Strong monotonicity and perturbation-proofness of exchange economies

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Abstract

We study the price impact of small perturbations to Walrasian equilibrium in exchange economies, as might be caused by agents' misreports, changes in the supply vector, or changes in the set of participants. A sequence of markets is *perturbation-proof* if the price impact of any perturbation is inversely proportional to the number of agents. Perturbation-proofness implies good large-market incentive properties of Walrasian equilibrium and robustness of prices to small misspecifications. Replica economies are perturbation-proof if and only if the base economy's demand correspondence is *strongly monotone*. When buyers' types are drawn identically and independently from a distribution with a strongly monotone expected demand correspondence, the resulting sequence of economies is perturbation-proof with high probability.

Keywords: Approximate incentive-compatibility, General equilibrium, Market design, Perturbation analysis, Prices, Strong convexity, Strong monotonicity

JEL Codes: C610, D400, D440, D470, D500, D510.

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1 Introduction

Consider a nested sequence of exchange economies indexed by N, the number of agents in the economy. Suppose we perturb each market slightly by changing the report of an agent; adding or removing some supply; or adding or removing some agents. This paper answers the following question: when does the effect of this perturbation on Walrasian equilibrium prices diminish rapidly in market size, namely at a rate inversely proportional to N?

We call such a sequence of markets *perturbation-proof*. Our motivation for studying perturbation-proofness is its relationship to incentives in Walrasian mechanisms: if a sequence of economies is perturbation-proof, then the benefit of unilateral misreporting in any Walrasian mechanism also diminishes rapidly in market size.

Walrasian mechanisms are of interest for two key reasons: first, Walrasian mechanisms are often used as a stylized model of real-world markets, and second, because market designers routinely implement mechanisms that choose or approximate Walrasian equilibria. A concern with these mechanisms, known since Hurwicz (1972), is that they may be vulnerable to strategic manipulation by agents with private information. While price-taking typically obtains in the limit as $N \to \infty$ (Roberts & Postlewaite, 1976), a key question is how these mechanisms perform in the modest-sized markets encountered in applications. Previous literature has demonstrated fast rates of convergence (in N) of incentives for price-taking behavior in narrow preference domains.[1] In this paper, we provide a general condition on demand called *strong monotonicity* that implies perturbation-proofness and the rapid convergence of reporting incentives.[2] These results may make it easier to assess the likely performance of Walrasian mechanisms—with respect to strategic incentives—in new economic environments.

Strong monotonicity is a condition on how quickly demand changes in response to price changes. With one consumption good, strong monotonicity requires that the slope of the demand curve is bounded away from zero.[3] The general definition appears in Section 3. Markets in which all buyers have strongly monotone demand (Theorem 1) and replica economies of a strongly monotone base economy (Theorem 2) are perturbation-proof. With one good, the intuition for this result is simple: as the number of agents with strongly monotone demand grows, the market demand curve

[1] Such as the unit-demand double auction of Satterthwaite and Williams (1989) and the linear-quadratic models surveyed by Rostek and Yoon (2020)

[2] Strong monotonicity and the related notion of strong convexity (discussed in Appendix A) are used routinely in perturbation analysis and in computer science for the analysis of algorithms.

[3] More accurately, all upper and lower derivatives of demand are no larger than $-m$ for some $m > 0$.

becomes increasingly steep and so a small movement in the supply curve[4] leads to increasingly small movements in the price coordinate of the intersection of supply and demand. For replica economies, we show that strong monotonicity is also *necessary* for perturbation-proofness.

We then study a private valuations model of markets, in which buyer types are drawn identically and independently according to a distribution with strongly monotone *expected* demand. By Theorem 3, the resulting sequence of markets is perturbation-proof with high probability and in expectation (over draws of the market). This implies that the ex post benefit of *any* misreport by a single agent is $O(1/N^{1-\varepsilon})$ with high probability (namely $1 - O(1/N^{1-\varepsilon})$) for any $\varepsilon > 0$. A corollary is that the interim expected benefit of the optimal misreport is $O(1/N^{1-\varepsilon})$, which is faster than the $O(1/N^{\frac{1}{2}-\varepsilon})$ rate of interim incentives implied by the "strategy-proofness in the large" results of Azevedo and Budish (2019).

We then apply our results to economic models with indivisibilities, in which strong monotonicity of expected demand is a condition only on the prices at which demand changes and not the size of these demand changes (each demand change is bounded below by the size of the indivisibility). We provide a simple characterization of strong monotonicity in this setting: expected demand is strongly monotone if the probability that demand changes between any two prices grows at least proportionally to the distance between the two prices. We interpret this as a condition on *variety* in the possible preferences of buyers and *uncertainty* about the reservation prices associated with demand changes (we formalize these notions below). We apply our results to derive new incentive properties of the Walrasian mechanism in a market with complementarities in buyers' preferences.

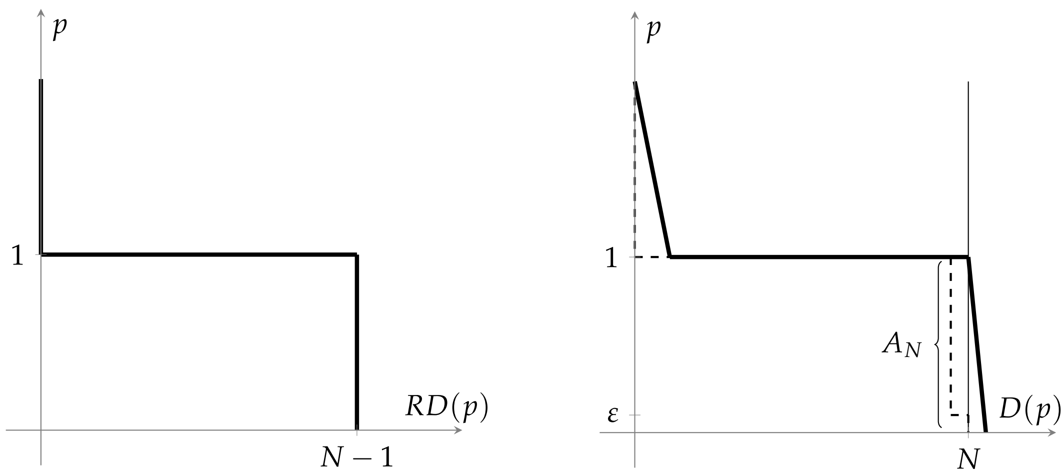
Examples In this section, we contrast two sequences of markets—one in which a buyer has a large influence on the price independently of the market size and one in which each buyer has a $O(1/N)$ influence on the price in expectation—to illustrate the important role of the demand curve's slope on the price impact of small perturbations.

Example 1.1. Consider an economy with a single consumption good and N buyers. The first $N-1$ buyers have unit demand for the good with value 1, while the N^{th} buyer's demand as a function of price is $D_N(p) = \max\{2-p, 0\}$. The mechanism designer uses a Walrasian mechanism in this market.

Suppose the supply is N and all buyers report their preferences truthfully. In that case, buyer N receives one unit in equilibrium at a price of 1. However, if buyer N misreports and claims to

[4] In Proposition 1, we show all perturbations in the first paragraph may be thought of as changes in supply.

have unit demand with value ε, the Walrasian equilibria prices are in $[0, \varepsilon]$, so buyer N may effect a price arbitrarily close to zero, regardless of the auctioneer's decision rule in the case of multiple equilibria. Thus, the set of Walrasian equilibrium prices attainable to buyer N by some report is $A_N = [0, 1]$. Since the buyer is assigned a single unit of the good under any of these reports, the buyer must be made better off by any report that lowers the price.



(a) Residual demand (without buyer N) (b) Truthful and perturbed demand with buyer N

Figure 1: Demand functions for Example 1.1

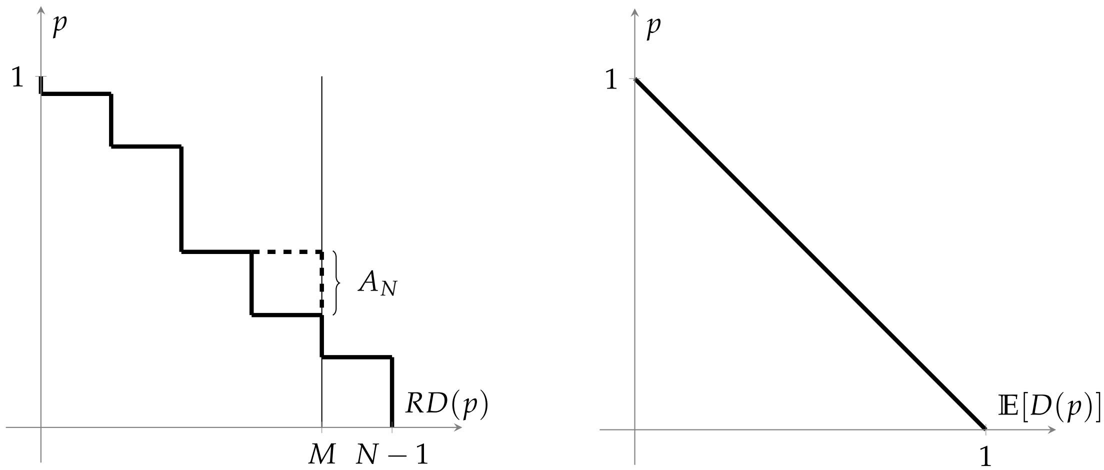
In this case, a small change in the report of one agent had a substantial impact on prices, even when that agent is small relative to the size of the economy. We demonstrate below that the main cause is that the residual demand curve, $RD(p)$, the sum of the demand curves of buyers 1 through $N - 1$, is flat near the equilibrium price (even in the limit as $N \to \infty$), as illustrated in Figure 1. This allows for a small change in the reported demand function of one agent to move the intersection with the supply curve a relatively large distance in price space.

Example 1.2. Again, consider an economy with N buyers and a single good with supply $M < N$. Buyer $n \in \{1, ..., N\}$ has unit demand for the good with value a_n, where a_n is drawn independently and uniformly on $[0, 1]$. We are interested now in the *expected* influence that any single agent may have on the Walrasian equilibrium price(s), where this expectation is taken over draws of the N agents.

Consider the problem from the perspective of agent N, supposing that all other agents truthfully report their values to the mechanism designer and that the agent is restricted to reporting unit demand. In this case, the set of prices that the agent may, by *some* report,[5] realize is $A_N :=$

[5]Not necessarily an optimal, or even beneficial, report.

$[a^{(M-1)}, a^{(M)}]$ where $a^{(i)}$ is the i^{th} order statistic of the $N-1$ *other* draws of the valuation distribution. Because the expected spacing of the uniform order statistics is $O(1/N)$, the expected maximum impact of agent 1 on the equilibrium price is $O(1/N)$ as well. The same logic holds for any valuation distribution which has full support on some interval, with a density uniformly bounded away from zero (that is, $f(x) > c > 0$ for all $x \in \operatorname{supp}(f)$).[6]



(a) Example of a residual demand curve and attainable prices for buyer N

(b) Expected demand curve for an individual agent (equivalently average demand as $N \to \infty$)

Figure 2: Demand functions for Example 1.2

While the residual demand curve has flat segments for finite N, as $N \to \infty$, these flat segments become small when normalized by supply, approaching the negative-sloping demand curve illustrated on the top-right panel of Figure 2. We say that this expected demand curve, which has a slope uniformly bounded away from zero, is *strongly monotone*, introduced formally in Section 3. We show in Section 4 that the strong monotonicity property of the expected demand is what drives the rapid convergence of incentives toward price-taking behavior in this example.

Related literature The motivating application of our perturbation analysis is to the study of *ex post* incentives in the Walrasian mechanism. Hurwicz (1972) first observed that agents with private information about their preferences may benefit from strategically misreporting demand in order to influence the price vector. This problem is more pervasive than just Walrasian equilibrium: the celebrated theorem of Green and Laffont (1979) implies that there is no mechanism in the quasilinear domain which is strategy-proof, efficient and budget-balanced. In large markets, however, Roberts and Postlewaite (1976) showed that the benefits of misreporting in a Walrasian mechanisms for

[6]See, for example, Satterthwaite and Williams (1989).

any individual agent must tend to zero, under the condition that the Walrasian equilibrium price correspondence (mapping measures over the function space of possible excess demand functions to prices) is continuous at the limit economy. Jackson (1992) extended this result to show that the L^∞ distance between the true preferences and an optimal report must also tend to zero under the same condition. He, Miralles, Pycia, and Yan (2018) employ a similar condition to establish approximate incentive compatibility in replica economies associated with pseudomarkets à la Hylland and Zeckhauser (1979). However, the *rates* of convergence are not studied in these papers, and so it may be difficult for practical market designers to assess whether to expect good reporting incentives in real-world applications. Furthermore, the regularity and continuity conditions used in these results can be challenging to apply, because they rely on attributes of the Walrasian equilibrium price correspondence rather than underlying properties of the agents' preferences.

