# Geometry II

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## Contents



**Remark 0.0.1 —** On the existence of triangulation

## §0.1 Homotopy

**Definition 0.1.1** (Homotopy)**.** Given two continuous maps $f, g : X \to Y$, if there exists a continuous map

$$H : X \times [0,1] \to Y$$

such that $f = H_0, g = H_1$, where $H_t = H\big|_{X\times\{t\}}$, then we say $f$ and $g$ are **homotopic**, denoted by $f \simeq g$, and the map $H$ is a **homotopy**.

**Definition 0.1.2** (Relative homotopy)**.** Let $A \subset X$, $f, g : X \to Y$, and $f\big|_A = g\big|_A$. We say $f$ and $g$ are homotopic relative to $A$ ($f \simeq g \; rel \; A$), if $H$ satisfies $H_t\big|_A = f\big|_A$.

More often we'll talk about homotopy between paths, here by path we mean a map $\gamma : [0,1] \to X$. We say two paths are homotopic if they are homotopic relative to the endpoints($i.e.\{0,1\}$)

**Proposition 0.1.3**

The homotopic relation is an equivalence relation.

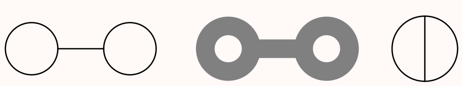
Besides studying the homotopy of maps, we can also consider the homotopy between spaces:

**Definition 0.1.4.** We say two topological spaces $X, Y$ are **homotopy equivalent** or have the same **homotopy type**, if there exists $f : X \to Y$, $g : Y \to X$, such that

$$f \circ g = \mathrm{id}_Y, \quad g \circ f = \mathrm{id}_X .$$

**Example 0.1.5**

The following spaces are homotopy equivalent:



**Definition 0.1.6** (Fundamental groups)**.** Let $\Omega(X, x_0)$ denote all the loops starting at $x_0$, i.e. $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

Define the **fundamental group** of $X$ to be:

$$\pi_1(X, x_0) = \Omega(X, x_0)/\simeq,$$

where $\simeq$ is the homotopy relative to $x_0$.

We define the group operation to be the *concatenation* of paths, denoted by $(a, b) \mapsto ab$, where

$$ab(t) = \begin{cases} a(2t), & t \in [0, \frac{1}{2}]; \\ b(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

**Proposition 0.1.7**

The concatenation descends to a well-defined group operation:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0).$$

*Proof.* Just some trivial checking. Note that the inverse of $a$ is just $\overline{a}(t) := a(1-t)$. □

**Proposition 0.1.8**

An homeomorphism $f : (X, x_0) \to (Y, y_0)$ will induce a group homomorphism $f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Note that $X$ may be disconnected, so the fundamental group is dependent of the base point $x_0$. If $\gamma = \langle c \rangle$ is a homotopy class of paths from $x_0$ to $x_1$, then $\gamma$ induces a group homomorphism:

$$\gamma_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1) : \langle a \rangle \mapsto \langle \bar{c}ac \rangle .$$

It's easy to see $\gamma_\sharp$ is an isomorphism.

Hence $\pi_1(X, x_0)$ only depends on the path connected components of $x_0$. Thus if $X$ is path connected, and $X, Y$ are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, or sometimes we can leave the base point out, just write $\pi_1(X) \cong \pi_1(Y)$.

**Remark 0.1.9 —** If $x_0 = x_1$, then $\gamma \mapsto \gamma_\#$ gives a homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_1(X, x_0))$.

**Example 0.1.10**

If $X \simeq \{pt\}$, then $\pi_1(X) \cong \{1\}$. In this case, $X$ is called a **contractible space**. Note that the inverse is not true, e.g. $X = S^n$ for $n \geq 2$. Path connected spaces with trivial fundamental group are called **simple connected** spaces.

Some classic contractible space includes convex sets in $\mathbb{R}^n$, trees in graph theory and cones $CX = X \times [0,1]/X \times \{1\}$.

Some more complex contractible examples including "a house with two rooms", the equitorial inclusion $S^\infty = \bigcup_{n=0}^\infty S^n$ with limit topology, i.e. the largest topology s.t. $S^n \to S^\infty$ continuous.

There are several concepts:

- Retraction: $f : X \to A$, $A \subset X$, $f\big|_A = \mathrm{id}_A$.
- Deformation retraction: $f$ as above with $i \circ f \simeq \mathrm{id}_X$, where $i : A \to X$ is the inclusion.
- Strong deformation retraction: $f$ as above with $i \circ f \simeq \mathrm{id}_X$ *rel* $A$.

