strong monotonicity and perturbation-proofness of exchange economies

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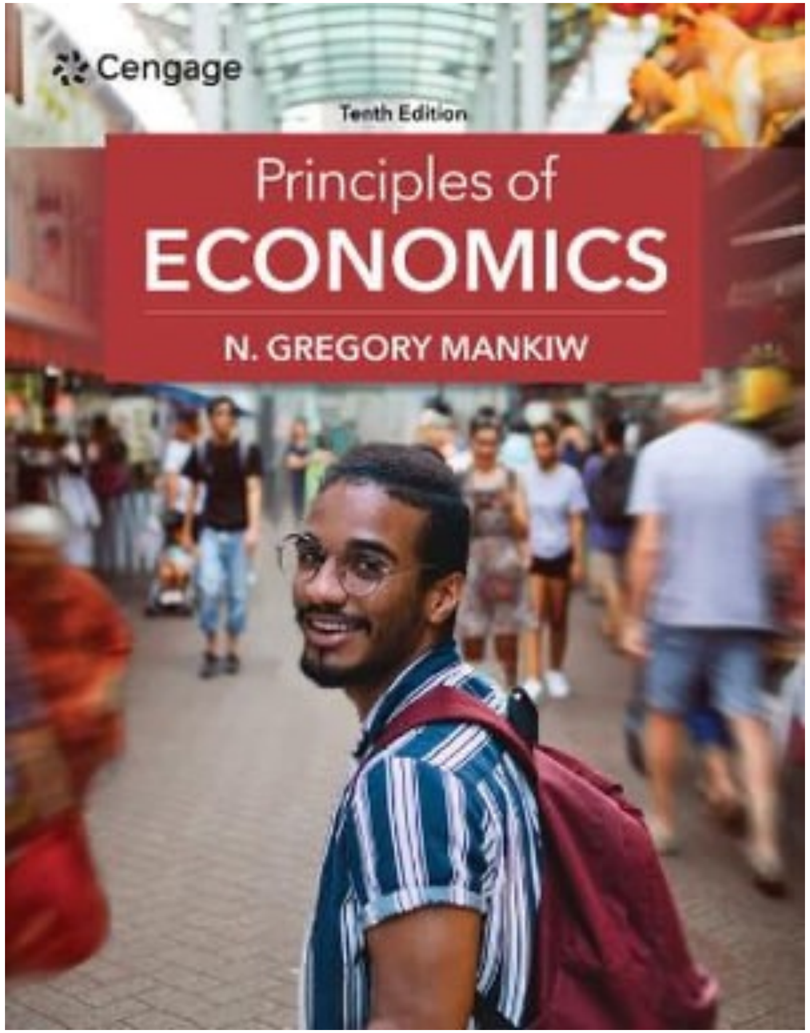
motivating question



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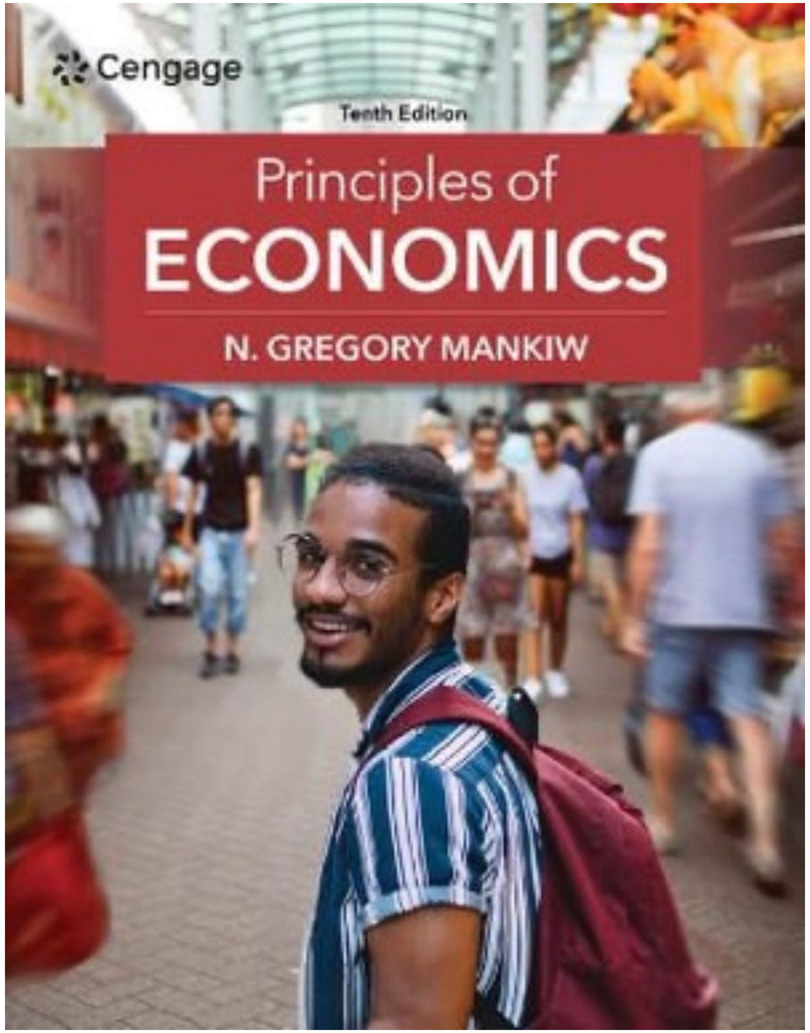
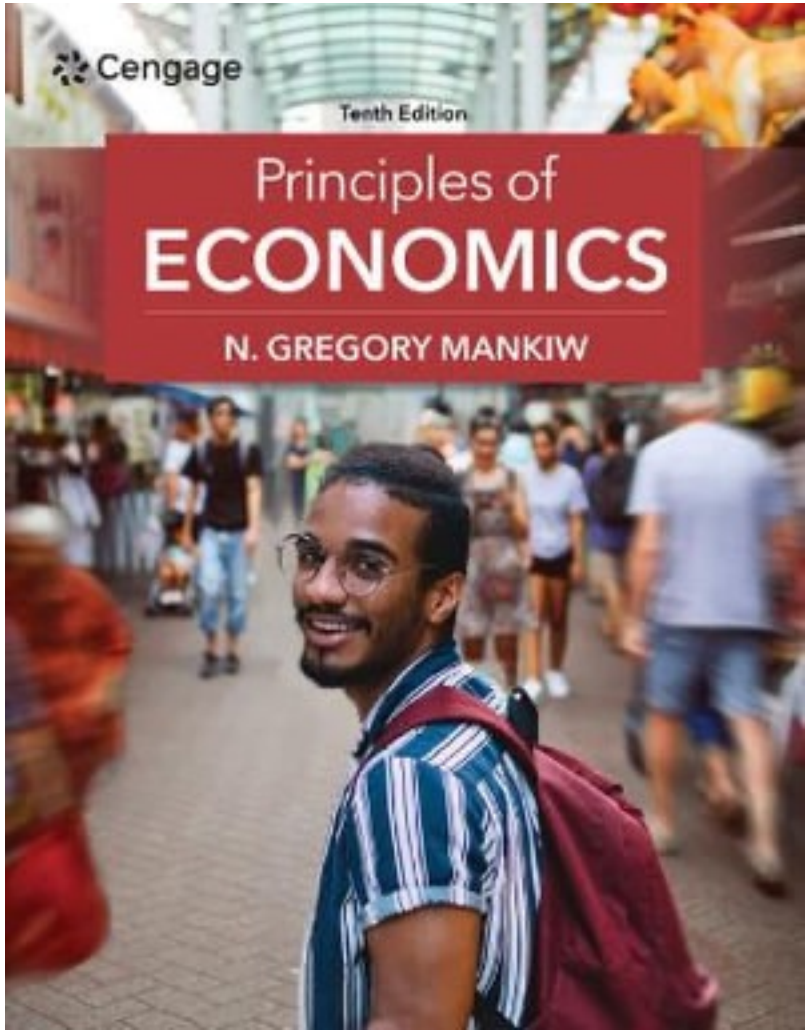


4-1b What Is Competition?

The market for ice cream, like most markets in the economy, is highly competitive. Each buyer knows that there are several sellers from which to choose, and each seller is aware that his product is similar to that offered by other sellers. As a result, the price and quantity of ice cream sold are not determined by any single buyer or seller. Rather, price and quantity are determined by all buyers and sellers as they interact in the marketplace.

Economists use the term **competitive market** to describe a market in which there are so many buyers and so many sellers that each has a negligible impact on the market price. Each seller of ice cream has limited control over the price because other sellers are offering similar products. A seller has little reason to charge less than the going price, and if he charges more, buyers will make their purchases elsewhere. Similarly, no single buyer of ice cream can influence the price of ice cream because each buyer purchases only a small amount.

motivating question



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1. How many buyers and sellers?
2. How big an impact on prices?
3. Under what conditions?

motivating question

Consider an exchange economy with N quasilinear buyers and supply $s \in \mathbb{R}_+^L$, and its Walrasian equilibria (WE).

Suppose

- a new buyer is added (or removed),
- new supply is added (or removed), or
- an agent misreports its type.

When is the impact of these **perturbations** on WE prices $O(1/N)$?

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A sequence of markets is **perturbation-proof** if for every *bounded* sequence of perturbations, the max. distance between WE prices in the perturbed and unperturbed economies is $O(1/N)$.

Implies $O(1/N)$-*ex post* incentive compatibility of Walrasian mechanisms.

example

example 1: market with M identical goods

Rustichini, Satterthwaite and Williams (1994)

N unit demand buyers

Value v_n of buyer $n = 1,2,\dots,N$ is drawn $\mathrm{Unif}[0,1]$

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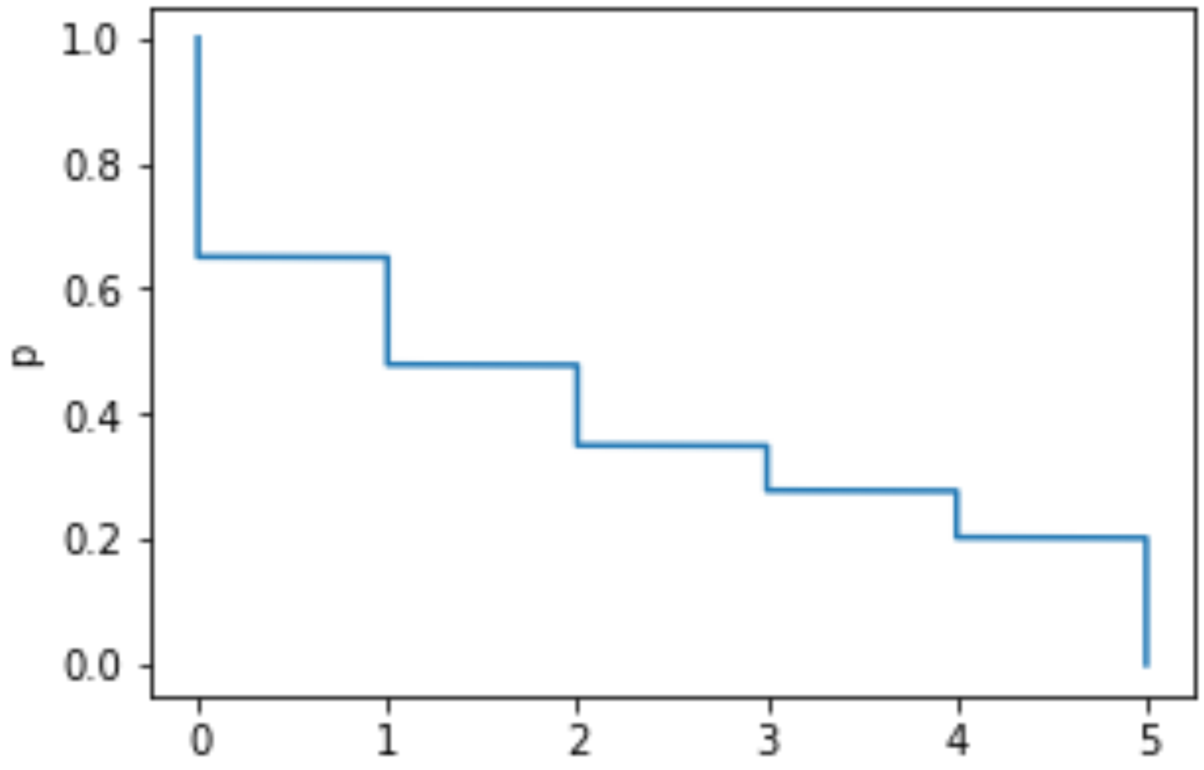

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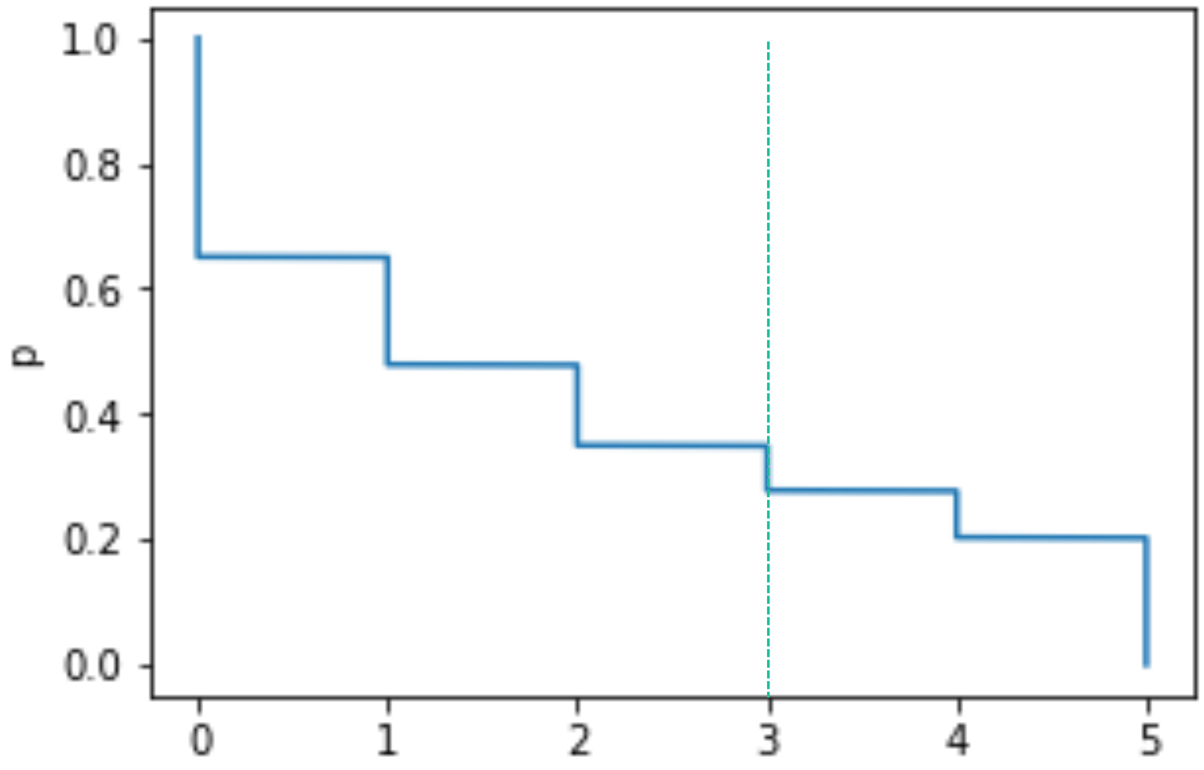

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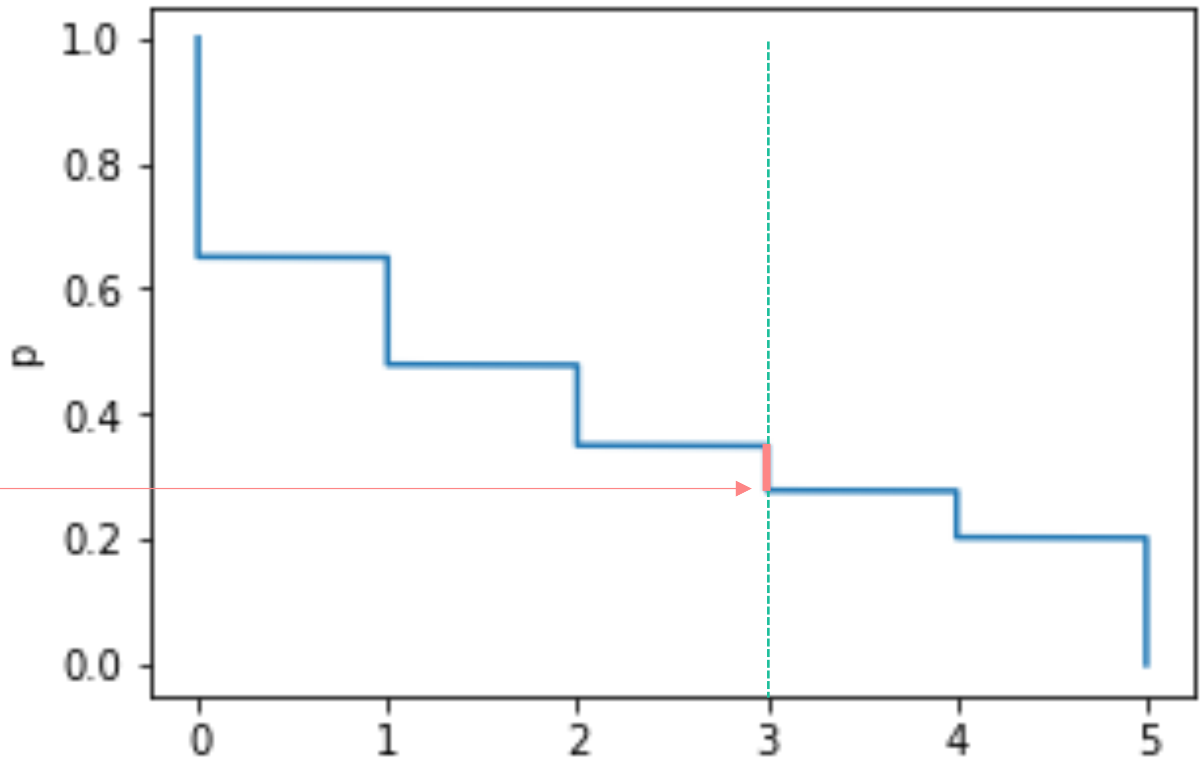
Any price between 0.29 and 0.34 is a WE price



Generally, WE prices are $\left[v^{(N-M;N)}, v^{(N-M+1;N)}\right]$.