The rates of convergence of *ex post* incentives have been studied in several specific models, including the unit-demand double auction of Satterthwaite and Williams (1989) and linear-quadratic finance models surveyed by Rostek and Yoon (2020). Satterthwaite and Williams (1989) show that as long as values and costs are drawn i.i.d. from a full-support distributions with a lower-bounded density, the maximum benefit from misreporting is $O(1/N)$ and the distance between the true and optimal reports is $O(1/N)$. Similar to this paper, the finance literature surveyed by Rostek and Yoon (2020) emphasizes the relationship between the slope of the aggregate demand and the incentives for price-taking behavior in Bayes-Nash equilibrium (again $O(1/N)$ as long as the slope grows with N), specialized to the case of linear-quadratic preferences. This paper identifies the general property of the demand curve that drives the incentive results in these specific models.

Azevedo and Budish (2019) show that all envy-free mechanisms, including Walrasian mechanisms, are "strategy-proof in the large", which means that the expected benefit to a single agent of misreporting in response to any full-support i.i.d. distribution of opponent reports is $O(1/N^{\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$. Our result is both stronger and weaker than that of Azevedo and Budish (2019): weaker, because we restrict attention to the Walrasian mechanism and require strong monotonicity, but stronger, because the rate of convergence is faster; our results apply to both ex post and interim incentives; and we do not require their assumption that the type space for agents is finite.

Al-Najjar and Smorodinsky (2007) take an alternative approach to studying the influence of strategic behavior on market mechanisms, focusing on the Bayes-Nash equilibria (BNE) of the associated reporting game. Al-Najjar and Smorodinsky (2007) show that for any level of approximation there exists a sufficiently large $\overline{N}$ such that the outcome associated with any BNE of a competitive mechanism with at least $\overline{N}$ participants is approximately efficient. Unlike this paper, their approach does not characterize the ability of an agent to influence on prices, rather the *number* of agents who can influence prices. Moreover, Al-Najjar and Smorodinsky (2007) require a finite type space and a small probability that agents are not strategic.

Organization The remainder of this paper is organized as follows. In Section 2, we describe the model and introduce the notion of perturbation-proofness of exchange economies. In Section 3, we introduce strong monotonicity of demand and establish perturbation-proofness in large markets in which *all* agents have strongly monotone demand (Theorem 1) and replica economies of markets with strongly monotone demand (Theorem 2). In Section 4, we consider the case where agents are drawn i.i.d. from a distribution with strongly monotone expected demand and show in Theorem 3 (the key technical result of this paper) that the resulting markets are perturbation-proof with high probability. Finally, in Section 5, we apply our results to markets with indivisibilities.

Notation We will model consumption bundles as vectors in Euclidean space $\mathbb{R}^L$, equipped with the standard inner product $x \cdot y = \sum_{\ell=1}^{L} x_\ell y_\ell$, and norm $\|x\| = \sqrt{x \cdot x}$. We use $\geq$ to denote the partial order on $\mathbb{R}^L$ so that $x \geq y$ if and only if $x_\ell \geq y_\ell$ for $\ell = 1, ..., L$. The set $\mathbb{R}^L_+$ is $\{x \in \mathbb{R}^L : x \geq 0\}$, while $\mathbb{R}^L_{++}$ is $\{x \in \mathbb{R}^L : x_\ell > 0 \text{ for } \ell = 1, ..., L\}$. The notation $|\cdot|$ represents either the absolute value (if its argument is a number) or the cardinality (if its argument is a set). The distance between $x \in \mathbb{R}^L$ and a set $S \subseteq \mathbb{R}^L_+$ will be $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$. The Hausdorff distance between two sets $S, S' \subseteq \mathbb{R}^L_+$ is $d_H(S, S') = \max\{\sup_{x \in S'} \text{dist}(x, S), \sup_{x \in S} \text{dist}(x, S')\}$.

For a convex function $f : S \to \mathbb{R}$, we say $v \in \mathbb{R}^L_+$ is a subgradient of f at x if for any $x' \in S$, $f(x') - f(x) \geq v \cdot (x' - x)$. The subdifferential $\partial f(x)$ is the nonempty, convex, compact set of subgradients of f at p. Where the gradient of f is well-defined (which, by Rademacher's Theorem, is almost everywhere), we have $\partial f(x) = \{\nabla f(x)\}$.

Finally, we use the asymptotic notation of Knuth (1976), where for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we say $f(x) = O(g(x))$ if $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$; $f(x) = o(g(x))$ if $\lim_{x \to \infty} \frac{|f(x)|}{g(x)} = 0$; $f(x) = \Omega(g(x))$ if $\liminf_{x \to \infty} \frac{f(x)}{g(x)} > 0$; and $f(x) = \Theta(f(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

2 Exchange economies and perturbations

We consider the setting of an exchange economy with L types of consumable good and a numeraire good, money.

There is a finite set of buyers $\mathscr{N}$ and N is its cardinality. Each buyer $n \in \mathscr{N}$ can consume any bundle of consumable goods $x_n \in X_n$ where X_n is a convex, compact subset of $\mathbb{R}^L_+$, called the *consumption possibility set*. Assume that $0 \in X_n$ for each $n \in \mathscr{N}$.

Each buyer has quasilinear preferences over commodity bundles in X_n with a *valuation function* $v_n : X_n \to \mathbb{R}$, so that the agent's *utility* associated with allocation x_n after payment t is $U_n(x_n, t) = v_n(x_n) - t$. We will assume that the valuation functions are drawn from a function space $\mathscr{V}$, such that each $v_n \in \mathscr{V}$ is monotone, concave[7] and satisfies the normalization $v_n(0) = 0$.

There is an exogenous *supply vector* $s \in \mathbb{R}^L_{++}$ for the consumable goods, which is nonnegative in all components.[8] Buyers are unconstrained in their spending of money.

We refer to $\mathscr{E} = \langle \mathscr{N}, s \rangle$ as a *market*.

Efficiency, equilibrium and mechanism design An *allocation* $\mathbf{x} = (x_n)_{n \in \mathscr{N}}$ is an assignment of consumption bundles $x_n \in X_n$ to each buyer $n \in \mathscr{N}$. Allocation $\mathbf{x}$ is *feasible* in $\mathscr{E}$ if $\sum_{n \in \mathscr{N}} x_n \leq s$. The set of all feasible allocations for $\mathscr{E}$ is denoted $\mathscr{X}$.

The *surplus* associated with allocation $\mathbf{x}$ is defined by $\mathcal{S}(\mathbf{x}) = \sum_{n \in \mathscr{N}} v_n(x_n)$. An *efficient allocation* for $\mathscr{E}$ is a feasible allocation $\mathbf{x}$ that solves the surplus maximization problem $\max_{\mathbf{x} \in \mathscr{X}} \mathcal{S}(\mathbf{x})$.

Let $p \in \mathbb{R}^L_+$ be a price vector and $D_n : \mathbb{R}^L_+ \rightrightarrows X_n$ the *demand correspondence* of buyer n, the set of maximizers of $U_n(x, p \cdot x)$. Throughout we will assume for each agent $n \in \mathscr{N}$ that $D_n(p) = \{0\}$ for prices p outside of a compact set $\mathscr{P} \subseteq \mathbb{R}^L_+$. The indirect utility function $u_n : \mathbb{R}^L_+ \to \mathbb{R}$ is $u_n(p) = \max_{x \in X_n} U_n(x, p \cdot x)$. The indirect utility function is convex and related to the demand correspondence by the identity $\partial u_n(p) = -D_n(p)$.

A *Walrasian equilibrium* of $\mathscr{E}$ is a feasible allocation $\mathbf{x} \in \mathscr{X}$ and a price vector $p \in \mathbb{R}^L_+$ such that

(a) markets clear—that is, $\sum_{n \in \mathscr{N}} x_n = s$, and

(b) assignments are demanded given the price vector—that is, $x_n \in D_n(p)$ for each $n \in \mathscr{N}$.

The assumptions made on preferences ensure that Walrasian equilibria exist and are efficient (see Appendix B). We write $W(\mathscr{E})$ for the set of Walrasian equilibrium price vectors of $\mathscr{E}$.

Buyers report their preferences to a mechanism designer who determines an outcome and transfers. By the revelation principle, we may restrict our attention to mechanisms in which

[7] The main function of this assumption is to ensure that Walrasian equilibria exist. For analysis of linear pricing mechanisms, including incentives, without the assumption of concavity, see Milgrom and Watt (2021).

[8] In Appendix C, we discuss extensions of our results to setting where supply decisions are made by participants in the mechanism.

buyers report their valuation functions v_n (and implicitly the domain X_n) to the market designer. In this analysis, we abstract away from the important question of how bidders communicate these potentially complicated objects to the mechanism designer.[9] In the *Walrasian mechanism*, the mechanism designer (or Walrasian auctioneer) determines[10] and implements the Walrasian equilibrium prices and allocations based on the reported valuation functions, with some predetermined decision rule if the Walrasian equilibrium is not unique. In this paper, this decision rule will not play a key role.

Perturbations and perturbation-proofness We now introduce formally the object of our analysis, perturbations of exchange economies, and introduce the concept of perturbation-proofness.

Definition 2.1. Let $\mathscr{E} = \langle \mathscr{N}, s \rangle$ be a market, which we call the *original* or *unperturbed market*. A *perturbation* is a vector $\delta s \in \mathbb{R}^L$ such that $s + \delta s \geq 0$. We refer to $\mathscr{E}' = \langle \mathscr{N}, s + \delta s \rangle$ as the *perturbed market* and $\|\delta s\|$ as the *size* of the perturbation.

Note here that we are interested in small but *finite* perturbations, distinguishing this analysis from the study of shadow prices, which are relevant only for infinitesimal perturbations. We will study the influence of these perturbations as the market becomes large, in the following sense.

Definition 2.2. A *nested market sequence* is a sequence of markets $(\mathscr{E}_t)_{t \in \mathbb{N}}$ where $\mathscr{E}_t = \langle \mathscr{N}_t, s_t \rangle$, and $\mathscr{N}_{t+1} \supseteq \mathscr{N}_t$ for all $t \in \mathbb{N}$ with $N_t \to \infty$.

We now introduce the concept of perturbation-proofness, which requires that small perturbations of markets in a nested market sequence lead to small changes in price.

Definition 2.3. A nested market sequence $(\mathscr{E}_t)_{t \in \mathbb{N}}$ is $O(f(t))$*-perturbation-proof* if for all sequences of perturbations $(\delta s_t)_{t \in \mathbb{N}}$ with size $\|\delta s_t\| \leq O(1)$, we have that $\|p_t - p'_t\| \leq O(f(t))$ for any $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$.

Note that this definition implicitly requires that the set of Walrasian equilibrium prices $W(\mathscr{E}_t)$ is small—that is, with diameter no larger than $O(f(t))$— since $\delta s_t = 0$ for all t is a valid perturbation.

In this paper, we will be almost exclusively interested in $O(1/N_t)$-perturbation-proofness (or for technical reasons in markets with incomplete information, the very close rate of $O(1/N_t^{1-\varepsilon})$

[9]The design of bidding languages to report complex preferences has been the subject of substantial study, including by Milgrom (2009), Bichler, Goeree, Mayer, and Shabalin (2014), Bichler, Milgrom, and Schwarz (2020).

[10]Here we are also implicitly assuming that the Walrasian equilibrium can be computed efficiently and exactly by the market designer, which is, in general, a non-trivial assumption given that the problem of computing Walrasian equilibrium is PPAD-complete. However, we show in Appendix D that Walrasian equilibrium *can* be approximated efficiently in the strongly convex case using tâtonnement (gradient) methods.

for all $\varepsilon > 0$). For expositional purposes (that is, except in the precise statements of theorems), we will use the term "perturbation-proof", with the big O notation omitted, to refer to these rates. However, we leave the definition above general to permit faster or slower rates, as discussed in Appendix A, and rates that depend on s_t or δs_t as well as N_t.

