Mathematical Analysis II

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Theorem 0.1 (Fubini's Theorem)
Let $f(x,y) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, and f is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

1. $f(x,y)$ as a function of y is integrable on $\mathbb{R}^{d_2}$ for $x \in \mathbb{R}^{d_1} \backslash Z$ with $m(Z) = 0$.
2. Let $g(x) = \int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y$, for $x \in \mathbb{R}^{d_1} \backslash Z$, where Z is a null set. We have g is integrable on $\mathbb{R}^{d_1}$.
3. $$\int_{\mathbb{R}^{d_1+d_2}} f(x,y) = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Proof. Let $\mathscr{F}$ be the space consisting of all the integrable functions that satisfy Fubini's theorem.

Lemma 0.2
$\mathscr{F}$ is a vector space. Furthermore, for non-negative monotone sequence $f_n \in \mathscr{F}$, if $\lim f_n$ is integrable, then $\lim f_n \in \mathscr{F}$ as well.

Proof of the lemma. First notice that $f \in \mathscr{F} \implies cf \in \mathscr{F}$.

If $f, g \in \mathscr{F}$, consider $f + g$:

By our conditions, there exists $X_f, X_g \subset \mathbb{R}^{d_1}$, s.t. $f(x,y)$ integrable on $\mathbb{R}^{d_2}$, $\forall x \notin X_f$, and $g(x,y)$ integrable on $\mathbb{R}^{d_2}$, $\forall x \notin X_g$.

This implies $f(x,y) + g(x,y)$ integrable on $\mathbb{R}^{d_2}$ for $x \notin X_f \cup X_g$, which proves (1).

$$\int_{\mathbb{R}^{d_2}} f(x,y) + g(x,y)\,\mathrm{d}y = \int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y + \int_{\mathbb{R}^{d_2}} g(x,y)\,\mathrm{d}y.$$

So the LHS is integrable on $\mathbb{R}^{d_1}$ (this is (2)), taking the integral we get

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x + \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} g(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y) + g(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Therefore $\mathscr{F}$ is a vector space.

For a monotone non-negative sequence f_n, $\exists X_n \subset \mathbb{R}^{d_1}$ s.t. f_n is integrable with respect to y for $x \notin X_n$.

Similarly, when $x \notin \bigcup_{n=1}^{\infty} X_n$, as a function of y, by Beppo-Levi (or Dominated convergence),

$$\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y = \lim_{n\to\infty} \int_{\mathbb{R}^{d_2}} f_n(x,y)\,\mathrm{d}y.$$

This equation holds when $\int f(x,y)\,\mathrm{d}y$ is finite, so we need to prove it is finite almost everywhere.
For $x \notin \bigcup X_n$, we have:

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f_n(x,y)\,\mathrm{d}y \right) \mathrm{d}x \to \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f_n(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1+d_2}} f_n \to \int_{\mathbb{R}^{d_1+d_2}} f$$

Compare these relations we deduce

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1+d_2}} f < +\infty.$$

so $\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y$ is finite almost everywhere. This gives (1), and (2), (3) follows immediatedly. $\square$

Back to the proof of the original theorem, we want to prove $\mathscr{F} = \mathcal{L}^1$.
We prove the indicator function of following sets are in $\mathscr{F}$:

- Cuboids;
- Finite open sets;
- G_δ sets;
- Null sets;
- General measurable sets.

Let I be a cuboid, $I = I_x \times I_y$, so $\chi_I = \chi_{I_x}\chi_{I_y}$.

$$\int \chi_I = |I| = |I_x||I_y| = \int \chi_{I_x}|I_y|\,\mathrm{d}x = \int\int (\chi_{I_x}\chi_{I_y}\,\mathrm{d}y)\,\mathrm{d}x.$$

Let O be a finite open set, $O = \bigcup_{n=1}^{\infty} I_n$, where I_n are pairwise disjoint cuboids.

