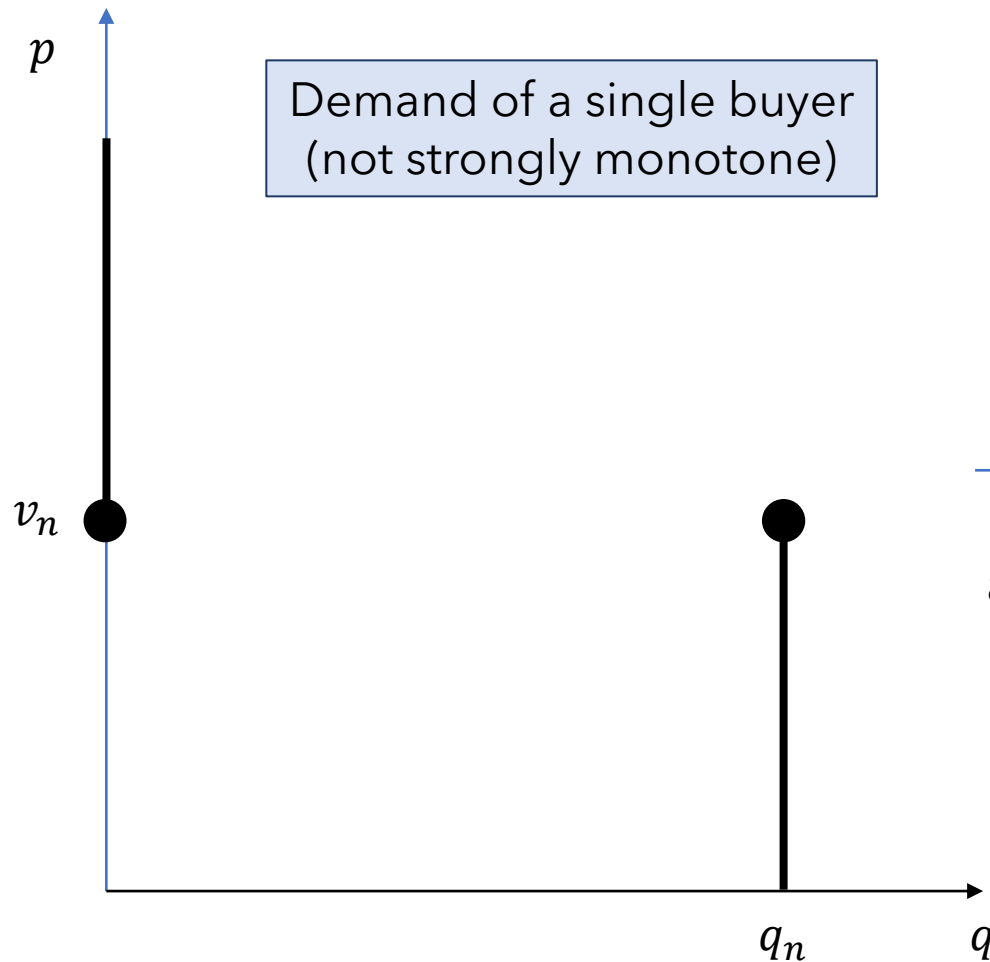
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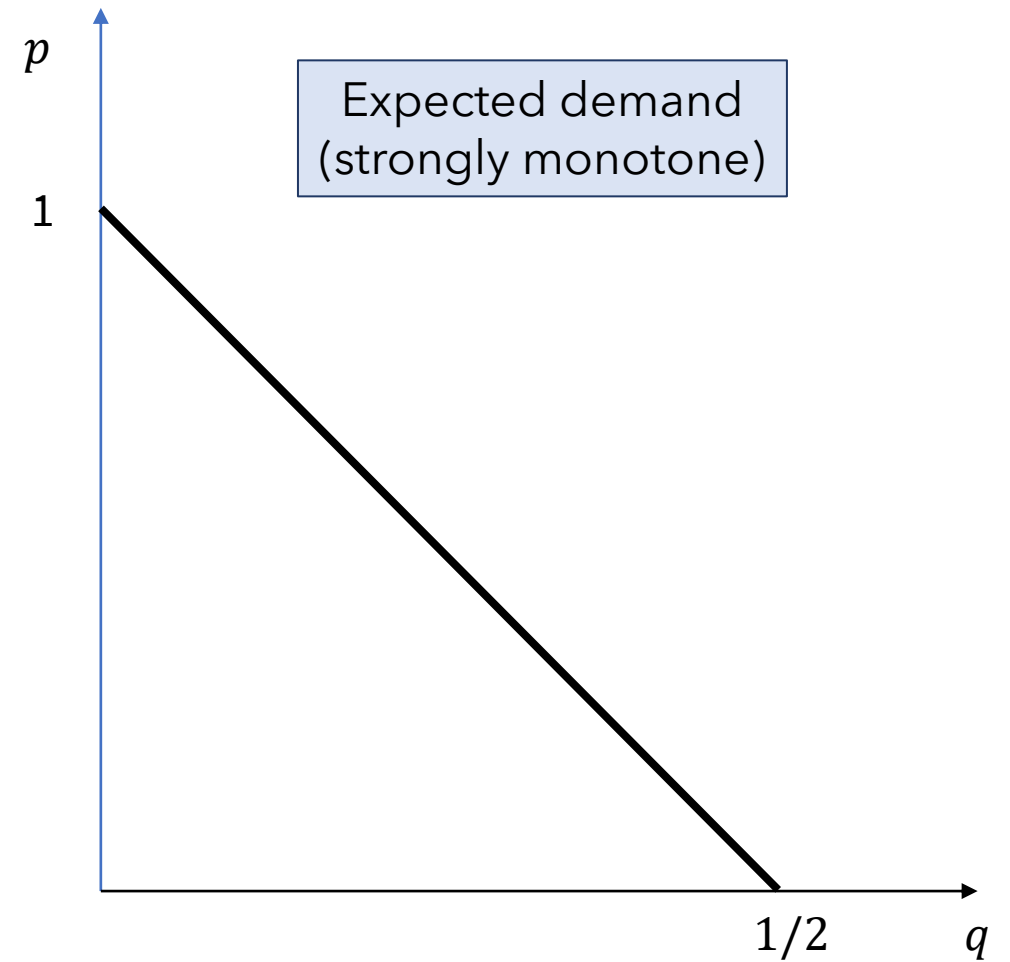
Moreover, if agents are drawn i.i.d. from distribution for which the *expected* demand/supply are strongly monotone, the Walrasian and markup mechanisms are **ex post  $O_p(1/N^{1-\varepsilon})$ -incentive-compatible**.

