Topping Up and Optimal Subsidies

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Introduction

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This paper: we characterize **optimal nonlinear subsidy programs** in presence of private markets.

Model

Model Overview

Eligible Consumers

type $\theta \sim F \implies$ demand $D(p, \theta)$

Producers

constant marginal cost c

▸ Model Details

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maximizing weighted total surplus:

$\omega(\theta)$ weight on type θ's consumer surplus

α opportunity cost of subsidy spending

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costless contracting

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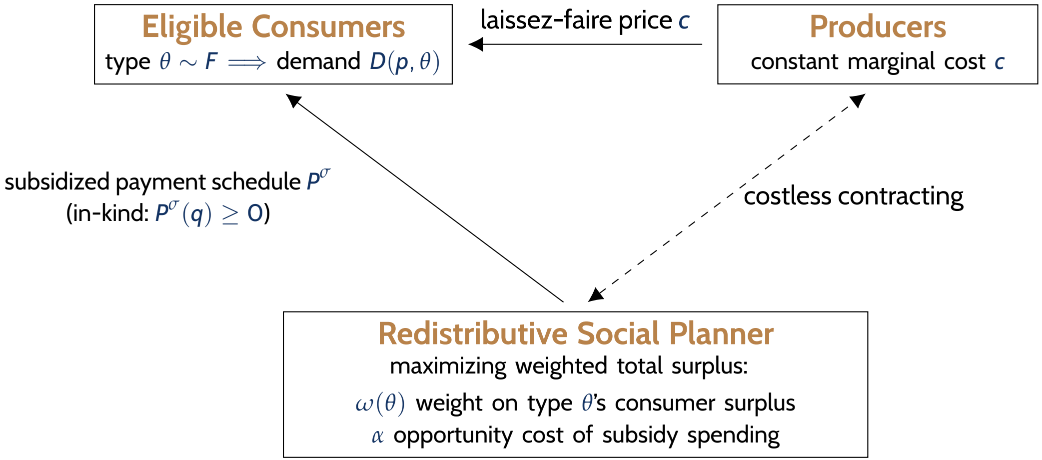

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subsidized payment schedule P^{σ}
(in-kind: $P^{\sigma}(q) \geq 0$)

costless contracting

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"Topping Up": Consumers can purchase from **both subsidized program and private market.**

"Opting Out": Consumers must **choose between** subsidized or private market allocation.

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Topping Up vs. No Topping Up

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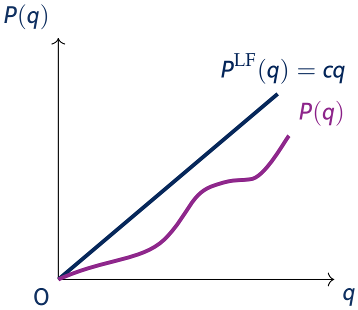

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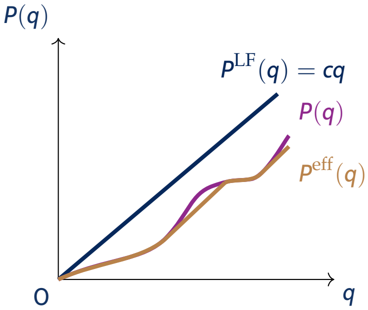

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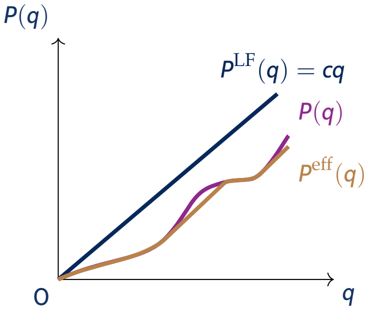

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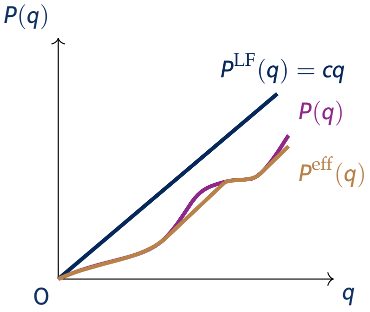

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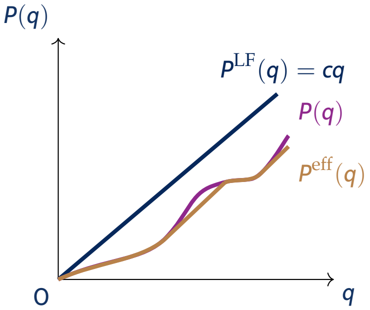

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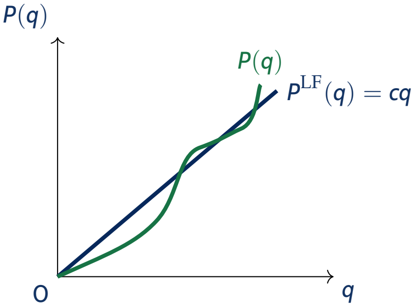

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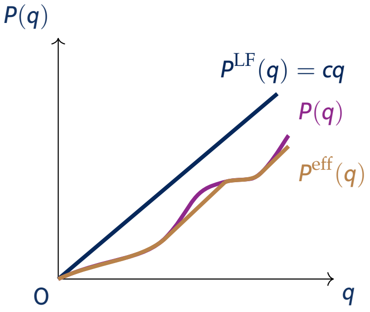

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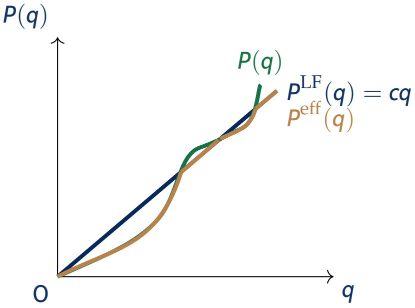

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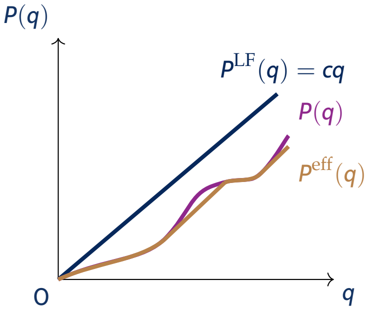

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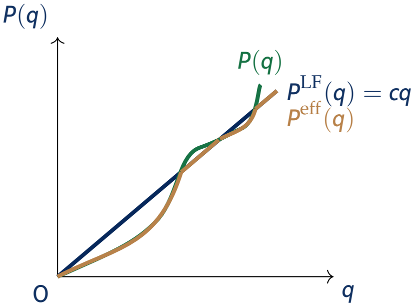

Direct mechanism: $P(q) \leq cq$

$\iff U(\theta) \geq U^{\mathrm{LF}}(\theta)$

$\iff \Sigma(q) \geq 0$.

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{(q,t)} \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega(\theta) \underbrace{\left[\theta v(q(\theta)) - t(\theta)\right]}_{\text{consumer surplus}} - \alpha \underbrace{\left[c q(\theta) - t(\theta)\right]}_{\text{net cost}} \right] dF(\theta),$$

subject to

- incentive compactibility, $\theta \in \arg\max_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} \left[\theta v(q(\hat{\theta})) - t(\hat{\theta})\right] \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \quad$ (IC)
- no lump-sum transfers, $t(\theta) \geq 0 \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \quad$ (LS)
- individual rationality, $\theta v(q(\theta)) - t(\theta) \geq U^{LF}(\theta) \quad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \quad$ (IR)
- topping up constraint, $q(\theta) \geq q^{LF}(\theta) \quad \forall \theta \in [\underline{\theta}, \overline{\theta}]. \quad$ (TU)

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{q \text{ non-decreasing}} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{[J(\theta)v(q(\theta)) - cq(\theta)]}_{\text{surplus of virtual type}} \, dF(\theta) + (\text{terms independent of } q),$$

subject to

with topping up:

$$q(\theta) \geq q^{\text{LF}}(\theta) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{FOSD}$$

without topping up:

$$\underline{U} + \int_{\underline{\theta}}^{\theta} v(q(s))ds \geq U^{\text{LF}}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q^{\text{LF}}(s))ds, \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]. \tag{SOSD}$$

Here, the virtual type is $J(\theta) = \underbrace{\theta}_{\text{efficiency}} + \underbrace{\frac{\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] dF(s)}{\alpha f(\theta)}}_{\text{redistributive motive}} + \underbrace{\frac{\max\{\mathbf{E}_{\theta}[\omega(\theta) - \alpha], 0\} \underline{\theta} \delta_{\theta = \underline{\theta}}}{\alpha f(\theta)}}_{\text{(LS) constraint at } \underline{\theta}}.$

Correlation Assumption

Redistributive motive $\propto \int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] \, \mathrm{d}F(s)$ depends on average welfare weight of types $\geq \theta$.

Two baseline cases:

"Negative Correlation": $\omega(\theta)$ is decreasing in θ.

- high-demand consumers tend to have lower need for redistribution.
- *e.g.*, food, education, and, if $\omega \propto 1/\text{Income}$, **normal** goods.

$\rightsquigarrow$ virtual surplus $J(\theta)$ is (positive then) negative.

"Positive Correlation": $\omega(\theta)$ is increasing in θ.

- high-demand consumers tend to have higher need for redistribution.
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When To Subsidize

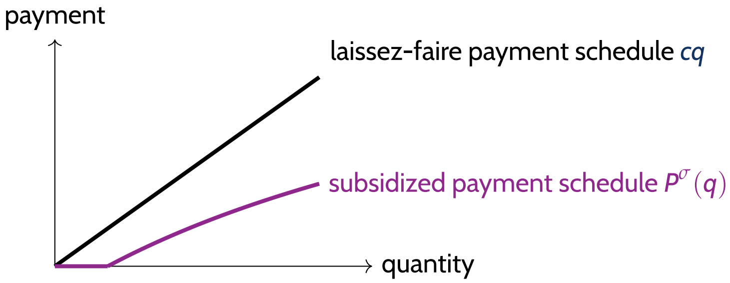
(And When Not To)

When (Not) To Subsidize?

With Topping Up:

topping up $\Longleftrightarrow$ marginal price $\leq c$

Without Topping Up:



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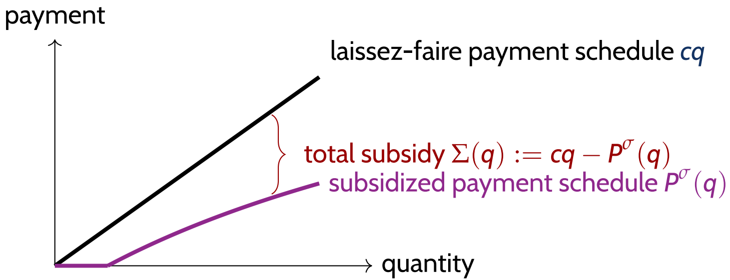

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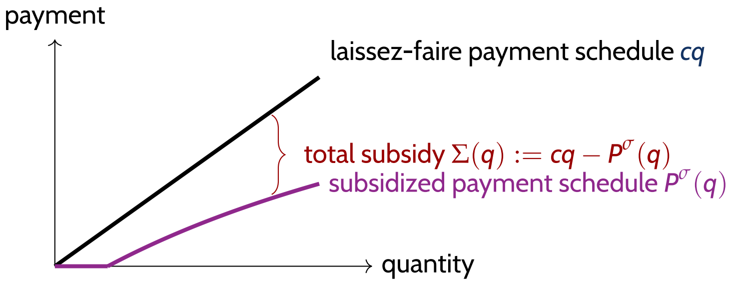

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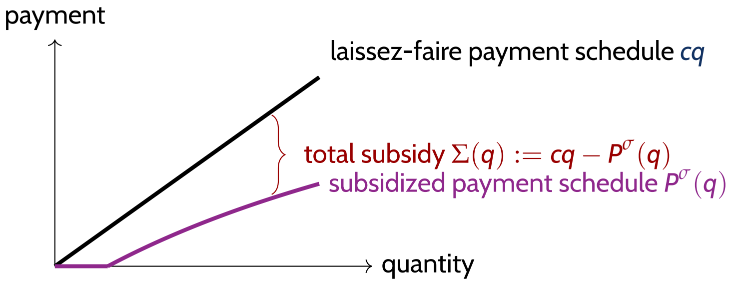

With topping up, subsidies are captured disproportionately by high θ consumers.

