Geometry II

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§1 Introduction

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This course covers the topic of elementary *differential geometry* and *fundamental groups in algebraic topology*.

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§1.1 Intro

Definition 1.1.1 (Manifold)**.** Let M be an open subset of $\mathbb{R}^n$, we call M an **m-dimensional regular manifold** of $\mathbb{R}^n$, if $\forall p \in M$, exists an open neighborhood $W \subset \mathbb{R}^n$ such that there exists open set $U \subset \mathbb{R}^m$ and homeomorphism $\phi : U \to M \cap W$, satisfying the Jabobi matrix of ϕ is injective everywhere, i.e.

$$\mathrm{D}\phi(x) = \begin{pmatrix} \frac{\partial \phi_1}{\partial u_1} & \cdots & \frac{\partial \phi_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial u_1} & \cdots & \frac{\partial \phi_n}{\partial u_m} \end{pmatrix}$$

has rank m for all $x \in U$.

When $n = 3$, we say M is a curve for $m = 1$, and a surface for $m = 2$.

Remark 1.1.2 — The term "C^r regular manifold" means ϕ is a C^r function.

Example 1.1.3

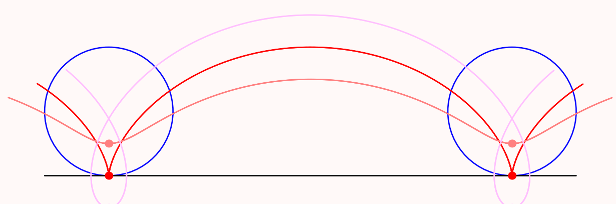
Some quadratic surfaces like cylinders, biparted hyperboloids and saddle surfaces are all regular 2-manifolds, but a cone is not a regular manifold.

Example 1.1.4

The curve $\phi(r) = (\cos(2\pi r), \sin(2\pi r), r)$ is a 1-manifold in $\mathbb{R}^3$. This curve is called a *helix*.

Example 1.1.5 (Cycloid)

A cycloid is the locus of a point on a circle while the circle "rolls" along a line. When the point lies inside resp. outside the circle, the curve is called curtate cycloid resp. procolate cycloid.



The cycloid is not a manifold because it has singularity where it touches the line, and procolate cycloids are also not manifolds as they have self-intersections.

Remark 1.1.6 — The regular manifolds we talk about are also called "embedded manifolds", the ones with self-intersections can be discribed as "immersed manifolds", such as the curves in the previous example. The immersed manifolds are complex and hence beyond the scope of this class.

However, it turns out that the curves or surfaces with self-intersections also have some properties, so we need to find a way to describe them. This induces:

Definition 1.1.7 (Regular parametrized curve)**.** Let $\gamma : J \to \mathbb{R}^3$ be a function, where J is an open interval. if for every point $p \in J$, there exists open neighborhood J' s.t. $\gamma\big|_{J'}$ is a regular 1-manifold, then we say $\gamma(J)$ is a **regular parametrized curve**, and $\gamma\big|_{J'}$ is called its **regular parametrization**.

Likewise, we have:

Definition 1.1.8 (Regular parametrizd surface)**.** Let $\phi : U \to \mathbb{R}^3$ be a function, where $U \in \mathbb{R}^2$ is an open set. If for every point $p \in U$, there exists open neighborhood U' s.t. $\phi\big|_{U'}$ is a regular 2-manifold, we say $\phi(U)$ is a **regular parametrized surface**.

§1.2 Prerequisites

Vector calculus Let $\vec{v}(t)$ be a 3-dimensional vector function, its derivative resp. integration is the vector formed by taking derivative resp. integration of each component, and the derivative satisfies Leibniz's rule with respect to both dot product and cross product.

Multi-variable calculus If $f : \mathbb{R}^m \to \mathbb{R}^n$ is C^2, then the partial derivative can change order with each other:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f.$$

Integration of surfaces has two types:

1. $\iint f \,\mathrm{d}x\,\mathrm{d}y$, multiple integrals.
2. $\iint f \,\mathrm{d}x \wedge \mathrm{d}y$, integrals with orientation.

Remark 1.2.1 — On how to construct regular manifolds:
Let $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be a smooth function, for a fixed $y \in \mathbb{R}^{n-m}$, if $\forall x \in f^{-1}(y)$, $Df(x)$ has rank $n-m$, then $M := f^{-1}(y)$ is an m-dimensional regular manifold.
In fact this is known as "Regular Value Theorem" in literarture, and y is called a regular value of f. This leads to a branch in mathematics, namely *differential topology*.

Remark 1.2.2 — On real/complex analysis: Holomorphic (which is the complex version of differentiable) is way stronger than smooth condition.

§2 Theory of space curve

In this section we mainly discuss the regular parametrized curves $\gamma : J \to \mathbb{E}^3$.

Our goal is to find some identities to describe the "shape" of the curves. Since the curve is 1-dimensional manifold in 3 dimensional space, somehow we should find 3 identities to describe it, including length and another two concerning how it "bends".

§2.1 Arc length

Definition 2.1.1 (Arc length)**.** Let $\gamma : J \to \mathbb{E}^3$ be a regular parametrized curve. In an interval $[a, b] \subset J$, we define its length to be

$$Length_\gamma([a,b]) := \int_a^b \|\gamma'(t)\|\,\mathrm{d}t.$$

where $\gamma'(t) \in V(\mathbb{E}^3) = \mathbb{R}^3$.

Proposition 2.1.2
Arc length is a **geometry quantity**, i.e. fixed under reparametrization.

Proof. For an arbitary regular reparametrization $t = t(\tilde{t})$, $\tilde{\gamma}(\tilde{t}) = \gamma(t)$, by Chain rule we get

$$\begin{aligned} Length_\gamma([a,b]) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))| \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}'(\tilde{t})|\,\mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{a},\tilde{b}]). \end{aligned}$$

However, here we used the fact that $\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}$ is positive, so when the reparametrization reverses the orientation, we need to take extra care of it.

$$\begin{aligned} Length_\gamma([a,b]) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))| \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{b}}^{\tilde{a}} |\tilde{\gamma}'(\tilde{t})|\,\mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{b},\tilde{a}]). \end{aligned}$$

□

The arc length induces a parametrization for regular curves, namely the **arc length parameter** $\gamma(s)$, with $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\| = 1$ everywhere.

§2.2 Curvature

Definition 2.2.1 (Curvature)**.** Let $\gamma(s)$ be a regular curve with arc length parameter, define its **curvature** to be

$$\mathrm{Curv}_\gamma(s) = \kappa(s) := \|\gamma''(s)\|.$$

Since it is deduced from arc length (which is a geometry quantity), it must be a geometry quantity as well.

Remark 2.2.2 — Sometimes $\gamma''(s)$ is called the curvature vector. It's parallel to the normal vector and can be interpreted as centripedal force.

Example 2.2.3
For a straight line, its curvature is always 0.
For a circle with radius R, $\gamma(s) = (R\cos(\frac{s}{R}), R\sin(\frac{s}{R}))$, so $\mathrm{Curv}_\gamma(s) = \frac{1}{R}$.

Proposition 2.2.4

When the parameter is a general parameter $\gamma(t)$, the curvature is equal to:

$$\mathrm{Curv}_\gamma(t) = \frac{\|\gamma''(t) \times \gamma'(t)\|}{\|\gamma'(t)\|^3}.$$

Example 2.2.5

Let $\Gamma : x^2 + k^2y^2 = 1$, calculate curvature of Γ at point (x, y).

Solution. First we take a parametrization for Γ: $(x, y) = (\cos t, \frac{1}{k}\sin t)$.

Then compute the derivatives:

$$(x', y') = (-\sin t, \frac{1}{k}\cos t) = (-ky, \frac{1}{k}x),$$

$$(x'', y'') = (-\cos t, -\frac{1}{k}\sin t) = (-x, -y).$$

$$\mathrm{Curv}_\Gamma = \frac{|ky^2 + \frac{1}{k}x^2|}{(k^2y^2 + \frac{1}{k^2}x^2)^{\frac{3}{2}}} = \frac{1}{k(\frac{1}{k^2}x^2 + k^2y^2)^{\frac{3}{2}}}.$$

When $(x, y) = (1, 0)$, $\mathrm{Curv} = k^2$; when $(x, y) = (0, \frac{1}{k})$, $\mathrm{Curv} = \frac{1}{k}$. □

Remark 2.2.6 — Osculating circle: A circle tangent to the curve with the same curvature as the curve at the tangent point. Specifically, its radius is equal to $\frac{1}{\mathrm{Curv}}$.

This is useful in engineering to indicate the curvature of a curve.

§2.3 Torsion and Frenet frame

Definition 2.3.1 (Torsion)**.** Let $\gamma(s)$ be a curve with arc length parameter.

Let $\vec{t} := \gamma'(s), \vec{n} := \frac{\gamma''(s)}{\|\gamma''(s)\|}$ be the tangent vector and normal vector.

Let $\vec{b} = \vec{t} \times \vec{n}$ be the **binormal vector**. Define the **torsion** to be

$$\mathrm{Tors}_\gamma(s) = \tau(s) := -\vec{b}' \cdot \vec{n}.$$

In fact $\vec{b}'$ is parallel to $\vec{n}$:

$$\vec{b}' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}' = \vec{t} \times \vec{n}' \perp \vec{t},$$

and $\|\vec{b}\| = 1$, so $\vec{b} \perp \vec{b}'$, so $\vec{b}' \parallel \vec{n}$.

The torsion's geometric meaning is less intuitive than the previous ones. It describes how much the curve is moving "out" the plane it currently lies in.

Proposition 2.3.2

Torsion can be represented in general parameter:

$$\mathrm{Tors}_\gamma(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

Remark 2.3.3 — The torsion can be negative (while curvature is always non-negative), and it is only defined at the points where the curvature is nonzero.

Note that the vectors $\vec{t}, \vec{n}, \vec{b}$ form a right-handed orthonormal basis in $\mathbb{R}^3$, and it's called the curve $\gamma(s)$'s **Frenet frame**.

The plane containing $\vec{n}$ and $\vec{b}$ is called **normal plane**, the plane containing $\vec{t}$ and $\vec{n}$ is called **osculating plane**, and the last plane which contains $\vec{t}, \vec{b}$ is called **rectifying plane**.

The Frenet frame is not a fixed frame, it's moving with the point along the curve. So we can compute its derivative (with respect to s, the arc length parameter):

$$(\vec{t}', \vec{n}', \vec{b}') = (\vec{t}, \vec{n}, \vec{b}) \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Example 2.3.4

When γ lies on the surface of a sphere, assmue $\kappa > 0$ on $\gamma\big|_J$, then

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = c^2.$$

where c is the radius of the sphere.

Proof. Let $\vec{u} = \gamma(s) - p$, then $\vec{u} \cdot \vec{u} = c^2$. To get a relation of κ and τ, we only need to represent $\vec{u}$ in terms of $\vec{t}, \vec{n}$ and $\vec{b}$.