At first glance, our definition of perturbations may appear very narrow, allowing only for changes in the supply vector. However, we now show that our definition of perturbation-proofness implies robustness—at the same rate—to two other important changes to the economy, namely additions to the set of agents and misreporting by agents.

Proposition 1. *Suppose that $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$-perturbation-proof, and consider the following changes to the economy:*

(a) Misreporting*: Let n be an agent in $\mathscr{N}_t$ for each $t \geq T$ and let v_n be its valuation. Obtain $\mathscr{E}'_t$ by replacing v_n by some $v'_t \in \mathscr{V}$ for $t \geq T$.*

(b) Addition of buyers*: Suppose $\mathscr{N}_0$, a finite subset of buyers from $\mathscr{V}$, is added to each $\mathscr{E}_t$ to obtain $\mathscr{E}'_t$.*

In both cases, $\|p_t - p'_t\| \leq O(f(t))$ for all $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$.

Note that we may consider the problem of removing agents by swapping the role of $\mathscr{E}_t$ and $\mathscr{E}'_t$ in (b) and the problem of misreporting by a bounded number of agents (that is, $O(1)$ in N_t) by repeated application of (a).

Proof. We omit the t index for clarity. The necessary and sufficient conditions for Walrasian equilibrium of the original economy is $D(p) = s$. For (a), suppose that agent n_0 receives allocation x_0 under truthful reporting, while obtains an allocation of $\tilde{x}$ under an announcement that induces Walrasian equilibrium price $\tilde{p}$. Buyer n_0's announcement must satisfy $\sum_{n\in\mathscr{N}\setminus\{n_0\}} D_n(\tilde{p}) + \tilde{x} = s$. But this is the same as the necessary and sufficient conditions for equilibrium of the problem $D(p) = s + x_0 - \tilde{x}$, which corresponds to a perturbation of $\mathscr{E}$ by $\delta s = x_0 - \tilde{x}$, which is $O(1)$ in N_t since X_n is bounded. Thus the effect of misreporting on prices may be thought of as a perturbation in the sense of Definition 2.1. The same general idea works for the addition of buyers; the equivalent perturbation is $-\sum_{n\in\mathscr{N}_0} D_n(\tilde{p})$ where $\tilde{p}$ is the induced price in the perturbed economy, which is $O(1)$ in N_t since N_0 and each X_n are bounded.[11] □

[11]Morover, the same idea works for the addition (or subtraction) of any Lipschitz convex function to the dual objective of the efficient allocation problem (discussed in Appendix B). Such functions have bounded subdifferentials, and so the effect of their addition on the necessary and sufficient conditions of the dual problem are equivalent to $O(1)$ changes in the supply vector. Each perturbation discussed above—supply vector changes, misreporting and the addition of agents—may be interpreted as Lipschitz perturbations of the dual objective.

Finally, we note the relationship between perturbation-proofness and approximate incentive compatibility of the Walrasian mechanism.

Definition 2.4. We say a mechanism is *ex post $O(f(t))$-incentive compatible* on $(\mathscr{E}_t)_{t\in\mathbb{N}}$ if for each agent $n \in N_t$, holding fixed the truthful reports of the other agents, the maximum benefit of misreporting under the mechanism in $\mathscr{E}_t$—the utility under the optimal misreport minus the utility received under truthful reporting—is $O(f(t))$.

Proposition 2. *If $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$-perturbation-proof, then the Walrasian mechanism is $O(f(t))$-incentive compatible on $(\mathscr{E}_t)_{t\in\mathbb{N}}$.*

Proof. Since $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is perturbation-proof, any misreport by an agent, including the optimal one, results in a price p'_t satisfying $\|p'_t - p_t\| \leq O(f(t))$, where $p_t \in W(\mathscr{E}_t)$. The utility realized by the agent under the misreport is no larger than $u_n(p'_t)$, while the utility realized under truthful reporting in the Walrasian mechanism is exactly $u_n(p_t)$, so that the benefit of misreporting is bounded above by $u_n(p'_t) - u_n(p_t)$. Since u_n is Lipschitz, this expression is at most a constant multiple of $\|p'_t - p_t\|$ which is $O(f(t))$.[12] $\square$

3 Strong monotonicity and complete information exchange economies

We now introduce the key condition on demand that results in perturbation-proofness of exchange economies.

Definition 3.1. Buyer n is *active* at price p if $D_n(p) \neq \{0\}$.

Definition 3.2. Buyer n's demand correspondence $D_n : \mathbb{R}^L_+ \rightrightarrows X_n$ is *strongly monotone* if there exists some $m > 0$ such that for all prices $p, p' \in \mathbb{R}^L_+$ where buyer n is active,

$$(d' - d) \cdot (p - p') \geq m\|p - p'\|^2, \text{ for all } d \in D_n(p) \text{ and } d' \in D_n(p'). \tag{SM}$$

For economy $\mathscr{E} = \langle \mathcal{N}, s\rangle$, we say the *total* demand correspondence $D = \sum_{n\in\mathcal{N}} D_n$ is strongly monotone if inequality (SM) holds for all p, p' such that *at least* one buyer $n \in \mathcal{N}$ is active and $d \in D(p), d' \in D(p')$.

[12]The Lipschitz property of u_n follows since the Lipschitz constant of a proper, convex function is the magnitude of the largest selection from a subderivative of that function (see Theorem 9.13 in Rockafellar and Wets (2009)). For u_n, this is a demand bundle, of bounded magnitude by the assumption that X_n is compact.

Definition 3.2 is an adaptation to the economic context of the concept of strong monotonicity which was developed in the mathematical study of optimization and used routinely in computer science for the study of algorithms. Definition 3.2 differs from the standard definition of strong monotonicity (in Appendix A) in two key ways: first, a sign change reflecting the fact that D_n is the *negative* subdifferential of the indirect utility function, and second, the requirement that (SM) apply only at prices p, p' where the buyer is active. This latter modification reflects the fact that in our context, demand for each good must be nonnegative.[13] If demand at some price p is zero, the law of demand implies that the demand at price $p' = \alpha p$ for $\alpha > 1$ must also be zero. This implies that the inequality in (SM) cannot be satisfied for prices p, p'.

Note the resemblance of (SM) to the law of demand (in which the right-hand side of (SM) is replaced with a zero). However, whereas the law of demand is a theorem that applies to *all* demand correspondences, not all demand correspondences are strongly monotone.

Markets with strongly monotone agents We first consider the case where *all* agents have strongly monotone demand. This assumption is strong (it is not satisfied, for example, in the unit demand valuations of Example 1.2), but the analysis in this setting provides intuition for other results.

Theorem 1. *Consider a nested market sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ in which all agents have strongly monotone demand with constant $m > 0$. Then $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O\left(1/N_t^a\right)$-perturbation-proof where N_t^a is the number of buyers who are active at some prices $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$.*

Note that we speak of 'the' equilibrium price in the original and perturbed economies under the assumption that all agents have strongly monotone demand. This is because strong monotonicity of demand implies that inverse demand is single-valued (except for $s = 0$).[14]

Before proving Theorem 1, we discuss the intuition for the result in the setting with a single consumable good. The strong monotonicity of each agent's demand implies that the slope of the market demand curve at any price (including the equilibrium price) grows proportionally to the number of active buyers at that price. As the demand curve becomes increasingly steep at the equilibrium price, small perturbations of the supply vector have a progressively smaller effect on the resulting price.[15]

[13] It is also possible to extend our results to settings in which agents may be "traders"—buyers at some prices and sellers at others—with a modified activity requirement, as discussed in Appendix C.

[14] Alternatively, using the notions developed in Appendix A and Appendix B, we have that the dual objective is strongly convex (and thus strictly convex) so that the equilibrium price is always unique.

[15] In Appendix D, we show that this same logic also implies that markets in which all agents have strongly monotone demand are tâtonnement-stable, with a fast (subpolynomial) rate of convergence of tâtonnement.

Proof. By assumption, for each active buyer n at prices p_t, p'_t, we have for any $d_n \in D_n(p_t)$ and $d'_n \in D_n(p'_t)$

$$(d'_n - d_n) \cdot (p_t - p'_t) \geq m\|p_t - p'_t\|^2, \tag{1}$$

so that

$$\sum_{n \in \mathcal{N}_t} (d'_n - d_n) \cdot (p_t - p'_t) \geq N_t^a m\|p_t - p'_t\|^2. \tag{2}$$

By definition of Walrasian equilibrium, we have $\sum_{n \in \mathcal{N}_t} d'_n = s'$ and $\sum_{n \in \mathcal{N}_t} d_n = s$, so that

$$(s' - s) \cdot (p_t - p'_t) \geq N_t^a m\|p_t - p'_t\|^2.$$

Since $s' - s = \delta s$, by the Cauchy-Schwarz inequality, we then have

$$\|\delta s\| \left\|p_t - p'_t\right\| \geq N_t^a m \left\|p_t - p'_t\right\|^2,$$

or on re-arranging,

$$\|p_t - p'_t\| \leq \frac{\|\delta s\|}{mN_t^a} \leq O\left(\frac{1}{N_t^a}\right).$$

$\square$

Necessary conditions and replica economies In the remainder of this section and Section 4, we weaken the assumption in Theorem 1 that *each* agent has strongly monotone demand. In doing so, we require more structure on the sequence of economies we analyze (as in Theorem 2) or accept weaker conclusions (in Section 4 we settle for probabilistic results).

An alternative, if imprecise, interpretation of the proof of Theorem 1 motivates our approach. Again, we focus on the case with one consumable good. When all agents have strongly monotone demand functions, the *average* (per-capita) demand curve (averaged over the number of active buyers at price p_t) is downward sloping. The market-clearing condition of the Walrasian mechanism must hold for the per-capita economy with per-capita supply. On the other hand, the per-capita perturbation diminishes at a rate inversely proportional to the number of active buyers at price p_t. So, if the average demand curve is bounded away from zero at p_t, the effect of a perturbation on prices diminishes at the same rate as the per-capita size of the perturbation.

This suggests the importance of the *average* demand correspondence for the analysis of perturbations. Given an average demand correspondence, the simplest associated sequence of economies is the replica economies of Debreu and Scarf (1963).

Definition 3.3. Let $\mathcal{N}$ be a set of buyers and define $\mathcal{N}_k$, its $k-$*fold replica*, as the set of kN buyers such that for each $n \in \mathcal{N}$, there are k buyers in $\mathcal{N}_k$ with the same preferences as n. The $k-$fold replica of a *base economy* $\mathscr{E} = \langle N, s \rangle$ is $\mathscr{E}_k = \langle \mathcal{N}_k, ks \rangle$.

In replica economies, the average demand correspondence is constant with respect to the number of replicas. In this setting, Theorem 2 states that strong monotonicity of the average demand correspondence is a necessary and sufficient condition for the conclusions of Theorem 1 to hold for all possible supply vectors s.

Theorem 2. *Let N be a market with total demand correspondence $D = \sum_{n \in \mathcal{N}} D_n$, and let $\mathcal{N}_k$ be its $k-$fold replica. Then $\mathscr{E}_k = \langle \mathcal{N}_k, ks \rangle$ is $O(1/N_k)$-perturbation-proof for all base economy supply vectors s if and only if D is strongly monotone.*

4 Markets with private valuations

For the remainder of this paper, we will study exchange economies that generalize the private value model of auctions. Instead of drawing a single value parameter according to a common prior distribution, agents draw preferences from an abstract (potentially infinite-dimensional) function space.

Definition 4.1. Let $\mathscr{V}$, equipped with an appropriate σ-algebra, be a measurable space of valuation functions, [16] and let ν be a distribution over $\mathscr{V}$, which is common knowledge. We say that a market $\mathscr{E} = \langle N, s \rangle$ has *private valuations* if the valuation function of each buyer in $\mathcal{N}$ is drawn identically and independently from ν.

In this section, we make one important assumption on the set of buyer types.

Assumption 1. *Given distribution ν on $\mathscr{V}$, there exists a compact set $\mathscr{P} \subseteq \mathbb{R}^L_+$ such that $D_n(p) = \{0\}$ ν-almost surely for $p \notin \mathscr{P}$.*

Assumption 1 is embedded in many auction models and will be required for the main results in this section.[17]

[16] For example, by a result of Aumann (1963), $\mathscr{V}$ could be taken as the set of bounded, continuous functions on a compact subset of $\mathbb{R}^L_+$, or the set of bounded functions with discontinuities of the first kind, or, more generally, any subset of a Baire class. In particular, we may further restrict to require that the valuation functions are monotone, concave and normalized in accordance with our previous assumptions.

