UNIVERSITY OF SOUTHAMPTON

Faculty of Engineering and Physical Sciences
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An Investigation into Holographic Entanglement Entropy PHYS6006

by

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Abstract

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We reproduce the equation of Ryu and Takayanagi's proposal for the holographic entanglement entropy. This uses the Ads/CFT correspondence to find an equation in the Ads space to get the entanglement entropy in the conformal field theory. This equation is a simple formula using the minimal area surface of the Ads space $S_{ee} = \frac{A_{min}}{4G_N^{(d+1)}}$. The solution for the CFT in two dimensions is well known, and the findings of the holographic formula agrees well with that solution. We produce a d-dimentional version of these with two different shapes on the bulk, a strip and a sphere. With the sphere using the conformal symmetries of the Ads space After this we found we needed numerical methods to add a balck hole to find the thermal state.

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Chapter 1

Introduction

There have been many advances in string theory over the last few decades, one of which is the theory of holography. This idea of holography claims that a d+1 dimensional quantum gravity has a direct duality to a d dimensional system of many quantum bodies called a conformal field theory (CFT). This idea was inspired by the Bekenstein-Hawking formula[2, 15]:

$$S_{BH} = \frac{Area}{4G_N} \tag{1.1}$$

where G_N is Newton's constant. This shows the entropy of a black hole is not proportional to its volume but to its area of the event horizon, while Ads/CFT[21] proposes d+1 dimensional Ads (anti-De-Sitter; the maximally symmetric space-time of constant negative curvature) has a direct duality to a d dimensional conformal field theory. A conformal field theory is a quantum field theory which conforms to conformal symmetries. These symmetries are translation, inversion, scale and Lorentz transformations.

As of late, Ads/CFT correspondence is just a conjecture and has no formal proof (its underlying mechanism being unknown) but has been extremely rigorously tested. This has a vast number of uses in condensed matter physics and QCD.

In condensed matter physics, entanglement entropy has emerged as a valuable tool for characterising and quantifying quantum correlations in many-body systems. Strongly correlated systems, such as those found in high-temperature superconductors, topological materials, and strongly interacting gases, exhibit rich quantum behaviours that are challenging to understand using traditional methods. Entanglement entropy provides a unique perspective for studying such systems, allowing researchers to probe the entanglement structure and correlations between different regions of a many-body system. This has led to important discoveries, such as identifying critical points, topological phase transitions, and quantum phase transitions, which have deepened our understanding of the underlying physics of condensed matter systems.[9]

Moreover, entanglement entropy has been utilised in designing quantum protocols for quantum computing, a rapidly developing field that leverages the principles of quantum mechanics to process information in a fundamentally different way from classical computers. Entanglement is a crucial resource in quantum computing, enabling quantum gates, error correction, and communication. Understanding the entanglement properties of quantum states is essential for optimising quantum algorithms, improving quantum error correction codes, and developing novel quantum information processing tasks. Entanglement entropy provides a quantitative measure of the entanglement content of quantum states, which is vital for characterising and optimising the performance of quantum computing devices[17].

The combination of entanglement entropy and AdS/CFT has been successfully applied to various condensed matter systems, including strongly correlated systems such as quantum critical points, topological phases, and quantum phase transitions. For example, the AdS/CFT correspondence has been used to study the entanglement properties of quantum Hall states[10], topological insulators, and spin liquids[7], shedding light on their exotic quantum behaviours. Additionally, the use of entanglement entropy and AdS/CFT has been employed in designing novel quantum computing protocols, such as the holographic quantum error correction codes, which exploit the holographic duality to improve the resilience of quantum information against errors.

In this paper, we describe findings by Shinsei Ryu and Tadashi Takayanagi[25]; they discuss holography in terms of entanglement entropy. The entanglement entropy, which we use the Von Neumann entropy when applied to a reduced density matrix, evaluates the amount of entanglement Between subsystems in a quantum system. We have a density matrix that traces out the part of the system to give our value of entanglement between the two subsystems. If we split up our system into A and B, we say this is the information for observer A who cannot access subsystem B as the information is lost during the tracing. Tantalisingly this leading divergence term of our entropy is proportional to the area leading Ryu and Takayanagi to propose the holographic law 5.3 with a (not rigorous) proof that has been developed by Maldacena [20] (the person who originally proposed the Ads/CFT correspondence). This law has us find the minimal area surface of an Ads space-time. This paper reproduces the equations for Ads_3 and larger dimensional systems. We then add a black brane to the Ads to see how it affects the entropy.

Chapter 2

Entanglement Entropy

In this section, we will look over and define some of the properties of Entanglement Entropy(EE). The EE measures the amount two subsystems of a quantum system that are entangled. In quantum information theory, it is interpreted as information in the quantum state.

2.1 What is Entanglement?

Quantum entanglement is the phenomenon that occurs when a group of particles are generated, interact, or share spatial proximity in a way such that the quantum state of each particle of the group cannot be described independently of the state of the others, including when a large distance separates the particles. This is described best by deriving the bell inequalities.

Consider a quantum system of two spins one half $S = \frac{1}{2}$ particles, when the system is in the singlet state $|0, 0_z\rangle$ where are second quantum number is the spin in the z-direction, finding $|0, 0_z\rangle$ in terms of there eigenstates

$$|0, 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \tag{2.1}$$

The first sign refers to particle one, and the second is particle 2. As is clear from the equation, the eigenstates are superposed; this tells us that we cannot have a definite spin of a particle. To know this, we must measure which will cause this singlet state to collapse to one of the two product states.

Deriving the inequalities from [3]. Now we can look at a two-particle system that will have three components in arbitrary directions ($S_{ni} = \pm\frac{\hbar}{2},\ for\ i = 1, 2, 3$), the two particles cannot have the same spin in the same direction as shown above because particles can't be in the same quantum state, we can generate a table of possible outcomes

Particle 1	Particle 2
(S_{n1}, S_{n2}, S_{n3})	$(-S_{n1}, -S_{n2}, -S_{n3})$
$(-S_{n1}, S_{n2}, S_{n3})$	$(S_{n1}, -S_{n2}, -S_{n3})$
$(S_{n1}, -S_{n2}, S_{n3})$	$(-S_{n1}, S_{n2}, -S_{n3})$
$(S_{n1}, S_{n2}, -S_{n3})$	$(-S_{n1}, -S_{n2}, S_{n3})$
$(-S_{n1}, -S_{n2}, -S_{n3})$	(S_{n1}, S_{n2}, S_{n3})
$(S_{n1}, -S_{n2}, -S_{n3})$	$(-S_{n1}, S_{n2}, S_{n3})$
$(-S_{n1}, S_{n2}, -S_{n3})$	$(S_{n1}, -S_{n2}, S_{n3})$
$(-S_{n1}, -S_{n2}, S_{n3})$	$(S_{n1}, S_{n2}, -S_{n3})$

If we look at the positive values of the spins in the table, we see that the number of states with certain positive spins are

$$N(S_{n1}, S_{n2}) \leq N(S_{n1}, S_{n3}) + N(S_{n3}, S_{n2}) \tag{2.2}$$

Because the probability is proportional to the number of times, it shows, we find

$$P(S_{n1}, S_{n2}) \leq P(S_{n1}, S_{n3}) + P(S_{n3}, S_{n2}) \tag{2.3}$$

We have used purely classical reasoning to derive the two forms of Bell's inequality we have encountered thus far. Recall that we measure along axes of the magnetic field in the context within which the above was derived. As such, there are angles between these various axes. With angles adjusting the probability of $P(S_{n1}, S_{n2}) = \frac{1}{2}\sin^2(\frac{\theta_{ij}}{2})$ with i,j being 1,2,3 and the angle being $\hat{n}_i \cdot \hat{n}_j = \cos\theta_{ij}$. This leads the inequality to become;

$$\frac{1}{2}\sin^2(\frac{\theta_{12}}{2}) \leq \frac{1}{2}\sin^2(\frac{\theta_{13}}{2}) + \frac{1}{2}\sin^2(\frac{\theta_{32}}{2}) \tag{2.4}$$

These inequalities allow us to test locality because if we break these, we will find that we can't have predefined variables. has shown the breaking of these inequalities citeFreedman1972. This means that quantum particles are indeed entangled.

