

Measure Theory

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Hence $(L_p/\sim, \|\cdot\|_p)$ is a normed vector space.

When $p = \infty$, define

$$\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(|f| > a) = 0\}, \quad L_\infty := \{f : \|f\|_\infty < \infty\}.$$

We call the functions in L_∞ **essentially bounded**.

Let $\mu(X) < \infty$, then $f \in L_\infty \implies f \in L_p$, and $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$: For all $0 < a < \|f\|_\infty$,

$$a^p \mu(|f| > a) \leq \int_X |f|^p \mathbf{I}_{|f| > a} d\mu \leq \int_X |f|^p d\mu \leq \|f\|_\infty^p \mu(X),$$

So taking the exponent $\frac{1}{p}$,

$$a \leftarrow a \mu(|f| > a)^{\frac{1}{p}} \leq \|f\|_p \leq \|f\|_\infty$$

But when $\mu(X) = \infty$, let $f \equiv 1$, then $f \in L_\infty$ but $f \notin L_p$.

Theorem 0.0.1

Let $f, g \in L_\infty$,

$$\|fg\| \leq \|f\| \|g\|_\infty,$$

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Proof.

$$\int_X |fg| d\mu \leq \int_X |f| \|g\|_\infty d\mu = \|f\| \|g\|_\infty.$$

Since $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$, a.e., we get the second inequality. \square

Similarly we get $(L_\infty, \|\cdot\|_\infty)$ is a normed vector space.

The norm can deduce a *distance*:

$$\rho(f, g) := \|f - g\|.$$

Theorem 0.0.2 (L_p space is complete)

Let $1 \leq p \leq \infty$. If $\{f_n\} \subset L_p$ satisfying $\lim_{n,m \rightarrow \infty} \|f_n - f_m\|_p = 0$, then there exist $f \in L_p$ s.t. $\lim_{n \rightarrow \infty} \|f - f_n\|_p = 0$.

Proof. Take $n_1 < n_2 < \dots$ such that

$$\|f_m - f_n\|_p \leq \frac{1}{2^k}, \quad \forall n, m \geq n_k.$$

Let $g = \uparrow \lim_{k \rightarrow \infty} g_k$, where

$$g_k := |f_{n_1}| + \sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}| \in L_p, \quad g_k \geq 0.$$

Since

$$\begin{aligned} \|g_k\|_p &\leq \|f_{n_1}\|_p + \sum_{i=1}^k \|f_{n_{i+1}} - f_{n_i}\|_p \leq \|f_{n_1}\|_p + 1. \\ \implies \|g\|_p &= \uparrow \lim_{k \rightarrow \infty} \|g_k\|_p \leq \|f_{n_1}\|_p + 1. \end{aligned}$$

Here we use the monotone convergence theorem. We can check the above also holds for $p = \infty$.

Therefore $g \in L_p \implies g < \infty, a.e..$ We have

$$f := f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i}) = \lim_{k \rightarrow \infty} f_k, a.e.$$

the series is absolutely convergent, so f exists a.e. and $|f| \leq g, a.e..$

Lastly we can check: when $p = \infty$,

$$\|f_n - f\|_{\infty} \leq \|f_n - f_{n_k}\|_{\infty} + \|f_{n_k} - f\|_{\infty},$$

where the both term approach to 0 as $n \rightarrow \infty$.

When $p < \infty$, by Fatou's lemma,

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p d\mu = \int_X \lim_{k \rightarrow \infty} |f_n - f_{n_k}|^p d\mu \leq \liminf_{k \rightarrow \infty} \int_X |f_n - f_{n_k}|^p d\mu \leq \varepsilon.$$

□

Remark 0.0.3 — Using the same technique we can prove that if f_n is Cauchy in measure, then f_n converge to some f in measure:

Let $A_i = \{|f_{n_{i+1}} - f_{n_i}| > 2^{-i}\}$ s.t. $\mu(A_i) < 2^{-i}$.

Define $f = f_{n_1} + \sum_{i \geq 1} (f_{n_{i+1}} - f_{n_i})$ on the set $\bigcup_{k \geq 1} \bigcap_{i \geq k} A_i^c$.

