LINEAR PRICING MECHANISMS FOR MARKETS WITHOUT CONVEXITY

Now: *Walrasian Mechanisms for Non-Convex Economies and the Bound-Form First Welfare Theorem*

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quasilinear exchange economy

L commodities (+ money)

$n = 1,2, \dots, N$ **buyers** with quasilinear preferences $v_n(x) - t$.
Valuations $v_n(x)$ are bounded, nondecreasing and $v_n(0) = 0$.

$f = 1,2, \dots, F$ **sellers** with profit $t - c_f(y)$.
Costs $c_f(y)$ are bounded, nondecreasing and $c_f(0) = 0$.

Valuations may be non-concave and costs may be non-convex:
$\Rightarrow$ **Walrasian equilibria may not exist**

non-convexities are pervasive

startup & switching costs, economies of scale, indivisibilities, complementarities, externalities

Goal: to offer extensions of the **Walrasian mechanism,** sharing its desirable properties, without the assumption of convexity

our new mechanisms

Markup Mechanisms (α, p, ω)

Sellers' prices p

Buyers' prices $(1+\alpha)p$

Allocation ω in supply and demand sets at respective prices, enforcing **physical feasibility**

$$\text{production} \geq \text{consumption}$$

and **budget feasibility**

$$\text{buyer payments} \geq \text{seller payments}$$

for markets with two-sided non-convexity

(there may be no feasible allocations that exactly clear the market)

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for markets with two-sided non-convexity

(there may be no feasible allocations that exactly clear the market)

Rationing Mechanisms (p, ω)

Buyers & sellers both face price p

Allocation ω may involve **rationing**: assignments that are not preferred given the price vector p.

(p, ω) chosen to satisfy **physical feasibility** and **exact budget balance**

for markets with one-sided convexity

(exact market-clearing may be guaranteed)

bound-form first welfare theorem

If no equilibrium exists, any (price, allocation) pair must feature either:

- supply $>$ demand $\Rightarrow$ **budget deficit**
- demand $>$ supply $\Rightarrow$ **rationing**

bound-form first welfare theorem

If no equilibrium exists, any (price, allocation) pair must feature either:

- supply > demand ⇒ **budget deficit**
- demand > supply ⇒ **rationing**

Given **any** feasible allocation $\omega = (x, y)$, **any** non-negative price vector $p \in \mathbb{R}_+^L$, and the surplus function $S: \Omega \to \mathbb{R}$, and an efficient $\omega^* \in \Omega$,

$$S(\omega^*) - S(\omega) \le p \cdot \left(\sum_f y_f - \sum_n x_n\right) + \left(\sum_f \mathcal{R}_f(p, y_f) + \sum_n \mathcal{R}_n(p, x_n)\right).$$

Welfare loss from ω $\le$ Revenue deficit + Rationing losses for sellers & buyers

where **rationing losses** are defined as

- $\mathcal{R}_f(p, y_f)$ = maximum profit at price p minus profit at y_f given p
- $\mathcal{R}_n(p, x_n)$ = maximum utility at price p minus utility at x_n given p

markup mechanisms: results

Markup Mechanisms (α, p, ω)

Sellers' prices p

Buyers' prices $(1+\alpha)p$

ω in supply and demand sets at respective prices

Enforcing **physical feasibility** and **budget feasibility**

Large market assumptions:

- Number of buyers N and sellers $\to \infty$ with $\frac{\#\text{ buyers}}{\#\text{ sellers}} \to \phi$
- Non-convexities are bounded by $\delta \sim O(1)$
  - Radius of (non-convex) demand/supply set
- Growing gains from trade: efficient surplus $\Omega(N)$
- Set of approximately clearing prices is bounded

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Theorem: There exists a markup mechanism (α, p, ω) with $\alpha \sim O(1/N)$, which is:

1. Nearly efficient: proportion of surplus lost cf. efficient allocation is $O(\delta/N)$.
2. Nearly incentive-compatible*: maximum gain to a misreport is approx. $O(1/N)$
3. Easy to compute* (using convex optimization & bisection search for α).

** Under additional assumption of strong monotonicity (as in Walrasian mechanism!)*

proof idea: finding (α, p, ω)

For a **fixed** α

- Convexify economy
- Scale down buyer values to $\frac{1}{1+\alpha}\mathrm{cav}(u(x))$
- Add δL units of 'auctioneer demand' for each good

Worst case allowance to be adjusted in practice

Find Walrasian equilibrium (p, ω^*).

proof idea: finding (α, p, ω)

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- Convexify economy
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Find Walrasian equilibrium (p, ω^*).

'Round': use Shapley-Folkman Theorem to find feasible allocation with $\left|\left|\widehat{\omega} - \omega^*\right|\right| \leq \delta L$.

Distance from $\sum S_n$ to $\sum \mathrm{co}(S_n)$ is no more than δL

proof idea: finding (α, p, ω)

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Find Walrasian equilibrium (p, ω^*).

'Round': use Shapley-Folkman Theorem to find feasible allocation with $||\widehat{\omega} - \omega^*|| \leq \delta L$.

Search to find the smallest $\alpha \geq 0$ such that budgets balance.

