Measure Theory

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Hence $(L_p/\sim, \|\cdot\|_p)$ is a normed vector space.

When $p = \infty$, define

$$\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(|f| > a) = 0\}, \quad L_\infty := \{f : \|f\|_\infty < \infty\}.$$

We call the functions in L_∞ **essentially bounded**.

Let $\mu(X) < \infty$, then $f \in L_\infty \implies f \in L_p$, and $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$: For all $0 < a < \|f\|_\infty$,

$$a^p\mu(|f| > a) \le \int_X |f|^p \mathbf{I}_{|f|>a}\,\mathrm{d}\mu \le \int_X |f|^p\,\mathrm{d}\mu \le \|f\|_\infty^p \mu(X),$$

So taking the exponent $\frac{1}{p}$,

$$a \leftarrow a\mu(|f| > a)^{\frac{1}{p}} \le \|f\|_p \le \|f\|_\infty$$

But when $\mu(X) = \infty$, let $f \equiv 1$, then $f \in L_\infty$ but $f \notin L_p$.

Theorem 0.0.1

Let $f, g \in L_\infty$,

$$\|fg\| \le \|f\|\|g\|_\infty,$$

$$\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

Proof.

$$\int_X |fg|\,\mathrm{d}\mu \le \int_X |f|\|g\|_\infty\,\mathrm{d}\mu = \|f\|\|g\|_\infty.$$

Since $|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$, *a.e.*, we get the second inequality. □

Similarly we get $(L_\infty, \|\cdot\|_\infty)$ is a normed vector space.

The norm can deduce a *distance*:

$$\rho(f, g) := \|f - g\|.$$

Theorem 0.0.2 (L_p space is complete)
Let $1 \le p \le \infty$. If $\{f_n\} \subset L_p$ satisfying $\lim_{n,m\to\infty}\|f_n - f_m\|_p = 0$, then there exist $f \in L_p$ s.t. $\lim_{n\to\infty}\|f - f_n\|_p = 0$.

Proof. Take $n_1 < n_2 < \cdots$ such that

$$\|f_m - f_n\|_p \le \frac{1}{2^k}, \quad \forall n, m \ge n_k.$$

Let $g =\uparrow \lim_{k\to\infty} g_k$, where

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \in L_p, \quad g_k \ge 0.$$

Since

$$\|g_k\|_p \le \|f_{n_1}\|_p + \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \le \|f_{n_1}\|_p + 1.$$

$$\implies \|g\|_p =\uparrow \lim_{k\to\infty} \|g_k\|_p \le \|f_{n_1}\|_p + 1.$$

Here we use the monotone convergence theorem. We can check the above also holds for $p = \infty$.

Therefore $g \in L_p \implies g < \infty, a.e.$. We have

$$f := f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i}) = \lim_{k\to\infty} f_k, a.e.$$

the series is absolutely convergent, so f exists a.e. and $|f| \le g, a.e.$.

Lastly we can check: when $p = \infty$,

$$\|f_n - f\|_\infty \le \|f_n - f_{n_k}\|_\infty + \|f_{n_k} - f\|_\infty,$$

where the both term approach to 0 as $n \to \infty$.

When $p < \infty$, by Fatou's lemma,

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} |f_n - f_{n_k}|^p \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_X |f_n - f_{n_k}|^p \,\mathrm{d}\mu \le \varepsilon.$$

□

Remark 0.0.3 — Using the same technique we can prove that if f_n is Cauchy in measure, then f_n converge to some f in measure:

Let $A_i = \{|f_{n_{i+1}} - f_{n_i}| > 2^{-i}\}$ s.t. $\mu(A_i) < 2^{-i}$.

Define $f = f_{n_1} + \sum_{i\ge 1}(f_{n_{i+1}} - f_{n_i})$ on the set $\bigcup_{k\ge 1}\bigcap_{i\ge k} A_i^c$.

This theorem implies that $(L_p, \|\cdot\|_p)$ is a Banach space. So we can try to define an *inner product* on L_p space:

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

We can check $\langle\cdot,\cdot\rangle$ is bilinear only if $p = 2$, so L_2 is actually a Hilbert space.

When $0 < p < 1$, let

$$\|f\|_p := \int_X |f|^p \,\mathrm{d}\mu, \quad L_p = \{f : \|f\|_p < \infty\}.$$

Lemma 0.0.4
Let $0 < p < 1$, $C_p = 1$, then
$$|a+b|^p \le C_p(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}.$$

So L_p is a vector space.