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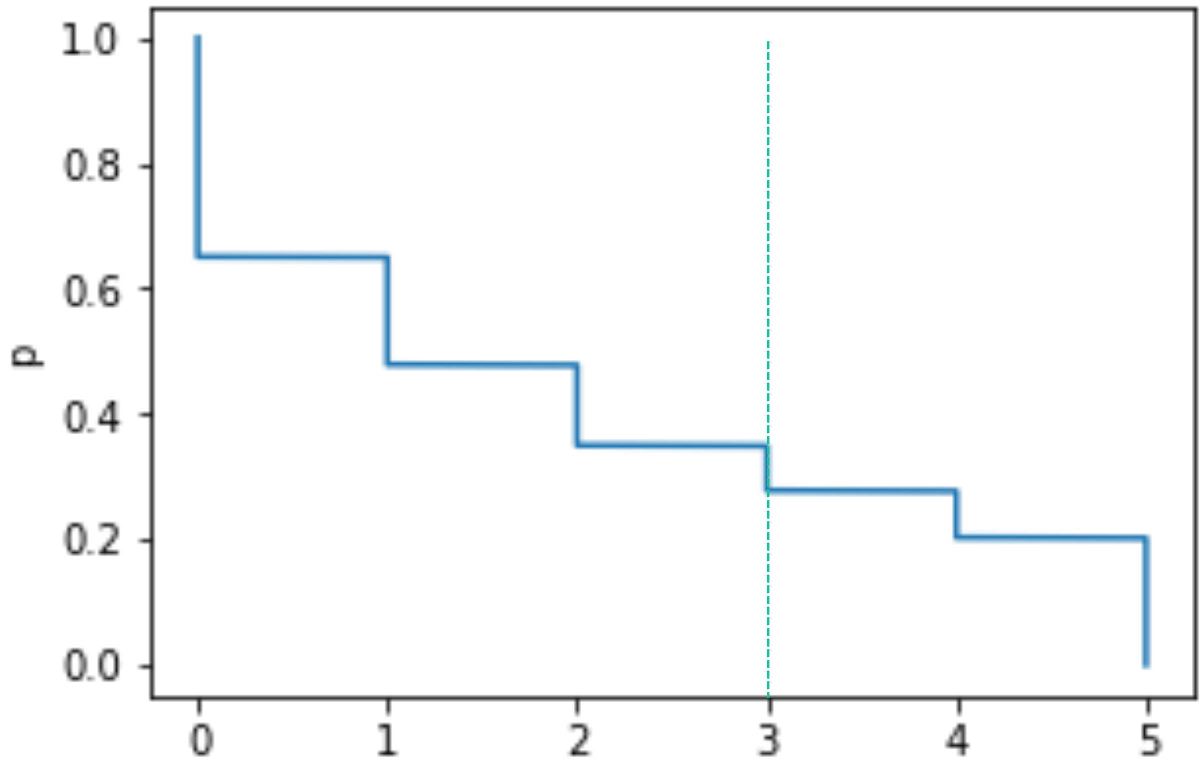
What WE prices can buyer 3 effect with *some report?*



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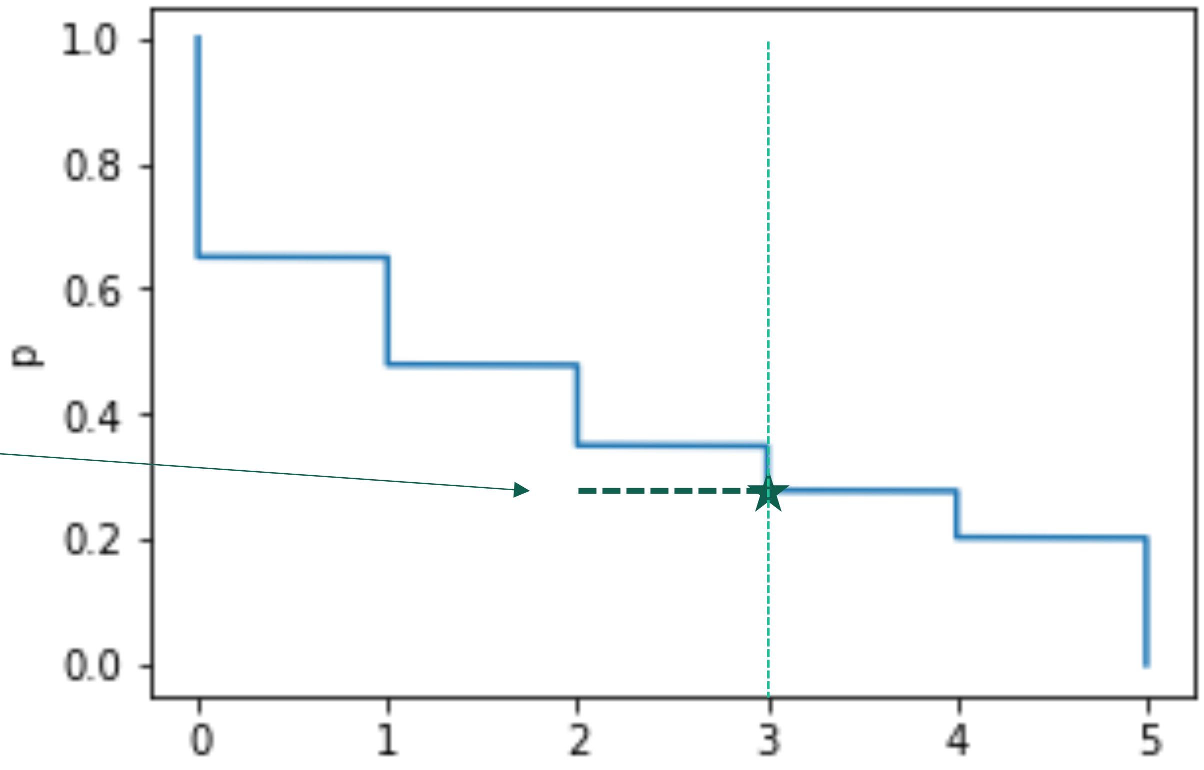

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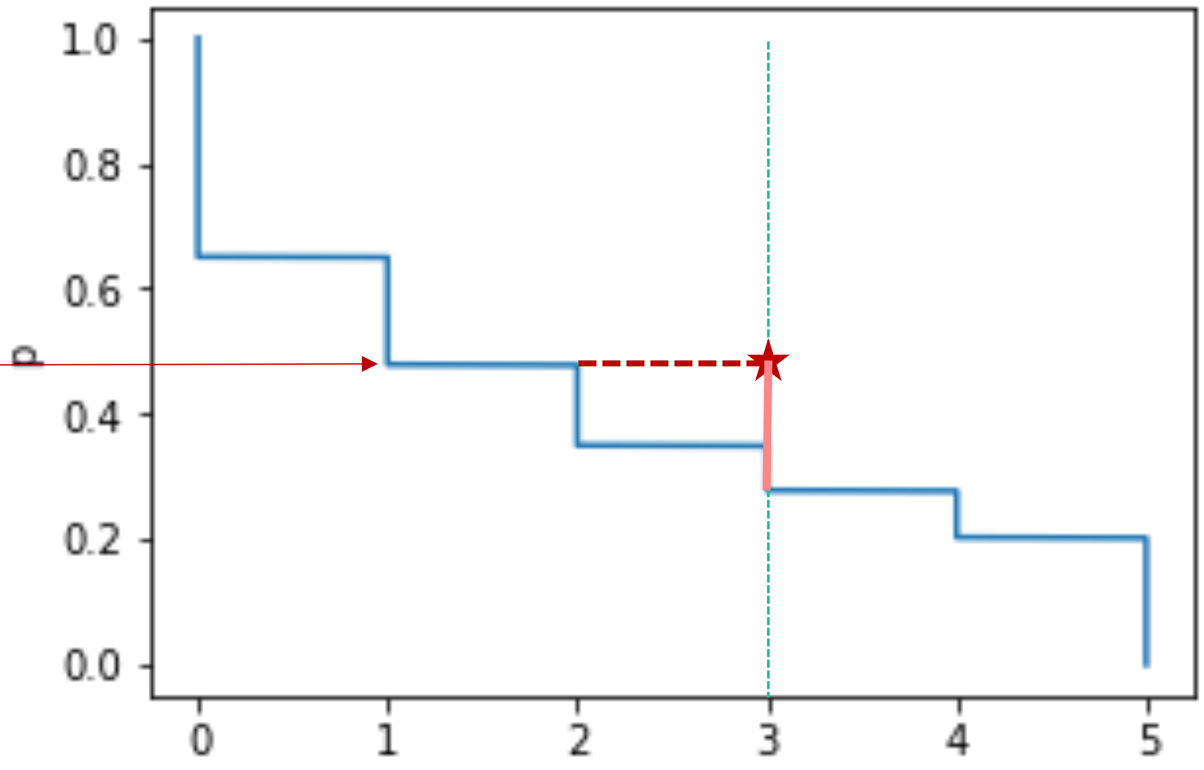

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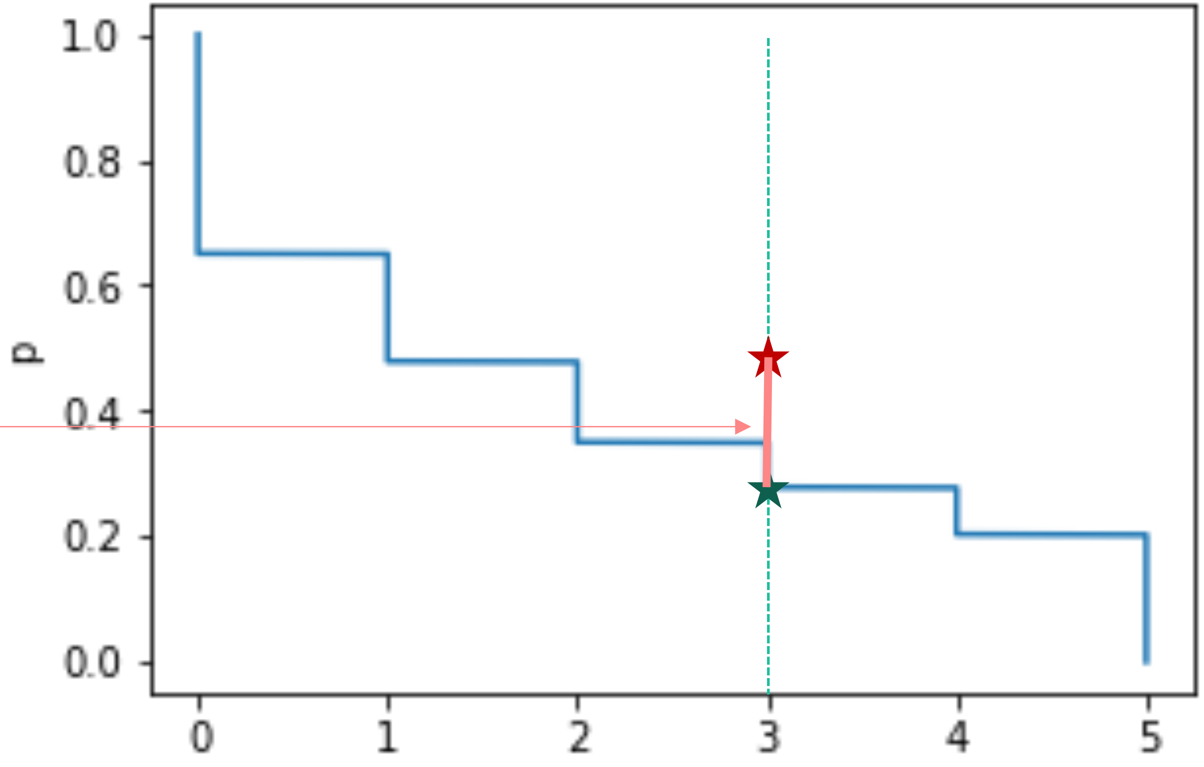

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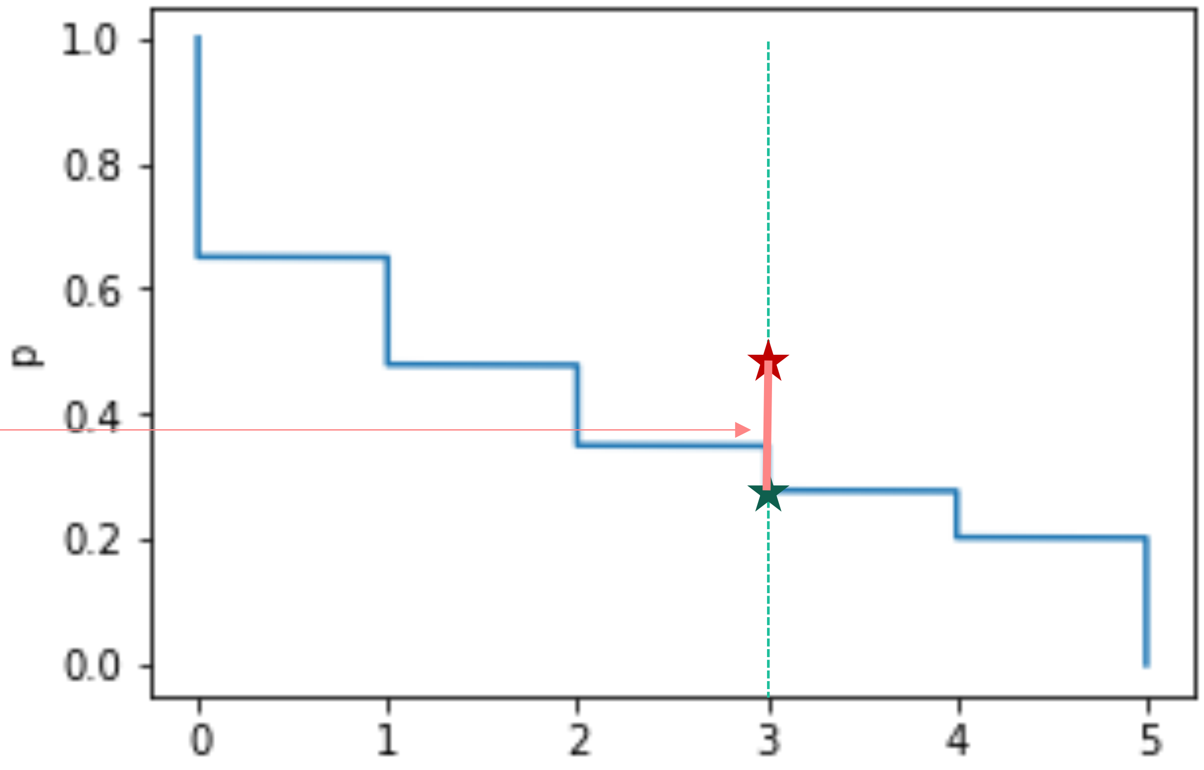

Holding fixed other agents' truthful reports, the prices an agent may realize by *some* report is $\left[v_{-i}^{(N-M;N-1)}, v_{-i}^{(N-M+1;N-1)}\right]$.

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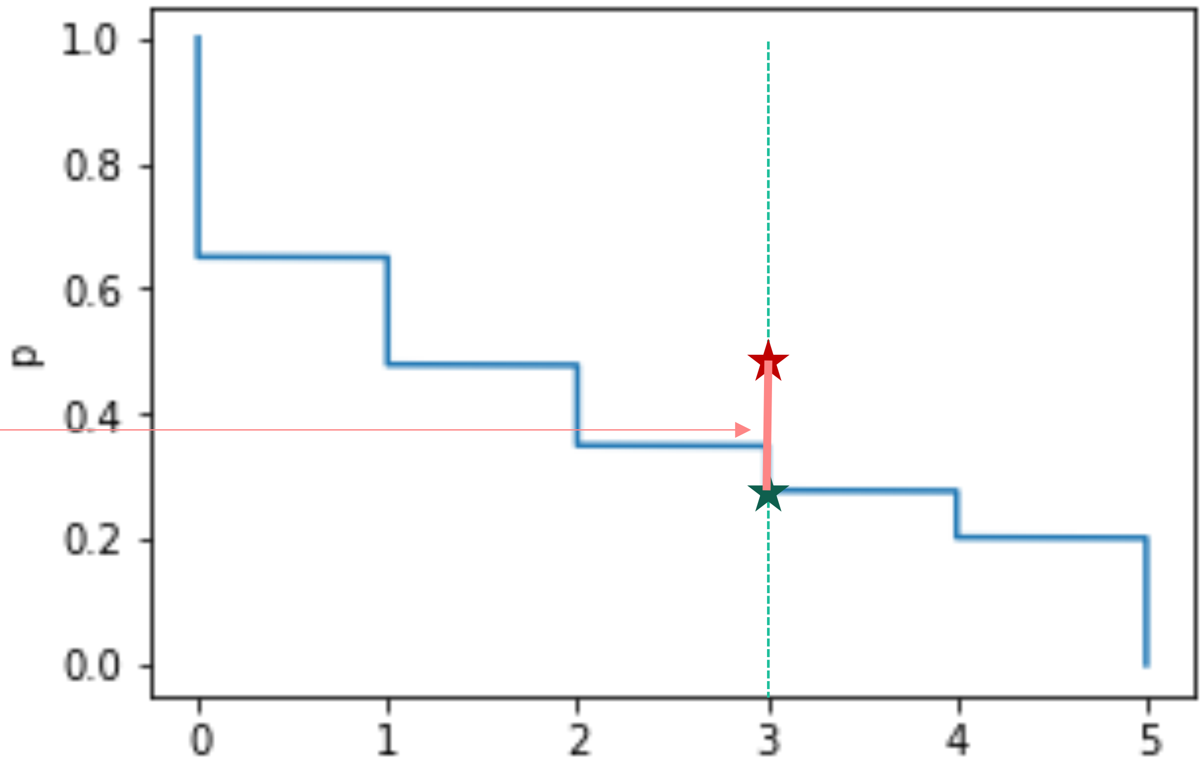

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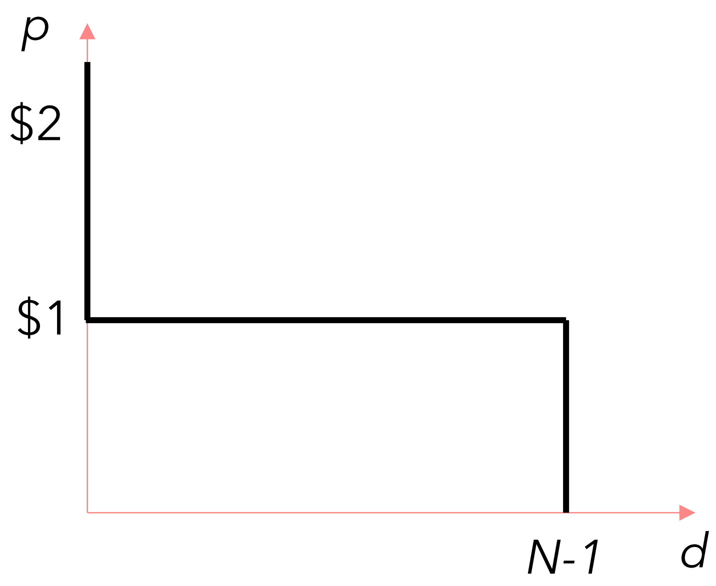
The expected maximum influence of any agent on prices is $O(1/N)$.