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# strong monotonicity of buyers



*in expectation* →  
if  $v_n \sim \text{Unif}[0,1]$ ,  
and  $q_n \sim \text{Unif}[0,1]$



# rationing mechanism: results

## **Rationing Mechanism** $(p, \omega)$

Buyers' and sellers' prices  $p$

Allocation  $\omega$  may involve **rationing**: assignments that are not preferred given the price vector  $p$ .

$(p, \omega)$  chosen to satisfy **physical feasibility** and **budget feasibility**

## **Large market assumptions:**

As previously

## **Additional assumption:**

Buyers have convex preferences, and (expected) demand is *strongly monotone*.

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**Theorem:** There exists a rationing mechanism  $(p, \omega)$  which is:

1. Nearly efficient: **total** surplus lost c.f. efficient allocation is  $O(\delta/N)$ .
2. Nearly incentive-compatible\*: maximum gain to a misreport is approx.  $O(1/N)$
3. Easy to compute\*: using two convex optimizations.

# rationing mechanism: proof idea

## Rationing mechanism

1. Find market-clearing prices  $p$  in convexified economy with  $\rho L$  units of **additional fictitious auctioneer supply** of each good.
2. Round sellers' allocation  $y \rightarrow$  feasible allocation **with excess demand**.
3. Reoptimize buyers given fixed supply from step 2 to determine buyer allocations  $x$ .

Advantage:  $O(1/N)$  **total** loss.      Disadvantage: Only  $O(1/N)$ -IR for buyers.

**Intuition:** Price  $p$  and shadow price for buyers are  $O(1/N)$  close by strong monotonicity so that the rationed loss of buyers is also  $O(1/N)$ .



# conclusion

Our **bound-form first welfare theorem** relates deadweight losses to rationing.

