Geometry II

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Remark 0.1 — Here we give a proof of $F \in \mathrm{SO}(3)$:

Proof. Note that

$$\left(FF^T\right)' = F'F^T + F(F')^T = F\left(H + H^T\right)F^T = 0.$$

thus $FF^T = I$ as it holds at $s_0 \implies F \in \mathrm{O}(3)$.
   Beisdes, it's easy to see that $\det(F)$ doesn't change sign, so $F \in \mathrm{SO}(3)$. $\square$

   In the words of tangent spaces or Lie groups, we can say that $T_I \mathrm{SO}(3) = \{X \mid X + X^T = 0\}$, and $T_F \mathrm{SO}(3) = \{FX \mid X + X^T = 0\}$.

Remark 0.2 — The above ODE cannot be solved explicitly, so here we introduce a method called "successive approximation". (For more details, see my notes of Analysis I)
   Let $F_0(s) = F_0$, $F_1(s) = F_0 + \int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t$, and define

$$F_{j+1}(s) = F_0 + \int_{s_0}^{s} F_j(t)H(t)\,\mathrm{d}t.$$

We can compute

$$|F_1(s) - F_0(s)| = \left|\int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t\right| \leq \int_{s_0}^{s} |F_0(t)H(t)|\,\mathrm{d}t \leq M(s - s_0) \cdot |F_0|.$$

$$|F_{j+1}(s) - F_j(s)| = \left|\int_{s_0}^{s} (F_j(t) - F_{j-1}(t))H(t)\,\mathrm{d}t\right| \leq M^{j+1}\frac{(s - s_0)^{j+1}}{(j+1)!}|F_0|.$$

Therefore F_j must uniformly converge to some function F on some small interval $[s_0 - \delta, s_0 + \delta]$.
   With some effort we can check F is differentiable and satisfies the ODE. Furthermore, F can extend to the entire interval J, and it's the *unique* solution.

§1 Theory of surfaces

§1.1 The first fundamental form

Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. We denote the point in $U \subset \mathbb{R}^2$ as $u = (s, t)$. Hence the partial derivative of ϕ gives:

$$\phi_s(u) = \frac{\partial \phi}{\partial s}(u), \quad \phi_t(u) = \frac{\partial \phi}{\partial t}(u).$$

Define the first fundamental quantities

$$E(u) = \phi_s(u) \cdot \phi_s(u), \quad F(u) = \phi_s(u) \cdot \phi_t(u), \quad G(u) = \phi_t(u) \cdot \phi_t(u).$$

The first fundamental form of the surface ϕ is the real bilinear form $T_u \phi(U) \times T_u \phi(U) \to \mathbb{R}$:

$$g(u) = E(u) \, \mathrm{d}s^2 + 2F(u) \, \mathrm{d}s \, \mathrm{d}t + G(u) \, \mathrm{d}t^2.$$

It can also be written as $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

> **Remark 1.1** (On partial detivatives) — When using the notation $\frac{\partial}{\partial x}$, we must declare the parameters we're using, e.g. (x, y), For example when $x' = x, y' = x + y$, the meaning of $\frac{\partial}{\partial x}$ is different from $\frac{\partial}{\partial x}$.

We say $\phi(s, t_0)$ is an s-curve and $\phi(s_0, t)$ is a t-curve. If s-curve and t-curve are orthogonal at every point $u \in U$, i.e. $F = 0$, or the matrix is diagonal, we say ϕ is an **orthogonal parametrization**, and s, t are **orthogonal parameters**.

Moreover, if $E = G, F = 0$ for all $u \in U$, (the matrix is a scalar) then we call ϕ an **isothermal parametrization**, and s, t are **isothermal parameters**. (Sometimes also called **comformal parameters**)

> **Example 1.2**
>
> The longtitude and latitude on a sphere are orthogonal parameters, but not isothermal parameters; While the stereographical projection is an isothermal parametrization of the sphere.

> **Remark 1.3** — The word "isothermal" is connected to thermology in a rather complicated way. The word "conformal" provides a more intuitive comprehension.

> **Remark 1.4** — A fun fact: Isothermal parameters always exist locally on regular parametrized surfaces. This only holds for 2-dimensional manifold.

§1.2 Linear algebra review

Let V be a vector space over $\mathbb{F}$.

Symmetrical bilinear form vs. quadratic form

A symmetrical bilinear form is a linear map $B : V \times V \to \mathbb{F}$ with $B(v, w) = B(w, v)$. A quadratic form is a map $Q : V \to \mathbb{F}$ with $Q(v) = B(v, v)$ for some symmetrical bilinear form B.