2.2 Entanglement entropy in Quantum mechanics

Following Nielsen and Chuang[23], the Density operator allows us to have an alternative formulation to quantum mechanics, which is mathematically equivalent to the state vector approach. The upside to this new formulation is a more convenient language for places in quantum mechanics, such as our entanglement entropy. The density operator is defined as:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{2.5}$$

which allows us to define the Von-Neumann entropy as simply

$$S = -tr(\rho \log \rho) \tag{2.6}$$

For any ρ when we apply this to density matrices, we call it entanglement entropy. As can be seen, if the states are pure (whose quantum state is known completely), they will have zero entropy because we are taking the entropy over the whole region. So we can divide the total system into two subsystems, A and B, generating a mixed state. Because we are working in a Hilbert space, we can write the total space as the direct product of 2 spaces $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponding to the subsystems A and B. An observer who can only access subsystem A will feel that the reduced density matrix describes the total system with that matrix defined by

$$\rho_A = tr_b \rho_{total} \tag{2.7}$$

where the trace is taken only over the Hilbert space $\mathcal{H}_B$. At this point, we are ready to define the entanglement entropy of subsystem A as the Von-Neumann entropy of the reduced density matrix ρ_A

$$S_A = -n\partial_n[\ln Z(n) - n\ln Z(1)]_{n=1} = -tr_A \rho_A log \rho_A \tag{2.8}$$

This allows us to measure the entanglement entropy. We can also define the EE for systems at a finite temperature by replacing our density matrix with a thermal one. $\rho_{thermal} = e^{-\beta H}$ with our H being the Hamiltonian.

2.2.1 Properties of The Density Matrix and EE

The properties we need for our purpose are:

- The density matrix has a unit trace
- The entropy is zero if and only if the system is in a pure state
- In a d-dimensional Hilbert space, the entropy is at most log(d)
- When B is a complement to A

$$S_A = S_B \tag{2.9}$$

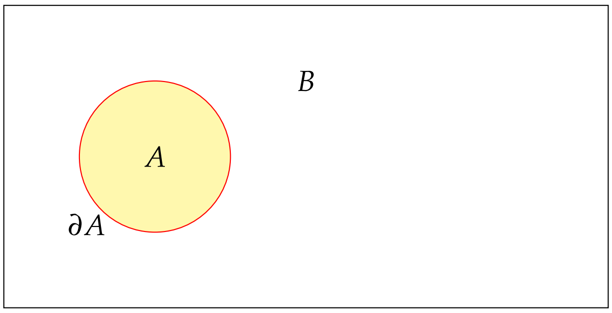

FIGURE 2.1: A QFT is divided into two subsystems, A and B. With A being an arbitrary shape, ∂A being the boundary of the subsystem. This boundary is also the boundary of B. This means that B is just the complement of A. This is a time-dependent background, so we need to specify the time $t = t_0$; this doesn't matter to us because we will only be looking at static systems.

- Entanglement entropy satisfies the subadditivity inequalities, which, when A is divided into two sub-manifolds A_1, A_2 it is found

  $$S_A \leq S_{A_1} + S_{A_2} \tag{2.10}$$

  so for any three subsystems A, B, and C that does not intersect, we can gain strong subadditivity

  $$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} \tag{2.11}$$

  so we can gain a stronger version of Eq 2.10

  $$S_A + S_B \leq S_{A\cup B} + S_{A\cap B} \tag{2.12}$$

2.3 QFT

Following Ryu and Takayanagi[25], in a Quantum field theory, calculations of the entanglement entropy are significantly more complicated than in regular quantum mechanics. Some of the significant differences in QFTs from quantum mechanics is that space is treated like a parameter, meaning that every space spot has its state. This leads to the Hilbert space of a QFT as an infinite direct product overall points in space.

As shown in figure 2.1, we can split the field into two segments at a single time slice($t - t_0$), A and B, with the two areas having the same boundary ∂A. This allows us to define our entropy using the formula 2.8, but there is a significant problem: we will always have divergences in any continuum theory. This requires us to introduce an ultraviolet cut-off ϵ. This then leads the leading term of the system to be the area of the boundary as shown numerically by [5, 26].

$$S_A = \gamma \cdot \frac{Area(\partial A)}{\epsilon^{d-1}} + subleadingterms \tag{2.13}$$

Where γ is a constant which depends on the system. This area of law does not hold in generic situations like 2D CFT. The entropy, in this case, there is a logarithmic scale with respect to the length so that we can write it as [16, 8]:

$$S_A = \frac{c}{3} log(\frac{l}{\epsilon}) \tag{2.14}$$

with c being our central charge, so, this for Ads_3 is [6]:

$$c = \frac{3R}{2G_N^{(3)}} \tag{2.15}$$

Chapter 3

An Introduction to General Relativity And The Ads Space Time

General relativity is a theory of gravitation developed by Albert Einstein between 1907 and 1915. The theory of general relativity says that the observed gravitational effect between masses results from their warping of space-time. This theory is governed by the Einstein field equations defined as

$$G_{\mu\nu} = T_{\mu\nu} \tag{3.1}$$

$T_{\mu\nu}$ is the stress-energy tensor containing the information about the energy in the system, and $G_{\mu\nu}$ is the shape of space-time. This contains a metric which describes how the manifold looks and is of the form

$$g_{\mu\nu}x^{\mu}x^{\nu} = ds^2 = -dt^2 + d\vec{x}^2 \tag{3.2}$$

these metrics can be transformed to different coordinates using

$$g'_{\alpha\beta} = g_{\mu\nu}\frac{dx^{\mu}}{dx'^{\alpha}}\frac{dx^{\nu}}{dx'^{\beta}} \tag{3.3}$$

3.1 Ads Spacetime

The Anti-de sitter space (Ads) is the maximally symmetric space time of constant negative curvature. This space can be viewed as the Lorentzian analogue of a sphere in a flat space that is one dimension larger. we can then find a metric with the form

$$ds^2 = \sum_{j=1}^{k} dx_j^2 - \sum_{i=1}^{z+1} dt_j^2 \tag{3.4}$$

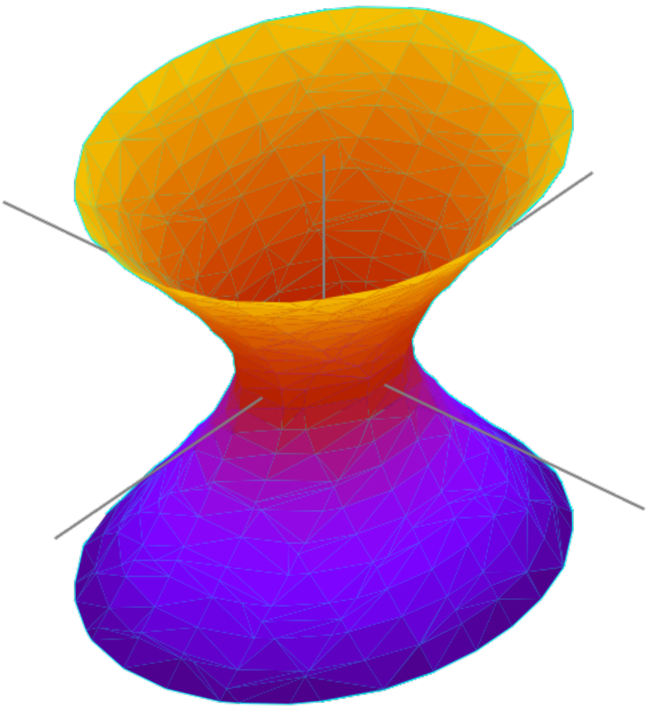

FIGURE 3.1: Image of (1 + 1)-dimensional anti-de Sitter space embedded in flat (1 + 2)-dimensional space.[19]

For a ant-de sitter space embedded in $\mathbb{R}^{k,z+1}$ with

$$\sum_{j=1}^{k} dx_j^2 - \sum_{i=1}^{z+1} = -R^2 \tag{3.5}$$

With R being a nonzero constant with the dimension of length, this value is considered the radius of curvature of the space. This general sphere ends up being the points at the same distance from the origin. This can be visualised as a hyperboloid like in figure 3.1

3.1.1 Coordinate patches

The coordinate patch covers part of the space and gives the half-space coordinates of the Ads space. The metric of this patch is

$$ds^2 = \frac{R}{z^2}(-dt^2 + dz^2 + \sum_i dx_i^2) \tag{3.6}$$

This patch has conformal symmetries given by

$$\text{Conformal Group}\begin{cases}\text{Poincare group}\begin{cases}\text{Translations } x^{\mu} = x\mu + c\mu\\ \text{lorentz Boosts } x^{\mu} = \Lambda\end{cases}\\ \text{Dilations } x^{\mu} = \lambda x^{\mu}\\ \text{Special Conformal Transformaion } x'^{i} = \frac{x^i + c^i x^2}{1+2c\cdot x+c^2x^2}\end{cases}$$

These Isometries are conformally (scale factor) equivalent to the half-space of Minkowski space-time. So at $ds = \infty$, where $z = 0$, we have a conformal Minkowski space-time. This leads us toward Ads/CFT due to the conformal symmetries of the QFT.