[17] It is simple to modify our results to require only that the equilibrium price belongs to a compact set $\mathscr{P}$ almost surely, but we have chosen the formulation of Assumption 1 because it does not require knowledge of supply vector s.

We now introduce the relevant notion of strong monotonicity in this environment.

Definition 4.2. Given distribution ν on $\mathscr{V}$, define the *expected indirect utility function* pointwise for $p \in \mathscr{P}$ by

$$\mathbb{E}_\nu[u(p)] = \int_{\mathscr{V}} u_n(p) d\nu(u_n).$$

The *expected demand correspondence* is defined by [18]

$$\mathbb{E}_\nu[D(p)] = -\partial \mathbb{E}_\nu[u(p)].$$

We say that ν on $\mathscr{V}$ has *strong monotonicity in expectation* if $\mathbb{E}_\nu[D(p)]$ is a strongly monotone demand correspondence in the sense of Definition 3.2.

Note that strong monotonicity in expectation does not require that the individual agents' demands are strongly monotone. For example, in Example 1.2, the individual demand correspondences are step functions (not strongly monotone), while the expected demand function is strongly monotone.

Perturbation-proofness in probability We now introduce a probabilistic notion of perturbation-proofness that is appropriate in economic settings with private valuations.

Definition 4.3. Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$. The sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(f(t))$*-perturbation-proof with probability* $g(t)$ if for all sequences $(\delta s_t)_{t\in\mathbb{N}}$ of perturbations with size $\|\delta s_t\| \leq O(1)$, we have that $\|p_t - p'_t\| \leq O(f(t))$ for $p_t \in W(\mathscr{E}_t)$ and $p'_t \in W(\mathscr{E}'_t)$, with probability $g(t)$ over draws of $\mathscr{E}_t$.

Under the assumption that ν has strong monotonicity in expectation, we obtain the following result.

Theorem 3. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$, and suppose that ν on $\mathscr{V}$ has strong monotonicity in expectation. Then for all $\varepsilon > 0$, we have that $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O(1/N_t^{1-\varepsilon})$-perturbation-proof with probability $1 - O(1/N_t^{1-\varepsilon})$.*

[18] The expected demand correspondence can be defined equivalently using the set-valued integral of Aumann (1965). That is, for any fixed p, the probability measure ν induces a probability measure over the sets $D_n(p)$ associated with valuation function $v_n \in \mathscr{V}$. A selection $\xi : \mathscr{V} \to X$ is a single-valued random vector such that $\xi(v_n)$ ν-almost surely belongs to $D_n(p)$ for each $v_n \in \mathscr{V}$. Then $\mathbb{E}_\nu[D_n(p)]$ is defined as $\mathrm{cl}(\{\mathbb{E}_\nu \xi\})$ over integrable selections ξ. A result of Rockafellar and Wets (1982) implies equivalence with Definition 4.2. Moreover, a law of large numbers applies to $\mathbb{E}_\nu[D_n(p)]$ pointwise so that for all $p \in \mathscr{P}$, $d_H\left(\frac{1}{|N_t|}\sum_{n\in N_t} D_n(p), \mathbb{E}_\nu[D_n(p)]\right) \to 0$ as $|N_t| \to \infty$, where N_t is a set of agents drawn i.i.d. from ν (Weil, 1982).

The proof of Theorem 3, presented in Appendix E, involves establishing the concentration of the empirical average demand correspondence around the expected demand correspondence, via an application of Bernstein's Inequality.

Reporting incentives We now describe the implications of Theorem 3 for reporting incentives in Walrasian mechanisms. Our results pertain to incentives under two informational structures, first, *ex post* incentive compatibility as in Definition 2.4 in which agents choose their reports with common knowledge of all agents' drawn valuations and, second, *interim* incentive compatibility, defined below.

Definition 4.4. Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$. Suppose that each agent chooses its report with knowledge of its own valuation, without knowing the valuations drawn by other agents, and with s_t, ν and the number of agents common knowledge. Then $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is *interim $O(f(t))$-incentive compatible* if, for each agent in the mechanism, the expected payoff of the optimal report minus the expected payoff of the truthful report is $O(f(t))$.

The following incentive properties of Walrasian mechanisms follow from Theorem 3.

Theorem 4. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a nested sequence of markets with private valuations drawn i.i.d. from distribution ν on $\mathscr{V}$, and suppose that ν on $\mathscr{V}$ has strong monotonicity in expectation. Then for all $\varepsilon > 0$, we have that a Walrasian mechanism on $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is*

(a) ex post $O(1/N_t^{1-\varepsilon})$-incentive compatible with probability $1 - O(1/N_t^{1-\varepsilon})$ over draws of $\mathscr{E}_t$.

(b) interim $O(1/N_t^{1-\varepsilon})$-incentive compatible.

5 Strong monotonicity with indivisible goods

In this section, we specialize our analysis to economies with indivisible goods, so that $X_n \subseteq \mathbb{Z}_+^L$. We study a model with indivisibilities for two key reasons: first, indivisibilities are a natural assumption in many important markets, and second, the change in demand associated with any price change is bounded below by the size of the indivisibility. This allows us to focus, for the purpose of establishing (expected) strong monotonicity, on the prices at which demand changes, rather than also concerning ourselves with the size of these demand changes.

In models with indivisibilities, strong monotonicity of individual demand cannot be observed since prices are a continuous variable while demand can take on only finitely many values. For this reason, we will focus on models with incomplete information, as in Section 4. The main goal is to establish conditions under which the expected demand correspondence is strongly monotone. In so doing, we will establish two secondary goals: first, we will exhibit applications of our results to settings in which $O(1/N)-$incentive-compatibility has not previously been established, and second, we will offer an interpretation of the strong monotonicity assumption in models with indivisibilities.

Recall that for economies with one good, unit demand buyers and uncertainty in valuations (as in Example 1.2), a sufficient condition for the expected maximum influence on price by any single buyer to be $O(1/N)$ is that a_n be drawn from a distribution with full support on an interval in $\mathbb{R}$ with density bounded below by $\alpha > 0$ (Rustichini, Satterthwaite, & Williams, 1994). We can see that this condition guarantees strong monotonicity of the expected demand since, for $p' > p$, the change in demand grows at least linearly with $p' - p$, that is

$$\mathbb{E}[d(p') - d(p)] \geq \int_{p}^{p'} \alpha d\tilde{p} = \alpha(p' - p).$$

This implies the required inequality, $\mathbb{E}[(d(p') - d(p))(p' - p)] \geq \alpha(p' - p)^2$.

It should be clear that the unit demand structure is not necessary for this result: all that is required is that, for any price p, there is a positive probability to draw marginal buyers and non-buyers of the good (that is, buyer types who would reduce demand in response to a price increase and types who would increase demand in response to a price decrease) and a condition that corresponds to a lower-bounded density. With more goods, we must also consider the many directions in which price changes can occur at any given price.

We formalize this intuition in the following proposition.

Proposition 3 (Expected strong monotonicity for multiple indivisible goods)**.** *Let $\mathscr{P}$ be a compact, convex subset of $\mathbb{R}^L_+$. Suppose there exists some $\alpha > 0$ such that for all $p, p' \in \mathscr{P}$ with $p \neq p'$ we have $\Pr_\nu[D_v(p) \neq D_v(p')] \geq \min\{\alpha\|p' - p\|, 1\}$ for some $\alpha > 0$. Then the expected demand correspondence associated with ν is strongly monotone.*

This condition may be interpreted in terms of two natural assumptions for economic models with indivisibilities. First, *uncertainty* about where (in price space) a demand change occurs: there must be some probability that demand changes (for a type drawn from ν) associated with *any* price change, and larger price changes must lead to a proportionately larger probability that demand changes. Second, and more subtly, the condition reflects *variety* in the preferences. To see this, fix a price p and consider small price changes in each of the coordinate directions from p. For each such price change, there must be some probability that demand changes and by the law of demand, these demand changes must be non-orthogonal to the price change.[19] Taking the limit as the price changes approach zero,[20] we must have that the possibility that demand changes at p in non-orthogonal directions. For example, at price p, some types in the support of ν might be indifferent to buying or not buying good x, while other types are indifferent to buying or not buying good y. In other words, each price p must be a kind of "marginal price" for different demand changes for various agent types. This interpretation is reminiscent of Hildenbrand's (1994) argument that the law of demand reflects primarily "heterogeneity" of the population of households.[21]

Finally, we show by example how Proposition 3 can be used to establish perturbation-proofness (and therefore incentive results) in a novel economic environment. We believe this is suggestive of many new economic environments where approximate incentive-compatibility results may be obtained which have not previously been established.

Example 5.1 (Complementarities)**.** Suppose there are two goods x and y, with $(x, y) \in \{0, 1\}^2$. Buyer n has the following valuation function for the goods

$$v_n(x, y) = a_n x + b_n y + c_n xy,$$

where each of a_n, b_n and c_n are strictly positive real numbers. The demand for such a buyer is illustrated in Figure 3.

Suppose that a_n and b_n are drawn independently from full support distributions on $[0, 1]$ and c_n is drawn from a full support distribution on $[0, 1 - a_n \wedge 1 - b_n]$, all with densities bounded below.

[19] This is a slightly stronger version of the usual law of demand that applies when demand strictly changes. If $x \in D(p)$ and $x \notin D(p') \ni x'$ then $u(x) - p \cdot x \geq u(x') - p \cdot x'$ and $u(x') - p' \cdot x' > u(x) - p' \cdot x$. Adding and rearranging gives $(p' - p) \cdot (x - x') > 0$.

[20] Assuming a sense of continuity of $\mathscr{V}$: namely that if there are demand correspondences $d_n \in \mathscr{V}$ approaching d in the L^∞-norm, then $d \in \mathscr{V}$. Otherwise this analysis applies generically.

[21] Hildenbrand (1994) argues that the law of demand may be derived at the market level from assumptions on the heterogeneity of household, rather than primarily reflecting the rationality of individual agents, as in the classical approach. While we maintain classical rationality assumptions, the forces driving his law of demand and our strong monotonicity of expected demand are similar.

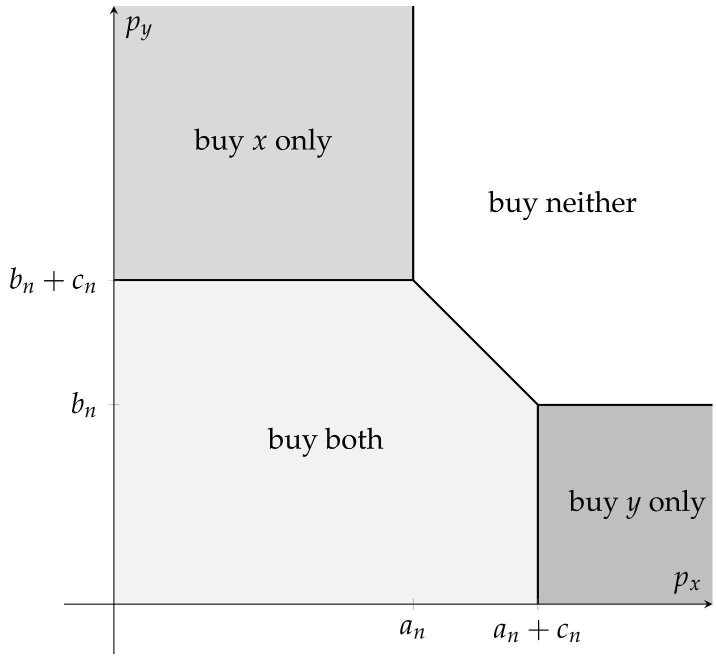


Figure 3: Demands for Example 5.1

Let $\mathscr{P} = [0,1]$ and prices $p, p' \in \mathscr{P}$ be given (without loss of generality, suppose $p_x \leq p'_x$). There are two possibilities, either

(a) $p_x \neq p'_x$. In this case, we have that buyers with $a_n \leq p_x$, $b_n \leq p'_y$ and $c_n \in [p'_x - a_n, 1 - a_n]$ change demand by a multiple of $(1,0)$ as p changes to p'.

(b) $p_x = p'_x$. In this case, we have that buyers with $b_n < p_y$ and $b_n + c_n < p'_y$ with $a_n > p_x$ change demand by a multiple of $(0,1)$ as p changes to p'.