Theorem 0.0.5 (Minkowski)
Let $0 < p < 1$ then
$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Remark 0.0.6 — When $\|f\|_p = (\int_X |f|^p \, \mathrm{d}\mu)^{\frac{1}{p}}$, $0 < p < 1$. then it won't satisfy Minkowski's inequality.

Thus L_p is only a metric space but not a normed vector space. Using the same method we can prove L_p is a complete metric space.

§0.1 Convergence in L_p space

Definition 0.1.1. Let $0 < p \le \infty$, $f, f_1, f_2, \cdots \in L_p$. When $\|f_n - f\|_p \to 0$, then we write $f_n \xrightarrow{L_p} f$, called **average converge of order p**.

Theorem 0.1.2
Let $0 < p < \infty$, $f, f_1, \cdots \in L_p$,

- If $f_n \xrightarrow{L_p} f$, then $f_n \xrightarrow{\mu} f$, and $\|f_n\|_p \to \|f\|_p$.
- If $f_n \to f, a.e.$ or in measure, then $\|f_n\|_p \to \|f\|_p \iff f_n \xrightarrow{L_p} f$.

Proof. When $f_n \xrightarrow{L_p} f$, let $A := \{|f_n - f| > \varepsilon\}$,
$$\mu(A) \le \frac{1}{\varepsilon^p} \int_X |f_n - f|^p \mathbf{I}_A \, \mathrm{d}\mu \le \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \to 0.$$
and obviously $\|f_n\|_p \to \|f\|_p$

On the other hand, when $f_n \to f, a.e.$ and $\|f_n\|_p \to \|f\|_p$, From $|a+b|^p \le C_p(|a|^p + |b|^p)$,
$$g_n := C_p(|f_n|^p + |f|^p) - |f_n - f|^p \ge 0.$$
$g_n \to 2C_p|f|^p, a.e.$, so
$$\int_X 2C_p |f|^p \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu = 2C_p \int_X |f|^p \, \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f|^p \, \mathrm{d}\mu.$$

When $f_n \to f$ in measure, for any subsequence there exist its subsequence $f_{n'} \to f, a.e.$, so $\|f_{n'} - f\|_p \to 0$, hence $\|f_n - f\|_p \to 0$. □

Remark 0.1.3 — This theorem implies for any L_p function f, we can take simple functions $f_1, f_2, \cdots \to f$ and $|f_n| \uparrow |f|$, so $f_n \xrightarrow{L_p} f$.

Definition 0.1.4 (Weak convergence)**.** Let $1 < p < \infty$, and $f_1, f_2 \cdots \in L_p$. If

$$\lim_{n\to\infty} \int_X f_n g \,\mathrm{d}\mu = \int_X fg \,\mathrm{d}\mu, \quad \forall g \in L_q.$$

Then we say f_n **weak convergent** to f, denoted by $f_n \xrightarrow{(w)L_p} f$.

When $p = 1$ and $(X, \mathscr{F}, \mu)$ is a σ-finite measure space, and the condition also holds, we say $\{f_n\}$ weak convergent to f in L_1.

Corollary 0.1.5

Let $1 \le p < \infty$, then

$$f_n \xrightarrow{L_p} f \implies f_n \xrightarrow{(w)L_p} f.$$

Proof. By Holder's inequality,

$$\left|\int_X (f_n - f)g \,\mathrm{d}\mu\right| \le \|f_n - f\|_p \|g\|_q \to 0.$$

□

If $\sup_{t\in T} \|f_t\|_p =: M < \infty$, then we say $\{f_t, t \in T\}$ is **bounded in L_p**.

Theorem 0.1.6

Let $1 < p < \infty$, $\{f_n\} \subset L_p$, there exists M s.t. $\|f_n\|_p \le M$, $\forall n$. If $f_n \to f$, *a.e.* or in measure, then $f \in L_p$ and $f_n \to f$ weakly.

Proof. First $\|f\|_p \le M$:

$$\int_X |f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \,\mathrm{d}\mu \le M^p.$$

Next we prove the weak convergence: For all $g \in L_q$, recall the bounded convergence theorem in probability, we can view M as a bound of f_n, and $\|g\|_q$ as P.

Let $B = \{|f_n - f| \le \hat{\varepsilon}\}$, consider

$$a := \int_B (f_n - f)g \,\mathrm{d}\mu, \quad b := \int_{B^c} (f_n - f)g \,\mathrm{d}\mu.$$

Note that

$$|a| \le \hat{\varepsilon} \int_X |g| \,\mathrm{d}\mu.$$

But $\int_X |g| \,\mathrm{d}\mu$ might be infinity, so let $A_k := \{\frac{1}{k} \le |g|^q \le k\}$, we have

$$\int_{A_k} |g| \,\mathrm{d}\mu \le k^{\frac{1}{q}} \mu(A_k) < \infty.$$

$(\frac{1}{k}\mu(A_k) < \int_{A_k} |g|^q \, \mathrm{d}\mu < \infty$ since $g \in L_q)$.