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Subsidies are more regressive than the equivalent lump-sum cash transfer.

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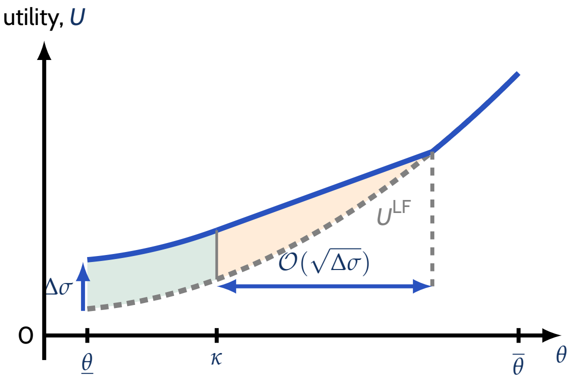

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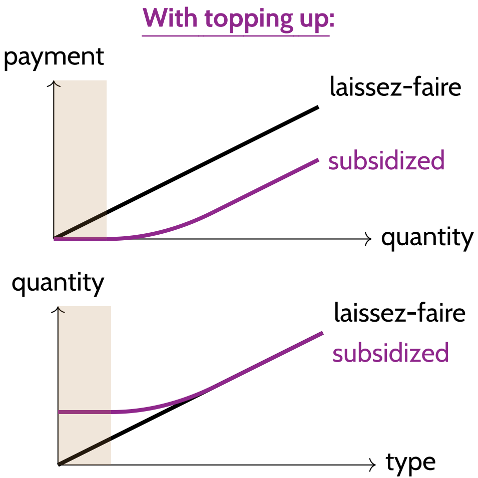
$\rightsquigarrow$ **Restricting topping up** enlarges the scope of redistribution with subsidies..

How To Subsidize

Optimal Subsidy Design



Free allocation with partial subsidies up to a cap
(cf. food stamps)

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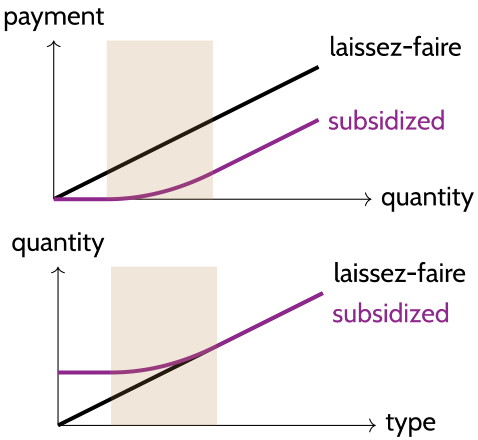

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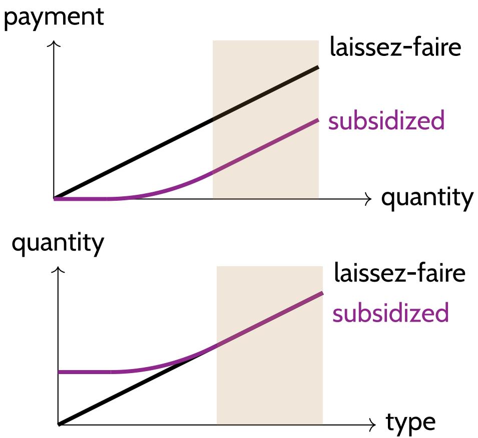

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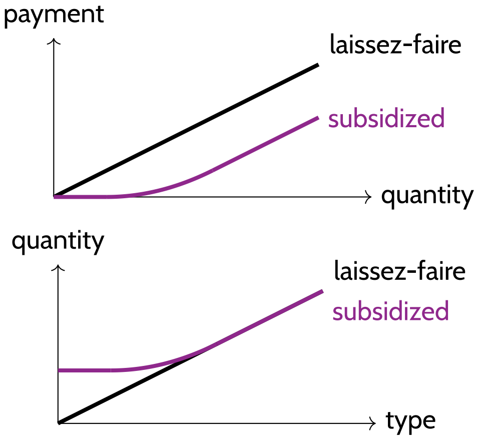

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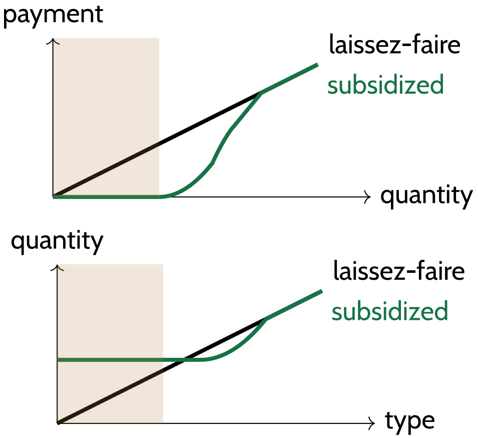

Free allocation and subsidies, intermediate consumption distorted down (cf. public housing)

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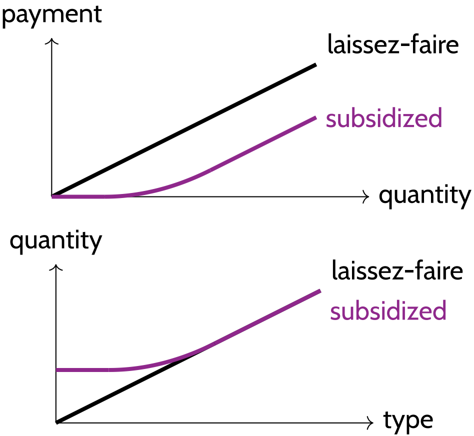

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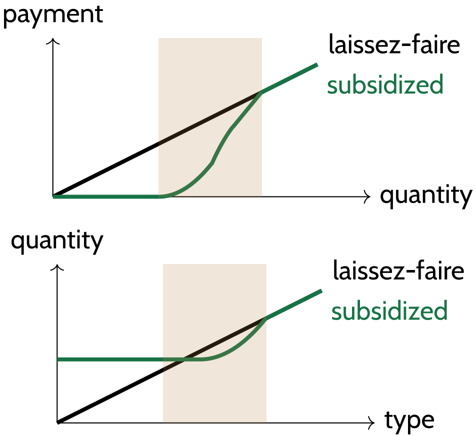

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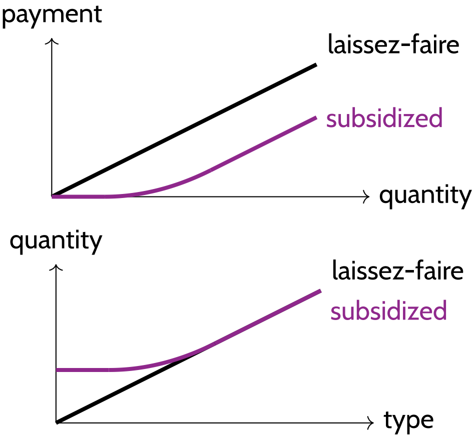

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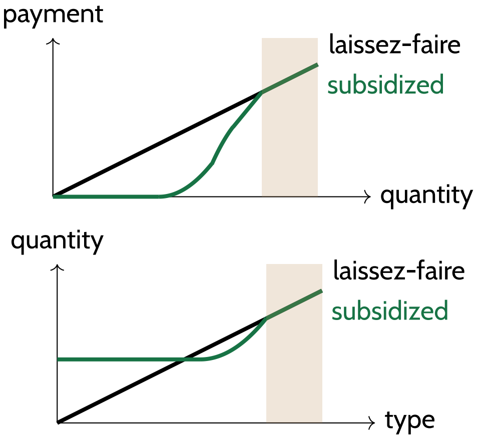

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Contrast With “Full” Mechanism Design (No Private Market Constraint)

#1. When should we redistribute in kind?

- **Full design:** always, because we can tax quality consumption of rich to subsidize poor.
- **With topping up:** whenever $\mathbf{E}[\omega(\theta)] \geq \alpha$.
- **Without topping up:** whenever $\max \omega > \alpha$.

$\rightsquigarrow$ Participation constraints reduce scope for redistribution, particularly if consumers can top up.

#2. When should we use a free public option?

- **Full design** / **Topping Up:** when $\mathbf{E}[\omega] > \alpha$.
- **Without topping up:** when $\mathbf{E}[\omega] > \alpha$ and sometimes when $\mathbf{E}[\omega] \leq \alpha$ (when $\mu^* > 0$).

$\rightsquigarrow$ Restricting private market access can increase scope for non-market allocations.

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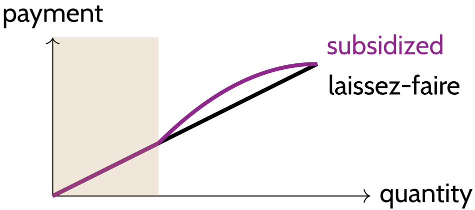

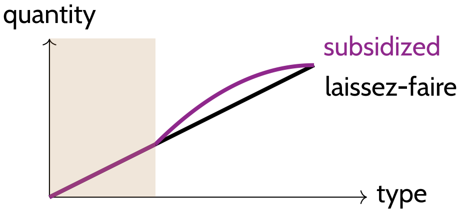

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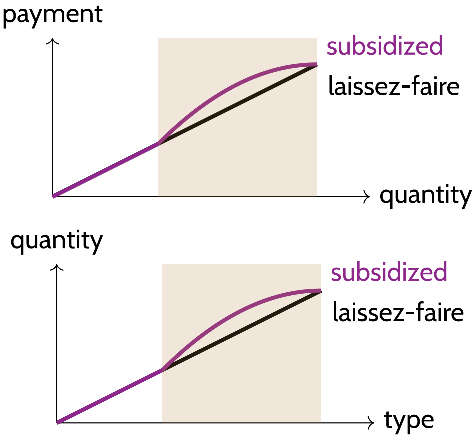

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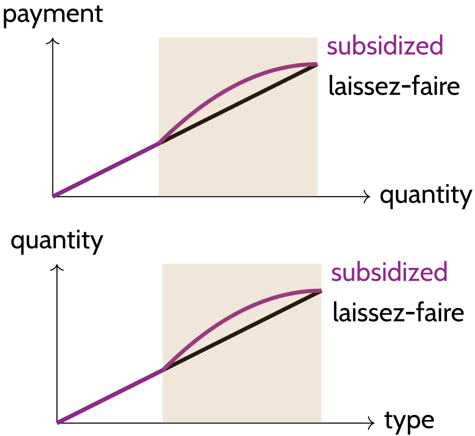

$\rightsquigarrow$ Topping up restrictions have no "bite."