In each case, we see that the probability of drawing such agents grows in $\|p - p'\|$. Note in both cases that there are other buyers who experience demand changes for this price change, we need only obtain a lower bound on the probability of demand changes associated with the price change. This implies the required condition in Proposition 3 and so markets consisting of buyers with such complementarities are perturbation-proof, and so the Walrasian mechanism is ex post $O(1/N^{1-\varepsilon})$-incentive compatible with high probability.

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A Strong monotonicity and related notions

We first introduce a stronger notion of convexity that is used in the perturbation analysis of convex programs. Strong convexity is used routinely in the analysis of convex optimization problems, see, for example, Boyd and Vandenberghe (2004). In the following definitions, K is a compact, convex subset of $\mathbb{R}^N$ and $f : K \to \mathbb{R}$ is a convex function defined on K.

Definition A.1 (Order γ-strong convexity)**.** For $\gamma > 0$, the function f is *order γ-strongly convex* with constant $m > 0$ if

$$f(y) \geq f(x) + s_x \cdot (y - x) + \frac{m}{2}\|y - x\|^{\gamma} \text{ for all } x, y \in K \text{ and } s_x \in \partial f(x).$$

If $\gamma = 2$, we simply say that f is strongly convex with constant m.

Note that by replacing m with zero in Definition A.1, we recover a definition of convexity of function f so that Definition A.1 is a stronger condition than convexity. Informally, a function is order γ-strongly convex if it is possible to fit an order γ polynomial between the function and all of its tangent planes.

In the same way that the convexity of function f is equivalent to the monotonicity of its subdifferential ∂f, the (order γ-)strong convexity of f is equivalent to the (order γ-)*strong monotonicity* of ∂f, as defined below.

Definition A.2 (Order γ-strong monotonicity)**.** Let $s : K \rightrightarrows \mathbb{R}$ be a nonempty-valued correspondence defined on K. For $\gamma > 0$, correspondence s is *order $\gamma-$strongly monotone* with constant $m' > 0$ if

$$(s_y - s_x) \cdot (y - x) \geq m'\|y - x\|^{\gamma}, \text{ for all } x, y \in K \text{ and } s_x \in s(x), s_y \in s(y).$$

For $\gamma = 2$, we just say that s is strongly monotone with constant m'.

Note that by replacing m' with zero in Definition A.2, we obtain the usual definition of a monotone correspondence.

Proposition 4. *Let $f : K \to \mathbb{R}$ be a convex function and $\partial f : K \rightrightarrows \mathbb{R}$ be its subdifferential mapping.*

(a) If f is order $\gamma-$strongly convex with constant $m > 0$, then ∂f is order $\gamma-$strongly monotone with constant m.

(b) If ∂f is order $\gamma-$strongly monotone with constant $m' > 0$, then f is order $\gamma-$strongly convex with constant $2m'/\gamma$.

Proof. First, suppose f is order $\gamma-$strongly convex. Then we have

$$f(y) \geq f(x) + s_x \cdot (y-x) + \frac{m}{2}\|y-x\|^\gamma$$
$$f(x) \geq f(y) + s_y \cdot (x-y) + \frac{m}{2}\|y-x\|^\gamma.$$

Adding these expressions and reorganizing obtains $(s_y - s_x) \cdot (y-x) \geq m\|y-x\|^\gamma$.

For the converse, define $\phi(\lambda) = f(x + \lambda(y-x))$ and $x_\lambda = x + \lambda(y-x)$. Then since $\frac{d\phi}{d\lambda}$ exists almost everywhere and equals $s_\lambda \cdot (y-x)$ for $s_\lambda \in \partial f(x_\lambda)$, we have by the fundamental theorem of calculus that

$$f(y) - f(x) = \phi(1) - \phi(0) = \int_0^1 s_\lambda \cdot (y-x) d\lambda.$$

By assumption, $(s_\lambda - s_x) \cdot (x_\lambda - x) \geq m\|x_\lambda - x\|^\gamma$. This implies that $\lambda(s_\lambda - s_x) \cdot (y-x) \geq m\lambda^\gamma \|y-x\|^\gamma$. Substituting into our expression above, we obtain

$$f(y) - f(x) \geq s_x \cdot (y-x) + \int_0^1 m\lambda^{\gamma-1}\|y-x\|^\gamma d\lambda = s_x \cdot (y-x) + \frac{m}{\gamma}\|y-x\|^\gamma.$$

□

There are several other well-known characterizations of (order 2-)strong convexity (see Boyd and Vandenberghe (2004)). Function f is strongly convex if and only if:

(a) $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{m}{2}\lambda(1-\lambda)\|y-x\|^2$ for all $x, y \in K$ and $\lambda \in [0,1]$.

(b) the function $g : K \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{m}{2}\|x\|^2$ is convex.

Wherever f is twice continuously differentiable, strong convexity requires that $D^2 f(x) - mI$ is positive semi-definite.

Strong convexity also has a dual formulation. Recall that the Fenchel dual of a proper convex function $f : K \to \mathbb{R} \cup \{\pm\infty\}$ defined on $K \subseteq \mathbb{R}^N$ is the function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ satisfying $f^*(x^*) = \sup_{x \in K} x^* \cdot x - f(x)$. The following dual characterization of strong convexity is known (see Borwein and Vanderwerff (2010)).

Proposition 5. *A proper convex function $f : K \to \mathbb{R}$ is strongly convex with constant m if and only if the Fenchel dual $f^* : \mathbb{R}^N \to \mathbb{R}$ is* strongly smooth, *that is,*

$$f^*(y^*) \leq f^*(x^*) + s \cdot (y^* - x^*) + \frac{1}{2m}\|y^* - x^*\|^2, \text{ for all } x^*, y^* \in \mathbb{R}^N \text{ and } s \in \partial f^*(x^*).$$

Equivalently, for all $x^*, y^* \in \mathbb{R}^N$, $s_x \in \partial f^*(x^*)$ and $s_y \in \partial f^*(y^*)$, $(s_y - s_x) \cdot (y - x) \leq \frac{1}{m}\|y^* - x^*\|^2$. This latter condition implies the Lipschitz-continuity of ∇f^* wherever it exists.

The following generalization of Theorem 1 applies for order γ-strong monotonicity.

Proposition 6. *Consider a nested market sequence $(\mathscr{E}_t)_{t\in\mathbb{N}}$ in which all agents have order γ-strongly convex preferences with constant $m > 0$. Then $(\mathscr{E}_t)_{t\in\mathbb{N}}$ is $O\left(1/N_t^{a\,\frac{1}{\gamma-1}}\right)$-perturbation-proof where N_t^a is the number of buyers who are active at some prices $p_t \in W(\mathscr{E}_t)$ and $p_t' \in W(\mathscr{E}_t')$.*

The proof is almost identical to the proof of Theorem 1 and is omitted. Our other main results, Theorem 3, Theorem 4 and the sufficiency direction of Theorem 2, can also be adapted in obvious ways for the alternative assumption of order γ-strong monotonicity.

A notion related to strong convexity that may be used when the set of minimizers of a function is not unique is the following.

Definition A.3 (Growth conditions)**.** Let S be the set of minimizers of f on K, which we suppose is non-empty, and let $f_0 = \min_{x\in K} f(x)$. For $\gamma > 0$, the function f satisfies the *order γ-growth condition* if there exists some constant $m > 0$ such that for all $x \in K$,

$$f(x) \geq f_0 + \frac{m}{2}[\mathrm{dist}(x, S)]^\gamma. \tag{GC}$$

For $\gamma = 2$, we call this the *quadratic growth condition*. If (GC) is satisfied only in some neighborhood of x, then we refer to it as the *local* order γ-growth condition at x.

Growth conditions were introduced by Shapiro (1992) and thoroughly studied in Bonnans and Shapiro (2013). Because the zero vector is in the subdifferential of f at any minimizer of f, it is clear that the order $\gamma-$growth condition is a weaker concept than order $\gamma-$strong convexity.

Proposition 6 is easily modified to apply to $d_H(P_t, P_t')$ under the assumption that N_t has the local order γ-growth condition at P_t for each t. Note, however, that the price selection rule for the Walrasian mechanism in the case of non-unique prices may now matter, since P_t and/or P_t' may not be $O(f(t))$ even when $d_H(P_t, P_t')$ is $O(f(t))$.

B Welfare theorems and equilibrium formulations

The fundamental theorems of welfare economics, as formalized by Arrow (1951), imply that the set of allocations associated with Walrasian equilibria coincide with the set of efficient allocations, which solve the problem

$$\max_{\mathbf{x}\in\mathcal{X}} \mathcal{S}(\mathbf{x}), \tag{OPT}$$

One way to see this is to consider the Lagrangian $\mathscr{L} : \mathcal{X} \times \mathbb{R}^L_+ \to \mathbb{R}$ associated with (OPT), given by

$$\mathscr{L}\left(\mathbf{x}, p\right) = \sum_{n\in N} v_n\left(x_n\right) + p\cdot\left(s - \sum_{n\in N} x_n\right). \tag{L}$$

Since Slater's constraint qualification[22] is satisfied in (OPT) (because the zero allocation is in the relative interior of the constraint space), any saddle point of $\mathscr{L}$, that is, any pair $\left(\mathbf{x}, p\right)$ that solves

$$\min_{p\in\mathbb{R}^L_+} \max_{\mathbf{x}\in\mathcal{X}} \mathscr{L}\left(\mathbf{x}, p\right), \tag{SP}$$

gives rise to a solution $\mathbf{x}$ to the convex program (OPT). Furthermore, the the values of programs (OPT) and (SP) are the same (this is the complementary slackness condition).

From (SP), we see that any saddle point $\left(\mathbf{x}, p\right)$ must satisfy $x_n \in \arg\max_{x\in X_n} v_n\left(x\right) - p\cdot x$ for each $n \in N$, which is the individual optimality property of Walrasian equilibrium. So the saddle points of (SP)–Walrasian equilibria–correspond to maximizers of (OPT)–efficient allocations–and *vice versa*. This is a statement of the fundamental welfare theorems for quasilinear economies. Moreover, since the objective in (OPT) is bounded and concave and the set $\mathcal{X}$ is compact, (OPT) has a solution and a Walrasian equilibrium exists.

The *dual problem*,

$$\min_{p\in\mathbb{R}^L_+} p\cdot s + \sum_{n\in N} u_n\left(p\right). \tag{D}$$

obtained by reorganizing (SP), plays a major role in the analysis of this paper. An advantage

[22]See, for example, Boyd and Vandenberghe (2004).

of studying the dual problem is that it is an unconstrained convex program. Writing $U(p) = \sum_{n \in N} u_n(p)$ for the total indirect utility function, the first-order (necessary) conditions of (D) are exactly the market-clearing conditions of the Walrasian equilibrium, $s \in -\partial U(p) = \sum_{n \in N} D_n(p)$.

C Production economies

In this section, we discuss extensions of our results to economic environments in which the supply vector is not fixed and instead determined by agents' production decisions. We consider two structures for such economies: first, two-sided markets in which agents are *either* buyers or sellers, and second, markets in which agents may be buyers *or* sellers of certain types of goods given the prices—we call such agents "traders". Most of the proofs are very similar to the exchange economy proofs of the main paper, and so many details are omitted.

Two-sided markets The buyer side of the economy is as in Section 2. We now add a set of sellers $\mathscr{F}$ of cardinality F. Each seller f can produce bundles Y_f, where Y_f is a convex, compact subset of $\mathbb{R}^L_+$ containing 0 called the production set. Each seller has cost function $c_f : Y_f \to \mathbb{R}$ resulting in profits $\Pi_f(y_f, t) = t - c_f(y_f)$. The cost functions are drawn from function space $\mathscr{C}$, such that each $c_f \in \mathscr{C}$ is monotone, convex and satisfies the normalization $c_f(0) = 0$. The pair $\mathscr{E} = \langle \mathscr{N}, \mathscr{F} \rangle$ is a market.

An allocation $\omega = ((x_n)_{n \in \mathscr{N}}, (y_f)_{f \in \mathscr{F}})$ maximizes the surplus $\mathcal{S}(\omega) = \sum_{n \in \mathscr{N}} v_n(x_n) - \sum_{f \in \mathscr{F}} c_f(y_f)$. Let $S_f(p)$ be the seller's supply correspondence and $\pi_f(p)$ the seller's indirect profit function, the maximizers and value function respectively of $\max_{y \in Y_f} \Pi_f(y, p \cdot y)$, related by $S_f(p) = \partial \pi_f(p)$.