Now we can proceed:

$$a := \int_{A_k B} (f_n - f) g \, \mathrm{d}\mu, \quad b := \int A_k^c \cup B^c (f_n - f) g \, \mathrm{d}\mu.$$

Now $|a| \le \hat{\varepsilon} k^{\frac{1}{q}} \mu(A_k) < \varepsilon$.

$$\left| \int_X (f_n - f) g \mathbf{I}_{A_k^c \cup B^c} \, \mathrm{d}\mu \right| \le \|f_n - f\|_p \|g \mathbf{I}_{A_k^c \cup B^c}\|_q \le 2M \left(\int_{A_k^c} |g|^q \, \mathrm{d}\mu + \int_{A_k \setminus B} |g|^q \, \mathrm{d}\mu \right).$$

By LDC(Dominated convergence), $A_k^c \to \{g = 0, \infty\}$, so $\int_{A_k^c} |g|^q \, \mathrm{d}\mu < \varepsilon$.

Since $\mu(A_k) < \infty$, $f_n \to f, a.e. \implies f_n \xrightarrow{\mu} f$. By the continuity of integrals, $\mu(A_k \setminus B) \le \mu(B^c) < \delta \implies \int_{A_k \setminus B} |g|^q \, \mathrm{d}\mu < \varepsilon$.

Now we can conclude: $\forall \varepsilon > 0$, first choose k large, then $\hat{\varepsilon}$ small, we get

$$\int_X (f_n - f) g \, \mathrm{d}\mu \le \varepsilon + 4M\varepsilon \implies f_n \xrightarrow{(w)L_p} f.$$

□

Remark 0.1.7 — The proof is a little complicated, we divide the entire integral to three part, and estimate them respectively.

When $p = 1$, f_n bounded in L_p *cannot* imply weak convergence.

Example 0.1.8

Let $X = \mathbb{N}$, $\mu(\{k\}) = 1, \forall k$, clearly it's σ-finite.

Let $f_n(k) = \mathbf{I}_{k=n}$, then $\|f_n\| = \sum_k \mu(k)|f_n(k)| = 1$, and $f_n \to 0, a.e.$.

But let $g = 1 \in L_\infty$, $\int_X (f_n - f) g \, \mathrm{d}\mu = 1 \not\to 0$.

Proposition 0.1.9

Let $f_1, f_2, \cdots \in L_1$, then:

$$\|f_n\| \to \|f\| \& f_n \to f, a.e. \implies f_n \xrightarrow{L_1} f \implies f_n \xrightarrow{(w)L_1} f \implies \int_A f_n \, \mathrm{d}\mu \to \int_A f \, \mathrm{d}\mu, \forall A.$$

Proof. For the last part let $g = \mathbf{I}_A$, the rest is trivial. □

§0.2 Integrals in probability space

We can also consider L_p space in probability space $(\Omega, \mathscr{F}, P)$.

Theorem 0.2.1

Let $0 < s < t < \infty$. Then $L_t \subset L_s$. If $s \ge 1$, we have $\|f\|_s \le \|f\|_t$, with equality f constant.

Proof. When $f \in L_t$, let $p = \frac{t}{s}, q = \frac{t}{t-s}$.

$$\int_\Omega |f|^s \cdot 1 \,\mathrm{d}P \le \||f|^s\|_p \|1\|_q = (E|f|^{sp})^{\frac{1}{p}} = (E|f|^t)^{\frac{1}{p}}.$$

So $f \in L_s \implies L_t \subset L_s$. When $s \ge 1$,

$$\|f\|_s^s \le (\|f\|_t)^{\frac{t}{p}} = \|f\|_t^s \implies \|f\|_s \le \|f\|_t.$$

□

From this we know $L_\infty \subset L_p$, and $\|f\|_p \uparrow \|f\|_\infty$.

Remark 0.2.2 — This theorem does not hold for general space. Let $X = \mathbb{N}$, $\mu(\{n\}) = 1$, $f(n) = \frac{1}{n}$, then $f \in L_2 \backslash L_1$.

The expectation Ef^k is called ***k*-order moment** of random variable f.

Definition 0.2.3 (Uniformly integrable)**.** Let $\{f_t, t \in T\}$ be r.v.'s, if $\forall \varepsilon > 0, \exists \lambda > 0$, such that

$$E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} < \varepsilon, \quad \forall t \in T,$$

then we say $\{f_t, t \in T\}$ **uniformly integrable**.