Chapter 4

Black Hole Thermodynamics

Viewing black hole thermodynamics from [1] and [22].

Black hole thermodynamics can be summarized by four laws which have analogues to the statistical thermodynamics. These also have natural interpretations in ADS/CFT. But we only need the first two to find the black hole entropy

4.1 Surface Gravity

first defining surface gravity as the gravitational force (per unit mass) or the gravitational acceleration on the horizon. In Newtonian gravity, the gravitational acceleration is given by

$$a = \frac{GM}{r^2} \tag{4.1}$$

so at $r = r_0$

$$\kappa = a(r = r_0) = \frac{c^4}{4GM} \tag{4.2}$$

We used Newtonian gravity to derive the surface gravity. Surface gravity is the force which is necessary to stay at the horizon. Because we are working in the GR framework, this concept is relative to the observer. The observer falling in cannot escape the horizon, no matter how large the force is. So, the necessary force diverges for the infalling observer himself. But if the asymptotic observer measures this force, the force remains finite and coincides with the Newtonian result. Two observers disagree with the acceleration values due to the redshift between them.

4.2 Zeroth Law

The four laws of black hole thermodynamics read as follows. The Zeroth law of black hole thermodynamics states that the surface gravity κ is constant over the horizon. This implies thermal equilibrium.

4.3 First Law

The first law of Black hole thermodynamics states the black hole mass increases by dM, and the horizon area increases by dA as well. So, one has a relation

$$dM \propto dA \tag{4.3}$$

For a precise formula, we look at the dimensions of both sides. First, the left-hand side must have GdM because Newton's constant and mass appear only in the combination GM in general relativity. This essentially acts as a dictionary G from mass to curvature.

Then we can see the right-hand side must have the dimension of acceleration, so we find

$$GdM = \frac{\kappa}{8\pi} dA \tag{4.4}$$

We have seen that black hole law are similar to thermodynamic laws. However, so far, this is just an analogy. The same expressions do not mean that they represent the same physics. Indeed, there are several problems in identifying black hole laws as thermodynamic laws:

- Even though the horizons act the same, they don't have the same dimension. One is dimensionless, and the other has the result of area
- , And we have nothing that escapes the black holes. or anything that radiates

4.4 Bekenstein-Hawking Entropy

The black hole is not an isolated object in our universe. For example, matter can make a black hole and matter quantum mechanics microscopically. So let's look at the quantum effect of matter. This is where hawking radiation comes in. We find the radiation and its temperature is given by

$$k_B T = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi G M} \text{(Schwarzschild black hole).} \tag{4.5}$$

So We find

$$d(Mc^2) = T\frac{k_B c^3}{4G\hbar} \tag{4.6}$$

So now, comparing this to the first law of thermodynamics $dE = Tds$, we obtain

$$S = \frac{A}{4G\hbar} k_B c^3 \tag{4.7}$$

This is the Bekenstein Hawking Entropy, which can also be known as the area law to the area being proportional to the entropy rather than the volume, as you would expect from standard statistical mechanics.

Looking at Bekenstein-Hawking entropy of black holes (area law), which is proportional to the event horizon's area. We can see the entropy S_A as the entropy of the observer who can only access subsystem A and not receive any signals from B. This will motivate our comparison to a CFT. Still, there are some differences we need to overcome, such as The entanglement entropy is proportional to the number of matter fields, while the black hole entropy is not. Also, the entanglement entropy includes ultraviolet divergences, as opposed to the gravity theory, which Contains IR divergences. These differences allowed us to find the correct description.[27, 12, 13, 18]

Chapter 5

Holographic Entanglement Entropy

The holographic calculation of entanglement entropy uses the setup of Ads/CFT correspondence; this can then be extended to other holographic setups that are more general to find the entanglement of the dual CFT.

As the introduction mentions, this correspondence argues that a gravity theory in a $d+1$ anti-De-Sitter space Ads_{d+1} equals a d dimensional conformal field theory. This will use the Poincare metric with the radius R:

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + d\vec{x}^2) \tag{5.1}$$

wth $d\vec{x}$ being the rest of the spatial dimensions. This Ads space has a boundary defined by a second-order pole at $Z = 0$. It means that we can see a parallel between our z direction and the energy scale of the CFT, with $Z = 0$ being the UV end of the scale, and as z tends toward infinity, we go into the IR region. The parallel is related to the isometries of the Ads space, which are equivalent to the conformal symmetries of the CFT. This allows us to have a cutoff identified as the UV cutoff in the dual CFT by imposing $z \leq \epsilon$, meaning the boundary is at $z = \epsilon$. Using this, we will arrive at the bulk-to-boundary relation expressed by the equivalence of the partition functions in both theories.

$$Z_{CFT} = Z_{AdsGravity} \tag{5.2}$$

Now, we can state the holographic entanglement formula proposed by Ryu and Takayanagi. We chose a time slice and found the minimal area of our Ads space-time, and the holographic formula for the entanglement entropy of the CFT is defined by:

$$S_{ee} = -n\partial_n[\ln Z(n) - n\ln Z(1)]_{n=1} = \frac{A_{min}}{4G_N^{(d+1)}} \tag{5.3}$$

With $4G_N^{(d+1)}$ being the (d+1)-dimensional Newtons constant of the Ads space-time and $\ln Z(n) = -S_{grav}$ being the classical gravity action. This minimal area surface hangs down into the bulk and is closed on the surface of the flat portion of the Ads space.

In our holographic theory, we have the UV/IR connection[28]. The UV region corresponds to the high-energy regime of the QFT, where short-distance physics dominates. In contrast, the IR region corresponds to the low-energy regime, where long-distance physics is relevant. In AdS/CFT, the UV/IR connection is a powerful duality that establishes a relationship between the physics of a strongly-coupled QFT in the UV and the gravitational dynamics of a weakly-coupled theory in the IR. Specifically, it implies that the UV properties of the QFT, such as its UV divergences and renormalization group flow, are dual to the IR properties of the gravitational theory, such as the geometry of the extra-dimensional space and the behaviour of fields at the AdS boundary. This deep connection between UV and IR physics in AdS/CFT provides valuable insights into the non-perturbative dynamics of strongly-coupled field theories. It has led to significant advancements in understanding various aspects of quantum gravity, quantum field theory, and condensed matter physics.

Chapter 6

Entanglement Entropy from Ads_3/CFT_2

In Ads_3/CFT_2 we need to find the minimal length like in figure 6.1 so with $\dot{x} = \frac{dx}{d\sigma}$, we define the length of the line in our space-time at a static time.

$$L = \int ds = \int d\sigma \sqrt{g_{\alpha\beta} x^{\alpha} x^{\beta}} = \int d\sigma \frac{R}{z} \sqrt{(\dot{x}^2 + \dot{z}^2)} \tag{6.1}$$

So to solve this, we do variations by solving the Euler-Lagrange equations.