example 2: market with N identical goods

N-1 unit demand buyers, each with value \$1 for a good



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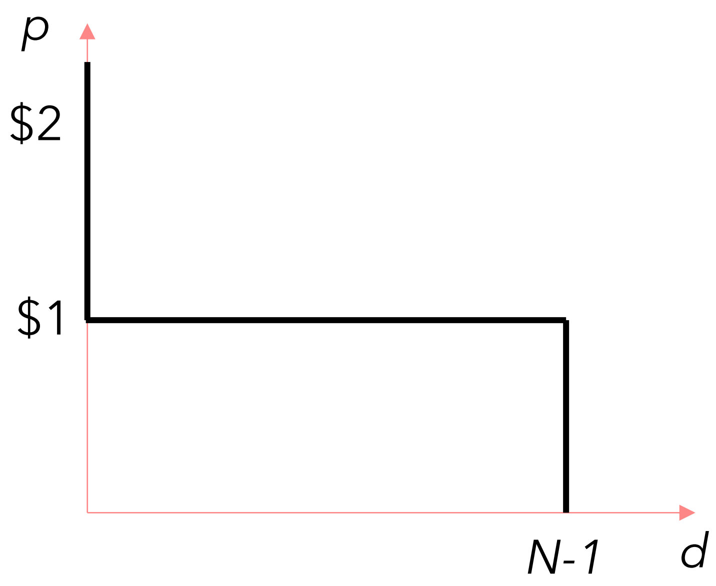
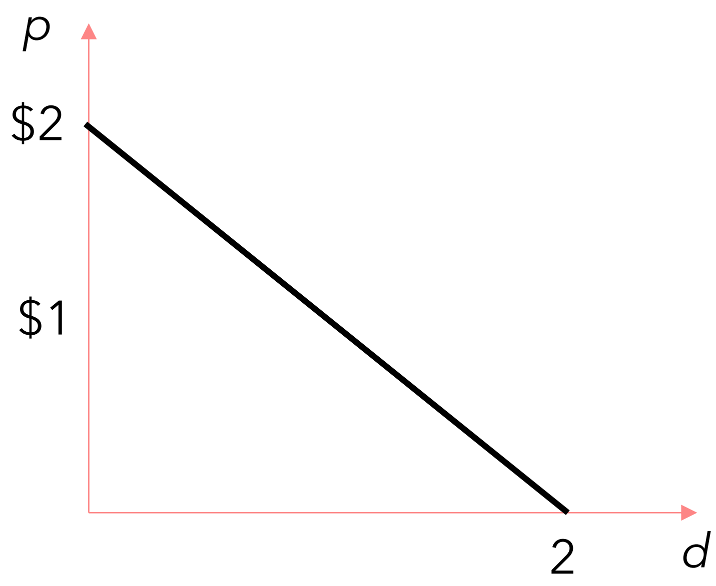


1 buyer with demand $d = (2 - p)_+$ for the good



example 2: market with *N* identical goods

N-1 unit demand buyers, each with value \$1 for a good

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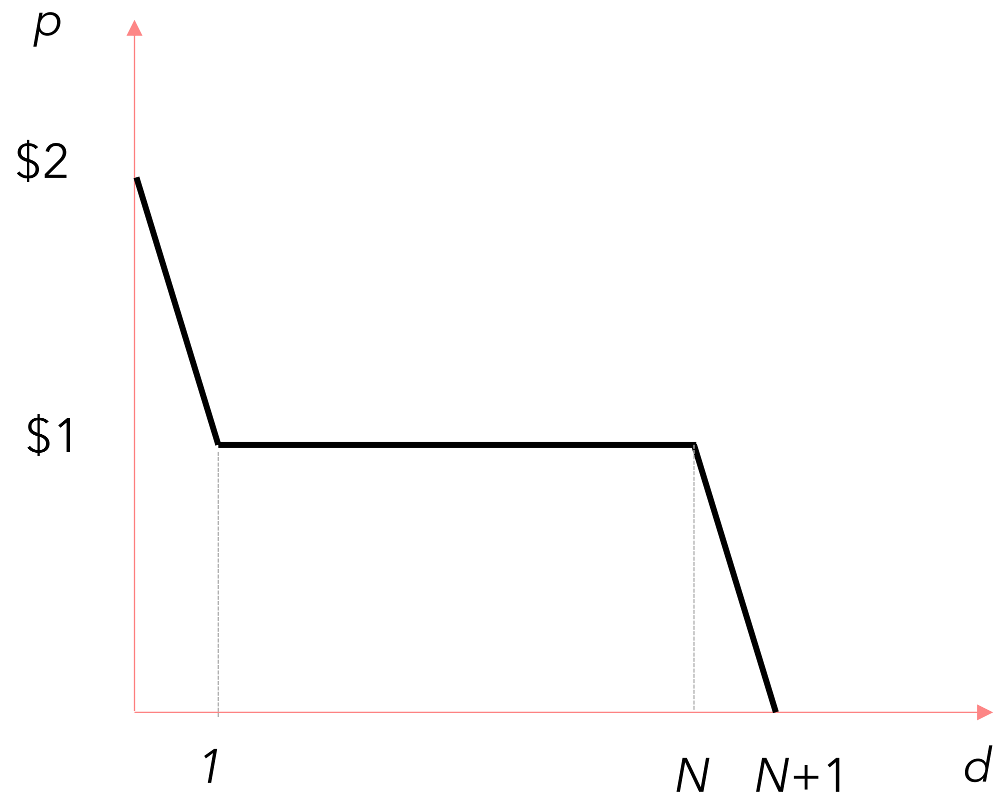

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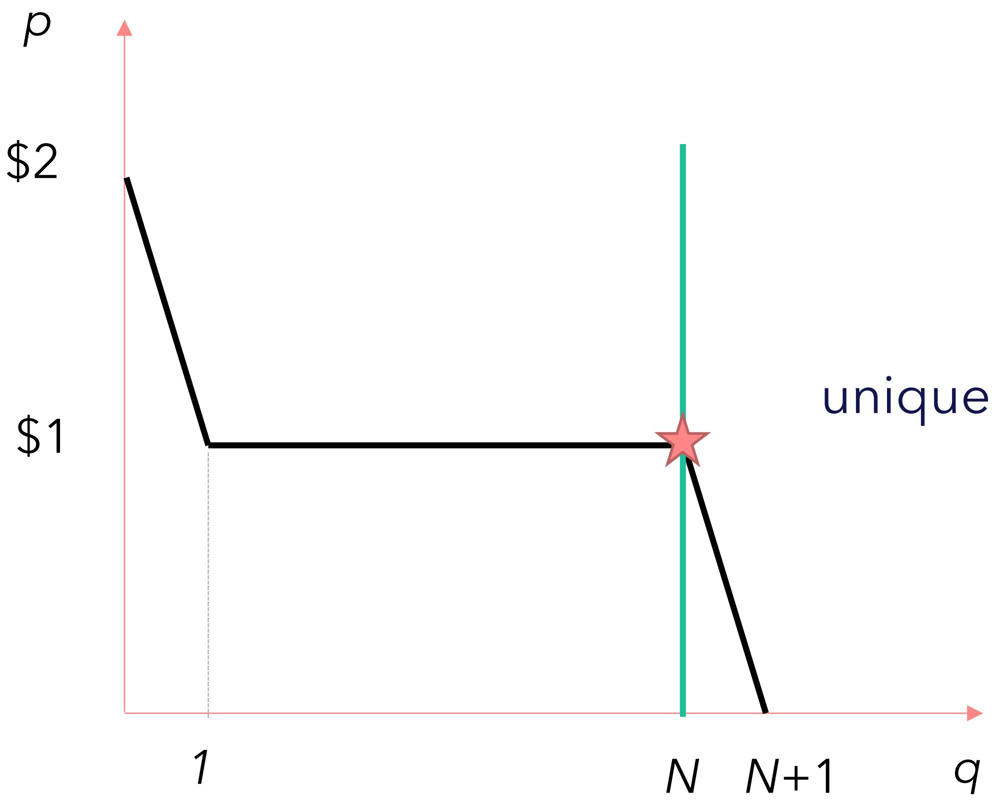

unique Walrasian equilibrium: p=\$1

example 2: **market with *N* identical goods**

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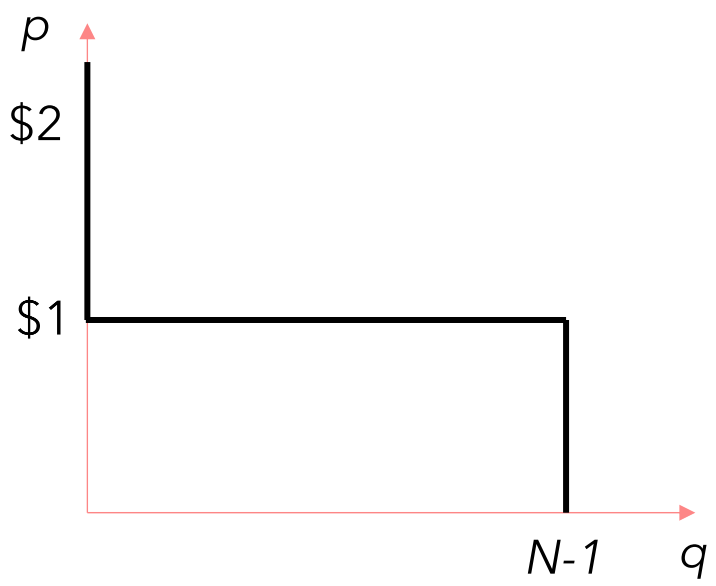
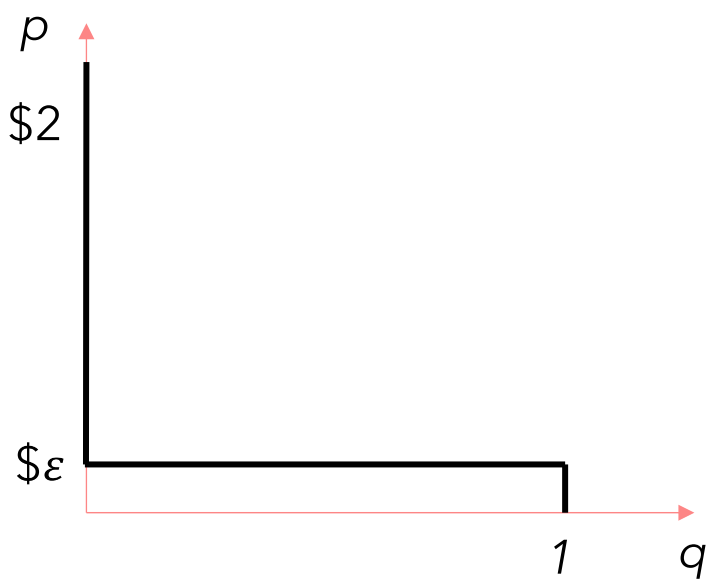


1 buyer **misreports** they have unit demand with value ε



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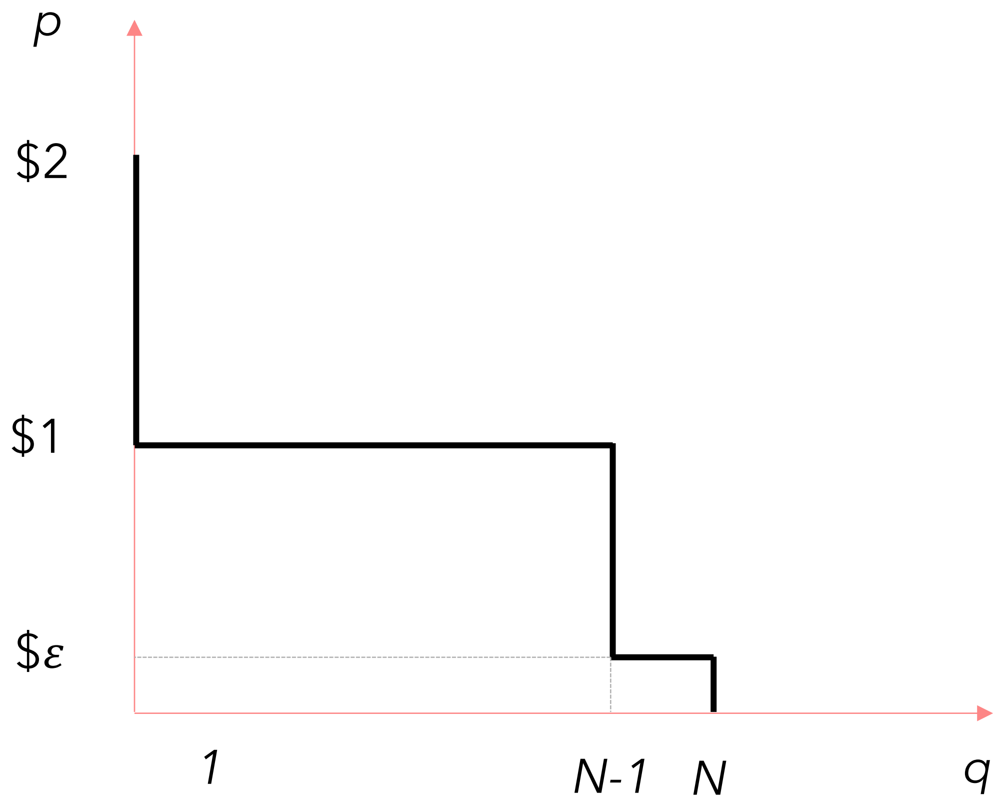

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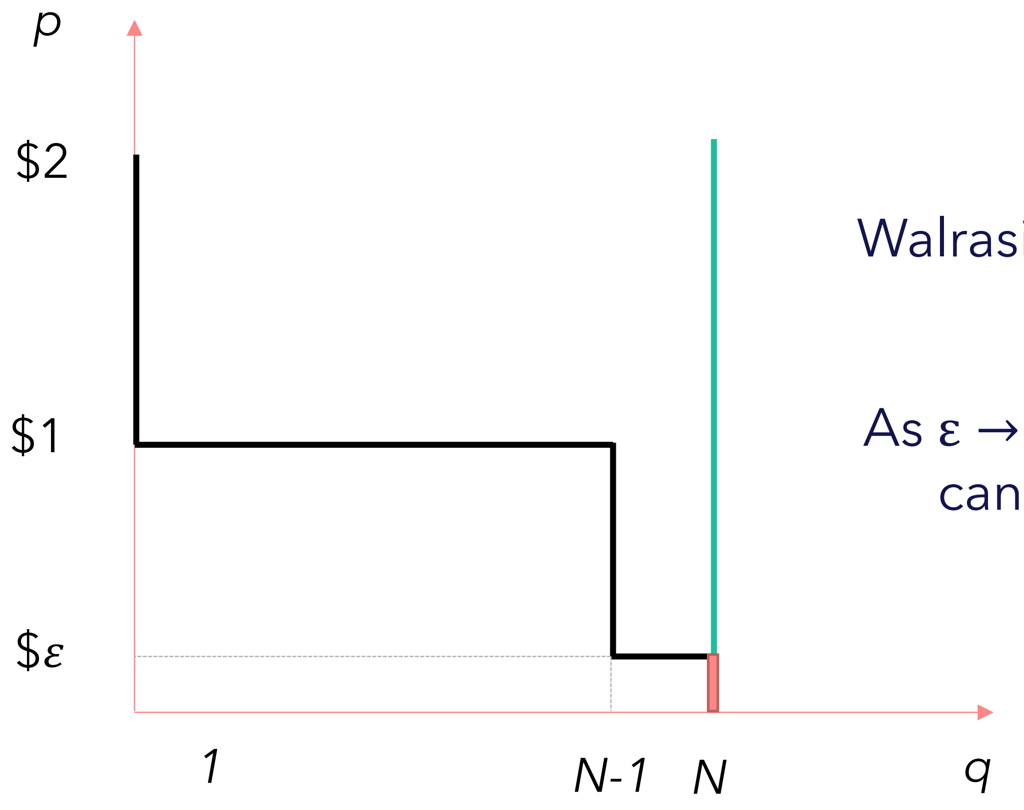
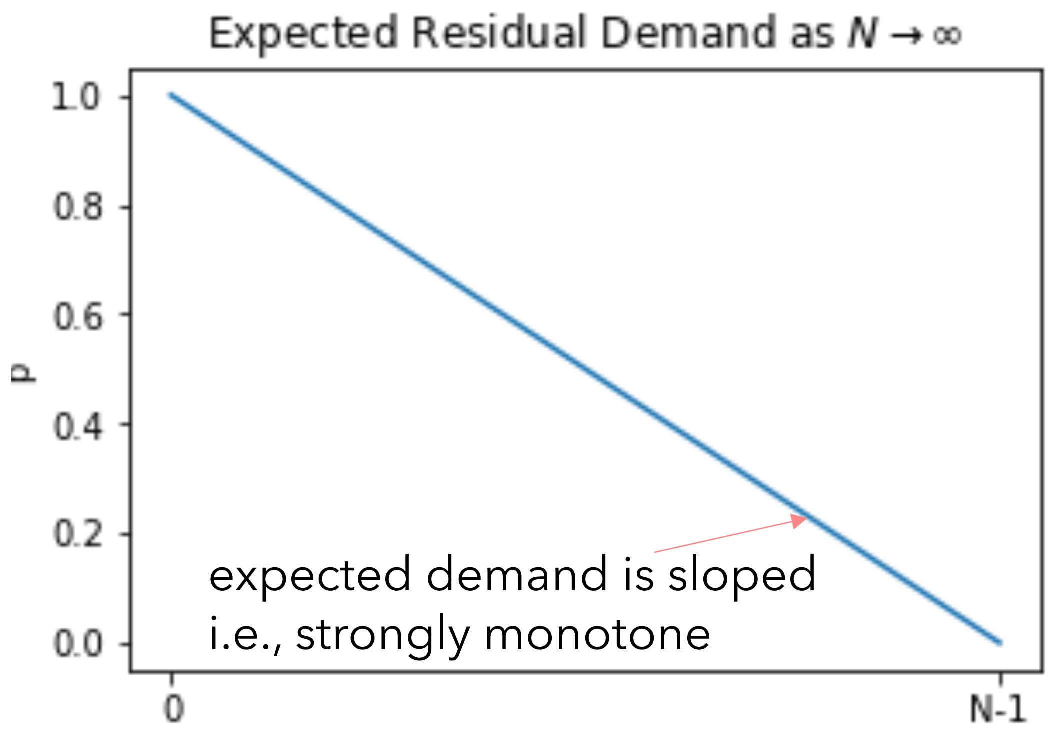


Walrasian equilibrium prices: $p \in [0, \varepsilon]$.