α needs to be large enough to cover excess supply ~1 buyer's demand $\Rightarrow O(1/N)$

Auctioneer's demand

proof idea: approximate efficiency

Use **bound-form first welfare theorem** with (p, ω).

$$\text{loss of } \omega \leq \text{rationing losses} + \text{budget deficits}$$

Each seller's rationing loss at p is 0.

Each buyer's rationing loss at p is $o(\alpha) = o(1/N)$ via envelope theorem.

- N buyers implies $o(1)$ total loss due to rationing.

Budget deficit from excess supply is $O(1)$.

Leads to $O(1)$ total loss

incentives: strong monotonicity (Watt, 2022)

For simplicity, suppose just 1 good. Demand is **strongly monotone** if there exists $m > 0$ such that $\frac{\partial D}{\partial p} \leq -m < 0$ for all p such that $D(p) \neq \{0\}$.*

Strongly monotone supply is defined analogously.

* *With L goods:* $(d - d') \cdot (p' - p) \geq \mu \|p - p'\|^2$ *for all* $d \in D(p), d' \in D(p')$

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Walrasian and markup mechanisms are **ex post $\boldsymbol{O(1/N)}$ –incentive compatible** if *each* agent has strongly monotone preferences.

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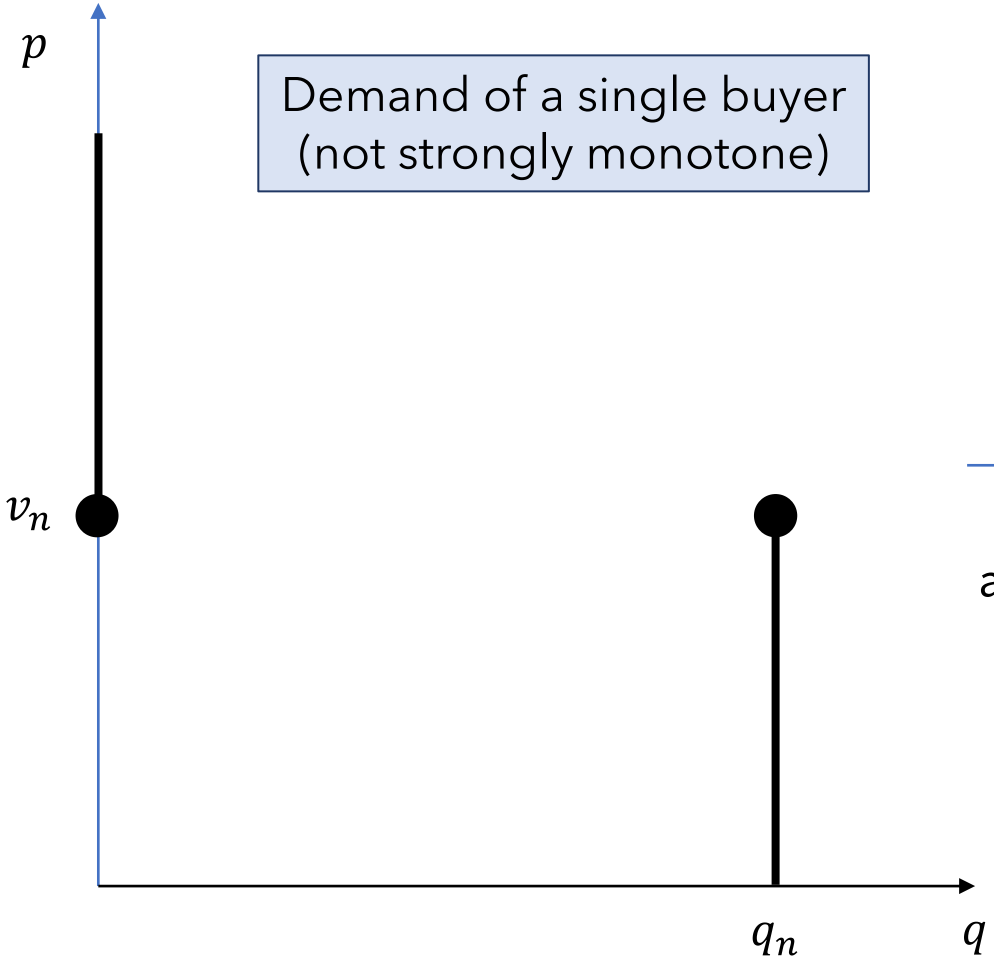
Walrasian and markup mechanisms are **ex post $\boldsymbol{O(1/N)}$ –incentive compatible** if *each* agent has strongly monotone preferences.