Taking derivative WRT s:

$$0 = 2\vec{u}' \cdot \vec{u} = 2\vec{t} \cdot \vec{u}.$$

and then by taking the second and third derivative,

$$0 = \vec{t}' \cdot \vec{u} + \vec{t}^2 = \kappa\vec{n} \cdot \vec{u} + 1.$$

We get $\vec{u} \cdot \vec{n} = -\frac{1}{\kappa}$.

$$(\kappa\vec{n})' = \kappa'\vec{n} + \kappa(-\kappa\vec{t} + \tau\vec{b}),$$

so the third derivative should be

$$0 = \kappa\vec{n} \cdot \vec{t} + \kappa'\vec{n} \cdot \vec{u} + \kappa(-\kappa\vec{t} + \tau\vec{b}) \cdot \vec{u} = -\frac{\kappa'}{\kappa} + \kappa\tau\vec{u} \cdot \vec{b},$$

hence $\vec{u} \cdot \vec{b} = \frac{1}{\tau}(\frac{1}{\kappa})'$.

At last we just plugged everything into $\vec{u}^2 = c^2$ to conclude. □

Note: the inverse statement does not hold, e.g. helix (which has constant curvature).

This example shows that Frenet frame is a powerful tool for handling the local properties of a curve. In fact, we could totally "determine" a curve near a point given the curvature and torsion.

Example 2.3.5

We can expand the curve $(\gamma(s), \vec{t}, \vec{u}, \vec{b})$ around $s = 0$:

$$\begin{cases} x(s) = x(0) + s - \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ y(s) = y(0) + \dfrac{\kappa(0)}{2}s^2 + \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ z(s) = z(0) + \dfrac{\kappa'(0)\tau(0)}{6}s^3 + o(s^3) \end{cases}$$

Remark 2.3.6 — By Frenet's formula, we can expand it to higher degrees, but the expansion need not converge to the original curve (similar reason as Taylor's formula). Also we can expand the curve with any parameter instead of arclength.

§2.4 Fundamental theorem of curve theory

Theorem 2.4.1 (Fundamental theorem of curve theory)

Let $\kappa, \tau : J \to \mathbb{R}$ be smooth functions, $\kappa(s) > 0$ on J. There exists a curve with arc length parameter $\gamma : J \to \mathbb{E}^3$, such that $\mathrm{Curv}_\gamma = \kappa$, $\mathrm{Tors}_\gamma = \tau$ holds on J.

Moreover, if $\tilde{\gamma}$ also satisfies above conditions, then exists $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ perserving orientation and distance s.t. $\tilde{\gamma} = \sigma \circ \gamma$.

Claim 2.4.2. Let $H = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

The ODE about $F : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$:

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}s}(s) = F(s)H(s) \\ F(s_0) = F_0 \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \end{cases}$$

always has unique solution. Moreover if $F(s_0) \in \mathrm{SO}(3)$, then $F(s) \in \mathrm{SO}(3)$ always holds.

Proof of the theorem. Since this claim requires some knowledge in ODE, which is beyond the scope of this course, we'll directly use it without proving.

WLOG $0 \in J$, let $s_0 = 0$ and $F(0) = I_3$.

Let $\mathcal{F} = (\vec{t}, \vec{n}, \vec{b}) := (\vec{e}_1, \vec{e}_2, \vec{e}_3)F(s)$ be a frame of $\mathbb{R}^3$.

Now we construct γ to be

$$\gamma(s_1) := \int_0^{s_1} \vec{t}\,\mathrm{d}s.$$

It's sufficient to prove that $\mathrm{Curv}_\gamma = \kappa$ and $\mathrm{Tors}_\gamma = \tau$.

Since $\mathcal{F}(0) = (e_1, e_2, e_3)$ is orthonormal frame, $\mathcal{F}(s)$ is orthonormal for all s.

Thus $|\vec{t}| = 1$, s is the arc length parameter. Some computation yields $\mathcal{F}$ is Frenet frame of γ. Compare its Frenet matrix to H, we get the desired result.

On the other hand, if $\tilde{\gamma}$ is as stated, take its Frenet frame $\tilde{\mathcal{F}}(s)$.

Let σ be the map which maps $\mathcal{F}(0)$ to $\tilde{\mathcal{F}}(0)$, $\gamma(0)$ to $\tilde{\gamma}(0)$. Then the Frenet frame of $\sigma \circ \gamma$ and $\tilde{\gamma}$ are the solution of the same ODE $\implies \sigma \circ \gamma = \tilde{\gamma}$ for all $s \in J$. ∎

Remark 2.4.3 — Here we give a proof of $F \in \mathrm{SO}(3)$:

Proof. Note that

$$\left(FF^T\right)' = F'F^T + F(F')^T = F\left(H + H^T\right)F^T = 0.$$

thus $FF^T = I$ as it holds at $s_0 \implies F \in \mathrm{O}(3)$.

Beisdes, it's easy to see that $\det(F)$ doesn't change sign, so $F \in \mathrm{SO}(3)$. □

In the words of tangent spaces or Lie groups, we can say that $T_I\mathrm{SO}(3) = \{X \mid X + X^T = 0\}$, and $T_F\mathrm{SO}(3) = \{FX \mid X + X^T = 0\}$.

Remark 2.4.4 — The above ODE cannot be solved explicitly, so here we introduce a method called "successive approximation". (For more details, see my notes of Analysis I)

Let $F_0(s) = F_0$, $F_1(s) = F_0 + \int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t$, and define

$$F_{j+1}(s) = F_0 + \int_{s_0}^{s} F_j(t)H(t)\,\mathrm{d}t.$$

We can compute

$$|F_1(s) - F_0(s)| = \left|\int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t\right| \le \int_{s_0}^{s} |F_0(t)H(t)|\,\mathrm{d}t \le M(s - s_0)\cdot|F_0|.$$

$$|F_{j+1}(s) - F_j(s)| = \left|\int_{s_0}^{s} (F_j(t) - F_{j-1}(t))H(t)\,\mathrm{d}t\right| \le M^{j+1}\frac{(s-s_0)^{j+1}}{(j+1)!}|F_0|.$$

Therefore F_j must uniformly converge to some function F on some small interval $[s_0 - \delta, s_0 + \delta]$.

With some effort we can check F is differentiable and satisfies the ODE. Furthermore, F can extend to the entire interval J, and it's the *unique* solution.

§3 Theory of surfaces

§3.1 The first fundamental form

Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. We denote the point in $U \subset \mathbb{R}^2$ as $u = (s,t)$. Hence the partial derivative of ϕ gives:

$$\phi_s(u) = \frac{\partial \phi}{\partial s}(u), \quad \phi_t(u) = \frac{\partial \phi}{\partial t}(u).$$

Define the first fundamental quantities

$$E(u) = \phi_s(u)\cdot\phi_s(u), \quad F(u) = \phi_s(u)\cdot\phi_t(u), \quad G(u) = \phi_t(u)\cdot\phi_t(u).$$

The first fundamental form of the surface ϕ is the real bilinear form $T_u\phi(U) \times T_u\phi(U) \to \mathbb{R}$:

$$g(u) = E(u)\,\mathrm{d}s^2 + 2F(u)\,\mathrm{d}s\,\mathrm{d}t + G(u)\,\mathrm{d}t^2.$$

It can also be written as $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

Remark 3.1.1 (On partial detivatives) — When using the notation $\frac{\partial}{\partial x}$, we must declare the parameters we're using, e.g. (x, y), For example when $x' = x, y' = x + y$, the meaning of $\frac{\partial}{\partial x}$ is different from $\frac{\partial}{\partial x}$.

Definition 3.1.2 (Length, angle and area)**.** Let $\gamma(r) = \phi(u(r))$ be a curve on ϕ, then its length is equal to

$$Length = \int_a^b \|s'(r)\phi_s(u(r)) + t'(r)\phi_t(u(r))\| \, \mathrm{d}r.$$

Let $\alpha, \beta : [0, \epsilon] \to \mathbb{E}^3$ be two curves on ϕ. Then the angle between α and β (in $\mathbb{E}^3$) is equal to

$$Angle = \arccos\left(\frac{\phi_s(u) \cdot \phi_t(u)}{\|\phi_s(u) \cdot \phi_t(u)\|}\right).$$

Lastly, let $R \subset U$ be a closed region whose boundary is a regular curve, the area of ϕ on R is defined as

$$Area = \iint_R \|\phi_s(u) \times \phi_t(u)\| \, \mathrm{d}s \, \mathrm{d}t.$$

We say $\phi(s, t_0)$ is an s-curve and $\phi(s_0, t)$ is a t-curve. If s-curve and t-curve are orthogonal at every point $u \in U$, i.e. $F = 0$, or the matrix is diagonal, we say ϕ is an **orthogonal parametrization**, and s, t are **orthogonal parameters**.

Moreover, if $E = G, F = 0$ for all $u \in U$, (the matrix is a scalar) then we call ϕ an **isothermal parametrization**, and s, t are **isothermal parameters**. (Sometimes also called **comformal parameters**)

Example 3.1.3

The longtitude and latitude on a sphere are orthogonal parameters, but not isothermal parameters; While the stereographical projection is an isothermal parametrization of the sphere.

Remark 3.1.4 — The word "isothermal" is connected to thermology in a rather complicated way. The word "conformal" provides a more intuitive comprehension.

Remark 3.1.5 — A fun fact: Isothermal parameters always exist locally on regular parametrized surfaces. This only holds for 2-dimensional manifold.

§3.2 Linear algebra review

Let V be a vector space over $\mathbb{F}$.

Symmetrical bilinear form vs. quadratic form

A symmetrical bilinear form is a linear map $B : V \times V \to \mathbb{F}$ with $B(v, w) = B(w, v)$. A quadratic form is a map $Q : V \to \mathbb{F}$ with $Q(v) = B(v, v)$ for some symmetrical bilinear form B.

By taking a basis of V, we can use the matrix to express them:

$$B(v, w) = vAw^T, \quad Q(v) = vAv^T.$$

where A is a symmetrical matrix. When we change the basis, the matrix A differs by a congruent transformation.

We could also write $B \in V^* \otimes V^*$ for a bilinear form B. All the symmetrical bilinear forms constitude a subspace of $V^* \otimes V^*$ of dimension $\frac{n(n+1)}{2}$. This is denoted by $\mathrm{Sym}^2(V)$.

Remark 3.2.1 — The subspace of anti-symmetric matrices is denoted by $\mathrm{Alt}^2(V)$, and $\mathrm{Alt}^2(V) \oplus \mathrm{Sym}^2(V) = V^* \otimes V^*$.

§3.3 Tangent spaces

A surface $\phi : U \to \mathbb{E}^3$ has a tangent space at every point $\phi(u)$, which is just the space (in this case, a plane) spanned by $\phi_s(u)$ and $\phi_t(u)$. We can prove that this tangent space is independent to the parameters. Furthermore, we can equip it with the inner product in $\mathbb{E}^3$, the matrix of this product is exactly $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

Remark 3.3.1 — In modern differential manifold theory, there's an intrinsic definition of tangent spaces, but this definition is too abstract.

Here we present one of these intrinsic definitions.