This theorem implies that $(L_p, \|\cdot\|_p)$ is a Banach space. So we can try to define an *inner product* on L_p space:

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).$$

We can check $\langle \cdot, \cdot \rangle$ is bilinear only if $p = 2$, so L_2 is actually a Hilbert space.

When $0 < p < 1$, let

$$\|f\|_p := \int_X |f|^p d\mu, \quad L_p = \{f : \|f\|_p < \infty\}.$$

Lemma 0.0.4

Let $0 < p < 1$, $C_p = 1$, then

$$|a + b|^p \leq C_p(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}.$$

So L_p is a vector space.

Theorem 0.0.5 (Minkowski)

Let $0 < p < 1$ then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark 0.0.6 — When $\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$, $0 < p < 1$. then it won't satisfy Minkowski's inequality.

Thus L_p is only a metric space but not a normed vector space. Using the same method we can prove L_p is a complete metric space.

§0.1 Convergence in L_p space

Definition 0.1.1. Let $0 < p \leq \infty$, $f, f_1, f_2, \dots \in L_p$. When $\|f_n - f\|_p \rightarrow 0$, then we write $f_n \xrightarrow{L_p} f$, called **average converge of order p** .

Theorem 0.1.2

Let $0 < p < \infty$, $f, f_1, \dots \in L_p$,

- If $f_n \xrightarrow{L_p} f$, then $f_n \xrightarrow{\mu} f$, and $\|f_n\|_p \rightarrow \|f\|_p$.
- If $f_n \rightarrow f, a.e.$ or in measure, then $\|f_n\|_p \rightarrow \|f\|_p \iff f_n \xrightarrow{L_p} f$.

Proof. When $f_n \xrightarrow{L_p} f$, let $A := \{|f_n - f| > \varepsilon\}$,

$$\mu(A) \leq \frac{1}{\varepsilon^p} \int_X |f_n - f|^p \mathbf{1}_A d\mu \leq \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \rightarrow 0.$$

and obviously $\|f_n\|_p \rightarrow \|f\|_p$

On the other hand, when $f_n \rightarrow f, a.e.$ and $\|f_n\|_p \rightarrow \|f\|_p$, From $|a + b|^p \leq C_p(|a|^p + |b|^p)$,

$$g_n := C_p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0.$$

$g_n \rightarrow 2C_p|f|^p, a.e.$, so

$$\int_X 2C_p|f|^p d\mu \leq \liminf_{n \rightarrow \infty} \int_X g_n d\mu = 2C_p \int_X |f|^p d\mu - \limsup_{n \rightarrow \infty} \int_X |f_n - f|^p d\mu.$$

When $f_n \rightarrow f$ in measure, for any subsequence there exist its subsequence $f_{n'} \rightarrow f, a.e.$, so $\|f_{n'} - f\|_p \rightarrow 0$, hence $\|f_n - f\|_p \rightarrow 0$. \square

Remark 0.1.3 — This theorem implies for any L_p function f , we can take simple functions $f_1, f_2, \dots \rightarrow f$ and $|f_n| \uparrow |f|$, so $f_n \xrightarrow{L_p} f$.

Definition 0.1.4 (Weak convergence). Let $1 < p < \infty$, and $f_1, f_2, \dots \in L_p$. If

$$\lim_{n \rightarrow \infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu, \quad \forall g \in L_q.$$

Then we say f_n **weak convergent** to f , denoted by $f_n \xrightarrow{(w)L_p} f$.

When $p = 1$ and (X, \mathcal{F}, μ) is a σ -finite measure space, and the condition also holds, we say $\{f_n\}$ weak convergent to f in L_1 .

Corollary 0.1.5

Let $1 \leq p < \infty$, then

$$f_n \xrightarrow{L_p} f \implies f_n \xrightarrow{(w)L_p} f.$$

Proof. By Holder's inequality,

$$\left| \int_X (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \|g\|_q \rightarrow 0.$$

□

If $\sup_{t \in T} \|f_t\|_p =: M < \infty$, then we say $\{f_t, t \in T\}$ is **bounded in L_p** .