If $\forall \varepsilon > 0, \exists \delta > 0$ s.t $\forall A \in \mathscr{F}$,

$$P(A) < \delta \implies E|f_t|\mathbf{I}_A < \varepsilon, \forall t \in T,$$

we say $\{f_t\}$ is uniformly absolutely continuous, which is abbreviated as absolutely continuous.

Theorem 0.2.4
Uniformly integrable $\iff$ absolute continuity and bounded in L_1.

Proof. Firstly when $\{f_t\}$ uniformly integrable, $\forall A \in \mathscr{F}, \lambda > 0$,

$$\begin{aligned} E|f_t|\mathbf{I}_A &= E|f_t|\mathbf{I}_{A\cap\{|f_t|\le\lambda\}} + E|f_t|\mathbf{I}_{A\cap\{|f_t|>\lambda\}} \\ &\le \lambda P(A) + E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \end{aligned}$$

Let $A = X$ we know $E|f_t| \le \lambda + \frac{\varepsilon}{2}, \forall t \in T$. Now let $\delta = \frac{\varepsilon}{2\lambda}$ we get AC property.

On the other hand,

$$\lambda P(|f_t| > \lambda) \le E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \le E|f_t| \le M, \forall t \in T.$$

So when $\lambda > \frac{M}{\delta}$, $P(|f_t| > \lambda) < \delta$, hence $E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \le \varepsilon$, $\forall t \in T$. □

Theorem 0.2.5
Let $0 < p < \infty$, and $f_n \to f$ in probability. TFAE:

(1) $\{|f_n|^p\}$ uniformly integrable;

(2) $f_n \xrightarrow{L_p} f$;

(3) $f \in L_p$ and $\|f_n\|_p \to \|f\|_p$.

Proof. (1) $\implies$ (2): Take subsequence $f_{n'} \to f, a.s.$,

$$E|f|^p \le \liminf_{n\to\infty} E|f_n|^p < \infty,$$

since $\{|f_n|^p\}$ is bounded in L_1. This means $f \in L_p$.

Let $A_n = \{|f_n - f| > \varepsilon\}$, now we compute

$$E|f_n - f|^p \le \varepsilon^p + E|f_n - f|^p \mathbf{I}_{A_n} \le \varepsilon^p + C_p E|f_n|^p \mathbf{I}_{A_n} + C_p E|f|^p \mathbf{I}_{A_n}$$

Since $P(A_n) \to 0$ and $\{|f_n|^p\}$ absolutely continuous (also note $E|f|^p\mathbf{I}_{A_n} \to 0$), RHS converges to 0. Therefore $f_n \xrightarrow{L_p} f$.

As for (3) $\implies$ (1), we'll prove a lemma:

Lemma 0.2.6
If $f_n \xrightarrow{P} f$, then $\forall 0 < p < \infty$,

$$|f_n|^p \mathbf{I}_{\{|f_n| \le \lambda\}} \xrightarrow{P} |f|^p \mathbf{I}_{\{|f| \le \lambda\}}, \quad \forall \lambda \in C(F_{|f|}).$$

By lemma and bounded convergence theorem, their expectation also converges. Note that $\|f_n\|_p \to \|f\|_p$, so

$$E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda\}} \to E|f|^p \mathbf{I}_{\{|f| > \lambda\}},$$

thus $\forall \varepsilon > 0, \exists \lambda_0 \in C(F_{|f|})$, s.t. $E|f|^p \mathbf{I}_{\{|f| > \lambda_0\}} < \frac{\varepsilon}{2}$, thus

$$\exists N, \quad E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda_0\}} < \varepsilon, \quad \forall n > N.$$

Now we can take $\lambda > \lambda_0$ such that $\max_{n \le N} E|f_n|^p \mathbf{I}_{\{|f_n|^p > \lambda\}} < \varepsilon$, and we're done.

Proof of the lemma. Since $|f_n| \to |f|$ in probability, WLOG $f_n, f \ge 0$. Define

$$A_n := (\{f_n \le \lambda\} \Delta \{f \le \lambda\}) \cap \{|f_n^p - f^p| > \varepsilon\}$$
$$B_n := \{f_n, f \le \lambda, |f_n^p - f^p| > \varepsilon\}.$$

Since x^p is uniformly continuous in $[0, \lambda]$, $B_n \subset \{|f_n - f| > \kappa_{\varepsilon,\lambda}\}$, $P(B_n) \to 0$.

