Linear Pricing Mechanisms without Convexity

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Abstract

We introduce two new linear pricing mechanisms that lead to feasible, budget-balanced and approximately efficient outcomes even when preferences or production sets are not convex and Walrasian equilibrium does not exist. One mechanism permits different prices for buyers and sellers; the other uses a single price vector but permits some agents to be rationed. In these mechanisms, both the inefficiency-to-value ratio and the maximum benefit any single agent can gain from false reporting tend quickly to zero as the numbers of producers and consumers increase.

Keywords: Approximate efficiency, Approximate incentive-compatibility, Market design, Non-convexity, Prices, Rationing

JEL Codes: C620, D400, D440, D450, D470, D500, D510, D610.

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1 Introduction

Auction markets are often designed to promote approximate competitive equilibrium outcomes because exact competitive outcomes have several desirable properties. First, a competitive equilibrium allocation is always feasible, no unwanted goods are produced, and no demand is left unfulfilled. Second, in the "large market" limit, no agent has much influence over the prices, which leads to the further conclusion that the mechanism is approximately incentive-compatible.[1] Third, suppliers' payments are equal to what consumers pay; this budget-balance is a common requirement for mechanism design because it means that no subsidy is required from a market operator or government. Fourth, in important special cases, the prices can be computed efficiently as the solution of a convex program or approximated as the rest point of a natural price adjustment process. Fifth, linear and anonymous pricing rules are familiar, simple and often seen as fairer than personalized payment rules.[2]

All of these properties, however, depend on the existence of a competitive equilibrium. With a traditional price-based mechanism, when a competitive equilibrium does not exist, the mechanism can either ration some agent or if supply exceeds demand, it can fill all demand. If all prices are positive, then the second option necessarily leads to paying more to suppliers than is received from buyers, so there is a budget deficit. If the mechanism instead rations some agent to avoid excess production, that may create an incentive for the rationed agent—even one whose report can have no effect on prices—to misreport its preferences to escape rationing.

To ensure the existence of equilibrium, traditional general equilibrium analysis assumes that preferences and production sets are convex, but non-convex production is a common feature in some auction markets. For example, electricity markets often use auction-like mechanisms, but generating plants typically incur fixed costs to operate or adjust production, and there are also fixed costs of maintaining unused capacity. Similarly, in fisheries, there are fixed costs of operating a boat. There are many more such examples.

Our goal in this paper is to introduce feasible mechanisms that achieve or approximate the desirable properties of price-guided allocations despite failures of convexity. In our featured new mechanism, the key innovation is to permit prices to differ between buyers and sellers, with neither side rationed at its own prices. We also study a linear-pricing mechanism in which all agents

[1]Certain assumptions are needed to reach conclusions about finite but large markets, which we discuss further in Section 4.3.

[2]With fixed linear prices, the allocation is unaffected if firms merge or break up, which is often seen both as fair and as providing a further good incentive property.

face the same prices but some may subjected to rationing. We show that, in large markets, both mechanisms approximate the desirable properties of the competitive pricing mechanism with convexity. We also explore how any approximation losses and gains to misreporting preferences depend on scale of the market and the sizes of any nonconvexities.

There are three kinds of challenges to designing mechanisms for markets without convexity, which we illustrate by example in Section 1.1. One is that in some efficient allocations, production strictly exceeds consumption, which makes *budget balance* with positive prices impossible.[3] Second, even when efficient allocations equalize production and consumption, there may be no prices such that all agents prefer the recommended allocation to the zero trade: *individual rationality* is violated. Third, even if prices exist that make participation individually rational, some agents may need to be *rationed* at those prices, that is, some agents may strictly prefer a different bundle over the prescribed one. This may create an incentive for even a price-taking agent to report falsely in order to affect its allocation. These complications can be avoided when preferences and production sets are convex, but they provide challenges that need to be overcome when designing pricing mechanisms for markets in which convexity is not guaranteed.

We address these challenges in two ways. Firstly, to achieve budget balance, we study *double-price* mechanisms in which buyers may be charged slightly higher prices than are paid to producers. The price premium can be used to cover the cost of any units which must be discarded by the market operator. Secondly, we also study mechanisms in which the mechanism designer may ration some agents.

Our analysis applies to economies in which all agents have quasilinear preferences. Our first key result provides an upper bound on the welfare loss from any feasible allocation. Given *any* feasible allocation and *any* price vector, the welfare loss from the allocation is at most equal to the sum of the budget deficit plus the agents' *rationed surplus* at the specified prices. We define an agent's rationed surplus to be the excess of the payoff (utility or profits) it would obtain from its most preferred bundle at the specified prices over its payoff at the given allocation. If the price and allocation pair is a competitive equilibrium, both the rationed surplus and the budget deficit are zero, so Theorem 1 implies that the allocation is efficient. With that special case in mind, we call this result the *Bound-Form First Welfare Theorem*.

We use this Bound-Form First Welfare Theorem to construct pricing mechanisms in which

[3]We assume that buyers and sellers have free disposal, so there is always *some* efficient allocation with exact market-clearing. However, this efficient allocation may include bundles that would never be chosen by price-taking agents.

losses are small, despite non-convexities and failures of market clearing. In Section 4, we introduce a mechanism that produces an allocation and different prices for producers and consumers for which the allocation is approximately efficient and budget-balanced and participation is individually-rational. We call this mechanism the *α-price mechanism*, because buyers pay prices that are $(1+\alpha)$ times the prices paid to sellers. We can choose α close to zero when the economy is large. Theorem 2 shows that the welfare loss of this mechanism is bounded by a constant independently of the market size, so the *fraction* of the efficient surplus lost, the *relative inefficiency* of the mechanism, is at most inversely proportional to the market size. Theorem 3 provides conditions under which the benefits of misreporting in this mechanism are arbitrarily small for sufficiently large markets, so that the mechanism has good incentive properties similar to those of the Walrasian mechanism demonstrated by Roberts and Postlewaite (1976), Azevedo and Budish (2019) and Watt (2021). The mechanism is also computationally tractable even in non-convex economies, because it computes optima only for convex problems, although computing *exactly* efficient allocation for non-convex economies is NP-hard.

In Section 5, we consider the special case when non-convexities can occur only on one side of the market – for producers but not consumers. For this case, we introduce a *single* price mechanism in which some consumers may be rationed. Assuming that consumer preferences are strongly convex (Watt, 2021), we find that the mechanism's *total* losses are inversely proportional to the market size. Consequently, their *relative inefficiency* is inversely proportional to the *square* of market size. This mechanism is budget-balanced, has incentive properties similar to the double-price mechanism, and may be efficiently computed solving only convex optimization problems. It is also approximately individually rational, and exactly so if buyers' preferences are separable.

1.1 An example

Consider a market with one good and a single producer. The producer can make only non-negative integer numbers of units at a marginal cost $c = \$0.50$, so the production technology is not convex.

There are N buyers indexed $n = 1, ..., N$. If the price is low enough, buyer n would like to purchase *exactly* q_n units of the good: buyers derive no value from receiving fewer than q_n units and no additional value for additional units beyond q_n. We refer to q_n as buyer n's *order size*. Buyers have quasilinear preferences, where buyer n's per-unit value for the good is v_n, so its utility $U_n(x_n, t)$ from paying $\$t$ for x_n units is

$$U_n(x_n, t) = \begin{cases} -t & \text{if } x_n < q_n \\ q_n v_n - t & \text{if } x_n \geq q_n. \end{cases}$$

This utility function is not quasiconcave, so that buyers' preferences do not satisfy the standard convexity assumption of general equilibrium theory.

The supply and demand curves for a specific instance of this market, using the buyer values and order sizes in Table 1, is illustrated in Figure 1.

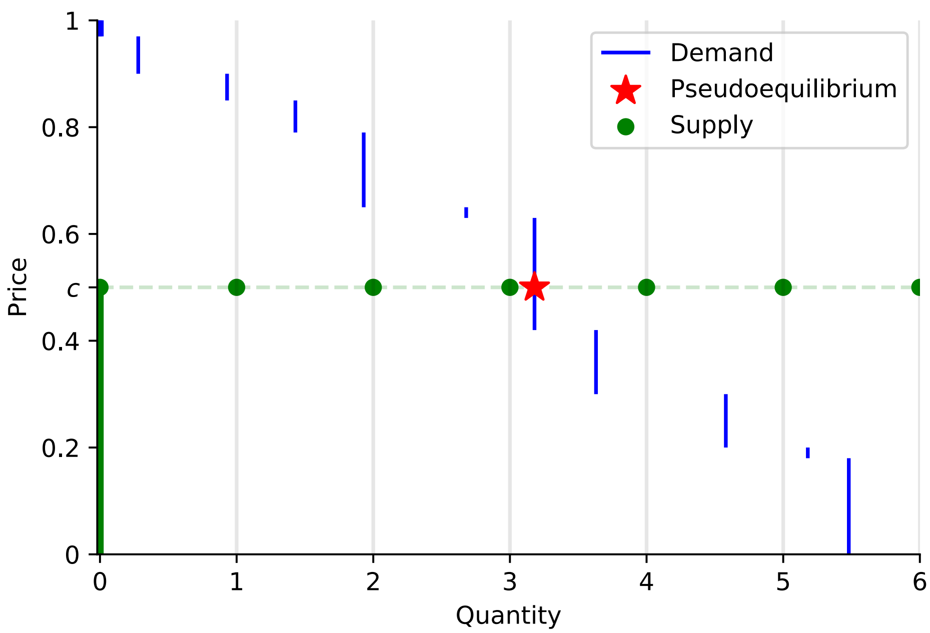

Figure 1: Example market

Buyer n	1	2	3	4	5	6	7	8	9	10
Unit value v_n	0.97	0.9	0.85	0.79	0.65	0.63	0.42	0.3	0.2	0.18
Order size q_n	0.28	0.65	0.5	0.5	0.75	0.5	0.45	0.95	0.6	0.3

Table 1: Buyer values

How might we design a market for this good? If a market-clearing price exists, then the allocation that maximizes aggregate welfare assigns each agent its preferred bundle for given the price. In a large market in which agents cannot much affect prices, there are good incentives for truthful reporting. In our example, if the sum of q_n over all agents with $v_n \geq 0.5$ is an integer, then \$0.50

is a competitive equilibrium price. However, if this sum is non-integer, as in Figure 1, there is no competitive equilibrium.

If a competitive equilibrium does not exist, a single price mechanism that selects an approximate equilibrium may seem attractive. A classic result of Starr (1969) implies that, in a non-convex market with one good, there exists a price and allocation such that at most one agent is assigned a bundle different to their most-preferred bundle given the price. We call the resulting price and allocation a *pseudoequilibrium.* In the example of Figure 1, the pseudoequilibrium allocates the first six buyers their full orders and the seller is asked to supply the resulting 3.18 units at a price \$0.50, which is infeasible given the seller's production technology.[4]

Given a pseudoequilibrium, an alternative is to round the allocation to one in which all agents are allocated a bundle of their choosing but there is an imbalance in supply and demand (in this case, an imbalance of no more than 1 unit of the good). The resulting price and allocation is a (Starr-)*approximate equilibrium*. In the market of Figure 1, at a price of \$0.50, the seller could fulfill the orders of the first six buyers by producing 4 units of the good, resulting in an oversupply of 0.82 units of the good. Alternatively, the seller could produce 3 units, resulting in an undersupply of 0.18 units. If the approximate equilibrium exhibits undersupply, the allocation is infeasible: buyers are being promised more units of the good than the seller is willing to provide. If the approximate equilibrium involves oversupply, it must pay sellers more than it receives from buyers, resulting in a budget deficit.