Economic Implications

With positive correlation between ω and θ:

1. The social planner derives no benefit from restricting topping up in the private market.

2. Optimal subsidies are self-targeting, with benefits flowing only to consumers with the highest need.

3. Social planner prefers subsidies to lump-sum cash transfers.

Differences In Practice

When? With topping up, scope of intervention larger with positive correlation ($\max \omega > \alpha$) than negative correlation ($\mathbf{E}[\omega] > \alpha$).

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In practice, many government programs focused on goods consumed disproportionately by needy.

How? Significant differences in marginal subsidy schedules observed in practice:

Larger subsidies for low q

- Food stamps (SNAP)
- Womens, Infants & Children (WIC) Program
- Housing Choice (Section 8) Vouchers
- Lifeline (Telecomm. Assistance) Program
- Public Housing Programs (no topping up)

Larger subsidies for high q

- Public transit fare capping
- Pharmaceutical subsidy programs
- Government-subsidized childcare places.

program details inferior programs

Conclusion

Concluding Remarks

Takeaways for Subsidy Policy:

- Linear subsidies are **never** optimal.
- When and how to subsidize depends on **correlation** between demand and whether topping up is possible/may be restricted:
  - With negative correlation (many goods), the social planner benefits from restricting top-up: e.g., public housing vs. rental assistance. Otherwise, why not lump-sum cash transfers? ("tortilla subsidy" vs. Progresa).
  - Goods with positive correlation are ideal candidates for subsidies (e.g., public transport), but these should have floors for optimal targeting.

Technical Contribution:

- We show how to solve mechanism design problems with FOSD and SOSD constraints caused by type-dependent outside options.
- Similar mechanism design problems arise in other contexts, e.g., subsidy design with other objectives (externalities, paternalism); exclusive contracting (topping up = non-exclusive contracting, no topping up = exclusive contracting.).

Related Literature

- **Public Finance.** Ramsey (1927), Diamond (1975), Mirrlees (1976, 1986), Atkinson & Stiglitz (1976), Nichols & Zeckhauser (1982), Hammond (1987), Blackorby & Donaldson (1988), Besley & Coate (1991), Blomquist & Christiansen (1998), Doligalski, Dworczak, Krysta & Tokarski (2025).
  - ⤳ **This paper:** allows for nonlinear subsidy designs.
- **Redistributive Mechanism Design.** Weitzman (1977), Condorelli (2013), Che, Gale & Kim (2013), Dworczak, Kominers & Akbarpour (2021, 2022), Kang (2023,2024), Akbarpour, Budish, Dworczak & Akbarpour (2024), Pai & Strack (2024).
  - ⤳ **This paper:** allows consumers to consume in private market outside of planner's control.
- **Partial Mechanism Design.** Jullien (2000), Philippon & Skreta (2012), Tirole (2012), Fuchs & Skrzypacz (2015), Dworczak (2020), Loertscher & Muir (2022), Loertscher & Marx (2022), Kang & Muir (2022), Kang (2023), Kang & Watt (2024).
  - ⤳ **This paper:** private market outside of planner's control, focus on benchmark where planner is as efficient as private market
- **Methodological Tools in Mechanism Design.** Jullien (2000), Amador, Werning, & Angeletos (2006), Toikka (2011), Amador & Bagwell (2013), Kleiner, Moldovanu, & Strack (2021), Corrao, Flynn & Sastry (2023), Dworczak & Muir (2024), Yang & Zentefis (2024), Valenzuela-Stookey & Poggi (2024).
  - ⤳ **This paper:** explicit characterization of solution with FOSD (topping up) and SOSD (private market access) constraints.

Fin

Appendix

Key Tradeoff

The optimal subsidy program trades off:

#1. screening, distorting consumption to redirect surplus to high-need consumers, versus

#2. heterogeneous outside options, consumers can access a private market.

Key Tradeoff

The **optimal subsidy** program trades off:

#1. **screening**, distorting consumption to redirect surplus to high-need consumers, versus

#2. **heterogeneous outside options**, consumers can access a private market.

Heterogeneous outside options are empirically relevant, *e.g.*,

- public housing (van Dijk, 2019; Waldinger, 2021),
- education (Akbarpour, Kapor, Neilson, van Dijk & Zimmerman, 2022; Kapor, Karnani & Neilson, 2024),
- healthcare (Li, 2017; Heim, Lurie, Mullen & Simon, 2021),
- SNAP (Haider, Jacknowitz & Schoeni, 2003; Ko & Moffitt, 2024; Rafkin, Solomon & Soltas, 2024).

Outside options lead to **constraints** in the mechanism design problem.

Results Overview

We provide an **explicit characterization** of:

(a) **when** the social planner strictly benefits from subsidizing consumption, and

(b) **how** the optimal subsidy depends on quantity consumed.

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- consumer's ability to access private market (**topping up** vs. **no topping up**).

With **negative correlation** between θ and ω, subsidies are targeted to low consumption levels, and

$$\text{no topping up} \succeq \text{lump-sum transfers} \succeq \text{topping up}$$

With **positive correlation** between θ and ω, subsidies are targeted to high consumption levels, and

$$(\text{no topping up} = \text{topping up}) \succeq \text{lump-sum transfers}$$

Setup

Consumers:

- There is a unit mass of risk-neutral consumers in market for a divisible, homogeneous good.
- Consumers differ in type $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} \geq 0$, and $\theta \sim F$, continuous with density $f > 0$.
- Each consumer derives utility $\theta v(q) - t$ from quantity $q \in [0, A]$ given payment t.
  $v : [0, A] \to \mathbb{R}$ is differentiable with $v' > 0$, $v'' < 0$ and $v' \to \infty$ as $q \downarrow 0$.

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Extensions (not today): equilibrium effects, observable characteristics, product choice and eligibility.

Model Overview

Laissez-Faire Equilibrium

- Perfectly competitive private market $\rightsquigarrow$ **laissez-faire price** $p^{\mathrm{LF}} = c$ per unit.
- Each consumer solves

$$U^{\mathrm{LF}}(\theta) := \max_{q \in [0,A]} \left[\theta v(q) - cq\right].$$

v is strictly concave $\rightsquigarrow$ unique maximizer:

$$q^{\mathrm{LF}}(\theta) = (v')^{-1}\left(\frac{c}{\theta}\right) = D(c, \theta).$$

- To simplify statements of some results, assume today that $q^{\mathrm{LF}}(\underline{\theta}) > 0$.

▸ Model Overview

Subsidy Design

Social planner costlessly contracts with firms and sells units at a **subsidized payment schedule** $P^{\sigma}(q)$.

$\rightsquigarrow \Sigma(q) = cq - P^{\sigma}(q)$ is the **total subsidy** as a function of q, and $\sigma(q) = \Sigma'(q)$ is the **marginal subsidy**.

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Implementation: Consumer θ solves $U^{\sigma}(\theta) := \max_q [\theta v(q) - P^{\sigma}(q)]$, leading to **subsidized demand** $q^{\sigma}(\theta)$.

Model Overview | NLS assumption | Taxation

Redistributive Objective

The social planner seeks to maximize **weighted total surplus**.

- **Consumer surplus**: social planner assigns a welfare weight $\omega(\theta) := \mathbf{E}[\omega|\theta]$ to consumer type θ.
  - $\rightsquigarrow$ $\omega(\theta)$: expected social value of giving consumer θ one unit of money.

▸ Endogenizing ω and α

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$\rightsquigarrow$ **Objective**:

$$\max_{P^\sigma(q)\geq 0 \text{ s.t. } \sigma(q)\geq 0} \int_\theta \left[\omega(\theta)U^\sigma(\theta) - \alpha\Sigma(q^\sigma(\theta))\right] \mathrm{d}F(\theta)$$

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Remarks:

- If $\omega(\theta) > \alpha$, social planner would want to transfer a dollar to type θ.
- If $\mathbf{E}_\theta[\omega(\theta)] > \alpha$, social planner would want to make a lump-sum cash transfer to all consumers.

Endogenizing ω and α

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{(q,t)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[\omega(\theta) \underbrace{[\theta v(q(\theta)) - t(\theta)]}_{\text{consumer surplus}} - \alpha \underbrace{[cq(\theta) - t(\theta)]}_{\text{net cost}} \Big] \, \mathrm{d}F(\theta),$$

subject to

- incentive compatibility, $$\theta \in \arg\max_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} \left[\theta v(q(\hat{\theta})) - t(\hat{\theta}) \right] \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \qquad \text{(IC)}$$
- no lump-sum transfers, $$t(\theta) \geq 0 \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}]; \qquad \text{(LS)}$$
- individual rationality, $$\theta v(q(\theta)) - t(\theta) \geq U^{\mathrm{LF}}(\theta) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \qquad \text{(IR)}$$
- topping up constraint, $$q(\theta) \geq q^{\mathrm{LF}}(\theta) \qquad \forall \theta \in [\underline{\theta}, \overline{\theta}]. \qquad \text{(TU)}$$

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subject to (IC), (LS), (IR), and (TU).

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subject to (IC), (LS), (IR), and (TU).

#1. Apply Myerson (1981) Lemma and Milgrom and Segal (2002) envelope theorem to express objective in terms of $\underline{U} := U(\underline{\theta})$ and $q(\theta)$ non-decreasing, substituting

$$t(\theta) = \theta v(q(\theta)) - \int_{\underline{\theta}}^{\theta} v(q(s))\,\mathrm{d}s - \underline{U}.$$

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{\underline{U},\, q \text{ non-decreasing}} \left\{ \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \int_{\underline{\theta}}^{\overline{\theta}} \left[\left[\alpha\theta + \frac{\int_{\theta}^{\overline{\theta}} \left[\omega(s) - \alpha\right] \, \mathrm{d}F(s)}{f(\theta)} \right] v(q(\theta)) - \alpha c q(\theta) \right] \, \mathrm{d}F(\theta) \right\},$$

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subject to (LS), (IR), and (TU).

#2. Suffices to enforce (LS) only for lowest type $\underline{\theta}$ because $t(\theta)$ is nondecreasing by (IC), so

$$\overline{U} \leq \underline{\theta} v(q(\underline{\theta})),$$

while (IR) for $\underline{\theta}$ implies

$$\overline{U} \geq U^{\mathrm{LF}}(\underline{\theta}).$$

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{\substack{U^{\mathrm{LF}}(\underline{\theta}) \leq \underline{U} \leq \underline{\theta} v(q(\underline{\theta})), \\ q \text{ non-decreasing}}} \left\{ \left[\mathbf{E}[\omega] - \alpha\right] \underline{U} + \int_{\underline{\theta}}^{\overline{\theta}} \left[\left[\alpha\theta + \frac{\int_{\theta}^{\overline{\theta}} [\omega(s) - \alpha] \, \mathrm{d}F(s)}{f(\theta)} \right] v(q(\theta)) - \alpha c q(\theta) \right] \mathrm{d}F(\theta) \right\},$$

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subject to (IR) and (TU).