Theorem 0.1.6

Let $1 < p < \infty$, $\{f_n\} \subset L_p$, there exists M s.t. $\|f_n\|_p \leq M, \forall n$. If $f_n \rightarrow f$, a.e. or in measure, then $f \in L_p$ and $f_n \rightarrow f$ weakly.

Proof. First $\|f\|_p \leq M$:

$$\int_X |f|^p \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p \, d\mu \leq M^p.$$

Next we prove the weak convergence: For all $g \in L_q$, recall the bounded convergence theorem in probability, we can view M as a bound of f_n , and $\|g\|_q$ as P .

Let $B = \{|f_n - f| \leq \varepsilon\}$, consider

$$a := \int_B (f_n - f) g \, d\mu, \quad b := \int_{B^c} (f_n - f) g \, d\mu.$$

Note that

$$|a| \leq \varepsilon \int_X |g| \, d\mu.$$

But $\int_X |g| \, d\mu$ might be infinity, so let $A_k := \{\frac{1}{k} \leq |g|^q \leq k\}$, we have

$$\int_{A_k} |g| \, d\mu \leq k^{\frac{1}{q}} \mu(A_k) < \infty.$$

$(\frac{1}{k}\mu(A_k) < \int_{A_k} |g|^q d\mu < \infty$ since $g \in L_q$).

Now we can proceed:

$$a := \int_{A_k B} (f_n - f)g d\mu, \quad b := \int_{A_k^c \cup B^c} (f_n - f)g d\mu.$$

Now $|a| \leq \hat{\varepsilon} k^{\frac{1}{q}} \mu(A_k) < \varepsilon$.

$$\left| \int_X (f_n - f)g \mathbf{I}_{A_k^c \cup B^c} d\mu \right| \leq \|f_n - f\|_p \|g \mathbf{I}_{A_k^c \cup B^c}\|_q \leq 2M \left(\int_{A_k^c} |g|^q d\mu + \int_{A_k \setminus B} |g|^q d\mu \right).$$

By LDC(Dominated convergence), $A_k^c \rightarrow \{g = 0, \infty\}$, so $\int_{A_k^c} |g|^q d\mu < \varepsilon$.

Since $\mu(A_k) < \infty$, $f_n \rightarrow f, a.e. \implies f_n \xrightarrow{\mu} f$. By the continuity of integrals, $\mu(A_k \setminus B) \leq \mu(B^c) < \delta \implies \int_{A_k \setminus B} |g|^q d\mu < \varepsilon$.

Now we can conclude: $\forall \varepsilon > 0$, first choose k large, then $\hat{\varepsilon}$ small, we get

$$\int_X (f_n - f)g d\mu \leq \varepsilon + 4M\varepsilon \implies f_n \xrightarrow{(w)L_p} f.$$

□

Remark 0.1.7 — The proof is a little complicated, we divide the entire integral to three part, and estimate them respectively.

When $p = 1$, f_n bounded in L_p *cannot* imply weak convergence.

Example 0.1.8

Let $X = \mathbb{N}$, $\mu(\{k\}) = 1, \forall k$, clearly it's σ -finite.

Let $f_n(k) = \mathbf{I}_{k=n}$, then $\|f_n\| = \sum_k \mu(k) |f_n(k)| = 1$, and $f_n \rightarrow 0, a.e..$

But let $g = 1 \in L_\infty$, $\int_X (f_n - f)g d\mu = 1 \not\rightarrow 0$.

Proposition 0.1.9

Let $f_1, f_2, \dots \in L_1$, then:

$$\|f_n\| \rightarrow \|f\| \& f_n \rightarrow f, a.e. \implies f_n \xrightarrow{L_1} f \implies f_n \xrightarrow{(w)L_1} f \implies \int_A f_n d\mu \rightarrow \int_A f d\mu, \forall A.$$

Proof. For the last part let $g = \mathbf{I}_A$, the rest is trivial. □

§0.2 Integrals in probability space

We can also consider L_p space in probability space (Ω, \mathcal{F}, P) .

Theorem 0.2.1

Let $0 < s < t < \infty$. Then $L_t \subset L_s$. If $s \geq 1$, we have $\|f\|_s \leq \|f\|_t$, with equality f constant.