In case the approximate equilibrium involves undersupply, a mechanism might use try to ration some agents to restore feasibility, assigning them allocations that are strictly worse than their preferred bundles.[5] In the market of Figure 1, we could assign the seller to produce 3 units but only fulfill the orders associated with five of the six buyers with values above \$0.50. In general, however, it is difficult to determine which agent to ration. In the market of Figure 1, the most efficient choice is to ration buyer 1, who happens to have the *highest* per-unit value of the good. In general, with non-convexities, multiple goods and multiple buyers and sellers, the problem of determining the most efficient allocation supported by fixed prices can be computationally challenging. Finally, and importantly for mechanism designers, the possibility of rationing presents

[4]Under the assumption of free disposal, it is feasible for the producer to supply 3.18 units (by producing 4 and discarding 0.82) but the producer would not be willing to do so at a price of \$0.50, and at higher prices, the seller would want to produce many more units.

[5]Our usage of the word 'rationing' differs somewhat to standard use in which 'rationing' is typically reserved for occasions when buyers are able to buy *less* of a good than they would like to purchase. We use rationing for both buyers and sellers, and for allocations involving less *or* more of a good (or the wrong combination of goods if there are multiple goods).

new challenges for incentives. In particular, an agent participating in a pricing mechanism with rationing may have an incentive to misreport its preferences to avoid being rationed, *even if its report does not influence the price*. For example, in the market of in Figure 1, if the mechanism designer selects an agent to ration to maximizes total welfare, then buyer 1 could report a value of 1.1 or an order size of 0.31 and avoid rationing, because its contribution to surplus would become larger than that of buyer 6. These are misreports that benefit the buyer without affecting the price of the good.

In the standard Walrasian mechanism, buyers have an incentive to misreport in order to influence the price, but, conditional on the price, the truthful report is the optimal one: there is no incentive to misreport to avoid rationing. Moreover, Roberts and Postlewaite (1976) show, under weak assumptions, that the ability of a single agent to influence the price tends to zero as the market becomes large, so the optimal report also tends towards the truthful one. However, the incentive to misreport to avoid rationing can remain intact even as the market grows large. When there are non-convexities on both sides of the market and agents may need to be rationed regardless of the size of the economy, the rationed agents may often have an incentive to misreport.

Another approach to mechanism design for such settings would abandon linear pricing, seeking to implement the total-value-maximizing allocation using a non-linear transfer rule. The theorem of Green and Laffont (1979) implies that any strategy-proof implementation of the efficient allocation rule must run a budget deficit, but even if a budget deficit is allowed or strategy-proofness is relaxed, the problem of identifying (or even approximating) an efficient allocation can be computationally challenging. While the efficient allocation in the market of Figure 1 can be obtained as the solutions to simple integer programming problems, in general, such optimization problems are NP-hard and become computationally intractable with more agents and goods.

To find a mechanism with the features we seek, including easy computability, good incentives, and linear prices, one needs to look beyond exact solutions of integer programming problems. Our featured approach, developed in Section 4, involves relaxing one key property of the standard competitive equilibrium model: the use of equal prices for buyers and sellers. We show that this approach can resolve the problems of budget imbalance, reporting incentives, and computational hardness, using only convex optimization procedures. In the market of Figure 1, this mechanism would choose the approximate equilibrium allocation in which the seller produces 4 units of the good, receiving \$0.50 per unit, and the orders of the first six buyers are filled at the price \$0.63. The revenue earned from sale to the buyers more than covers the cost of production of the sellers, and no agents are rationed at their respective prices. In Section 4, we show that it is always possible to find an allocation and two price vectors with this property such that the total welfare differs from the maximum total welfare by at most a constant and, in particular, does not depend on number of agents. In general, the required price differences vanish as the market grows large. Moreover, because none of the agents are rationed given their price vector and because buyers and sellers have little effect on their prices, this mechanism has large-market incentive properties similar to those of the Walrasian mechanism.

1.2 Related literature

The problem of non-convexity for the existence of competitive equilibrium was discussed in a series of papers by Farrell (1959), Rothenberg (1960), Koopmans (1961) and Bator (1961). Much of the subsequent classical literature on non-convexity in general equilibrium theory focused on concepts of *approximate* equilibria which replace aggregate feasibility requirements with approximate feasibility, measured in terms of distance in the commodity space between the aggregate supply and demand, while maintaining the requirement that individual agents act optimally given the prices. Starr (1969) showed the existence of such an approximate equilibrium in non-convex production economies, in which the maximum imbalance is proportional to the number of goods and a measure of non-convexity. Heller (1972) proved a similar result with an alternative measure of non-convexity. More recently, Nguyen and Vohra (2020) proved a bound for markets with indivisible goods that depends only on a measure of preference complementarity of agents. We build on some of these results (summarizing the key results we employ in Appendix A), but depart from this literature by requiring that any feasible mechanism must always specify a feasible outcome. Influenced by computer scientists' approaches to approximations in mechanism design, we will be interested in approximations in surplus space, rather than in commodity space.[6]

A substantial literature has focused on identifying various *substitutes* conditions on valuation functions in quasilinear markets with indivisibilities, under which competitive equilibria exist despite non-convexities. Contributors include Bikhchandani and Mamer (1997), Gul and Stacchetti (1999), Danilov, Koshevoy, and Murota (2001), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Baldwin and Klemperer (2019) and Nguyen and Vohra (2020). None of these results apply to markets with fixed costs such as those described above, in which case competitive equilibria do not generally exist. Our analysis

[6] This is also the approach to approximation taken by Scarf (1967).

seeks to develop practical mechanisms for those settings.

An alternative approach to establish equilibrium existence in markets with non-convexities is to study the large market limit with a continuum of agents. Aumann (1966) showed that the convexity assumption is not necessary for equilibrium existence in an economy with a continuum of traders and divisible goods, while Azevedo, Weyl, and White (2013) demonstrated a similar result for an economy with indivisibilities. In this paper, we study economies with a finite number of agents, which allows us to investigate properties of not only the limiting economy, but also rates of convergence, which are of practical significance to market designers who wish to apply mechanisms to finite economies. Another approach is to allow nonlinear pricing rules, as explored by Wilson (1993), Chavas and Briec (2012), Azizan, Su, Dvijotham, and Wierman (2020) and others, but we focus on what can be achieved by linear pricing rules, which seem to be desired for many applications.

Our study is motivated by other studies of market design with linear prices and non-convex production. In particular, we have taken inspiration from the novel market design for fisheries rights in New South Wales, Australia, introduced by Bichler, Fux, and Goeree (2018, 2019). Other markets with non-convexities that have been studied in the market design literature include electricity markets, where large start-up and ramping costs are the source of the non-convexity, and spectrum auctions, where geographical complementarities cause exposure problems.[7]

1.3 Notation

In this paper, we will view consumption and production bundles as vectors in Euclidean space $\mathbb{R}^L$, and let $x \cdot y$ denote the standard inner product of x and y and $\|x\| = \sqrt{x \cdot x}$ denote the Euclidean norm of x. We use $\geq$ to denote the partial order on $\mathbb{R}^L$ so that $x \geq y$ if and only if $x_\ell \geq y_\ell$ for $\ell = 1, ..., L$. The set $\mathbb{R}^L_+$ is $\{x \in \mathbb{R}^L : x \geq 0\}$. The radius of a set $S \subseteq \mathbb{R}^L$ is defined by $\text{rad}(S) = \inf_{x \in \mathbb{R}^L} \sup_{y \in S} \|x - y\|$. The notation $|\cdot|$ represents either the absolute value (if its argument is a number) or the cardinality (if its argument is a set). The Hausdorff distance between sets $S, S' \subseteq \mathbb{R}^L$ is denoted d_H and is defined by letting $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$ and $d_H(S, S') = \sup_{x \in S'} \text{dist}(x, S)$.

The convex hull of a set $S \subseteq \mathbb{R}^L$ is denoted by $\text{co}(S) = \{x \in \mathbb{R}^L : x = \lambda y + (1 - \lambda)z \text{ for } y, z \in S \text{ and } \lambda \in [0, 1]\}$. The upper concave envelope of $f : S \to \mathbb{R}$ is the function $\text{cav}(f) : \text{co}(S) \to \mathbb{R}$

[7] See Liberopoulos and Andrianesis (2016) for a summary of pricing mechanisms used in electricity markets with non-convexities, most of which include "uplift" (or side-payments) in addition to linear pricing, and Ausubel and Milgrom (2002) for a discussion of complementarities in spectrum auctions.

which is the (pointwise) smallest concave function g on $\text{co}(S)$ that satisfies $g(x) \geq f(x)$ for all $x \in S$. The lower convex envelope of $f : S \to \mathbb{R}$ is the function $\text{vex}(f) : \text{co}(S) \to \mathbb{R}$ which is the largest convex function g on $\text{co}(S)$ that satisfies $g(x) \leq f(x)$ for all $x \in S$.

We make regular use of the asymptotic notation due to Knuth (1976). For functions f and g both mapping $\mathbb{R}$ to $\mathbb{R}$, we write $f(x) = O(g(x))$ if $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$. We write $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$. Finally, we write $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

2 Model and preliminaries

2.1 Model

We employ a Walrasian model with a set of **buyers** N and a set of **firms** or **sellers** F, both finite. Together we refer to $A = N \cup F$ as the set of **agents**. There are L varieties of consumable **goods** and a numeraire good, money.

Each buyer $n \in N$ can choose consumption bundles in X, a compact subset of $\mathbb{R}^L_+$ containing 0, called the **consumption possibility set**. Buyer n has quasilinear preferences[8] over bundles in X with a continuous **valuation function** $u_n : X \to \mathbb{R}$, so that the buyer's **utility** associated with allocation x_n after payment t is $U_n(x_n, t) = u_n(x_n) - t$. We suppose that the valuation functions are bounded, nondecreasing[9] with respect to the partial order $\geq$ on $\mathbb{R}^L_+$ and normalized so that $u_n(0) = 0$. We let $\mathscr{U}$ be the space of possible valuation functions for the buyers, which we assume to be an admissible function space in the sense of Aumann (1963) (that is, it is possible to define a measure on $\mathscr{U}$).[10]

Each seller $f \in F$ can choose a production bundle in the **production possibility set** Y, a compact subset of $\mathbb{R}^L_+$ containing 0. Seller f has a **cost function**[11] $c_f : Y \to \mathbb{R}_+$ which allows us to write f's **profit** from producing $y_f \in Y$ and receiving payment t as $\pi_f(y_f, t) = t - c_f(y_f)$. The cost functions are nondecreasing with respect to the partial order $\geq$ on $\mathbb{R}^L_+$ and normalized so that $c_f(0) = 0$. Let $\mathscr{C}$ be the space of sellers' cost functions, which we also assume to be admissible.

[8]The quasilinearity assumption allows our analysis to abstract from income effects, as is usual in mechanism design analyses. For more discussion of the role of income effects see Morimoto and Serizawa (2015).

[9]This form of monotonicity captures our assumption of free disposal.

[10]For example the set of bounded, continuous functions on a compact subset of $\mathbb{R}^L$ is admissible, as is the set of bounded functions with discontinuities of the first kind, or more generally, any subset of a Baire class (Aumann, 1963).

[11]Note that sellers in this economy could equivalently be thought of as buyers with valuations $-c_f(y_f)$ and payments $-t$. However, we will be interested in mechanisms that may charge buyers and sellers different prices, and so it is convenient to distinguish the two groups in our notation.