$$\chi_O = \lim_{n\to\infty} \chi_{\bigcup_{k=1}^{n} I_k} \in \mathscr{F},$$

as it's an inceasing sequence.
For $G_\delta = \bigcap_{n=1}^{\infty} O_n$, $\chi_{O_n} \searrow \chi_{G_\delta}$. $\implies \chi_{G_\delta} \in \mathscr{F}$.
For null set E, if $\chi_E \in \mathscr{F}$, $\forall A \subset E$,

$$0 = \int \chi_E = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} \chi_E\,\mathrm{d}y \right) \mathrm{d}x.$$

hence $\int_{\mathbb{R}^{d_2}} \chi_E\,\mathrm{d}y = 0$, for $x, a.e.$ $\implies \int_{\mathbb{R}^{d_2}} \chi_A\,\mathrm{d}y = 0$ for $x, a.e.$.
Taking the integral with respect to x, we have $\chi_A \in \mathscr{F}$.
Therefore if E is a null set, by taking its equi-measure hull we deduce $\chi_E \in \mathscr{F}$.
Finally, for a general measurable set E, let O be its equi-measure hull, and $E = O \backslash A$. since $\mathscr{F}$ is a vector space, $\chi_E \in \mathscr{F}$.
The rest is trival now: Because all the simple functions are in $\mathscr{F}$, and any measurable functions can be expressed as limits of increasing simple functions, so $\mathscr{F} = \mathcal{L}^1(\mathbb{R}^{d_1+d_2})$. $\square$

Theorem 0.3 (Tonelli's theorem)

Let f be a non-negative measurable function on $\mathbb{R}^d$.

- $f(x, y)$ is measurable on $\mathbb{R}^{d_2}$ for x almost everywhere;
- $\int_{\mathbb{R}^{d_2}} f(x, y) \, \mathrm{d}y$ as a function of x is measurable;
- The integral satisfies:
$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y) \, \mathrm{d}y \right) \mathrm{d}x.$$

Proof. Consider the truncation function $f(x, y)\chi_{|x|+|y|<k}\chi_{f<k}$. □

Proposition 0.4

Let E be a measurable set on $\mathbb{R}^d$. For x almost everywhere, $E^x = \{y \mid (x, y) \in E\}$ is measurable on $\mathbb{R}^{d_2}$.

As a function of x, $m(E^x)$ satisfies
$$m(E) = \int_{\mathbb{R}^{d_1}} m(E^x).$$

Proof. Consider $f = \chi_E$ and use Tonelli's theorem. □

§0.1 Applications of Fubini's theorem

Definition 0.5 (Product measure)**.** Let $(X, \mathscr{F}, m)$ and $(Y, \mathscr{G}, m)$ be measure spaces, define a measure on $X \times Y$: The measure m induces an outer measure on $X \times Y$, and complete it to a normal measure by using Caratheodory conditions. This measure is called the **product measure** on $X \times Y$.

Theorem 0.6

Let $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, E_1, E_2 are subsets of $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$, respectively.

- If E_1, E_2 are measurable, then E is measurable as well, and $m(E) = m(E_1)m(E_2)$.
- If E is measurable, then E_1, E_2 are measurable, and $m(E) = m(E_1)m(E_2)$, unless one of E_1, E_2 is null set, which means E is null as well.

Corollary 0.7

Let $f(x)$ be a measurable function on $\mathbb{R}^{d_1}$, we have $g(x, y) = f(x)$ is measurable on $\mathbb{R}^{d_2}$.

Proposition 0.8
Let L be a linear map $\mathbb{R}^d \to \mathbb{R}^d$, $E \subset \mathbb{R}^d$ a measurable set, then $L(E)$ is measurable, and

$$m(L(E)) = |\det L| m(E).$$

From this transformation formula we deduce the integral version:
Let f be an integrable function on $\mathbb{R}^d$, then $f(L(x))$ is also integrable, and

$$\int f(L(x)) = \frac{1}{|\det L|} \int f(x).$$

Here we require $L \in \mathrm{GL}(n)$, since if $\det L = 0$, the function $f(L(x))$ need not be measurable.
At last we take a look at Fubini's theorem with the convolution product.
Recall the Gauss integral:

$$2 \int_0^\infty e^{-x^2} \,\mathrm{d}x = \sqrt{\pi}.$$

Here we give a different proof:

$$\begin{aligned} \int e^{-x^2} \,\mathrm{d}x \int e^{-y^2} \,\mathrm{d}y &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_0^{+\infty} e^{-r^2} \,\mathrm{d}r^2 \cdot \pi \\ &= \pi. \end{aligned}$$

§1 Lebesgue differentiation

The most important theorem in calculus is no doubt the Fundamental theorem of Calculus (which is also called Newton-Lebniz formula). Since we generalized the integrals, there must be a generalized version of this theorem:

Theorem 1.1 (Lebesgue differentiation theorem, part 1)
If f is integrable on $\mathbb{R}^d$, for any ball $B \subset \mathbb{R}^d$, we have

$$\lim_{x \in B, |B| \to 0} \frac{1}{m(B)} \int_B f(y) \,\mathrm{d}y = f(x), a.e.$$

This theorem clearly holds for continuous points of f.
Our basic idea is to take a continuous g, such that $\|g - f\|_{\mathcal{L}^1} < \varepsilon$.
and to prove

$$\left\{ x : \limsup_{x \in B, |B| \to 0} \frac{1}{m(B)} \int_B |f(y) - f(x)| \,\mathrm{d}y \geq \varepsilon_0 \right\}$$

is a null set.
Now we estimate

$$\begin{aligned} \frac{1}{m(B)} \int_B |f(y) - f(x)| \,\mathrm{d}y &\leq \frac{1}{m(B)} \int_B (|f(y) - g(y)| + |g(y) - g(x)| + |g(x) - f(x)|) \,\mathrm{d}y \\ &= |f(x) - g(x)| + \varepsilon + \frac{1}{m(B)} \int_B |f(y) - g(y)| \,\mathrm{d}y \end{aligned}$$

We find that the last term is pretty hard to deal with, so we'll introduce some new tools:

Definition 1.2 (Hardy-Littlewood maximal function)**.** Let f be an integrable function on $\mathbb{R}^d$. Define

$$Mf(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, \mathrm{d}y.$$

to be the **maximal function** of f.

Theorem 1.3 (Hardy-Littlewood)

The maximal function Mf satisfies:

- Mf is measurable;
- For x almost everywhere, $|f(x)| \leq Mf(x) < +\infty$.
- There exists a constant C s.t.

$$|\{x : Mf > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{\mathcal{L}^1}.$$

Proof. First we prove $\{Mf > \alpha\}$ is measurable. If $Mf(x_0) > \alpha$, then exists an open ball $B \ni x_0$,

$$\int_B |f(y)| \, \mathrm{d}y > \alpha m(B).$$

This implies that $B \subset \{Mf > \alpha\} \implies \{Mf > \alpha\}$ is an open set.

For the second part, we'll prove for $\forall \varepsilon_0 > 0, N > 0$,

$$m(\{x : Mf(x) + \varepsilon_0 < |f(x)| \leq N\}) = 0.$$

Otherwise denote the above set as E, for $\forall 0 < \lambda < 1$, $\exists B$ s.t. $|E \cap B| > \lambda |B|$.

Thus for $x \in E$,

$$\begin{aligned}
Mf(x) &\geq \frac{1}{m(B)} \int_B |f(y)| \, \mathrm{d}y \\
&\geq \frac{1}{m(B)} \int_{E \cap B} |f(y)| \, \mathrm{d}y \\
&\geq \frac{1}{m(B)} \int_{E \cap B} \varepsilon_0 + Mf(y) \, \mathrm{d}y \\
&= \frac{m(E \cap B)}{m(B)} \varepsilon_0 + \frac{1}{|B|} \int_{E \cap B} Mf(y) \, \mathrm{d}y.
\end{aligned}$$

Taking the integral with respect to x:

$$\left(1 - \frac{|E \cap B|}{|B|}\right) \int_{E \cap B} Mf \geq \frac{|E \cap B|^2}{|B|} \varepsilon_0.$$

This implies $(1 - \lambda)N \geq \lambda \varepsilon_0$, which is impossible as $\lambda \to 1$.

Now for the last part, since $\{Mf > \alpha\}$ is open, $\forall x \in \{Mf > \alpha\}$, $\exists B$ s.t.

$$\int_B |f(y)| \, \mathrm{d}y > \alpha m(B).$$

Hence for disjoint balls B_{i_k},

$$\|f\|_{\mathcal{L}^1} \geq \sum_{l=1}^{k} \int_{B_{i_l}} |f(y)| \, \mathrm{d}y > \alpha \sum_{l=1}^{k} |B_{i_l}|.$$

If we could select B_{i_l}'s such that their measure achieves say 1% of E, then we're done.

Lemma 1.4

Let $B_1, \dots, B_n$ be open balls in $\mathbb{R}^d$. There exists $i_1, \dots, i_k$ such that B_{i_j}'s are pairwise disjoint, and

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{j=1}^{k} 3B_{i_j}.$$

Here $3B$ means to multiply the radius of the ball by 3.

□