Geometry II

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Contents

Remark 0.1 — On how to construct regular manifolds:

Let $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be a smooth function, for a fixed $y \in \mathbb{R}^{n-m}$, if $\forall x \in f^{-1}(y)$, $Df(x)$ has rank $n-m$, then $M := f^{-1}(y)$ is an m-dimensional regular manifold.

In fact this is known as "Regular Value Theorem" in literarture, and y is called a regular value of f. This leads to a branch in mathematics, namely *differential topology*.

Remark 0.2 — On real/complex analysis: Holomorphic (which is the complex version of differentiable) is way stronger than smooth condition.

§1 Theory of space curve

In this section we mainly discuss the regular parametrized curves $\gamma : J \to \mathbb{E}^3$.

Our goal is to find some identities to describe the "shape" of the curves. Since the curve is 1-dimensional manifold in 3 dimensional space, somehow we should find 3 identities to describe it, including length and another two concerning how it "bends".

§1.1 Arc length

Definition 1.1 (Arc length)**.** Let $\gamma : J \to \mathbb{E}^3$ be a regular parametrized curve. In an interval $[a,b] \subset J$, we define its length to be

$$Length_\gamma([a,b]) := \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

where $\gamma'(t) \in V(\mathbb{E}^3) = \mathbb{R}^3$.

Proposition 1.2

Arc length is a **geometry quantity**, i.e. fixed under reparametrization.

Proof. For an arbitary regular reparametrization $t = t(\tilde{t})$, $\tilde{\gamma}(\tilde{t}) = \gamma(t)$, by Chain rule we get

$$\begin{aligned} Length_\gamma([a,b]) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))| \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}'(\tilde{t})|\,\mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{a},\tilde{b}]). \end{aligned}$$

However, here we used the fact that $\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}$ is positive, so when the reparametrization reverses the orientation, we need to take extra care of it.

$$\begin{aligned} Length_\gamma([a,b]) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))| \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{b}}^{\tilde{a}} |\tilde{\gamma}'(\tilde{t})|\,\mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{b},\tilde{a}]). \end{aligned}$$

□

The arc length induces a parametrization for regular curves, namely the **arc length parameter** $\gamma(s)$, with $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\| = 1$ everywhere.

§1.2 Curvature

Definition 1.3 (Curvature)**.** Let $\gamma(s)$ be a regular curve with arc length parameter, define its **curvature** to be

$$\mathrm{Curv}_\gamma(s) = \kappa(s) := \|\gamma''(s)\|.$$

Since it is deduced from arc length (which is a geometry quantity), it must be a geometry quantity as well.

Remark 1.4 — Sometimes $\gamma''(s)$ is called the curvature vector. It's parallel to the normal vector and can be interpreted as centripedal force.

Example 1.5

For a straight line, its curvature is always 0.

For a circle with radius R, $\gamma(s) = (R\cos(\frac{s}{R}), R\sin(\frac{s}{R}))$, so $\mathrm{Curv}_\gamma(s) = \frac{1}{R}$.

Proposition 1.6

When the parameter is a general parameter $\gamma(t)$, the curvature is equal to:

$$\mathrm{Curv}_\gamma(t) = \frac{\|\gamma''(t) \times \gamma'(t)\|}{\|\gamma'(t)\|^3}.$$

Example 1.7

Let $\Gamma : x^2 + k^2y^2 = 1$, calculate curvature of Γ at point (x, y).

Solution. First we take a parametrization for Γ: $(x, y) = (\cos t, \frac{1}{k} \sin t)$.

Then compute the derivatives:

$$(x', y') = (-\sin t, \frac{1}{k} \cos t) = (-ky, \frac{1}{k} x),$$

$$(x'', y'') = (-\cos t, -\frac{1}{k} \sin t) = (-x, -y).$$

$$\mathrm{Curv}_\Gamma = \frac{|ky^2 + \frac{1}{k}x^2|}{(k^2y^2 + \frac{1}{k^2}x^2)^{\frac{3}{2}}} = \frac{1}{k(\frac{1}{k^2}x^2 + k^2y^2)^{\frac{3}{2}}}.$$

When $(x, y) = (1, 0)$, Curv $= k^2$; when $(x, y) = (0, \frac{1}{k})$, Curv $= \frac{1}{k}$. $\square$

Remark 1.8 — Osculating circle: A circle tangent to the curve with the same curvature as the curve at the tangent point. Specifically, its radius is equal to $\frac{1}{\mathrm{Curv}}$.