The set $A$ is called (strong) deformation kernel of $f$.

**Example 0.1.11** (Differences between deformation and strong deformation)
Let $X$ be the follwing space:

$$([0,1] \times \{0\}) \cup ([0,1]_{\mathbb{Q}} \times [0,1])$$

We know $X \simeq \{pt\}$, but $\{q\} \times [0,1]$ is deformation kernel but not strong deformation kernel.

## §0.2 Fundamental groups

After introducing the fundamental groups, a natural question arises: how to compute the fundamental group of a given space? We first state the main result of this section:

**Theorem 0.2.1** (Van Kampen)
Let $X = U' \cup U''$ be a topology space such that $U', U''$ are open and $W = U' \cap U''$ path connected, then for $x_0 \in W$, we have

$$\pi_1(X, x_0) \cong \pi_1(U', x_0) * \pi_1(U'', x_0)/N,$$

where $N$ is the smallest normal subgroup generated by

$$i'_\sharp(\delta) i''_\sharp(\delta^{-1}) : \delta \in \pi_1(W, x_0),$$

$$\begin{tikzcd} & U' \arrow[dr, "j'"] & \\ W \arrow[ur, "i'"] \arrow[dr, "i''"'] & & X \\ & U'' \arrow[ur, "j''"'] & \end{tikzcd}$$

and $*$ means free product.

Note that this theorem is useless when both $U', U''$ have trivial fundamental groups. Thus we need to find a space with nontrivial fundamental group first. One of the simplest example is $S^1$ :

$$\pi_1(S^n, x_0) \cong \begin{cases} \{1\}, & n \geq 2, \\ \mathbb{Z}, & n = 1. \end{cases}$$

Let $X \vee Y := X \sqcup Y/(x_0 = y_0)$, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. Thus $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{k}) = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_k$, the free group of rank $k$.

**Example 0.2.2**

Since $nT^2$ is formed by $2n$ loops(borders of the polynomial representation) fused with a disk. Note that $W = U' \cap U'' \cong S^1$, so

$$\pi_1(nT^2) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle .$$

Similarly,

$$\pi_1(mP^2) = \left\langle c_1, \ldots, c_m \mid c_1^2 \cdots c_m^2 = 1 \right\rangle .$$

**Example 0.2.3**

For product spaces, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

In fact the isomorphism can be written down with $i_x, i_y, p_x, p_y$, i.e. $p_{x\sharp} \times p_{y\sharp}$ and $(i_{x\sharp}, i_{y\sharp})$.

**Theorem 0.2.4**

$\pi_1(S^1) \cong \mathbb{Z}$, where the generating element is id.

*Proof.* Consider the map $p : \mathbb{R} \to S^1$, with $t \mapsto e^{2\pi i t}$.

Given any path $\gamma : [0,1] \to S^1$, we can find a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}$, s.t. $\tilde{\gamma}(0) \in \mathbb{Z}$ is any given base point. We denote this map by $\Phi$, $\gamma \mapsto \tilde{\gamma}(1)$, where we require $\tilde{\gamma}(0) = 0$.

We can prove that $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$, and $\Phi$ only depends on the homotopy class of $\gamma$, so $\Phi$ induces a homomorphism of $\pi_1(S^1) \to \mathbb{Z}$.

**Remark 0.2.5** — Since every homotopy $[0,1] \times [0,1] \to S^1$ can be lifted uniquely, and the endpoints of each path form a path in $\mathbb{R}$, but it's always contained in $\mathbb{Z}$, hence it must be constant.

Note that

- $\Phi$ is surjective since $s \mapsto e^{2\pi i m s}$ is mapped to $m$ under $\Phi$;
- $\Phi$ is injective since $\ker \Phi = \{1\}$: if $\tilde{\gamma}(1) = 0$, then $\tilde{\gamma} \simeq const$, so $\gamma = p \circ \tilde{\gamma} \simeq const$.

So $\Phi$ is an isomorphism, $\pi_1(S^1) \cong \mathbb{Z}$. □

Next we'll prove Van Kampen theorem (0.2.1). In fact we only need to prove that:

**Claim 0.2.6.** The map

$$j'_\sharp * j''_\sharp : \pi_1(U', x_0) * \pi_1(U'', x_0) \to \pi_1(X, x_0)$$

is a surjective homomorphism, and its kernel is the normal closure generated by $i'_\sharp(\delta) i''_\sharp(\bar{\delta})$.