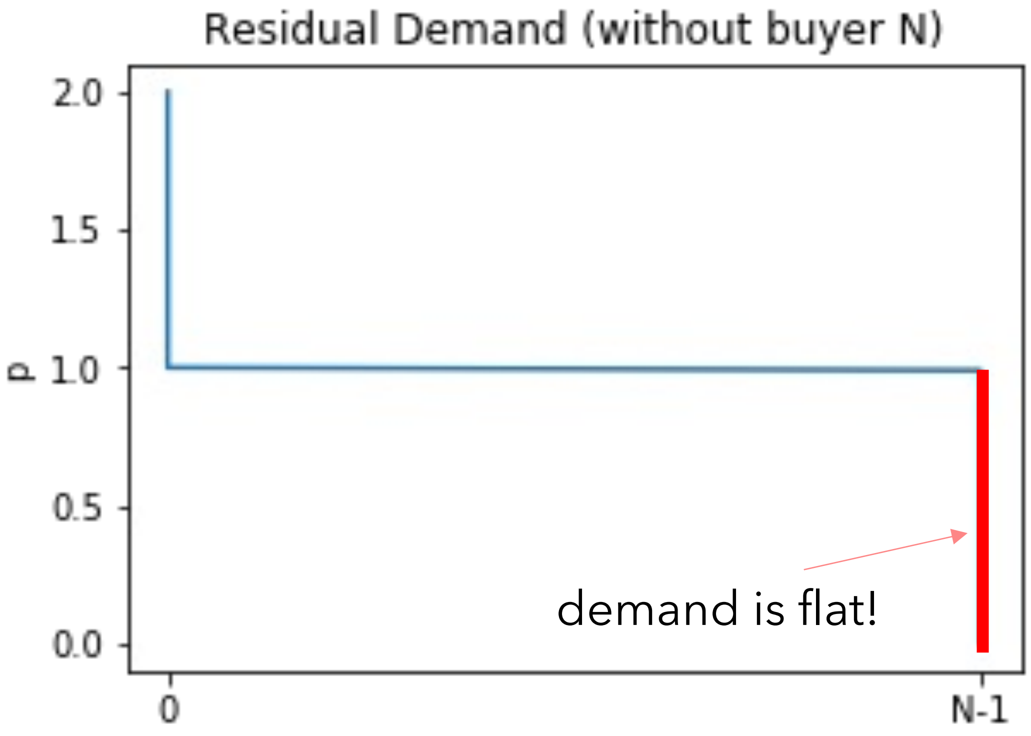
As $\varepsilon \to 0$, one buyer's misreport can erode entire revenue

Independent of N

price-taking behavior depends on the ***slope*** of the residual demand curve



Example 1



Example 2

key results

If each agent has **strongly monotone** demand, the resulting sequence of economies is perturbation-proof.

Strong monotonicity: $\left(d(p) - d(p')\right) \cdot (p' - p) \geq m||p - p'||^2$ for p, p' such that $d(p) \neq 0$

Replica economies are perturbation-proof **if and only if** the base economy is strongly monotone.

If buyers are drawn i.i.d. from a value distribution with strongly monotone **expected** demand, the resulting economies are perturbation-proof with high probability.

→ The Walrasian mechanism is $\boldsymbol{O(N^{-1+\varepsilon})}$**-incentive-compatible** (*ex post* with high probability and *interim*).

related literature

Ex post Incentives in Walrasian mechanisms

Hurwicz (1972)

Roberts and Postlewaite (1976), Jackson (1992) →

If WE correspondence (mapping economies as measures over valuation space to prices) is continuous at limit economy, then price impact tends to zero and optimal report tends to truth.

Green and Laffont (1979), Holmström (1979)

Rustichini, Satterthwaite and Williams (1994), Rostek and Yoon (survey, 2020)

Interim Incentives in Walrasian mechanisms

Azevedo and Budish (2019) →

*With a finite type space, Walrasian mechanisms are 'strategy-proof in the large': **interim** expected gains from misreporting bounded by $O(N^{-\frac{1}{2}+\varepsilon})$.*

Law of demand in non-quasilinear economies

Hildenbrand (1983, 1994), Chiappori (1985), Grandmont (1987), Jerison (1999), Quah (2000)

model and preliminaries

setup: exchange economy $\mathcal{E}_N$

N buyers, $n = 1, \ldots, N$, supply $s \in \mathbb{R}_+^L$.
Convex, compact set $X \subset \mathbb{R}_+^L$ of consumption bundles.

Buyer n has quasilinear utility $V_n(x, t) = v_n(x) + t$, where v_n is bounded, monotone, concave and satisfies $v_n(0) = 0$.
Leads to indirect utility $u_n(p)$ and demand $D_n(p) = -\partial u_n$.

Walrasian equilibrium: a price $p \in \mathbb{R}_{++}^L$ and allocation $x \in \mathcal{X}$ such that $\sum x_n = s$ and $x_n \in D_n(p)$.

perturbations and perturbation-proofness

Let $\mathcal{E} = (N, s)$ be an economy with Walrasian equilibrium prices $W(\mathcal{E})$.

Perturbed economy: $\mathcal{E}' = (N, s + \delta s)$

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Perturbed economy: $\mathcal{E}' = (N, s + \delta s)$

A sequence of **nested markets** $\mathcal{E}_t = (N_t, s_t)$ with $N_{t+1} \supseteq N_t$, with $N_t \to \infty$, is $O\big(f(t)\big)$ **−perturbation-proof** if for all sequences of perturbations (δs_t) with $||\delta s_t|| < O(1)$, we have $||p_t - p_t'|| \leq O(f(t))$ for all $p_t \in W(\mathcal{E}_t)$, and $p_t' \in W(\mathcal{E}_t')$.

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Lemmata: $O\big(f(t)\big)$ –perturbation proofness implies:

- Effect of adding or removing agents, or misreport by an agent is $O(f(t))$
- $O(f(t))$-*ex post* incentive compatibility: *benefit* of any unilateral misreport is $O(f(t))$

strong monotonicity

strong monotonicity

$D: K \rightrightarrows \mathbb{R}$ a correspondence on compact, convex $K \subseteq \mathbb{R}_+^L$.

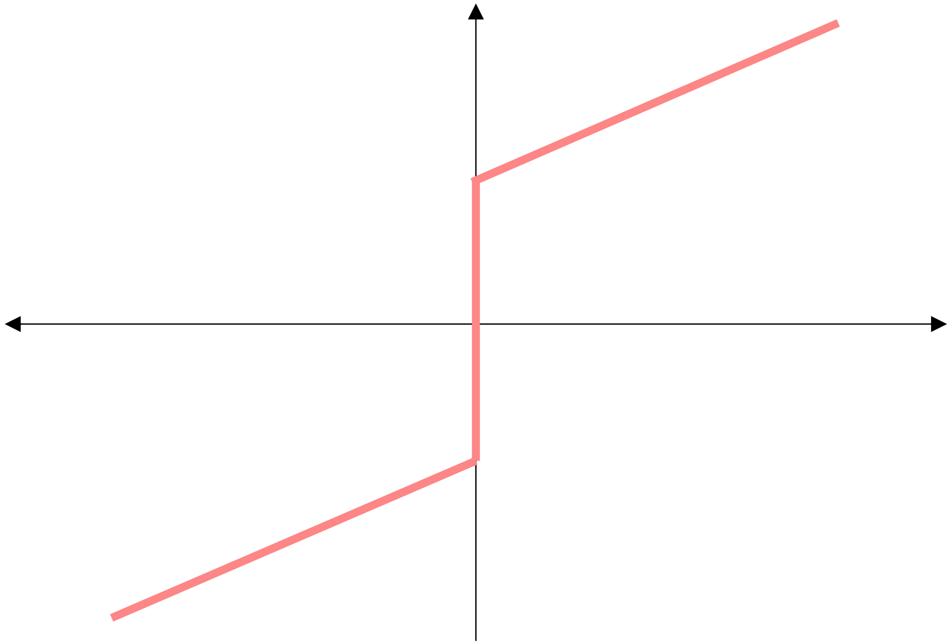
D is **strongly monotone** with constant $m > 0$ if

$$(d - d') \cdot (x - x') \geq m \|x' - x\|^2$$

for all $x, x' \in K$ and $d \in D(x), d' \in D(x')$.

Equivalently:

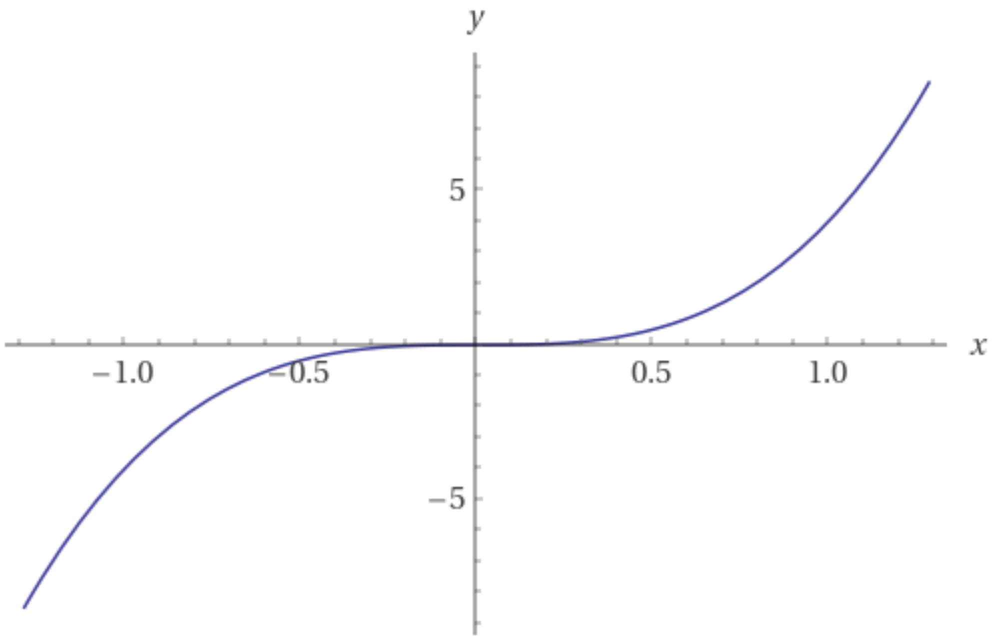
- all directional derivatives of D bounded below by m
- D is a subdifferential of a *strongly convex* function

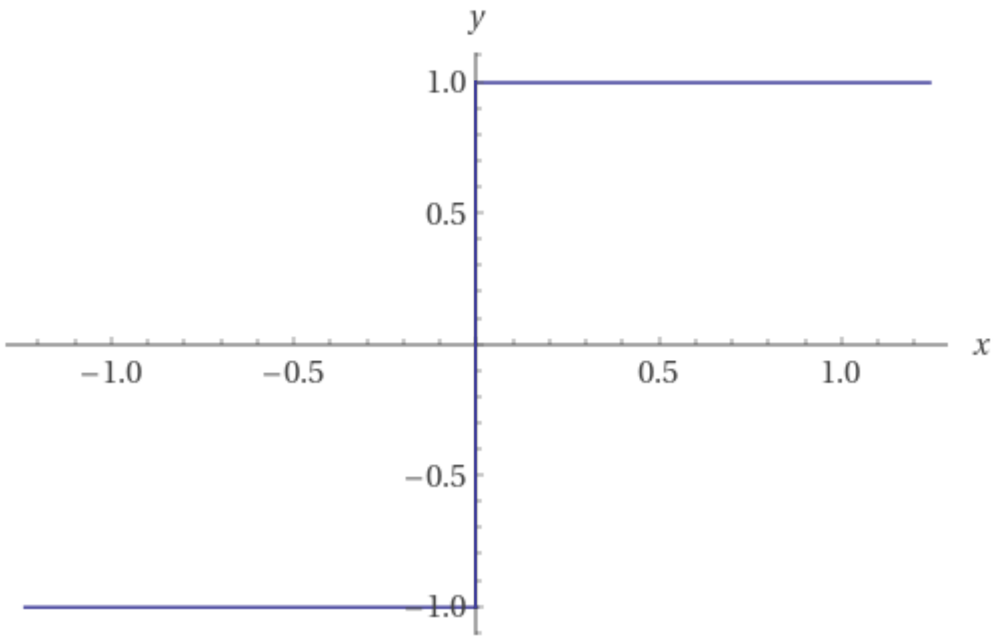


$D = 2x - 0.5 + 1_{x \geq 0}$

☞ *strong convexity*

not strongly monotone





$y = 4x^3$ is strictly but
not strongly monotone
NB: $\{y = 4x^3\} = \mathrm{gr}\big(\partial(x^4)\big)$

$\{y = \mathrm{cl}(\mathrm{sgn}(x))\} = \mathrm{gr}\big(\partial(|x|)\big)$
is not strongly monotone

strong monotonicity of demand

Minor complication: D_n is (naturally) flat around $D_n = 0$.

Thus, we say that n has **strongly monotone demand** if

$$(d - d') \cdot (p' - p) \geq m||p - p'||^2$$

for all prices p, p' and $d \in D_n(p)$, $d' \in D_n(p')$ where the agent is **active**, i.e., where demands are *both* not $\{0\}$.

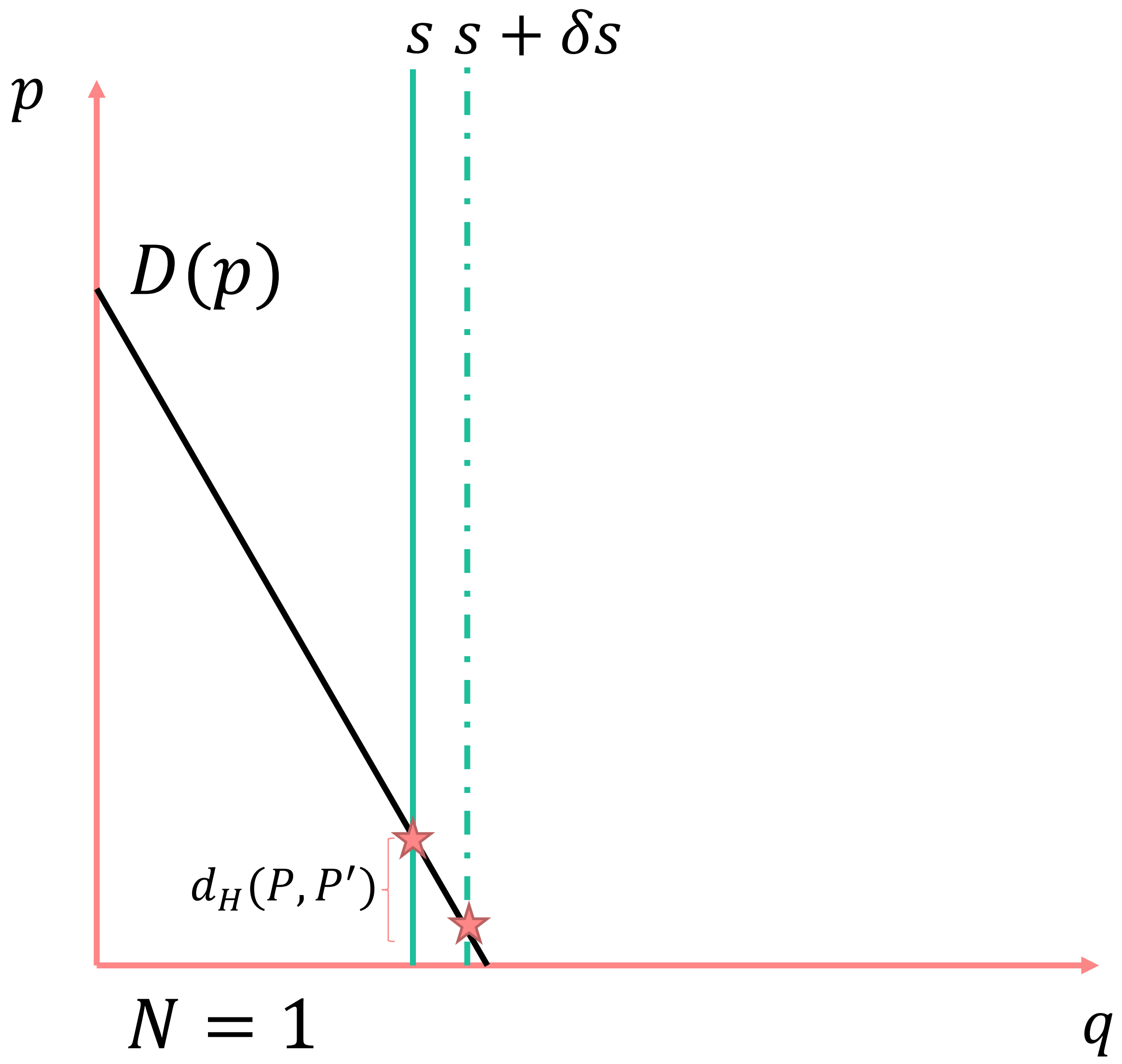
Notice resemblance to the **law of demand** $(d - d') \cdot (p' - p) \geq 0$.

perturbation-proofness in deterministic economies

theorem 1: strongly monotone individual demand

Let $\mathcal{E}_N$ be such that **each** agent $n \in N$ has strongly monotone demand with constant $m > 0$. Then $\mathcal{E}_N$ is $O(1/N^a)$ $-$perturbation-proof, where N^a is the number of active agents at the WE price.

intuitive proof



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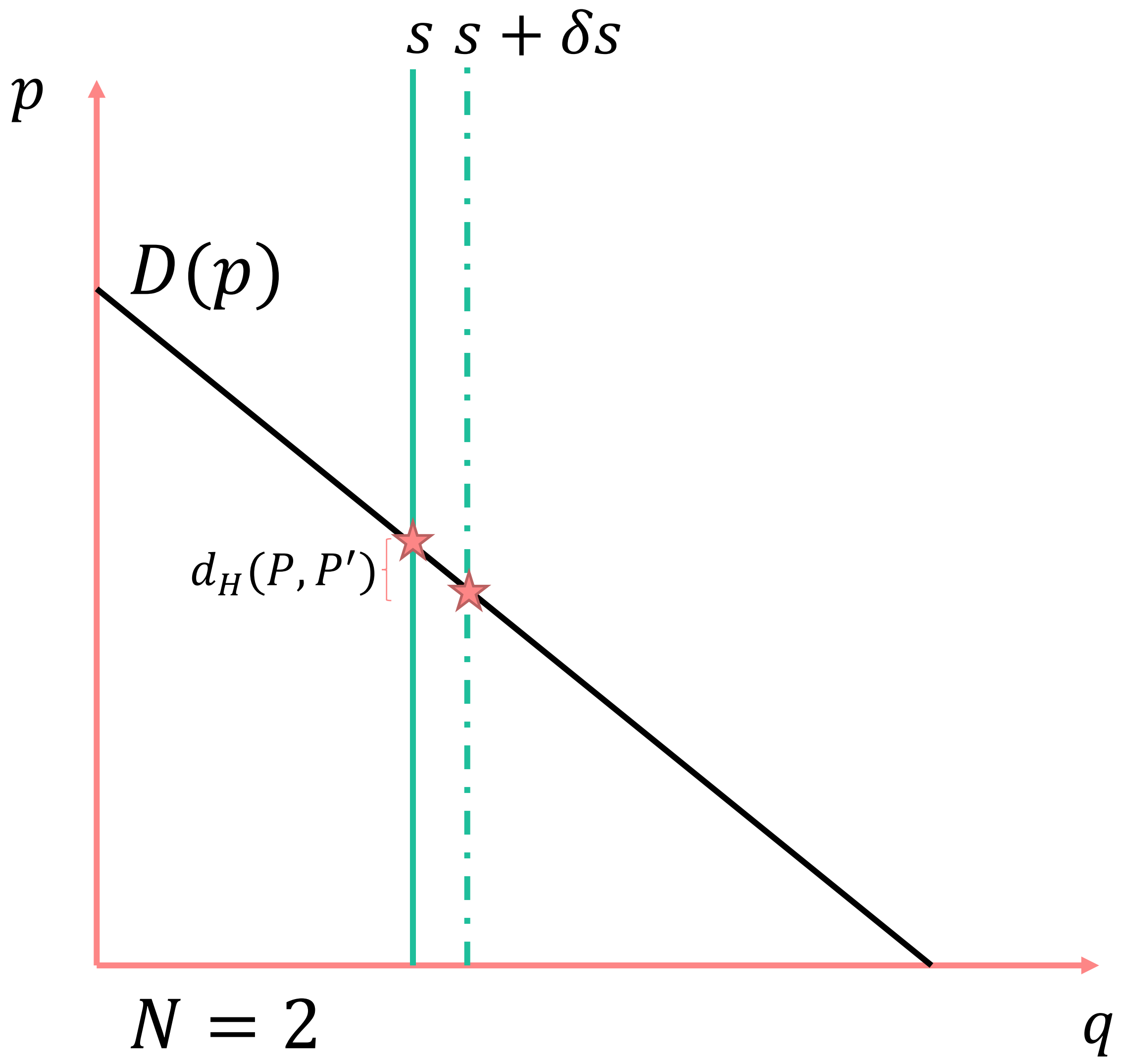