For the x Euler-Lagrange equations:

$$\frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{6.2}$$

$$\frac{d}{d\sigma} \left(\frac{\frac{R}{z} \dot{x}}{\sqrt{(\dot{x}^2 + \dot{z}^2)}} \right) - 0 = 0 \tag{6.3}$$

$$\frac{\frac{R}{z} \dot{x}}{\sqrt{(\dot{x}^2 + \dot{z}^2)}} = C \tag{6.4}$$

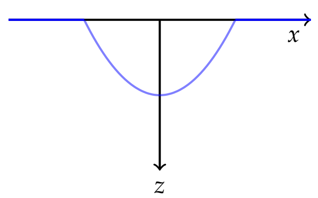

FIGURE 6.1: The minimal line in Ads_3

For the z Euler-Lagrange equations:

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{6.5}$$

$$\frac{d}{d\sigma}\left(\frac{\frac{R}{z}\dot{z}}{\sqrt{(\dot{x}^2+\dot{z}^2)}}\right) - \left(-\frac{R}{z^2}\sqrt{\dot{x}^2+\dot{z}^2}\right) = 0 \tag{6.6}$$

$$\frac{d}{d\sigma}\left(\frac{\frac{R}{z}\dot{z}}{\sqrt{(\dot{x}^2+\dot{z}^2)}}\right) + \frac{R}{z^2}\sqrt{\dot{x}^2+\dot{z}^2} = 0 \tag{6.7}$$

$$\tag{6.8}$$

Using the x Euler-Lagrange equation to simplify:

$$C^2\frac{d}{d\sigma}\left(\frac{\dot{z}}{\dot{x}}\right) + \frac{R^2}{z^3}\dot{x} = 0 \tag{6.9}$$

$$C^2\frac{d}{d\sigma}\left(\frac{\dot{z}}{\dot{x}}\right) = -\frac{R^2}{z^3}\dot{x} \tag{6.10}$$

$$\tag{6.11}$$

Now using a parameterisation of the line (fixing the gauge)

$$z = \sigma \tag{6.12}$$

$$\dot{z} = 1 \tag{6.13}$$

Simplifying the x Euler-Lagrange equation and solving for $\dot{x}$:

$$\frac{\frac{R}{z}\dot{x}}{\sqrt{(\dot{x}^2+1)}} = C \tag{6.14}$$

$$\dot{x}^2 = \frac{(C)^2}{(\frac{R}{z})^2 - (C)^2} \tag{6.15}$$

$$\dot{x} = \frac{(C)}{\sqrt{(\frac{R}{z})^2 - (C)^2}} \tag{6.16}$$

Now plugging it into the z Euler-Lagrange equation to show the chosen gauge satisfies the equation

$$\frac{d}{dz}\left(\sqrt{(\frac{R}{z})^2 - (C)^2}\right) + \frac{R^2}{z^3}\frac{1}{\sqrt{(\frac{R}{z})^2 - (C)^2}} = 0 \tag{6.17}$$

$$-\frac{R^2}{z^3}\frac{1}{\sqrt{(\frac{R}{z})^2 - (C)^2}} + \frac{R^2}{z^3}\frac{1}{\sqrt{(\frac{R}{z})^2 - (C)^2}} = 0 \tag{6.18}$$

As this is zero, we have a solution for this parameterisation. Finding an equation with just x and z.

$$\dot{x} = \frac{C}{\sqrt{(\frac{R}{z})^2 - C^2}} \tag{6.19}$$

$$x = \sqrt{-z^2 + \frac{R^2}{c^2}} \tag{6.20}$$

$$x^2 + z^2 = \frac{R^2}{c^2} \tag{6.21}$$

We can attempt to perform the integral using the solutions we found. However, because this is the half circle, we must split the interval in half because we can't overlap bounds, so we see.

$$\begin{aligned} L = 2\int_0^{Z_{max}} dz\frac{R}{z}\sqrt{\frac{1}{(\frac{R}{Cz})^2 - 1} + 1} &= 2\int_0^{Z_{max}} dz\frac{R}{z}\sqrt{\frac{(\frac{R}{C})^2}{(\frac{R}{C})^2 - z^2} + 1} \\ &= 2\int_0^{Z_{max}} dz\frac{R}{z}\sqrt{\frac{1}{(1-(\frac{Cz}{R})^2}} \end{aligned} \tag{6.22}$$

Let $z_* = \frac{R}{C}$. With all ADS spaces, as we approach the boundary, it diverges to infinity, so we require a UV/IR cutoff ϵ.

$$2R\int_0^{Z_{max}} dz\frac{1}{z}\sqrt{\frac{1}{(1-(\frac{z}{z_*})^2}} = [\tanh^{-1}(\sqrt{1-(\frac{z}{z_*})^2})]_\epsilon^{z_{max}} \tag{6.23}$$

Placing into logarithmic form

$$\begin{aligned} 2R[\tanh^{-1}(\sqrt{1-(\frac{z}{z_*})^2})]_\epsilon^{z_{max}} &= [\frac{1}{2}\ln(\frac{\sqrt{1-(\frac{z}{z_*})^2}+1}{1-\sqrt{1-(\frac{z}{z_*})^2}})]_\epsilon^{z_{max}} \\ &= [\frac{1}{2}\ln(\frac{\sqrt{1-(\frac{z_{max}}{z_*})^2}+1}{1-\sqrt{1-(\frac{z_{max}}{z_*})^2}}) - \frac{1}{2}\ln(\frac{\sqrt{1-(\frac{\epsilon}{z_*})^2}+1}{1-\sqrt{1-(\frac{\epsilon}{z_*})^2}})] \end{aligned} \tag{6.24}$$

for Z_{max} we need to use Eq 6.21 setting $x = 0$, so we have

$$z_{max} = \frac{R}{C} = z_* \tag{6.25}$$

if we plug in to eq 6.24 we get

$$L = 2R\left(-\frac{1}{2}\ln\left(\frac{1+\sqrt{1-(\frac{\epsilon}{z_*})^2}}{1-\sqrt{1-(\frac{\epsilon}{z_*})^2}}\right)\right) = 2R\left(\frac{1}{2}\ln\left(\frac{(\frac{\epsilon}{z_*})^2}{-(\frac{\epsilon}{z_*})^2+2\sqrt{1-(\frac{\epsilon}{z_*})^2}+2}\right)\right)$$
$$= 2R\ln\left(\frac{z_*}{\epsilon}\right) + \left(\frac{1}{2}\ln\left(\sqrt{1-(\frac{\epsilon}{z_*})^2}+1\right)^2\right) \quad (6.26)$$

Any parts of this solution that aren't divergent are unphysical because the UV/IR cutoff we want is close to zero.

$$L = 2R\ln\left(\frac{z_*}{\epsilon}\right) \quad (6.27)$$

finding the length of the interval of the dip into the ads space, which is also the length of the interval in the CFT; using Eq 6.21 and setting $z = 0$ we have $x^2 = (\frac{R}{c})^2$ so $x = \pm\frac{R}{c}$ the length is double this so $\ell = 2\frac{R}{c}$. Using the holographic formula and the fact $\ell = 2\frac{R}{c} = 2z_*$, we find:

$$S_{ee} = \frac{2R\ln(\frac{\ell}{2\epsilon})}{4G_N^{(3)}} \quad (6.28)$$

$$S_{ee} = \frac{R\ln(\frac{\ell}{2\epsilon})}{2G_N^{(3)}} \quad (6.29)$$

Using Eq 2.15:

$$S_{ee} = \frac{c\ln(\frac{\ell}{2\epsilon})}{3} \quad (6.30)$$

Comparing this to the Eq 2.14 we see we get the same result as the known QFT case.

Chapter 7

Entanglement Entropy In Higher Dimensions

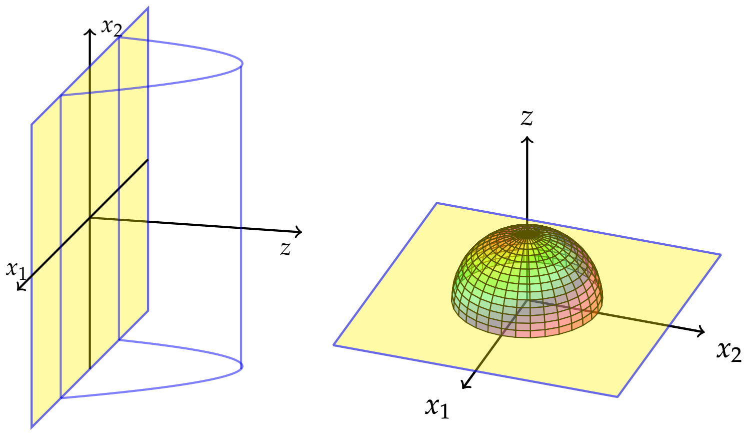

FIGURE 7.1: The minimal area of a strip in Ads_{d+1} and The minimal area of a Sphere in Ads_{d+1}

When finding the surfaces for the minimal area in more dimensions, there are more ways we can perform the calculation; we can split up the surface that is not affected by the warp factor in two ways. On the strip, and the other is a d-sphere. First, we will start with the strip.