Definition 3.3.2 (Tangent vectors)**.** Define an equivalence relation on smooth curves in $\phi(U)$:

Let $\gamma(r) = \phi(s(r), t(r))$ be a smooth curve $(-\epsilon, \epsilon) \to \phi(U)$. Two curves γ_1, γ_2 are equivalent iff $s_1'(0) = s_2'(0)$ and $t_1'(0) = t_2'(0)$.

Each equivalence class is a "tangent vector" at point $\phi(s_0, t_0)$.

§3.4 The second fundamental form

Since the first fundamental form is not sufficient to describe all the properties of the surface (it can only describe the curves lying on it and the area), we thereby introduce the second fundamental form.

Definition 3.4.1 (The second fundamental form)**.** Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. The normal vector at the point $u = (s, t)$ is defined as

$$\vec{n} = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}.$$

Since the cross product cares about orientation, so the normal vector is only fixed under orientation-preserving reparametrization.

Now we expand ϕ to the second derivative:

$$\phi(s + \Delta s, t + \Delta t) = \phi(s, t) + (\phi_s, \phi_t) \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + \frac{1}{2}(\Delta s, \Delta t) \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + o(|\Delta s|^2 + |\Delta t|^2).$$

Hence we define

$$L = \phi_{ss} \cdot \vec{n}, \quad M = \phi_{st} \cdot \vec{n}, \quad N = \phi_{tt} \cdot \vec{n}.$$

The second fundamental form is defined as $h = L\,\mathrm{d}s^2 + M\,\mathrm{d}s\,\mathrm{d}t + N\,\mathrm{d}t^2$.

Remark 3.4.2 — Another expression of L, M, N:

$$L = -\phi_s \cdot \vec{n}_s, \quad M = -\phi_s \cdot \vec{n}_t = -\phi_t \cdot \vec{n}_s, \quad N = -\phi_t \cdot \vec{n}_t.$$

Intuitively, the second fundamental form describes how much the surface is "going out" of the tangent plane.

Since the first fundamental form g gives an inner product of the tangent space, so we can compute the "canonical form" of h with respect to g, this process will generate some geometric quantities.

Definition 3.4.3. Define the **average curvature** and **Gaussian curvature**:

$$H := \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}, \quad K := \frac{LN - M^2}{EG - F^2}.$$

These expressions look complicated and ugly, the reason is that we didn't choose the right parameters. Indeed, if at some point $u = (s, t)$ we have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Definition 3.4.4 (Principal curvatures)**.** The characteristic polynomial $\lambda^2 - 2H\lambda + K$ has two real roots, they are called the **principal curvatures** of ϕ. The **principal directions** are defined as the directions of eigenvectors of $h : T_uU \times T_uU \to \mathbb{R}$ WRT the inner product g.

Now we'll dig deeper into the geometric meaning of these formulas.

Proposition 3.4.5

H and K are geometric quantities.

Proof. For any reparametrization $s = s(\tilde{s}, \tilde{t}), t = t(\tilde{s}, \tilde{t})$, we have

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J \begin{pmatrix} E & F \\ F & G \end{pmatrix} J^{-1}.$$

Similarly we can verify that

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = J \begin{pmatrix} L & M \\ M & N \end{pmatrix} J^{-1}.$$

Since

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Thus H, K are fixed under orientation-preserving reparametrization. $\square$

Remark 3.4.6 — When the reparametrization is orientation-reversing, L, M, N all differ by a sign, thus H will change while K is still fixed.

Some examples about curvatures:

Example 3.4.7

A sphere with radius R:

$$H = \frac{1}{R}, \quad K = \frac{1}{R^2}.$$

Note that

$$\int_{S^2} K \,\mathrm{d}Area = \frac{1}{R^2} 4\pi R^2 = 2\pi\chi(S^2).$$

This is an example of Gauss-Bonnet formula which we'll cover later.

Example 3.4.8

The conical and cylindrical surfaces have Gaussian curvature 0.

For a general ruled surface, we can prove that $K \leq 0$ everywhere.

Example 3.4.9

Minimal surfaces(like soap bubbles) have $H = 0$ and $K \leq 0$ everywhere.

Example 3.4.10 (Dupin canonical form)

Let $\phi : U \to \mathbb{E}^3$ be a regular surface, then at the neighborhood of any point, there exists a parameter s.t. $\phi(s,t) = (s, t, \kappa_1 s^2 + \kappa_2 t^2) + o(|s|^2 + |t|^2)$, where κ_1, κ_2 are principal curvatures of ϕ.

In this case we can talk about concepts like "elliptic point", "parabolic point" and "hyperbolic point".

Next we'll going to switch to a more intrinsic view to study the meaning of those definitions again.

If we look at a curve γ on a surface ϕ, let r be the arc length parameter, then $\|\gamma'\| = 1$, $\|\gamma''\| = \kappa(r)$, note that γ'' can be decompose with respect to the normal vector and tangent plane:

$$\gamma'' = \kappa_n \vec{n} + \kappa_g \vec{n} \times \gamma'.$$

Here κ_n is called **normal curvature**, and κ_g is called **geodesic curvature** of γ WRT ϕ.

Moreover we have *Euler's formula*:

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

If we compute the normal curvature in terms of $u = (s,t)$:

$$\gamma' = \phi_s s' + \phi_t t'$$

$$\gamma'' = (s', t') \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} + \phi_s s'' + \phi_t t''$$

Hence

$$\kappa_n = \gamma'' \cdot \vec{n} = (s', t') \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} = L(s')^2 + 2Ms't' + N(t')^2.$$

This is the formula under the arc length parameter.

Remark 3.4.11 — The general formula of κ_n:

$$\kappa_n = \frac{Ls'^2 + 2Ms't' + Nt'^2}{Es'^2 + 2Fs't' + Gt'^2}.$$

The normal plane of γ intersects the surface ϕ, the section curve is called a **normal section**.

Oberserve that: if $\|\gamma'\| = 1$, and the tangent vector is $\vec{t}$, then $\kappa_n(r)$ is the curvature of the normal section at u in the plane spanned by $\vec{n}, \vec{t}$.

Hence κ_n can be viewed as a quadratic form $\vec{n}^\perp \to \mathbb{R}$ which sends a vector $\vec{t}$ to the curvature of the normal section with tangent vector $\vec{t}$.

Furthermore, the principal directions are the "eigen-directions" of κ_n, which are the directions where the curvature of normal section attains its extremum.

Example 3.4.12

Consider the helix and the cylinder

$$\gamma(t) = (\cos t, \sin t, at), \quad S : x^2 + y^2 = 1.$$

It's easy to verify that $\kappa = \kappa_n = \frac{1}{1+a^2}$ as γ'' is always perpendicular to z-axis.

Note that $\kappa_g = 0$ everywhere, curves satisfying $\kappa_g = 0$ are called **geodesic line**.

§3.5 Gauss map and Weingarten map

The strange definition of those curvatures don't come from nothing, in this section we'll cover this topic and give a geometric interpretation.

Definition 3.5.1 (Gauss map)**.** Let Σ be a regular surface in $\mathbb{E}^3$, denote its normal vector at x by $\vec{n}(x)$. Then this map $\mathcal{G} : \Sigma \to \mathbb{S}^2$ by $x \mapsto \vec{n}(x)$ is called the **Gaussian map**.

In terms of a parametrized surface $\phi : U \to \mathbb{E}^3$, we can compute that

$$\mathcal{G} : U \to \mathbb{S}^2 : \quad \vec{n}(u) = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}$$

But each vector has a normal plane, namely $\vec{n}^\perp$, and this derives the **Weingarten map**:

Definition 3.5.2 (Weingarten map)**.** For all $u \in U$, define $W : \vec{n}(u)^\perp \to \vec{n}(u)^\perp$: $\vec{v} \mapsto W(\vec{v})$, where

$$W(\vec{v}) = -\left.\frac{\mathrm{d}(\mathcal{G} \circ \gamma)}{\mathrm{d}u}\right|_{u=0}, \quad \gamma := \phi(u(r)) \text{ is a curve on the surface.}$$

Remark 3.5.3 — In the language of modern differetial manifolds, Weingarten map is just the tangent map of Gauss map with a negative sign.

Since $\vec{n}^\perp$ has a basis ϕ_s, ϕ_t, we can compute the matrix of Weingarten map:

$$(\phi_s, \phi_t)W = (-\vec{n}_s, -\vec{n}_t).$$

Note that $-\vec{n}_s \cdot \phi_s = \vec{n} \cdot \phi_{ss} = L$, so if we take the inner product of (ϕ_s, ϕ_t) on both sides, we get

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} W = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \implies W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Since W is clearly a geometric quantity, so its trace and determinant are also geometric:

$$\operatorname{tr} W = \frac{GL - 2FM + EN}{EG - F^2} = 2H, \quad \det W = \frac{LN - M^2}{EG - F^2} = K,$$

which gives the average curvature and Gauss curvature.

Moreover, the principal curvatures are the eigenvalues of W, and principal directions are just the eigenspaces of W.

Let $\vec{v} = (\phi_s, \phi_t)X$, then its normal section has curvature

$$\kappa_n = \frac{X^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} X}{X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} X}.$$

When $\|\vec{v}\| = 1$, we can change a parameter s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = I_2$, in this case we can observe that when κ_n attains its extremum, $\vec{v}$ is precisely the eigenvector of W, i.e. lies on the principal directions.

Definition 3.5.4 (Curvature line)**.** A curve is called a **curvature line** if its tangent vector is the same as principal directions everywhere.

Example 3.5.5

Every curve on a sphere is curvature line.

Around a point where the principal curvatures are different, there exists a orthogonal grid of curvature lines.

Example 3.5.6

monkey saddle surface, "prong singularity"

In the case when the s-curve and t-curve are precisely the curvature lines, then we say this is a **curvature grid parameter**, and here we have $g = E\,\mathrm{d}s^2 + G\,\mathrm{d}t^2$ and $h = L\,\mathrm{d}s^2 + N\,\mathrm{d}t^2$.

Remark 3.5.7 — The geometric interpretation of Gauss curvature: For $u \in D \subset U$,

$$|K(u)| = \lim_{\text{“}D \to u\text{”}} \frac{Area_{\mathbb{S}^2}(\mathcal{G}(D))}{Area_{\mathbb{E}^3}(\phi(D))}$$

while $sgn(K(u))$ is the orientation of $\mathcal{G}$ at point u.

Example 3.5.8

Consider the Gauss map of a torus, the "outer" part and the "inner" part of the torus maps to $\mathbb{S}^2$ bijectively. If we compute

$$\int_{T^2} K\,\mathrm{d}Area_E = \int_{\mathbb{S}^2} (1 + (-1))\,\mathrm{d}Area_S = 0 = 2\pi\chi(T^2),$$

as Gauss-Bonnet formula implies.

§3.6 Fundamental equation of surfaces

Like the Fundamental theorem and Frenet frame in curve theory, we want to develop a theorem for describing surfaces using only fundamental forms.