#3. Writing virtual type

$$J(\theta) = \underbrace{\theta}_{\text{efficiency}} + \underbrace{\frac{\int_{\theta}^{\overline{\theta}} [\omega(s)-\alpha]\mathrm{d}F(s)}{\alpha f(\theta)}}_{\text{redistributive motive}} + \underbrace{\frac{\max\{\mathbf{E}_{\theta}[\omega(\theta)-\alpha], 0\}\underline{\theta}\delta_{\theta=\underline{\theta}}}{\alpha f(\theta)}}_{\text{(LS) constraint at } \underline{\theta}}$$

Call $J(\theta)-\theta$ the **distortion term**. Its sign depends on $\int_{\theta}^{\overline{\theta}} \omega(s)-\alpha\,\mathrm{d}F(s)$.

Mechanism Design Problem

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{q \text{ non-decreasing}} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left[J(\theta)v(q(\theta)) - cq(\theta)\right]}_{\text{surplus of virtual type}} \, \mathrm{d}F(\theta) + (\text{terms independent of } q),$$

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subject to (IR) and (TU).

#4. By envelope theorem, (TU) and (IR) for $\underline{\theta}$ implies (IR) for all θ.

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subject to

with topping up:

$$q(\theta) \geq q^{\mathrm{LF}}(\theta) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{FOSD}$$

without topping up:

$$\underline{U} + \int_{\underline{\theta}}^{\theta} v(q(s))\mathrm{d}s \geq U^{\mathrm{LF}}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q^{\mathrm{LF}}(s))\mathrm{d}s, \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{SOSD}$$

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In tariff space, these constraints are equivalent to marginal price $\leq$ c and average price $\leq$ c.

Assumption: No Lump-Sum Cash Transfers

Note: This constraint only binds if $\mathbf{E}_\theta[\omega(\theta)] > \alpha$.

Possible reasons:

- Institutional: subsidies designed by government agency without tax/transfer powers.
- Political: Liscow and Pershing (2022) find U.S. voters prefer in-kind redistribution to cash transfers.
- Household Economics: Currie (1994) finds in-kind redistribution has stronger benefits for children than cash transfer programs.
- Pedagogical: to contrast when the assumption is binding ($\leadsto$ cash transfers preferred to subsidies) versus non-binding (*vice versa*).
- Model: without NLS constraint, the social planner would want to make unbounded cash transfers when $\mathbf{E}[\omega] > \alpha$.

return

When Not To Subsidize?

Recall the "negative correlation" assumption: high θ consumers have lower ω.

Proposition. For any subsidy P^{σ}, the social planner would prefer to make a lump-sum transfer of $\mathbf{E}_{\theta}[\Sigma(q^{\sigma}(\theta))]$ to all consumers than the subsidy outcome.

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Proof: By definition of U^{LF} and correlation inequality,

$$\underbrace{\int_\Theta \omega(\theta) U^\sigma(\theta) - \alpha\Sigma(q^\sigma(\theta))\, \mathrm{d}F(\theta)}_{\text{objective given } P^\sigma} = \int_\Theta \omega(\theta)[\theta v(q^\sigma(\theta)) - cq^\sigma(\theta) + \Sigma(q^\sigma(\theta))] - \alpha\Sigma(q^\sigma(\theta))\, \mathrm{d}F(\theta)$$

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$$\begin{aligned}
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&\le \int_\Theta \omega(\theta)[U^{\mathrm{LF}}(\theta) + \Sigma(q^\sigma(\theta))] - \alpha\Sigma(q^\sigma(\theta))\,\mathrm{d}F(\theta) \\
&\le \underbrace{\int_\Theta \omega(\theta)[U^{\mathrm{LF}}(\theta) + \mathbf{E}_\theta[\Sigma(q^\sigma(\theta))]]\,\mathrm{d}F(\theta) - \alpha\,\mathbf{E}_\theta[\Sigma(q^\sigma(\theta))]}_{\text{objective given cash payment } \mathbf{E}_\theta\Sigma(q^\sigma(\theta))}. \quad \square
\end{aligned}$$

Theorem 1 (Negative Correlation, part). The social planner subsidizes consumption **only if** $\mathbf{E}_\theta[\omega(\theta)] > \alpha$ (and cash transfers are unavailable).

return

When to Subsidize (General): Proof by Picture

Theorem 1. Social planner subsidizes **if and only if** there exists a type $\hat{\theta}$ for which

$$\mathbf{E}_{\theta}[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha.$$

Suppose $\mathbf{E}_{\theta}[\omega(\theta)|\theta \geq \hat{\theta}] > \alpha$: we construct a subsidy schedule increasing weighted surplus.

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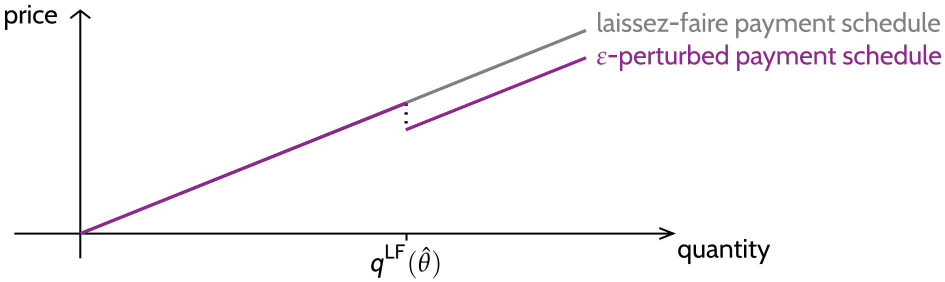

$\varepsilon-$perturbation increases utility of types $\geq \hat{\theta}$, net benefit $\varepsilon\, \mathbf{E}_{\theta}[\omega(\theta) - \alpha|\theta \geq \hat{\theta}]$.

return

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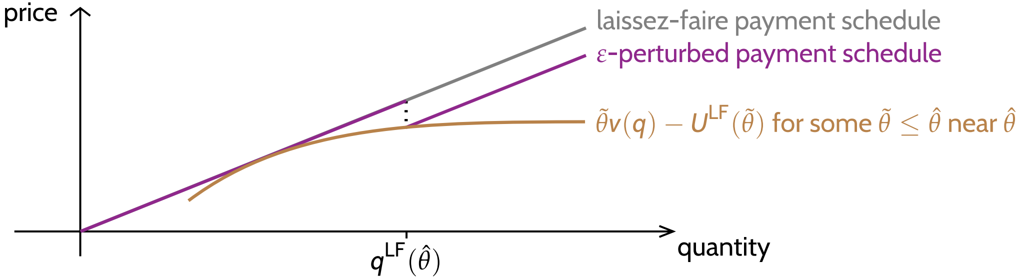
Theorem 1. Social planner subsidizes if and only if there exists a type $\hat{\theta}$ for which

$$\mathbf{E}_\theta[\omega(\theta) \mid \theta \geq \hat{\theta}] > \alpha.$$

Suppose $\mathbf{E}_\theta[\omega(\theta)|\theta \geq \hat{\theta}] > \alpha$: we construct a subsidy schedule increasing weighted surplus.



ε—perturbation increases utility of types $\geq \hat{\theta}$, net benefit $\varepsilon\, \mathbf{E}_\theta[\omega(\theta) - \alpha|\theta \geq \hat{\theta}]$.

But consumption is distorted for $O(\sqrt{\varepsilon})$ set of types near (but below) $\hat{\theta}$, at cost $\leq O(\sqrt{\varepsilon})\varepsilon$.

$\rightsquigarrow$ Benefits $>$ costs for small enough ε. **Note: Argument relies on nonlinearity.**

return

Topping Up $\Longleftarrow$ Lower-Bound (1/2)

Suppose $q(\theta) \geq q^{\text{LF}}(\theta)$. We want to show total subsidies $S(z)$ is increasing in z.

1. $t(\underline{\theta}) \leq cq(\underline{\theta})$ by (IR):

$$t(\underline{\theta}) \leq \underline{\theta}v(q(\underline{\theta})) - \underline{\theta}v(q^{\text{LF}}(\underline{\theta})) + cq^{\text{LF}}(\underline{\theta}),$$

and $\underline{\theta}v(q^{\text{LF}}(\underline{\theta})) - cq^{\text{LF}}(\underline{\theta}) \geq \underline{\theta}v(q(\underline{\theta})) - cq(\underline{\theta})$ by definition of q^{LF}, so $t(\underline{\theta}) \leq cq(\underline{\theta})$.

Topping Up $\Longleftarrow$ Lower-Bound (2/2)

 # 2. The *marginal* price of any units purchased is no greater than c by (IC):

$$t(\theta') - t(\theta) = \left[\theta' v(q(\theta')) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta'} v(q(s))\,\mathrm{d}s\right] - \left[\theta v(q(\theta)) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v(q(s))\,\mathrm{d}s\right]$$

$$= \theta' v(q(\theta')) - \theta v(q(\theta)) - \int_{\theta}^{\theta'} v(q(s))\,\mathrm{d}s$$

$$= \int_{\theta}^{\theta'} s v'(q(s))\,\mathrm{d}q(s).$$

But if $q(\theta) \geq q^{\mathrm{LF}}(\theta)$, then concavity of v implies $v'(q(\theta)) \leq v'(q^{\mathrm{LF}}(\theta)) = c/\theta$, so
$t(\theta') - t(\theta) \leq c[q(\theta') - q(\theta)]$.

return

Solving for the Optimal Mechanism

return to summary

$$\max_{q} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[J(\theta) v(q(\theta)) - c q(\theta) \right] \, \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Solving for the Optimal Mechanism

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$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} [J(\theta)v(q(\theta)) - cq(\theta)] \; \mathrm{d}F(\theta),$$

s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 1: Pointwise maximizer

$$q(\theta) = (v')^{-1}\left(\frac{c}{J(\theta)}\right) = D(c, J(\theta)).$$

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Demand $D(c, \cdot)$ is increasing, so:
q nondecreasing $\iff J(\theta)$ nondecreasing.
$q \geq q^{\mathrm{LF}} \iff D(c, J(\theta)) \geq D(c, \theta) \iff J(\theta) \geq \theta$.

Solving for the Optimal Mechanism

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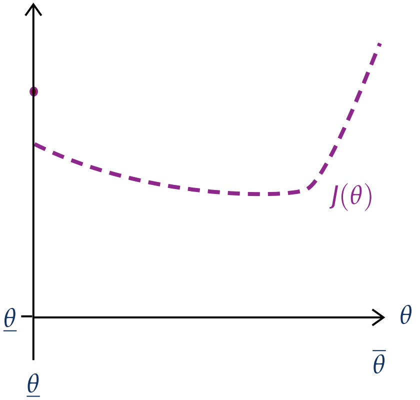

$J(\theta)$ may be non-monotone.

Solving for the Optimal Mechanism

return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} [J(\theta)v(q(\theta)) - cq(\theta)] \, dF(\theta),$$

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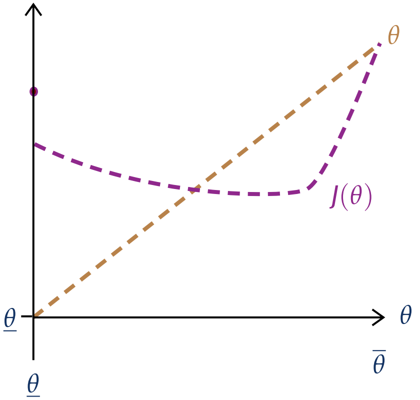


$J(\theta)$ may be smaller than θ.