An **economy** $\mathcal{E}$ consists of buyers with their valuation functions and sellers with their cost functions, so that we may write $\mathcal{E} = \langle N, (u_n)_{n\in N}, F, (c_f)_{f\in F}\rangle$. When it is clear, we use the shorthand $\mathcal{E} = \langle N, F\rangle$. At times, it is also convenient to associate $\mathcal{E}$ with the normalized counting measures μ on $\mathcal{U}$ and ν on $\mathcal{C}$ defined by

$$\mu(u_n) = \frac{\text{\# of buyers in } \mathcal{E} \text{ with valuation function } u_n}{|N|},$$
$$\chi(c_f) = \frac{\text{\# of sellers in } \mathcal{E} \text{ with cost function } c_f}{|F|},$$

and to let $\phi = \frac{|F|}{|N|}$, so that $\langle N, \mu, \phi, \chi\rangle$ is an alternative specification of economy $\mathcal{E}$.

Allocations and efficiency An **allocation** $\omega = ((x_n)_{n\in N}, (y_f)_{f\in F})$ is an assignment of consumption bundles $x_n \in X$ to each buyer $n \in N$ and production bundles $y_f \in Y$ to each seller $f \in F$. An allocation is **feasible** if $\sum_{n\in N} x_n \le \sum_{f\in F} y_f$. We denote by Ω the set of all feasible allocations.

We define the **surplus** $\mathcal{S}(\omega)$ associated with allocation $\omega \in \Omega$ by

$$\mathcal{S}(\omega) = \sum_{n\in N} u_n(x_n) - \sum_{f\in F} c_f(y_f).$$

The **efficient allocation** problem is to solve

$$\max_{\omega\in\Omega} \mathcal{S}(\omega), \tag{P}$$

with the resulting surplus denoted by $\mathcal{S}^*$.

For any allocation $\omega \in \Omega$, we will refer to $\mathcal{S}(\omega) - \mathcal{S}^*$ as the **deadweight loss** of ω and the ratio $\frac{\mathcal{S}(\omega)-\mathcal{S}^*}{\mathcal{S}^*}$ as the **relative inefficiency** of ω.[12]

Pricing rules We are interested in settings where buyers and sellers face a payment rule in the form of **linear prices**.

If a buyer n faces price $p \in \mathbb{R}^L_+$ and purchases bundle x, the resulting payment is $t = p \cdot x$, leading to utility $U_n(x, p\cdot x) = u_n(x) - p\cdot x$. The **demand correspondence** $D_n : \mathbb{R}^L_+ \rightrightarrows X$ maps each price vector p to the set of utility-maximizing bundles $D_n(p)$ which result in the **indirect utility**, $\hat{u}_n(p) = \max_{x\in X} u_n(x) - p\cdot x$.

[12] Later we will make assumptions to rule out cases where $\mathcal{S}^* = 0$ so that this ratio is well-defined.

Similarly, a seller f facing price p and supplying bundle y receives profit $\pi_f(y, p \cdot y)$. Seller f's **supply correspondence** $S_f : \mathbb{R}^L_+ \rightrightarrows Y$ maps each price vector p to the set of profit-maximizing bundles $S_f(p)$ which result in **producer surplus** $\hat{\pi}_f(p) = \max_{y \in Y} p \cdot y - c_f(y)$.

A pricing rule is **anonymous** if for all $n, n' \in N$, $p_n = p'_n$ and for all $f \in F$, $p_f = p'_f$. A pricing rule which is not anonymous is called **personalized**. A **single-pricing** rule imposes the same price vector p on all agents. Single-pricing has been the traditional emphasis of general equilibrium theory.

2.2 Convex quasilinear economies

Convexity is defined with respect to the set of payoff-improving allocations for an agent in the economy. The $\bar{u}$-**upper contour set** of buyer $n \in N$ is defined by

$$UC_n^{\bar{u}} = \{(x, t) \in X \times \mathbb{R} : U_n(x, t) \geq \bar{u}\},$$

while the $\bar{\pi}$**-upper contour set** of seller $f \in F$ is given by

$$UC_f^{\bar{\pi}} = \{(y, t) \in Y \times \mathbb{R} : \pi_f(y, t) \geq \bar{\pi}\}.$$

We say that buyer n has **convex preferences** if X is convex and $UC_n^{\bar{u}}$ is convex for all $\bar{u} \in \mathbb{R}$, while we say that seller f has **convex technology** if Y is convex and $UC_f^{\bar{\pi}}$ is convex for all $\bar{\pi} \in \mathbb{R}$. Equivalently, buyer n has convex preferences if and only if the utility function U_n is quasiconcave, and seller f has convex technology if and only if the profit function π_f is quasiconcave. In our quasilinear environment, convexity of preferences and technologies has the following well-known characterization:

Proposition 1. *The utility function $U_n(x, t) = u_n(x) - t$ is quasiconcave if and only if the valuation function $u_n(x)$ is concave. Similarly, the profit function $\pi_f(y, t) = t - c_f(y)$ is quasiconcave if and only if the cost function $c_f(y)$ is convex.*

Under the assumption of quasilinearity and the convexity of agents' preferences and technologies, we have the following statement of the fundamental welfare theorems of Arrow (1951) and Debreu (1951). [13]

[13] The statement of Proposition 2 is stronger than the classic statements of the welfare theorems in the 'only if' direction, which is possible due to the quasilinear form of the utility and profit functions. Without quasilinearity or

Proposition 2. *Suppose in (quasilinear) economy $\mathscr{E}$ that all buyers $n \in N$ have convex preferences and all sellers $f \in F$ have convex technology. Then a feasible allocation $\omega \in \Omega$ is efficient if and only if there exists $p \in \mathbb{R}^L_+, p > 0$ such that for all $n \in N$, $x_n \in D_n(p)$; for all $f \in F$, $y_f \in S_f(p)$; and $\sum_{n\in N} p \cdot x_n = \sum_{f\in F} p \cdot y_f$. The pair (p, ω) is a* ***competitive*** *or* ***Walrasian equilibrium****.*

2.3 Measures of non-convexity and approximate equilibria

The non-convexity of a set S can be measured in several ways by comparing S and $\text{co}(S)$. We will work with the following measures of non-convexity of a set:

- The **inner radius** of S is $r(S) = \sup_{x\in\text{co}(S)} \inf_{T\subseteq S: x\in\text{co}(T)} \text{rad}(T)$.
- The **inner distance** of S is $\rho(S) = \sup_{x\in\text{co}(S)} \inf_{y\in S} \|x - y\|$.

For a convex set S, it is clear that $r(S) = 0 = \rho(S)$, so that these are appropriate measures of the non-convexity of S. These measures are illustrated in Figure 2.

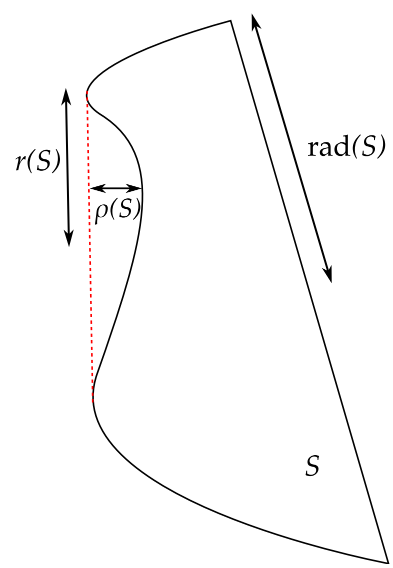

Figure 2: Measures of non-convexity of a set

The non-convexity of the preferences of buyer $n \in N$ may be measured by the largest inner radius or inner distance of their upper contour sets, that is $r_n = \sup_{\bar{u}\in\mathbb{R}} r(UC_n^{\bar{u}})$ or $\rho_n =$

an alternative assumption, it may only be possible to find prices so that agents are expenditure-minimizing for a given level of utility or profit, that is, a price quasiequilibrium with transfers. With quasilinearity and convexity, Proposition 1 implies that the efficient allocation program is convex. Since there exists a feasible allocation, Slater's Theorem (see, for example, Boyd and Vandenberghe (2004)) implies strong duality. A solution p^* to the dual program, $\inf_{p\in\mathbb{R}^L_+} \sum_{n\in N} \hat{u}_n(p) + \sum_{f\in F} \hat{\pi}_f(p)$, and an efficient allocation ω^* comprise a saddle point for the Lagrangian $\mathscr{L}(\omega, p) = \sum_{n\in N} u_n(x_n) - \sum_{f\in F} c_f(y_f) - p \cdot \left(\sum_{n\in N} x_n - \sum_{f\in F} y_f\right)$, so that for any $\omega \in \Omega$, $\mathscr{L}(\omega, p^*) \leq \mathscr{L}(\omega^*, p^*)$. Because the Lagrangian is separable across agents, the saddle point condition implies $x_n^* \in D_n(p^*)$ and $y_f^* \in S_f(p^*)$.

$\sup_{\bar{u}\in\mathbb{R}} \rho(UC_n^{\bar{u}})$. Similarly, the non-convexity of the technology of seller $f \in F$ may be measured by $r_f = \sup_{\bar{\pi}\in\mathbb{R}} r(UC_f^{\bar{\pi}})$ or $\rho_f = \sup_{\bar{\pi}\in\mathbb{R}} \rho(UC_f^{\bar{\pi}})$. Let $r_{\mathscr{E}}$ and $\rho_{\mathscr{E}}$ denote the largest of such measures among all the buyers and sellers in economy $\mathscr{E}$.

When agents' upper contour sets are not convex, the second welfare theorem may not hold and there may be no competitive equilibrium. Proposition 3, the Shapley-Folkman Lemma, assists in identifying allocations which are nearly competitive equilibria.

Proposition 3 (Shapley-Folkman Lemma[14])**.** *Let $S_i \subseteq \mathbb{R}^L$ for $i = 1, ..., M$, $S = \bigoplus_{i=1}^M S_i$ and $L' = \min(L, M)$. Then for any $x \in \text{co}(S)$, $x = \sum_{i=1}^N x_i$ where $x_i \in \text{co}(S_i)$ and $|i : x_i \in \text{co}(S_i) \setminus S_i| \leq L'$. Moreover, there exists $y, y' \in S$ such that $\|x - y\| \leq r_{\mathscr{E}}\sqrt{L'}$ and $\|x - y'\| \leq \rho_{\mathscr{E}} L'$.*

Proposition 3 has been used to establish results about what are called *approximate equilibria*, which are constructed as follows. First, consider a convexified version of the non-convex economy in which the upper contour sets of all agents are replaced by their convex hulls. This is equivalent to replacing the buyers' valuation functions by their upper concave envelopes and sellers' cost functions by their lower convex envelopes. The **convexified economy** $\hat{\mathscr{E}}$ is then

$$\hat{\mathscr{E}} = \langle N, (\text{cav}(u_n))_{n\in N}, F, (\text{vex}(c_f))_{f\in F}\rangle.$$

The convexified economy has a competitive equilibrium which is efficient by Proposition 2. We call the resulting price-allocation pair (p, ω) a **pseudoequilibrium** of the actual economy $\mathscr{E}$. Proposition 3 implies that ω can be chosen so at most L' agents in $\mathscr{E}$ are not utility- or profit-maximizing at ω given prices p. Proposition 3 implies that there is a nearby allocation ω' such that *all* agents are maximizing given prices p,[15] but markets may not exactly clear at ω'. The price-allocation pair (p, ω') is called an **approximate equilibrium**.

Practical market design requires allocations to be feasible and voluntary, so that pseudoequilibria and approximate equilibria are often insufficient. We modify the approach to approximation of equilibria in Section 4 to obtain mechanisms whose allocations satisfy these requirements.