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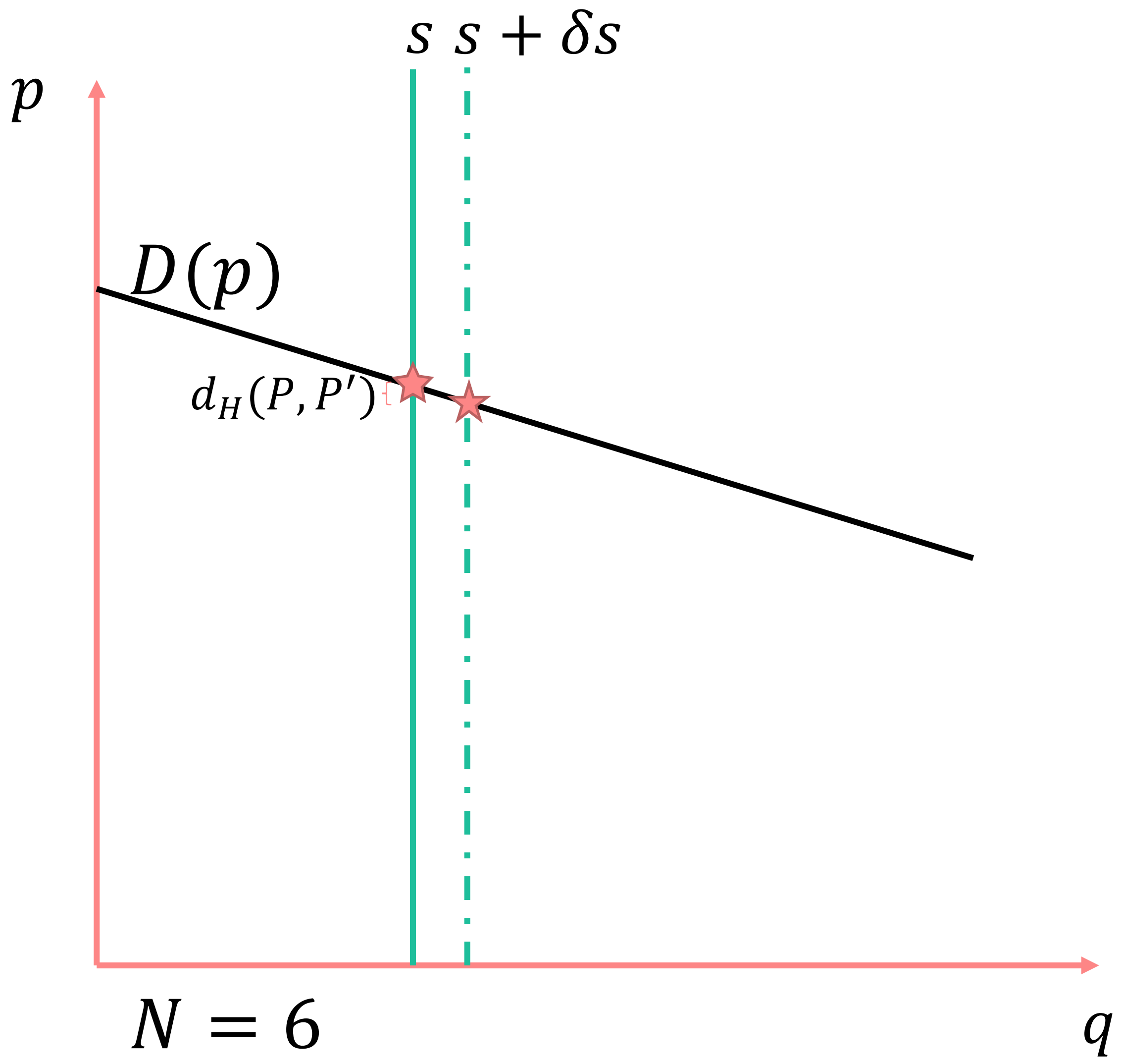

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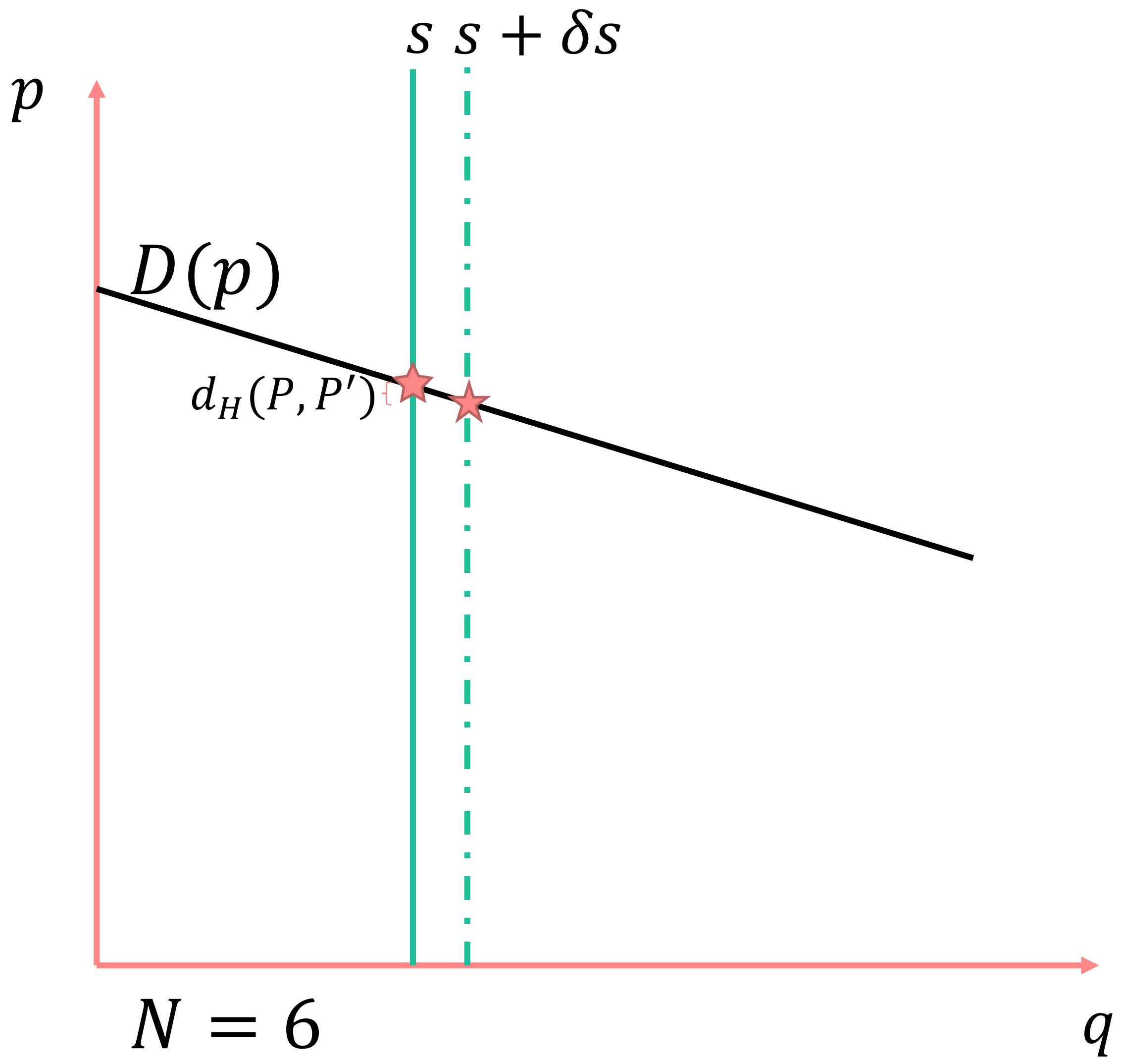

$$\left(d_n(p') - d_n(p)\right) \cdot (p - p') \geq m\|p - p'\|^2$$

$$\sum_{n \in N_t} \left(d_n(p') - d_n(p)\right) \cdot (p - p') \geq m N_t^a \|p - p'\|^2$$

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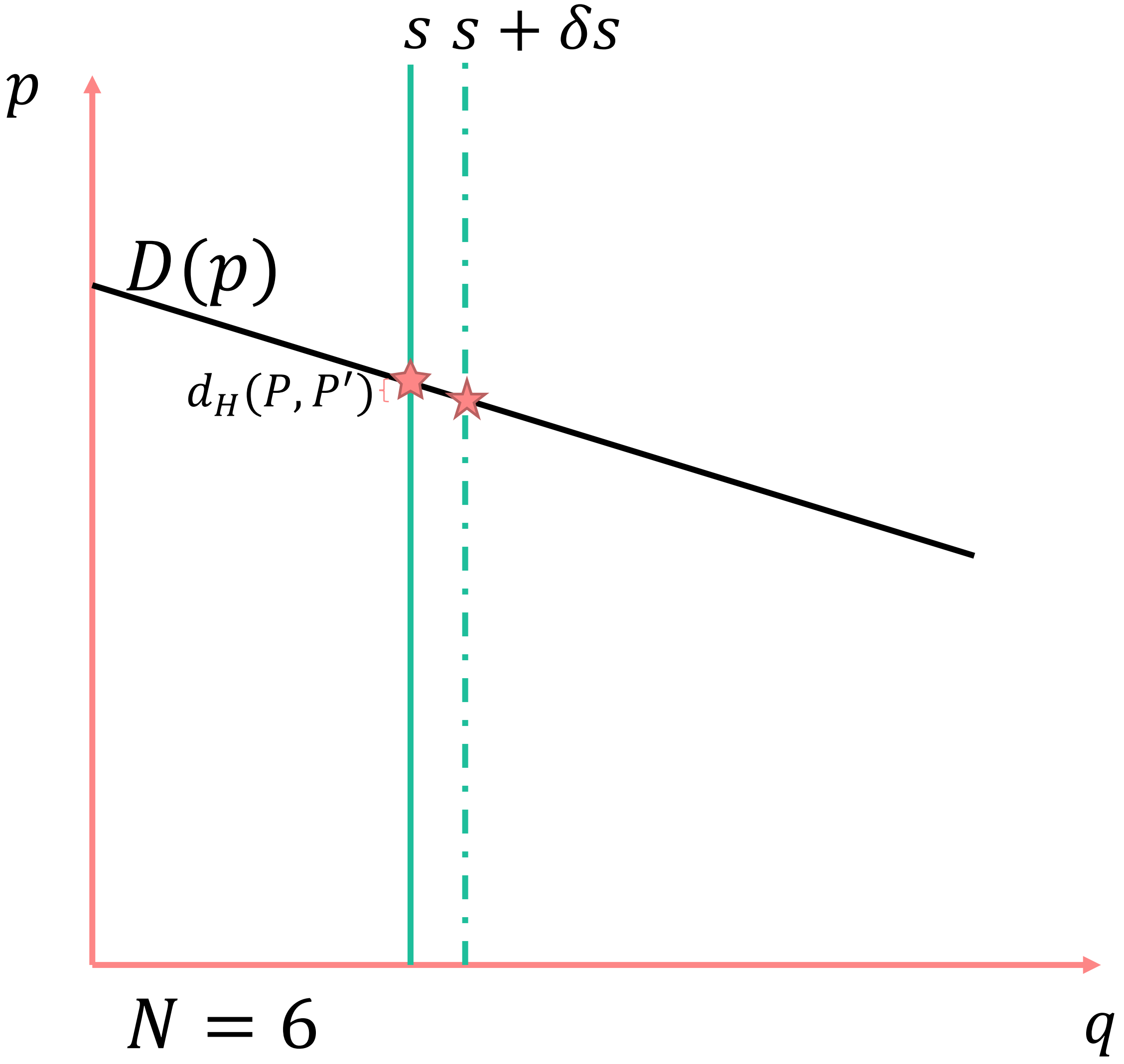

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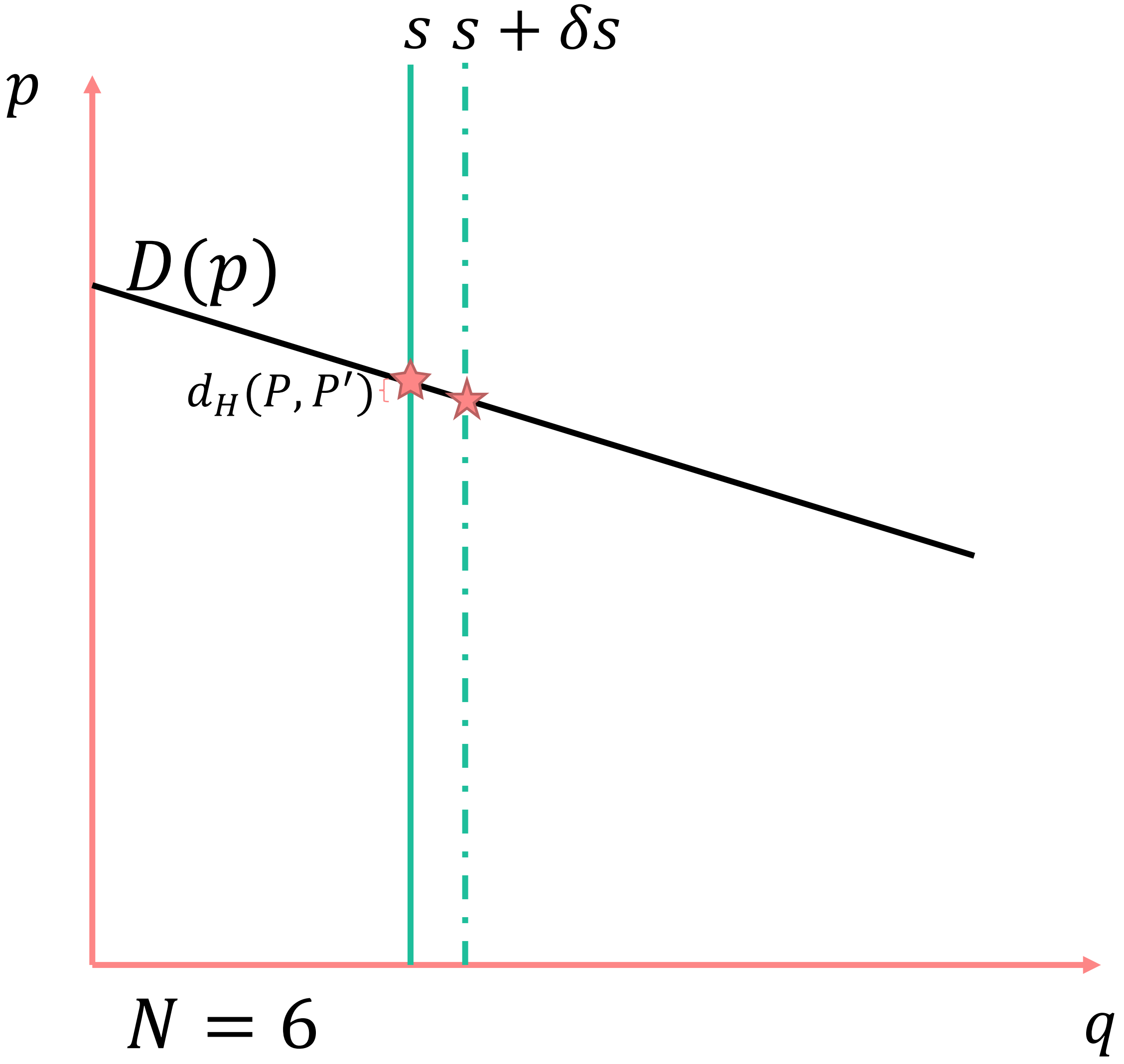

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$$\underbrace{\sum_{n \in N_t} \left(d_n(p') - d_n(p)\right)}_{\|\delta s\|} \cdot (p - p') \geq mN_t^a\|p - p'\|^2$$

$$\|\delta s\| \, \|p - p'\| \geq mN_t^a\|p - p'\|^2$$

$$\|p - p'\| \leq \|\delta s\|/mN_t^a$$

Small changes in supply lead to smaller changes in price at rate $1/N^a$.

theorem 2: replica economies

Let (N, s) be a base economy and consider $\mathcal{E}_k = (kN, ks)$, the k −fold replica economy.

Then $\mathcal{E}_k$ is $O(1/N)$ −perturbation-proof **if and only if** the demand correspondence of the base economy is strongly monotone.

☞ *proof approach*

economies with incomplete information

incomplete information setup

Let $\mathcal{V}$ be an admissible space of bounded, monotone and concave functions on X. Assume $D(p) = 0$ a.s. outside of compact $\mathcal{P} \subseteq \mathbb{R}^L_+$.

Let ν be measure on $\mathcal{V}$ and define the **expected indirect utility**

$$\mathbb{E}_\nu[u(p)] = \int_{\mathcal{V}} u_n(p)\, d\nu(u_n)$$

and the **expected demand** using Aumann's set-valued integral

$$\mathbb{E}_\nu[D(p)] = \int_{\mathcal{V}} \partial u_n(p)\, d\nu(u_n).$$

Rockefellar and Wets (1982) show $\partial \mathbb{E}_\nu[u(p)] = \mathbb{E}_\nu[D(p)]$.

theorem 3

Suppose each agent in $\mathcal{E}$ is drawn from a distribution ν over $\mathcal{V}$ such that $\mathbb{E}_\nu[D(p)]$ is strongly monotone. Then $\mathcal{E}$ is $O_p(N^{-1+\varepsilon})$-perturbation-proof for all $\varepsilon > 0$.

That is, with probability $1 - O(N^{-1+\varepsilon})$ over draws of $\mathcal{E}$, the maximum influence of any perturbation on prices is $O(N^{-1+\varepsilon})$.