This is useful in engineering to indicate the curvature of a curve.

§1.3 Torsion and Frenet frame

Definition 1.9 (Torsion)**.** Let $\gamma(s)$ be a curve with arc length parameter.

Let $\vec{t} := \gamma'(s), \vec{n} := \frac{\gamma''(s)}{\|\gamma''(s)\|}$ be the tangent vector and normal vector.

Let $\vec{b} = \vec{t} \times \vec{n}$ be the **binormal vector**. Define the **torsion** to be

$$\mathrm{Tors}_\gamma(s) = \tau(s) := -\vec{b}' \cdot \vec{n}.$$

In fact $\vec{b}'$ is parallel to $\vec{n}$:

$$\vec{b}' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}' = \vec{t} \times \vec{n}' \perp \vec{t},$$

and $\|\vec{b}\| = 1$, so $\vec{b} \perp \vec{b}'$, so $\vec{b}' \parallel \vec{n}$.

The torsion's geometric meaning is less intuitive than the previous ones. It describes how much the curve is moving "out" the plane it currently lies in.

Proposition 1.10

Torsion can be represented in general parameter:

$$\mathrm{Tors}_\gamma(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

Remark 1.11 — The torsion can be negative (while curvature is always non-negative), and it is only defined at the points where the curvature is nonzero.

Note that the vectors $\vec{t}, \vec{n}, \vec{b}$ form a right-handed orthonormal basis in $\mathbb{R}^3$, and it's called the curve $\gamma(s)$'s **Frenet frame**.

The plane containing $\vec{n}$ and $\vec{b}$ is called **normal plane**, the plane containing $\vec{t}$ and $\vec{n}$ is called **osculating plane**, and the last plane which contains $\vec{t}, \vec{b}$ is called **rectifying plane**.

The Frenet frame is not a fixed frame, it's moving with the point along the curve. So we can compute its derivative (with respect to s, the arc length parameter):

$$(\vec{t}', \vec{n}', \vec{b}') = (\vec{t}, \vec{n}, \vec{b}) \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Example 1.12

When γ lies on the surface of a sphere, assmue $\kappa > 0$ on $\gamma\big|_J$, then

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = c^2.$$

where c is the radius of the sphere.

Proof. Let $\vec{u} = \gamma(s) - p$, then $\vec{u} \cdot \vec{u} = c^2$. To get a relation of κ and τ, we only need to represent $\vec{u}$ in terms of $\vec{t}, \vec{n}$ and $\vec{b}$.

Taking derivative WRT s:

$$0 = 2\vec{u}' \cdot \vec{u} = 2\vec{t} \cdot \vec{u}.$$

and then by taking the second and third derivative,

$$0 = \vec{t}' \cdot \vec{u} + \vec{t}^2 = \kappa\vec{n} \cdot \vec{u} + 1.$$

We get $\vec{u} \cdot \vec{n} = -\frac{1}{\kappa}$.

$$(\kappa\vec{n})' = \kappa'\vec{n} + \kappa(-\kappa\vec{t} + \tau\vec{b}),$$

so the third derivative should be

$$0 = \kappa\vec{n} \cdot \vec{t} + \kappa'\vec{n} \cdot \vec{u} + \kappa(-\kappa\vec{t} + \tau\vec{b}) \cdot \vec{u} = -\frac{\kappa'}{\kappa} + \kappa\tau\vec{u} \cdot \vec{b},$$

hence $\vec{u} \cdot \vec{b} = \frac{1}{\tau}(\frac{1}{\kappa})'$.

At last we just plugged everything into $\vec{u}^2 = c^2$ to conclude. $\square$

Note: the inverse statement does not hold, e.g. helix (which has constant curvature).

This example shows that Frenet frame is a powerful tool for handling the local properties of a curve. In fact, we could totally "determine" a curve near a point given the curvature and torsion.

Example 1.13

We can expand the curve $(\gamma(s), \vec{t}, \vec{u}, \vec{b})$ around $s = 0$:

$$\begin{cases} x(s) = x(0) + s - \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ y(s) = y(0) + \dfrac{\kappa(0)}{2}s^2 + \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ z(s) = z(0) + \dfrac{\kappa'(0)\tau(0)}{6}s^3 + o(s^3) \end{cases}$$

Remark 1.14 — By Frenet's formula, we can expand it to higher degrees, but the expansion need not converge to the original curve (similar reason as Taylor's formula). Also we can expand the curve with any parameter instead of arclength.