7.1 A Strip in ADS_4/CFT_3

With

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + dx_1^2 + dx_2^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{7.1}$$

For our holographic formula, we have

$$S_{ee} = \frac{A_{min}}{4G_N} \tag{7.2}$$

$$A = \int d^2\xi \sqrt{det\hat{g}_{ab}} \tag{7.3}$$

$$\text{with } \hat{g}_{ab} \text{ being} \tag{7.4}$$

$$\hat{g}_{\alpha\beta} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b} \tag{7.5}$$

Making one coordinate depends on the others. Fixing the gauge. Fixing the gauge this early could cause missed terms, but this has been verified by fixing the gauge later.

$$x_1 = (z, x_2) \tag{7.6}$$

this leads to

$$\hat{g}_{zz} = \frac{R^2}{z^2}(\frac{\partial x_1}{\partial z})^2 + \frac{R^2}{z^2} \tag{7.7}$$

$$\hat{g}_{zx_2} = \hat{g}_{x_2 z} = \frac{R^2}{z^2}\frac{\partial x_1}{\partial z}\frac{\partial x_1}{\partial x_2} \tag{7.8}$$

$$\tag{7.9}$$

$$\hat{g}_{x_2x_2} = \frac{R^2}{z^2}(\frac{\partial x_1}{\partial x_2})^2 + \frac{R^2}{z^2} \tag{7.10}$$

$$\tag{7.11}$$

$$\begin{pmatrix} g_{zz} & g_{z2} \\ g_{2z} & g_{22} \end{pmatrix} \tag{7.12}$$

$$g = (\frac{R^2}{z^2}(\frac{\partial x_1}{\partial z})^2 + \frac{R^2}{z^2})(\frac{R^2}{z^2}(\frac{\partial x_1}{\partial x_2})^2 + \frac{R^2}{z^2}) - (\frac{R^2}{z^2}\frac{\partial x_1}{\partial z}\frac{\partial x_1}{\partial x_2})^2) \tag{7.13}$$

$$g = \frac{R^4}{z^4}((\frac{\partial x_1}{\partial z})^2 + (\frac{\partial x_1}{\partial x_2})^2 + 1) \tag{7.14}$$

Now plugging into the area formula

$$A = \int dz dx_2 \sqrt{\frac{R^4}{z^4}((\frac{\partial x_1}{\partial z})^2 + (\frac{\partial x_1}{\partial x_2})^2 + 1)} \tag{7.15}$$

Due to the fact we are minimising, and because the derivatives in directions other than the z direction can only increase, we can state $\frac{\partial x_1}{\partial x_2} = 0$.

$$A = V_2 \int dz \frac{R^2}{z^2}\sqrt{(\frac{\partial x_1}{\partial z})^2 + 1} \tag{7.16}$$

now we can express $\frac{\partial x_1}{\partial z} = \dot{x}$

$$A = V_2 \int dz \frac{R^2}{z^2} \sqrt{\dot{x}^2 + 1} \tag{7.17}$$

Now finding the Euler-Lagrange equation

$$\partial_z \left(\frac{R^2}{z^2} \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} \right) = 0 \tag{7.18}$$

$$\dot{x}^2 = \frac{\frac{z^4 C^2}{R^4}}{1 - \frac{z^4 C'^2}{R^4}} \tag{7.19}$$

$$\dot{x} = \frac{\frac{z^2 C}{R^2}}{\sqrt{1 - \frac{z^4 C^2}{R^4}}} \tag{7.20}$$

Now placing into our area equation

$$A = V_2 \int dz \frac{R^2}{z^2} \sqrt{\frac{\frac{z^4 C^2}{R^4}}{1 - \frac{z^4 C^2}{R^4}} + 1} \tag{7.21}$$

$$A = V_2 \int dz \frac{R^2}{z^2} \sqrt{\frac{1}{1 - \frac{z^4 C^2}{R^4}}} \tag{7.22}$$

We are performing the integral with the bounds $[\epsilon, z_*]$ in which $z_*^2 = \frac{R^2}{C}$ from Eq 7.20 because we need to split this into two halves due to its nature as seen in figure 7.1.

$$A = 2V_2 R^2 \int_\epsilon^{z_*} dz \frac{1}{z^2} \sqrt{\frac{1}{1 - \frac{z^4}{z_*^4}}} = [-\frac{2F1[-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{z^4}{z_*^4}]}{z}]_\epsilon^{z_*} \tag{7.23}$$

With $2F1[-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{z^4}{z_*^4}]$ being the hypergeometric function. This solution comes about because we have a function with three singularities. These functions have special solutions, so looking for when $z = z_*$ we find

$$2F1[-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{z_*^4}{z_*^4}] = 2F1[-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1] = \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \tag{7.24}$$

now looking for the solution to the hypergeometric function when $z = \epsilon$, we require the series expansion.

$$\frac{2F1[-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{\epsilon^4}{z_*^4}]}{\epsilon} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^2)) \tag{7.25}$$

Due to the fact we take the limit as $\epsilon \to 0$, we only require the divergent term. The minimal area in ADS_4/CFT_3 is

$$A = 2V_2R^2(-\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{z_*\Gamma(\frac{1}{4})} + \frac{1}{\epsilon}) \tag{7.26}$$

Finding the strip width, we integrate Eq 7.20:

$$\dot{x} = \frac{\frac{z^2}{z_*^2}}{\sqrt{1 - \frac{z^4}{z_*^4}}} \tag{7.27}$$

$$L = 2\int_0^{z_*} \frac{\frac{z^2}{z_*^2}}{\sqrt{1 - \frac{z^4}{z_*^4}}} \tag{7.28}$$

$$L = 2\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\sqrt{\pi}z_* \tag{7.29}$$

So our minimal area is:

$$A = 2V_2R^2(-2\frac{\sqrt{\pi}^2(\Gamma(\frac{3}{4})}{z_*\Gamma(\frac{1}{4})})^2\frac{1}{L} + \frac{1}{\epsilon}) \tag{7.30}$$

This means the entanglement entropy becomes

$$S_{EE} = \frac{2V_2R^2}{4G_4}(-2\frac{\sqrt{\pi}^2(\Gamma(\frac{3}{4})}{z_*\Gamma(\frac{1}{4})})^2\frac{1}{L} + \frac{1}{\epsilon}) \tag{7.31}$$

7.2 A Strip in ADS_{d+1}/CFT_d

Now we can generalise this for d dimensions. With

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + d\vec{x}^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{7.32}$$

For our holographic formula, we have

$$S_{ee} = \frac{A_{min}}{4G_N} \tag{7.33}$$

$$A = \int d^2\xi\sqrt{det\hat{g}_{ab}} \tag{7.34}$$

$$\text{with } \hat{g}_{ab} \text{ being} \tag{7.35}$$

$$\hat{g}_{\alpha\beta} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b} \tag{7.36}$$

Making one coordinate depend on the others, fixing the gauge.

$$x_1 = (z, \vec{x}) \tag{7.37}$$

this leads to

$$\hat{g}_{zz} = \frac{R^2}{z^2}(\frac{\partial x_1}{\partial z})^2 + \frac{R^2}{z^2} \tag{7.38}$$

$$\hat{g}_{x_i x_j} = \hat{g}_{x_i x_j} = \frac{R^2}{z^2}\frac{\partial x_1}{\partial x_i}\frac{\partial x_1}{\partial x_j} \tag{7.39}$$

$$\tag{7.40}$$

$$\hat{g}_{x_i x_i} = \frac{R^2}{z^2}(\frac{\partial x_1}{\partial x_i})^2 + \frac{R^2}{z^2} \tag{7.41}$$

$$\tag{7.42}$$

Because if there is movement in the other direction, we will not get a minimal surface, so all the off-axis components will be zero and $\hat{g}_{x_i x_i} = \frac{R^2}{z^2}$, so the determinant becomes:

$$g = \frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + (\sum^{d-2}(\frac{\partial x_1}{\partial x_i})^2) + 1) \tag{7.43}$$

this leads to

$$A = \int dz d\vec{x}\sqrt{\frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + (\sum^{d-2}(\frac{\partial x_1}{\partial x_i})^2) + 1)} \tag{7.44}$$

because if a derivative is non-zero in any direction will increase the area. Leading to the derivatives in that direction to be zero, so we can state

$$A = V_{d-2}\int dz\sqrt{\frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + 1)} \tag{7.45}$$

now following the same logic from the ADS_4 case, we find our minimal integral being

$$A = 2V_{d-2}R^{(d-1)}\int_{\epsilon}^{z_*} dz\frac{1}{z^{d-1}}\sqrt{\frac{1}{1-\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}} = [-\frac{2F1[\frac{1}{2},\frac{1}{2}(1-\frac{1}{d-1}),\frac{d-1}{2(d-1)},\frac{z^{2(d-1)}}{z_*^{2(d-1)}}]}{(d-2)z^{d-2}}]_{\epsilon}^{z_*} \tag{7.46}$$

with z_* being the turnaround point in the bulk. Using the same expansion as the ADS_4 case. we find our minimal area as

$$A = 2V_{d-2}R^{d-1}(-\frac{\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})}{(d-2)z_*^{d-2}\Gamma(\frac{1}{2(d-1)})} + \frac{1}{(d-2)\epsilon^{d-2}}) \tag{7.47}$$