Corollary: The Walrasian mechanism applied to $\mathcal{E}$ is $O_p(N^{-1+\varepsilon})$-IC (ex post and interim)

☞ proof

intuition

Consider normalized problems

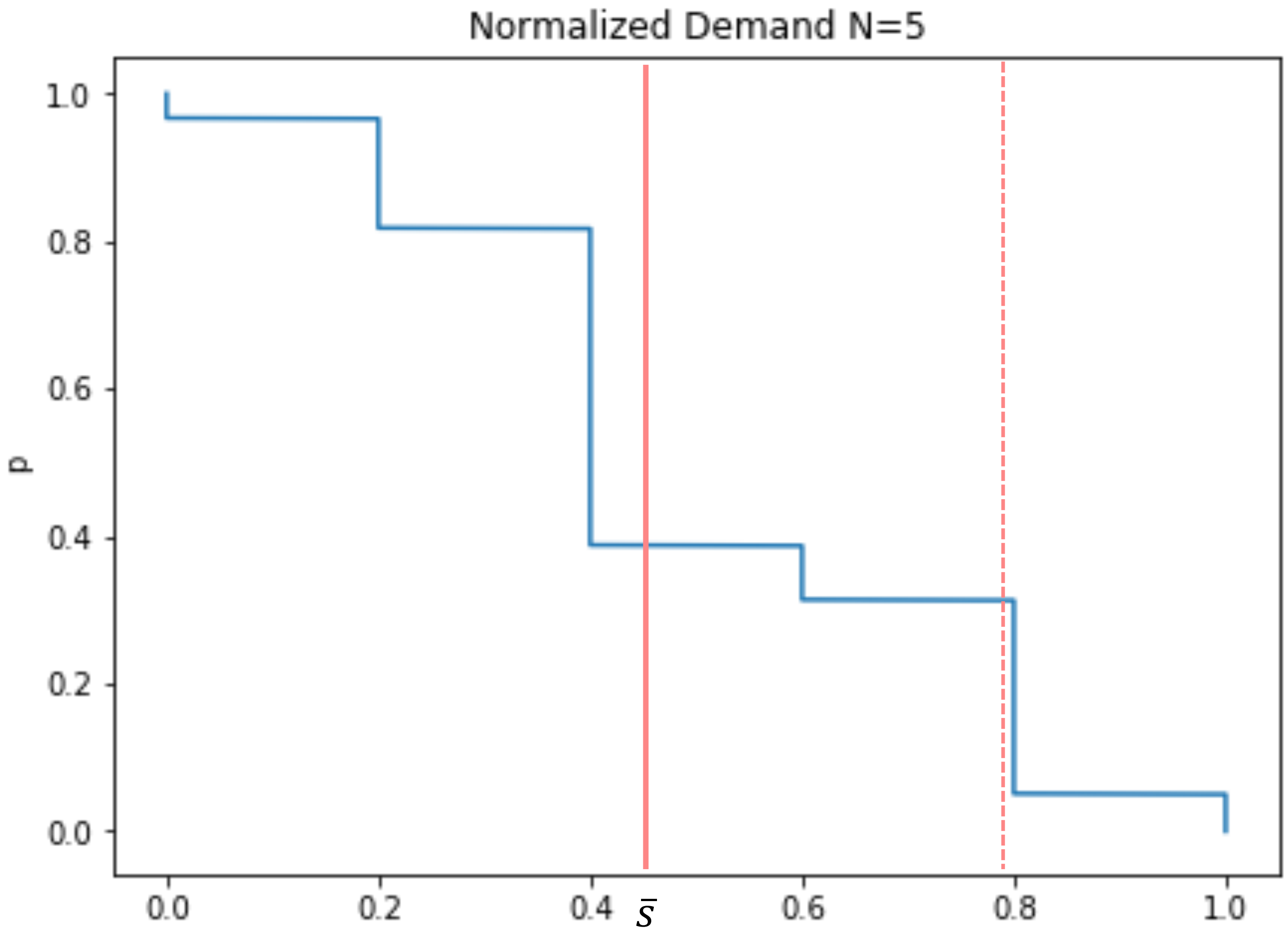
$$\bar{s} \in \frac{1}{N}\sum_{n} D_n(p)$$

and

$$\bar{s} + \frac{\delta s}{N} \in \frac{1}{N}\sum_{n} D_n(p)$$

A law of large numbers implies

$$\frac{1}{N}\sum_{n} D_n(p) \to \mathbb{E}_{\nu}[D(p)]$$



intuition

Consider normalized problems

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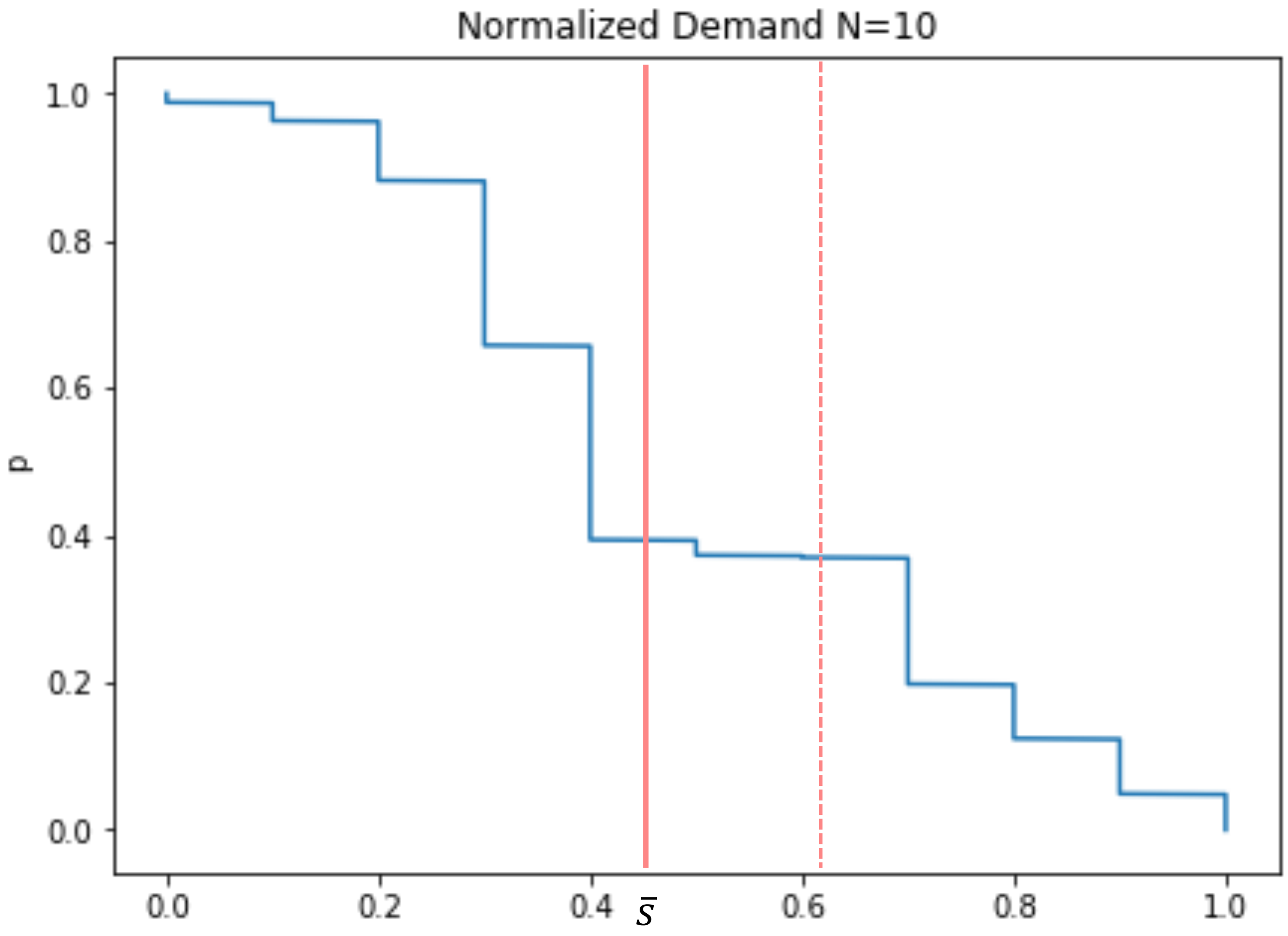

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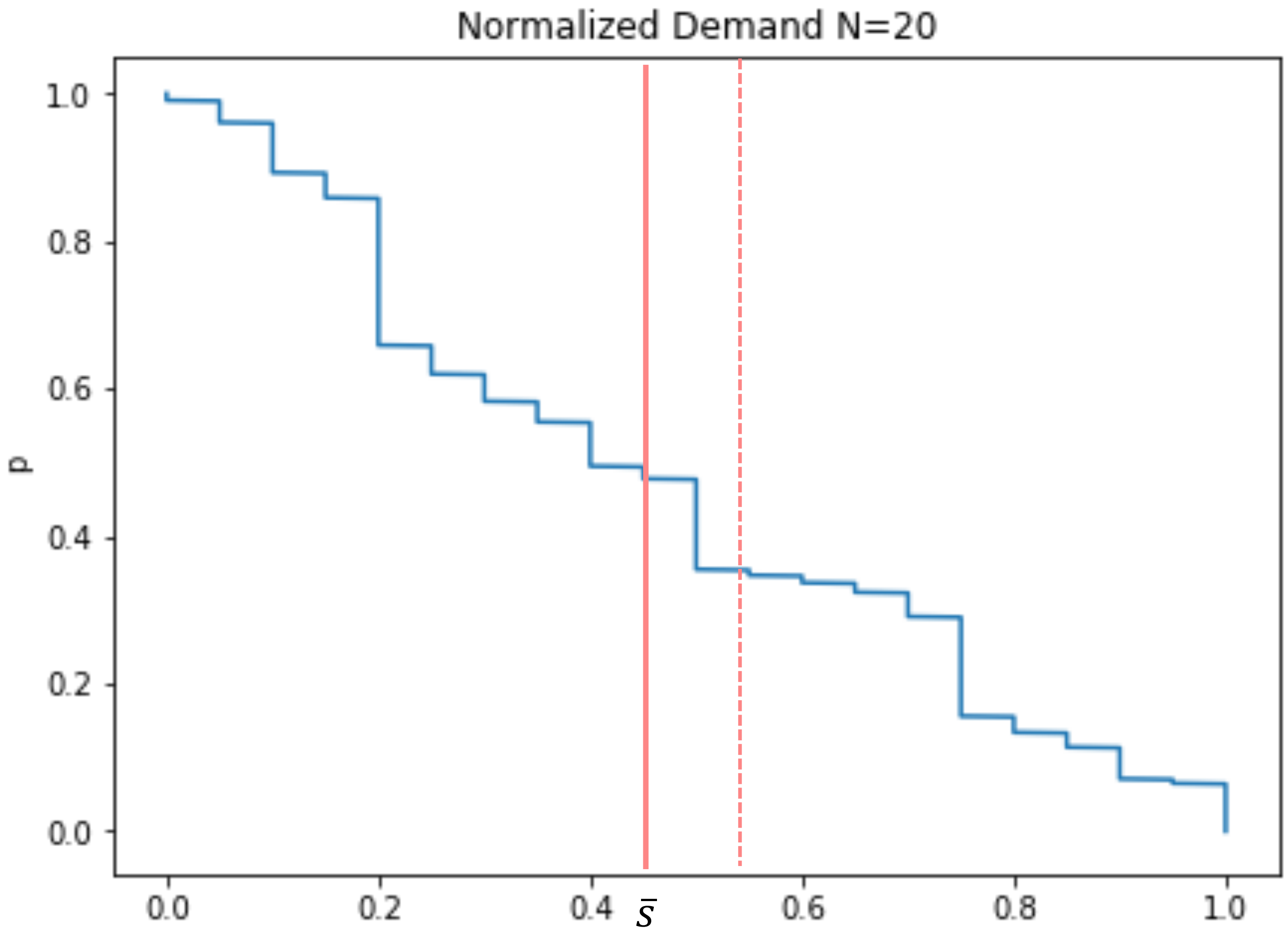

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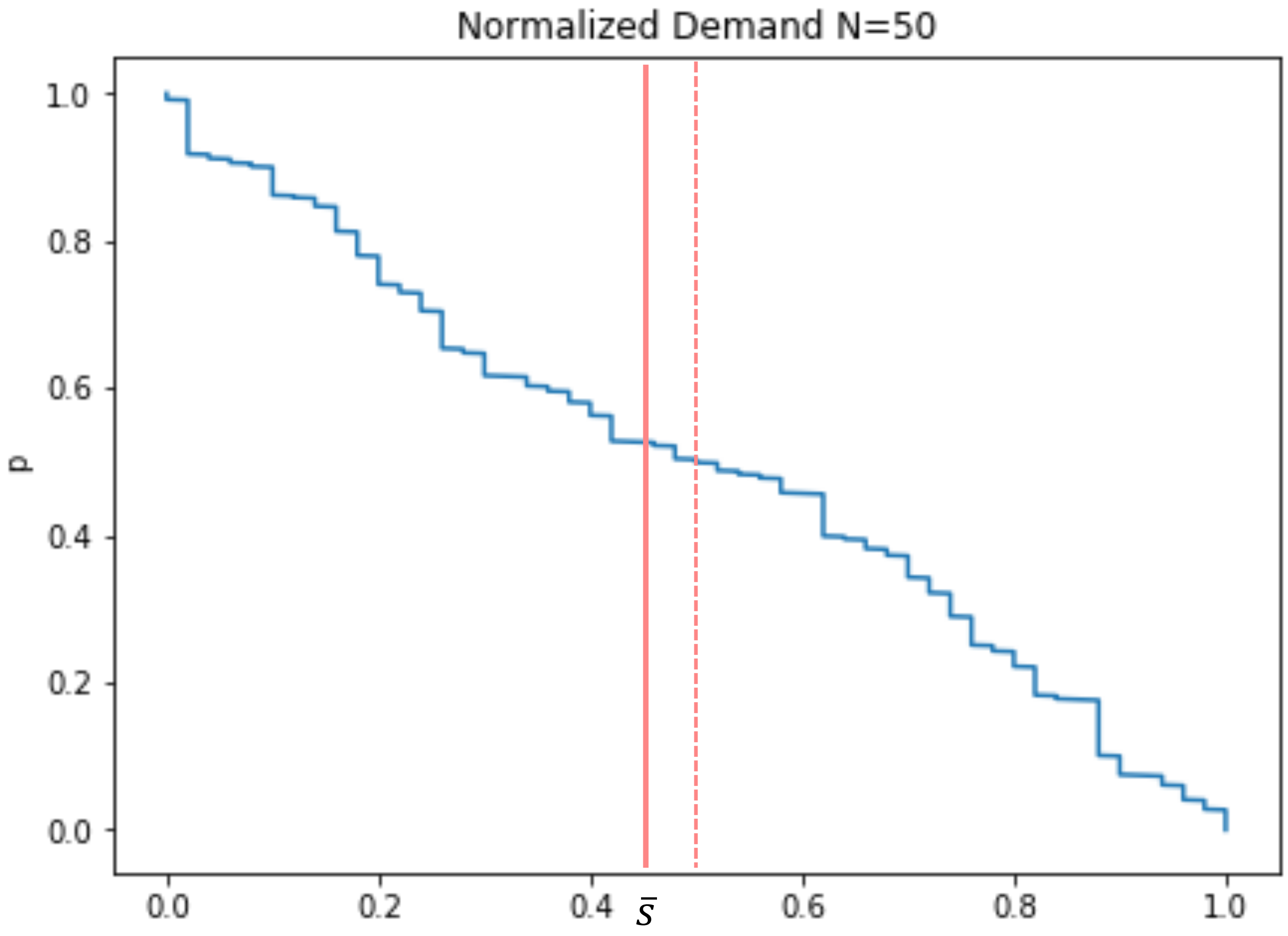

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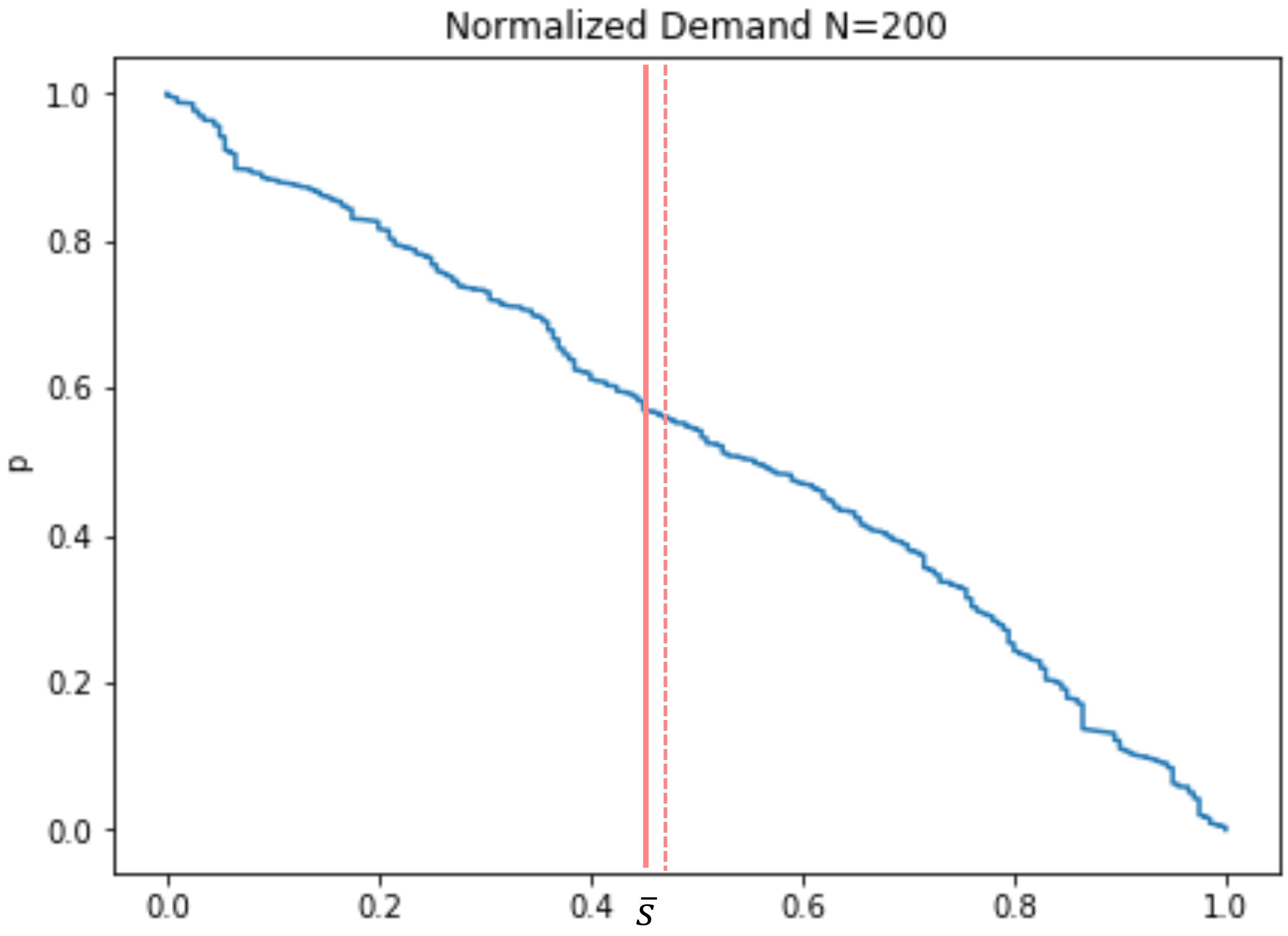