§1.4 Fundamental theorem of curve theory

Theorem 1.15 (Fundamental theorem of curve theory)

Let $\kappa, \tau : J \to \mathbb{R}$ be smooth functions, $\kappa(s) > 0$ on J. There exists a curve with arc length parameter $\gamma : J \to \mathbb{E}^3$, such that $\mathrm{Curv}_\gamma = \kappa$, $\mathrm{Tors}_\gamma = \tau$ holds on J.

Moreover, if $\tilde{\gamma}$ also satisfies above conditions, then exists $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ perserving orientation and distance s.t. $\tilde{\gamma} = \sigma \circ \gamma$.

Claim 1.16. Let $H = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

The ODE about $F : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$:

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}s}(s) = F(s)H(s) \\ F(s_0) = F_0 \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \end{cases}$$

always has unique solution. Moreover if $F(s_0) \in \mathrm{SO}(3)$, then $F(s) \in \mathrm{SO}(3)$ always holds.

Proof of the theorem. Since this claim requires some knowledge in ODE, which is beyond the scope of this course, we'll directly use it without proving.

WLOG $0 \in J$, let $s_0 = 0$ and $F(0) = I_3$.

Let $\mathcal{F} = (\vec{t}, \vec{n}, \vec{b}) := (\vec{e}_1, \vec{e}_2, \vec{e}_3)F(s)$ be a frame of $\mathbb{R}^3$.

Now we construct γ to be

$$\gamma(s_1) := \int_0^{s_1} \vec{t}\,\mathrm{d}s.$$

It's sufficient to prove that $\mathrm{Curv}_\gamma = \kappa$ and $\mathrm{Tors}_\gamma = \tau$.

Since $\mathcal{F}(0) = (e_1, e_2, e_3)$ is orthonormal frame, $\mathcal{F}(s)$ is orthonormal for all s.

Thus $|\vec{t}| = 1$, s is the arc length parameter. Some computation yields $\mathcal{F}$ is Frenet frame of γ. Compare its Frenet matrix to H, we get the desired result.

On the other hand, if $\tilde{\gamma}$ is as stated, take its Frenet frame $\tilde{\mathcal{F}}(s)$.

Let σ be the map which maps $\mathcal{F}(0)$ to $\tilde{\mathcal{F}}(0)$, $\gamma(0)$ to $\tilde{\gamma}(0)$. Then the Frenet frame of $\sigma \circ \gamma$ and $\tilde{\gamma}$ are the solution of the same ODE $\implies \sigma \circ \gamma = \tilde{\gamma}$ for all $s \in J$. $\square$

Remark 1.17 — Here we give a proof of $F \in \mathrm{SO}(3)$:

Proof. Note that

$$\left(FF^T\right)' = F'F^T + F(F')^T = F\left(H + H^T\right)F^T = 0.$$

thus $FF^T = I$ as it holds at $s_0 \implies F \in \mathrm{O}(3)$.

Beisdes, it's easy to see that $\det(F)$ doesn't change sign, so $F \in \mathrm{SO}(3)$. $\square$

In the words of tangent spaces or Lie groups, we can say that $T_I\mathrm{SO}(3) = \{X \mid X + X^T = 0\}$, and $T_F\mathrm{SO}(3) = \{FX \mid X + X^T = 0\}$.

Remark 1.18 — The above ODE cannot be solved explicitly, so here we introduce a method called "successive approximation". (For more details, see my notes of Analysis I)

Let $F_0(s) = F_0$, $F_1(s) = F_0 + \int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t$, and define

$$F_{j+1}(s) = F_0 + \int_{s_0}^{s} F_j(t)H(t)\,\mathrm{d}t.$$

We can compute

$$|F_1(s) - F_0(s)| = \left|\int_{s_0}^{s} F_0(t)H(t)\,\mathrm{d}t\right| \le \int_{s_0}^{s} |F_0(t)H(t)|\,\mathrm{d}t \le M(s - s_0)\cdot|F_0|.$$

$$|F_{j+1}(s) - F_j(s)| = \left|\int_{s_0}^{s} (F_j(t) - F_{j-1}(t))H(t)\,\mathrm{d}t\right| \le M^{j+1}\frac{(s - s_0)^{j+1}}{(j+1)!}|F_0|.$$

Therefore F_j must uniformly converge to some function F on some small interval $[s_0 - \delta, s_0 + \delta]$.

With some effort we can check F is differentiable and satisfies the ODE. Furthermore, F can extend to the entire interval J, and it's the *unique* solution.