For a fixed market $\mathscr{E}$, let $Z(p) = \sum_{n \in \mathscr{N}} D_n(p) - \sum_{f \in \mathscr{F}} S_f(p)$ be the excess demand correspondence. We assume that for any $M > 0$, $\|z\| < -M$ or $\|z\| > M$ for all $z \in Z(p)$ and p outside of a compact set $\mathscr{P}_M$. This is analogous to our assumption that $D(p) = \{0\}$ outside of $\mathscr{P}$ in the main text.

A Walrasian equilibrium is a pair (ω, p) such that $\sum_{n \in \mathscr{N}} x_n = \sum_{f \in \mathscr{F}} y_f$, $x_n \in D_n(p)$ and $y_f \in S_f(p)$. Walrasian equilibria exist in this setting and are efficient.

Whereas Walrasian equilibria solve $Z(p) = 0$, a perturbation in this setting is now a solution to $Z(p) = \delta s$. Definition 2.3 now applies to prices $p_t \in W(\mathscr{E}_t)$ and prices p'_t with $Z_t(p') = \delta s_t$. The direct analogies of Proposition 1 and Proposition 2 apply to this altered definition of perturbation-proofness.

Seller f is active at price p if $S_f(p) \neq \{0\}$ and it is not the case that for all $\beta > 1$, $S_f(\beta p) = S_f(p)$. This latter case reflects the possibility that the seller is producing at a boundary of its production possibility set and is needed because upper bounds are not imposed on prices.[23] The seller has *strongly monotone supply* if there is some $m^F > 0$ for which $(y - y') \cdot (p - p') \geq m^F \|p - p'\|^2$ for all active prices p, p' and $y \in S_f(p), y' \in S_f(p')$.

Theorem 1 now applies to economies in which all buyers have strongly monotone demand, all sellers have strongly monotone supply, and N_t^a is the total number of active buyers and sellers at prices $p_t \in W(\mathscr{E}_t)$ and p'_t with $Z(p'_t) = \delta s_t$. The proof is similar, where now the expressions for strong monotonicity of demand and supply are added in order to obtain expressions for excess demand. That is, we add

$$(y_f - y'_f) \cdot (p_t - p'_t) \geq m^F \|p_t - p'_t\|^2$$

for all sellers to (2) to obtain

$$\left(\sum_{n \in \mathscr{N}_t} (d'_n - d_n) + \sum_{f \in \mathscr{F}_t} (y_f - y'_f) \right) \cdot (p_t - p'_t) \geq \min\{m^N, m^F\}(N_t^a + F_t^a)\|p_t - p'_t\|^2,$$

with the term in parentheses on the left corresponding to the new definition of perturbation δs_t.

Theorem 2, Theorem 3 and Theorem 4 also apply with the appropriate modifications, where in all cases the proofs are modified to exploit strong monotonicity of the (expected) *excess* demand functions, and the (expected) dual objective now taking the form $\sum_{n \in \mathscr{N}} u_n(p) + \sum_{f \in \mathscr{F}} \pi_f(p)$.

Markets with traders We now replace the set of buyers with a set $\mathscr{T}$ of traders of cardinality T. Each trader $n \in \mathscr{T}$ has access to bundles X_n, where X_n is a convex, compact subset of $\mathbb{R}^L$ containing 0, called the netput set. Vectors $x \in X_n$ may be positive in some components and negative in others. Each trader has a net-value function $v_n : X_n \to \mathbb{R}$ which may be positive or negative, monotone, concave and satisfying $v_n(0) = 0$, resulting in payoffs $U_n(x, t) = v_n(x) - t$. The remaining formulation is the same as in the main text, except:

(a) Walrasian equilibrium is now defined as $\sum_{n \in \mathscr{T}} x_n = 0$ and $x_n \in D_n(p)$ for each $n \in \mathscr{T}$.

(b) an agent is 'active' if it is *not* the case that $D_n(p) = D_n(\beta p)$ for all $\beta > 1$, replacing Definition 3.1.

[23] A similar condition is not needed in the buyers' case because prices are bounded below by zero. An alternative would be to allow sellers to produce unbounded quantities of goods.

(c) for all $M > 0$, there exists a compact set $\mathscr{P}_M$ such that for all $p \notin \mathscr{P}_M$ and $x \in D_n(p)$, $\|x\| < -M$ or $\|x\| > M$.

D Tâtonnement stability of strongly monotone economies

Recall the continuous-time tâtonnement process in which prices are adjusted in proportion to the excess demand for the relevant good:

$$\frac{dp}{dt} = \alpha[D(p(t)) - s], \text{ with } p(0) = p_0$$

for some adjustment speed $\alpha > 0$ and starting price $p_0 \in \mathscr{P}$. Here we assume $D(p)$ is single-valued, as is in our case of interest where D is a strongly monotone demand correspondence.

It is well-known that in quasilinear economies (and other economies in which there is a representative consumer) that $\lim_{t\to\infty} p(t)$ is a Walrasian equilibrium price for any starting price p_0.

However, in general, the rate of convergence of prices to equilibrium may be arbitrarily slow. The intuition for this is as follows: there may in general exist prices p at large distance from Walrasian equilibrium price p^* for which the excess demand is very small. This implies that the speed of adjustment of prices is very small, while the distance from equilibrium is very large.

However, under the assumption of strong monotonicity of demand, we have (via the Cauchy-Schwarz inequality) that $\|p - p^*\| \leq \frac{1}{m}\|D(p) - s\|$ so that prices cannot be large when excess demand is small. This will imply that the price adjustment process cannot slow down at prices a long distance from Walrasian equilibrium.

We have the following theorem about the convergence of the continuous-time tâtonnement process to Walrasian equilibrium.

Proposition 7. *Consider the continuous-time tâtonnement process applied to a strongly monotone demand correspondence. The time to convergence to p within an ε-ball[24] of a Walrasian equilibrium price p^* is subpolynomial in ε.*

Proof. This involves a simple modification of the classical proof of convergence of tâtonnement for

[24]For the analysis of tâtonnement as an algorithm, the ε-neighborhood of p^* is the appropriate subject of study. One reason as to why: p^* may be irrational, and thus we can never expect a computer to converge exactly to p^*.

quasilinear economies. Consider the Lyapunov function for the differential equation

$$L(t) = \|p(t) - p^*\|^2.$$

We have by definition of the tâtonnement process that

$$\frac{dL}{dt} = 2(p(t) - p^*) \cdot \frac{dp}{dt} = 2(p(t) - p^*) \cdot \alpha(D(p(t)) - s).$$

By the definition of strong monotonicity, we then have

$$\frac{dL}{dt} \geq -2\alpha m\|p(t) - p^*\|^2.$$

Solving this differential inequality gives

$$\|p(t) - p^*\| \leq e^{-2\alpha m t}.$$

But then for $t \geq \frac{-1}{2\alpha m}\log(\varepsilon)$, we must have that $\|p(t) - p^*\| \leq \varepsilon$.

□

This proof can also be adapted to the discrete-time version of the tâtonnement process.

Despite this, Budish, Cramton, Kyle, Lee, and Malec (2020) find that even under the strong monotonicity assumption, the tâtonnement algorithm may be too slow for practical identification of prices (in their setting, they hope to solve for prices in very large markets once per second). This illustrates the importance of the constant on the practical usefulness of the algorithm. Instead, Budish et al. (2020) find greater success in the use of an interior-point method for the convex program.

E Proofs omitted from the main text

E.1 Proof of Theorem 2

We begin with a helpful lemma.

Lemma 1. *Suppose $d \in D(p)$ and $d' \in D(p')$, with $(d - d') \cdot (p' - p) = 0$. Then $d \in D(p')$ and $d' \in D(p)$.*

Proof. Since $d \in D(p)$, by strong duality, we have from the dual objective associated with supply vector d that

$$\sum_{n \in N} u_n(p) + p \cdot d \leq \sum_{n \in N} u_n(p') + p' \cdot d.$$

Similarly,

$$\sum_{n \in N} u_n(p') + p' \cdot d' \leq \sum_{n \in N} u_n(p) + p \cdot d'.$$

Rearranging, and combining these inequalities, we obtain

$$(p' - p) \cdot d' \leq \sum_{n \in N} u_n(p) - \sum_{n \in N} u_n(p') \leq (p' - p) \cdot d.$$

However, by assumption $(p' - p) \cdot d' = (p' - p) \cdot d$, so that

$$\sum_{n \in N} u_n(p) - \sum_{n \in N} u_n(p') = (p' - p) \cdot d = (p' - p) \cdot d'.$$

Thus, we have that

$$\sum_{n \in N} u_n(p) + p \cdot d = \sum_{n \in N} u_n(p') + p' \cdot d, \text{ and}$$
$$\sum_{n \in N} u_n(p) + p \cdot d' = \sum_{n \in N} u_n(p') + p' \cdot d'.$$

Strong duality then implies that $d \in D(p')$ and $d' \in D(p)$. □

We now proceed to the proof of the Theorem 2.

The sufficiency proof of Theorem 2 follows almost identically to the proof of Theorem 1, except that the demand selections on the left-hand side of (1) are replaced by selections from the total demand correspondence of the base economy, and the number of active buyers N_t^a on the right-hand side of (2) is replaced by the number of replicas k. Since k is $\Theta(|N_k|)$, the conclusion follows.

For necessity, we consider the contrapositive: let $\mathscr{E}$ be a base economy which fails to be strongly monotone and let $D = \sum_{n \in N} D_n(p)$ be its total demand correspondence. Consider any real-valued sequence m_t with $m_t \to 0$, and let sequences of prices p_t, p_t' be such that $p_t \neq p_t'$ and $(d_t - d_t') \cdot (p_t' - p_t) < m_t \|p_t - p_t'\|^2$ for $d_t \in D(p_t)$ and $d_t' \in D(p_t')$ (the existence of such a sequence is assured by the failure of strong monotonicity). By the Bolzano-Weierstrass theorem, it is without loss to assume that $p_t \to p$ and $p_t' \to p'$ for some $p, p' \in \mathscr{P}$. There are two cases:

(1) $p \neq p'$. By Berge's Theorem, D is upper-hemicontinuous so that $d_t \to d \in D(p)$ and $d'_t \to d' \in D(p')$, and we must have $(d - d') \cdot (p' - p) = 0$. Let $s = d$, then p must be a Walrasian equilibrium price in the sequence of economies $\mathscr{E}_k = \langle N_k, kd \rangle$. By Lemma 1, we also have that p' is a Walrasian equilibrium price for $\mathscr{E}_k$.

Without loss of generality,[25] consider any perturbation δs such that $\delta s \cdot (p' - p) > 0$. Note that p' cannot be an equilibrium price of the perturbed economies $\mathscr{E}'_k$ since

$$\sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) - \left(\sum_{n \in N_k} u_n(p') + p' \cdot (kd + \delta s) \right) = \delta s \cdot (p - p') < 0.$$

On the other hand, in the limit as $k \to \infty$, the set of equilibrium prices of $\mathscr{E}'_k$ must approach a (closed, proper) subset of the equilibrium prices of the base economy (also the equilibrium prices of $\mathscr{E}_k$) since

$$\arg\min_p \sum_{n \in N_k} u_n(p) + p \cdot (kd + \delta s) = \arg\min_p \sum_{n \in N} u_n(p) + p \cdot \left(d + \frac{\delta s}{k} \right),$$

and the objective $\sum_{n \in N} u_n(p) + p \cdot (d + \delta s / k)$ epi-converges (as $k \to \infty$) to the objective of the unperturbed base economy, so that Theorem 7.33 of Rockafellar and Wets (2009) applies. But then p' is a Walrasian equilibrium price of $\mathscr{E}_k$ but not $\mathscr{E}'_k$, and $d_H(P_k, P'_k) \not\to 0$, so cannot be $O(1/|N_k|)$.

(2) $p = p'$. It suffices to consider the case when for all t, $d_t \neq d'_t$, otherwise the argument in the previous case works as well. So, without loss of generality (restricting to a subsequence if necessary), assume that $p_t \to p$ and $p'_t \to p$ in such a way that the angle between $p'_t - p_t$ and $d'_t - d_t$ converges to a constant. *For now, let us assume that $\sum_{n \in N} u_n(p) + d \cdot p$ is twice continuously differentiable at p.* In this case, we have by assumption that

$$\lim_{t \to \infty} \frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} = 0 \tag{*}$$

and this limit is the (negative of the) second directional derivative of $\sum_{n \in N} u_n(p)$ at p in the limiting direction of $p'_t - p_t$. That is, the failure of strong convexity implies a zero second derivative of the objective in some direction at some point.

