Geometry II

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Here we'll prove part of this theorem (since the other part needs further knowledge).

> **Remark 0.0.1 —** X has a triangulation means that X is homeomorphic to finitely many n-simplex fused together at the boundary linearly, and the *link* of each vertical is a triangulation of S^{n-1}.

Proof. Observe that given a triangulation, we can get a polygon fusing presentation of the surface by adding the triangles one by one, fusing only one edge each time.

If we write down the edges of this polygon at a certain order, using letters to indicate differet edges and bars for direction, we can get something like $ab\bar{a}\bar{b}$ for a torus.

TODO: pictures!

In fact, nT^2 can be presented as $[a_1, b_1][a_2, b_2] \dots [a_n, b_n]$, where $[a, b] = ab\bar{a}\bar{b}$. Likely, mP^2 is $c_1^2 c_2^2 \dots c_m^2$ since P^2 is c^2. So our goal is to say that any given "edge words" can be reformed to one of the above standard forms.

Note that $(A) : Wa\bar{a} = W$, and $(B) : aUV\bar{a}U'V' = bVU\bar{b}V'U'$. The second operation is cut the polygon in the middle to get b, and fuse two parts together to eliminate a. There's also a reversed version: $aUVaU'V' = bV'VbUU'$. Also note that the word is cyclic, so $(C) : UV = VU$.

TODO: pictures!

This is kind of like Olympiad combinatorics problem. So we need techniques like:

- A "complexity" to measure how close we are to destination:

  vertical numbers(verticals fused together are regarded as one) and egde pair numbers

- Some labels to control different branches:

  whether it has edges with the same direction

- Some efficient "combo moves"

Observe that

- (A) will reduce vetical and edge pair by 1,
- (B) won't effect edge pairs, but may change vertical numbers,
- (C) won't change anything.

In fact we can reduce the vertical number to 1, i.e. all the verticals are fused to one point in the surface. If we have at least 2 verticals, say P and Q, and PQ is an edge. There must be another edge connecting P, Q, If those two P are different in the polygon, we can use (B) to eliminate one P vertical (by adding edge pair of QQ), and use (A) to eliminate they're the same.

TODO: pictures!!!

Repeating above process we can make the vertical number become 1.

If we have $aUbV\overline{a}U'\overline{b}V'$, we can use (B) twice to reform it to $cd\overline{c}\overline{d}W$.

TODO: pictures!!!

So we can achive nT^2 from a word with no same-direction-pairs. Techniquely we still need to prove that we can always find $a \dots b \dots \overline{a} \dots \overline{b}$ in original word, but this can be proved easily otherwise we can perform (A) to reduce edges.

Now for mP^2 :

After some fancy operations we're done. □

Remark 0.0.2 — On the existence of triangulation

§0.1 Homotopy

Definition 0.1.1 (Homotopy)**.** Given two continuous maps $f, g : X \to Y$, if there exists a continuous map

$$H : X \times [0, 1] \to Y$$

such that $f = H_0, g = H_1$, where $H_t = H\big|_{X \times \{t\}}$, then we say f and g are **homotopic**, denoted by $f \simeq g$, and the map H is a **homotopy**.

Definition 0.1.2 (Relative homotopy)**.** Let $A \subset X$, $f, g : X \to Y$, and $f\big|_A = g\big|_A$. We say f and g are homotopic relative to A ($f \simeq g \ rel \ A$), if H satisfies $H_t\big|_A = f\big|_A$.

More often we'll talk about homotopy between paths, here by path we mean a map $\gamma : [0, 1] \to X$. We say two paths are homotopic if they are homotopic relative to the endpoints($i.e.\{0, 1\}$)

Proposition 0.1.3

The homotopic relation is an equivalence relation.