Given a parameter on a surface, there's a natural frame $(\phi_s, \phi_t, \vec{n})$. If we take the derivative of the frame, we'll get

$$(\phi_s, \phi_t, \vec{n})_{st} = (\phi_s, \phi_t, \vec{n})_{ts}.$$

Taking the inner product with $(\phi_s, \phi_t, \vec{n})^T$ and apply the product rule:

$$\left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_s\right)_t - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_t (\phi_s, \phi_t, \vec{n})_s = \left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_t\right)_s - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_s (\phi_s, \phi_t, \vec{n})_t$$

This equation will give us some relations between the fundamental quantities. In literature these relations are known as Gauss equation and Codazzi equations.

Gauss equation can be written as:

$$(\phi_s \cdot \phi_{ts})_t - (\phi_s \cdot \phi_{tt})_s = \phi_{st} \cdot \phi_{st} - \phi_{ss} \cdot \phi_{tt}.$$

Codazzi equations are related to $\vec{n}$ and more complicated.

From Gauss equation we can deduce a famous theorem:

Theorem 3.6.1 (Gauss' Theorema Egregium)
The Gauss curvature K is determined by the first fundamental form.

Proof. Note that $(\phi_s \cdot \phi_{ts})_t = \frac{1}{2}E_{tt}$, and $(\phi_s \cdot \phi_{tt})_s = (F_t - \frac{1}{2}G_s)_s = F_{ts} - \frac{1}{2}G_{ss}$.

Suppose $\phi_{ss} = x\phi_s + y\phi_t + L\vec{n}$, then

$$\frac{1}{2}E_s = \phi_s \cdot \phi_{ss} = Ex + Fy, \quad F_s - \frac{1}{2}G_t = \phi_t \cdot \phi_{ss} = Fx + Gy$$

So x, y is determined by E, F, G.

Similarly, we get

$$\begin{aligned}\phi_{ss} &= *\phi_s + *\phi_t + L\vec{n}\\ \phi_{st} &= *\phi_s + *\phi_t + M\vec{n}\\ \phi_{tt} &= *\phi_s + *\phi_t + N\vec{n}\end{aligned}$$

where $*$ are determined by E, F, G.

By Gauss equation, we get $* = -(LN - M^2) + *$, and $*$ is determined by E, F, G and their partial derivatives. □

Remark 3.6.2 — The computation looks messy, but in modern mathematics, we have a systematic notation which is more simplified.

Definition 3.6.3 (Isometries)**.** Let $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. If a map $\psi : \widetilde{U} \to U$ satisfies $\psi^*(g) = \widetilde{g}$, then it's called an **isometry**.

Let $\mathcal{F} = (\phi_s, \phi_t, \vec{n})$. Suppose $\mathcal{F}_s = \mathcal{F}A$, and $\mathcal{F}_t = \mathcal{F}B$. Taking the second derivative we get $\mathcal{F}_{st} = \mathcal{F}(BA + A_t)$, $\mathcal{F}_{ts} = \mathcal{F}(AB + B_s)$. Thus we have

$$A_t - B_s = AB - BA.$$

But we want to express things in terms of E, F, G, so we can compute the dot product of $\mathcal{F}^T$:

$$P := \mathcal{F}^T \cdot \mathcal{F} = \begin{pmatrix} E & F & \\ F & G & \\ & & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F & \\ -F & E & \\ & & 1 \end{pmatrix}$$

Substituting into $\mathcal{F}_s = \mathcal{F}A$ we get

$$PA = \begin{pmatrix} \frac{E_s}{2} & \frac{E_t}{2} & -L \\ F_s - \frac{E_t}{2} & \frac{G_s}{2} & -M \\ L & M & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{F}^T \mathcal{F}_{st} &= (\mathcal{F}^T \mathcal{F}_s)_t - \mathcal{F}_t^T \mathcal{F}_s = (PA)_t - B^T PA. \\ \implies (PA)_t - (PB)_s &= (PB)^T P^{-1}(PA) - (PA)^T P^{-1}(PB). \end{aligned}$$

Gauss equation corresponds to the $(1, 2)$ entry of the matrix equation:

$$\frac{1}{2}(E_{tt} - 2F_{st} + G_{ss}) = -(LN - M^2) + \frac{p}{4(EG - F^2)},$$

where p is a polynomial in fundamental quantities and their derivatives.

These equations still looks complicated, so we'll "package" them:

$$-(LN - M^2) = R_{1212}.$$

Let

$$A = \begin{pmatrix} \Gamma^1_{\_11} & \Gamma^1_{\_12} & h^1_{\_1} \\ \Gamma^2_{\_11} & \Gamma^2_{\_12} & h^2_{\_1} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{\_12} & \Gamma^1_{\_22} & h^1_{\_2} \\ \Gamma^2_{\_12} & \Gamma^2_{\_22} & h^2_{\_2} \\ h_{12} & h_{22} & 0 \end{pmatrix}$$

Here the Γ's are called Christoffel notations.

Codazzi equations correspond to the $(1, 3), (2, 3)$ enties:

$$\begin{aligned} L_t - M_s &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \\ M_t - N_s &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{21}) - N\Gamma^2_{21}. \end{aligned}$$

Remark 3.6.4 — The index notations for computation can be defined as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij}), \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = (g^{ij}),$$

and h is defined similarly. If we use Einstein summation notation, we can write $g_{ij}g^{jk} = \delta_i^k$.

Let $\vec{v}_1 := \phi_s, \vec{v}_2 = \phi_t$, and

$$\frac{\partial \vec{v}_\alpha}{\partial \vec{u}^\beta} = \sum_\gamma \Gamma^\gamma_{\_\alpha\beta} \vec{v}_\gamma + h_{\alpha\beta}\vec{n}, \quad \frac{\partial \vec{n}}{\partial \vec{u}^\beta} = -\sum_\gamma h^\gamma_{\_\beta} \vec{v}_\gamma.$$

Here the upper index is defined as:

$$h^\gamma_{\_\beta} := \sum_\delta g^{\gamma\delta} h_{\delta\beta}.$$

From this we can write Γ out explicitly:

$$\Gamma^{\gamma}_{\text{-}\alpha\beta} = \sum_{\delta} \frac{g^{\gamma\delta}}{2} \left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}} \right)$$

This is called *Christoffel notations*.

$$R^{\delta}_{\text{-}\alpha\beta\gamma} := \frac{\partial \Gamma^{\delta}_{\text{-}\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\text{-}\alpha\gamma}}{\partial u^{\beta}} + \sum_{\eta} (\Gamma^{\eta}_{\text{-}\alpha\beta}\Gamma^{\delta}_{\text{-}\eta\gamma} - \Gamma^{\eta}_{\text{-}\alpha\gamma}\Gamma^{\delta}_{\text{-}\eta\beta}).$$

This is called *Riemann symbols*. Another type is defined as:

$$R_{\delta\alpha\beta\gamma} = \sum_{\eta} g_{\delta\eta} R^{\eta}_{\text{-}\alpha\beta\gamma}.$$

In surface theory, only R_{1212} is nontrivial.

Using these notations, we can write the equations as:

- Gauss:
$$R_{\delta\alpha\beta\gamma} = -(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}).$$
- Codazzi:
$$\frac{\partial h_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial h_{\alpha\gamma}}{\partial u^{\beta}} = \sum_{\delta} (h_{\beta\delta}\Gamma^{\delta}_{\text{-}\alpha\gamma} - h_{\gamma\delta}\Gamma^{\delta}_{\text{-}\alpha\beta}).$$

Here we explain the above computation a little.
The vector

$$\sum_{\gamma} \Gamma^{\gamma}_{\text{-}\alpha\beta} v_{\gamma} =: \nabla_{\beta}\vec{v}_{\alpha}$$

is called the covariant derivative of $\vec{v}_{\alpha}$. It's projection of the derivative of $\vec{v}_{\alpha}$ onto the tagent space.

$$\frac{\partial}{\partial u^{\beta}} \frac{\partial}{\partial u^{\gamma}}(v_{\alpha}) = \frac{\partial}{\partial u^{\gamma}} \frac{\partial}{\partial u^{\beta}}(v_{\alpha})$$
$$\implies -\nabla_{\beta}\nabla_{\gamma}v_{\alpha} + \nabla_{\gamma}\nabla_{\beta}v_{\alpha} = h_{\alpha\beta}\nabla_{\gamma}\vec{n} - h_{\alpha\gamma}\nabla_{\gamma}\vec{n}.$$

So the covariant derivative is not commutative, and the "curvature" or the second fundamental form basically measures this discommutation.

Now if we look at

$$\frac{\partial g_{\delta\alpha}}{\partial u^{\beta}} = \frac{\partial v_{\delta} \cdot v_{\alpha}}{\partial u^{\beta}} = \frac{\partial v_{\delta}}{\partial u^{\beta}} v_{\alpha} + v_{\delta} \frac{\partial v_{\alpha}}{\partial u^{\beta}} = \sum_{\gamma} g_{\alpha\gamma}\Gamma^{\gamma}_{\text{-}\delta\beta} + \sum_{\gamma} g_{\delta\gamma}\Gamma^{\gamma}_{\text{-}\alpha\beta},$$

similarly, by symmetry, computing

$$\frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}, \quad \frac{\partial g_{\delta\beta}}{\partial u^{\alpha}},$$

will yield

$$\Gamma^{\gamma}_{\text{-}\alpha\beta} = \sum_{\delta} \frac{g^{\gamma\delta}}{2} \left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}} \right).$$

In fact this is more intuitive in Einstein summation notation.

§3.7 Fundamental theorem of surface theory

Theorem 3.7.1 (Fundamental theorem of surface theory)
Let $D \subset \mathbb{R}^2$, $u = (u^1, u^2)$ is the coordinate. Let $g_{\alpha\beta}, h_{\alpha\beta} : D \to \mathbb{R}$ be C^3 functions, and the matrix $(g_{\alpha\beta})$ is symmetrical anf positive definite, $(h_{\alpha\beta})$ is symmetrical.

Let $g^{\alpha\beta}$ be the inverse matrix of $g_{\alpha\beta}$, and $R_{\delta\alpha\beta\gamma}$ is as above. If these functions satisfies Gauss equation and Codazzi equation, then:

For all $p \in D$, exists a neighborhood $U = U(p) \subset D$ and a regular surface $\phi : U \to \mathbb{E}^3$, such that $g_{\alpha\beta}, h_{\alpha\beta}$ are the first and second fundamental quantities of ϕ.

Furthermore, if $\widetilde{\phi} : U \to \mathbb{E}^3$ also satisfies above conditions, then $\widetilde{\phi} = \sigma \circ \phi$, where σ is an isometry of $\mathbb{E}^3$.

Basically we need to solve a partial differential equation, and we need to consider how to construct this equation.

Proof. Let $\phi : U \to \mathbb{E}^3$, $v_\alpha, \vec{n} : D \to V(\mathbb{E}^3)$ be unknown functions satisfying

$$\begin{cases} v_\alpha = \dfrac{\partial \phi}{\partial u^\alpha} \\ \dfrac{\partial v_\alpha}{\partial u^\beta} = \displaystyle\sum_\gamma \Gamma^\gamma_{\ \alpha\beta} v_\gamma + h_{\alpha\beta}\vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = -\displaystyle\sum_\gamma h^\gamma_{\ \beta} v_\gamma \end{cases}$$

This is a linear homogeneous PDE of degree 1, and it actually has a unique solution.