Proof. When $f \in L_t$, let $p = \frac{t}{s}, q = \frac{t}{t-s}$.

$$\int_{\Omega} |f|^s \cdot 1 \, dP \leq \| |f|^s \|_p \|1\|_q = (E|f|^{sp})^{\frac{1}{p}} = (E|f|^t)^{\frac{1}{p}}.$$

So $f \in L_s \implies L_t \subset L_s$. When $s \geq 1$,

$$\|f\|_s^s \leq (\|f\|_t)^{\frac{t}{p}} = \|f\|_t^s \implies \|f\|_s \leq \|f\|_t.$$

□

From this we know $L_{\infty} \subset L_p$, and $\|f\|_p \uparrow \|f\|_{\infty}$.

Remark 0.2.2 — This theorem does not hold for general space. Let $X = \mathbb{N}$, $\mu(\{n\}) = 1$, $f(n) = \frac{1}{n}$, then $f \in L_2 \setminus L_1$.

The expectation $E f^k$ is called **k -order moment** of random variable f .

Definition 0.2.3 (Uniformly integrable). Let $\{f_t, t \in T\}$ be r.v.'s, if $\forall \varepsilon > 0, \exists \lambda > 0$, such that

$$E|f_t| \mathbf{I}_{\{|f_t| > \lambda\}} < \varepsilon, \quad \forall t \in T,$$

then we say $\{f_t, t \in T\}$ **uniformly integrable**.

If $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall A \in \mathcal{F}$,

$$P(A) < \delta \implies E|f_t| \mathbf{I}_A < \varepsilon, \forall t \in T,$$

we say $\{f_t\}$ is uniformly absolutely continuous, which is abbreviated as absolutely continuous.

Theorem 0.2.4

Uniformly integrable \iff absolute continuity and bounded in L_1 .

Proof. Firstly when $\{f_t\}$ uniformly integrable, $\forall A \in \mathcal{F}, \lambda > 0$,

$$\begin{aligned} E|f_t| \mathbf{I}_A &= E|f_t| \mathbf{I}_{A \cap \{|f_t| \leq \lambda\}} + E|f_t| \mathbf{I}_{A \cap \{|f_t| > \lambda\}} \\ &\leq \lambda P(A) + E|f_t| \mathbf{I}_{\{|f_t| > \lambda\}} \end{aligned}$$

Let $A = X$ we know $E|f_t| \leq \lambda + \frac{\varepsilon}{2}, \forall t \in T$. Now let $\delta = \frac{\varepsilon}{2\lambda}$ we get AC property.

On the other hand,

$$\lambda P(|f_t| > \lambda) \leq E|f_t| \mathbf{I}_{\{|f_t| > \lambda\}} \leq E|f_t| \leq M, \forall t \in T.$$

So when $\lambda > \frac{M}{\delta}$, $P(|f_t| > \lambda) < \delta$, hence $E|f_t| \mathbf{I}_{\{|f_t| > \lambda\}} \leq \varepsilon, \forall t \in T$.

□

Theorem 0.2.5

Let $0 < p < \infty$, and $f_n \rightarrow f$ in probability. TFAE:

- (1) $\{|f_n|^p\}$ uniformly integrable;
- (2) $f_n \xrightarrow{L_p} f$;
- (3) $f \in L_p$ and $\|f_n\|_p \rightarrow \|f\|_p$.

Proof. (1) \implies (2): Take subsequence $f_{n'} \rightarrow f, a.s.$,

$$E|f|^p \leq \liminf_{n \rightarrow \infty} E|f_n|^p < \infty,$$

since $\{|f_n|^p\}$ is bounded in L_1 . This means $f \in L_p$.

Let $A_n = \{|f_n - f| > \varepsilon\}$, now we compute

$$E|f_n - f|^p \leq \varepsilon^p + E|f_n - f|^p \mathbf{I}_{A_n} \leq \varepsilon^p + C_p E|f_n|^p \mathbf{I}_{A_n} + C_p E|f|^p \mathbf{I}_{A_n}$$

Since $P(A_n) \rightarrow 0$ and $\{|f_n|^p\}$ absolutely continuous (also note $E|f|^p \mathbf{I}_{A_n} \rightarrow 0$), RHS converges to 0. Therefore $f_n \xrightarrow{L_p} f$.

