Measure Theory

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Lemma 0.0.1
$|\varphi|(A) = 0 \iff \varphi(B) = 0, \forall B \subset A$.

Proof. Just write $|\varphi| = \varphi^+ + \varphi^-$, we know $\varphi(B) = 0$.
Conversely, $\varphi(X^\pm \cap A) = 0 \implies |\varphi|(A) = 0$. □

§0.1 Radon-Nikodym theorem

We assume the functions and sets below are all measurable. Let $(X, \mathscr{F})$ be a measurable space, φ a signed measure.

Definition 0.1.1 (R-N derivative)**.** If there exists a a.e. unique function f s.t.

$$\varphi(A) = \int_A f \, \mathrm{d}\mu, \quad \forall A \in \mathscr{F},$$

we say f is the **Radon-Nikodym derivative** of φ with respect to μ, abbreviated by R-N derivative or derivative, denoted by $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$.

Remark 0.1.2 — When μ is σ-finite, then f must be unique a.e..

Definition 0.1.3 (Absolute continuity)**.** If $\forall A \in \mathscr{F}$,

$$\mu(A) = 0 \implies \varphi(A) = 0,$$

then we say φ is **absolutely continuous** with respect to μ, denoted by $\varphi \ll \mu$.

Observe that

$$\mu(A) = 0 \implies \mu(A \cap X^\pm) = 0 \implies \varphi^\pm(A) = 0,$$

so $\varphi \ll \mu \iff \varphi^\pm \ll \mu \iff |\varphi| \ll \mu$.

It's obvious that $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists only if $\varphi \ll \mu$, but it turns out that this is also the sufficient condition when μ is a σ-finite measure.

We can't prove this directly, so we'll prove some easy cases first.

Lemma 0.1.4

Let φ, μ be finite measures. Then

$$\exists f \in \mathscr{L} := \left\{ g \in L_1 : g \geq 0, \int_A g \, \mathrm{d}\mu \leq \varphi(A), \forall A \right\},$$

such that $\int_X f \, \mathrm{d}\mu = \sup \int_X g \, \mathrm{d}\mu$.

Proof. This is somehow similar to find simple functions approaching non-negative measurable functions.

First let $\beta = \sup \int_X g \, \mathrm{d}\mu$, and choose g_k s.t. $\int_X g_k \, \mathrm{d}\mu \to \beta$.

Let $f_n := \max_{k \leq n} g_k$, and $f_n \uparrow f$. By Levi's theorem, $\int_A f \, \mathrm{d}\mu = \lim_{n\to\infty} f_n \, \mathrm{d}\mu$, so if $f_n \in \mathscr{L}$, $f \in \mathscr{L}$ as well. Let $A_k = A \cap \{f_n = g_k, f_n \neq g_j, j < k\}$ be a partition of A,

$$\int_A f_n \, \mathrm{d}\mu = \sum_{k=1}^n \int_{A_k} g_k \, \mathrm{d}\mu \leq \sum_{k=1}^n \varphi(A_k) = \varphi(A).$$

Thus $f_n \in \mathscr{L}$, we have $\int_X f \, \mathrm{d}\mu = \beta \geq \int_X g \, \mathrm{d}\mu$, for all $g \in \mathscr{L}$. □

Proposition 0.1.5

Suppose φ, μ are both finite, then $\varphi \ll \mu \implies \frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists.

Proof. Decompose φ to $\varphi^+ - \varphi^-$, we may assume $\varphi \geq 0$.

Starting from previous lemma, we'll prove that $\int_A f \, \mathrm{d}\mu = \varphi(A)$. Let $\nu(A) = \varphi(A) - \int_A f \, \mathrm{d}\mu$ be a measure.