Finding the strip width as in ADS_4

$$L = \int_0^{z_*} dz\frac{z^{d-1}}{z_*^{d-1}}\sqrt{\frac{1}{1-\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}} \tag{7.48}$$

$$L = 2\frac{\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}z_* \tag{7.49}$$

So plugging this into our area, we get

$$A = 2V_{d-2}R^{d-1}(-2^{d-2}\frac{\sqrt{\pi}^{d-1}\Gamma(\frac{d}{2(d-1)})^{d-1}}{(d-2)L^{d-2}\Gamma(\frac{1}{2(d-1)})^{d-1}} + \frac{1}{(d-2)\epsilon^{d-2}}) \tag{7.50}$$

This means that our entanglement entropy becomes

$$S_{ee} = \frac{2V_{d-2}R^{d-1}}{4G_4}(-2^{d-2}\frac{\sqrt{\pi}^{d-1}\Gamma(\frac{d}{2(d-1)})^{d-1}}{(d-2)L^{d-2}\Gamma(\frac{1}{2(d-1)})^{d-1}} + \frac{1}{(d-2)\epsilon^{d-2}}) \tag{7.51}$$

7.3 ADS_{d+1}/CFT_d sphere

With

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + d\vec{x}_d^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{7.52}$$

for the sphere we can find sphere $dx_1^2 + d\vec{x}_{d-1}^2 = dr^2 + r^2 d\Omega_d^2 - 1$

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + dr^2 + r^2 d\Omega_2^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{7.53}$$

for our holographic formula, we have

$$S_{ee} = \frac{A_{min}}{4G_N} \tag{7.54}$$

$$A = \int d^2\xi \sqrt{det\hat{g}_{ab}} \tag{7.55}$$

$$\text{with } \hat{g}_{ab} \text{ being} \tag{7.56}$$

$$\hat{g}_{\alpha\beta} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b} \tag{7.57}$$

now letting z be a function of r leads to the off-axis components being zero because we are finding a minimal area surface, so we get

$$\hat{g}_{zz} = \frac{R^2}{z^2}(\frac{dr}{dz})^2 + \frac{R^2}{z^2} \tag{7.58}$$

$$\hat{g}_{z\Omega_i} = \hat{g}_{\Omega_i z} = 0 \tag{7.59}$$

$$\hat{g}_{\Omega_i\Omega_j} = \hat{g}_{\Omega_j\Omega_i} = 0 \tag{7.60}$$

$$\hat{g}_{\Omega_i\Omega_i} = \frac{R^2r^2}{z^2} \tag{7.61}$$

$$\tag{7.62}$$

using these to find the determinate

$$g = \frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + 1) \tag{7.63}$$

$$\tag{7.64}$$

$$A = \int dz d\Omega_{d-1}\sqrt{\frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + 1)} \tag{7.65}$$

$$A = V(S^{d-2})\int dr\sqrt{\frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + 1)} \tag{7.66}$$

letting $\frac{dr}{dz} = \dot{r}$

$$A = V(S^{(d-2)})R^{(d-1)}\int dr\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2 + 1} \tag{7.67}$$

Solving the Euler-Lagrange equations

$$\frac{d}{dr}(\frac{\partial}{\partial \dot{r}}(\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2 + 1})) - \frac{\partial}{\partial z}\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2 + 1} = 0 \tag{7.68}$$

$$\frac{d}{dr}(\frac{r^{(d-2)}}{z^{(d-1)}}\frac{\dot{r}}{\sqrt{\dot{r}^2 + 1}}) - \frac{\partial}{\partial z}\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2 + 1} = 0 \tag{7.69}$$

$$\frac{(d-2)r^{d-3}z^{1-d}\dot{r}^2}{\sqrt{\dot{r}^2 + 1}} + \frac{(d-1)z^{-d}r^{d-2}\dot{r}}{\sqrt{\dot{r}^2 + 1}} + \frac{z^{1-d}r^{d-2}\ddot{r}}{\sqrt{\dot{r}^2 + 1}} - \frac{z^{1-d}r^{d-2}\dot{r}^2\ddot{r}}{(\dot{r}^2 + 1)^{\frac{3}{2}}} - (d-2)r^{d-3}z^{1-d}\sqrt{\dot{r}^2 + 1} = 0 \tag{7.70}$$

This second-order non-linear differential equation cannot be solved using standard methods, but we can use the special conformal transformation looking in Euclidean space. We get a transformation given by [4]

$$x'^i = \frac{x^i + c^i x^2}{1 + 2c\cdot x + c^2x^2} \tag{7.71}$$

for a flat surface, we have a unit normal of the form $n^i = (1, 0, 0, ..., 0)$ with x^1 being zero extended along all the other directions. Also we define a vector $c^i = (\frac{1}{2R}, 0, 0, ..., 0)$. Thus a sphere of radius R will have its centre at $r^i = \frac{1}{2}\frac{c^i}{c^2} = (R, 0, 0, ..., 0)$ so we have

$$(x'^i - r^i)^2 = \frac{(x^2 + 2(c\cdot x)x^2 - r^i(1 + 2c\cdot x + c^2x^2))^2}{(1 + 2c\cdot x + c^2x^2)^2} \tag{7.72}$$

$$= \frac{(x^2 - 2(r\cdot x + (r\cdot c)x^2)(1 + 2c\cdot x + c^2x^2)}{(1 + 2c\cdot x + c^2x^2)^2} + R^2 \tag{7.73}$$

$$= \frac{(x^2 - 2r\cdot x - 2(r\cdot c)x^2)}{(1 + 2c\cdot x + c^2x^2)} + R^2 \tag{7.74}$$

because all points in the surface will obey the fact that $r \cdot x = 0$ and $c \cdot r = \frac{1}{2}$ this leads to

$$(x'^i - r^i)^2 = R^2 \tag{7.75}$$

now we can repeat this for the Ads case

$$x'^i = \frac{x^i + c^i(x^2 + z^2)}{1 + 2c \cdot x + c^2(x^2 + z^2)} \tag{7.76}$$

$$z' = \frac{z}{1 + 2c \cdot x + c^2(x^2 + z^2)} \tag{7.77}$$

for a flat surface, we have a unit normal of the form $n^i = (1, 0, 0, ..., 0)$ with x^1 being zero extended along all the other directions. Also we define a vector $c^i = (\frac{1}{2R}, 0, 0, ..., 0)$. Thus a sphere of radius R will have its centre at $r^i = \frac{1}{2}\frac{c^i}{c^2} = (R, 0, 0, ..., 0)$ and because all points in the surface will obey the fact that $r \cdot x = 0$ and $c \cdot r = \frac{1}{2}$ this leads to

$$(x'^i - r^i)^2 = \frac{x^i + c^i(x^2 + z^2) - r^i(1 + 2c \cdot x + c^2(x^2 + z^2)))^2}{(1 + 2c \cdot x + c^2(x^2 + z^2))^2} \tag{7.78}$$

$$= \frac{(x^2 + z^2)[1 + 2c \cdot x + c^2(x^2 + z^2)] - z^2}{(1 + 2c \cdot x + c^2(x^2 + z^2))^2} - \frac{x^2 + z^2}{1 + 2c \cdot x + c^2(x^2 + z^2)} + R^2 \tag{7.79}$$

$$= \frac{-z^2}{(1 + 2c \cdot x + c^2(x^2 + z^2))^2} - \frac{x^2 + z^2}{1 + 2c \cdot x + c^2(x^2 + z^2)} + R^2 = -z'^2 + R^2 \tag{7.80}$$