Moreover, if agents are drawn i.i.d. from distribution for which the *expected* demand/supply are strongly monotone, the Walrasian and markup mechanisms are **ex post $\boldsymbol{O_{\mathrm{P}}(1/N^{1-\varepsilon})}$-incentive-compatible.**

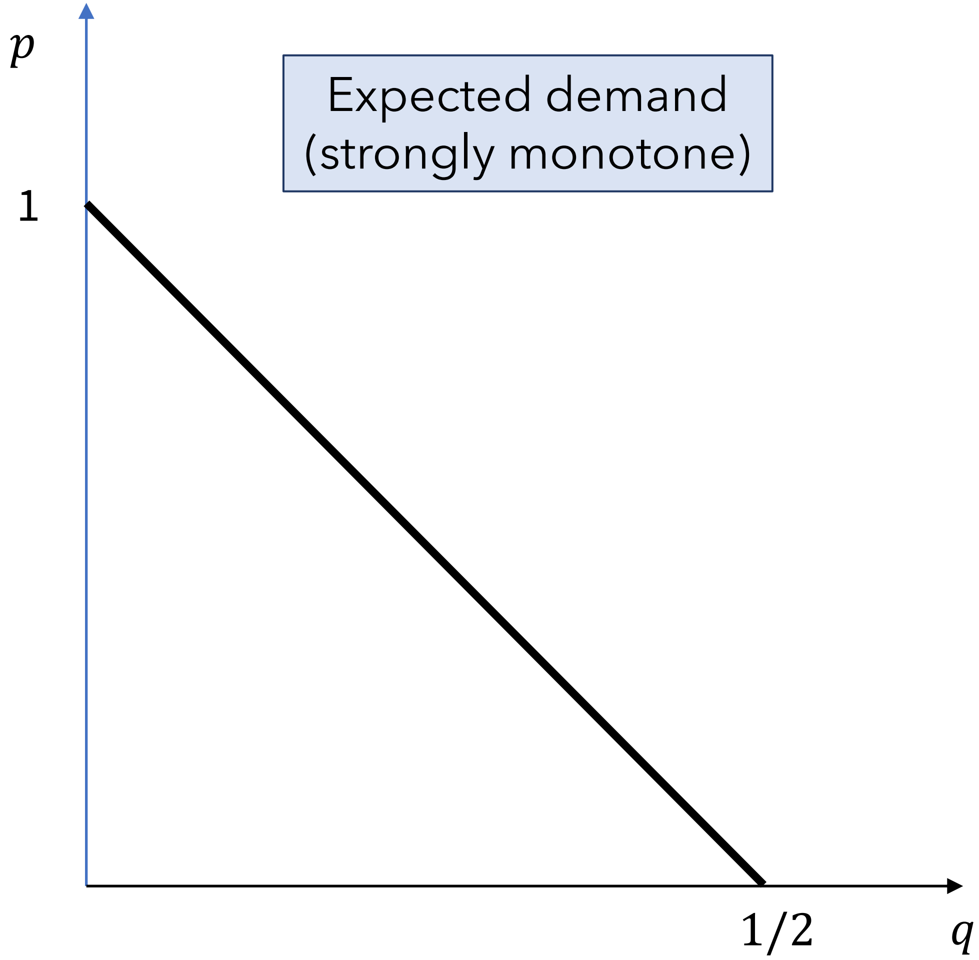
** With L goods: $(d - d') \cdot (p' - p) \geq \mu \|p - p'\|^2$ for all $d \in D(p), d' \in D(p')$*

strong monotonicity of buyers



in expectation

if $v_n \sim \mathrm{Unif}[0,1]$,
and $q_n \sim \mathrm{Unif}[0,1]$



rationing mechanism: results

Rationing Mechanism (p, ω)

Buyers' and sellers' prices p

Allocation ω may involve **rationing**: assignments that are not preferred given the price vector p.

(p, ω) chosen to satisfy **physical feasibility** and **budget feasibility**

Large market assumptions:
As previously

Additional assumption:
Buyers have convex preferences, and (expected) demand is *strongly monotone.*

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Theorem: There exists a rationing mechanism (p, ω) which is:

1. Nearly efficient: **total** surplus lost c.f. efficient allocation is $O(\delta/N)$.
2. Nearly incentive-compatible*: maximum gain to a misreport is approx. $O(1/N)$
3. Easy to compute*: using two convex optimizations.

rationing mechanism: proof idea

Rationing mechanism

1. Find market-clearing prices p in convexified economy with ρL units of **additional fictitious auctioneer supply** of each good.
2. Round sellers' allocation $y \rightarrow$ feasible allocation **with excess demand**.
3. Reoptimize buyers given fixed supply from step 2 to determine buyer allocations x.

Advantage: $O(1/N)$ **total** loss. Disadvantage: Only $O(1/N)$-IR for buyers.

Intuition: Price p and shadow price for buyers are $O(1/N)$ close by strong monotonicity so that the rationed loss of buyers is also $O(1/N)$.

conclusion

Our **bound-form first welfare theorem** relates deadweight losses to rationing.

Our **markup mechanism** generalizes the Walrasian mechanism to work with non-convexities and provides:

- Approximate efficiency
- Good incentives
- Individually rational
- Easy computations
- Linear pricing
- No budget deficit

With **one-sided convexity** and strong monotonicity, our **rationing mechanism** offers most of the same features, with a smaller loss but (potentially) some small violations of individual rationality.