Also $P(A_n) \to 0$ as

$$A_n \subset \{\lambda - \delta < f \le \lambda + \delta\} \cup \{|f_n - f| > \delta\},$$

and $F_{|f|}$ continuous at λ. □

□

§1 Signed measure

§1.1 Definitions

Let $(X, \mathscr{F}, \mu)$ be a measure space, consider

$$\varphi(A) := \int_A f \,\mathrm{d}\mu, \quad \forall A \in \mathscr{F}.$$

If the integral of f exists, then φ has countable additivity. Also note $\varphi(\emptyset) = 0$, so φ looks like a measure, except it can take negative values.

In fact, denote $X^+ = \{f \ge 0\}, X^- = \{f < 0\}$, then $\varphi(A) = \varphi(AX^+) + \varphi(AX^-)$.

Definition 1.1.1 (Signed measure)**.** If a set function $\varphi : \mathscr{F} \to \overline{\mathbb{R}}$ which satisfies countable additivity and $\varphi(\emptyset) = 0$, then we call φ a **signed measure**.

If $|\varphi(A)| < \infty, \forall A \in \mathscr{F}$, then φ is **finite**; Similarly we define **σ-finite**.

Since $\int_A f \, \mathrm{d}\mu$ can't reach both $\pm\infty$(otherwise the integral doesn't exist), so

Proposition 1.1.2
Let φ be a signed measure, then:

$$\varphi(A) < \infty, \quad \forall A \in \mathscr{F}, \quad or \quad \varphi(A) > -\infty, \quad \forall A \in \mathscr{F}.$$

Proof. Assume that $\varphi(A) = \infty, \varphi(B) = -\infty$, then:

$$\varphi(A \cup B) = \varphi(A) + \varphi(A \backslash B) = +\infty,$$

and similarly $\varphi(A \cup B) = -\infty$, contradiction! □

Remark 1.1.3 — From now on we may assmue $\varphi(A) > -\infty$.

Proposition 1.1.4
If $A \supseteq B$, and $|\varphi(A)| < \infty$, then $|\varphi(B)| < \infty$.

Proof. Trivial, same as above proposition. □

Proposition 1.1.5
Let $A_1, A_2, \dots$ be pairwise disjoint sets, and $|\varphi(\sum_{n=1}^{\infty} A_n)| < \infty$, then

$$\sum_{n=1}^{\infty} |\varphi(A_n)| < \infty.$$

Proof. Let $I = \{n : \varphi(A_n) > 0\}$, $J = \{n : \varphi(A_n) < 0\}$,

$$B = \sum_{n \in I} A_n, \quad C = \sum_{n \in J} A_n,$$

since $B, C \subset \sum_{n=1}^{\infty} A_n$, thus $\varphi(B), \varphi(C) \in \mathbb{R}$.

Note that $\sum_{n \in I} |\varphi(A_n)| = |\varphi(B)|$, $\sum_{n \in J} \varphi(A_n) = |\varphi(C)|$, and we're done. □

§1.2 Hahn decomposition and Jordan decomposition

Let's look at the indefinite integral again, notice that

$$\varphi(A) = \int_{A \cap \{f > 0\}} f \, \mathrm{d}\mu + \int_{A \cap \{f < 0\}} f \, \mathrm{d}\mu = \int_A f^+ \, \mathrm{d}\mu - \int_A f^- \, \mathrm{d}\mu.$$

It turns out that this property holds for any signed measure.

Definition 1.2.1 (Hahn decomposition)**.** If a patition $\{X^+, X^-\}$ of X satisfies:

$$\varphi(A) \geq 0, \forall A \subset X^+, \quad \varphi(A) \leq 0, \forall A \subset X^-,$$

then $\{X^+, X^-\}$ is called a **Hahn decomposition** of φ.

Definition 1.2.2 (Jordan decomposition)**.** Let $\varphi^\pm = \int_A f^\pm \, \mathrm{d}\mu$ be measures, if

$$\varphi = \varphi^+ - \varphi^-,$$

then it's called a **Jordan decomposition** of φ.

We're going to find X^+, or equivalently, find φ^+. Let $\varphi^*(A) := \sup\{\varphi(B) : B \subseteq A\}$.
It's clear that φ^* is non-negative, monotone, and $\varphi^*(\emptyset) = 0$.
Consider $\mathscr{F}^- = \{A : \varphi^*(A) = 0\}$. Intuitively, this is all the subsets of X^-, unioned with "null sets" in X^+.

Theorem 1.2.3 (Hahn decomposition)

Let X^- be a set with maximum $|\varphi|$ in $\mathscr{F}^-$, (since $\varphi > -\infty$, X^- must exist) and $X^+ = X \backslash X^-$ doesn't contain any set A with $\varphi(A) < 0$.