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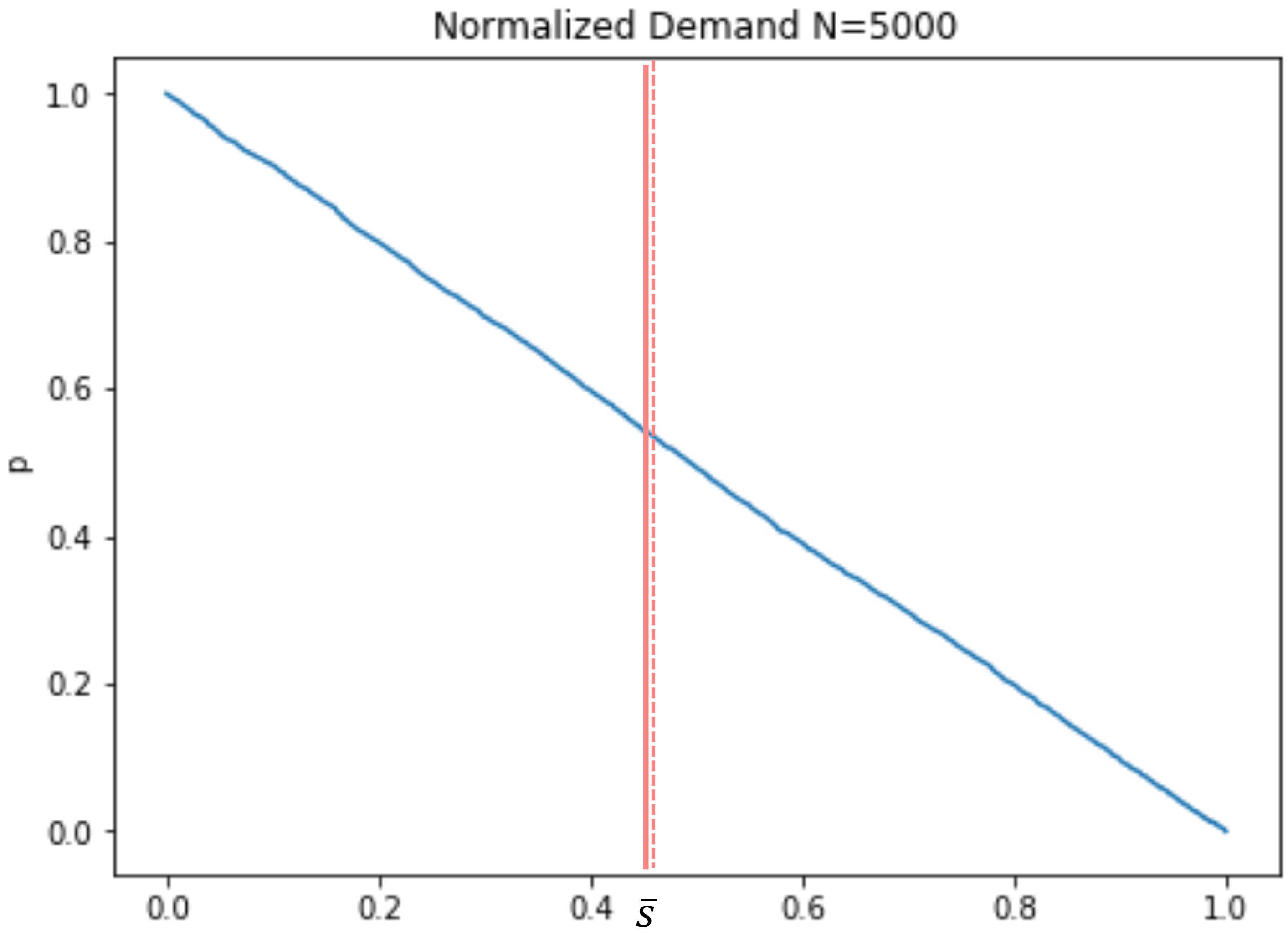

why the simple approach does not work

We are interested in properties of *inverse* demand $\mathbb{E}_\nu[\|P(N\bar{s}) - P(N\bar{s} + \delta s)\|]$.

While subdifferentiation (demand) is preserved under expectation,

$$\mathbb{E}_\nu[D(p)] = \partial \mathbb{E}_\nu[u(p)]$$

inverse subdifferentiation is not,

$$\mathbb{E}_\nu[P(d)] \neq \partial^{-1} \mathbb{E}_\nu[u](d)$$

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Example:
Consider the sale of k identical goods to N buyers with values $v \sim \mathrm{Unif}[0,1]$.
Normalized demand is in expectation $1 - p$, **independent** of N.
Inverse demand depends on N: expected WE prices are

$$\left[\mathbb{E}v^{(N-k;N)}, \mathbb{E}v^{(N-k+1;N)}\right]$$

economies with indivisibilities

Consider an exchange economy with indivisible goods $X \subseteq \mathbb{Z}_+^L$.

Proposition: a simple test for $O(1/N)$ –IC for Walrasian mechanism

The expected demand correspondence is strongly monotone in expectation if and only if there exists $\alpha > 0$ such that for all $p, p' \in \mathcal{P}$,

$$\Pr_{n \sim \nu}[D_n(p) \neq D_n(p')] \geq \min\{\alpha||p - p'||,1\}.$$

example

Suppose $X = \{0,1\}^2$.

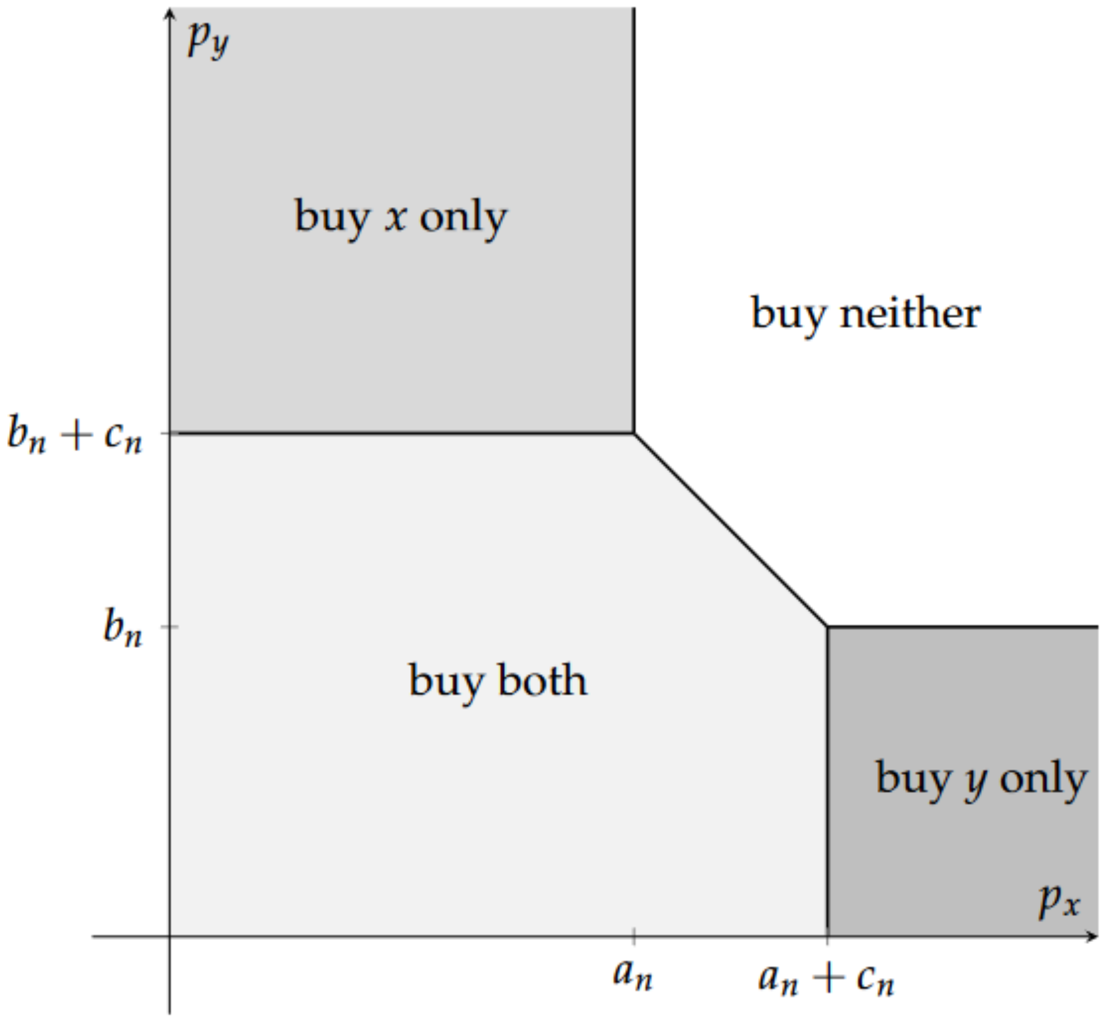
Buyers have *additive preferences* with a *complementarity* term:

$v_n(x, y) = a_n x + b_n y + c_n xy$ with

$a_n, b_n \sim \text{Unif}[0,1]$,

$c_n \sim \text{Unif}[0, \min\{1 - a_n, 1 - b_n\}]$.

Easy to show previous condition satisfied: implies that Walrasian mechanism in this setting would be $O(1/N)$-IC.



conclusion

- **Strong monotonicity** implies **perturbation-proofness**: small perturbations in an economy leads to **small changes in the set of prices**: rate approx. O(1/N).
- This implies approximate incentive compatibility of Walrasian mechanism.
- Strong monotonicity is **required** for this rate in replica economies.
- In economies with **indivisibilities**, there is a simple test for strong monotonicity of expected demand.

appendix

equivalent characterizations: strong convexity

$f: K \to \mathbb{R}$ a proper, convex function on a compact, convex set K, with $D = \partial f$.

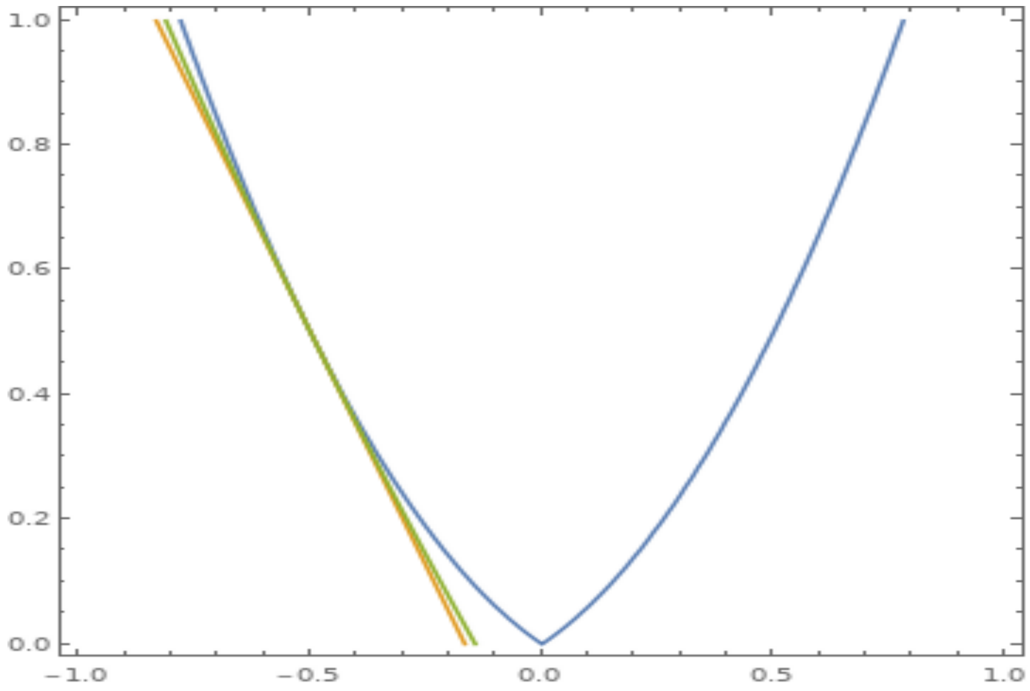
f is **strongly convex** with constant $m > 0$ if

$$f(y) \geq f(x) + s \cdot (y - x) + \frac{m}{2} \|y - x\|^2$$

for all $x, y \in K$ and $s \in \partial f(x)$.

Equivalently, $g(x) = f(x) - m||x||^2$ is convex, or if $f \in C^2$, each second derivative of f is bounded below by m.

f strongly convex **iff** ∂f is strongly monotone



$y = x^2 + 0.5|x|$ is strongly convex on $[-1,1]$ with modulus 2

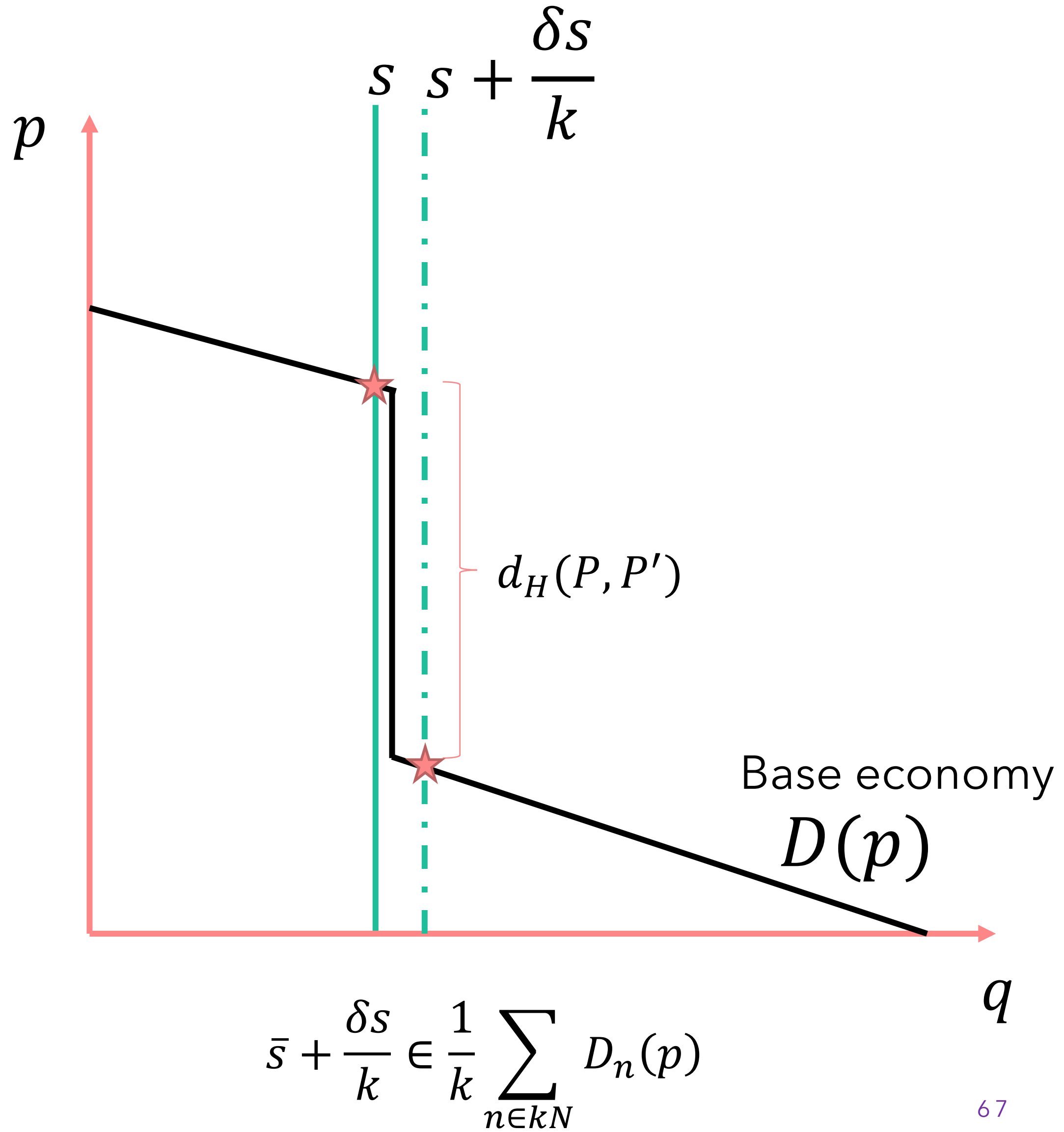
replica economies - proof sketch (1/3)

Sufficiency: analogous to Theorem 1

Necessity: We show failure of strong monotonicity implies there exist perturbations leading to $\omega(1/N)$ price changes.

For $m_t \downarrow 0$, let $p_t, p_t{}'$ be given with $(d_t - d_t') \cdot (p_t' - p_t) < m_t \left|\left|p_t - p_t'\right|\right|^2$. WLOG $p_t \to p$, $p_t' \to p_t'$.

Case 1: $p \neq p'$. Then $d_t \to d$ and $d_t' \to d'$ with $(d - d') \cdot (p' - p) = 0$.



replica economies - proof sketch (2/3)

Case 2: $p = p'$. Assume $\sum u_n(p)$ is C^2 in a neighborhood of p.

We have $\lim_{t\to\infty} \frac{(d_t - d_t')\cdot(p_t' - p_t)}{\|p_t - p_t'\|^2} = 0.$ (*)

Now show that the angle between $d_t - d_t{}'$ and $p_t - p_t{}'$ cannot approach orthogonality.

Intuition: suppose otherwise, i.e. that $p_t - p_t{}'$ approaches x unit vector while $d_t - d_t{}'$ approaches y unit vector.

Then $\frac{dd_x}{dp_x}(p) = 0$, $\frac{dd_y}{dp_x}(p) \neq 0$. By symmetry of the Slutsky matrix, $\frac{dd_x}{dp_y}(p) \neq 0$.
But then the Slutsky matrix is not negative semidefinite!

replica economies - proof sketch (3/3)

Thus, we have $(d_t - d_t') \cdot (p_t' - p_t) \geq c\|d_t - d_t'\|\|p_t' - p_t\|$ for some $c > 0$.

Then for $\lim_{t\to\infty} \frac{(d_t - d_t')\cdot(p_t' - p_t)}{\|p_t - p_t'\|^2} = 0$ on subsequence of *normalized* perturbations, that is, $d_t - d_t' = O(1/N)$, we must have that $||p_t - p_t'|| \geq \omega\,(1/N)$.

For non-C^2 objectives, we need to be more careful:

- first consider a regularized economy with $C^{1,1}$ objective (Moreau-Yosida)
- show that regularized economy perturbation-proof iff original economy
- an analogous result to Slutsky symmetry + negative semidefiniteness holds for $C^{1,1}$ objectives.

☞ return

main theorem - proof approach (1/6)

Consider $\mathcal{E} = \langle N, s \rangle$ with N agents drawn iid from ν with strongly monotone $\mathbb{E}D$. Let P be the set of prices in WE, and δs any perturbation.

Goal: show with high probability over draws of $\mathcal{E}$, that for *all* p, p' with

$$||p - p'|| > c/N^{1-\varepsilon}$$

that $||d - d'|| > ||\delta s||$ for $d \in D(p), d' \in D(p')$. This will imply $d_H(P, P') \leq c/N^{1-\varepsilon}$.

Five steps to the proof (not in detail):

1. **Concentration:** for any *fixed* p, p' at distance $c/N^{1-\varepsilon}$, use Bernstein Inequality to show that with subexponential probability, i.e. $\sim 1 - \exp(-kN^{\varepsilon})$
$$\min_{\substack{d \in D(p),\\ d' \in D(p')}} (d - d') \cdot (p' - p) \geq mN||p - p'||^2/2$$

main theorem - proof approach (2/6)

Bernstein Inequality: for independent X_i with $|X_i| \leq B$,

$$\Pr\left[\left|\sum_i X_i - \sum_i \mathbb{E}[X_i]\right| \geq t\right] \leq 2\exp\left(\frac{-\frac{1}{2}t^2}{\sum_i \mathbb{E}\left[X_i^2\right] + \frac{1}{3}Bt}\right).$$

Applying this to $M_n(p, p') = \min_{d \in D(p), d' \in D(p')} (d - d') \cdot (p' - p)$

Let its mean be $\mu_{p,p'}$.