Consider the initial-value problem in the neighborhood of a given point $p \in D$.

We hope to prove that

- The above PDE initial-value problem has a unique solution under the Gauss-Codazzi equations;
- If initially (i.e. at p) we have
$$\vec{n} = \frac{v_1 \times v_2}{\|v_1 \times v_2\|},$$
then it holds for all $p' \in U(p)$.

For the second statement, we can compute $\frac{\partial}{\partial u^\beta}(\vec{n} \cdot v_\alpha) = 0$, so they are constant.

For the PDE part, if we want a C^2 solution of some linear PDE of degree 1:

$$\frac{\partial y^j}{\partial x^\alpha} = f^j_\alpha(x^1, \ldots, x^n, y^1, \ldots, y^m)$$

There's a necessary condition that the partial derivatives are commutative, i.e.

$$\frac{\partial}{\partial x^\beta}\frac{\partial y^j}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta}.$$

This expands to

$$\frac{\partial f^j_\alpha}{\partial x^\beta} + \sum_k f^k_\beta \frac{\partial f^j_\alpha}{\partial y^k} = \frac{\partial f^j_\beta}{\partial x^\alpha} + \sum_k f^k_\alpha \frac{\partial f^j_\beta}{\partial y^k}.$$

In fact this is also the sufficient condition of the existence local solution.

Remark 3.7.2 — The proof is beyond the scope of this course, but the basic idea is to build the y^j's dimension by dimension (from curve to surfaces to 3d manifolds ...). The 1d part can be constructed using solutions of ODE, and the compatibility follows by our condition.

In the language of differential forms, let $y = (y^1, \dots, y^m)$, we are given $\mathrm{d}y$, since the condition says $\mathrm{d}(\mathrm{d}y) = 0$, i.e. $\mathrm{d}y$ is a *closed form*, so we always have local solution of y.

Returning to our original problem, this condition is actually what we used to deduce the Gauss-Codazzi equations, so our PDE must have a unique solution on a neighborhood of p. $\square$

Example 3.7.3

We can't grant that the global function exists. For example, let $D = \{x^2 + y^2 \in [a^2, b^2]\}$, and M be a helicoid.

Since there's a natural map $\phi : D \backslash ([a, b] \times \{0\}) \to M$ (projection), let g, h be the fundamental forms of ϕ, by the symmetry we can extend g, h to entire D.

It's clear that there exists local solutions but the global solutions does't exist. (In theory of differential forms, this is similar to closed forms may not be exact)

But if the region D is *simply connected*, the global solution always exist.

§3.8 Isometric, conformal and area-perserving maps

Let $U, \widetilde{U} \subset \mathbb{R}^2$, and $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. Let $f : \widetilde{U} \to U$ be a map between two surfaces.

Earlier we introduced isometric maps (isometry), i.e. $f^*(g) = \widetilde{g}$. Since the length depends only on the first fundamental form, the isometry perserves the length, angles and areas on surfaces.

The **conformal** maps perserves the angles on the surfaces, and it's easy to imply this is equivalent to $f^*(g) = \lambda \widetilde{g}$ for some $\lambda \in \mathbb{R}$.

As the name suggests, the **area-perserving** maps perserves the areas on two surfaces, which is saying $\det f^*(g) = \det \widetilde{g}$.

It's easy to prove that isometric = conformal + area-perserving. These three properties induce Riemann geometry, complex geometry and sympletic geometry, respectively (in two dimensional).

§3.8.1 Isometries

Firstly by Gauss' Theorema Egregium, Isometries perserves Gaussian curvature.

Example 3.8.1

Let $S_{a,b} : \frac{x^2}{a} + \frac{y^2}{b} = 2z$ be a saddle surface. Let $(x, y, z) = (as, bt, \frac{as^2+bt^2}{2})$ be a parametrization.

We can compute the fundamental forms:

$$g = a^2(1 + s^2)\,\mathrm{d}s^2 + 2abst\,\mathrm{d}s\,\mathrm{d}t + b^2(1 + t^2)\,\mathrm{d}t^2,$$

$$h = \frac{a\,\mathrm{d}s^2 + b\,\mathrm{d}t^2}{\sqrt{1 + s^2 + t^2}}.$$

So $K = \frac{1}{ab(1+s^2+t^2)^2}$. In fact the Gaussian curvature of some different surfaces, say $S_{2,3}$ and $S_{1,6}$ are the same.

But there is not an isometry between them:

If $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then τ fixes the circles centered at $(0, 0)$ as their Gaussian curvature are the same. Then $\tau_* = \mathrm{d}\tau : T_{(0,0)}\mathbb{R}^2 \to T_{(0,0)}\mathbb{R}^2$ can only be rotation or reflection. (If τ_* is not orthogonal, it will map small circles to ellipse)

While $g(0, 0) = a^2\,\mathrm{d}s^2 + b^2\,\mathrm{d}s^2$, which has eigenvalue a^2 and b^2, and they're fixed under τ_*, so $S_{2,3}$ isn't isometric to $S_{1,6}$.

Remark 3.8.2 — Given $E, F, G : D \to \mathbb{R}$ s.t. $g = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ positive definite, is there a surface $D \to \mathbb{E}^3$ can have g as its first fundamental form locally?

When we require E, F, G to be C^ω(analytic), the answer is "yes", but if we only require C^∞, it's still an open problem.

Even though we don't know the situation in 3 dimensional space, we can study the case in higher dimensions:

Theorem 3.8.3

It's always possible to construct $\phi : D \to \mathbb{E}^4$ to have E, F, G as its first fundamental form.

Surfaces with Gaussian curvature 0 everywhere are called **developable surfaces**. Developable surfaces can only be cylinder, cone, tangent surface of a curve and their concatenation.

Example 3.8.4 (Pseudosphere)

Let $\phi(x, y) = (\frac{\cos x}{y}, \frac{\sin x}{y}, \cosh^{-1}(y) - \frac{\sqrt{y^2-1}}{y})$, where $(x, y) \in (-\pi, \pi) \times [1, +\infty)$.

It's obtained by rotating a *tractrix* around its asymptote. We can calculate its Gaussian curvature, whichis a constant -1. This is where the name comes from.

Recall that hyperbolic plane also has constant curvature -1, in fact they are locally isometric. In 1901, Hilbert proved a theorem that there exists an isometry $\mathbb{H}^2 \to \mathbb{E}^3$.

At last we'll prove an interesting fact:

Proposition 3.8.5 (The local existence of isothermal parameters)
Let $\phi : U \to \mathbb{E}^3$, for all $\hat{u} \in U$, there exists a neighborhood $\widetilde{U}$ and a reparametrization $u = u(\widetilde{u})$, such that
$$g(\widetilde{u}) = \rho^2(\widetilde{u})(\widetilde{E}\,\mathrm{d}\widetilde{s}^2 + \widetilde{G}\,\mathrm{d}\widetilde{t}^2).$$

Remark 3.8.6 — Note that the right hand side is clearly conformal to regions in $\mathbb{E}^2$, so this in fact implies that any surfaces is locally conformal to $\mathbb{E}^2$.

Proof. The critical idea is to realize $\mathbb{R}^2$ as $\mathbb{C}$. To be more precise, we'll follow the steps below:

- Find a way to express $E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ as $(a\,\mathrm{d}s + b\,\mathrm{d}t)(\overline{a}\,\mathrm{d}s + \overline{b}\,\mathrm{d}t)$, where a, b are functions with complex value.
- If there exists a complex function f s.t. $\mathrm{d}f(s+it) = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$, then $g = \frac{1}{|\rho|^2}\,\mathrm{d}f\,\mathrm{d}\overline{f}$.
- Assume further that f is *holomorphic* and non-degenerate, then $f(u) = \widetilde{x}(u) + i\widetilde{y}(u)$ is locally inversible, i.e. exists $u = u(\widetilde{x}, \widetilde{y})$, then
$$g = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x} + i\,\mathrm{d}\widetilde{y})(\mathrm{d}\widetilde{x} - i\,\mathrm{d}\widetilde{y}) = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x}^2 + \mathrm{d}\widetilde{y}^2).$$

Let $a = \sqrt{E}$, $b = \frac{-F + i\sqrt{EG - F^2}}{\sqrt{E}}$. (Note $EG - F^2 > 0$ as g is positive definite)
Next we'll choose suitable f, ρ. Consider the differential equation $T = T(s,t)$:
$$\frac{\partial T}{\partial s} = -\frac{a(s,T)}{b(s,T)}, \quad T(\hat{s}, t) = t.$$

From the relation $f(s, T(s,t)) = t - \hat{t}$ and implicit function theorem we can uniquely determine f.

Remark 3.8.7 — The detail of the solution to this equation in complex functions is beyond the scope of this class.

Such f satisfies $\mathrm{d}f = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$.
When $f(s,t) = (\widetilde{x}, \widetilde{y})$, the Jacobian determinant is
$$\widetilde{x}_s\widetilde{y}_t - \widetilde{x}_t\widetilde{y}_s = -|\rho|^2(a\overline{b} - b\overline{a}) = |\rho|^2\sqrt{EG - F^2} > 0.$$
so f must be non-degenerate. □

§4 Algebraic topology

§4.1 A bit of manifold

First we'll introduce a few concepts before we move on.

- We say a topological space is an n-dimensional **topological manifold** if it's Hausdorff and locally homeomorphic to $\mathbb{R}^n$. Sometimes we also require manifolds to be compact / paracompact / C_2. Here paracompact means that any open covering has a locally finite subcovering.

- Manifolds with boundary: locally homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$.
- When we talk about the regularity of manifolds, we must appoint an atlas first. Let $\phi_i : U_i \to E_i \subset \mathbb{R}^n$ be homeomorphisms mentioned above, then each ϕ_i is a **chart**, and $\{(U_i, \phi_i)\}_{i \in I}$ is the **atlas**. The map
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
is called **transition functions**.

  The regularity of the manifold is actually the regularity of transition functions, such as C^r, C^∞, piecewise linear, etc.

Example 4.1.1

The sphere $\mathbb{S}^2$ and projective plane $\mathbb{R}P^2$ are 2d manifolds. But they're different since $\mathbb{R}P^2$ is not *orientable*. In fact $\mathbb{R}P^2$ can be obtained by fusing the edge of a Mobius band to a disk(keep in mind that Mobius band has only one edge!).

There are many manifolds which looks wired, but I can't draw them on the computer ;)

Example 4.1.2 (Projective curves)

Consider a quadratic equation

$$C : z^2 + w^2 = 1, \quad (z, w) \in \mathbb{C}^2.$$

What does this surface look like?

Let $Z = z + iw, W = z - iw$, the equation becomes $ZW = 1$, hence the surface is $(\zeta, \frac{1}{\zeta}), \zeta \in \mathbb{C}\backslash\{0\}$. So C is homeomorphic to $\mathbb{C}\backslash\{0\}$.