[14] It is perhaps most accurate to refer only to the result in the second sentence of Proposition 3 as the Shapley-Folkman Lemma, althought it was first reported by Starr (1969) as a result of private communication with Lloyd Shapley and Jon Folkman. Starr (1969) then proved the first half of sentence three of Proposition 3, while Heller (1972) proved the second half. For simplicity, we will refer to the whole of Proposition 3 as the Shapley-Folkman Lemma.

[15] To see this, note that if a buyer is assigned a bundle x_n in ω that is not utility-maximizing at p, then x_n must be the convex combination of bundles (x_n') in X which are *exposed points* in u_n (i.e. where $\text{cav}(u_n) = u_n$), and that the agents in the convexified economy must be indifferent between x_n and these bundles. That is, the concavified portions of buyers' utility functions consist of (patches of) hyperplanes, and if an agent is assigned a bundle on such a patch, then the price vector must be normal to that hyperplane. This implies that the original buyer must be maximizing at bundles in (x_n'), which are on the relative boundaries of the patch of hyperplane.

3 Rationing, inefficiency and incentives

When competitive equilibrium does not exist, allocations supported by a single price vector must, by definition, entail rationing or non-exact market-clearing (and therefore budget imbalance), or both. Rationing occurs at a given price and feasible allocation if the demand of some buyers is unfulfilled or the supply of certain sellers is unneeded. For our purposes, it will be helpful to measure the extent of rationing for buyers and sellers by what we call the *rationed utility* and *rationed profit*, which are both defined by the difference between the payoff an agent would obtain at their preferred bundle given the prices and the payoff they instead receive in the prescribed allocation at those prices.

Definition 3.1. The **rationed utility** $\mathscr{R}_n(p, x)$ **of buyer** n at price p and allocation x is

$$\mathscr{R}_n(p, x) = \hat{u}_n(p) - U_n(x, p \cdot x).$$

The **rationed profit** $\mathscr{R}_f(p, y)$ **of seller** f is

$$\mathscr{R}_f(p, y) = \hat{\pi}_f(p) - \pi_f(y, p \cdot y).$$

The **rationed surplus** of allocation $\omega = ((x_n)_{n \in N}, (y_f)_{f \in F})$ at price p is defined by

$$\mathscr{R}(p, \omega) = \sum_{n \in N} \mathscr{R}_n(p, x_n) + \sum_{f \in F} \mathscr{R}_f(p, y_f).$$

With this definition, we can state our first main result.

Theorem 1 (Bound-Form First Welfare Theorem). *Let $p \in \mathbb{R}^L_+$ be a price vector and $\omega = ((x_n)_{n \in N}, (y_f)_{f \in F})$ be any feasible allocation. Then, the deadweight loss of allocation ω satisfies*

$$\underbrace{\mathcal{S}^* - \mathcal{S}(\omega)}_{\text{deadweight loss}} \leq \underbrace{\mathscr{R}(p, \omega)}_{\text{rationed surplus}} + \underbrace{p \cdot \left(\sum_{f \in F} y_f - \sum_{n \in N} x_n \right)}_{\text{budget deficit}}.$$

Proof. Fix any efficient allocation ω^*. By definition of the indirect utility and consumer surplus functions, the following must hold for any prices:

$$\hat{u}_n(p) \geq u_n(x_n^*) - p \cdot x_n^*$$
$$\hat{\pi}_f(p) \geq p \cdot y_f^* - c_f(y_f^*).$$

Summing these inequalities, we obtain

$$\sum_{n \in N} \hat{u}_n(p) + \sum_{f \in F} \hat{\pi}_f(p) + p \cdot \left(\sum_{n \in N} x_n^* - \sum_{f \in F} y_f^* \right) \geq \mathcal{S}^*.$$

Since ω^* is feasible, the third term is non-positive, which implies

$$\sum_{n \in N} \hat{u}_n(p) + \sum_{f \in F} \hat{\pi}_f(p) \geq \mathcal{S}^*$$

Subtracting $\mathcal{S}(\omega)$ and using our rationing definitions,

$$\mathcal{S}^* - \mathcal{S}(\omega) \leq \sum_{n \in N} \hat{u}_n(p) + \sum_{f \in F} \hat{\pi}_f(p) - \mathcal{S}(\omega) = \mathscr{R}(p, \omega) + p \cdot \left(\sum_{f \in F} y_f - \sum_{n \in N} x_n \right),$$

which is what we sought to prove. □

The Bound-Form First Welfare Theorem is a generalization of the First Welfare Theorem for quasilinear economies. Indeed, if p and ω are the price and allocation associated with a Walrasian equilibrium, then the right-hand side of the above inequality is zero and we recover the statement that Walrasian equilibria are efficient. One interpretation of the First Welfare Theorem is that prices act as a "certificate of optimality": if supporting prices exist for some allocation, then that allocation is efficient. As Scarf (1994) lamented, in the absence of convexity, there is, in general, no such optimality test. However, one interpretation of Theorem 1 is as an *approximate* optimality test: if we can show that the rationed surplus plus net subsidies to sellers is small, then the given allocation will be approximately efficient. Theorem 1 is also useful in the way that it links incentives to efficiency: since for any fixed price, agents would prefer not to be rationed, Theorem 1 suggests that a pricing mechanism with little rationing and in which individual agents have little influence over prices will have good incentive properties and small deadweight loss. We will exploit these properties of Theorem 1 to design pricing mechanisms in Section 4.

4 Two-sided non-convexities: the α^*-price mechanism

4.1 Pricing mechanisms

Taking a market design perspective, we study direct pricing mechanisms that map profiles of reports of sellers' cost functions $(c_f)_{f\in F}$ and buyers' value functions $(u_n)_{n\in N}$ to a feasible allocation $\omega \in \Omega$ and prices for each agent $\boldsymbol{p} = ((p_n)_{n\in N}, (p_f)_{f\in F})$.[16] We do not delve into the complicated question of how agents communicate these potentially complicated costs and values to the mechanism.[17]

We use familiar definitions to describe certain properties of the mechanisms. A pricing mechanism is **incentive-compatible** if truthful reporting is a Nash equilibrium of the reporting game induced by the mechanism, while it is ε-**incentive-compatible** if truthful reporting is an ε-Nash equilibrium. The mechanism is **efficient** if the output allocation ω is an efficient allocation given the reported value functions and cost functions. It is **budget-balanced** if, for all report profiles, $\sum_{f\in F} p_f \cdot y_f = \sum_{n\in N} p_n \cdot x_n$, while it is **weakly budget-balanced** if $\sum_{f\in F} p_f \cdot y_f \leq \sum_{n\in N} p_n \cdot x_n$. Finally, the pricing mechanism is ε-**individually-rational** if, given reported value and cost functions, the allocation and price determined by the mechanism delivers each agent a payoff (utility or profit) of at least $-\varepsilon$. For $\varepsilon = 0$, we simply say the mechanism is **individually-rational**.

4.2 Approximately-efficient pricing mechanisms

In this subsection, we introduce a weakly budget-balanced double-pricing mechanism (one set of prices for buyers and another for sellers) which is *approximately* efficient, which means that the relative inefficency of the mechanism approaches zero as the economy becomes large, even if the preferences and/or production sets are not convex, provided that the radius of any non-convexity is bounded. We also provide a worst-case bound on the loss of surplus compared to the first-best allocation. Before providing a technical description of this pricing mechanism and its analysis, we sketch intuitively the steps of our approach.

For a weakly budget-balanced mechanism to allow free disposal, it must charge higher prices to buyers than are paid to the sellers. With that in mind, we focus on the class of *α-pricing mechanisms*, where payments to sellers are determined by a price vector p, while payments by buyers

[16] For simplicity, we do not consider randomized mechanisms as we will be able to achieve our objectives using deterministic mechanisms.

[17] The design of reporting languages to report complex preferences for economic mechanisms has been studied by Milgrom (2009), Bichler, Goeree, Mayer, and Shabalin (2014), Bichler, Milgrom, and Schwarz (2020) and others.

are determined by a price vector $(1+\alpha)p$ for some scalar $\alpha > 0$ to be chosen later. We select p to be the equilibrium price vector of a related economy with three changes from the actual economy: (1) every buyers' value function is replaced by the smaller function $u_n/(1+\alpha)$, (2) all values and costs are then replaced by their concave or convex hulls, respectively, and (3) we add demand for each good by the auctioneer in the amount of $R := \max\{r_{\mathscr{G}}\sqrt{L}, \rho_{\mathscr{G}} L\}$. We call the resulting price and allocation a *pseudoequilibrium* of our economy.

Our next step shows that this pseudoequilibrium allocation is nearly optimal for the adjusted economy without the $(1+\alpha)$ rescaling of buyers' values. Indeed, we show that the welfare loss from using the pseudoequilibrium allocation instead of the optimum with the unscaled values is of lower order than α. To prove that, we apply the Bound-Form First Welfare Theorem to the pseudoequilibrium prices and quantities and the Milgrom and Segal (2002) envelope theorem.

By construction, if the step (1) rescaling is omitted but buyers' prices are set to $(1+\alpha)p$ instead of p, the buyers' demands are unchanged, so with those prices the pseudoequilibrium allocation is still individually rational.

If the convexification of the economy matters for an agent, then its supply or demand without convexification is set-valued. Our next step is to apply the Shapley-Folkman lemma to round the pseudoequilibrium allocation to some alternative demanded allocations for each agent, but in which the result changes the net demand for each good by at most R units – an amount that we propose to take from the units otherwise allocated to the auctioneer. The final allocation always has supply greater than or equal to demand and its excess supply is no more than $2R$ units of each good. This excess supply can result in a loss of efficiency, but as we have just seen, this loss is bounded by a constant, independently of the size of the market.[18]

Since the excess supply of goods is bounded, the budget imbalance at price p is bounded as well. As trade increases with the size of the economy, the buyers' premium, α, needed to guarantee budget balance is inversely related to market size. Thus, the total welfare loss from this double-price mechanism is bounded by a constant plus a term that is inversely proportional to market size.

We are now ready to present the technical version of our main result. Let $\mathscr{E}_t = \langle N_t, \mu_t, \phi_t, \chi_t \rangle$ be

[18] The choice of R units of each good as a set-aside for the auctioneer in step 1 is useful as a theoretical guarantee, although it might be possible to allocate fewer units to the auctioneer in step 1 and arrive at a more efficient feasible allocation using the same approach. We note that an alternative approach could be to start by checking for feasible allocations with zero units set aside (these would correspond to competitive equilibria) and then increase the set-aside gradually until a budget-balanced market-clearing alpha-price mechanism is identified, but we do not discuss this alternative mechanism further in this paper.

a sequence of economies indexed by $t = 1, 2, ...$ For technical purposes, we assume the following

Assumption 1 (Existence of limit economy)**.** *As $t \to \infty$, $|N_t| \to \infty$ and $\phi_t \to \phi \in (0, 1)$. Furthermore, μ_t converges weakly to probability measure μ_∞ on $\mathscr{U}$ and χ_t converges weakly to measure χ_∞ on $\mathscr{C}$.*

We make the following economic assumptions:

Assumption 2 (Individual non-convexities are bounded)**.** *There exists $R > 0$ with $R_{\mathscr{E}_t} < R$ for all t.*

Assumption 3 (Growing gains from trade)**.** *As $t \to \infty$, the efficient surplus $\mathcal{S}_t^*$ grows at least as quickly as $|N_t|$ asymptotically, that is, $\mathcal{S}_t^* = \Omega(|N_t|)$. Since utilities are bounded, equivalently, $\mathcal{S}_t^* = \Theta(|N_t|)$.*

Assumption 4 (Prices are bounded)**.** *There exists $M > 0$ such that $d_H\left(\sum_{n\in N_t} D_n(p), \sum_{f\in F_t} S_f(p)\right) \geq R$ for sufficiently large t and $p \in \mathbb{R}_+^L$ such that $\|p\| < \frac{1}{M}$ or $\|p\| > M$.*

Assumption 1 ensures that we may discuss meaningfully the mathematical properties of the mechanism as the number of agents becomes large. Assumption 2 prevents the economy from becoming more non-convex as the number of agents grow, while Assumption 3 reflects the natural economic assumption that the possible gains from trade grow with the size of the economy. Assumption 4 implies the compactness of the set of possible prices for which the aggregate demand and supply to be close to equal, so that pseudoequilibrium prices do not limit to zero or infinity as the economy grows. An alternative to Assumption 4 is to assume that all $u \in \mathscr{U}$ satisfy $\max_{x\in X\setminus\{0\}} \frac{u(x)}{\|x\|} < M$ and for all $c \in \mathscr{C}$, $\min_{y\in Y\setminus\{0\}} \frac{c(y)}{\|y\|} > \frac{1}{M}$.