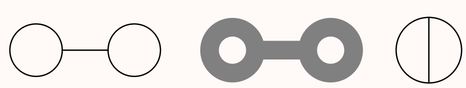
Besides studying the homotopy of maps, we can also consider the homotopy between spaces:

Definition 0.1.4. We say two topological spaces X, Y are **homotopy equivalent** or have the same **homotopy type**, if there exists $f : X \to Y$, $g : Y \to X$, such that

$$f \circ g = \mathrm{id}_Y, \quad g \circ f = \mathrm{id}_X .$$

Example 0.1.5

The following spaces are homotopy equivalent:



Definition 0.1.6 (Fundamental groups)**.** Let $\Omega(X, x_0)$ denote all the loops starting at x_0, i.e. $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

Define the **fundamental group** of X to be:

$$\pi_1(X, x_0) = \Omega(X, x_0)/\simeq,$$

where $\simeq$ is the homotopy relative to x_0.

We define the group operation to be the *concatenation* of paths, denoted by $(a, b) \mapsto ab$, where

$$ab(t) = \begin{cases} a(2t), & t \in [0, \frac{1}{2}]; \\ b(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proposition 0.1.7

The concatenation descends to a well-defined group operation:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0).$$

Proof. Just some trivial checking. Note that the inverse of a is just $\overline{a}(t) := a(1-t)$. $\square$

Proposition 0.1.8

An homeomorphism $f : (X, x_0) \to (Y, y_0)$ will induce a group homomorphism $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Note that X may be disconnected, so the fundamental group is dependent of the base point x_0. If $\gamma = \langle c \rangle$ is a homotopy class of paths from x_0 to x_1, then γ induces a group homomorphism:

$$\gamma_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1) : \langle a \rangle \mapsto \langle \overline{c}ac \rangle .$$

It's easy to see $\gamma_\sharp$ is an isomorphism.

Hence $\pi_1(X, x_0)$ only depends on the path connected components of x_0. Thus if X is path connected, and X, Y are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, or sometimes we can leave the base point out, just write $\pi_1(X) \cong \pi_1(Y)$.

Remark 0.1.9 — If $x_0 = x_1$, then $\gamma \mapsto \gamma_\#$ gives a homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_1(X, x_0))$.

Example 0.1.10

If $X \simeq \{pt\}$, then $\pi_1(X) \cong \{1\}$. In this case, X is called a **contractible space**. Note that the inverse is not true, e.g. $X = S^n$ for $n \geq 2$. Path connected spaces with trivial fundamental group are called **simple connected** spaces.

Some classic contractible space includes convex sets in $\mathbb{R}^n$, trees in graph theory and cones $CX = X \times [0, 1]/X \times \{1\}$.

Some more complex contractible examples including "a house with two rooms", the equitorial inclusion $S^\infty = \bigcup_{n=0}^{\infty} S^n$ with limit topology, i.e. the largest topology s.t. $S^n \to S^\infty$ continuous.

There are several concepts:

- Retraction: $f : X \to A$, $A \subset X$, $f\big|_A = \mathrm{id}_A$.
- Deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$, where $i : A \to X$ is the inclusion.
- Strong deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$ *rel* A.

The set A is called (strong) deformation kernel of f.

Example 0.1.11 (Differences between deformation and strong deformation)
Let X be the follwing space:

$$([0,1] \times \{0\}) \cup ([0,1]_{\mathbb{Q}} \times [0,1])$$

We know $X \simeq \{pt\}$, but $\{q\} \times [0,1]$ is deformation kernel but not strong deformation kernel.

§0.2 Fundamental groups

After introducing the fundamental groups, a natural question arises: how to compute the fundamental group of a given space? We first state the main result of this section:

Theorem 0.2.1 (Van Kampen)
Let $X = U' \cup U''$ be a topology space such that U', U'' are open and $W = U' \cap U''$ path connected, then for $x_0 \in W$, we have

$$\pi_1(X, x_0) \cong \pi_1(U', x_0) * \pi_1(U'', x_0)/N,$$

where N is the smallest normal subgroup generated by

$$i'_\sharp(\delta) i''_\sharp(\delta^{-1}) : \delta \in \pi_1(W, x_0),$$

$$\begin{tikzcd}[row sep=small] & U' \arrow[dr, "j'"] & \\ W \arrow[ur, "i'"] \arrow[dr, "i''"'] & & X \\ & U'' \arrow[ur, "j''"'] & \end{tikzcd}$$

and $*$ means free product.

