

Mathematical Analysis II

Felix Chen

Contents

1	Introduction	1
1.1	Recap	1
2	Lebesgue integrals	2
2.1	Definition of Lebesgue integrals	3
2.2	Dominated convergence theorem	3

§1 Introduction

Teacher: Yang Shiwu

Grading: Homework-Midterm-Endterm: 30-30-40

Contents of this course: Real analysis

§1.1 Recap

Definition 1.1 (Measurable space). Let X be a set and \mathcal{A} be a σ -algebra, we say (X, \mathcal{A}) is a measurable space if

- $\emptyset \in \mathcal{A}$;
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- If $A_k \in \mathcal{A}$, then $\bigcup_{k=1}^{+\infty} A_k \in \mathcal{A}$.

Outer measure m^* :

- $m^*(A) \geq 0$;
- $m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k)$;
- $m^*(A) \leq m^*(B)$ when $A \subset B$.

Caratheodry condition:

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c), \quad \forall T \subset X.$$

We define the measurable sets to be all the sets E satisfying above condition.

This implies the Lebesgue measure space $(\mathbb{R}^n, \mathcal{U}, m)$. It is a complete measure space, i.e. null sets are measurable.

Proposition 1.2 (Properties of measurable sets)

Let E be a measurable set, there exists G_δ set and F_σ set such that

$$E = G_\delta \setminus Z_1 = F_\sigma \cup Z_2.$$

where Z_1, Z_2 are null sets.

$$E_k \nearrow E \implies \lim_{k \rightarrow \infty} m(E_k) = m(E) \text{ and}$$

$$m\left(\liminf_{k \rightarrow \infty} E_k\right) \leq \liminf_{k \rightarrow \infty} m(E_k).$$

(Fauto's Lemma)

Definition 1.3 (Measurable function). Let f be a map from measurable space (X, \mathcal{A}) to (Y, \mathcal{B}) . We say f is measurable if

$$\forall B \in \mathcal{B}, \quad f^{-1}(B) \in \mathcal{A}.$$

In the case of real functions, this is equivalent to

$$\forall t \in \mathbb{R}, \quad \{x; f(x) > t\} \text{ is a measurable set.}$$

Proposition 1.4

Let f be a non-negative measurable function, $\exists \varphi_k \nearrow f$, where φ_k are simple functions.

For a general measurable function f , decompose it to $f = f_+ - f_-$.

Theorem 1.5 (Egorov)

Let E be a measurable set and $m(E) < \infty$, $f_n \rightarrow f, a.e.$, Then $\forall \varepsilon > 0$, there exists a closed set F_ε s.t. $m(E \setminus F_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on F_ε .

Theorem 1.6 (Lusin)

Let E be a measurable set and $m(E) < \infty$. Then $\forall \varepsilon > 0, \exists F_\varepsilon$ such that $f|_{F_\varepsilon}$ is continuous.

Convergence patterns:

- $f_n \rightarrow f, a.e.$
- $f_n \rightarrow f, a.u.$
- $f_n \xrightarrow{m} f$

§2 Lebesgue integrals

The critical question of this section: If a sequence of functions f_n converges to f , when does their integrals $\int f_n$ converge to $\int f$?

§2.1 Definition of Lebesgue integrals

- Simple functions: $f = \sum_{k=1}^N a_k \chi_{E_k}$, define

$$\int f = \sum_{k=1}^N a_k m(E_k).$$

- $f : E \rightarrow \mathbb{R}^n$, where $m(E) < \infty$, f bounded. These functions form the set \mathcal{L}_0 . Then $\epsilon \varphi_k \rightarrow f$, φ_k simple, define

$$\int f = \lim_{k \rightarrow \infty} \int \varphi_k.$$

- Non-negative function:

$$\int f = \sup \left\{ \int g \mid 0 \leq g \leq f, g \in \mathcal{L}_0 \right\}.$$

- General functions:

$$\int f = \int f_+ - \int f_-.$$

$$\text{Integrable} \iff \int f_+, \int f_- < \infty.$$

Relations between Riemann integrals and Lebesgue integrals:

- f is integrable on $[a, b]$ iff f bounded and the discontinuous points form a null set.
- If f Riemann integrable on $[a, b]$, then two types of integral yield the same result.

§2.2 Dominated convergence theorem

Theorem 2.1

Let E be a bounded and measurable set, $f_n \rightarrow f, a.e.$ on E . Furthermore, f_n is uniformly bounded almost everywhere ($|f_n| < M, a.e.$) Then

$$\int_E |f_n - f| \rightarrow 0 \implies \lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Proof. By Egorov's Theorem □

Lemma 2.2 (Fatou's Lemma)

If $f_n \geq 0$, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Proof. For any $g \in \mathcal{L}_0$, $0 \leq g \leq \liminf_{n \rightarrow \infty} f_n$, we need to prove $\int g \leq \liminf_{n \rightarrow \infty} \int f_n$.

Let $g_k = \min\{f_k, g\}$, then $g_k \in \mathcal{L}_0$. We'll prove $g_k \rightarrow g$:

Assume by contradiction that $\exists \epsilon_0 > 0, \exists x_0$ s.t.

$$g(x_0) - g_{k'}(x_0) > \epsilon_0.$$

then $g(x_0) - f_{k'}(x_0) > \epsilon_0$, □

Theorem 2.3 (Beppo-Levi)

$f_n \nearrow f$, non-negative,

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof.

$$f_n \leq f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f.$$

By Fatou's Lemma,

$$\begin{aligned} \int \liminf_{n \rightarrow \infty} f_n &\leq \liminf_{n \rightarrow \infty} \int f_n. \\ \implies \int f &\leq \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

Corollary 2.4

Let f_n be non-negative functions,

$$\int \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \int f_k(x).$$

Proposition 2.5

Let f be an integrable function, $\forall \varepsilon > 0$, we have:

- Exists finite measurable set B s.t.

$$\int_{B^c} |f| < \varepsilon.$$

- (ab continuity) $\exists \delta > 0$ s.t. $\forall B, m(B) < \delta$, we have

$$\int_B |f| < \varepsilon.$$

This is equivalent to

$$\lim_{m(B) \rightarrow 0} \int_B f = 0.$$

Proof. ...

□