The α^*-price mechanism is defined for an economy $\mathscr{E}$ as follows.

Definition 4.1 (α^*-price mechanism)**.** Let $\mathscr{E}$ be given. For $\alpha > 0$ to be chosen, define the convex program

$$\min_{p\in\mathbb{R}_+^L} \max_{x_n\in\mathrm{co}(X), y_f\in\mathrm{co}(Y)} \sum_{n\in N} \frac{\mathrm{cav}(u_n)(x_n)}{1+\alpha} - \sum_{f\in F} \mathrm{vex}(c_f)(y_f) - p \cdot \left(\sum_{n\in N} x_n + R1_L - \sum_{f\in F} y_f\right)$$

where 1_L is the vector of ones in $\mathbb{R}^L$. Let $(p^\alpha, \tilde{\omega}^\alpha)$ be a[19] solution to this program (it is a pseudoequilibrium of an economy with buyers' demand scaled down by $(1 + \alpha)$ and R units of each good demanded by the mechanism designer).

From $\tilde{\omega}^\alpha$, obtain, via Proposition 3, an allocation ω^α with $\|\omega^\alpha - \tilde{\omega}^\alpha\| \leq R$ such that $x_n^\alpha \in \max_{x\in X} \frac{1}{1+\alpha} u_n(x) - p \cdot x$ for each $n \in N$ and $y_f^\alpha \in S_f(p)$. By construction, this ω^α will be feasible in $\mathscr{E}$.

[19] If there are multiple solutions to the convex program, we suppose there is some fixed way of choosing a solution (e.g., the lexicographically smallest solution), but our argument will not depend on this.

Let α^* be the smallest α satisfying[20]

$$\alpha \sum_{n \in N} p^\alpha \cdot x_n^\alpha \geq p^\alpha \cdot \left(\sum_{f \in F} y_f^\alpha - \sum_{n \in N} x_n^\alpha \right). \tag{A}$$

Implement the allocation ω^{α^*} charging buyers the price $(1+\alpha^*)p^{\alpha^*}$ and paying sellers according to the price p^{α^*}.

A few properties of this mechanism follow directly from the definition. First, the mechanism is feasible and weakly budget-balanced.[21] Second, the allocation and payments are individually rational for each agent. For sellers, this follows since the price and production allocations obtain equal profits to the pseudoequilibrium[22] allocations of the adjusted convexified economy, in which the production bundles are individually optimal. For buyers, the pseudoequilibrium price p^α and consumption allocation in $\tilde{\omega}^\alpha$ satisfy

$$\frac{1}{1+\alpha} u_n(\tilde{x}_n^\alpha) - p^\alpha \cdot \tilde{x}_n^\alpha = \frac{1}{1+\alpha} u_n(x_n^\alpha) - p^\alpha \cdot x_n^\alpha \geq 0$$

so that $u_n(x_n^\alpha) - (1+\alpha)p^\alpha \cdot x_n^\alpha \geq 0$ as well.

Theorem 2, proved in Appendix B.1, concerns the approximate efficiency of this mechanism.

Theorem 2. *Let $\mathscr{E}_t$ be a sequence of economies satisfying Assumptions 1–4. Then the deadweight loss of the allocation associated with the α^*-price mechanism is $O(1)$ as $|N_t| \to \infty$, so that the relative inefficiency is $O\left(\frac{1}{|N_t|}\right)$.*

Remark. Although the rate of convergence in Theorem 2 is stated in terms of $|N_t|$, by Assumption 1, the same asymptotic rate of convergence holds with respect to $|F_t|$ or $|A_t|$.

4.3 Incentives

Under the α^*-price mechanism, although sellers receive their optimal bundle given the price and buyers receive a nearly optimal bundle, agents may have incentives to misreport preferences to

[20] Note that such an α^* always exists since as $\alpha \to \infty$, there is zero trade (which satisfies the inequality).

[21] Note that in general the mechanism *can* result in a budget surplus, because there may be no α that exactly equates the expressions in the inequality defining α^* (the expressions are not continuous, as p^α and ω^α depend on α in some complicated way).

[22] More accurately, because the constraint in the convex program defining $(p^\alpha, \tilde{\omega}^\alpha)$ is not exact market-clearing, this is not exactly a pseudoequilibrium, but we may imagine it as such under the interpretation that the mechanism designer acts as a consumer purchasing R units of each good.

influence the prices. In this section, we show that these reporting incentives are effectively the same as those of the usual Walrasian equilibrium mechanism: under certain assumptions on the underlying preferences, the benefit from deviating to an alternative report tends quickly to zero.

The **benefit of misreporting** for an agent in some mechanism is defined as the payoff obtained by the agent under their optimal report, under the assumption that all other agents report truthfully. A mechanism is **limiting incentive-compatible** in a sequence of economies $(\mathscr{E}_t)_{t\in\mathbb{N}}$ if the maximum benefit of misreporting among agents in $\mathscr{E}_t$ tends to zero as $t \to \infty$.

Because the prices obtained in the α^*-price mechanism can be viewed as Walrasian equilibrium prices of a related economy (and in the continuum limit the two mechanisms coincide), many of the key results on incentives in the Walrasian equilibrium can be adapted to the α^*-price mechanism. We now briefly discuss the key results on incentives in the Walrasian mechanism which we adapt to our setting.

For exchange economies, Roberts and Postlewaite (1976) showed that if the Walrasian price correspondence, mapping economies, viewed as measures μ_t on $\mathscr{U}$, to the set of Walrasian equilibrium prices, is continuous at the limiting measure $\lim_{t\to\infty} \mu_t$, then each agent's influence on the price must limit to zero as the number of agents approaches infinity. Jackson (1992) showed that an agent's optimal reported demand converges in the L^∞ norm to the demand associated with that agent's true preferences under the same continuity assumption. Continuity of the Walrasian price correspondence holds, for example, if the limiting economy has a unique clearing price. However, Roberts and Postlewaite (1976) provided an example of a sequence of economies where lower hemicontinuity fails at the limiting economy, and the Walrasian equilibrium mechanism is not limiting incentive-compatible.

Watt (2021) studied the rate of this convergence, showing that a condition on the demand and supply correspondences, called **strong monotonicity**, ensures the fast convergence of incentives in the Walrasian mechanism. [23]

Definition 4.2. The demand correspondence $D : \mathbb{R}^L_+ \rightrightarrows X$ is **strongly monotone** if for all p, p' and $d \in D(p)$, $d' \in D(p')$ we have

$$(p - p') \cdot (d' - d) \geq m\|p - p'\|^2 \text{ for some } m > 0.$$

[23] Furthermore, Watt (2021) showed that a slightly weakened version of the strong convexity property, called the local quadratic growth condition, is, in some sense, *necessary* for this rate of convergence. The incentive results in this paper can be easily adapted under this condition as well.

The supply correspondence $S : \mathbb{R}^L_+ \rightrightarrows Y$ is strongly monotone if for all p, p' and $s \in S(p)$, $s' \in S(p')$, we have

$$(p - p') \cdot (s - s') \geq m\|p - p'\|^2.$$

An economy is strongly monotone if both the demand correspondence and the supply correspondence are strongly monotone.

Strongly monotone demand correspondences generalize the concept of a *strictly* downward-sloping demand curve to setting with more than one good. If all agents have strongly monotone preferences, Watt (2021) shows that the influence of any one agent on prices is $O\left(\frac{1}{|N_t|}\right)$. Furthermore, if each buyer is drawn independently at random from some distribution μ over $\mathscr{U}$ and each seller is drawn independently at random from χ over $\mathscr{C}$ for which the *expected* demand and supply correspondences are strongly monotone and Lipschitzian,[24] the maximum benefit of misreporting under the Walrasian mechanism is $O\left(|N_t|^{-1+\varepsilon}\right)$ in expectation for all $\varepsilon > 0$.[25]

While the above results considered the incentives of agents under complete information, Azevedo and Budish (2019) studied interim incentives and show that the Walrasian mechanism (with a finite set of buyer and seller types) is **strategy-proof in the large**. This implies that the benefit of misreporting for any agent against any full-support, independent and identically-distributed distribution of agent types tends to zero at $O(|N_t|^{-\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$.

Theorem 3, proved in Appendix B.2, adapts these results to the α^*-price mechanism.

Theorem 3. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}} = (\langle N_t, \mu_t, \phi_t, \chi_t\rangle)_{t\in\mathbb{N}}$ be a sequence of economies satisfying Assumptions 1–4.*

(a) Let P be the set of Walrasian equilibrium prices for $\langle \mu_\infty, \phi_\infty, \chi_\infty\rangle$. Then the set P' of attainable prices for any agent in the α^*-price mechanism satisfies $d_H(P, P') \to 0$. In addition, if the α^*-price correspondence, mapping economies $\langle N, \mu, \phi, \chi\rangle$ into the set of prices under the α^*-price mechanism, is continuous[26] at the limit economy, then the mechanism is limiting incentive-compatible on $(\mathscr{E}_t)_{t\in\mathbb{N}}$.

(b) Suppose the type spaces $\mathscr{U}$ and $\mathscr{C}$ are finite sets and let full-support probability distributions μ and χ be defined (respectively) on them. Suppose that each buyer's type is drawn i.i.d. from $\mathscr{U}$ according to μ, the type of each seller is drawn i.i.d. from $\mathscr{C}$ according to χ, and the distributions μ and χ are

[24] A correspondence $D : \mathbb{R}^L_+ \rightrightarrows X$ is Lipschitzian if there exists an $L > 0$ such that for all p, p', $d_H(D(p), D(p')) \leq L\|p - p'\|$.

[25] Here the expectation is with respect to the measure over economies induced by draws of agents from μ and χ. We clarify the meaning of the "expected demand and supply correspondences" in Appendix B.2.

[26] We formalize both the α^*-price correspondence and the sense of continuity in Appendix B.2

common knowledge while realized types are private information. Then, for any agent, the expected benefit of misreporting under the α^*-price mechanism is $O(|N_t|^{-\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$.