Our **markup mechanism** generalizes the Walrasian mechanism to work with non-convexities and provides:

- Approximate efficiency
- Good incentives
- Individually rational
- Easy computations
- Linear pricing
- No budget deficit

With **one-sided convexity** and strong monotonicity, our **rationing mechanism** offers most of the same features, with a smaller loss but (potentially) some small violations of individual rationality.

# **appendix: example**

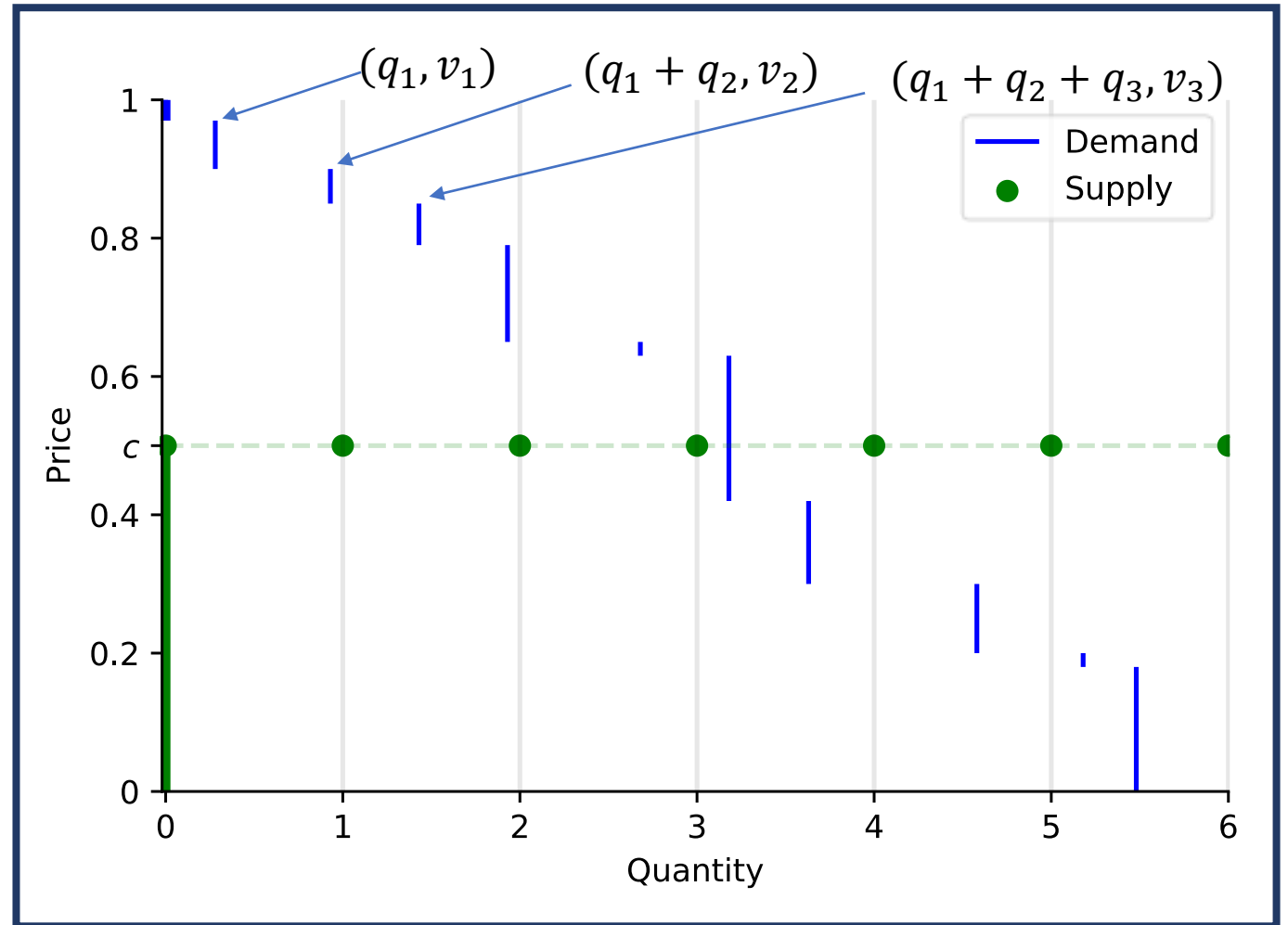
# a non-convex market

Money + one other good.