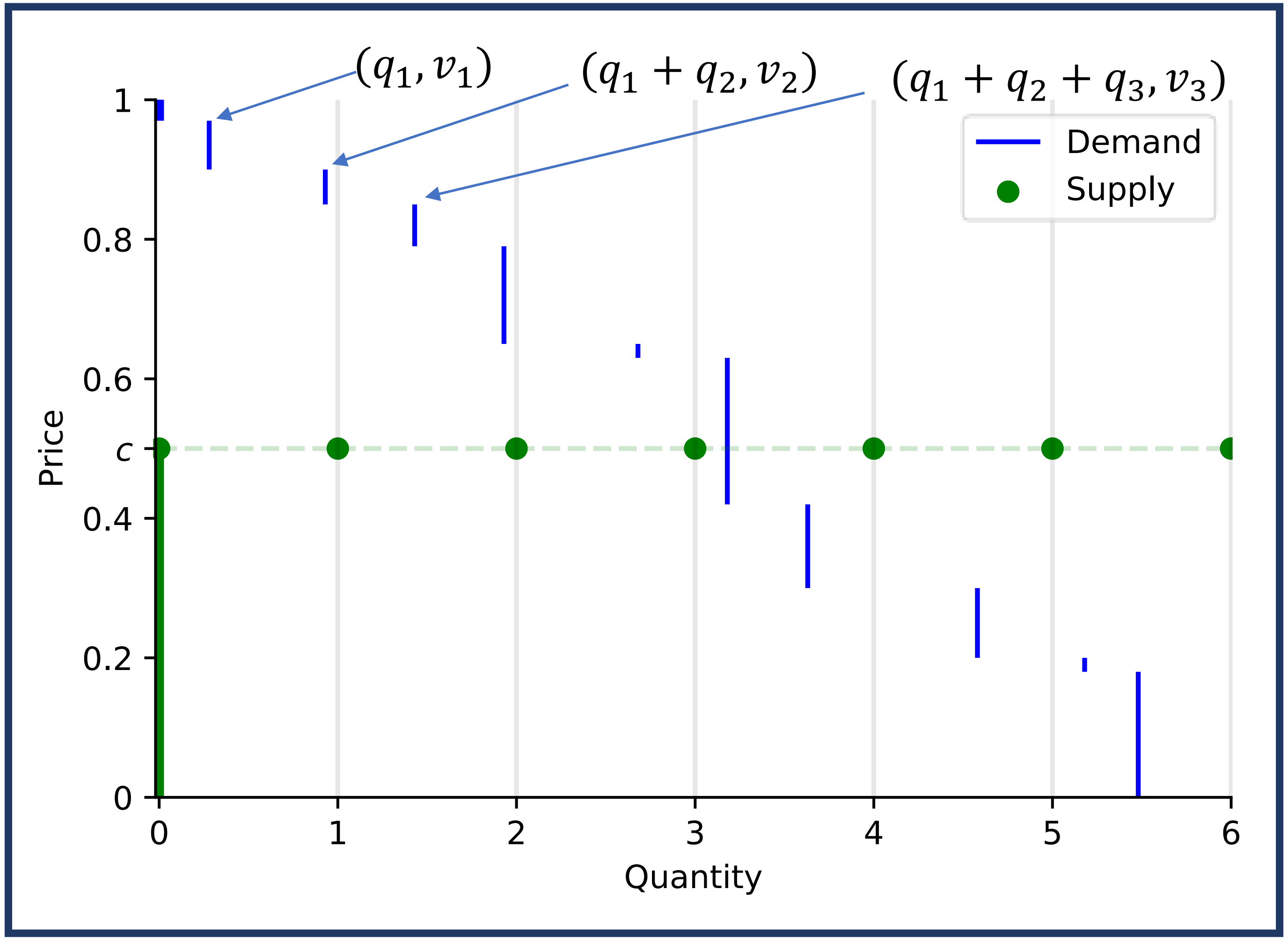
appendix: example

a non-convex market

Money + one other good.

One seller, who can produce any **integer** quantity of the good at marginal cost $c = \$0.50$.

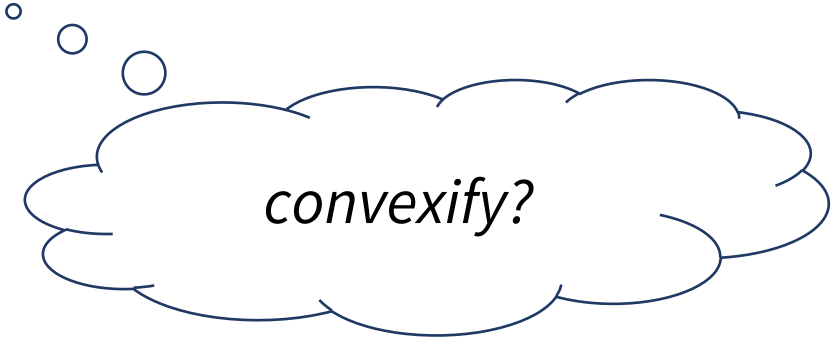
Buyers $n = 1, \dots, N$.
Buyer n has use only for **exactly** $q_n \in \mathbb{R}_+$ units with total value $q_n v_n$, so v_n per unit