We now argue that the limiting angle between $d'_t - d_t$ and $p_t - p'_t$ cannot be 90° (that is, the

[25]Relabeling p, p' if necessary.

demand change cannot approach orthogonality with the price change). To see this, without loss of generality (changing orthonormal coordinates if necessary) suppose that $p_t - p'_t$ approaches unit vector in the direction of the first coordinate (say p_x) and $d_t - d'_t$ approaches the unit vector in the direction of the second coordinate (say p_y). In this case, we must have that $\frac{\partial d_x}{\partial p_x}(p) = 0$ and $\frac{\partial d_y}{\partial p_x}(p) \neq 0$. But then, by symmetry of the Slutsky matrix (equivalently, recognizing that these are mixed partials in the same coordinates and by Schwarz's Theorem), we must have $\frac{\partial d_x}{\partial p_y}(p) \neq 0$. But this would imply that the Hessian of the objective at p is not positive semidefinite, which contradicts the convexity of the objective.

Thus (restricting to a subsequence if necessary), we have that $(d_t - d'_t) \cdot (p'_t - p_t) \geq c\|d_t - d'_t\|\|p'_t - p_t\|$ for some $c > 0$ and for all t. Then (restricting to a subsequence if necessary), we may take $\|d_t - d'_t\| = O\left(1/N\right)$ and the only way that

$$\frac{(d_t - d'_t) \cdot (p'_t - p_t)}{\|p_t - p'_t\|^2} \geq \frac{c\|d_t - d'_t\|}{\|p_t - p'_t\|}$$

can tend to zero is if $\|p_t - p'_t\| \geq \Omega(1/N)$.

We now adapt the argument to the case that $\sum u_n(p) + d \cdot p$ is not twice continuously differentiable at p. In this case, we consider the sequence of $1/N^2$-Moreau-Yosida regularized economies (see Appendix E.2 below), notating the corresponding quantities in the regularized economies by tildes. Because $\tilde{d}_t - \tilde{d}'_t = d_t - d'_t + 1/N^2(p_t - p'_t)$ by Proposition 8 below, the limit in (*) must hold for a sequence of $\tilde{d}_t \in \tilde{D}(p), \tilde{d}'_t \in \tilde{D}(p')$ in the regularized economy. But then since the regularized objective is $C^{1,1}$ (continuous with Lipschitz continuous gradient), the limit of $(d_t - d'_t)/\|p_t - p'_t\|$ must approach a symmetric matrix by Theorem 13.52 of Rockafellar and Wets (2009) (a generalization of second-derivative symmetry for $C^{1,1}$ functions). The remainder of the above proof then follows through for the regularized economy. Finally, as argued in Appendix E.2, the sequence of regularized economies is perturbation-proof if and only if the original sequence is perturbation-proof.

E.2 Proof of Theorem 3

Notation We omit the t index except where necessary for clarity. We write $V(p) = \sum_{n \in N} u_n(p)$ for the realized total indirect utility and $D(p) = \partial V(p)$ for the realized demand correspondence. Recall that P is the set of minimizers of (D) which has the objective $V(p) + s \cdot p$, while P' is the set of minimizers of the objective $V(p) + (s + \delta s) \cdot p$. We will abuse notation to write inequalities like

$\|D(p) - D(p')\| \geq \|\delta s\|$ as shorthand for $\|d - d'\| \geq \|\delta s\|$ for all $d \in D(p)$ and $d' \in D(p')$.

Proof approach Consider economy $\mathscr{E} = \langle N, s \rangle$ obtained by drawing $N := |N|$ buyers from distribution ν over $\mathscr{V}$ which satisfies the conditions of Theorem 3.

Our approach will be to show that with high probability (henceforth, w.h.p.) [26] over draws of the economy $\mathscr{E}$, that for *all* price vectors p with $\text{dist}(p, P) > c/N^{1-\varepsilon}$ (for a constant c to be chosen later), we have that $\|D(p) - s\| > \|\delta s\|$. That is, w.h.p., the demand at prices p outside a neighborhood of size $c/N^{1-\varepsilon}$ from P must differ (in magnitude) from the supply vector s by more than the size of the perturbation $\|\delta s\|$. This will imply on that measure of economies that any price in P' must be within distance $c/N^{1-\varepsilon}$ of P, so that w.h.p. $d_H(P, P')$ will be less than $\frac{c}{N^{1-\varepsilon}}$.

Before completing the proof, we offer some high-level intuition for our approach and divide the proof into a number of steps.

1. **Concentration**: For any fixed p, p' at a distance of $c/N^{1-\varepsilon}$, we show using the Bernstein Inequality that w.h.p. $M(p, p') := \min_{d \in D(p), d' \in D(p')} (d - d') \cdot (p' - p)$ is at least $\frac{mN}{2} \|p - p'\|^2$. That is, w.h.p., the definition of strong monotonicity with constant $m/2$ holds for fixed prices p, p'. This will imply via the Cauchy-Schwarz Inequality logic used in Theorem 1 that for large enough c, w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for $k > 1$.

2. **Extension to discretized sphere**: Fixing p, we then extend the result that $\|D(p) - D(p')\| > k\|\delta s\|$ to *all* prices p' at distance of at least $c/N^{1-\varepsilon}$. To do so, we first discretize the $c/N^{1-\varepsilon}$ unit sphere and employ a union bound, which critically relies on the subexponential tail bound obtained from the Bernstein Inequality in Step 1.[27]

3. **Extension to sphere via regularization**: We then extend the result to the full $c/N^{1-\varepsilon}-$sphere centered at p under the assumption that the *realized* correspondence is Lipschitz continuous. At the end of the proof (in the paragraph titled *Regularization*), we show that this additional assumption is without loss of generality because in economies with non-Lipschitz demand correspondences, it is possible to analyze a *regularized* version of the economy with Lipschitz demand for which $d_H(P, P')$ is approximately (up to $o\left(1/N^{1-\varepsilon}\right)$) equal to the original economy.

[26]Throughout, we will use the term 'high probability' to refer to a probability that tends to 1 as $N \to \infty$. Then, X_t is $O_p(f(t))$ if $\left|\frac{X(t)}{f(t)}\right| < c$ w.h.p..

[27]This is why the Bernstein Inequality is used rather than the simpler Chebyshev's Inequality, which is sufficient to obtain result in Step 1.

4. **Extension to exterior of sphere**: Using convexity, we then show that this implies $\|D(p) - D(p')\| > k\|\delta s\|$ for all p' with distance at least $c/N^{1-\varepsilon}$ from p.

5. **Uniformization over** p: Finally, we extend the result of Step 4 to a fine grid of prices over $\mathscr{P}$, and use the Lipschitzian property of demand in the regularized economy to establish that w.h.p. $\|D(p) - D(p')\| > k\|\delta s\|$ for all p, p' at distance of at least $c/N^{1-\varepsilon}$. This concludes the proof.

This proof resembles the "Fano method" used for proofs in the study of stochastic processes, but we have not been able to adapt known results to obtain our conclusions. We now fill in the details in these steps to complete the proof.

Step 1: Concentration Consider any fixed $p, p' \in \mathscr{P}$ with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. Define for each $n \in N$, $M_n(p, p') = \min_{d \in D_n(p), d' \in D_n(p')} (d - d') \cdot (p' - p)$ and let $M(p, p') = \sum_{n \in N} M_n(p, p')$. By the strong monotonicity assumption, $M_n(p, p')$ is a real-valued random variable satisfying $\mathbb{E}_\nu \left[M_n(p, p')\right] \geq m\|p - p'\|^2$. It will help us to write $\mu_{p,p'} := \mathbb{E}_\nu \left[M_n(p, p')\right]$.

We will apply the Bernstein Inequality[28]: given independent real-valued random variables $X_1, X_2, ..., X_N$ with $|X_i| \leq B$, we have

$$\Pr\left[\left|\sum_i X_i - \sum_i \mathbb{E}[X_i]\right| \geq t\right] \leq 2\exp\left(\frac{-\frac{1}{2}t^2}{\sum_i \mathbb{E}[X_i^2] + \frac{1}{3}Bt}\right).$$

To apply the Bernstein Inequality to $M_n(p, p')$, we require an estimate of the second moment of $M_n(p, p')$. We use the Bhatia and Davis (2000) inequality to obtain an upper bound: for any real-valued random variable X with mean μ and $m \leq X \leq M$ a.s., $\mathrm{Var}[X] \leq (M - \mu)(\mu - m)$. Since $M_n(p, p')$ is bounded below by zero (by the monotonicity of d_n) and $M_n(p, p')$ is bounded a.s. above by $2X_{max}\|p - p'\|$ (using the Cauchy-Schwarz inequality), we have that

$$\mathbb{E}_\nu[M_n(p, p')^2] \leq 2X_{max}\|p - p'\|\mu_{p,p'}.$$

[28]See, for example Boucheron, Lugosi, and Massart (2013).

Thus, applying the Bernstein Inequality to $M_n(p, p')$, we obtain

$$\Pr\left[M(p,p') \geq \frac{1}{2}N\mu_{p,p'}\right] \geq 1 - 2\exp\left(\frac{-\frac{1}{8}N^2\mu_{p,p'}^2}{2NX_{max}\|p-p'\|\mu_{p,p'} + \frac{1}{3}NX_{max}\|p-p'\|\mu_{p,p'}}\right)$$
$$= 1 - 2\exp\left(\frac{-3N\mu_{p,p'}}{56X_{max}\|p-p'\|}\right).$$

Since $\mu_{p,p'} \geq m\|p-p'\|^2$ and $\|p-p'\| \geq c/N^{1-\varepsilon}$, we have

$$\Pr\left[M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2\right] \geq 1 - 2\exp\left(\frac{-3Nm\|p-p'\|^2}{56X_{max}\|p-p'\|}\right)$$
$$\geq 1 - 2\exp\left(\frac{-3cN^{\varepsilon}m}{56X_{max}}\right)$$

The above probability tends to 1 as $N \to \infty$. Note that the event $M(p,p') \geq \frac{mN}{2}\|p-p'\|^2$ for $\|p-p'\| = \frac{2k\|\delta s\|}{mN^{1-\varepsilon}}$ (that is, $c = 2k\|\delta s\|/m$, in our previous notation) is equivalent to the event that $(D(p) - D(p')) \cdot (p-p') \geq k\|\delta s\|N^{\varepsilon}\|p-p'\|$. By the Cauchy-Schwarz Inequality, this implies $\|d-d'\| \geq k\|\delta s\|N^{\varepsilon}$. For $k > 1$ and sufficiently large N, if $p \in P$, this implies the event that p' could not be in P'. In later arguments, it will help to choose k larger than 1 to leave room for other approximations.

Step 2: Extension to discretized sphere With the same fixed p as in Step 1, we now consider $\mathbb{S}_c(p)$, the $c/N^{1-\varepsilon}$-sphere around p. By standard covering arguments, it is possible to identify $O(N^{(3+\varepsilon)L})$ points on the sphere of radius $O(1/N^{1-\varepsilon})$ such that the distance between each pair is at most $O(N^{-4})$. Let such a discretization be $\mathbb{D}_c(p)$.

Note that the number of pairs p, p' with $p' \in \mathbb{D}_c(p)$ is polynomial in N. A union bound over the events $M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2$ over $p' \in \mathbb{D}_c(p)$ thus implies

$$\Pr\left[M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2 \textbf{ for all } p' \in \mathbb{D}_c(p)\right] \geq 1 - 2O(N^{(3+\varepsilon)L})\exp\left(\frac{-3cN^{\varepsilon}m}{56X_{max}}\right),$$

which also tends to 1 as $N \to \infty$. Thus $\|D(p) - D(p')\| > k\|\delta s\|$ for all p' in $\mathbb{D}_c(p)$ w.h.p. for large enough N (where again, we have set $c = 2k\|\delta s\|/m$ in the above).

Assumption: In Steps 3 and 5, we assume that the realized demand correspondence is $O(N^2)$-Lipschitzian. We justify this assumption in our discussion on regularization below.