Solving for the Optimal Mechanism

▸ return to summary

$$\max_q \alpha \int_{\underline{\theta}}^{\overline{\theta}} [J(\theta)v(q(\theta)) - cq(\theta)] \; \mathrm{d}F(\theta),$$

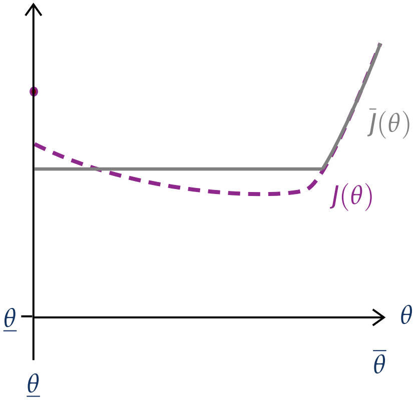
s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 2: Relaxing the (LB) constraint

Toikka (2011); Akbarpour, Dworczak, Kominers (2021)

$$\rightsquigarrow q(\theta) = (v')^{-1}\left(\frac{c}{\bar{J}(\theta)}\right) = D(c, \bar{J}(\theta)),$$

where $\bar{J}$ is ironing of J, pooling types in any non-monotonic interval of J at its F-weighted average.



Ironing deals with non-monotonicity.

▸ Ironing

Solving for the Optimal Mechanism

return to summary

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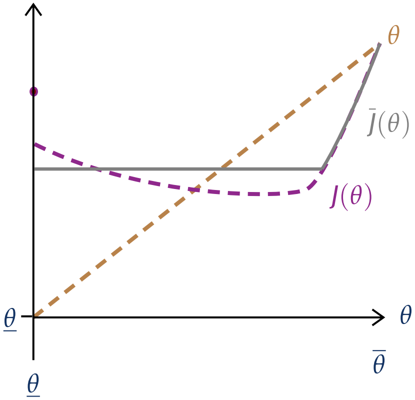

But not lower-bound constraint $\rightsquigarrow$ interaction.

Ironing

Solving for the Optimal Mechanism

return to summary

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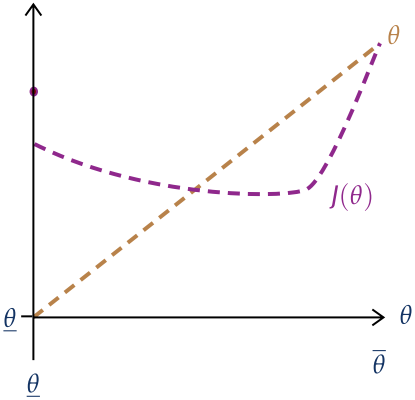
s.t. q nondecreasing and $q(\theta) \geq q^{\mathrm{LF}}(\theta)$.

Guess 3: Our approach

Suppose solution is of the form

$$q(\theta) = D(c, H(\theta)).$$

Feasibility requires H to be nondecreasing and satisfy $H(\theta) \geq \theta$.



Need to identify nondecreasing $H \geq \theta$.

Ironing

Characterizing the Optimal Subsidy Allocation

Theorem. The optimal allocation rule is unique, continuous and satisfies

$$q^*(\theta) = D(c, H(\theta)),$$

where the subsidy type $H(\theta)$ is defined by

$$H(\theta) := \begin{cases} \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta) & \text{for } \theta \leq \theta_\alpha \\ \theta & \text{for } \theta \geq \theta_\alpha, \end{cases}$$

and θ_α is defined by

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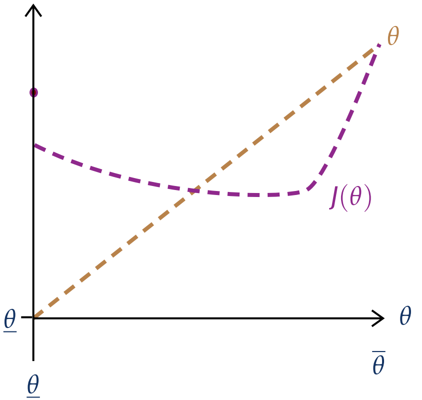

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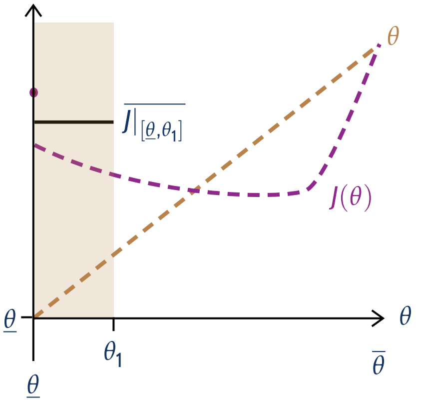

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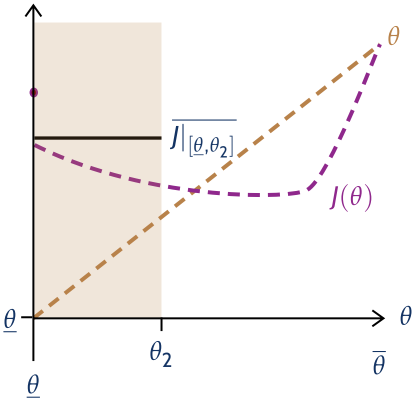

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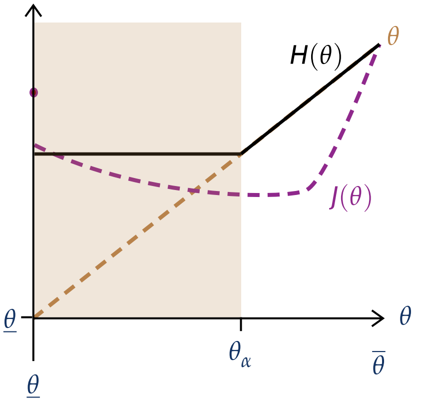


construction $\rightsquigarrow$ pooling condition and continuity

▸ Ironing

Characterizing the Optimal Subsidy With Topping Up

Theorem. The optimal allocation rule is unique, continuous and satisfies

$$q^*(\theta) = \begin{cases} D\left(c, \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta)\right) & \text{for } \theta \leq \theta_\alpha \\ q^{\mathrm{LF}}(\theta) & \text{for } \theta \geq \theta_\alpha, \end{cases}$$

where θ_α is defined by

$$\theta_\alpha = \inf\left\{\theta \in \Theta : \overline{J|_{[\underline{\theta},\theta]}}(\theta) \leq \theta\right\}.$$

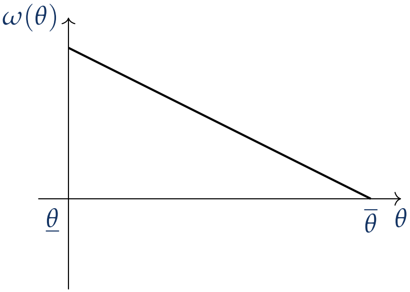
Intuition: there exists a type $\theta_\alpha \in \Theta$ (possibly $\underline{\theta}$ or $\overline{\theta}$) such that

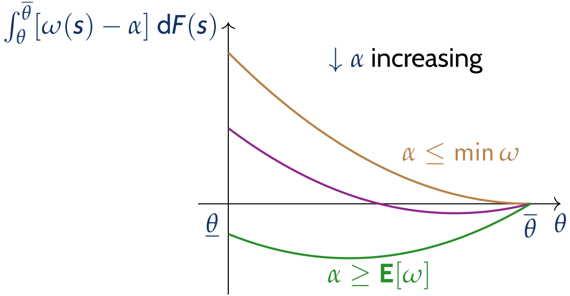
$$q^*(\theta) > q^{\mathrm{LF}}(\theta) \text{ for all } \theta < \theta_\alpha, \text{ and}$$
$$q^*(\theta) = q^{\mathrm{LF}}(\theta) \text{ for all } \theta \geq \theta_\alpha.$$

Ironing

Intuition

Negative correlation $\rightsquigarrow$ $\omega(\theta)$ decreasing $\rightsquigarrow$ distortion is single-crossing zero from above.





Social planner wants to distort consumption of all types down, low-demand types up and high-demand types down, or all types upwards.

Optimal Marginal Subsidy Schedule

Case 1: $\alpha \leq \min \omega \leq \mathbf{E}[\omega]$ (upward distortion for all)

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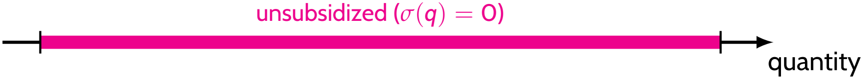
Case 2: $\min \omega \leq \alpha \leq \mathbf{E}[\omega]$ (upward distortion for low types, downward distortion for high types)

Case 3: $\min \omega \leq \mathbf{E}[\omega] \leq \alpha$ (downward distortion for all)



Proof

Economic Implications

With topping up and **negative correlation** between ω and θ:

1. Lump-sum cash transfers are always **more progressive** than subsidies.

2. The optimal subsidy progam is **never linear**, with higher marginal subsidies for low levels of consumption.

  # 2a. Optimal subsidies are "all or none": active subsidy programs should always incorporate a **free allocation** ("public option").

  # 2b. If *any* consumer has $\omega < \alpha$, optimal (marginal) subsidies are **capped** in quantity.

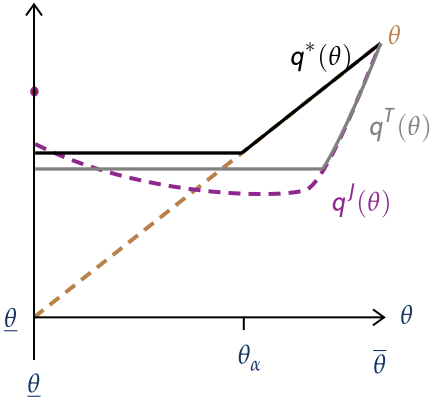
Role of The Private Market

Comparing optimum with and without (LB) constraint, $q^*(\theta)$ can exceed $q^T(\theta)$ for *all* types.

$\rightsquigarrow$ Inability to tax can cause upward distortion, even for consumers who would be subsidized in the absence of the (LB) constraint.

It is not optimal to calculate optimal subsidy/tax and set taxes to zero.

Highlights distinction from Mirrleesian marginal approach (FOC $\not\rightsquigarrow$ optimum).



Subsidy Design without Topping Up

Scope of In-Kind Redistribution

Recall: Mechanism Design Problem Without Topping Up

The social planner chooses **total allocation function** q and **total payment function** t to maximize weighted total surplus:

$$\max_{q \text{ non-decreasing}} \alpha \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left[J(\theta)v(q(\theta)) - cq(\theta)\right]}_{\text{surplus of virtual type}} \mathrm{d}F(\theta) + (\text{terms independent of } q),$$

subject to

$$\underline{U} + \int_{\underline{\theta}}^{\theta} v(q(s))\mathrm{d}s \geq U^{\mathrm{LF}}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(q^{\mathrm{LF}}(s))\mathrm{d}s, \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \tag{SOSD}$$

In tariff space, this constraint is equivalent to average price $\leq$ c $\rightsquigarrow$ some marginal units may be taxed.