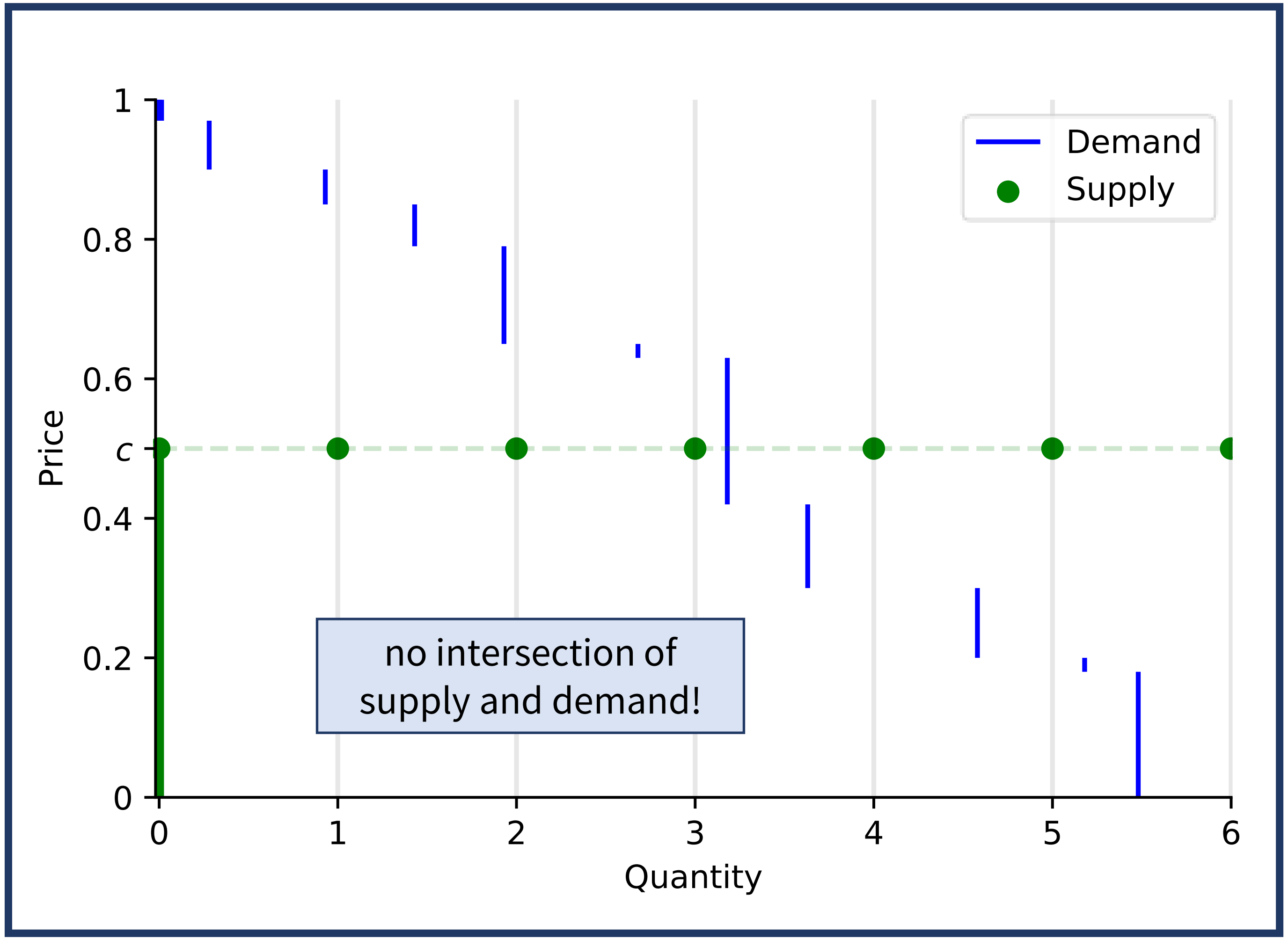


Walrasian equilibrium does not exist

Problem

non-existence of Walrasian equilibrium





pseudoequilibrium via convexification

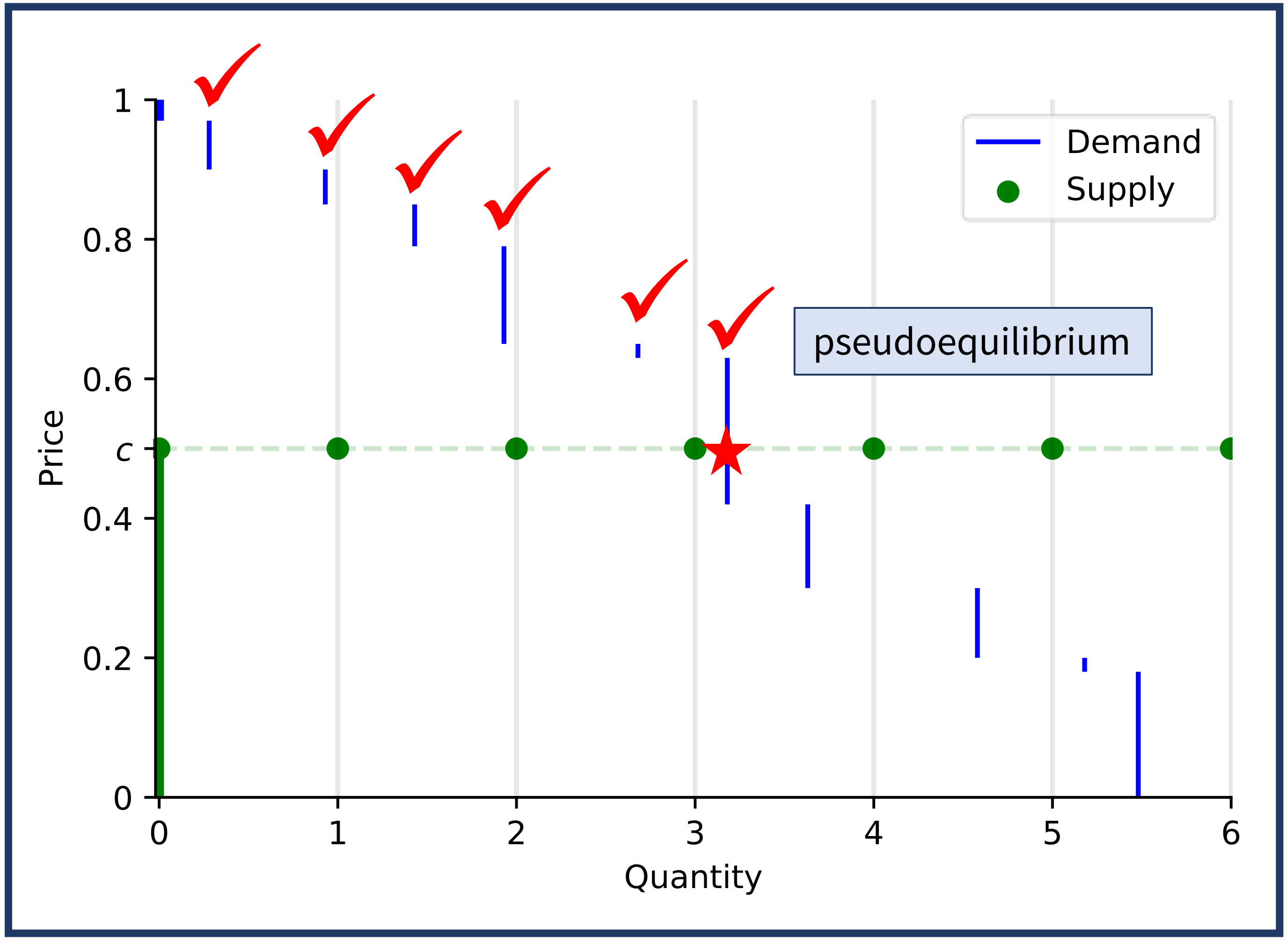
Starr (1969), Shapley-Folkman, Heller (1972),
Nguyen & Vohra (2021)