Let ν_n be increasing signed measures.

$$\nu_n(A) := \nu(A) - \frac{1}{n}\mu(A), \quad \forall A \in \mathscr{F}.$$

Let $X_n^\pm$ be the Hahn decomposition of ν_n, and

$$X^+ = \bigcup_{n=1}^\infty X_n^+, \quad X^- = \bigcap_{n=1}^\infty X_n^-.$$

First since $X^- \subset X_n^-$,

$$\nu(X^-) = \nu_n(X^-) + \frac{1}{n}\mu(X^-) \leq \frac{1}{n}\mu(X^-) \to 0.$$

We have $f + \frac{1}{n}\mathbf{I}_{X_n^+} \in \mathscr{L}$ since

$$\begin{aligned}\int_A \left(f + \frac{1}{n}\mathbf{I}_{X_n^+} \right) \mathrm{d}\mu &= \varphi(A) - \nu(A) + \frac{1}{n}\mu(X_n^+ \cap A) \\ &\leq \varphi(A) - \nu(X_n^+ \cap A) + \frac{1}{n}\mu(X_n^+ \cap A) \\ &= \varphi(A) - \nu_n(X_n^+ \cap A) \leq \varphi(A).\end{aligned}$$

So we have $\int_X f \, \mathrm{d}\mu \geq \int_X (f + \frac{1}{n}\mathbf{I}_{X_n^+}) \, \mathrm{d}\mu$, $\mu(X_n^+) = 0 \implies \mu(X^+) = 0$.

Since $\varphi \ll \mu$, $\varphi(X^+) = 0 \implies \nu(X^+) = 0$. □

Proposition 0.1.6

Let φ be a σ-fintie signed measure, μ be a finite measure, if $\varphi \ll \mu$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists and its integral exists.

Proof. Let $X = \sum_{n=1}^{\infty} A_n$, $|\varphi(A_n)| < \infty$, then the R-N derivative f_n exists on A_n,

Let $f = \sum_{n=1}^{\infty} f_n \mathbf{I}_{A_n}$, then f finite a.e.,

$$\varphi(A \cap A_n) = \int_{A \cap A_n} f_n \,\mathrm{d}\mu = \int_{A \cap A_n} f \,\mathrm{d}\mu.$$

WLOG φ^- finite, then

$$\varphi(\{f < 0\} \cap A_n) = \int_{A_n} f^- \,\mathrm{d}\mu = \int_{A_n} f_n^- \,\mathrm{d}\mu \geq -\varphi^-(A_n)$$

So the integral of f exists.

Since φ is countably additive and the integral of f exists, we can add the above equality to get the desired. □

Proposition 0.1.7

Let φ be an arbitary signed measure, the above conclusion also holds.

Proof. Let

$$\mathscr{G} := \left\{ \sum_{n=1}^{\infty} A_n : |\varphi(A_n)| < \infty, n = 1, 2, \ldots \right\}.$$

Since $\emptyset \in \mathscr{G}$, and it's closed under set difference:

$$\sum_{n=1}^{\infty} A_n \backslash \sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} (A_n \backslash B)$$

by $A_n \backslash B \subset A_n$, we have $|\varphi(A_n \backslash B)| < \infty$.

Clearly it's closed under countable disjoint union, combined with difference sets we deduce it's closed under countable union, thus $\mathscr{G}$ is a σ-ring.

Note that there exists B s.t. $\mu(B) = \gamma := \sup\{\mu(A) : A \in \mathscr{G}\}$. (Since we can take $\mu(B_n) \to \gamma, B = \bigcup_{n=1}^{\infty} B_n$.)

So φ is σ-finite on $(B, B \cap \mathscr{F})$, the R-N derivative exists.

For all $C \subset B^c$, we must have $\varphi(C) = 0$ or ∞. TODO!! □

At last we come to the full statement:

Theorem 0.1.8

Let φ be a signed measure, μ a σ-finite measure, if $\varphi \ll \mu$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists.

Example 0.1.9

Let $X = \mathbb{R}$, $\mu(A) = \#A$, μ is not σ-finite. Let $\varphi(A) = 0$ when A countable, 1 otherwise.
In this case the R-N derivative doesn't exist, otherwise

$$0 = \varphi(\{x\}) = \int_{\{x\}} f \, \mathrm{d}\mu = f(x)\mu(x) = f(x),$$

contradiction!

Remark 0.1.10 — If μ, ν are σ-finite measures, $\nu \ll \mu$, then

$$\int_X \mathbf{I}_A \, \mathrm{d}\nu = \int_X \mathbf{I}_A \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \implies \int_X f \, \mathrm{d}\nu = \int_X f \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.$$

§0.2 The dual space of L_p

Let $(X, \mathscr{F}, \mu)$ be a measure space, $1 < p < \infty$.
Recall that $f_n \xrightarrow{(w)L_p} f$ is defined as

$$\lim_{n\to\infty} \int_X f_n g \, \mathrm{d}\mu = \int_X fg \, \mathrm{d}\mu, \quad \forall g \in L_q.$$

By Holder's inequality,

$$\left| \int_X fg \, \mathrm{d}\mu \right| \le \|g\|_q \|f\|_p, \quad \forall f \in L_p, g \in L_q.$$

Thus given any $g \in L_q$, we can induce a **funtional** on L_p, moreover it's linear and bounded.