Greater Scope for In-Kind Redistribution

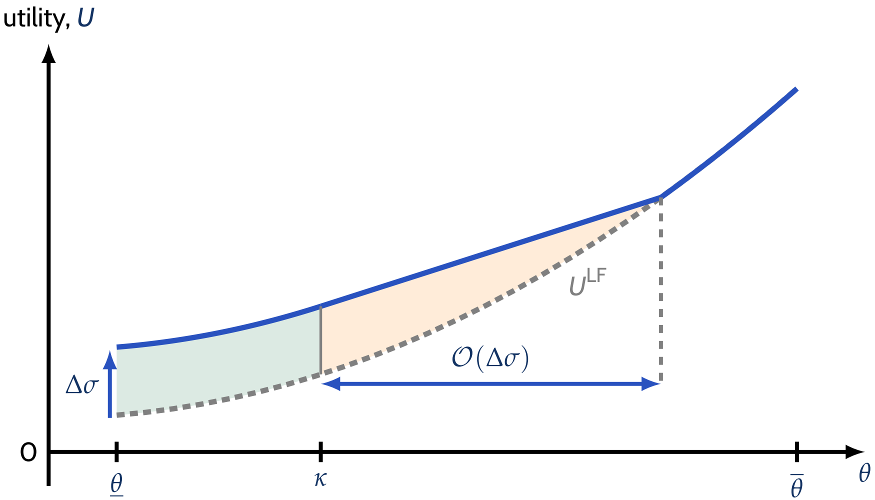
Theorem (No Topping Up). With negative correlation between ω and θ, the social planner has an active in-kind subsidy program if and only if $\omega(\underline{\theta}) \geq \alpha$.

- $\rightsquigarrow$ Subsidy program without topping up may outperform lump-sum cash transfers.
- $\rightsquigarrow$ There is a greater scope for redistribution than in the case with topping up ($\mathbf{E}[\omega] \geq \alpha$).

Note: without a private market outside option, the social planner intervenes whenever $\omega(\theta) \not\equiv \alpha$.

Intuition

Without topping up, social planner can target subsidies toward consumers with low levels of consumption.



Subsidy Design without Topping Up

Optimal Subsidy Design

Characterization of Optimal Mechanism

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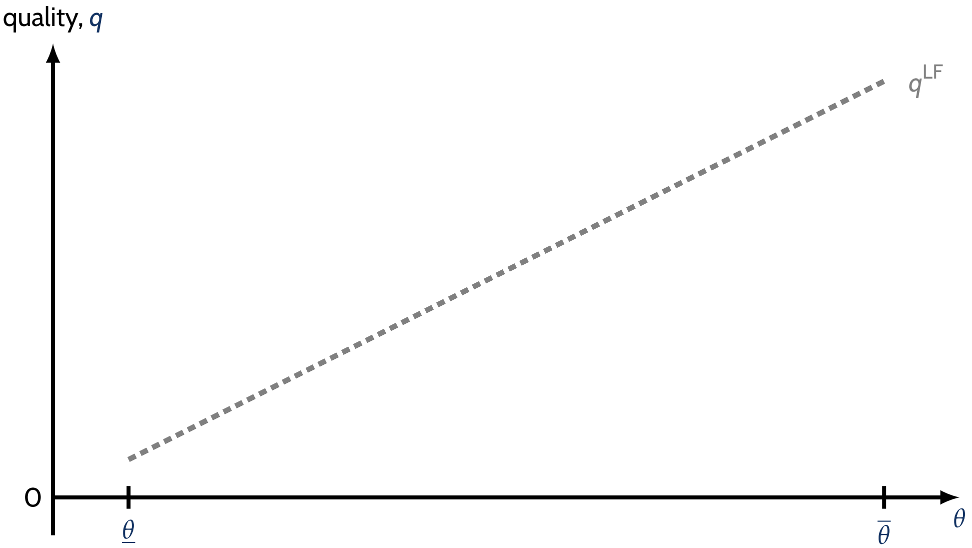

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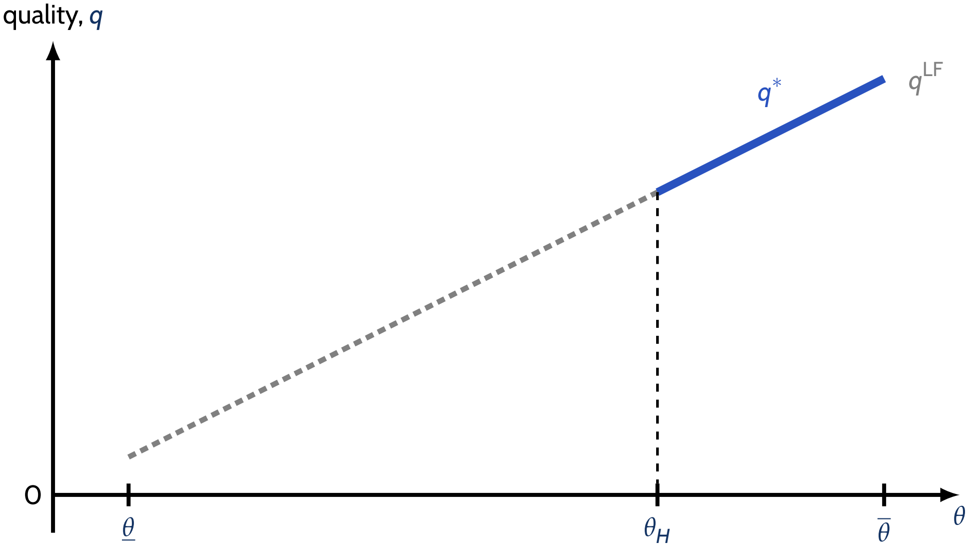

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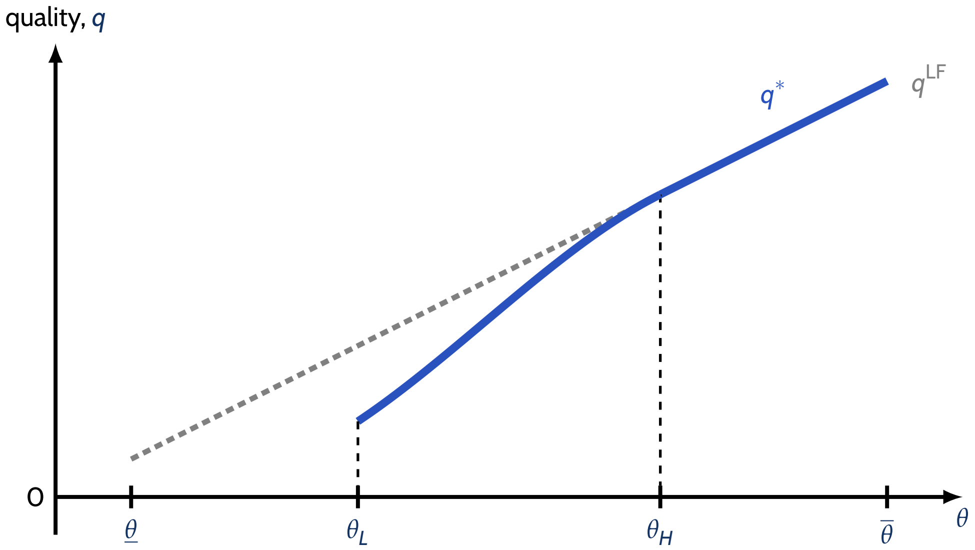

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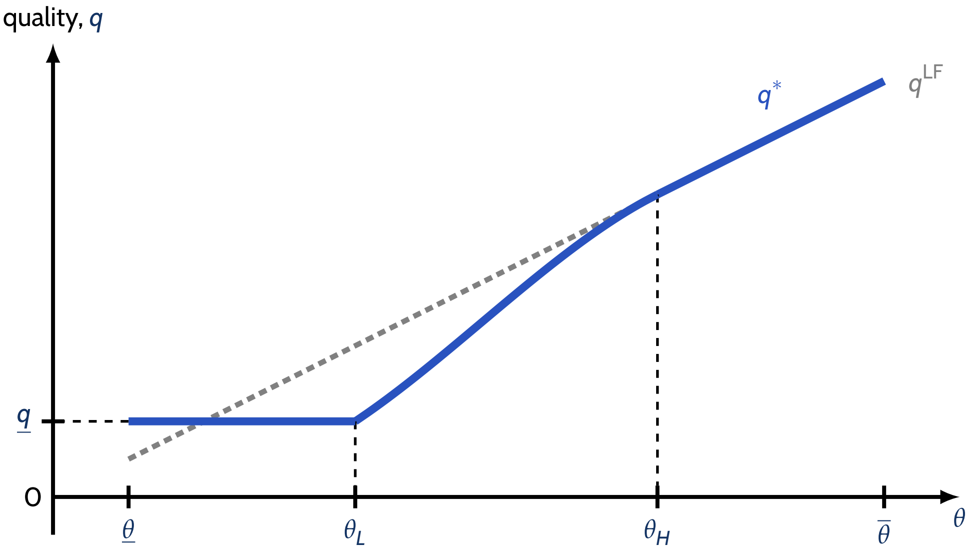

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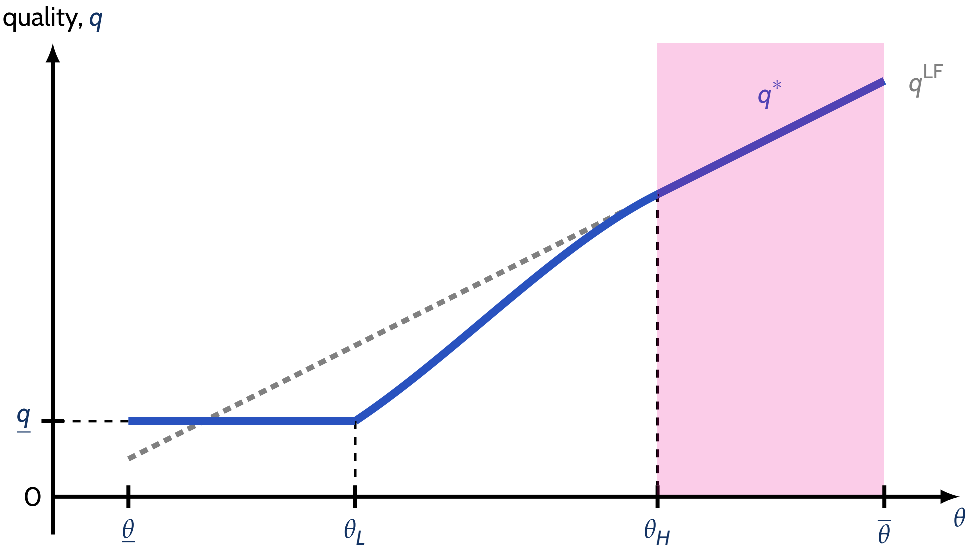

Ⓐ Which consumers go to the private market?

Theorem 2(a). Under the optimal mechanism:

- If $\mathbf{E}[\omega] \leq \alpha$, then there exists $\mu^* \geq 0$ such that the (IR) constraint binds exactly for consumers with types in $[\theta_H, \overline{\theta}]$, where

$$\theta_H := \max\left\{\theta \in [\underline{\theta}, \overline{\theta}] : \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \, \mathrm{d}F(s) + \mu^* \leq 0\right\}.$$

- If $\mathbf{E}[\omega] > \alpha$, then $\theta_H = \overline{\theta}$.