Convexify costs and values and find resulting WE.
At most L agents are assigned non-preferred bundles.

Problems

infeasible assignments

round allocations?



approximate equilibrium via Shapley-Folkman

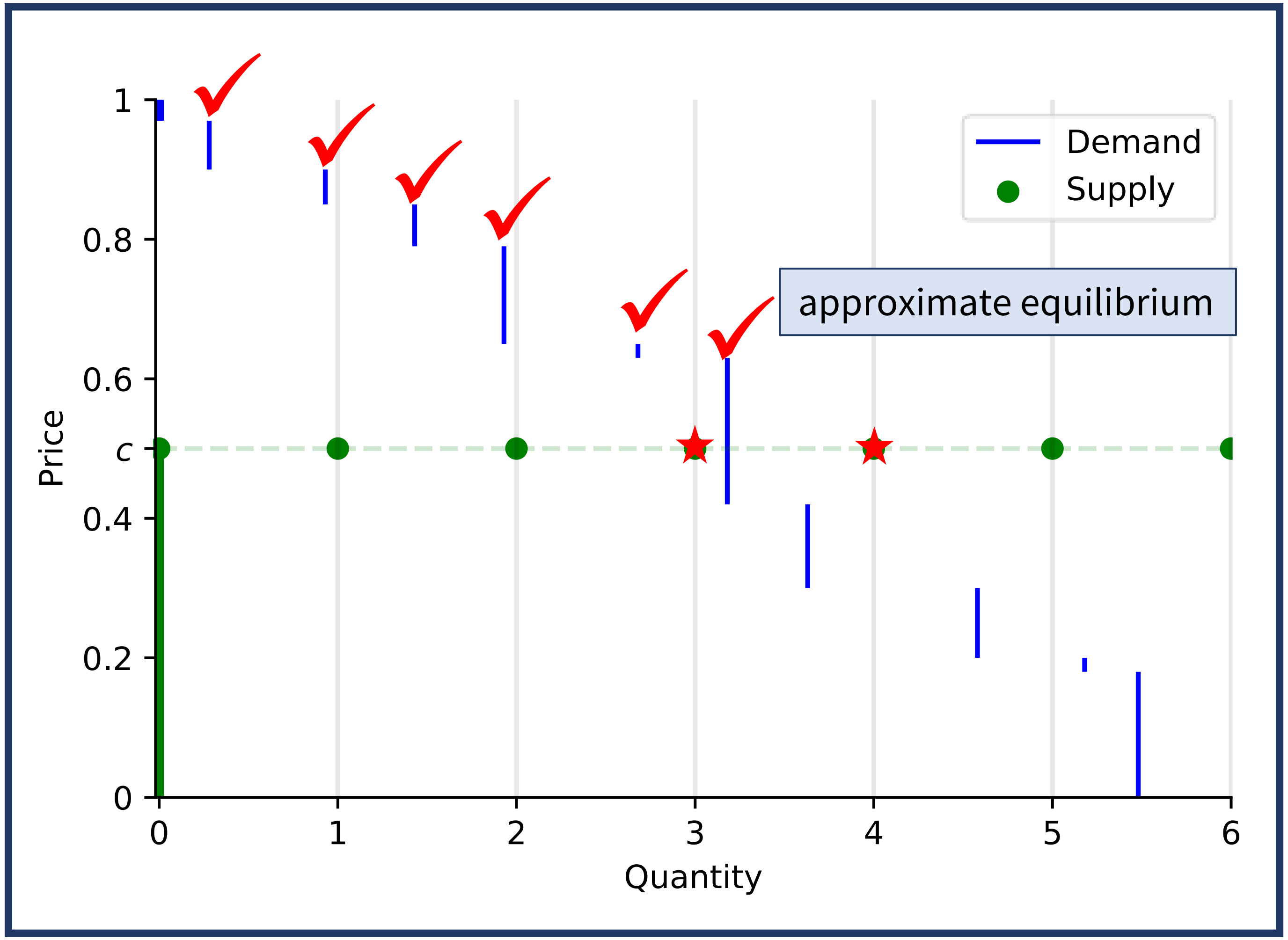
Starr (1969), Shapley-Folkman, Heller (1972), Nguyen & Vohra (2021)