We can also discuss this in $\mathbb{C}P^2 = \mathbb{C}P^1 \cup \mathbb{C}^2$, where $\mathbb{C}P^1 = \{\infty\} \cup \mathbb{C} \cong \mathbb{S}^2$.

So in homogeneous coordinate, the equation can be written as $ZW = T^2$. The surface is consisting of $(1, 0, 0), (0, 1, 0), (\zeta, \frac{1}{\zeta}, 0)$. Thus the projective completion of C is homeomorphic to $\mathbb{S}^2$, which is $\mathbb{C}\backslash\{0\}$ appending with two points.

Example 4.1.3 (Elliptic curves)

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ pairwise different.

$$E : w^2 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3).$$

What does E looks like in $\mathbb{C}P^2$?

Observe that for $z \in \mathbb{C}\backslash\{\lambda_1, \lambda_2, \lambda_3\}$, there're 2 values for w. So the image of E is two planes($\mathbb{C}$) fused together at $\lambda_1, \lambda_2, \lambda_3$ and ∞ with some adjust.

In fact this can be realized as two cylinder fused together at their edges.

$E \cong T^2\backslash\{pt\}$ in $\mathbb{C}^2$, and T^2 in $\mathbb{C}P^2$.

In fact $\mathbb{C}P^2$ is a 4-dimensional closed manifold, and it's also a 2-dimensional complex manifold. $PSL(3, \mathbb{C})$ acts transitively on $\mathbb{C}P^2$.

Example 4.1.4

We can fuse the edges of polygons to get manifolds: By fusing together opposite edges of a square, we can get torus or Klein bottle.

We'll use the word "fuse" frequently in the future, so here we'll make it clear what we mean by "fusing" things together.

Definition 4.1.5 (Quotient maps)**.** A continuous map $f : X \to Y$ is called a **quotient map**, if it's surjective, and $\forall B \subset Y$, $f^{-1}(B)$ open $\implies B$ open.

This is saying that the topology on Y is the "largest" topology (or quotient topology) while keeping f continuous.

So when we "fusing" things together, we're actually giving an equivalence relation on the original space, and the result is the quotient topology induced from the natural projection map.

Now we look at the elliptic curves again, let $U = \mathbb{C}\backslash([\lambda_1, \lambda_2] \cup [\lambda_3, \infty])$. Let X be the path end compactification of U, then $X \simeq S^1 \times [0, 1]$.

Let X_1, X_2 be two copies of X, and fusing the corresponding circles at the end in the reversed direction, we'll get a torus without 4 points, by adding $\lambda_1, \lambda_2, \lambda_3$ back we'll get $T^2\backslash\{pt\}$.

Remark 4.1.6 — The quotient topology may have some bad properties, like not being Hausdorff: Consider $\mathbb{R}^2\backslash\{(0, 0)\}$ with connected vertical lines as equivalence class, then we'll get a line with 2 points at the origin, which is a typical non-Hausdorff space.

A closed surface is a connected compact 2-dimensional manifold with no edges. We have the following classification theorem:

Theorem 4.1.7

All the closed surfaces must be homeomorphic to $nT^2(n \geq 0)$ or $mP^2(m \geq 1)$. Here n is called the **genus** of orientable surfaces.

nT^2 can be viewed as S^2 fused with n handles (torus), and mP^2 can be viewed as S^2 fused with m crosscaps (Mobius strip).

In this course we mainly talk about surfaces with triangulation, i.e. we take it for granted that all surfaces has triangulation.

Here we'll prove part of this theorem (since the other part needs further knowledge).

Remark 4.1.8 — X has a triangulation means that X is homeomorphic to finitely many n-simplex fused together at the boundary linearly, and the *link* of each vertical is a triangulation of S^{n-1}.

Proof. Observe that given a triangulation, we can get a polygon fusing presentation of the surface by adding the triangles one by one, fusing only one edge each time.

If we write down the edges of this polygon at a certain order, using letters to indicate differet edges and bars for direction, we can get something like $ab\bar{a}\bar{b}$ for a torus.

TODO: pictures!

In fact, nT^2 can be presented as $[a_1, b_1][a_2, b_2] \ldots [a_n, b_n]$, where $[a, b] = ab\bar{a}\bar{b}$. Likely, mP^2 is $c_1^2 c_2^2 \ldots c_m^2$ since P^2 is c^2. So our goal is to say that any given "edge words" can be reformed to one of the above standard forms.

Note that $(A) : Wa\overline{a} = W$, and $(B) : aUV\overline{a}U'V' = bVU\overline{b}V'U'$. The second operation is cut the polygon in the middle to get b, and fuse two parts together to eliminate a. There's also a reversed version: $aUVaU'V' = bV'VbUU'$. Also note that the word is cyclic, so $(C) : UV = VU$.

TODO: pictures!

This is kind of like Olympiad combinatorics problem. So we need techniques like:

- A "complexity" to measure how close we are to destination:

  vertical numbers(verticals fused together are regarded as one) and egde pair numbers

- Some labels to control different branches:

  whether it has edges with the same direction

- Some efficient "combo moves"

Observe that

- (A) will reduce vetical and edge pair by 1,
- (B) won't effect edge pairs, but may change vertical numbers,
- (C) won't change anything.

In fact we can reduce the vertical number to 1, i.e. all the verticals are fused to one point in the surface. If we have at least 2 verticals, say P and Q, and PQ is an edge. There must be another edge connecting P, Q, If those two P are different in the polygon, we can use (B) to eliminate one P vertical (by adding edge pair of QQ), and use (A) to eliminate they're the same.

TODO: pictures!!!

Repeating above process we can make the vertical number become 1.

If we have $aUbV\overline{a}U'\overline{b}V'$, we can use (B) twice to reform it to $cd\overline{c}\overline{d}W$.

TODO: pictures!!!

So we can achive nT^2 from a word with no same-direction-pairs. Techniquely we still need to prove that we can always find $a \dots b \dots \overline{a} \dots \overline{b}$ in original word, but this can be proved easily otherwise we can perform (A) to reduce edges.

Now for mP^2 :

After some fancy operations we're done. □

Remark 4.1.9 — On the existence of triangulation

§4.2 Homotopy

Definition 4.2.1 (Homotopy)**.** Given two continuous maps $f, g : X \to Y$, if there exists a continuous map

$$H : X \times [0, 1] \to Y$$

such that $f = H_0, g = H_1$, where $H_t = H\big|_{X\times\{t\}}$, then we say f and g are **homotopic**, denoted by $f \simeq g$, and the map H is a **homotopy**.

Definition 4.2.2 (Relative homotopy)**.** Let $A \subset X$, $f, g : X \to Y$, and $f\big|_A = g\big|_A$. We say f and g are homotopic relative to A ($f \simeq g\ rel\ A$), if H satisfies $H_t\big|_A = f\big|_A$.

More often we'll talk about homotopy between paths, here by path we mean a map $\gamma : [0, 1] \to X$. We say two paths are homotopic if they are homotopic relative to the endpoints($i.e.\{0, 1\}$)

Proposition 4.2.3

The homotopic relation is an equivalence relation.

Besides studying the homotopy of maps, we can also consider the homotopy between spaces:

Definition 4.2.4. We say two topological spaces X, Y are **homotopy equivalent** or have the same **homotopy type**, if there exists $f : X \to Y$, $g : Y \to X$, such that

$$f \circ g = \mathrm{id}_Y, \quad g \circ f = \mathrm{id}_X .$$

Example 4.2.5

The following spaces are homotopy equivalent:

Definition 4.2.6 (Fundamental groups)**.** Let $\Omega(X, x_0)$ denote all the loops starting at x_0, i.e. $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

Define the **fundamental group** of X to be:

$$\pi_1(X, x_0) = \Omega(X, x_0)/ \simeq,$$

where $\simeq$ is the homotopy relative to x_0.

We define the group operation to be the *concatenation* of paths, denoted by $(a, b) \mapsto ab$, where

$$ab(t) = \begin{cases} a(2t), & t \in [0, \frac{1}{2}]; \\ b(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proposition 4.2.7

The concatenation descends to a well-defined group operation:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0).$$

Proof. Just some trivial checking. Note that the inverse of a is just $\overline{a}(t) := a(1-t)$. □

Proposition 4.2.8

An homeomorphism $f : (X, x_0) \to (Y, y_0)$ will induce a group homomorphism $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Note that X may be disconnected, so the fundamental group is dependent of the base point x_0. If $\gamma = \langle c \rangle$ is a homotopy class of paths from x_0 to x_1, then γ induces a group homomorphism:

$$\gamma_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1) : \langle a \rangle \mapsto \langle \overline{c}ac \rangle .$$

It's easy to see $\gamma_\sharp$ is an isomorphism.

Hence $\pi_1(X, x_0)$ only depends on the path connected components of x_0. Thus if X is path connected, and X, Y are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, or sometimes we can leave the base point out, just write $\pi_1(X) \cong \pi_1(Y)$.

Remark 4.2.9 — If $x_0 = x_1$, then $\gamma \mapsto \gamma_\#$ gives a homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_1(X, x_0))$.

Example 4.2.10

If $X \simeq \{pt\}$, then $\pi_1(X) \cong \{1\}$. In this case, X is called a **contractible space**. Note that the inverse is not true, e.g. $X = S^n$ for $n \geq 2$. Path connected spaces with trivial fundamental group are called **simple connected** spaces.

Some classic contractible space includes convex sets in $\mathbb{R}^n$, trees in graph theory and cones $CX = X \times [0,1]/X \times \{1\}$.

Some more complex contractible examples including "a house with two rooms", the equitorial inclusion $S^\infty = \bigcup_{n=0}^\infty S^n$ with limit topology, i.e. the largest topology s.t. $S^n \to S^\infty$ continuous.

There are several concepts:

- Retraction: $f : X \to A$, $A \subset X$, $f\big|_A = \mathrm{id}_A$.
- Deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$, where $i : A \to X$ is the inclusion.
- Strong deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$ *rel* A.

The set A is called (strong) deformation kernel of f.

Example 4.2.11 (Differences between deformation and strong deformation)

Let X be the follwing space:

$$([0,1] \times \{0\}) \cup ([0,1]_{\mathbb{Q}} \times [0,1])$$

We know $X \simeq \{pt\}$, but $\{q\} \times [0,1]$ is deformation kernel but not strong deformation kernel.

§4.3 Fundamental groups

After introducing the fundamental groups, a natural question arises: how to compute the fundamental group of a given space? We first state the main result of this section:

Theorem 4.3.1 (Van Kampen)
Let $X = U' \cup U''$ be a topology space such that U', U'' are open and $W = U' \cap U''$ path connected, then for $x_0 \in W$, we have

$$\pi_1(X, x_0) \cong \pi_1(U', x_0) * \pi_1(U'', x_0)/N,$$

where N is the smallest normal subgroup generated by

$$i'_\sharp(\delta) i''_\sharp(\delta^{-1}) : \delta \in \pi_1(W, x_0),$$

$$\begin{tikzcd} & U' \arrow[dr, "j'"] & \\ W \arrow[ur, "i'"] \arrow[dr, "i''"'] & & X \\ & U'' \arrow[ur, "j''"'] & \end{tikzcd}$$

and $*$ means free product.