As for (3) \implies (1), we'll prove a lemma:

Lemma 0.2.6

If $f_n \xrightarrow{P} f$, then $\forall 0 < p < \infty$,

$$|f_n|^p \mathbf{I}_{\{|f_n| \leq \lambda\}} \xrightarrow{P} |f|^p \mathbf{I}_{\{|f| \leq \lambda\}}, \quad \forall \lambda \in C(F_{|f|}).$$

By lemma and bounded convergence theorem, their expectation also converges. Note that $\|f_n\|_p \rightarrow \|f\|_p$, so

$$E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda\}} \rightarrow E|f|^p \mathbf{I}_{\{|f| > \lambda\}},$$

thus $\forall \varepsilon > 0, \exists \lambda_0 \in C(F_{|f|})$, s.t. $E|f|^p \mathbf{I}_{\{|f| > \lambda_0\}} < \frac{\varepsilon}{2}$, thus

$$\exists N, \quad E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda_0\}} < \varepsilon, \quad \forall n > N.$$

Now we can take $\lambda > \lambda_0$ such that $\max_{n \leq N} E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda\}} < \varepsilon$, and we're done.

Proof of the lemma. Since $|f_n| \rightarrow |f|$ in probability, WLOG $f_n, f \geq 0$. Define

$$A_n := (\{f_n \leq \lambda\} \Delta \{f \leq \lambda\}) \cap \{|f_n^p - f^p| > \varepsilon\}$$

$$B_n := \{f_n, f \leq \lambda, |f_n^p - f^p| > \varepsilon\}.$$

Since x^p is uniformly continuous in $[0, \lambda]$, $B_n \subset \{|f_n - f| > \kappa_{\varepsilon, \lambda}\}$, $P(B_n) \rightarrow 0$.

Also $P(A_n) \rightarrow 0$ as

$$A_n \subset \{\lambda - \delta < f \leq \lambda + \delta\} \cup \{|f_n - f| > \delta\},$$

and $F_{|f|}$ continuous at λ . □

□

§1 Signed measure

§1.1 Definitions

Let (X, \mathcal{F}, μ) be a measure space, consider

$$\varphi(A) := \int_A f d\mu, \quad \forall A \in \mathcal{F}.$$

If the integral of f exists, then φ has countable additivity. Also note $\varphi(\emptyset) = 0$, so φ looks like a measure, except it can take negative values.

In fact, denote $X^+ = \{f \geq 0\}$, $X^- = \{f < 0\}$, then $\varphi(A) = \varphi(AX^+) + \varphi(AX^-)$.

Definition 1.1.1 (Signed measure). If a set function $\varphi : \mathcal{F} \rightarrow \overline{\mathbb{R}}$ which satisfies countable additivity and $\varphi(\emptyset) = 0$, then we call φ a **signed measure**.

If $|\varphi(A)| < \infty, \forall A \in \mathcal{F}$, then φ is **finite**; Similarly we define **σ -finite**.

Since $\int_A f d\mu$ can't reach both $\pm\infty$ (otherwise the integral doesn't exist), so

Proposition 1.1.2

Let φ be a signed measure, then:

$$\varphi(A) < \infty, \quad \forall A \in \mathcal{F}, \quad \text{or} \quad \varphi(A) > -\infty, \quad \forall A \in \mathcal{F}.$$

Proof. Assume that $\varphi(A) = \infty, \varphi(B) = -\infty$, then:

$$\varphi(A \cup B) = \varphi(A) + \varphi(A \setminus B) = +\infty,$$

and similarly $\varphi(A \cup B) = -\infty$, contradiction! \square

Remark 1.1.3 — From now on we may assume $\varphi(A) > -\infty$.

Proposition 1.1.4

If $A \supseteq B$, and $|\varphi(A)| < \infty$, then $|\varphi(B)| < \infty$.