There exists (p, ω) such that supply ≈ demand, but not exactly equal.

Problems

demand > supply and/or payments > receipts

ration some agents?



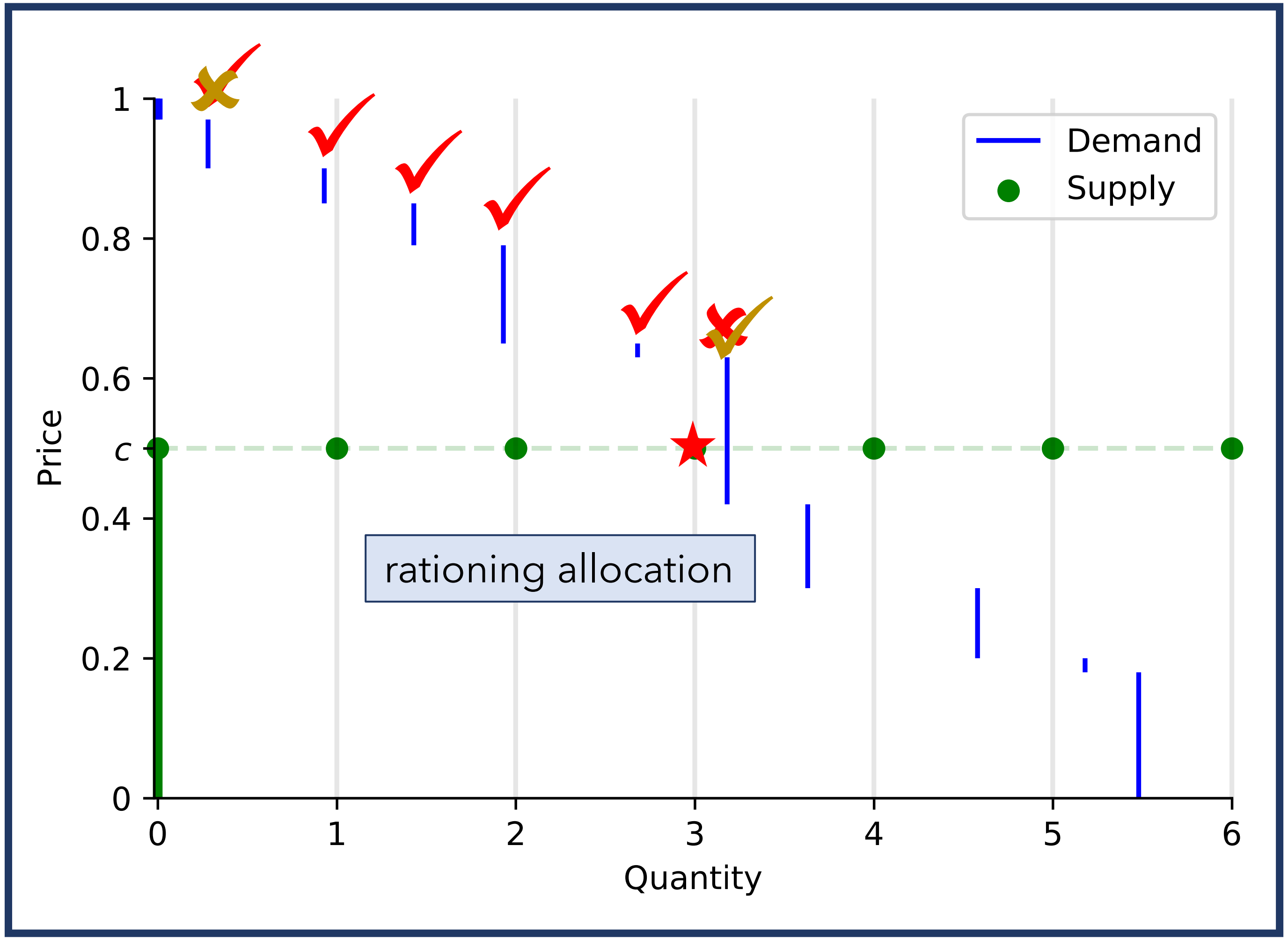
rationing in two-sided non-convex markets