Note that this theorem is useless when both U', U'' have trivial fundamental groups. Thus we need to find a space with nontrivial fundamental group first. One of the simplest example is S^1 :

$$\pi_1(S^n, x_0) \cong \begin{cases} \{1\}, & n \geq 2, \\ \mathbb{Z}, & n = 1. \end{cases}$$

Let $X \vee Y := X \sqcup Y/(x_0 = y_0)$, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. Thus $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{k}) = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_k$, the free group of rank k.

Example 4.3.2
Since nT^2 is formed by $2n$ loops(borders of the polynomial representation) fused with a disk. Note that $W = U' \cap U'' \cong S^1$, so

$$\pi_1(nT^2) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle .$$

Similarly,

$$\pi_1(mP^2) = \left\langle c_1, \ldots, c_m \mid c_1^2 \cdots c_m^2 = 1 \right\rangle .$$

Example 4.3.3
For product spaces, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

In fact the isomorphism can be written down with i_x, i_y, p_x, p_y, i.e. $p_{x\sharp} \times p_{y\sharp}$ and $(i_{x\sharp}, i_{y\sharp})$.

Theorem 4.3.4
$\pi_1(S^1) \cong \mathbb{Z}$, where the generating element is id.

Proof. Consider the map $p : \mathbb{R} \to S^1$, with $t \mapsto e^{2\pi i t}$.

Given any path $\gamma : [0,1] \to S^1$, we can find a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}$, s.t. $\tilde{\gamma}(0) \in \mathbb{Z}$ is any given base point. We denote this map by Φ, $\gamma \mapsto \tilde{\gamma}(1)$, where we require $\tilde{\gamma}(0) = 0$.

We can prove that $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$, and Φ only depends on the homotopy class of γ, so Φ induces a homomorphism of $\pi_1(S^1) \to \mathbb{Z}$.

> **Remark 4.3.5** — Since every homotopy $[0,1] \times [0,1] \to S^1$ can be lifted uniquely, and the endpoints of each path form a path in $\mathbb{R}$, but it's always contained in $\mathbb{Z}$, hence it must be constant.

Note that

- Φ is surjective since $s \mapsto e^{2\pi i m s}$ is mapped to m under Φ;
- Φ is injective since $\ker \Phi = \{1\}$: if $\tilde{\gamma}(1) = 0$, then $\tilde{\gamma} \simeq \mathit{const}$, so $\gamma = p \circ \tilde{\gamma} \simeq \mathit{const}$.

So Φ is an isomorphism, $\pi_1(S^1) \cong \mathbb{Z}$. □

Next we'll prove Van Kampen theorem (4.3.1). In fact we only need to prove that:

> **Claim 4.3.6.** The map
> $$j'_\sharp * j''_\sharp : \pi_1(U', x_0) * \pi_1(U'', x_0) \to \pi_1(X, x_0)$$
> is a surjective homomorphism, and its kernel is the normal closure generated by $i'_\sharp(\delta) i''_\sharp(\bar{\delta})$.

CLearly it's a group homomorphism.

For any $\gamma \in \pi_1(X, x_0)$, it can be decompose to $a_1 b_1 a_2 \cdots a_k b_k$, where $a_i \subset U', b_i \subset U''$, let the partition points be $p_1, \dots, p_k, q_1, \dots, q_k \in W$, and denote s_i, t_i the path from x_0 to p_i, q_i. So we have
$$\gamma = \underbrace{a_1 \bar{s}_1}_{\in \pi_1(U', x_0)} \cdot \underbrace{s_1 b_1 \bar{t}_1}_{\in \pi_1(U'', x_0)} \cdots$$
Thus $j'_\sharp * j''_\sharp$ is indeed surjective.

At last we'll study its kernel, let $\gamma \in \ker j'_\sharp * j''_\sharp$. Since $\gamma \simeq \{x_0\}$, say the homotopy is $H : [0,1] \times [0,1] \to U' \cup U''$.

We can partition $[0,1] \times [0,1]$ to many small cells such that each cell's image is completely contained in either U' or U''.

TODO

Using the "word processing" method, since we've showed that $\gamma = \alpha_1 \beta_1 \cdots$ where $\alpha_i \subset U', \beta_i \subset U''$. So actually we're saying that
$$\gamma = i'_\sharp(\alpha_1) i''_\sharp(\beta_1) \cdots$$
if we some $\delta \subset U' \cap U''$, then the conjugate of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$ can change $\cdots i'_\sharp(\delta) \cdots$ to $\cdots i''_\sharp(\delta) \cdots$.

Thus if γ is in the kernel, it can indeed be written as a product of conjugates of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$.

> **Remark 4.3.7** — A more frequently used version is that W is a strong deformation kernel of some open neighborhood in X.

Example 4.3.8

For any finite representation of a group

$$G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle ,$$

G can be realized as the fundamental groups of a space: Let X be a CW-complex with a single 0-cell, n 1-cells corresponding to x_i, and m 2-cells corresponding to R_i.

Remark 4.3.9 — The path connected condition of W can't be removed, e.g. two segments can fuse to S^1.

Example 4.3.10

Let $f : S \to S$ be a homeomorphism, where S is a closed surface. Consider the *mapping torus*:

$$M_f = S \times [0,1]/\sim,$$

where $(0,0) \sim (f(x),1)$.

Let $Y = S \times \{0\} \cup (\{x_0\} \times [0,1])$, U' is an open neighborhood of Y, $U'' = M_f \backslash Y$.

Observe that $U' \simeq S \vee circle$, and $U'' \simeq (S\backslash disk) \times (\varepsilon, 1-\varepsilon) \simeq S\backslash disk$.

$$\pi_1(M_f) \cong \pi_1(X) * \langle t \rangle / (g \sim t f_\sharp(g) t^{-1}) \cong \pi_1(S) \rtimes_{f_\sharp} \langle t \rangle$$

Seifent-vanKampen: if $i'_\sharp, i''_\sharp$ are both injective, then $j'_\sharp, j''_\sharp$ are also injective.

Next we'll see some applications of fundamental groups:

- Bronwer fixed point theorem: A continuous map $f : D^n \to D^n$ must have a fixed point.
- Invariance of the boundary: If $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$, s.t. $x \in \mathbb{R}^{n-1} \times \{0\}$, then x doesn't have a neighborhood homeomorphic to $\mathbb{R}^n$.
- Invariance of regions: If $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is also open, i.e. $f : U \to f(U)$ is a homeomorphism.

Here we can only prove the case $n = 2$, since the complete proof need homotopy groups of rank n (i.e. π_n), but here we only introduced π_1. However, the idea is nearly identical.

Proof. Assmue by contradiction that f has no fixed points, let $g(x) = \frac{x-f(x)}{\|x-f(x)\|}$, then $g : D^n \to S^{n-1}$ is a deformation. Thus $g_\sharp : \pi_1(D^2) \to \pi_1(S^1)$ surjective, but $\pi_1(D^2) = \{1\}, \pi_1(S^1) = \mathbb{Z}$, contradiction! □

Proof. If x is an interior point, $x \in U$ and U homeomorphic to $\mathbb{R}^n$, then $U\backslash\{x\}$ can deform to a $n-1$ dimensional sphere, thus $\pi_1(U\backslash\{x\}) \neq \{1\}$.

But if x is a boundary point, then $\pi_1(U\backslash\{x\}) = \{1\}$, contradiction! □

Proof. Assume by contradiction that there exists $0 \in U$ s.t. $f(0) \in \mathbb{R}^n$ has no open neighborhood lying completely in $f(U)$.

We can construct a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\|x - g(f(x))\| \leq 1, \quad x \in B(0,1); \quad g(f(x)) \neq 0.$$

Then by Bronwer fixed point theorem on $x \mapsto x - g(f(x))$ we get a contradiction.

The construction is as below:

Since $f(\partial B(0,1))$ must be at least say 10ε away from 0, and $B(f(0),\varepsilon)$ has a point outside of the image of f, so we have a map $P : B(f(0),\varepsilon)\backslash\{p\} \to \partial B(f(0),\varepsilon)$.

Then consider $g = f^{-1} \circ P$, since f^{-1} may not exist on every point, so we need Tietze extension theorem to get an extension h. In $B(f(0), 2\varepsilon)$, we'll change h a little (i.e. take a polynomial approximation) to ensure $g(f(x)) \in B(0,1)$. □

§4.4 Covering spaces

Except van Kampen's theorem, there's another way to compute fundamental groups.

Definition 4.4.1 (Covering maps)**.** Let $p : \widetilde{X} \to X$ be a continuous map. If

- p is surjective;
- For any $x \in X$, there exists an open neighborhood $U = U(x) \subset X$, such that $p^{-1}(U)$ is a union of disjoint open sets $\{U_\alpha\}$, and p is homeomorphism from U_α onto U for each α.

Then we say p is a **covering map**, and $\widetilde{X}$ is a **covering space** of X. $p^{-1}(x)$ is called a **fiber**.

Remark 4.4.2 — Often we'll require $\widetilde{X}, X$ are path connected to ensure the relations with fundamental groups. In this case $\#p^{-1}(x)$ is constant.

Definition 4.4.3. We say two covering is **isomorphic** if exists homeomorphism $\tau : \widetilde{X} \to \widetilde{X}'$ s.t. $p' \circ \tau = p$. Two covering is **equivalent** if $p' \circ \widetilde{\sigma} = \sigma \circ p$. The difference is shown in the diagram.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tau"] \arrow[dr, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\sigma}"] \arrow[d, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ X \arrow[r, "\sigma"'] & X' \end{tikzcd}$$

Example 4.4.4

The map $x \mapsto e^{ix}$ is a covering map from $\mathbb{R}$ to S^1. Also $\mathbb{R}^2$ is a covering space of T^2, since T^2 can be represented as $\mathbb{R}^2/\mathbb{Z}^2$.

Example 4.4.5

The surface $2T^2$ can be viewed as an octagon with edges fused together, (an octagon with each angle $45°$) which can be realized in hyperbolic plane $\mathbb{H}^2$.

In fact, $\mathbb{H}^2$ is always the covering space of kT^2 when $k \geq 2$, and kP^2 when $k \geq 3$.

From the examples we can see that covering spaces are the "expanded" spaces of original spaces, i.e. the structures are "flattened" in covering spaces, so that we can study the structure of original spaces more easily.

An important application is that we can "lift" the maps to covering spaces.

Theorem 4.4.6 (Map lifting theorem)

Let $p : \widetilde{X} \to X$ be a covering map, X is path connected. Let A be a path connected space, $f : A \to X$ has a **lifting** $\widetilde{f} : A \to \widetilde{X}$ s.t. $\widetilde{f}(a) \in p^{-1}(f(a)), \forall a \in A$ if and only if there exists a homomorphism Φ s.t. $f_\sharp = p_\sharp \circ \Phi$:

$$\begin{tikzcd} & \pi_1(\widetilde{X}, e_0) \arrow[d, "p_\sharp"] \\ \pi_1(A, a_0) \arrow[ur, dashed, "\Phi"] \arrow[r, "f_\sharp"'] & \pi_1(X, x_0) \end{tikzcd}$$

This is equivalent to $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{X}, e_0))$.