By taking a basis of V, we can use the matrix to express them:

$$B(v, w) = vAw^T, \quad Q(v) = vAv^T.$$

where A is a symmetrical matrix. When we change the basis, the matrix A differs by a congruent transformation.

We could also write $B \in V^* \otimes V^*$ for a bilinear form B. All the symmetrical bilinear forms constitute a subspace of $V^* \otimes V^*$ of dimension $\frac{n(n+1)}{2}$. This is denoted by $\mathrm{Sym}^2(V)$.

> **Remark 1.5 —** The subspace of anti-symmetric matrices is denoted by $\mathrm{Alt}^2(V)$, and $\mathrm{Alt}^2(V) \oplus \mathrm{Sym}^2(V) = V^* \otimes V^*$.

§1.3 Tangent spaces

A surface $\phi : U \to \mathbb{E}^3$ has a tangent space at every point $\phi(u)$, which is just the space (in this case, a plane) spanned by $\phi_s(u)$ and $\phi_t(u)$. We can prove that this tangent space is independent to the parameters. Furthermore, we can equip it with the inner product in $\mathbb{E}^3$, the matrix of this product is exactly $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

> **Remark 1.6 —** In modern differential manifold theory, there's an intrinsic definition of tangent spaces, but this definition is too abstract.

Here we present one of these intrinsic definitions.

Definition 1.7 (Tangent vectors)**.** Define an equivalence relation on smooth curves in $\phi(U)$:

Let $\gamma(r) = \phi(s(r), t(r))$ be a smooth curve $(-\epsilon, \epsilon) \to \phi(U)$. Two curves γ_1, γ_2 are equivalent iff $s_1'(0) = s_2'(0)$ and $t_1'(0) = t_2'(0)$.

Each equivalence class is a "tangent vector" at point $\phi(s_0, t_0)$.

§1.4 The second fundamental form

Since the first fundamental form is not sufficient to describe all the properties of the surface (it can only describe the curves lying on it and the area), we thereby introduce the second fundamental form.

Definition 1.8 (The second fundamental form)**.** Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. The normal vector at the point $u = (s, t)$ is defined as

$$\vec{n} = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}.$$

Since the cross product cares about orientation, so the normal vector is only fixed under orientation-preserving reparametrization.

Now we expand ϕ to the second derivative:

$$\phi(s + \Delta s, t + \Delta t) = \phi(s, t) + (\phi_s, \phi_t) \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + \frac{1}{2}(\Delta s, \Delta t) \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + o(|\Delta s|^2 + |\Delta t|^2).$$

Hence we define

$$L = \phi_{ss} \cdot \vec{n}, \quad M = \phi_{st} \cdot \vec{n}, \quad N = \phi_{tt} \cdot \vec{n}.$$

The second fundamental form is defined as $h = L \, \mathrm{d}s^2 + M \, \mathrm{d}s \, \mathrm{d}t + N \, \mathrm{d}t^2$.

Intuitively, the second fundamental form describes how much the surface is "going out" of the tangent plane.

Since the first fundamental form g gives an inner product of the tangent space, so we can compute the "canonical form" of h with respect to g, this process will generate some geometric quantities.

Definition 1.9. Define the **average curvature** and **Gaussian curvature**:

$$H := \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}, \quad K := \frac{LN - M^2}{EG - F^2}.$$

These expressions look complicated and ugly, the reason is that we didn't choose the right parameters. Indeed, if at some point $u = (s, t)$ we have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Definition 1.10 (Principal curvatures)**.** The characteristic polynomial $\lambda^2 - 2H\lambda + K$ has two real roots, they are called the **principal curvatures** of ϕ. The **principal directions** are defined as the directions of eigenvectors of $h : T_uU \times T_uU \to \mathbb{R}$ WRT the inner product g.

Now we'll dig deeper into the geometric meaning of these formulas.

Proposition 1.11

H and K are geometric quantities.

Proof. For any reparametrization $s = s(\tilde{s}, \tilde{t}), t = t(\tilde{s}, \tilde{t})$, we have

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J\begin{pmatrix} E & F \\ F & G \end{pmatrix}J^{-1}.$$

Similarly we can verify that

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = J\begin{pmatrix} L & M \\ M & N \end{pmatrix}J^{-1}.$$

Since

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Thus H, K are fixed under reparametrization. $\qquad\square$

Example 1.12

Some examples about curvatures:

- A sphere with radius R:
$$H = \frac{1}{R}, \quad K = \frac{1}{R^2}.$$

  Note that
$$\int_{S^2} K \, \mathrm{d}Area = \frac{1}{R^2} 4\pi R^2 = 2\pi\chi(S^2).$$

  This is an example of Gauss-Bonnet formula which we'll cover later.