Rationing = deny some agents their preferred bundles

Problems

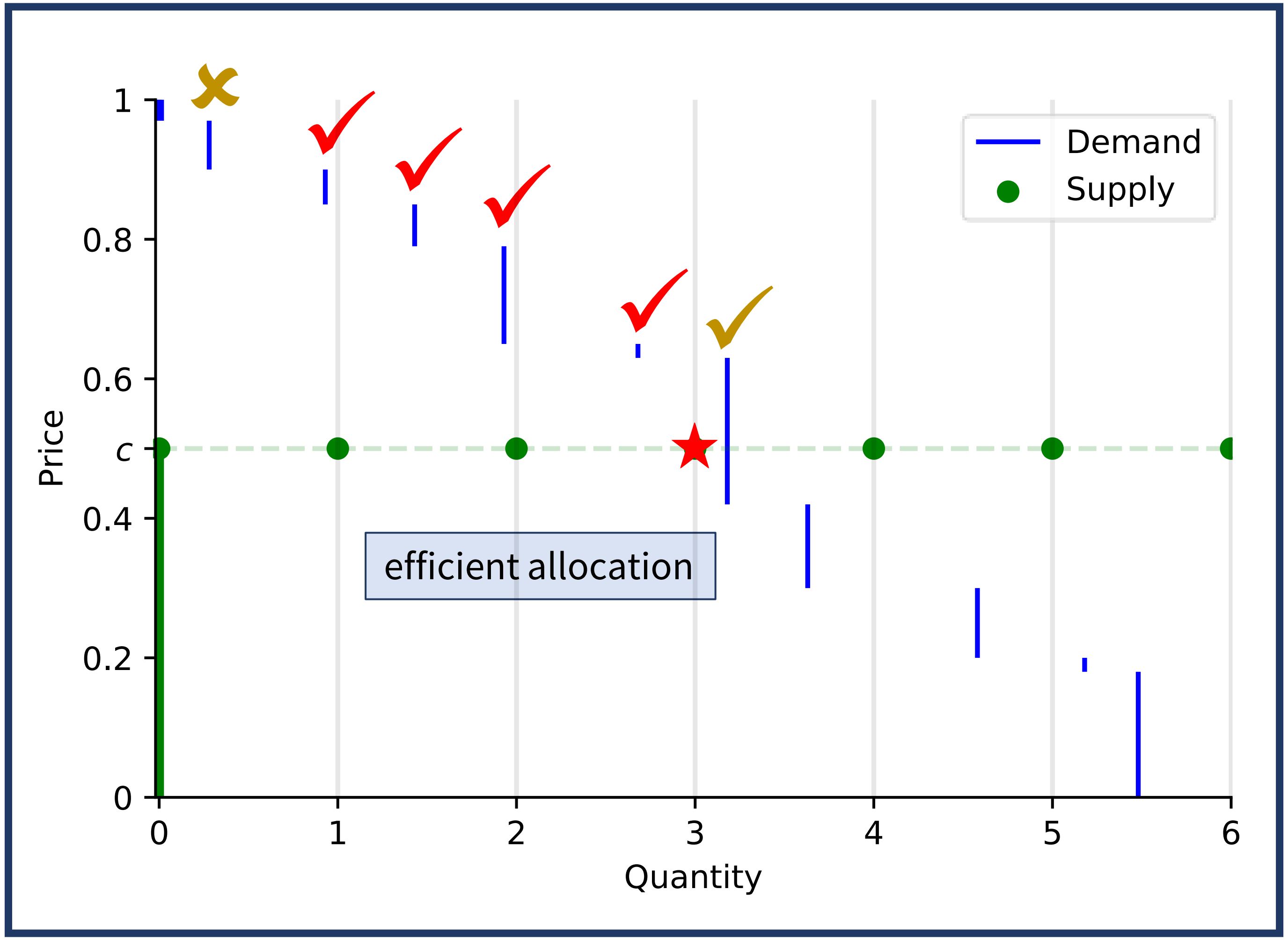
budget deficit
who should be rationed?
new incentive to misreport to avoid rationing

abandon prices?



efficient allocation problem

Solve exactly the efficient allocation problem.



Problems

budget deficit or bad incentives, computational challenge.

…add inefficient entry, exit & integration decisions, political resistance