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$$\theta \mapsto \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \; dF(s)$$

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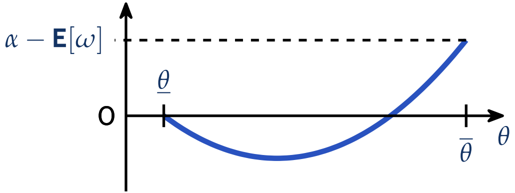

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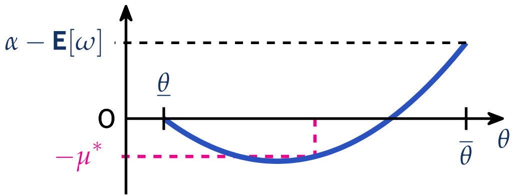

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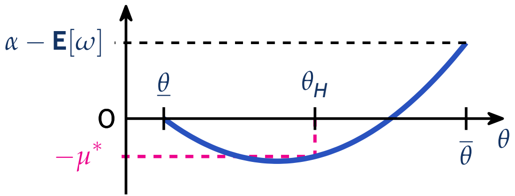

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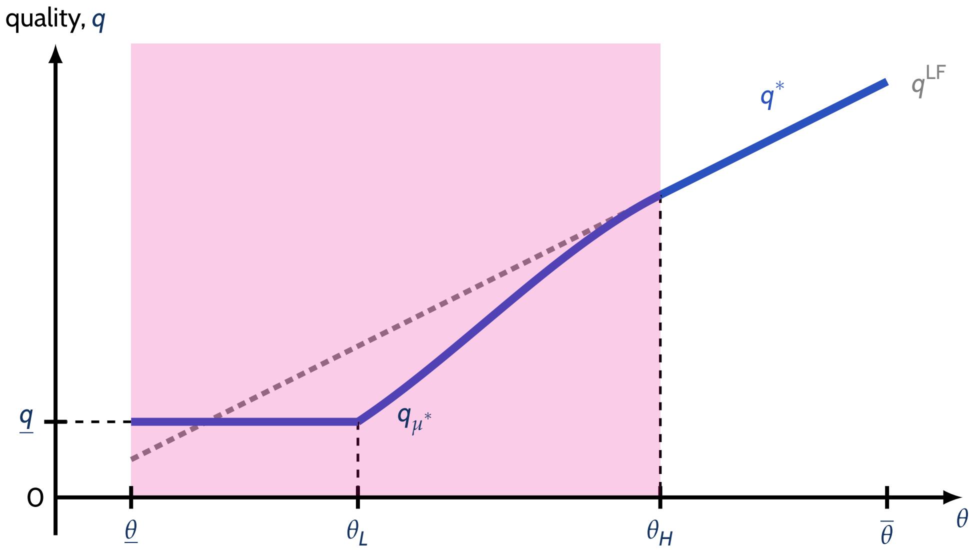
- If $\mathbf{E}[\omega] > \alpha$, then $\theta_H = \overline{\theta}$ (this holds even if $\omega(\overline{\theta}) < \alpha$!).

B Which consumers benefit from in-kind redistribution?



B Which consumers benefit from in-kind redistribution?

Theorem 2(b). For any $\mu \geq 0$, define

$$q_\mu(\theta) := D(c, \overline{H_\mu}(\theta)), \quad \text{where } H_\mu(\theta) := \frac{\theta}{c} + \frac{\mu\underline{\theta} \cdot \delta_{\theta=\underline{\theta}} + \mu + \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \; dF(s)}{\alpha c f(\theta)},$$

$$\theta_H(\mu) := \begin{cases} \max\left\{\theta \in [\underline{\theta}, \overline{\theta}] : \int_{\underline{\theta}}^{\theta} [\alpha - \omega(s)] \; dF(s) + \mu \leq 0\right\} & \text{if } \mathbf{E}[\omega] \leq \alpha, \\ \overline{\theta} & \text{if } \mathbf{E}[\omega] > \alpha. \end{cases}$$

Under the optimal mechanism, consumers with types in $[\underline{\theta}, \theta_H(\mu^*)]$ consume $q^*(\theta) = q_{\mu^*}(\theta)$, where

$$\mu^* := \min\left\{\mu \in \mathbb{R}_+ : \int_{\underline{\theta}}^{\theta_H(\mu)} v(q_\mu(s)) \; ds + \underline{\theta} v(q_\mu(\underline{\theta})) - U^{\text{LF}}(\theta_H(\mu)) \geq 0\right\}.$$

Optimal Subsidy Design Without Topping Up

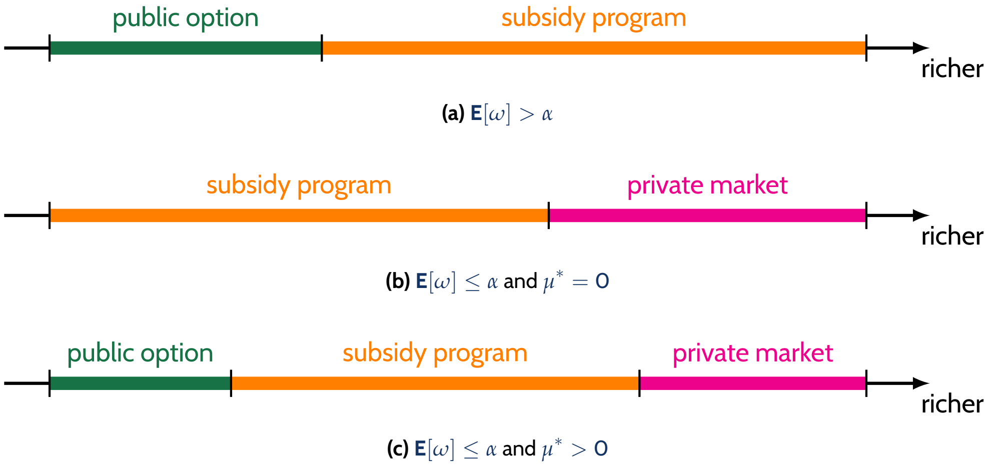


Figure Optimal in-kind redistribution programs under negative correlation.

Proof Sketch

Economic Implications

Without topping up and with **negative correlation** between ω and θ:

#1. Subsidies are preferred to lump-sum cash transfers, and can be targeted to consumers with high ω.

#2. The optimal subsidy program is **never linear**, with higher marginal subsidies for low consumption levels.

   a. The optimal subsidy can involve a **public option** (always if $\mathbf{E}[\omega] \geq \alpha$ and sometimes if $\mathbf{E}[\omega] \leq \alpha$).

   b. If $\mathbf{E}[\omega] \leq \alpha$, high θ (low ω) consumers consume **only** in the private market.

   c. Allocations are always distorted **downwards** for high θ consumers in the subsidy program.

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   c. Allocations are always distorted **downwards** for high θ consumers in the subsidy program.

For a fixed α, compared to the optimal subsidy program **with topping up**:

- The set of subsidized consumers is larger.
- Low θ consumers receive a (weakly) larger subsidy, and high θ consumers a (weakly) smaller subsidy.

Discussion

Theorem 1 $\rightsquigarrow$ scope of intervention larger for “inferior goods” than “normal goods.”

In practice, many government programs focused on goods consumed disproportionately by needy:

Examples:

- Egyptian *Tamween* food subsidy program subsidizes five loaves of *baladi* bread/day at AUD 0.01/loaf, with a cap on weights and quality of bread.
- CalFresh Restaurant Meals Program subsidizes fast food restaurants not dine-in restaurants.
- Indonesian Fuel Subsidy Program subsidizes low-octane fuel (for motorbikes) and not high-octane fuel (for cars).
- Until $\sim$2016, UK’s NHS subsidized amalgam fillings and not composite (tooth-coloured) fillings.

Verifying H from Theorem 2

Because $q^*(\theta) = D(c, H(\theta))$, for any feasible q

$$\int_\Theta \underbrace{\left[H(\theta)v(q^*(\theta)) - cq^*(\theta)\right]}_{\text{surplus of type } H(\theta) \text{ at } D(c,H(\theta))} \mathrm{d}F(\theta) \geq \int_\Theta \underbrace{\left[H(\theta)v(q(\theta)) - cq(\theta)\right]}_{\text{surplus of type } H(\theta) \text{ at } q(\theta)} \mathrm{d}F(\theta).$$

skip

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$$\underbrace{\int_\Theta [J(\theta)v(q^*(\theta)) - cq^*(\theta)] \, \mathrm{d}F(\theta)}_{\text{objective at } q^*} \geq \underbrace{\int_\Theta [J(\theta)v(q(\theta)) - cq(\theta)] \, \mathrm{d}F(\theta)}_{\text{objective at feasible } q}.$$

▸ skip

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Subtracting, it suffices to show, for any feasible q

$$\int_\Theta [J(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))] \, \mathrm{d}F(\theta) \geq 0.$$

skip

Verifying the Variational Inequality

To show $\int_{\Theta}[J(\theta) - H(\theta)][v(q^*(\theta)) - v(q(\theta))]\,\mathrm{d}F(\theta) \geq 0$.

▶ skip

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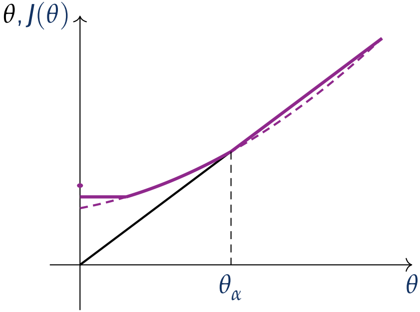
There are three possibilities for H, partitioning Θ into intervals:



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Verifying the Variational Inequality

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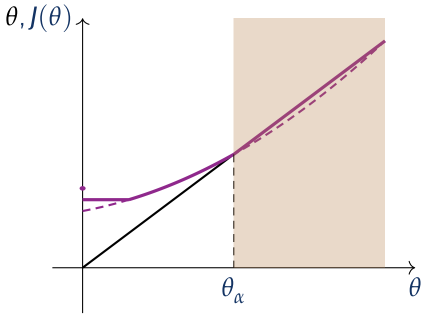
\# 1. $H(\theta) = \theta$: by construction $J(\theta) \leq \theta = H(\theta)$ and $v(q(\theta)) \geq v(q^*(\theta)) \rightsquigarrow$ integrand ≥ 0.



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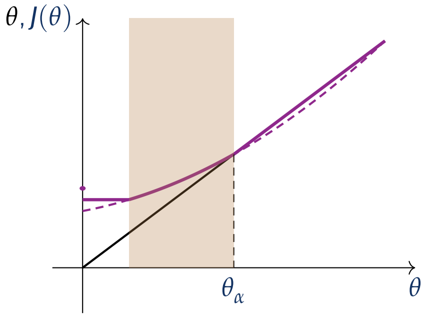

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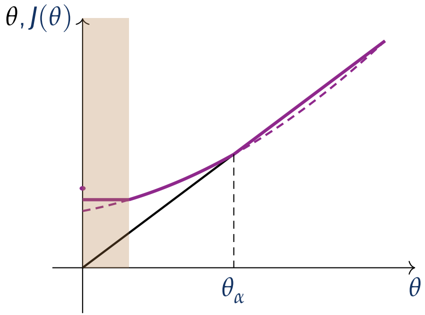
\# 2. $H(\theta) = J(\theta)$: integrand $= 0$.