One seller, who can produce any **integer** quantity of the good at marginal cost  $c = \$0.50$ .

Buyers  $n = 1, \dots, N$ .

Buyer  $n$  has use only for **exactly**  $q_n \in \mathbb{R}_+$  units with total value  $q_n v_n$ , so  $v_n$  per unit

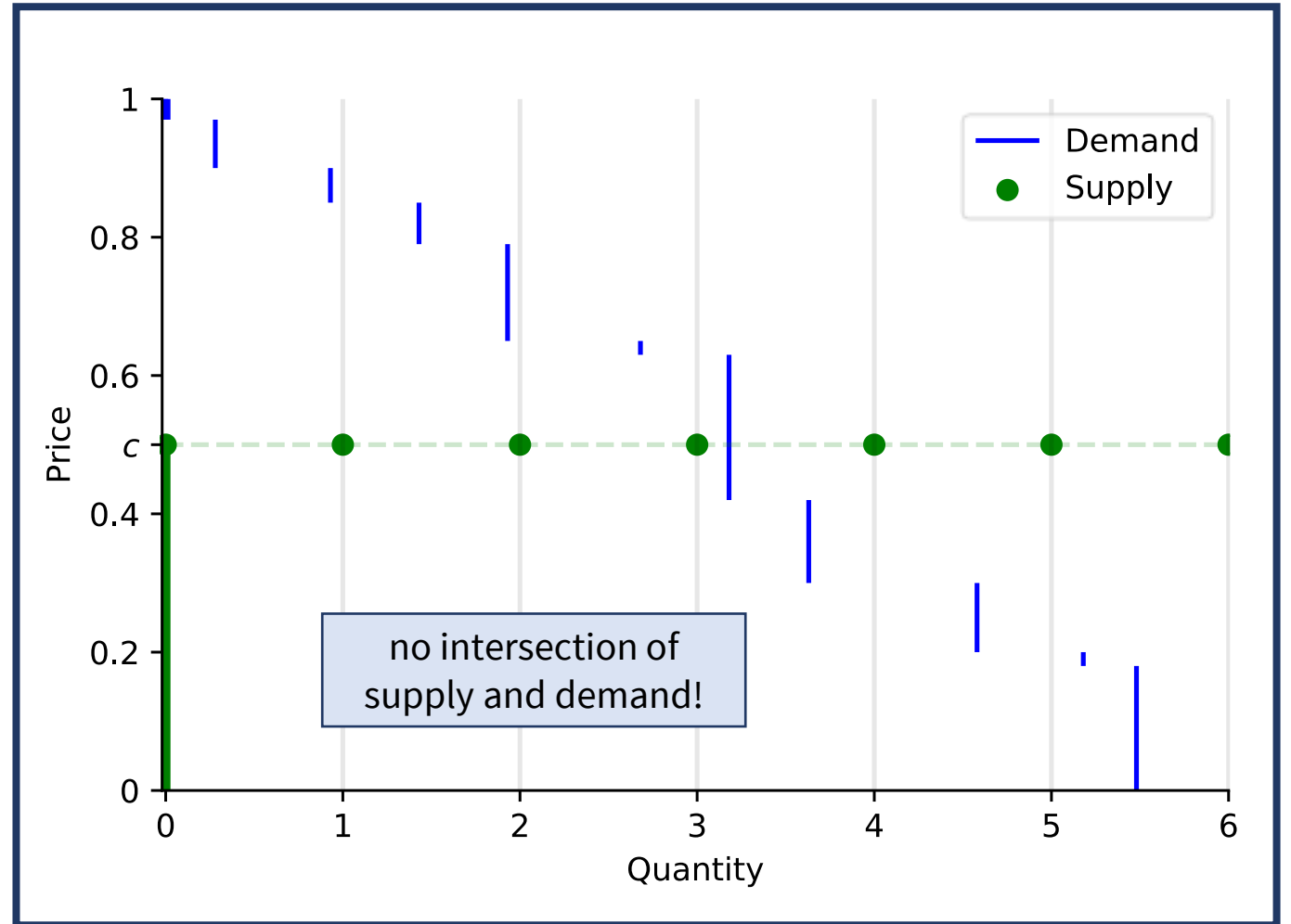


# Walrasian equilibrium does not exist

## Problem

non-existence of Walrasian equilibrium

*convexify?*



# pseudoequilibrium via convexification

Starr (1969), Shapley-Folkman, Heller (1972),  
Nguyen & Vohra (2021)