(c) *Suppose that buyers are drawn i.i.d. from $\mathcal{U}$ according to some probability measure μ and that the type of each seller in F_t is drawn i.i.d. from $\mathcal{C}$ according to a probability measure χ (here $\mathcal{U}$ and $\mathcal{C}$ need not be finite) for which the expected demand and supply correspondences are strongly monotone and Lipschitzian. The realized types are known to agents but not to the auctioneer. In expectation over draws of the economy, the maximum benefit for misreporting for any agent is $O\left(|N_t|^{-1+\varepsilon}\right)$, for any $\varepsilon > 0$.*

4.4 Computational properties

In general, the problem of identifying a Walrasian equilibrium is known to be computationally hard. Even in the case of an exchange economy with additively separable, piecewise-linear and concave utility functions, Chen, Dai, Du, and Teng (2009) show that the problem of calculating Walrasian equilibrium is PPAD-complete.[27] This complexity class[28] has three important properties: a solution (e.g. a Walrasian equilibrium in convex economies) is known to exist, solutions may be easily verified (in deterministic polynomial time) but can be computationally intensive to identify. For Walrasian equilibrium, a solution is easily verified by checking that all buyers and sellers are optimizing with respect to the identified prices. These prices act as a "certificate of optimality" for the efficient allocation problem in convex economies.

In contrast, the efficient allocation problem in non-convex economies has no such certificate of optimality. This implies immediately that the problem of identifying efficient allocations in non-convex economies is in a higher complexity class than PPAD. This problem is of practical significance for market designers: for example, the problem of identifying an optimal allocation in the fisheries market of Bichler et al. (2018) involved solving a large integer programming problem, which is known to be NP-complete. No efficient algorithms are known for such problems, although a variety of heuristic approaches may make specific instances of such problems computationally tractable.

The α^*-price mechanism involves solving, for *fixed α*, a convex optimization problem, which

[27] PPAD stands for 'polynomial parity arguments on directed graphs', and is a complexity class containing many fixed point problems (including the approximation of Nash equilibria) and is conjectured to be harder than P. We refer the reader to Daskalakis, Goldberg, and Papadimitriou (2009) for further information.

[28] More accurately, these three properties roughly define the complexity class TFNP of which PPAD is the subclass in which the existence proof relies on a certain topological fixed point argument.

offers prices as a certificate of optimality. Therefore, for fixed α, known algorithms for the identification of Walrasian equilibria may be used, for example tâtonnement. In important special cases, for example, weak gross substitutes (Codenotti, McCune, & Varadarajan, 2005) and markets with strongly monotone preferences (Watt, 2021), tâtonnement may be used to efficiently approximate Walrasian equilibrium. The identification of α^* may be more challenging, although a bisection search algorithm might be employed, in practice, to identify *some* α that satisfies equation (A) for sufficiently large economies (and therefore, *some* approximately-efficient allocation). Unfortunately, we are not able to identify theoretical guarantees related to the computation of α^*, the *smallest* such α, although the proof of Theorem 2 provides an upper bound on the search space for α^*.

4.5 Alternative mechanisms

The α^*-price mechanism is suggestive of several alternative mechanisms that could be employed in non-convex economies. The first is the **maximum surplus anonymous pricing** mechanism which was introduced by Bichler et al. (2018) and adapted for use in an exchange for fisheries licenses in New South Wales. The maximum surplus anonymous pricing mechanism solves the surplus optimization problem

$$\max_{\omega \in \Omega} \sum_{n \in N} u_n(x_n) - \sum_{f \in F} c_f(y_f),$$

subject to the constraint that there exist prices p_N and p_F satisfying

(a) Individual rationality: for all $n \in N, f \in F$, $u_n(x_n) - p_N \cdot x_n \geq 0$ and $p_F \cdot y_f - c_f(y_f) \geq 0$.

(b) Budget balance: $\sum_{n \in N} p_N \cdot x_n \geq \sum_{f \in F} p_F \cdot y_f$.

Another possibility is be the **maximum surplus $\alpha-$price mechanism**, which imposes the additional constraint that $p_N = (1+\alpha)p_F$ for some $\alpha > 0$.

A corollary of Theorem 2 is that the deadweight loss of both of these mechanisms is $O(1)$ in $|N|$. However, we have focused our analysis on the α^*-price mechanism for several reasons. Firstly, the alternative mechanisms require the solution of non-convex optimization problems, while the α^*-mechanism involves an inner loop of a convex optimization problem (for fixed α). This offers a theoretical benefit of obtaining prices as the dual variables of a convex optimization problem (as in Walrasian equilibrium) and may have the added advantage of improved compuation, as discussed in Section 4.4. Secondly, optimizers of the alternative mechanisms may involve rationing at the prevailing prices, which may result in inferior incentive performance. With rationing, there may be an incentive to misreport to affect not only prices but also the quantity received. We show that it is possible to obtain many of the desirable properties of Walrasian equilibrium in non-convex economies without the need to ration agents at the prevailing prices. In the next section, we will identify some settings in which rationing may be employed without substantial adverse effects on incentives.

5 One-sided non-convexities: the single-price mechanism with rationing

In this section, we consider the case where one side of the market—for simplicity, the buyers—has convex preferences, while the other side of the market has non-convex preferences. There are two reasons to conjecture that convexity on one side of the market might lead to more efficient pricing mechanisms. The first is that convexity on one side of the market makes it possible to obtain exact market-clearing outcomes via prices, since all feasible bundles are in the exposed set for the convex side of the market.[29] This eliminates one possible source of inefficiency in the market mechanism: unused supply.[30]The second reason is that in many familiar models, faster rates of convergence have been identified. For example, in the double auction model of Rustichini, Satterthwaite, and Williams (1994), the expected loss under the mechanism is $O\left(\frac{1}{|N|}\right)$, while simulation studies of the maximum surplus anonymous pricing mechanism due to Bichler et al. (2018) suggested similarly small deadweight losses.

We show that a **single-price mechanism with rationing** can lead to rapid convergence to efficiency under certain additional assumptions on buyers' preferences.

Definition 5.1 (Single-price mechanism with rationing)**.** Let $(p, \tilde{\omega})$ be a solution to

$$\min_{p \in \mathbb{R}^L_+} \max_{\omega \in X^{|N|} \times Y^{|F|}} \sum_{n \in N} u_n(x_n) - \sum_{f \in F} \mathrm{vex}(c_f)(y_f) - p \cdot \left(\sum_{n \in N} x_n - \sum_{f \in F} y_f - R_{\mathscr{E}} \right). \tag{SPM}$$

Using Proposition 3, identify ω satisfying $\|\tilde{\omega} - \omega\| \leq R_{\mathscr{E}}$ with $y_f \in S_f(p)$ for each $f \in F$ and let $s = \sum_{f \in F} y_f$ be the resulting supply vector.

[29]That is, given any feasible consumption bundle x, there is some price p such that agent n demands x at p.

[30]Note that in order to apply Theorem 1 with a single price and obtain an $O\left(\frac{1}{|N_t|}\right)$ bound, we effectively require that the identified allocation is exactly market-clearing, else the budget deficit term in the First Welfare Bound is $O(1)$.

Let $\mathbf{x}' = (x'_n)_{n \in N}$ solve

$$\max_{(x_n)_{n \in N} \in X^N} \sum_{n \in N} u_n(x_n) \text{ subject to } \sum_{n \in N} x_n \leq s. \tag{SPM2}$$

The single-price mechanism with rationing purchases y_f from seller f at price p, and allocates x'_n to buyer n at price p.

As in the α^*-price mechanism, we use the pseudoequilibrium of a perturbed economy to identify a price vector. The key difference to the α^*-price mechanism is that in this case we solve for the pseudoequilibrium with an additional R units of *supply* rather than demand. This allows us to obtain, by applying Proposition 3, an allocation ω at which all agents optimize with respect to p, but with (at most) $2R$ units of excess demand. While we previously sought to avoid excess demand, here we use the existence of excess demand to our advantage. We then ration buyers in order to attain exact market-clearing. To do so in the least-cost way, we calculate the most efficient allocation of the fixed supply vector to buyers. If the dual variables of this optimization problem are close to the first price identified, then buyers will be approximately optimizing at the new allocation according to the first prices identified. Under the assumption of strong monotonicity, Watt (2021) shows that the effect of a small change in supply on the price vector is $O\left(\frac{1}{|N_t|^{1-\varepsilon}}\right)$. The intuition is that only buyers with values near the prices will be rationed so that the resulting allocation is, via Theorem 1, approximately-efficient. We formalize this result in Theorem 4, which is proved in Appendix B.3.

Theorem 4. *Let $(\mathscr{E}_t)_{t \in \mathbb{N}}$ be a sequence of economies satisfying Assumptions 1–4 and that all buyers have convex preferences. Suppose in addition that the type of each buyer is drawn i.i.d. from $\mathscr{U}$ according to some probability measure μ and that the type of each seller in F_t is drawn i.i.d. from $\mathscr{C}$ according to a probability measure χ (here $\mathscr{U}$ and $\mathscr{C}$ need not be finite) for which the expected demand and supply correspondences are strongly monotone and Lipschitzian. Then the expected deadweight loss of the allocation associated with the single-price mechanism with rationing is $O\left(|N_t|^{-1+\varepsilon}\right)$ for any $\varepsilon > 0$, so that the relative inefficiency is $O\left(|N_t|^{-2+\varepsilon}\right)$. The expected maximum benefit of misreporting for any agent under the mechanism is $O\left(|N_t|^{-1+\varepsilon}\right)$ and the mechanism is individually rational for all sellers and $O\left(|N_t|^{-1+\varepsilon}\right)$-individually rational for buyers. If the buyers' preferences are, in addition, separable across goods, the mechanism is exactly individually rational for buyers.*

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A Non-convexity and approximate equilibria

We begin with a slightly stronger statement of the Shapley-Folkman Lemma that is used in general equilibrium theorem with non-convexities.

Proposition 4. *Let $S_i \subseteq \mathbb{R}^L$ for $i = 1, ..., N$, $S = \bigoplus_{i=1}^N S_i$ and $L' = \min(L, N)$. Then for any $x \in \mathrm{co}(S)$:*

(a) (Shapley-Folkman Lemma) $x = \sum_{i=1}^N x_i$ where $x_i \in \mathrm{co}(S_i)$ and $|i : x_i \in \mathrm{co}(S_i) \setminus S_i| \leq L'$.

(b) (Starr, 1969) If S_i is ordered so that $r(S_i)$ is nonincreasing in i, then there is $y \in S$ such that $|x - y| \leq \sqrt{\sum_{i=1}^{L'} r(S_i)^2}$.

(c) (Heller, 1972) If S_i is ordered so that $\rho(S_i)$ is nonincreasing in i, then there is $y \in S$ such that $|x - y| \leq \sqrt{\sum_{i=1}^{L'} \rho(S_i)^2}$.

These results have been used in the general equilibrium context to obtain **approximate equilibria**, which are price-allocation pairs (p, ω) such that $x_n \in D_n(p)$ for all n, $y_f \in S_f(p)$ for all f but $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq s$ for some $s > 0$. In particular, the allocation associated with an approximate equilibrium may have excess demand and therefore be infeasible. The approximate equilibrium is obtained by identifying the competitive equilibrium associated with a convexified version of the economy (in which each agent's upper contour set is replaced by its convex hull) and applying the results of Proposition 4 to the resulting allocation. The approximate equilibrium analogues of Proposition 4 are contained in Proposition 5 below.

Proposition 5. *For economy $\mathscr{E} = (N, F)$:*

(a) (Starr, 1969) There is $\omega \in \mathrm{co}(\Omega)$ and $p \in \mathbb{R}_+^L, p > 0$ such that $|n : x_n \in \mathrm{co}(X)| + |f : y_f \in \mathrm{co}(Y)| \leq L$ and for all other agents, $x_n \in D_n(p)$ and $y_f \in S_f(p)$.