\# 3. $H(\theta) = \overline{J|_{[\underline{\theta},\theta_\alpha]}}(\theta) \neq J(\theta)$:
technical lemma $\rightsquigarrow$ on any such interval Θ_i, $H = \overline{J|_{\Theta_i}}$
$\rightsquigarrow$ optimality of $D(c, H(\theta))$ in problem on Θ_i *without* (LB)
$\Longrightarrow$ same variational inequality characterizes optimality. $\square$



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Summing Up

Proof approach:

- Guess form of solution $q^*(\theta) = D(c, H(\theta))$.
- Identify $H(\theta)$ which is continuous, $\geq \theta$, and satisfies the **pooling condition**.
- Verify optimality using **variational inequalities**.

Same method of solution works for general $\omega \rightsquigarrow$ see paper.

Generalization

Solving the Mechanism Design Problem

Let us focus on the negative correlation case. We form the Lagrangian:

$$\mathcal{L}(q,\lambda) = \alpha \int_{\underline{\theta}}^{\overline{\theta}} J(\theta)v(q(\theta)) - cq(\theta) - \lambda(\theta)[U(\theta) - U^{\mathsf{LF}}(\theta)]dF(\theta)$$

One possibility: if $q(\theta) = D(\bar{J}(\theta), c)$ is feasible (i.e., if $\underline{\theta}v(D(\bar{J}(\underline{\theta}), c)) + \int_{\underline{\theta}}^{\theta} D(\bar{J}(s), c)\, \mathrm{d}s \geq U^{\mathsf{LF}}(\theta)$ for all $\theta \in \Theta$), then it must be optimal.

Else Lagrangian duality $\leadsto$ (IR) must bind on some interval. We show it must include $\overline{\theta}$ (else a redistributive reallocation downwards is possible).

Integrating the constraint by parts and letting $\Lambda(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)\, \mathrm{d}F(\theta)$, we get

$$\mathcal{L}(q,\lambda) = \alpha \int_{\underline{\theta}}^{\overline{\theta}} (J(\theta) + \Lambda(\underline{\theta})\underline{\theta}\delta_{\theta=\underline{\theta}})v(q(\theta)) - cq(\theta) + \frac{\Lambda(\theta)}{f(\theta)}[v(q(\theta)) - v(q^{\mathsf{LF}}(\theta))]dF(\theta)$$

Note, wherever (IR) is non-binding, Λ is constant! Find unique μ^* such that

$D(J + \frac{\mu^*}{f} + \mu^*\underline{\theta}\delta_{\theta=\underline{\theta}}(\theta^*), c) = D(\theta^*, c)$, where $\mu^* = (\theta^* - J(\theta^*))f(\theta^*)$.

Return to Summary

Comparative Statics of Subsidies

Question: How do optimal subsidies change when

(a) the social planner's desire to redistribute to each consumer increases?

(b) the correlation between demand and welfare weight increases?

(c) the marginal cost of production decreases?

▸ Details

Comparative Statics of Subsidies

Question: How do optimal subsidies change when

(a) the social planner's desire to redistribute to each consumer increases?

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(c) the marginal cost of production decreases?

▸ Details

Short Answer: Each cause the optimal subsidy program to be more generous.

But (a) and (b) cause $J(\theta)$ to increase for each $\theta \rightsquigarrow$ a larger set of consumers subsidized. (c) does not.

Equilibrium Effects

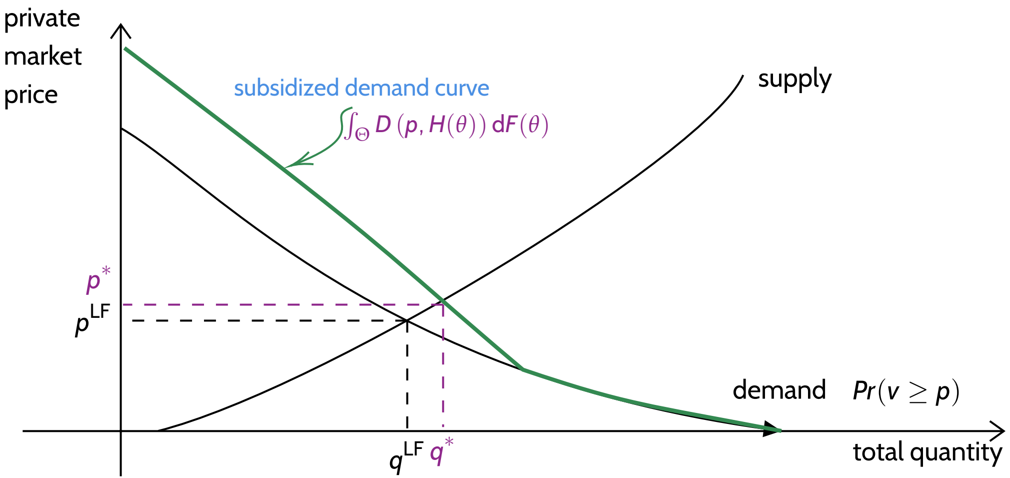
Our baseline model shuts down equilibrium effects of government subsidies on private market prices.

Empirical evidence of price effects from government subsidy programs, *e.g.*:

- public housing (Diamond and McQuade, 2019; Baum-Snow and Marion, 2009)
- pharmaceuticals (Atal et al., 2021)
- public schools (Dinerstein and Smith, 2021)
- school lunches (Handbury and Moshary, 2021)

Equilibrium Effects

Our results extend directly to imperfectly elastic supply curves:



Private Market Taxation

Our baseline model assumes the planner cannot tax the private market.

Taxation of private market reduces consumers' outside option, relaxing the (LB) constraint. If taxation is costly (*e.g.*, because of distortions on ineligible consumers):

Proposition. Suppose the planner faces a convex cost $\Gamma(\tau)$ for taxation of the private market. Then there exists an optimal tax level τ^* and subsidy program for eligible consumers satisfying

$$q^*(\theta) = D(H_{\tau^*}(\theta)),$$

where $H_{\tau^*}(\theta) \leq H(\theta)$.

Return to Model

Budget Constraints and Endogenous Welfare Weights

In our baseline model, $\omega(\cdot)$ and α are taken exogenously.

Our model can be extended to allow weights to be endogenous (cf. Pai and Strack, 2024):

- $\alpha \iff$ Lagrange multiplier on the social planner's budget constraint.
- $\omega(\theta) \iff$ the marginal value of money for a consumer with concave preferences

$$\varphi\left(\theta v(q) + I - t\right),$$

and income $I \sim G_\theta$, known but not observed by the social planner, then

$$\omega(\theta) = \mathbf{E}_{I \sim G_\theta}[\varphi'(\theta v(q(\theta)) + I - t(\theta))].$$

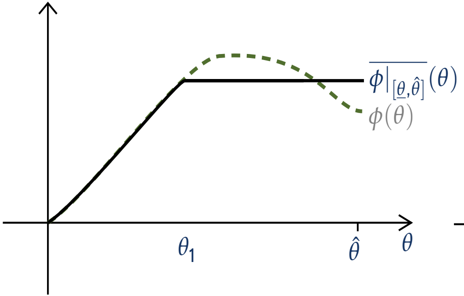
Return to Model

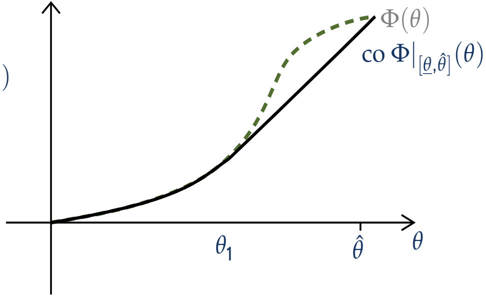
Ironing

Let ϕ be a (generalized) function and $\Phi : \theta \mapsto \int_{\underline{\theta}}^{\theta} \phi(s)\, \mathrm{d}F(s)$. Then $\overline{\phi}$ is the monotone function satisfying

$$\text{for all } \theta \in [\underline{\theta}, \hat{\theta}], \qquad \int_{\underline{\theta}}^{\theta} \overline{\phi}(s)\, \mathrm{d}F(s) = \mathrm{co}\, \Phi(\theta).$$

Intuitively, $\overline{\phi}$ replaces non-monotone intervals of ϕ with F-weighted averages.





▸ Proof ▸ Statement

How to Subsidize?

Positive Correlation

Theorem. Regardless of the consumer's ability to top up, the optimal subsidy allocation rule is unique, continuous and satisfies

$$q^*(\theta) = D\left(c, H(\theta)\right), \text{ where } H(\theta) = \begin{cases} \theta & \text{if } \theta \leq \theta_\alpha, \\ \overline{J_{[\theta_\alpha, \bar{\theta}]}}(\theta) & \text{if } \theta \geq \theta_\alpha, \end{cases}$$

where $\theta_\alpha = \inf\{\theta \in \Theta : J(\theta) \geq \theta\}$.

Intuition: there exists a type $\theta_\alpha \in \Theta$ (possibly $\underline{\theta}$ or $\bar{\theta}$) such that

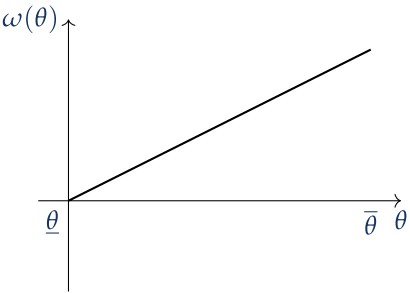
$$q^*(\theta) = q^{\text{LF}}(\theta) \text{ for all } \theta \leq \theta_\alpha, \text{ and}$$
$$q^*(\theta) \geq q^{\text{LF}}(\theta) \text{ for all } \theta > \theta_\alpha.$$

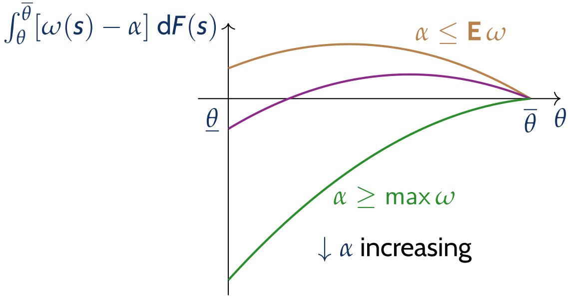
Arbitrary Correlation

How to Subsidize?

Positive Correlation

Positive correlation $\rightsquigarrow \omega(\theta)$ increasing $\rightsquigarrow$ distortion is single-crossing zero from below.





Social planner wants to distort consumption of all types down, high-demand types up and low-demand types down, or all types upwards.

Proof Intuition: q^* is unconstrained optimal where $J(\theta) \geq \theta$, and the (IR) and (TU) constraints bind exactly where $J(\theta) \leq \theta$.

Optimal Subsidy Schedule

Positive Correlation

Case 1: E$[\omega] \geq \alpha$ (upward distortion for all)

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Optimal Subsidy Schedule

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Case 3: $\max \omega \leq \alpha$ (downward distortion for all)