Note that this theorem is useless when both U', U'' have trivial fundamental groups. Thus we need to find a space with nontrivial fundamental group first. One of the simplest example is S^1 :

$$\pi_1(S^n, x_0) \cong \begin{cases} \{1\}, & n \geq 2, \\ \mathbb{Z}, & n = 1. \end{cases}$$

Let $X \vee Y := X \sqcup Y/(x_0 = y_0)$, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. Thus $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{k}) = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_k$, the free group of rank k.

Example 0.2.2

Since nT^2 is formed by $2n$ loops(borders of the polynomial representation) fused with a disk. Note that $W = U' \cap U'' \cong S^1$, so

$$\pi_1(nT^2) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle .$$

Similarly,

$$\pi_1(mP^2) = \left\langle c_1, \ldots, c_m \mid c_1^2 \cdots c_m^2 = 1 \right\rangle .$$

Example 0.2.3

For product spaces, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

In fact the isomorphism can be written down with i_x, i_y, p_x, p_y, i.e. $p_{x\sharp} \times p_{y\sharp}$ and $(i_{x\sharp}, i_{y\sharp})$.

Theorem 0.2.4

$\pi_1(S^1) \cong \mathbb{Z}$, where the generating element is id.

Proof. Consider the map $p : \mathbb{R} \to S^1$, with $t \mapsto e^{2\pi i t}$.

Given any path $\gamma : [0, 1] \to S^1$, we can find a unique path $\tilde{\gamma} : [0, 1] \to \mathbb{R}$, s.t. $\tilde{\gamma}(0) \in \mathbb{Z}$ is any given base point. We denote this map by Φ, $\gamma \mapsto \tilde{\gamma}(1)$, where we require $\tilde{\gamma}(0) = 0$.

We can prove that $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$, and Φ only depends on the homotopy class of γ, so Φ induces a homomorphism of $\pi_1(S^1) \to \mathbb{Z}$.

Remark 0.2.5 — Since every homotopy $[0, 1] \times [0, 1] \to S^1$ can be lifted uniquely, and the endpoints of each path form a path in $\mathbb{R}$, but it's always contained in $\mathbb{Z}$, hence it must be constant.

Note that

- Φ is surjective since $s \mapsto e^{2\pi i m s}$ is mapped to m under Φ;
- Φ is injective since $\ker \Phi = \{1\}$: if $\tilde{\gamma}(1) = 0$, then $\tilde{\gamma} \simeq const$, so $\gamma = p \circ \tilde{\gamma} \simeq const$.

So Φ is an isomorphism, $\pi_1(S^1) \cong \mathbb{Z}$. □

Next we'll prove Van Kampen theorem (0.2.1). In fact we only need to prove that:

Claim 0.2.6. The map

$$j'_\sharp * j''_\sharp : \pi_1(U', x_0) * \pi_1(U'', x_0) \to \pi_1(X, x_0)$$

is a surjective homomorphism, and its kernel is the normal closure generated by $i'_\sharp(\delta) i''_\sharp(\bar{\delta})$.

CLearly it's a group homomorphism.

For any $\gamma \in \pi_1(X, x_0)$, it can be decompose to $a_1 b_1 a_2 \cdots a_k b_k$, where $a_i \subset U', b_i \subset U''$, let the partition points be $p_1, \dots, p_k, q_1, \dots, q_k \in W$, and denote s_i, t_i the path from x_0 to p_i, q_i. So we have

$$\gamma = \underbrace{a_1 \bar{s}_1}_{\in \pi_1(U', x_0)} \cdot \underbrace{s_1 b_1 \bar{t}_1}_{\in \pi_1(U'', x_0)} \cdots$$

Thus $j'_\sharp * j''_\sharp$ is indeed surjective.

At last we'll study its kernel, let $\gamma \in \ker j'_\sharp * j''_\sharp$. Since $\gamma \simeq \{x_0\}$, say the homotopy is $H : [0,1] \times [0,1] \to U' \cup U''$.

We can partition $[0,1] \times [0,1]$ to many small cells such that each cell's image is completely contained in either U' or U''.