Definition 0.2.1. We say a funtional $\Phi : L_p \to \mathbb{R}$ is bounded linear if:

$$|\Phi(f)| \le C\|f\|_p, \quad \Phi(f_1 + cf_2) = \Phi(f_1) + c\Phi(f_2).$$

We can easily see that Φ is continuous:

$$\|f_n - f\|_p \to 0 \implies |\Phi(f_n) - \Phi(f)| \to 0.$$

Let $\|\Phi\| := \inf C = \sup_{\|f\|_p=1} |\Phi(f)|$.
For all $A \in \mathscr{F}$, $\Phi_A := \Phi(f\mathbf{I}_A)$ is also a linear and bounded functional. It's clear that $\|\Phi_A\| \le \|\Phi\|$.
Let Φ_g denote the functional induced by $g \in L_q$:

$$\Phi_g : f \mapsto \int_X fg \, \mathrm{d}\mu, \quad |\Phi_g(f)| \le \|g\|_q \|f\|_p.$$

Moreover, take $f = |g|^{q-1}\mathrm{sgn}(g)$, we found that $\|\Phi_g\| = \|g\|_q$. We check it here:

$$\int_X |f|^p \, \mathrm{d}\mu = \int_X |g|^{p(q-1)} \, \mathrm{d}\mu = \int_X |g|^q \, \mathrm{d}\mu,$$

so $f \in L_p$, $\|f\|_p = \|g\|_q^{\frac{q}{p}} = \|g\|_q^{q-1}$. Thus the equality of Holder's inequality holds.
In fact L_q contains all the bounded linear functionals of L_p:

Theorem 0.2.2

The dual space of L_p is L_q, i.e. $L_p^* = L_q$.

The critical part is to use a signed measure φ to determine g:

$$\varphi(A) = \int_A g \,\mathrm{d}\mu = \int_X \mathbf{I}_A g \,\mathrm{d}\mu = \Phi(\mathbf{I}_A), \quad A \in \mathscr{F}.$$

We're faced with two main problems:

- $\mathbf{I}_A$ may not be in L_p.
- μ may not be σ-finite, so the derivative may not be unique.

To solve these problem, we'll start from finite measure, and proceed by finite $\to$ σ-finite $\to$ arbitary.

Proposition 0.2.3

If μ is a finite measure, then $L_p^* = L_q$.

Proof. For any bounded linear functional Φ, let $\varphi(A) = \Phi(\mathbf{I}_A)$,

$$|\varphi(A)| \le C\|\mathbf{I}_A\|_p = C\mu(A)^{\frac{1}{p}},$$

so φ is finite and $\varphi \ll \mu$.

Clearly $\varphi(\emptyset) = 0$, and $\varphi(A+B) = \varphi(A) + \varphi(B)$.

For countable additivity, let $A = \sum_{n=1}^{\infty} A_n$, $B_N = \sum_{n=N+1}^{\infty} A_n$, since $\mu(A)$ finite,

$$\left|\varphi(A) - \sum_{n=1}^{N} \varphi(A_n)\right| = |\varphi(B_N)| \le C\mu(B_N)^{\frac{1}{p}} \to 0.$$

By $\varphi \ll \mu$, let $g = \frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$. We have $|g| < \infty, a.e.$ and $g \in L^1$, so

$$\Phi(\mathbf{I}_A) = \varphi(A) = \int_A g \,\mathrm{d}\mu = \int_X \mathbf{I}_A g \,\mathrm{d}\mu, \quad \forall A \in \mathscr{F}.$$

By the linearity of Φ, we know for simple functions the above equation holds.

For $f \in L_p$ non-negative, we can take simple $f_n \uparrow f$, so $\int f_n^p \,\mathrm{d}\mu \uparrow \int f^p \,\mathrm{d}\mu \implies f_n \xrightarrow{L_p} f$.

By the continuity of Φ, $\Phi(f_n) \to \Phi(f)$.