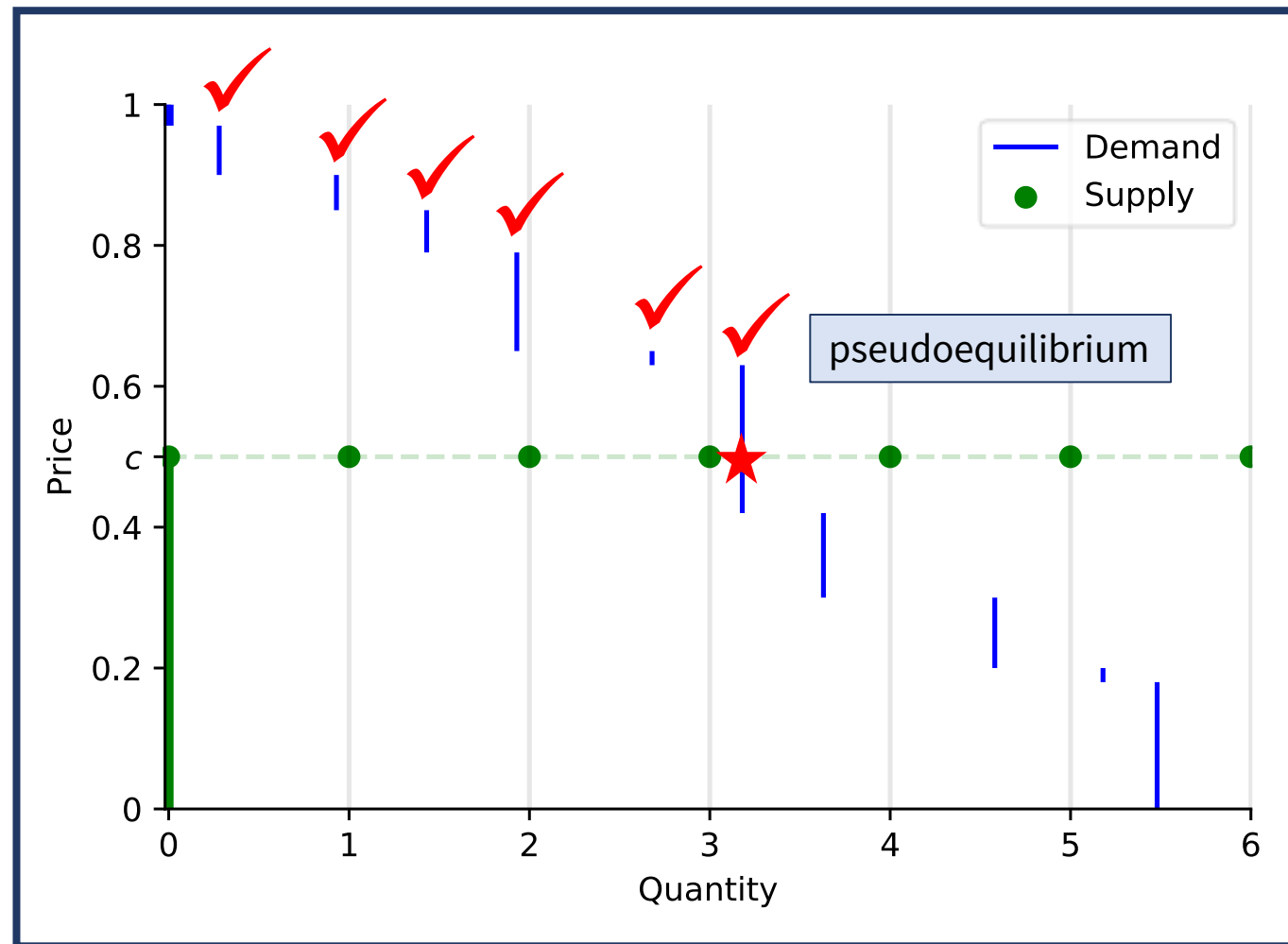
Convexify costs and values  
and find resulting WE.