$$(x'^i - r^i)^2 + z'^2 = L^2 \tag{7.81}$$

This meaning

$$r^2 + z^2 = L^2 \tag{7.82}$$

using the conformal calculation, we see there is a solution, so by finding the area integral, we get

$$A = V(S^{(d-2)})R^{(d-1)} \int dz \frac{r^{(d-2)}}{z^{(d-1)}} \sqrt{\dot{r}^2 + 1}$$
$$= V(S^{(d-2)})R^{(d-1)} \int dz \frac{(L^2 - z^2)^{\frac{(d-2)}{2}}}{z^{(d-1)}} \sqrt{\frac{z^2}{L^2 - z^2} + 1} \tag{7.83}$$

$$A = V(S^{(d-2)})R^{(d-1)} \int dz \frac{(L^2 - z^2)^{\frac{(d-2)}{2}}}{z^{(d-1)}} \sqrt{\frac{L^2}{L^2 - z^2}} \tag{7.84}$$

changing variables with $z = L\sqrt{1-y^2}$ and $dz = \frac{-Ly}{\sqrt{1-y^2}}dy$ this lead to the bounds as $z \to L$ $y \to 0$ and as $z \to \epsilon$ $y \to \sqrt{1-(\frac{\epsilon}{L})^2}$

$$A = V(S^{(d-2)})R^{(d-1)} \int dy \frac{(L\sqrt{1-y^2})^{(d-2)}}{(L^2 - L^2(1-y^2))^{\frac{(d-1)}{2}}} \sqrt{\frac{L^2}{L^2 - L^2(1-y^2)}} \frac{-Ly}{\sqrt{1-y^2}} \tag{7.85}$$

$$A = V(S^{(d-2)})R^{(d-1)} \int_{\frac{\epsilon}{L}}^{1} dy \frac{(\sqrt{1-y^2})^{(d-3)}}{y^{(d-1)}} \tag{7.86}$$

By doing this integral, we get a regularised hypergeometric function. so with $V(S^{(d-2)}) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1)}{2}}$ we find the entanglement entropy to be

$$S = \frac{\pi^{\frac{d-1}{2}}}{4G\Gamma(\frac{d-1)}{2}} R^{(d-1)} \begin{Bmatrix} \sum_{j=1}^{\frac{d-2}{2}} c_j (\frac{L}{\epsilon})^{d-2j} + c_L \ln(\frac{L}{\epsilon}) + c_0 + \mathcal{O}(\frac{\epsilon^2}{L^2}), & \text{if d is even} \\ \sum_{j=1}^{\frac{d-1}{2}} c_j (\frac{L}{\epsilon})^{d-2j} + \tilde{c}_0 + \mathcal{O}(\frac{\epsilon}{L}), & \text{if d is odd} \end{Bmatrix} \tag{7.87}$$

$$\begin{array}{ll} c_j = \frac{(-1)^{j-1}\Gamma[\frac{d-1}{2}]}{(d-2j)\Gamma[\frac{d-2j+1}{2}]\Gamma[j]}, & c_L = \frac{(-1)^{\frac{(d-2)}{2}}\Gamma[\frac{d-1}{2}]}{\sqrt{\pi}\Gamma[\frac{d}{2}]} \\ c_0 = \frac{(-1)^{\frac{d-1}{2}}\sqrt{2}\Gamma[\frac{d-1}{2}]}{2\Gamma[\frac{d}{2}]}, \quad \tilde{c}_0 = \frac{(-1)^{\frac{d-2}{2}}\Gamma[\frac{d-1}{2}]}{2\sqrt{\pi}\Gamma[\frac{d}{2}]}(\psi(\frac{d}{2}) + \gamma_E + 2\ln(2)) & \end{array} \tag{7.88}$$

Chapter 8

Adding A Black Brane To The Ads Space

We will now add a black brane to the Ads space-time. As seen in section 4, this adds a thermodynamic element and temperature to the space. This is analogous to adding temperature to the CFT[14].

8.1 A Strip in ADS_{d+1}/CFT_d With a Generic Black Brane

Now we can solve for ADS with a black hole for d dimensions with

$$ds^2 = \frac{R^2}{z^2}(\frac{dz^2}{g(z)} - f(z)dt^2 + d\vec{x}^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{8.1}$$

with

$$g = 1 - mz^d..., \qquad f = 1 - mz^d... \tag{8.2}$$

Where m is a constant, the ellipsis characterises the components of z that go to zero faster. For our holographic formula, we have

$$S_{ee} = \frac{A_{min}}{4G_N} \tag{8.3}$$

$$A = \int d^2\xi \sqrt{det\hat{g}_{ab}} \tag{8.4}$$

$$\text{with } \hat{g}_{ab} \text{ being} \tag{8.5}$$

$$\hat{g}_{\alpha\beta} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b} \tag{8.6}$$

making one coordinate depend on the others

$$x_1 = (z, \vec{x}) \tag{8.7}$$

This leads to

$$g = \frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + (\sum^{d-2}(\frac{\partial x_1}{\partial x_i})^2) + \frac{1}{g(z)}) \tag{8.8}$$

this leads to

$$A = \int dzd\vec{x}\sqrt{\frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + (\sum^{d-2}(\frac{\partial x_1}{\partial x_i})^2) + \frac{1}{g(z)})} \tag{8.9}$$

because if a derivative in any direction will increase the result, the derivatives in that direction are zero, so we can state

$$A = V_{d-2}\int dz\sqrt{\frac{R^{2(d-1)}}{z^{2(d-1)}}((\frac{\partial x_1}{\partial z})^2 + \frac{1}{g(z)})} \tag{8.10}$$

now following the same logic from the ADS_4 case, we find the differential equation of the $\dot{x}$ and the minimal integral being

$$\frac{\partial x_1}{\partial z} = (\sqrt{\frac{\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}{(1 - \frac{z^{2(d-1)}}{z_*^{2(d-1)}})}}\frac{1}{\sqrt{g(z)}} \tag{8.11}$$

$$L = 2\int_0^{z^*}\sqrt{\frac{\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}{(1 - \frac{z^{2(d-1)}}{z_*^{2(d-1)}})}}\frac{1}{\sqrt{g(z)}}dz \tag{8.12}$$

$$A = 2V_{d-2}R^{(d-1)}\int_\epsilon^{z_*} dz\frac{1}{z^{d-1}}\sqrt{\frac{1}{1 - \frac{z^{2(d-1)}}{z_*^{2(d-1)}}}}\frac{1}{\sqrt{g(z)}} \tag{8.13}$$

8.2 A Sphere in ADS_{d+1}/CFT_d With a Generic Black Brane

For the black brane with a sphere, we start where we always do with the metric

$$ds^2 = \frac{R^2}{z^2}(\frac{dz^2}{g(z)} - f(z)dt^2 + d\vec{x}^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{8.14}$$

with

$$g = 1 - mz^d..., \qquad f = 1 - mz^d... \tag{8.15}$$

for the sphere we can find sphere $dx_1^2 + d\vec{x}_{d-1}^2 = dr^2 + r^2 d\Omega_d^2 - 1$

$$ds^2 = \frac{R^2}{z^2}(\frac{dz^2}{g(z)} - f(z)dt^2 + dr^2 + r^2 d\Omega_2^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{8.16}$$

for our holographic formula, we have

$$S_{ee} = \frac{A_{min}}{4G_N} \tag{8.17}$$

$$A = \int d^2\xi \sqrt{det\hat{g}_{ab}} \tag{8.18}$$

$$\text{with } \hat{g}_{ab} \text{ being} \tag{8.19}$$

$$\hat{g}_{\alpha\beta} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b} \tag{8.20}$$

now letting z be a function of r leads to the off-axis components being zero because we are finding a minimal area surface, so we get

$$\hat{g}_{zz} = \frac{R^2}{z^2}(\frac{dr}{dz})^2 + \frac{R^2}{z^2}\frac{1}{g(z)} \tag{8.21}$$

$$\hat{g}_{z\Omega_i} = \hat{g}_{\Omega_i z} = 0 \tag{8.22}$$

$$\hat{g}_{\Omega_i\Omega_j} = \hat{g}_{\Omega_j\Omega_i} = 0 \tag{8.23}$$

$$\hat{g}_{\Omega_i\Omega_i} = \frac{R^2 r^2}{z^2} \tag{8.24}$$

$$\tag{8.25}$$

using these to find the determinate

$$g = \frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + \frac{1}{g(z)}) \tag{8.26}$$

$$\tag{8.27}$$

$$A = \int dz d\Omega_{d-1}\sqrt{\frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + \frac{1}{g(z)})} \tag{8.28}$$

$$A = V(S^{d-2})\int dr\sqrt{\frac{R^{2(d-1)}r^{2(d-2)}}{z^{2(d-1)}}((\frac{dr}{dz})^2 + \frac{1}{g(z)})} \tag{8.29}$$

letting $\frac{dr}{dz} = \dot{r}$

$$A = V(S^{(d-2)})R^{(d-1)}\int dr\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2 + \frac{1}{g(z)}} \tag{8.30}$$