For the integral part, let $X^+ = \{g \ge 0\}, X^- = \{g < 0\}$. Then $f_n^{\pm} := f_n \mathbf{I}_{X^{\pm}}$ non-negative simple, and $f_n^{\pm} \xrightarrow{L_p} f^{\pm} := f\mathbf{I}_{X^{\pm}}$.

Now we can use Levi's theorem to get

$$\int_X f_n^{\pm} g \,\mathrm{d}\mu \to \int_X f^{\pm} g \,\mathrm{d}\mu.$$

Note since LHS is $\Phi(f_n^{\pm})$, RHS must be $\Phi(f^{\pm}) \in \mathbb{R}$, so we can safely apply $f = f^+ + f^-$. At last f non-negative $\implies$ f measurable is easy, so we've proven

$$\Phi(f) = \int_X fg \,\mathrm{d}\mu, \quad \forall f \in L_p.$$

Next we'll prove $g \in L_q$. Let $A_n = \{|g| \le n\}$, let $g_n := g\mathbf{I}_{A_n}$, clearly $g_n \in L_q$ as the base measure is finite.

Since $\Phi_{g_n} = \Phi_{A_n}$, so

$$\|g_n\|_q = \|\Phi_{A_n}\| \le \|\Phi\|.$$

Now $|g_n| \uparrow |g|, a.e.$, by Levi $\|g_n\|_q \to \|g\|_q$, so $\|g\|_q < \infty$. □

Proposition 0.2.4

When μ is σ-finite, $L_p^* = L_q$.

Proof. Let $X = \sum_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$.

There exists g_n on X_n s.t. $\Phi_{X_n} = \Phi_{g_n}$. Let $g = \sum_{n=1}^{\infty} g_n \mathbf{I}_{X_n}$.

For $f \in L_p$, $\sum_{n=1}^{N} f\mathbf{I}_{X_n} \xrightarrow{L_p} f$, we have

$$\Phi(f) \leftarrow \Phi\left(\sum_{n=1}^{N} f\mathbf{I}_{X_n}\right) = \sum_{n=1}^{N} \Phi_{X_n}(f) = \sum_{n=1}^{N} \int_{X_n} fg \,\mathrm{d}\mu.$$

Similarly, let $A^+ = \{fg \ge 0\}, A^- = \{fg < 0\}$, $f^{\pm} = f\mathbf{I}_{A^{\pm}}$, we know the integral converges.

$g \in L_q$ is also the same as before.TODO

$$\|g\|_q = \lim_{N\to\infty} \left\|\sum_{n=1}^{N} g_n \mathbf{I}_{X_n}\right\| \le \|\Phi_g\| = \|\Phi\|.$$

□

Proposition 0.2.5

μ is an arbitary measure.

Proof. If $\mu(A) < \infty$, consider $\Phi_A : f \mapsto \Phi(f\mathbf{I}_A)$, we can get g_A.

If $A \subset B$, $\mu(B) < \infty$, then $g_B \mathbf{I}_A = g_A, a.e.$, $\|\Phi_A\| \le \|\Phi_B\|$.

We can take $A_n \uparrow, \mu(A_n) < \infty$ s.t.

$$\sup_n \|\Phi_{A_n}\| = \sup\{\|\Phi_A\| : \mu(A) < \infty\}.$$

Remark 0.2.6 — Here we're using A_n to replace $X_1 + \dots X_n$ in the previous proof.

Let $g_n := g_{A_n} \uparrow g$, then $g \in L_q$:

$$\|g\|_q^q = \int_X \lim_{n\to\infty} |g_n|^q \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |g_n|^q \,\mathrm{d}\mu \le \|\Phi\|^q.$$

Let $A = \bigcup_{n=1}^{\infty} A_n$, since $g \in L_q$, by Holder and LDC,

$$\int_X fg \,\mathrm{d}\mu \leftarrow \int_X fg_n \,\mathrm{d}\mu = \Phi_{A_n}(f) = \Phi(f\mathbf{I}_{A_n}) \to \Phi(f\mathbf{I}_A).$$

The last part is to prove $\Phi(f\mathbf{I}_{A^c}) = 0$. Otherwise let $D_n = \{|f| > \frac{1}{n}\} \cap A^c$, then $\mu(D_n) < \infty$ since

$$\mu(D_n) \le \mu\left(|f| > \frac{1}{n}\right) \le \int_X (n|f|\mathbf{I}_{D_n})^p \, \mathrm{d}\mu < \infty.$$