(b) Let $r_n = \sup_{\bar{u} \in \mathbb{R}} r(UC_n^{\bar{u}})$ and $r_f = \sup_{\bar{\pi} \in \mathbb{R}} r(UC_f^{\bar{\pi}})$. Let $\delta \geq 0$ satisfy $r_n \leq \delta$ for all $n \in N$ and $r_f \leq \delta$ for all $f \in F$. Then there exists $p \in \mathbb{R}_+^L, p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_f(p)$ for all $f \in F$ and $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq \delta\sqrt{L}$.

(c) Let $\rho_n = \sup_{\bar{u} \in \mathbb{R}} \rho(UC_n^{\bar{u}})$ and $\rho_f = \sup_{\bar{\pi} \in \mathbb{R}} \rho(UC_f^{\bar{\pi}})$. Let $\delta' \geq 0$ satisfy $\rho_n \leq \delta'$ for all $n \in N$ and $\rho_f \leq \delta'$ for all $f \in F$. Then there exists $p \in \mathbb{R}_+^L, p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_f(p)$ for all $f \in F$ and $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq \delta' L$.

The statement of Proposition 5(a) is standard, but the statements of Proposition 5(b) and (c) are stronger than the classical statement due to Starr (1969) and Heller (1972). Again, the quasi-linearity of agent preferences allows us to conclude that agents are utility- and profit-maximizing, rather than just expenditure-minimizing.

Finally, we introduce a more general class of quasilinear preferences to which many of our results also apply and which offer a meaningful interpretation in terms of perceived complementarity and substitutability of goods. Nguyen and Vohra (2020) introduced the concept of the **generalized Δ-single improvement property**, which is a generalization of the well-known single improvement property.

Definition A.1. The preferences of buyer $n \in N$ satisfies the generalized Δ-single improvement property for some $\Delta > 0$ if for any price vector $p > 0$, any two bundles $x, y \in D_n(p)$ and any price change δp such that $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x-y)^+$ and $b \leq (y-x)^+$ such that:

(a) $|a| + |b| \leq \Delta$

(b) $\delta p \cdot a > \delta p \cdot b$, and

(c) $x - a + b \in D_n(p)$.

Here $(x-y)^+$ denotes the vector whose ℓ^{th} component is $\max(x_\ell - y_\ell, 0)$.

The Δ in this definition captures a measure of the substitutability and complementarity between goods. Preferences with the single improvement property of Gul and Stacchetti (1999) are contained in the class with $\Delta = 2$.

By our assumption on the compactness of X and Y, all preferences and technologies satisfy the general Δ-improvement property for some Δ. But the following stronger relationship between the inner radii of preferences and the Δ-single improvement property also holds.

Proposition 6. *Let $r_n = \sup_{\bar{u} \in \mathbb{R}} r(UC_n^{\bar{u}})$. Then the preferences of buyer $n \in N$ satisfy the generalized $\Delta-$single improvement property for all $\Delta > 2\sqrt{2}r_n$.*

Proof. Let the preferences of buyer n satisfy $r_n = \sup_{\bar{u} \in \mathbb{R}} r(UC_n^{\bar{u}})$. Let $x, y \in X$ and $p \in \mathbb{R}_+^L$ be given such that $x, y \in D_n(p)$. Suppose $|(x-y)^+| + |(y-x)^+| \geq 2r_n$ (else the preferences immediately satisfy the Δ improvement property for $\Delta = 2r_n$).

For any $\epsilon > 0$, let $z \in \mathbb{R}_+^L$ be the unique convex combination of x and y such that $|x - z| = r_n + \epsilon$ and write $z = \lambda x + (1-\lambda)y$. By construction $(z, p \cdot z) \in \text{co}(UC_n^{u_n(x) - p \cdot x})$. Then by the bound on the non-convexity of the preferences, there is a set $T \subseteq UC_n^{u(x)-p\cdot x}$ with $\text{rad}(T) \leq r_n$ such that $(z, p\cdot z) = \sum_{(x',t')\in T} \alpha_{(x',t')}(x',t')$ where $\sum_{(x',t')\in T} \alpha_{(x',t')} = 1$.

We now argue that for all $(x',t') \in T$, $x' \in D_n(p)$ and $t' = p \cdot x'$. To see this, note that $x \in D_n(p)$ implies $u_n(x') - p\cdot x' \leq u_n(x) - p \cdot x$. Summing, we have

$$\begin{aligned} u_n(x) - p\cdot x &\geq \sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - p\cdot x'] \\ &= \sum_{(x',t')\in T} \alpha_{(x',t')} u_n(x') - p \cdot z \\ &= \sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - t'] \end{aligned}$$

On the other hand, since $(x',t') \in UC_n^{u(x)-p\cdot x}$ we have $u_n(x') - t' \geq u(x) - p\cdot x$. The only way these can simultaneously hold is if $u_n(x') - t' = u(x) - p \cdot x$ for all $(x',t') \in T$.

However, we then have $\sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - p\cdot x'] = u(x) - p\cdot x$. This implies that at least one of $u_n(x') - p\cdot x' \geq u_n(x) - p\cdot x$. But then $x \in D_n(p)$ implies that $u_n(x') - p\cdot x' = u_n(x) - p\cdot x$ for all x', so $x' \in D_n(p)$.

By construction, $|x - x'| \leq 2r_n + \epsilon$. But then $||x - x'||_1 \leq 2\sqrt{2}r_n + \epsilon$ as well. □

Clearly, the generalized Δ-single property can be readily extended to sellers, by replacing the expressions for utility with those for profits, and an analogue of Proposition 6 also holds.

Nguyen and Vohra (2020) demonstrate the following approximate equilibrium result in a setting with indivisibilities (so that $X \subseteq \mathbb{Z}_+^L$ and $Y \subseteq \mathbb{Z}_+^L$).

Proposition 7. *Suppose all buyers' preferences and sellers' technologies satisfy the generalized Δ-improvement property and that $X \subseteq \mathbb{Z}_+^L$ and $Y \subseteq \mathbb{Z}_+^L$. Then there exists $p \in \mathbb{R}_+^L$, $p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_n(p)$ and for each $\ell \in L$, $\left|\sum_{n\in N} x_{nl} - \sum_{f\in F} y_{fl}\right| \leq \Delta - 1$.*

Note that the concept of approximate equilibrium in this result is somewhat stronger than the previous results since the maximum imbalance in supply and demand is bounded good-by-good, rather than in terms of Euclidean distance in commodity space. However, depending on the relative size of Δ, the inner radii of non-convexity and the breadths of non-convexity of preferences, any of the approximate equilibrium bounds in Proposition 5(b), 5(c) or 7 may be strongest for our purposes.

B Proofs omitted from the main text

B.1 Proof of Theorem 2

Theorem 2 is proved in two parts. First, in Lemma 1, we establish that the premium α_t^* is $O\left(\frac{1}{|N_t|}\right)$. Then we use Theorem 1 to show that this implies that the loss of the associated allocation is $O(1)$.

Lemma 1. *Under Assumptions 1–4, the α_t^* in Definition 4.1 is $O\left(\frac{1}{|N_t|}\right)$.*

Proof. For notational simplicity we drop the index for t in the prices, premiums and allocations.

The construction in Definition 4.1 ensures, via Proposition 3, that $\sum_{f\in F} y_f^\alpha - \sum_{n\in N} x_n^\alpha \leq (2R)1_L$. Thus, it suffices to show that for sufficiently large $|N_t|$, there is an α such that

$$\alpha \sum_{n\in N} p^\alpha \cdot x_n^\alpha \geq (2R)p^\alpha \cdot 1_L.$$

Moreover, if this $\alpha = O\left(\frac{1}{|N_t|}\right)$, then since $\alpha^* < \alpha$, Lemma 1 will follow. To arrive at this result, we will show that for fixed $\alpha > 0$ close enough to zero, $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$, while $p^\alpha \cdot 1_L$ is $O(1)$.

Let $\mathcal{S}^\alpha$ be the value of the saddle point problem

$$\min_{p\in\mathbb{R}_+^L} \max_{\omega\in\Omega} \sum_{n\in N} \frac{\mathrm{cav}(u_n)(x_n)}{1+\alpha} - \sum_{f\in F} \mathrm{vex}(c_f)(y_f) - p\cdot\left(\sum_{n\in N} x_n + R1_L - \sum_{f\in F} y_F\right).$$

First, note that $\mathcal{S}^\alpha$ is $\Theta(|N_t|)$ for sufficiently small α by Assumption 3. To see this, denote by $f(|N_t|) = \sum_{n\in N_t} u_n(x_n^*)$ and $g(|N_t|) = \sum_{f\in F_t} c_f(y_f^*)$. Then by Assumption 3 and the definition of $\Theta(\cdot)$, $\liminf_{N\to\infty} \frac{f(N)}{N} = u > 0$, say, and $\limsup_{N\to\infty} \frac{g(N)}{N} = c > 0$, with $u - c > 0$. Then $\mathcal{S}^\alpha \geq \liminf_{N\to\infty} \frac{f(N)}{(1+\alpha)N} - g(N) = \frac{u}{1+\alpha} - c$, which is positive for sufficiently small α.

Now we show that this implies $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$ for small, fixed α. To see this, note that since $\sum_{f\in F_t} c_f(y_f^\alpha)$ is $\Omega(|N_t|)$, individual rationality of the sellers (in the perturbed economy) implies that $\sum_{f\in F_t} p^\alpha \cdot y_f^\alpha$ is $\Omega(|N_t|)$. But then by complementary slackness $\sum_{n\in N_t} p^\alpha \cdot x_n^\alpha = \sum_{f\in F_t} p^\alpha \cdot y_f^\alpha - Rp^\alpha \cdot 1_L$, and since Assumption 4 implies $\|p\| \leq M$, we must have that $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$.

Since for α near zero, $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$ and $(2R)p^\alpha \cdot 1_L$ is $O(1)$, for sufficiently large $|N_t|$, there is a least α such that

$$\alpha \sum_{n\in N} p^\alpha \cdot x_n^\alpha \geq (2R)p^\alpha \cdot 1_L,$$

and furthermore, this α is $O\left(\frac{1}{|N_t|}\right)$. Since $\alpha^* < \alpha$, we have that α^* is $O\left(\frac{1}{|N_t|}\right)$, as required. $\square$

We now apply the First Welfare Bound to show that the allocation ω^{α^*} is approximately efficient. In order to satisfy the assumptions on prices in Theorem 1, we imagine ω^{α^*} was implemented with a single price vector p^{α^*} and (therefore) a budget deficit. Theorem 1 tells us that

$$\mathcal{S}(\omega^*) - \mathcal{S}(\omega^{\alpha^*}) \leq \mathcal{R}(p^{\alpha^*}, \omega^{\alpha^*}) + p^{\alpha^*} \cdot \left(\sum_{f \in F_t} y_f^{\alpha^*} - \sum_{n \in N_t} x_n^{\alpha^*} \right). \tag{1}$$

By construction, in ω^{α^*} at prices p^{α^*}, no sellers are rationed, while at prices $(1+\alpha^*)p^{\alpha^*}$, no buyers are rationed. We now use Lemma 1 to show that since α^* is small for large $|N_t|$, that the total rationing of buyers at (counterfactual) price p^{α^*} is $O(1)$. To do so, we prove in Lemma 2 that $\mathcal{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O\left(\frac{1}{|N_t|}\right)$ so that $\sum_{n \in N_t} \mathcal{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O(1)$.