Solving the Euler-Lagrange equations

$$\frac{d}{dr}(\frac{\partial}{\partial \dot{r}}(\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2+\frac{1}{g(z)}})) - \frac{\partial}{\partial z}\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2+\frac{1}{g(z)}} = 0 \quad (8.31)$$

$$\frac{d}{dr}(\frac{r^{(d-2)}}{z^{(d-1)}}\frac{\dot{r}}{\sqrt{\dot{r}^2+\frac{1}{g(z)}}}) - \frac{\partial}{\partial z}\frac{r^{(d-2)}}{z^{(d-1)}}\sqrt{\dot{r}^2+\frac{1}{g(z)}} = 0 \quad (8.32)$$

$$\frac{(d-2)r^{d-3}z^{1-d}\dot{r}^2}{\sqrt{\dot{r}^2+\frac{1}{g(z)}}} + \frac{(d-1)z^{-d}r^{d-2}\dot{r}}{\sqrt{\dot{r}^2+\frac{1}{g(z)}}} + \frac{z^{1-d}r^{d-2}\ddot{r}}{\sqrt{\dot{r}^2+\frac{1}{g(z)}}} - \frac{z^{1-d}r^{d-2}\dot{r}(2\dot{r}\ddot{r}-\frac{\dot{g}(z)}{g(z^2)})}{2(\dot{r}^2+\frac{1}{g(z)})^{\frac{3}{2}}} - (d-2)r^{d-3}z^{1-d}\sqrt{\dot{r}^2+1} = 0 \quad (8.33)$$

This second-order non-linear differential equation cannot be solved using Any analytical methods, so we require numerics, i.e. the shooting method.

8.3 Schwarzschild Black Brane

From [14], we use the solution to the action
$S_{Bulk} = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-det(g_{nm})}(\mathcal{R}+\frac{d(d-1)}{R^2})$ they find the black brane describe by :

$$f(z) = g(z) = 1 - mz^d \quad (8.34)$$

So using the equation 8.13 and 8.33, we now state the equations for both the strip and the sphere

$$A = 2V_2R^{(3)}\int_{\epsilon}^{z_*} dz\frac{1}{z^{d-1}}\sqrt{\frac{1}{1-\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}}\frac{1}{\sqrt{1-mz^d}} \quad (8.35)$$

$$L = 2\int_0^{z^*}\sqrt{\frac{\frac{z^{2(d-1)}}{z_*^{2(d-1)}}}{(1-\frac{z^{2(d-1)}}{z_*^{2(d-1)}})}}\frac{1}{\sqrt{1-mz^d}}dz \quad (8.36)$$

and

$$\frac{(d-2)r^{d-3}z^{1-d}\dot{r}^2}{\sqrt{\dot{r}^2+\frac{1}{1-mz^d}}}+\frac{(d-1)z^{-d}r^{d-2}\dot{r}}{\sqrt{\dot{r}^2+\frac{1}{1-mz^d}}}+\frac{z^{1-d}r^{d-2}\ddot{r}}{\sqrt{\dot{r}^2+\frac{1}{1-mz^d}}}$$
$$-\frac{z^{1-d}r^{d-2}\dot{r}(2\dot{r}\ddot{r}-\frac{dmz^{d-1}}{(1-mz^d)^2})}{2(\dot{r}^2+\frac{1}{1-mz^d})^{\frac{3}{2}}}-(d-2)r^{d-3}z^{1-d}\sqrt{\dot{r}^2+1}=0 \quad (8.37)$$

Now we can solve both numerically. We find the two figures below using the RK45 method to generate the path for the shape of the strip case.

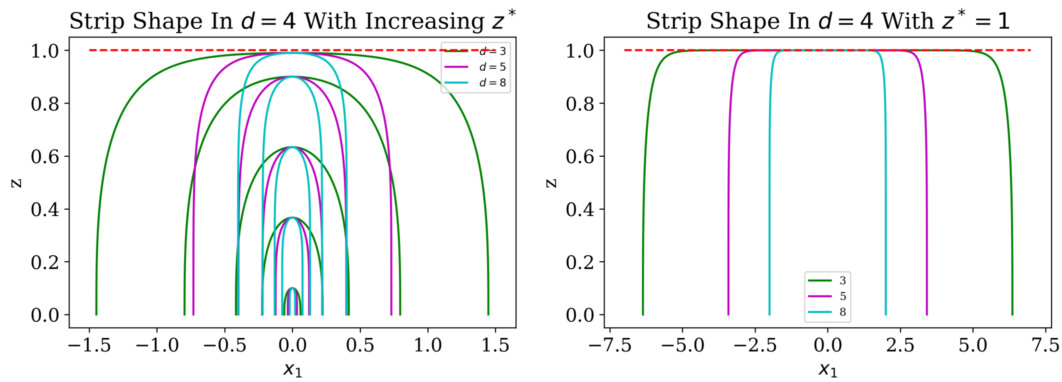

FIGURE 8.1: On the left, we have the shape of the strip with z^* being 0.1,0.4,0.6,0.9, and on the right, when z^* is one, both with varying dimension

Veiwing these figures, we see that as z^* increases, L increase dramatically. As we send z^* off to infinity, the minimal area surface goes along the black brane and becomes the thermal surface.

Now, looking at the sphere in Figure, we see that the sphere gets dragged toward the black brane, and as r increases, we would get the thermal surface.

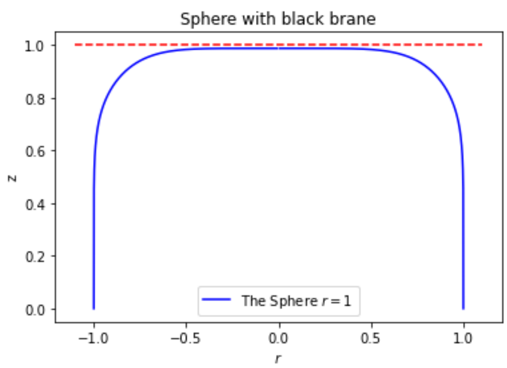

FIGURE 8.2: The minimal surface sphere with r from 0 to 1 at d=4

Chapter 9

Conclusions

In this paper, we re-derived the equations from Ryu and Takayanagi's proposal in Ads_3/CFT_2 and for higher dimensional systems for the strip and the sphere case. We used the holographic formula in the setup of Ads/CFT because the calculations of the entanglement entropy from a Quantum field theory perspective are pretty tricky, making it more manageable. The solution for the CFT in 2 dimensions is well known, and the finding of the holographic formula agrees well with that solution. We also produced the equations for the higher dimensional models. We also placed a black brane into the solution, finding that as we increase z^* we get a thermal space.

The Ryu and Takayanagi proposal has emerged as a powerful tool for understanding the holographic duality between quantum field theories and gravitational theories in higher-dimensional spacetimes. Extending the proposal to solve d-dimensional cases by including temperature effects through black holes.

The insights from the Ryu and Takayanagi proposal have also found applications in condensed matter physics. The proposal has been used to study the entanglement properties of strongly correlated quantum systems, providing valuable insights into quantum phase transitions, topological order, and quantum information processing in condensed matter systems. For example, holographic entanglement entropy has been employed to characterize the behaviour of entangled states in condensed matter systems, such as fractional quantum Hall states[10] and quantum spin liquids[7]. This leads to many more avenues for further research in the condensed matter field.

What is next, and what can we look at in the field of Ads/CFT and Entanglement entropy? There is a proof by Maldacena [20] of the RT proposal. Further, there are specific calculations for different hall states in condensed matter physics. More on the Ads/CFT side, we can find the proof for this conjecture by generating the Einstein equations from the CFT [11]. And due to its power, we can use it for lattice simulations [24]. The future of Ads/CFT is promising, with a massive amount of applicability.

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