Step 3: Extension to sphere via regularization Consider $p'' \in \mathbb{S}_c(p) \setminus \mathbb{D}_c(p)$. Since p'' is at a distance of at most $O(N^{-4})$ from p' in $\mathbb{D}_c(p)$ and $(D(p) - D(p')) \cdot (p - p') \geq k\|\delta s\| N^{\varepsilon} \|p - p'\|$ w.h.p. for all p' in $\mathbb{D}_c(p)$, using the Cauchy Schwarz inequality, we obtain

$$\begin{aligned}
&(D(p) - D(p'')) \cdot (p - p'') \\
&\geq (D(p) - D(p')) \cdot (p - p') - \|D(p') - D(p'')\| \|p - p'\| - \|D(p) - D(p')\| \|p' - p''\| \\
&\geq k\|\delta s\| N^{\varepsilon} \|p - p'\| - O(N^2) \cdot O(N^{-4}) \|p - p'\| - \|D(p) - D(p')\| O(N^{-4}) \\
&\geq k' \|\delta s\| N^{\varepsilon} \|p - p'\|
\end{aligned}$$

for any $k' < k$ and sufficiently large N, where the second line uses the $O(N^2)$-Lipschitz property of demand. This implies that all $p'' \in \mathbb{S}_c(p)$ have $\|D(p'') - D(p)\| \geq k'\|\delta s\|$ w.h.p. for sufficiently large N.

Step 4: Extension to exterior of sphere Now let p'' be a point outside of $\mathbb{S}_c(p)$ and let p' be the point on $\mathbb{S}_c(p)$ which is on the line between p and p''. By convexity, we have for all $d' \in D(p')$ and $d'' \in D(p'')$ that $(d'' - d') \cdot (p' - p'') \geq 0$. Since $p' - p'' = \frac{\|p' - p''\|}{\|p - p'\|}(p - p')$, we also have $(d'' - d') \cdot (p - p') \geq 0$. But then since for all $d' \in D(p')$, we have $(d' - d) \cdot (p - p') \geq k'\|\delta s\| N^{\varepsilon} \|p - p'\|$ w.h.p., by adding the previous expression, we obtain $(d'' - d) \cdot (p - p') \geq k'\|\delta s\| N^{\varepsilon} \|p - p'\|$ with the same probability. But this implies that $\|d'' - d\| \geq k'\|\delta s\| N^{\varepsilon}$, as required. Thus, we have for all p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$ that $\|d - d'\| \geq k'\|\delta s\| N^{\varepsilon}$ with high probability.

Step 5: Uniformization over p Up until now, we have held p fixed, but we now wish to extend the conclusion of Step 4 above to any realized $p \in P$. To do so, we apply another discretization of $\mathscr{P}$ with points at distance $\Theta(N^{-4})$. Again, by standard covering arguments (since $\mathscr{P}$ is compact), $O(N^{4L})$ points are required for such a covering of $\mathscr{P}$. We may again apply a union bound to obtain the conclusions of Step 4 for *all* p in the discretization. Because the realized demand is $O(N^2)$-Lipschitz, this implies (via the same logic as in Step 3) the same result for $p \in P$ not in the covering for sufficiently large N.

This implies that with probability approaching 1 as $N \to \infty$, for any $p \in P$ and p' with $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$, that $\|s - d'\| \geq k'\|\delta s\| N^{\varepsilon}$ for all $d' \in D(p')$. By choosing c large enough for $k' > 1$, we then have $\|s - d'\| > \delta s$ for all $d' \in D(p')$, which means p' cannot be in P'. This implies that $d_H(P, P') < \frac{c}{N^{1-\varepsilon}}$ with probability approaching 1 for sufficiently large c, that is the random variable

$d_H(P, P')$ is $O_p\left(\frac{1}{N^{1-\varepsilon}}\right)$, as is required.

Regularization In Steps 3 and 5 above, we assumed that the realized demand correspondence is an $O(N^2)-$Lipschitz. Here, we show that this assumption is without loss of generality by exploiting the Moreau-Yosida regularization of convex functions. The explicit construction of the Moreau-Yosida approximation will not be important for our argument (although it is not complicated—see, for example, Rockafellar and Wets (2009)), so instead we state the result as an existence theorem.

Proposition 8 (Moreau-Yosida). *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semi-continuous function defined on a convex, compact subset X of a Hilbert space. Then for all $\gamma > 0$, there exists a function $\tilde{f} : X \to \mathbb{R} \cup \{\infty\}$, the γ-*Moreau envelope *of f, with the following properties:*

- *$\tilde{f}$ is convex, $\frac{1}{\gamma}$-Lipschitz-continuous and Fréchet-differentiable with gradient $\nabla \tilde{f}$ which is $\frac{1}{\gamma}$-Lipschitz continuous*
- *f and $\tilde{f}$ have the same minimizers.*

Furthermore, if f is L-Lipschitz continuous, then $\tilde{f}$ is also L-Lipschitz, and for all $x \in X$,

$$\tilde{f}(x) \leq f(x) \leq \tilde{f}(x) + \frac{\gamma L^2}{2}.$$

The inverse mapping of the gradient of $\tilde{f}$ and the inverse mapping of the subdifferential of f are related by

$$(\nabla \tilde{f})^{-1}(x^*) = \gamma x^* + (\partial f)^{-1}(x^*).$$

Note that V is proper, convex and $X_{max}-$Lipschitz, where X_{max} is defined as the maximum magnitude demand vector, $\max_{v_n \in \text{supp}(\nu)} \max_{x \in \text{dom}(v_n)} \|x\|$ (which exists by the assumption of compactness of the consumption possibility sets). The $\frac{1}{N^2}-$Moreau envelope of V, $\tilde{V}$, is thus convex, $\max\{X_{max}, N^2\}$-Lipschitz continuous and Fréchet differentiable with gradient (i.e. demand function) which is N^2-Lipschitz.

We now show that it suffices for us to analyze the $1/N^2$ regularized dual objective. Let $\tilde{P}$ and $\tilde{P}'$ be the unperturbed and perturbed Walrasian prices (respectively) for the regularized demand.

First, we note that $d_H(P, P') = d_H(\tilde{P}, \tilde{P}') + O(1/N^2)$, which implies that if $d_H(\tilde{P}, \tilde{P}')$ is $O(1/N^{1-\varepsilon})$, so is $d_H(P, P')$. To see this, note that $\tilde{P} = (\nabla \tilde{V})^{-1}(s)$, $\tilde{P}' = (\nabla \tilde{V})^{-1}(s + \delta s)$, $P = (\partial V)^{-1}(s)$ and $P' = (\partial V)^{-1}(s + \delta s)$ so that by the last identity in Proposition 8, $\tilde{P} = P + \frac{1}{N^2}s$ and $\tilde{P}' = P' + \frac{1}{N^2}(s + \delta s)$ which, since $\|s\| < X_{max}$ and $\|s + \delta s\| < X_{max}$, implies the first claim.

Second we claim that $\mathbb{E}[\nabla\tilde{V}]$ is m'-strongly monotone for all $m' < m$ and sufficiently large N and $\|p - p'\| \geq \frac{c}{N^{1-\varepsilon}}$. To see this, consider the expression $\mathbb{E}[(p-p')\cdot(\tilde{d}' - \tilde{d})]$ for $\tilde{d} = \nabla V(p)$ and $\tilde{d}' = \nabla V(p')$. Note $p \in (\nabla\tilde{V})^{-1}(\tilde{d})$ and $p' \in (\nabla\tilde{V})^{-1}(\tilde{d}')$ so that $p - \frac{1}{N^2}\tilde{d} \in (\partial V)^{-1}(\tilde{d})$ and $p' - \frac{1}{N^2}\tilde{d}' \in (\partial V)^{-1}(\tilde{d}')$. But then

$$\begin{aligned}
\mathbb{E}[(p-p')\cdot(\tilde{d}'-\tilde{d})] &= \mathbb{E}[(p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}')\cdot(\tilde{d}'-\tilde{d}) + \frac{1}{N^2}(\tilde{d}-\tilde{d}')\cdot(\tilde{d}'-\tilde{d})] \\
&\geq m\|p - \frac{1}{N^2}\tilde{d} - p' + \frac{1}{N^2}\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{m}{N^2}\|\tilde{d}-\tilde{d}'\|^2 - \frac{1}{N^2}\|\tilde{d}-\tilde{d}'\|^2 \\
&\geq m\|p-p'\|^2 - \frac{1+m}{N^2}X_{max}^2,
\end{aligned}$$

where the second line above follow by the strong monotonicity property of ∂V. So, for $\|p-p'\| \geq \frac{c}{N^{1-\varepsilon}}$ the second term is asymptotically dominated by the first, and the claim follows. Together, these two claims imply that we may replicate the arguments in Steps 1-2 above for the regularized demand $\nabla\tilde{V}$ (which is Lipschitz) and Steps 3-5 imply the required result.

E.3 Proof of Theorem 4

Part (a) follows by direct combination of Theorem 3 with Proposition 2.

For part (b), note that in the proof of Theorem 3, we showed that with subexponential probability—in fact, with probability $1 - O(1/N)$, we have that the maximum distance between the price associated with the truthful report of an agent and any alternative report of that agent is $O(1/N^{1-\varepsilon})$. In the complementary $O(1/N)$ measure of draws of economies, we have that the maximum influence on price is $O(1)$, since by assumption the set of possible prices $\mathscr{P}$ is compact. Thus, we have that the expected maximum influence of any report, including the *interim* optimal report, on price is $(1 - O(1/N))O(1/N^{1-\varepsilon}) + O(1/N)O(1) = O(1/N^{1-\varepsilon})$

E.4 Proof of Proposition 3

We use some concepts introduced by Baldwin and Klemperer (2019) and refer readers to Baldwin and Klemperer (2019) for a complete treatment.

Definition E.1. For buyer n with demand correspondence D_n, the *locus of indifference prices (LIP)* is $\mathscr{L}_n = \left\{p \in \mathbb{R}_+^L : |D_n(p)| > 1\right\}$.

The LIP divides price space into *unique demand regions* in which demand is constant, so that demand can only change as prices change through the $(L-1)-$dimensional facets that comprise the LIP.[29] Moreover, Baldwin and Klemperer (2019) show that as prices change between adjacent unique demand regions, demand changes by an integer multiple of the "primitive" normal vector of the associated facet(s) separating the regions. Here, a primitive vector is one in which the greatest common divisor of its entries is 1.

Definition E.2. The *demand type* $\mathscr{D}_n$ of buyer n is the set of primitive facet normal vectors of the LIP. We refer to an element of $\mathscr{D}_n$ as a *demand subtype*.[30]

Consider any price change $p \mapsto p'$. For all the demand subtypes δ associated with buyers in $\mathscr{V}$ (note there are finitely many possible subtypes for L goods with bounded demand), we have either that $\delta \cdot (p'-p) = 0$ or $\delta \cdot (p'-p) > 0$. In the first case, p and p' must both lie on the same facet of the LIP, so that demand does not change along p to p'. In the other case, since the number of possible demand subtypes is finite, there is a least $\delta \cdot (p'-p)$ among them: let δ' be that subtype and let $\delta' \cdot (p'-p) = k\|p'-p\|$ for some $k > 0$. (In other words, k is the least product of $\|\delta\|$ among the demand subtypes and the cosine of the angle between δ and $p'-p$. This is why our expression maintains $\|p'-p\|$ as a constant of proportionality.)

For now, suppose that $\alpha\|p-p'\| \leq 1$. In this case, we have that a lower bound on $\mathbb{E}[(D(p) - D(p')) \cdot (p'-p)]$ is given by the $\Pr_\nu[D(p) \neq D(p')]$ multiplied by the least value of $(D(p) - D(p')) \cdot (p'-p)$ conditional on a demand change. Since by assumption $\Pr_\nu[D(p) \neq D(p')] \geq \alpha\|p-p'\|$ and the least value of the projected demand change is $k\|p'-p\|$, we obtain the lower bound on $\mathbb{E}[(D(p)-D(p')) \cdot (p'-p)]$ of $\alpha k\|p'-p\|^2$, which is the required inequality for strong monotonicity.

If $\alpha\|p-p'\| \geq 1$, we can divide the line segment up into pieces $p, p_1, p_2, ..., p_N, p'$ where between p and p_1, p_1 and p_2, p_2 and p_3 etc., demand changes occur with probability 1, and between p_N and p', demand changes occur with at least $\alpha\|p_N - p'\|$. In this case, since the demand changes are lower bounded by the size of an indivisibility, it is still clear that the size of the demand change is proportional to the distance $\|p-p'\|$, as is required for strong monotonicity in this setting.

[29]Note that the cyclic monotonicity of demand implies that the change in demand as p changes to p' is independent of the path in price space between p and p'. So, unless otherwise specified, when we say a price change from p to p', we will be referring to straight line paths between p and p'.

[30]Note that the "subtype" terminology is not used by Baldwin and Klemperer (2019).