By LDC, $f\mathbf{I}_{D_n} \xrightarrow{L_p} f\mathbf{I}_{A^c}$, so $\Phi(f\mathbf{I}_{D_n}) \neq 0$ for some n. But $\mu(D) < \infty$, let $B_n = A_n + D$ we'll find a contradiction on $\sup_n \|\Phi_{B_n}\| > \sup_n \|\Phi_{A_n}\|$. □

When $p = 1$, we can prove for σ-fintie measure μ that $L_1^* = L_\infty$. The method is the same as above.

§0.3 Lebesgue decomposition

Let φ, ϕ be two signed measures.

If $\varphi \ll |\phi|$, then we say φ is absolute continuous with respect to ϕ, denoted by $\varphi \ll \phi$. We can see that $\varphi \ll \phi \iff |\varphi| \ll |\phi|$.

Definition 0.3.1. If $\exists N \in \mathscr{F}$ such that

$$|\varphi|(N^c) = |\phi|(N) = 0,$$

then we say φ and ϕ are **mutually singular**, denoted by $\varphi \perp \phi$.

Lemma 0.3.2

$\varphi \perp \phi$ iff there exists $N \in \mathscr{F}$ such that

$$\varphi(A \cap N^c) = \phi(A \cap N) = 0, \quad \forall A.$$

Proof. This is trivial by $|\varphi|(A) = 0 \iff \varphi(B) = 0, \forall B \subset A$. □

Two measures are mutually singular is to say their supports are disjoint.

Lemma 0.3.3

If $\varphi \ll \phi$ and $\varphi \perp \phi$, then $\varphi \equiv 0$.

Proof. Take N s.t. $|\varphi|(N^c) = |\phi|(N) = 0$, since $\varphi \ll \phi$, $|\varphi|(N) = 0$ as well, thus $|\varphi|(X) = 0$. □

Theorem 0.3.4 (Lebesgue decomposition)

Let φ, ϕ be σ-finite signed measures, there exists unique σ-finite signed measures φ_c, φ_s s.t.

$$\varphi = \varphi_c + \varphi_s, \quad \varphi_c \ll \phi, \varphi_s \perp \phi.$$

Again, we'll start from finite measures, and reach σ-finite signed measures step by step.

Proposition 0.3.5

Let φ, μ be finite measures, then the Lebesgue decomposition holds.

Proof. Since $\varphi \ll \varphi + \mu$, let $f = \frac{\mathrm{d}\varphi}{\mathrm{d}(\varphi+\mu)}$, note that $0 \le f \le 1$, $(\varphi + \mu)$-a.e. (here we use the finite condition) and $1 - f = \frac{\mathrm{d}\mu}{\mathrm{d}(\varphi+\mu)}$.

Let $N = \{f = 1\}$,

$$\varphi_c(A) = \varphi(A \cap N^c), \quad \varphi_s(A) = \varphi(A \cap N).$$

Clearly $\varphi_s(N^c) = 0$,

$$\varphi(N) = \int_N f \,\mathrm{d}(\varphi + \mu) = \int_N 1 \,\mathrm{d}(\varphi + \mu) = \varphi(N) + \mu(N)$$

so $\mu(N) = 0, \varphi_s \perp \mu$.

On the other hand, if $\mu(A) = 0$, since $1 - f > 0$,

$$0 = \mu(AN^c) = \int_{AN^c} (1 - f) \,\mathrm{d}(\varphi + \mu) \implies \varphi_c(A) \le (\varphi + \mu)(AN^c) = 0.$$

Thus $\varphi_c \ll \mu$, we're done. □

From this proof, we can see that the critical point is to find a set N, s.t. $\mu(N) = 0$ and $\varphi_c = \varphi(\cdot \cap N^c) \ll \mu$, i.e. in some sense the "largest" null set of μ.

So this can give another proof:

Proof. Let $\gamma := \sup\{\varphi(A) : A \in \mathscr{F}, \mu(A) = 0\}$.

Let $A_n \in \mathscr{F}, \mu(A_n) = 0$ and $\varphi(A_n) \to \gamma$. Let $N = \bigcup A_n$, then $\varphi(N) = \gamma, \mu(N) = 0$.