Furthermore, the Hahn decomposition is unique:

$$\varphi(A) = 0, \quad \forall A \in X_1^+ \Delta X_2^+ = X_1^- \Delta X_2^-.$$

The critical part of this theorem is:

Lemma 1.2.4

If $\varphi(A) < 0$, then we can find $A_0 \subset A$ s.t. $\varphi^*(A_0) = 0$, $\varphi(A_0) < 0$.

To prove this lemma, we need another lemma:

Lemma 1.2.5

If $\varphi(A) < \infty$, then $\forall \varepsilon > 0$, $\exists A_\varepsilon \subset A$ s.t.

$$\varphi(A_\varepsilon) \geq 0, \quad \varphi^*(A \backslash A_\varepsilon) \leq \varepsilon.$$

Proof. Assume by contradiction that $\exists \varepsilon_0 \geq 0$ s.t. $\forall A_0 \subset A$, $\varphi(A_0) < 0$ or $\varphi^*(A \backslash A_0) > \varepsilon_0$, this means,

$$\varphi(A_0) \geq 0 \implies \varphi^*(A \backslash A_0) > \varepsilon_0.$$

This will clearly yield a contradiction:

Take any $\varphi(A_0) \geq 0$(say $A_0 = \emptyset$), then exists $A_1 \subset A \backslash A_0$ s.t. $\varphi(A_1) > \varepsilon_0$, and $\varphi(A_0 \cup A_1) \geq 0$, continuing this process we can get infinitely many pairwise disjoint sets $A_1, A_2, \ldots$, with $\varphi(A_n) > \varepsilon_0$, so $\varphi(\sum_{i=1}^\infty A_n) = \infty \implies \varphi(A) = \infty$, contradiction! $\square$

Proof of Lemma 1.2.4. Applying above lemma repeatedly and take a limit:

Take $C_1 \subset A$ s.t. $\varphi(C_1) \geq 0$ and $\varphi^*(A \backslash C_1) \leq 1$. Let $A_1 = A \backslash C_1$, $\varphi(A_1) < 0$.

Again take

$$C_{k+1} \subset A_k, A_{k+1} = A_k \backslash C_{k+1} \implies \varphi^*(A_{k+1}) \leq \frac{1}{k+1}, \varphi(A_{k+1}) < 0.$$

Since $A_k \downarrow$, let $A_0 = \lim_{k\to\infty} A_k$, note $\varphi^*(A_k) \downarrow 0$, we must have $\varphi^*(A_0) = 0$.
Also $\varphi(\sum C_k) = \sum \varphi(C_k) \geq 0$, so $\varphi(A_0) < 0$. □

Proof of Theorem 1.2.3. First we prove that $\mathscr{F}^-$ is a σ-ring: $\emptyset \in \mathscr{F}^-$, if $A_1, A_2 \in \mathscr{F}^-$,

$$0 \leq \varphi^*(A_1 \backslash A_2) \leq \varphi(A_1) = 0.$$

Thus $A_1 \backslash A_2 \in \mathscr{F}^-$.
If $A_1, A_2, \cdots \in \mathscr{F}^-$ pairwise disjoint,

$$\varphi(B) = \sum_{n=1}^{\infty} \varphi(B \cap A_n) \leq 0, \quad \forall B \subset \sum_{n=1}^{\infty} A_n.$$

Hence $\sum_{n=1}^{\infty} A_n \in \mathscr{F}^-$.
Next we'll prove Hahn decomposition exists:
Let $\alpha := \inf\{\varphi(A) : A \in \mathscr{F}^-\}$, $\alpha \leq 0$.
Let $\{A_n\} \in \mathscr{F}^-$ s.t. $\varphi(A_n) \to \alpha$, then $X^- := \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}^-$.

$$\varphi(X^-) = \varphi(A_n) + \varphi(X^- \backslash A_n) \leq \varphi(A_n) + \varphi^*(X^- \backslash A_n) = \varphi(A_n) \to \alpha.$$

Therefore $-\infty < \varphi(X^-) = \alpha$.
Hence $\forall A, \varphi(AX^-) \leq \varphi^*(X^-) = 0$. By Lemma 1.2.4 we get $\forall A, \varphi(AX^+) \geq 0$, otherwise $\exists A_0 \subset A$ s.t. $\varphi^*(A_0) = 0, \varphi(A_0) < 0$. Then $\varphi(X^- \cup A_0) = \alpha + \varphi(A_0) < \alpha$, contradiction!
At last we'll prove the uniqueness:
If $X_1^{\pm}, X_2^{\pm}$ are both Hahn decompositions, then $A \in X_1^+ \cap X_2^- + X_1^- \cap X_2^+$, it's clear $\varphi(A) = 0$. □