At most  $L$  agents are assigned  
non-preferred bundles.

## Problems

infeasible assignments

*round allocations?*



# approximate equilibrium via Shapley-Folkman

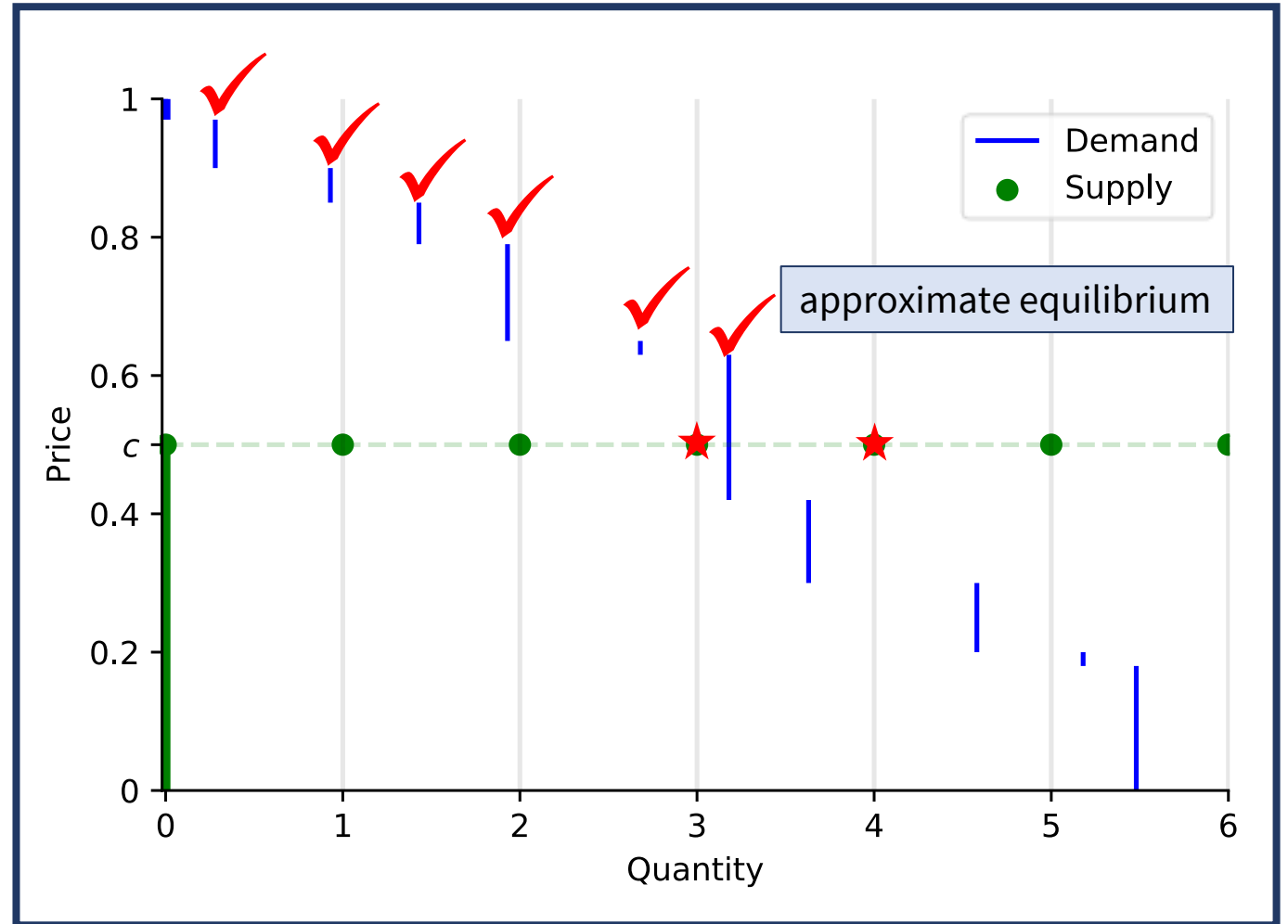
Starr (1969), Shapley-Folkman, Heller (1972),  
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There exists  $(p, \omega)$  such  
that supply  $\approx$  demand, but  
not exactly equal.