Lemma 2. *Let (p, ω) be a competitive equilibrium and $p' \gg p$ be an alternative price. Then the rationing loss of buyer n at allocation ω given price p' is $O(\|p - p'\|)$*

Proof. Let x_n be buyer n's consumption bundle in ω. Since (p, ω) is a competitive equilibrium, we have that $x_n \in D_n(p)$, the agent's demand correspondence at price p. By the definition of rationing,

$$\begin{aligned}
\mathcal{R}_n(p', x_n) &= \hat{u}_n(p') - (u_n(x_n) - p' \cdot x_n) \\
&= \hat{u}_n(p') - (u_n(x_n) - p \cdot x_n) - p \cdot x_n + p' \cdot x_n \\
&= \hat{u}_n(p') - \hat{u}_n(p) + (p' - p) \cdot x_n.
\end{aligned}$$

Now let $p(t) = (1-t)p + tp'$ for $t \in [0,1]$ and consider the parametrized utility maximization problem

$$\hat{u}_n(p(t)) = \max_{x \in X} u_n(x) - p(t) \cdot x.$$

The conditions for the Milgrom and Segal (2002) envelope theorem hold, namely the objective function is differentiable in t given any $x \in X$ and the demand correspondence is always non-empty. The derivative of the objective with respect to t is $(p - p') \cdot x$, which is bounded (since X is assumed compact)

Then for any selection of $d(t) \in D_n(p(t))$, we have that

$$\hat{u}_n(p') = \hat{u}_n(p) - \int_0^1 (p' - p) \cdot d(t)dt \leq \hat{u}_n(p) - (p' - p) \cdot d',$$

for any d' in $D_n(p')$ since D_n is nonincreasing along $p(t)$.

Substituting into the expression for $\mathscr{R}_n$ above, we obtain

$$\mathscr{R}_n(p', x_n) \leq (p' - p) \cdot (x_n - d').$$

With no assumptions on demand, we see that $\mathscr{R}_n(p', x_n)$ is $O(\|p - p'\|)$, since the consumption set is bounded. $\square$

Since α^* is $O\left(\frac{1}{|N_t|}\right)$ and $\|p^{\alpha^*}\|$ is bounded, this implies that $\mathscr{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O\left(\frac{1}{|N_t|}\right)$, and so $\mathscr{R}(p^{\alpha^*}, \omega^{\alpha^*})$ is $O(1)$.

Finally, we note that the budget deficit (the second term on the right of Equation (1)) is $O(1)$ since the excess supply is bounded by construction and each component of p^{α^*} is $O(1)$ (as argued previously). Thus Theorem 2 follows.

B.2 Proof of Theorem 3

Most of the results in Theorem 3 follow by simple application of the corresponding theorems for Walrasian equilibrium, noting that at the limiting economy, the prices determined by the Walrasian mechanism and the α^*-price mechanism coincide (the R units purchased by the mechanism designer do not matter in the limit). Moreover, since α^* is $O(1/|N_t|)$, agents' misreports have little influence on α^* in large markets as well. Therefore, any conditions that ensure that agents have diminishing influence over prices in Walrasian equilibrium imply that agents have diminishing influence over prices in the α^*-price mechanism. Since the indirect utility function is locally Lipschitz (and represents an upper bound on the utility achievable under any price change caused by misreporting), upper bounds on changes in the price (and α^*) imply upper bounds on increases in utility. We will only provide sufficient details to make clear the adaptations of the arguments required for the α^*-price mechanism and refer readers to Roberts and Postlewaite (1976), Azevedo and Budish (2019) and Watt (2021) for more details.

Part (a) Let Q be the Walrasian equilibrium correspondence mapping normalized economies $\langle \mu, \phi, \chi \rangle$ to Walrasian prices, which we assume is continuous at the limiting normalized economy of $(\mathscr{E}_t)_{t\in\mathbb{N}}$, $\langle \mu_\infty, \phi_\infty, \chi_\infty \rangle$. Let Q_α map economies $\langle N, \mu, \phi, \chi \rangle$ to the set of (seller) prices under the $\alpha-$price mechanism for fixed $\alpha > 0$. By dividing the objective in Definition 4.1 by $|N|$ and rewriting, $Q_\alpha(\langle N, \mu, \phi, \chi \rangle)$ are the solutions to the normalized dual problem

$$\min_{p\in\mathcal{Q}} \left\{ \int_{\mathscr{U}} \frac{\hat{u}_n\left(\frac{p}{1+\alpha}\right)}{1+\alpha} d\mu + \phi \int_{\mathscr{C}} \hat{\pi}_f(p) d\nu - \frac{p \cdot R1_L}{|N|} \right\},$$

where $\mathcal{Q} = \{p \in \mathbb{R}^L_+ : \frac{1}{M} \leq \|p\| \leq M\}$ (the optimizers in $\mathbb{R}^L_+$ lie in $\mathcal{Q}$ by Assumption 4). Note that in the limit as $t \to \infty$, since $|N| \to \infty$ and $\alpha \to 0$, the objective of Q_{α_t} converges pointwise and therefore uniformly on P (by compactness) to the dual objective of the Walrasian equilibrium problem (even if a single agent misreports). But this implies that $d_H(P, P') \to 0$ (by, for example, Sendov (1990) Corollary 2.1).

To discuss continuity of Q_α, we need to describe a topology on the domain of Q_α, which is the product space $\mathbb{R}_+ \times (\mathbb{N} \cup \{\infty\}) \times \mathscr{P}(\mathscr{U}) \times \mathbb{R}_+ \times \mathscr{P}(\mathscr{C})$ (corresponding to $\alpha, N, \mu, \phi, \chi$), where $\mathscr{P}(\mathscr{M})$ is the set of probability measures on a measurable space $\mathscr{M}$. We use the product metric, where we use the Lévy-Prokhorov metric[31] on $\mathscr{P}(\mathscr{U})$ and $\mathscr{P}(\mathscr{C})$, the usual metric on $\mathbb{R}_+$ and the metric on $\mathbb{N} \cup \infty$ given by $d(M, N) = \left| \frac{1}{M} - \frac{1}{N} \right|$. We now assume that Q_α is continuous at $(0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$ in this product topology. Since for the true economy $(\alpha_t, |N_t|, \mu_t, \phi_t, \chi_t) \to (0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$ in the product metric and for any misreport of a single agent, the resulting $(\alpha'_t, |N_t|, \mu'_t, \phi_t, \chi'_t) \to (0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$, the triangle inequality will imply that the resulting prices will be arbitrarily close in the Hausdorff metric (see Roberts and Postlewaite (1976) for this argument). Finally, noting that for each agent $\hat{u}_n(p)$ and $\hat{\pi}_f(p)$ are (locally) Lipschitz continuous functions of p, this implies that the maximum benefit of misreporting must tend to zero as well.

Part (b) This follows by simply noting that the α^*-price mechanism is envy-free, and so Theorem 1 of Azevedo and Budish (2019) implies the result.

[31] Recall the Lévy-Prokhorov metric on $\mathscr{P}(\mathscr{M})$ on measurable space $\mathscr{M}$ with Borel sigma algebra $\mathscr{B}(\mathscr{M})$ is defined by $\rho(\nu, \mu) = \inf \left\{ \varepsilon > 0 \mid \nu(E) \leq \mu\left(B_\varepsilon(E)\right) + \varepsilon \text{ and } \mu(E) \leq \nu\left(B_\varepsilon(E)\right) + \varepsilon \text{ for every Borel set } E \subseteq \mathscr{B}(\mathscr{M}) \right\}$, where $B_\varepsilon(E)$ denotes the ε -neighborhood of E

Part (c) First we formally define the expected supply and demand correspondences, using the concept of a set-valued integral due originally to Aumann (1965). For any fixed p, the probability measure μ induces a probability measure over sets $D_n(p)$ (associated with the utility function $u_n \in \mathscr{U}$). A selection $\xi : \mathscr{U} \to X$ is a single-valued random vector such that $\xi(u_n)$ almost surely belongs to $D_n(p)$ for each $u_n \in \mathscr{U}$. Then $\mathbb{E}_\mu[D_n(p)]$ is defined as $\text{cl}(\{\mathbb{E}\xi\})$ for integrable selections ξ. We assume that the monotonicity property applies to this correspondence (and analogously for the supply correspondence).

Firstly, note that regardless of the report of any single agent (since their total demand is bounded), Lemma 1 applies so that α^* is $O(1/|N_t|)$. Secondly, conditional on any α, the prices are determined as the Walrasian equilibrium of a related economy, and Watt (2021) shows that under the strong monotonicity assumption, any single agent's misreport has at most a $O(|N_t|^{-1+\varepsilon})$ influence on the resulting price. This implies that the total influence on price of any agent is $O(|N_t|^{-1+\varepsilon})$ for any $\varepsilon > 0$. Since the indirect utility (or consumer surplus) functions of any agent is locally Lipschitz and agents will do no better under any misreport causing prices to change to p than their indirect utility at p, this implies the resulting impact on agent payoffs is $O(|N_t|^{-1+\varepsilon})$.

B.3 Proof of Theorem 4

Under strong monotonicity, Theorem 2 in Watt (2021) implies that the price p from the saddle point of Equation (SPM) and the shadow price p' which is the dual variable associated with Equation (SPM2) satisfy $\|p - p'\| \leq O\left(|N_t|^{-1+\varepsilon}\right)$ for all $\varepsilon > 0$.

Then by the expression obtained for $\mathscr{R}_n(p, x_n')$ in Lemma 2, we have for all $d_n \in D_n(p)$ and $d_n' \in D_n(p')$

$$\mathscr{R}_n(p, x_n') \leq (p' - p) \cdot (d_n - d_n').$$

Note that demand selections $d_n \in D_n(p)$ and $d_n' \in D_n(p')$ can be chosen to satisfy $\sum_n d_n = s + R1_L$ and $\sum_n d_n' = s$ by complementary slackness and the monotonicity of buyers' preferences (if these equations are not satisfied for some good for some demand selections, complementary slackness implies that the price of that good is zero and any agent would be willing to accept the excess supply of that good for free by monotonicity). Thus, by summing, we have

$$\mathscr{R}(p, x_n') \leq (p' - p) \cdot R1_L \leq O\left(|N_t|^{-1+\varepsilon}\right).$$

Employing Theorem 1 with the prices and allocation from the single-price mechanism with rationing (noting that it is budget-balanced and no sellers are rationed) thus implies that the deadweight loss is $O\left(|N_t|^{-1+\varepsilon}\right)$.

The proof of incentive-compatibility is almost identical to the proof of Theorem 3(c), with the added complication that a buyer may have an incentive to misreport in order to avoid rationing. However, since each buyer's rationing is $O\left(|N_t|^{-1+\varepsilon}\right)$, the maximum expected benefit of misreporting is $O\left(|N_t|^{-1+\varepsilon}\right)$ as well (comprising the $O\left(|N_t|^{-1+\varepsilon}\right)$ capacity to influence prices and the $O\left(|N_t|^{-1+\varepsilon}\right)$ maximum benefit of avoiding rationing conditional on any price). Finally, the approximate individual rationality follows from the observation that $x_n' \in \max_x u_n(x) - p' \cdot x$ so that $u_n(x_n') - p' \cdot x \geq 0$. Then $u_n(x_n') - p \cdot x = u_n(x_n') - p' \cdot x + (p' - p) \cdot x \geq (p' - p) \cdot x$, which may be less than zero. However, since $\|p' - p\|$ is $O\left(|N_t|^{-1+\varepsilon}\right)$ and $\|x\| \leq O(1)$, at worst we have that the resulting allocation is $O\left(|N_t|^{-1+\varepsilon}\right)$ $-$individually rational.