Theorem 1.2.6 (Jordan decomposition)
The Jordan decomposition exists and is unique:

$$\varphi = \varphi^+ - \varphi^-, \quad \varphi^+ = \varphi^*, \varphi^- = (-\varphi)^*.$$

Proof. Let $\varphi^{\pm}$ be measures with $\varphi^{\pm} = \pm\varphi(A \cap X^{\pm})$. It's clear that this is a Jordan decomposition.
Now given any Jordan decomposition $\varphi^{\pm}$.
Since

$$\forall B \subset A, \varphi(B) \leq \varphi^+(B) \leq \varphi^+(A),$$

so $\varphi^* \leq \varphi^+$. But $A \cap X^+ \subset A$, so $\varphi^* \geq \varphi^+$, which proves the result.
Similarly $\varphi^- = (-\varphi)^*$, so it is unique. □

Remark 1.2.7 — The support of $\varphi^{\pm}$ are disjoint, but if $\phi \neq 0$, then the support of $\varphi^{\pm} + \phi$ intersects. $\varphi^{\pm}$ are called the **upper variation** and **lower variation**, respectively, and $|\varphi| = \varphi^+ + \varphi^-$ is called the **total variation**.

Lemma 1.2.8
$|\varphi|(A) = 0 \iff \varphi(B) = 0, \forall B \subset A.$

Proof. Just write $|\varphi| = \varphi^+ + \varphi^-$, we know $\varphi(B) = 0$.
Conversely, $\varphi(X^{\pm} \cap A) = 0 \implies |\varphi|(A) = 0$. □

§1.3 Radon-Nikodym theorem

We assume the functions and sets below are all measurable. Let $(X, \mathscr{F})$ be a measurable space, φ a signed measure.

Definition 1.3.1 (R-N derivative)**.** If there exists a a.e. unique function f s.t.

$$\varphi(A) = \int_A f \, \mathrm{d}\mu, \quad \forall A \in \mathscr{F},$$

we say f is the **Radon-Nikodym derivative** of φ with respect to μ, abbreviated by R-N derivative or derivative, denoted by $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$.

Remark 1.3.2 — When μ is σ-finite, then f must be unique a.e..

Definition 1.3.3 (Absolute continuity)**.** If $\forall A \in \mathscr{F}$,

$$\mu(A) = 0 \implies \varphi(A) = 0,$$

then we say φ is **absolutely continuous** with respect to μ, denoted by $\varphi \ll \mu$.

Observe that

$$\mu(A) = 0 \implies \mu(A \cap X^{\pm}) = 0 \implies \varphi^{\pm}(A) = 0,$$

so $\varphi \ll \mu \iff \varphi^{\pm} \ll \mu \iff |\varphi| \ll \mu$.

It's obvious that $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists only if $\varphi \ll \mu$, but it turns out that this is also the sufficient condition when μ is a σ-finite measure.

We can't prove this directly, so we'll prove some easy cases first.

Lemma 1.3.4

Let φ, μ be finite measures. Then

$$\exists f \in \mathscr{L} := \left\{ g \in L_1 : g \geq 0, \int_A g \, \mathrm{d}\mu \leq \varphi(A), \forall A \right\},$$

such that $\int_X f \, \mathrm{d}\mu = \sup \int_X g \, \mathrm{d}\mu$.

Proof. This is somehow similar to find simple functions approaching non-negative measurable functions.

First let $\beta = \sup \int_X g \, \mathrm{d}\mu$, and choose g_k s.t. $\int_X g_k \, \mathrm{d}\mu \to \beta$.

Let $f_n := \max_{k \leq n} g_k$, and $f_n \uparrow f$. By Levi's theorem, $\int_A f \, \mathrm{d}\mu = \lim_{n\to\infty} f_n \, \mathrm{d}\mu$, so if $f_n \in \mathscr{L}$, $f \in \mathscr{L}$ as well. Let $A_k = A \cap \{f_n = g_k, f_n \neq g_j, j < k\}$ be a partition of A,

$$\int_A f_n \, \mathrm{d}\mu = \sum_{k=1}^{n} \int_{A_k} g_k \, \mathrm{d}\mu \leq \sum_{k=1}^{n} \varphi(A_k) = \varphi(A).$$

Thus $f_n \in \mathscr{L}$, we have $\int_X f \, \mathrm{d}\mu = \beta \geq \int_X g \, \mathrm{d}\mu$, for all $g \in \mathscr{L}$. $\square$

Proposition 1.3.5

Suppose φ, μ are both finite, then $\varphi \ll \mu \implies \frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists.