## Problems

demand > supply and/or  
payments > receipts

*ration some agents?*





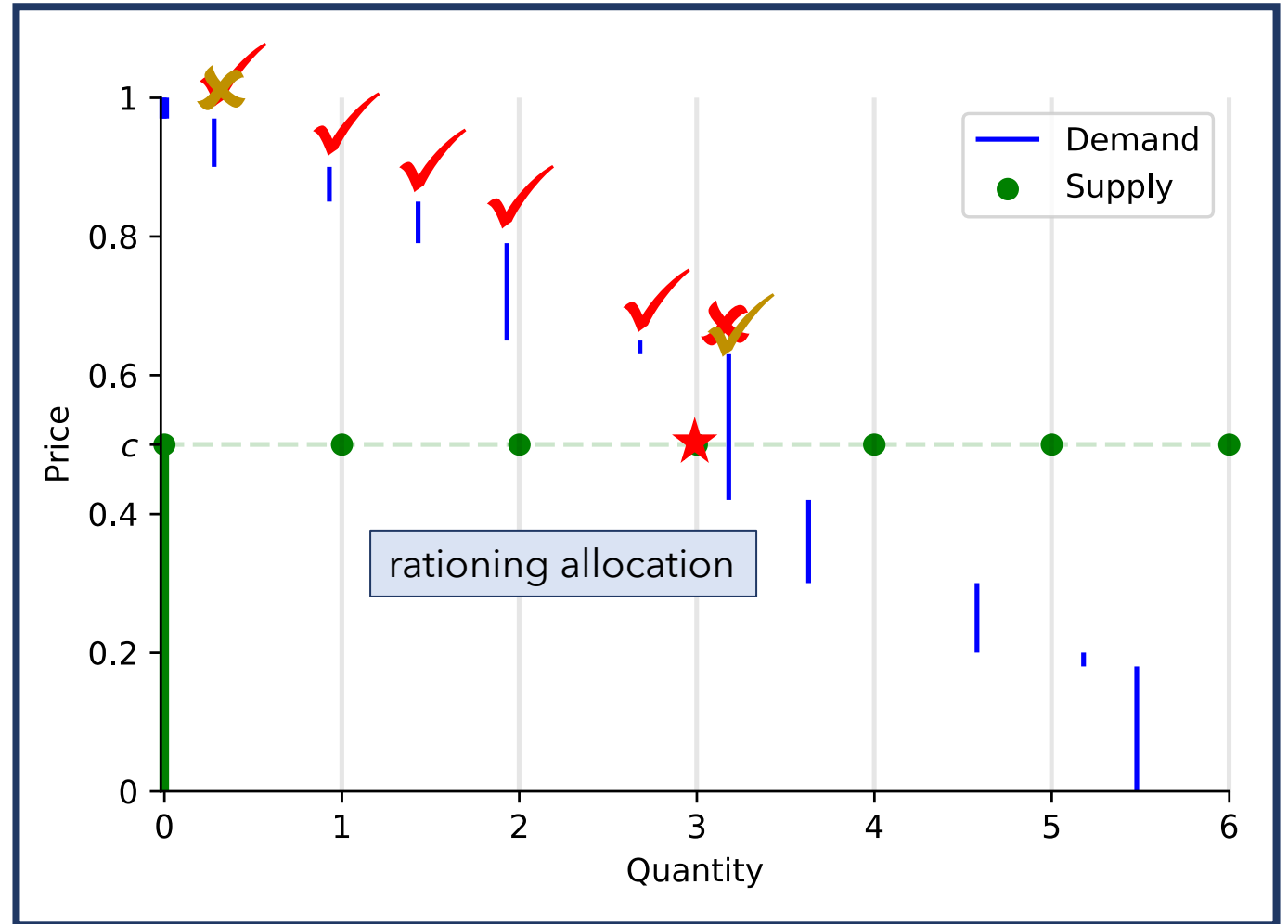
# rationing in two-sided non-convex markets

Rationing = deny some agents their preferred bundles

## Problems

budget deficit  
who should be rationed?  
new incentive to misreport to avoid rationing

*abandon prices?*



# efficient allocation problem

Solve exactly the efficient allocation problem.

## Problems

budget deficit or bad incentives, computational challenge.

...add inefficient entry, exit & integration decisions, political resistance