Proof. Trivial, same as above proposition. \square

Proposition 1.1.5

Let A_1, A_2, \dots be pairwise disjoint sets, and $|\varphi(\sum_{n=1}^{\infty} A_n)| < \infty$, then

$$\sum_{n=1}^{\infty} |\varphi(A_n)| < \infty.$$

Proof. Let $I = \{n : \varphi(A_n) > 0\}, J = \{n : \varphi(A_n) < 0\}$,

$$B = \sum_{n \in I} A_n, \quad C = \sum_{n \in J} A_n,$$

since $B, C \subset \sum_{n=1}^{\infty} A_n$, thus $\varphi(B), \varphi(C) \in \mathbb{R}$.

Note that $\sum_{n \in I} |\varphi(A_n)| = |\varphi(B)|, \sum_{n \in J} \varphi(A_n) = |\varphi(C)|$, and we're done. \square

§1.2 Hahn decomposition and Jordan decomposition

Let's look at the indefinite integral again, notice that

$$\varphi(A) = \int_{A \cap \{f > 0\}} f d\mu + \int_{A \cap \{f < 0\}} f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu.$$

It turns out that this property holds for any signed measure.

Definition 1.2.1 (Hahn decomposition). If a partition $\{X^+, X^-\}$ of X satisfies:

$$\varphi(A) \geq 0, \forall A \subset X^+, \quad \varphi(A) \leq 0, \forall A \subset X^-,$$

then $\{X^+, X^-\}$ is called a **Hahn decomposition** of φ .

Definition 1.2.2 (Jordan decomposition). Let $\varphi^\pm = \int_A f^\pm d\mu$ be measures, if

$$\varphi = \varphi^+ - \varphi^-,$$

then it's called a **Jordan decomposition** of φ .

We're going to find X^+ , or equivalently, find φ^+ . Let $\varphi^*(A) := \sup\{\varphi(B) : B \subseteq A\}$.

It's clear that φ^* is non-negative, monotone, and $\varphi^*(\emptyset) = 0$.

Consider $\mathcal{F}^- = \{A : \varphi^*(A) = 0\}$. Intuitively, this is all the subsets of X^- , unioned with “null sets” in X^+ .

Theorem 1.2.3 (Hahn decomposition)

Let X^- be a set with maximum $|\varphi|$ in \mathcal{F}^- , (since $\varphi > -\infty$, X^- must exist) and $X^+ = X \setminus X^-$ doesn't contain any set A with $\varphi(A) < 0$.

Furthermore, the Hahn decomposition is unique:

$$\varphi(A) = 0, \quad \forall A \in X_1^+ \Delta X_2^+ = X_1^- \Delta X_2^-.$$

The critical part of this theorem is:

Lemma 1.2.4

If $\varphi(A) < 0$, then we can find $A_0 \subset A$ s.t. $\varphi^*(A_0) = 0$, $\varphi(A_0) < 0$.

To prove this lemma, we need another lemma:

Lemma 1.2.5

If $\varphi(A) < \infty$, then $\forall \varepsilon > 0$, $\exists A_\varepsilon \subset A$ s.t.

$$\varphi(A_\varepsilon) \geq 0, \quad \varphi^*(A \setminus A_\varepsilon) \leq \varepsilon.$$

Proof. Assume by contradiction that $\exists \varepsilon_0 \geq 0$ s.t. $\forall A_0 \subset A$, $\varphi(A_0) < 0$ or $\varphi^*(A \setminus A_0) > \varepsilon_0$, this means,

$$\varphi(A_0) \geq 0 \implies \varphi^*(A \setminus A_0) > \varepsilon_0.$$

This will clearly yield a contradiction:

Take any $\varphi(A_0) \geq 0$ (say $A_0 = \emptyset$), then exists $A_1 \subset A \setminus A_0$ s.t. $\varphi(A_1) > \varepsilon_0$, and $\varphi(A_0 \cup A_1) \geq 0$, continuing this process we can get infinitely many pairwise disjoint sets A_1, A_2, \dots , with $\varphi(A_n) > \varepsilon_0$, so $\varphi(\sum_{i=1}^\infty A_n) = \infty \implies \varphi(A) = \infty$, contradiction! \square

Proof of Lemma 1.2.4. Applying above lemma repeatedly and take a limit:

Take $C_1 \subset A$ s.t. $\varphi(C_1) \geq 0$ and $\varphi^*(A \setminus C_1) \leq 1$. Let $A_1 = A \setminus C_1$, $\varphi(A_1) < 0$.