Proof. Decompose φ to $\varphi^+ - \varphi^-$, we may assume $\varphi \geq 0$.

Starting from previous lemma, we'll prove that $\int_A f \,\mathrm{d}\mu = \varphi(A)$. Let $\nu(A) = \varphi(A) - \int_A f \,\mathrm{d}\mu$ be a measure.

Let ν_n be increasing signed measures.

$$\nu_n(A) := \nu(A) - \frac{1}{n}\mu(A), \quad \forall A \in \mathscr{F}.$$

Let $X_n^{\pm}$ be the Hahn decomposition of ν_n, and

$$X^+ = \bigcup_{n=1}^{\infty} X_n^+, \quad X^- = \bigcap_{n=1}^{\infty} X_n^-.$$

First since $X^- \subset X_n^-$,

$$\nu(X^-) = \nu_n(X^-) + \frac{1}{n}\mu(X^-) \leq \frac{1}{n}\mu(X^-) \to 0.$$

We have $f + \frac{1}{n}\mathbf{I}_{X_n^+} \in \mathscr{L}$ since

$$\begin{aligned}\int_A \left(f + \frac{1}{n}\mathbf{I}_{X_n^+}\right) \mathrm{d}\mu &= \varphi(A) - \nu(A) + \frac{1}{n}\mu(X_n^+ \cap A) \\ &\leq \varphi(A) - \nu(X_n^+ \cap A) + \frac{1}{n}\mu(X_n^+ \cap A) \\ &= \varphi(A) - \nu_n(X_n^+ \cap A) \leq \varphi(A).\end{aligned}$$

So we have $\int_X f \,\mathrm{d}\mu \geq \int_X (f + \frac{1}{n}\mathbf{I}_{X_n^+}) \,\mathrm{d}\mu$, $\mu(X_n^+) = 0 \implies \mu(X^+) = 0$.

Since $\varphi \ll \mu$, $\varphi(X^+) = 0 \implies \nu(X^+) = 0$. $\square$

Proposition 1.3.6

Let φ be a σ-fintie signed measure, μ be a finite measure, if $\varphi \ll \mu$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists and its integral exists.

Proof. Let $X = \sum_{n=1}^{\infty} A_n$, $|\varphi(A_n)| < \infty$, then the R-N derivative f_n exists on A_n,

Let $f = \sum_{n=1}^{\infty} f_n \mathbf{I}_{A_n}$, then f finite a.e.,

$$\varphi(A \cap A_n) = \int_{A \cap A_n} f_n \,\mathrm{d}\mu = \int_{A \cap A_n} f \,\mathrm{d}\mu.$$

WLOG φ^- finite, then

$$\varphi(\{f < 0\} \cap A_n) = \int_{A_n} f^- \,\mathrm{d}\mu = \int_{A_n} f_n^- \,\mathrm{d}\mu \geq -\varphi^-(A_n)$$

So the integral of f exists.

Since φ is countably additive and the integral of f exists, we can add the above equality to get the desired. $\square$

Proposition 1.3.7

Let φ be a signed measure, the above conclusion also holds.

Proof. Let
$$\mathscr{G} := \left\{ \sum_{n=1}^{\infty} A_n : |\varphi(A_n)| < \infty, n = 1, 2, \dots \right\}.$$
Since $\emptyset \in \mathscr{G}$, and it's closed under set difference:
$$\sum_{n=1}^{\infty} A_n \backslash \sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} (A_n \backslash B)$$
by $A_n \backslash B \subset A_n$, we have $|\varphi(A_n \backslash B)| < \infty$.

$\mathscr{G}$ is a σ-ring.

There exists B s.t. $\mu(B) = \gamma := \sup\{\mu(A) : A \in \mathscr{G}\}$. Just take $\mu(B_n) \to \gamma, B = \bigcup_{n=1}^{\infty} B_n$.

So φ is σ-finite on $(B, B \cap \mathscr{F})$, the R-N derivative exists.

For all $C \subset B^c$, we must have $\varphi(C) = 0$ or ∞. TODO!! $\square$

At last we come to the full statement:

Theorem 1.3.8

Let φ be a signed measure, μ a σ-finite measure, if $\varphi \ll \mu$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$ exists.

Example 1.3.9

Let $X = \mathbb{R}$, $\mu(A) = \#A$, μ is not σ-finite. Let $\varphi(A) = 0$ when A countable, 1 otherwise.

In this case the R-N derivative doesn't exist, otherwise
$$0 = \varphi(\{x\}) = \int_{\{x\}} f \,\mathrm{d}\mu = f(x)\mu(x) = f(x),$$
contradiction!