If $\mu(A) = 0, \varphi_c(A) > 0$ for some A, then $\mu(N \cup A) = 0$,

$$\varphi(N \cup A) = \varphi(N) + \varphi(A \cap N^c) > \varphi(N) = \gamma,$$

contradiction!

Hence $\varphi_c \ll \mu$. □

Proposition 0.3.6

Let φ, μ be σ-finite measures, the Lebesgue decomposition holds.

Proof. Let $\{A_n\}$ be a partition of X, $\varphi(A_n) < \infty, \mu(A_n) < \infty$.

On $(A_n, A_n \cap \mathscr{F})$, there exists Lebesgue decomposition $\varphi_{n,c}, \varphi_{n,s}$, let $\varphi_c(A) = \sum_{n=1}^{\infty} \varphi_{n,c}(A \cap A_n)$, φ_s similarly defined, we can easily check that $\varphi_c \ll \mu$ and $\varphi_s \perp \mu$. □

At last we prove the Lebesgue decomposition: Let X^+, X^- be the Hahn decomposition of φ, WLOG φ^- finite.

By previous propositions, we have $\varphi_c^{\pm}, \varphi_s^{\pm}$, since φ_s^-, φ_c^- finite, so φ_c, φ_s is well-defined. The rest is some trivial work to check they satisfy the condition.

Now it remains to check the uniqueness. Suppose $\varphi_{c,i}, \varphi_{s,i}$ are two decompositions, $i = 1, 2$.

Let N_i be sets s.t. $\mu(N_i) = |\varphi_{s,i}|(N_i^c) = 0$, let $N = N_1 \cup N_2$, we have

$$\mu(N) = 0 \implies \varphi_{c,i}(N) = 0; \quad |\varphi_{s,i}|(N^c) = 0, i = 1, 2.$$

Thus $\varphi_{c,i}(A) = \varphi_{c,i}(AN^c) = \varphi(AN^c)$, and $\varphi_{s,i}(A) = \varphi_{s,i}(AN) = \varphi(AN)$.

At last we take $\mu = |\phi|$ to finally conclude.

Example 0.3.7

Let μ be a probability on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, λ is Lebesgue measure.

If $\mu \ll \lambda$, we say μ is continuous, and $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is the density function of μ.

If $\mu(\{x\}) > 0$, then we say x is an atom of μ,

$$D = D_\mu := \{x \in \mathbb{R} : \mu(\{x\}) > 0\},$$

then μ finite $\implies$ D countable.

If $\mu(D) = 1$, then we say μ is discrete.

If $\mu \perp \lambda$ and $D_\mu = \emptyset$, then we say μ is singular.

Then for any finite measure μ, let $\mu = \mu_c + \mu_s$ be the Lebesgue decomposition with respect to λ. Let $\mu_1 = \mu_c, \mu_2 = \mu(\cdot \cap D_\mu), \mu_3 = \mu_s - \mu_2$.

Then μ_1, μ_2, μ_3 are pairwise singular.

§0.4 Conditional expectations

Let $(X, \mathscr{F}, P)$ be a probability space. Let $\mathscr{G}$ be a sub σ-algebra of $\mathscr{F}$. Then we have another probability space $(X, \mathscr{G}, P)$.

Recall that $L_2(\mathscr{G}) \subset L_2(\mathscr{F})$ are Hilbert spaces.

Let $g \in \mathscr{G}$ be a function, $g \geq 0$, then $\int_X g \,\mathrm{d}P$ is the same in two spaces.(By Levi's theorem)

By linear algebra, for any $f \in \mathscr{F}$, there's a unique optimal approximation (or orthogonal projection) $f^* \in \mathscr{G}$ s.t.

$$\|f - f^*\|_2 = \inf_{g \in L_2(\mathscr{G})} \|f - g\|_2.$$

Therefore by orthogonality,

$$Efg = Ef^*g, \forall g \in L_2(\mathscr{G}) \iff Ef\mathbf{I}_A = Ef^*\mathbf{I}_A, \forall A \in \mathscr{G}.$$

Let $\varphi(A) = Ef\mathbf{I}_A$, $\varphi \ll P$, in fact we have $f^* = \frac{\mathrm{d}\varphi}{\mathrm{d}P}$.