Proof. If we fixed $\tilde{f}(a_0) = e_0$, then for a neighborhood V of e_0, there's a unique map $\tilde{f} : U \to V$, where U is a neighborhood of a_0. This is because p restricted on V is a homeomorphism, and f continuous implies U is open, U is called a *basic neighborhood* of a_0.

For any $b \in A$, there's a path γ from a_0 to b. Since γ is compact, it can be splited to several segments, where each segment lies inside a basic neighborhood of some point.

Therefore the lifting of γ can be uniquely determined by the lifting of one point. Hence $\tilde{f}(b)$ is also determined.

Next we'll show that this $\tilde{f}$ is well-defined and continuous. Let α, β be two paths from a_0 to b. Then $f \circ \alpha, f \circ \beta$ are two paths from x_0 to $f(b)$.

When $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{(X)}, e_0))$, let $w = \alpha\beta^{-1} \in \pi_1(A)$, then there exists $\varphi \in \pi_1(\widetilde{X})$ s.t. $f \circ w = p \circ \varphi$.

But there's a unique lifting for $f \circ \alpha, f \circ \beta$, so $\tilde{f}(\alpha)\tilde{f}(\beta)^{-1} = \varphi$, thus $\tilde{f}(b)$ is well-defined.

Clearly $\tilde{f}$ is continuous, so we're done. □

Remark 4.4.7 — Different base points will result in the image $p_\sharp$ and $\tilde{f}$.

Example 4.4.8

Let M be a closed surface, $M \neq S^2, \mathbb{RP}^2$. Note that M has a contractible covering space, so any map $S^n \to M$ is always homotopic to constant, where $n \geq 2$.

Now if we look at the definition of isomorphic coverings, we'll find that this is just a map lifting, where τ is a lifting of p, τ^{-1} is a lifting of p'. By map lifting theorem we get:

Corollary 4.4.9

Two covering spaces $\widetilde{X}, \widetilde{X}'$ of X are isomorphic iff $p_\sharp(\pi_1(\widetilde{X})) = p'_\sharp(\pi_1(\widetilde{X}'))$.

From this we discover that each covering of X corresponds to a subgroup of $\pi_1(X)$. In fact the inverse is also true:

Theorem 4.4.10 (Existence theorem of covering spaces)

Let X be a path connected and locally path connected space, then for all subgroups $G \leqslant \pi_1(X, x_0)$, there exists a covering $p : \widetilde{X} \to X$ s.t.

$$p_\sharp(\pi_1(\widetilde{X}, e_0)) = G.$$

Remark 4.4.11 — This implies that **universal coverings** always exists, i.e. the covering space $\widetilde{X}$ which has trivial fundamental group.

The proof is quite complex, so we'll put it off here.

Definition 4.4.12 (Regular covering space)**.** If $p_\sharp(\pi_1(\widetilde{X}, e_0))$ is a normal subgroup of $\pi_1(X, x_0)$, then we say it's a **regular covering** of X.

In this case the base point will not change the image of $p_\sharp$.

We say the lifting of p with respect to itself is a **deck transformation**. In fact, deck transformations are just automorphisms of covering spaces, and they constitude a group $Deck_X(\widetilde{X})$ or $Deck_{\widetilde{X}/X}$.

Here's another definition of regular covering: If the group action $Deck_{\widetilde{X}/X}$ onto $\widetilde{X}$ are transitive in $p^{-1}(x_0)$, then we say the covering is **regular covering**.

There should be some pictures of regular and non-regular coverings of $S^1 \vee S^1$, but I'm a bit lazy :-)

Now we'll prove this two definitions are equivalent.

Proposition 4.4.13

Let $p : \widetilde{X} \to X$ be a covering, p is regular iff $p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$ is a normal subgroup.

Proof. When $p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \lhd \pi_1(X, x_0)$, for $\tilde{x}_0, \tilde{x}_0' \in \widetilde{X}$, we need to prove that there exists $\tau \in Deck_{\widetilde{X}/X}$ s.t. $\tau(\tilde{x}_0) = \tilde{x}_0'$.

We'll use lifting theorem on p, thus we only need to show

$$p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \subset p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0').$$

Let γ be a path from $\tilde{x}_0$ to $\tilde{x}_0'$, and $\alpha \in \pi_1(\widetilde{X}, \tilde{x}_0)$. Note that $\alpha \simeq \gamma\overline{\gamma}\alpha\gamma\overline{\gamma}$, $\alpha' = \overline{\gamma}\alpha\gamma \in \pi_1(\widetilde{X}, \tilde{x}_0')$. Hence

$$p_\sharp(\alpha) = p_\sharp(\gamma)p_\sharp(\alpha')p_\sharp(\overline{\gamma}) \in hp_\sharp\pi_1(\widetilde{X}, \tilde{x}_0')h^{-1} = p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0').$$

The converse is the same. □

Now we'll prove Theorem 4.4.10: First we'll handle the case of universal covering.

Theorem 4.4.14

Universal covering space and is unique under isomorphism for path connected and locally path connected space X. If X is also locally semi-simply connected, then universal covering exists.

Proof. If $\widetilde{x}, \widetilde{X}'$ are both universal coverings, by map lifting theorem, since $\pi_1(\widetilde{X})$ is trivial, $p : \widetilde{X} \to X$ can be lifted to $\sigma : \widetilde{X} \to \widetilde{X}'$, similarly we have σ', and it's easy to see σ and σ' are inverse maps, so they are isomorphic.

For existence part, X locally semi-simply connected means for $\forall x \in X$, there exists a neighborhood basis $\{U_i\}$ s.t. $\pi_1(U_i, x) \to \pi_1(X, x)$ is trivial.

Let $P(X, x_0)$ be all paths in X starting from x_0, and $\mathscr{X}$ is the homology equivalent classes (with fixed endpoints) of $P(X, x_0)$.

Let $p : \mathscr{X} \to X$ by $\langle a \rangle \mapsto a(1)$, and $\tilde{x}_0$ denote the constant path.

Next we'll define the topology on $\mathscr{X}$:

Let $\{U_\alpha\}$ be a topology basis of X, consider the following sets:

$$U(U_\alpha, a) = \{\langle ac \rangle \mid c \in P(U_\alpha, a(1))\}.$$

Let the topology basis on $\mathscr{X}$ be the above sets. We claim $p : \mathscr{X} \to X$ is indeed a covering. □

Example 4.4.15

A counter example of above theorem when X is not locally semi-simply connected: Hawaiian earrings (a family of tangent circles with radius $\to 0$).

§4.5 Covering spaces and group actions

Now we can view all these things from group actions.

Let X be a topological space, G is a group acting on X. We say the action is **freely discontinuous** if for all $x \in X$, there's a neighborhood U s.t. $gU \cap U \neq \emptyset$ only holds for $g = e$.

Proposition 4.5.1

Let $G \curvearrowright X$ be a freely discontinuous action, then the quotient map $X \to X/G$ by $x \mapsto Gx$ is a regular covering, and the group action is just deck transformations.

Example 4.5.2

The antipodal map in S^n generates a group $\{\pm 1\}$, and the action is freely discontinuous, so $S^n \to S^n/\{\pm 1\} = \mathbb{R}P^n$ is a covering.

Let $\alpha : (x, y) \mapsto (x, y + 1)$ and $\beta : (x, y) \mapsto (x + 1, -y)$ on $\mathbb{E}^2$ generates a group action $G \curvearrowright \mathbb{E}^2$. This is also freely discontinuous, and $\mathbb{E}^2/G$ is a Klein bottle.

Let X be a topological space, G is a group acting on X. We say the action is **properly discontinuous** if for all compact set $K \subset X$, $gK \cap K \neq \emptyset$ only holds for finitely many g.

Usually we suppose X is a locally compact Hausdorff space.

Example 4.5.3

Let $\mathbb{Z}$ acts on $\mathbb{C}$ by $\sigma : x + iy \mapsto \lambda x + i\lambda^{-1}y$. Then it's not properly discontinuous.

Proposition 4.5.4

Let G acts on X properly discontinuously. If X is locally compact Hausdorff, then so is X/G.

Proof. For $\overline{x} \neq \overline{y} \in X/G$, take compact neighborhoods $K(x), K(y) \subset X$, since $gK(x) \cap K(y) \neq \emptyset$ only holds for finitely many g, we can "shrink" $K(x)$ and $K(y)$ so that $gK(x) \cap K(y) = \emptyset$ for all $g \in G$. Thus X/G is Hausdorff.

Clearly X/G is locally compact. □

Proposition 4.5.5

Let X be a locally compact Hausdorff space, G act on X. The action is properly discontinuous + free $\iff$ it's freely discontinuous.

Proof. Trivial. □

Corollary 4.5.6

Let X be a locally compact Hausdorff space, and G acts on X properly discontinuously. If G has no torsion, then the action is freely discontinuous.

Proof. If the action is not free, there exists $g \neq \mathrm{id}$, $gx = x$. Thus $\{x\} \cap \{gx\} \neq \emptyset$ holds for any g^n.

Since g is not a torsion, this contradicts with proper discontinuity. □

Example 4.5.7

Consider the action of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ on the space $UHP = \{z \in \mathbb{C} \mid Im(z) > 0\}$ by fractional linear transformation. (hyberbolic transformation)

This action is properly discontinuous, let $\Gamma(2) = \{A \in \Gamma \mid A \equiv I(\mathrm{mod} 2)\}$, which has no torsion, thus it's freely discontinuous. Note that $[\Gamma : \Gamma(2)] = 6$, and $UHP \to UHP/\Gamma(2)$ is a covering map.

At last, we'll combined what we've learned and prove a well-known theorem:

Theorem 4.5.8

Simply connected surfaces with complete metric and constant curvature -1 are globally isometrically isomorphic to $\mathbb{H}^2$.

Remark 4.5.9 — Here the curvature is the Gauss curvature. The proof is similar for $\mathbb{S}^2 (k = 1)$ and $\mathbb{E}^2 (k = 0)$.

Sketch of the proof. The surface can be viewed as a manifold, whose charts are assigned the first fundamental form. The proof can be spilt to 2 parts, one for local properties and one for global properties.

If we have the local result, i.e. each point has an open neighborhood homeomorphic to an open disk in $\mathbb{H}^2$, we'll prove the theorem:

• There exists a unique well-defined locally isometric extension $f : M \to \mathbb{H}^2$. (Here we need M simply connected)

Since f is locally isometric, f is a covering map. But $\pi_1(M) = \{1\}$, by the uniqueness of universal covering, there exists an isomorphism of coverings σ s.t. $f = \mathrm{id} \circ \sigma = \sigma$. Thus f is a homeomorphism.

• Locally, we'll take a geodesic parallel parameter, i.e. the y-axis and x-curves are geodesic lines. We have $I = \mathrm{d}x^2 + G(x, y)\,\mathrm{d}y^2$, where $G(0, y) = 1$, $G_x(0, y) = 0$. □