Use Bhatia-Davis Inequality for $m \leq X \leq M$ a.s.: $\mathrm{Var}[X] \leq (M - \mu)(\mu - m)$.

$$\mathbb{E}[M_n(p, p')^2] \leq 2X_{max}\|p - p'\|\mu_{p,p'}$$

$$\Pr\left[M(p,p') \geq \frac{1}{2}N\mu_{p,p'}\right] \geq 1 - 2\exp\left(\frac{-\frac{1}{8}N^2\mu_{p,p'}^2}{2NX_{max}\|p-p'\|\mu_{p,p'} + \frac{1}{3}NX_{max}\|p-p'\|\mu_{p,p'}}\right)$$

$$= 1 - 2\exp\left(\frac{-3N\mu_{p,p'}}{56X_{max}\|p-p'\|}\right).$$

Since $\mu_{p,p'} \geq m\|p-p'\|^2$ and $\|p-p'\| \geq c/N^{1-\varepsilon}$, we have

$$\Pr\left[M(p,p') \geq \frac{1}{2}mN\|p-p'\|^2\right] \geq 1 - 2\exp\left(\frac{-3Nm\|p-p'\|^2}{56X_{max}\|p-p'\|}\right)$$

$$\geq 1 - 2\exp\left(\frac{-3cN^{\varepsilon}m}{56X_{max}}\right)$$

Note that the event $M(p,p') \geq \frac{mN}{2}\|p-p'\|^2$ for $\|p-p'\| = \frac{c}{N^{1-\varepsilon}}$ is equivalent to the event that $\big(D(p) - D(p')\big) \cdot (p-p') \geq k \parallel \delta s \parallel N^{\varepsilon}\|p-p'\|$.
By the Cauchy-Schwarz Inequality, this implies $\|d-d'\| \geq k \parallel \delta s \parallel N^{\varepsilon}$.

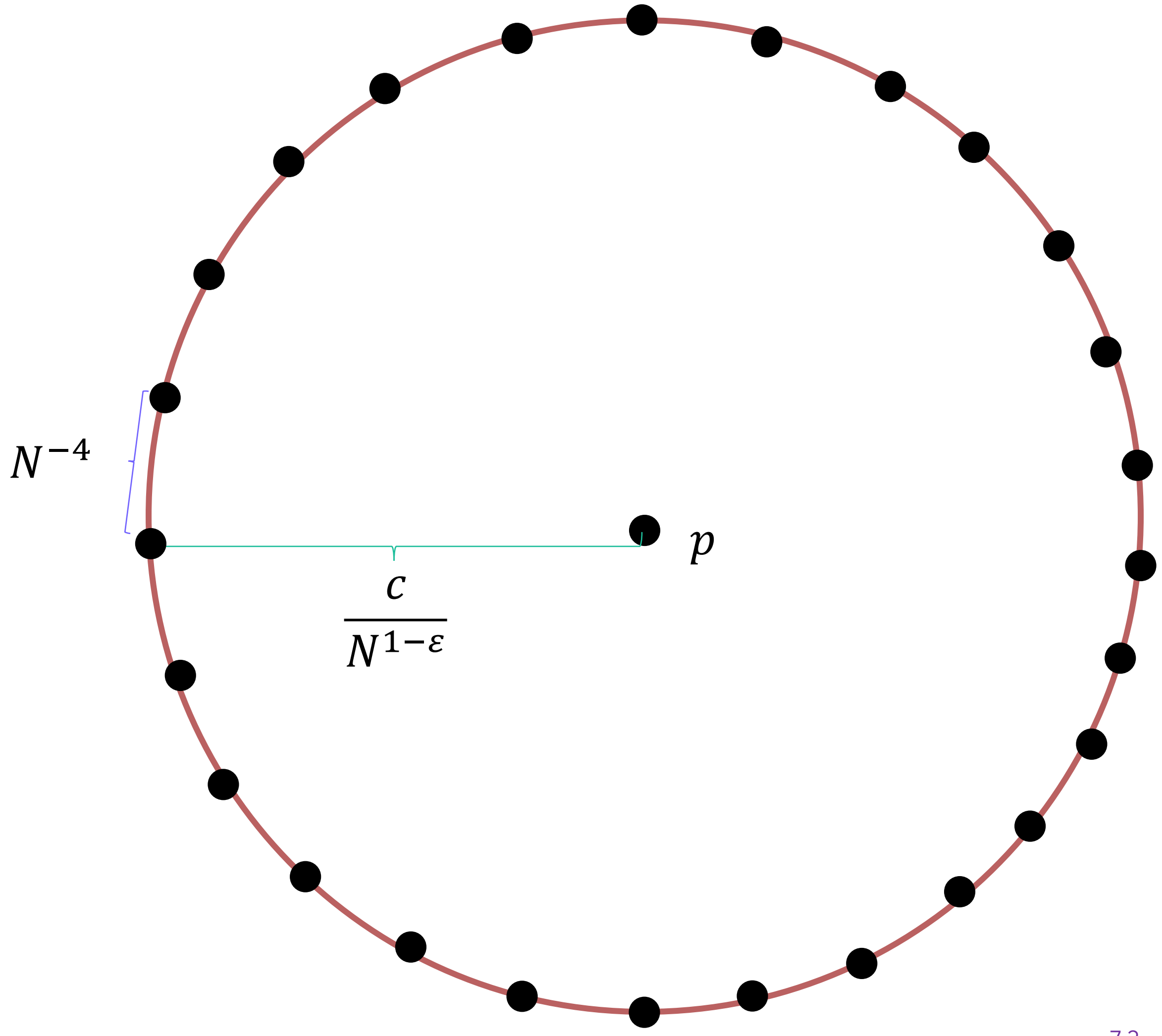
main theorem - proof approach (4/6)

2. **Extend to discretized sphere:** fix p.

   Consider a discretized sphere of radius $c/N^{1-\varepsilon}$ with grid at distance $1/N^4$.

   $O\left(N^{(3+\epsilon)L}\right)$ points in discretization.

   Can use a union bound on the <u>subexponential</u> probability obtained in step 1.



main theorem - proof approach (5/6)

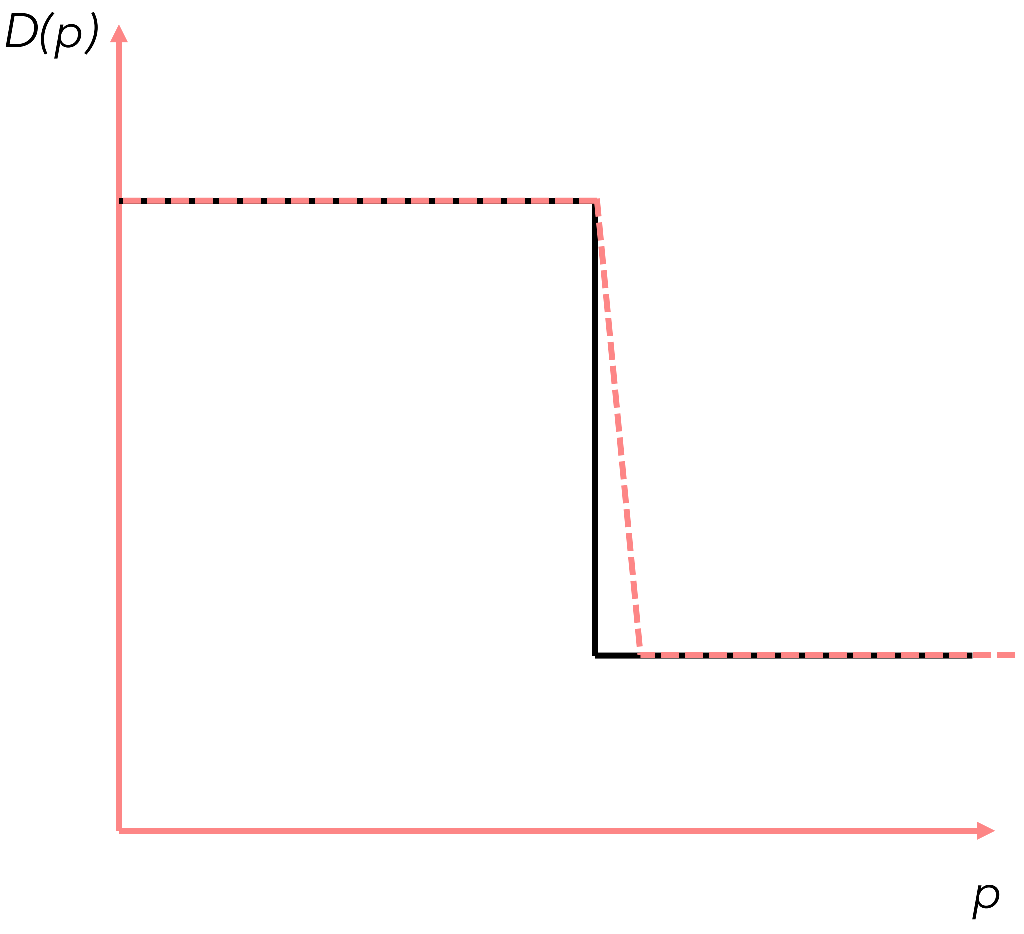
3. Extend to sphere via regularization:

Take the γ –**Yosida regularization** of V to obtain a $1/\gamma$ *Lipschitz* demand function arbitrarily close to D which satisfies

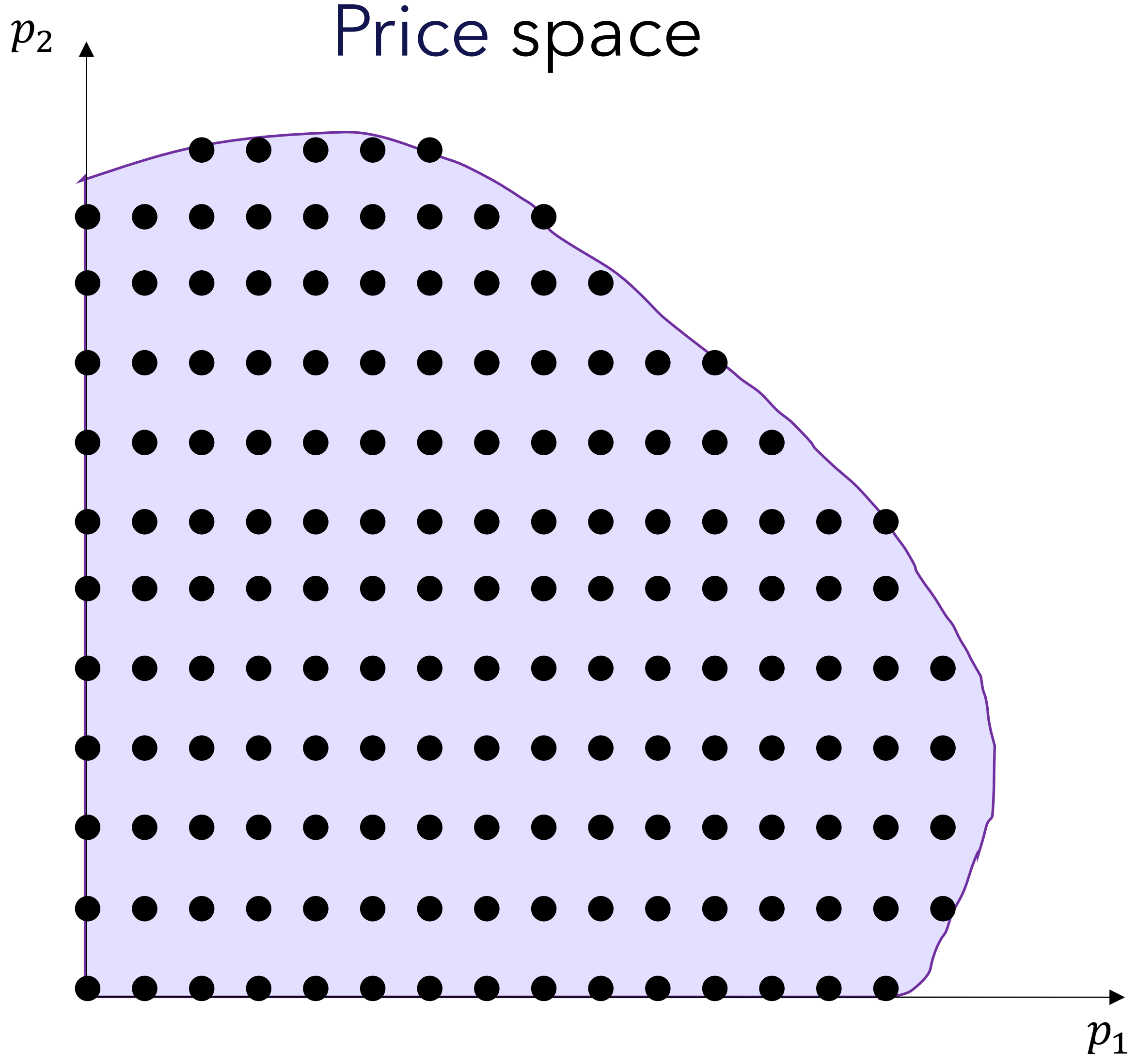
$$(\nabla V)^{-1}(d) = \gamma d + (\partial V)^{-1}(d)$$

Suffices to take $\gamma = \frac{1}{N^2}$ and $d_H(P, P')$ changes by $O\left(\frac{1}{N^2}\right)$.

Extend result using Lipschitz property to *all* p on the sphere



main theorem - proof approach (6/6)



4. **Extend to exterior of sphere:** By monotonicity, $(d - d') \cdot (p' - p)$ only increases radially.

5. **Uniformization over p:** Use another union bound over a grid of p and the Lipschitz property to fill in gaps (as above).

☞ return

tâtonnement

Recall the **tâtonnement process**

$$\frac{dp}{dt} = \alpha(D(p(t)) - s)$$

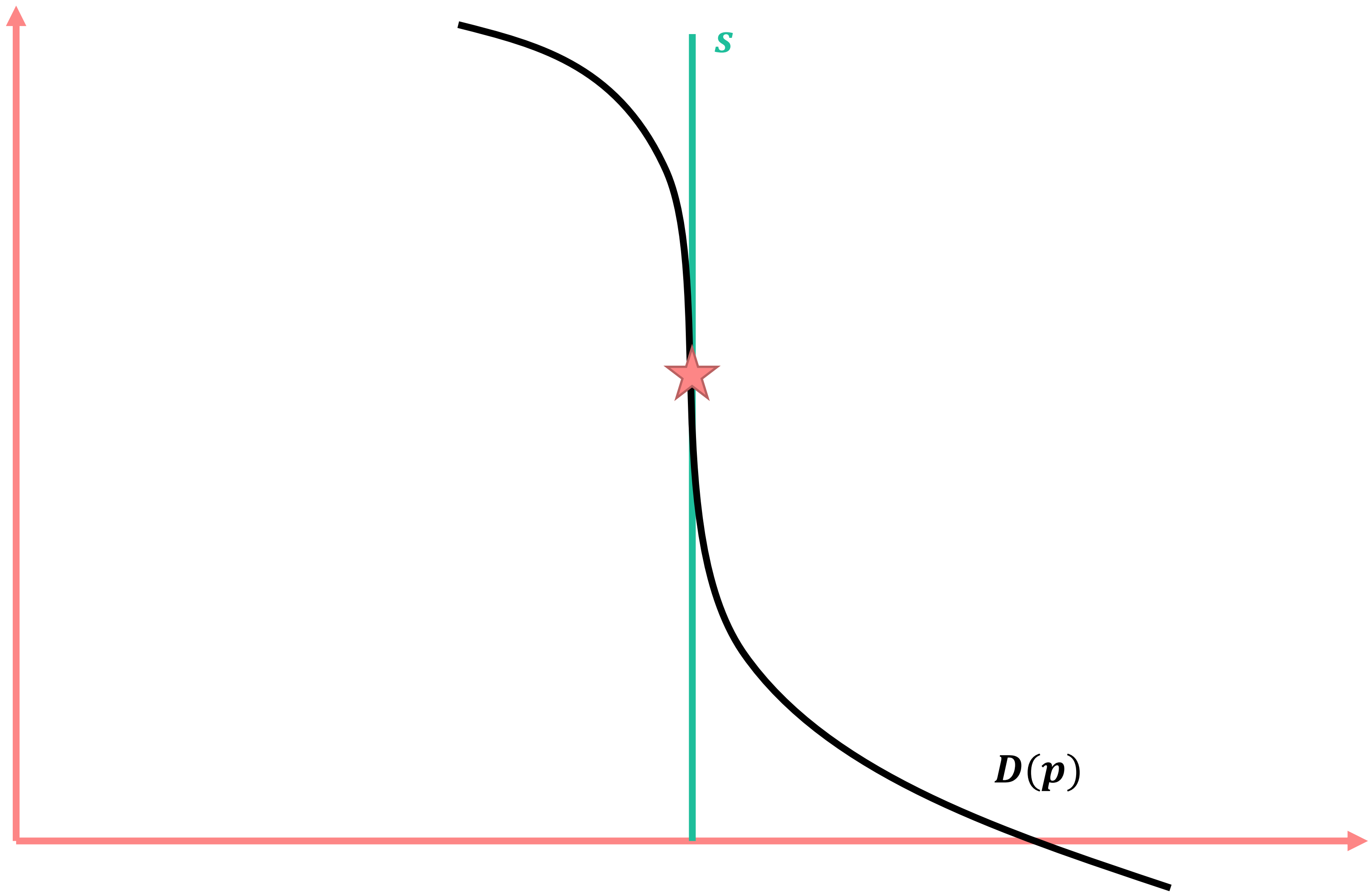
for $\alpha > 0$. (Here assume $D(p(t))$ is single-valued or take any selection).

$\lim_{t \to \infty} p(t)$ is a Walrasian equilibrium price in quasilinear economies.

However, in general, the **rate of convergence** of prices to p^* may be slow.

CS technicality: p^* may be irrational, so computational properties always of ε −approximation of p^* (identification of p such that $p \in B_\varepsilon(p^*)$ for given ε).

tâtonnement – what can go wrong



tâtonnement theorem (continuous time)

Strong convexity implies tâtonnement converges quickly to p^*

Lyapunov function $L(t) = \|p(t) - p^*\|^2$

$$\frac{dL}{dt} = 2(p(t) - p^*) \cdot \frac{dp}{dt}$$

$$= 2(p - p^*) \cdot \alpha\big(d(p) - d(p^*)\big)$$

$$\geq -2m\ \|p - p^*\|^2 \text{ by strong convexity}$$

$$\|p - p^*\| \leq e^{-mt} \longrightarrow t = -\frac{1}{m}\log(\epsilon)\text{: subpolynomial in } \varepsilon$$

Without strong convexity, convergence to ϵ −ball around p^* can be **arbitrarily slow**.