Again take

$$C_{k+1} \subset A_k, A_{k+1} = A_k \setminus C_{k+1} \implies \varphi^*(A_{k+1}) \leq \frac{1}{k+1}, \varphi(A_{k+1}) < 0.$$

Since $A_k \downarrow$, let $A_0 = \lim_{k \rightarrow \infty} A_k$, note $\varphi^*(A_k) \downarrow 0$, we must have $\varphi^*(A_0) = 0$.

Also $\varphi(\sum C_k) = \sum \varphi(C_k) \geq 0$, so $\varphi(A_0) < 0$. □

Proof of Theorem 1.2.3. First we prove that \mathcal{F}^- is a σ -ring: $\emptyset \in \mathcal{F}^-$, if $A_1, A_2 \in \mathcal{F}^-$,

$$0 \leq \varphi^*(A_1 \setminus A_2) \leq \varphi(A_1) = 0.$$

Thus $A_1 \setminus A_2 \in \mathcal{F}^-$.

If $A_1, A_2, \dots \in \mathcal{F}^-$ pairwise disjoint,

$$\varphi(B) = \sum_{n=1}^{\infty} \varphi(B \cap A_n) \leq 0, \quad \forall B \subset \sum_{n=1}^{\infty} A_n.$$

Hence $\sum_{n=1}^{\infty} A_n \in \mathcal{F}^-$.

Next we'll prove Hahn decomposition exists:

Let $\alpha := \inf\{\varphi(A) : A \in \mathcal{F}^-, \alpha \leq 0\}$.

Let $\{A_n\} \in \mathcal{F}^-$ s.t. $\varphi(A_n) \rightarrow \alpha$, then $X^- := \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}^-$.

$$\varphi(X^-) = \varphi(A_n) + \varphi(X^- \setminus A_n) \leq \varphi(A_n) + \varphi^*(X^- \setminus A_n) = \varphi(A_n) \rightarrow \alpha.$$

Therefore $-\infty < \varphi(X^-) = \alpha$.

Hence $\forall A, \varphi(AX^-) \leq \varphi^*(X^-) = 0$. By Lemma 1.2.4 we get $\forall A, \varphi(AX^+) \geq 0$, otherwise $\exists A_0 \subset A$ s.t. $\varphi^*(A_0) = 0, \varphi(A_0) < 0$. Then $\varphi(X^- \cup A_0) = \alpha + \varphi(A_0) < \alpha$, contradiction!

At last we'll prove the uniqueness:

If X_1^\pm, X_2^\pm are both Hahn decompositions, then $A \in X_1^+ \cap X_2^- + X_1^- \cap X_2^+$, it's clear $\varphi(A) = 0$. □

Theorem 1.2.6 (Jordan decomposition)

The Jordan decomposition exists and is unique:

$$\varphi = \varphi^+ - \varphi^-, \quad \varphi^+ = \varphi^*, \varphi^- = (-\varphi)^*.$$

Proof. Let φ^\pm be measures with $\varphi^\pm = \pm \varphi(A \cap X^\pm)$. It's clear that this is a Jordan decomposition.

Now given any Jordan decomposition φ^\pm .

Since

$$\forall B \subset A, \varphi(B) \leq \varphi^+(B) \leq \varphi^+(A),$$

so $\varphi^* \leq \varphi^+$. But $A \cap X^+ \subset A$, so $\varphi^* \geq \varphi^+$, which proves the result.

Similarly $\varphi^- = (-\varphi)^*$, so it is unique. □

Remark 1.2.7 — The support of φ^\pm are disjoint, but if $\phi \neq 0$, then the support of $\varphi^\pm + \phi$ intersects. φ^\pm are called the **upper variation** and **lower variation**, respectively, and $|\varphi| = \varphi^+ + \varphi^-$ is called the **total variation**.