CLearly it's a group homomorphism.

For any $\gamma \in \pi_1(X, x_0)$, it can be decompose to $a_1 b_1 a_2 \cdots a_k b_k$, where $a_i \subset U', b_i \subset U''$, let the partition points be $p_1, \ldots, p_k, q_1, \ldots, q_k \in W$, and denote $s_i, t_i$ the path from $x_0$ to $p_i, q_i$. So we have

$$\gamma = \underbrace{a_1 \bar{s}_1}_{\in \pi_1(U', x_0)} \cdot \underbrace{s_1 b_1 \bar{t}_1}_{\in \pi_1(U'', x_0)} \cdots$$

Thus $j'_\sharp * j''_\sharp$ is indeed surjective.

At last we'll study its kernel, let $\gamma \in \ker j'_\sharp * j''_\sharp$. Since $\gamma \simeq \{x_0\}$, say the homotopy is $H : [0,1] \times [0,1] \to U' \cup U''$.

We can partition $[0,1] \times [0,1]$ to many small cells such that each cell's image is completely contained in either $U'$ or $U''$.

TODO

Using the "word processing" method, since we've showed that $\gamma = \alpha_1 \beta_1 \cdots$ where $\alpha_i \subset U', \beta_i \subset U''$. So actually we're saying that

$$\gamma = i'_\sharp(\alpha_1) i''_\sharp(\beta_1) \cdots$$

if we some $\delta \subset U' \cap U''$, then the conjugate of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$ can change $\cdots i'_\sharp(\delta) \cdots$ to $\cdots i''_\sharp(\delta) \cdots$.

Thus if $\gamma$ is in the kernel, it can indeed be written as a product of conjugates of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$.

**Remark 0.2.7** — A more frequently used version is that $W$ is a strong deformation kernel of some open neighborhood in $X$.

**Example 0.2.8**

For any finite representation of a group

$$G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle,$$

$G$ can be realized as the fundamental groups of a space: Let $X$ be a CW-complex with a single 0-cell, $n$ 1-cells corresponding to $x_i$, and $m$ 2-cells corresponding to $R_i$.

**Remark 0.2.9** — The path connected condition of $W$ can't be removed, e.g. two segments can fuse to $S^1$.

**Example 0.2.10**

Let $f : S \to S$ be a homeomorphism, where $S$ is a closed surface. Consider the *mapping torus*:

$$M_f = S \times [0,1] / \sim,$$

where $(0,0) \sim (f(x), 1)$.

Let $Y = S \times \{0\} \cup (\{x_0\} \times [0,1])$, $U'$ is an open neighborhood of $Y$, $U'' = M_f \backslash Y$.

Observe that $U' \simeq S \vee circle$, and $U'' \simeq (S \backslash disk) \times (\varepsilon, 1-\varepsilon) \simeq S \backslash disk$.

$$\pi_1(M_f) \cong \pi_1(X) * \langle t \rangle / (g \sim t f_\sharp(g) t^{-1}) \cong \pi_1(S) \rtimes_{f_\sharp} \langle t \rangle$$

Seifent-vanKampen: if $i'_\sharp, i''_\sharp$ are both injective, then $j'_\sharp, j''_\sharp$ are also injective.

Next we'll see some applications of fundamental groups:

- Bronwer fixed point theorem: A continuous map $f : D^n \to D^n$ must have a fixed point.
- Invariance of the boundary: If $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$, s.t. $x \in \mathbb{R}^{n-1} \times \{0\}$, then $x$ doesn't have a neighborhood homeomorphic to $\mathbb{R}^n$.
- Invariance of regions: If $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is also open, i.e. $f : U \to f(U)$ is a homeomorphism.

Here we can only prove the case $n = 2$, since the complete proof need homotopy groups of rank $n$ (i.e. $\pi_n$), but here we only introduced $\pi_1$. However, the idea is nearly identical.

*Proof.* Assmue by contradiction that $f$ has no fixed points, let $g(x) = \frac{x-f(x)}{\|x-f(x)\|}$, then $g : D^n \to S^{n-1}$ is a deformation. Thus $g_\sharp : \pi_1(D^2) \to \pi_1(S^1)$, but $\pi_1(D^2) = \{1\}, \pi_1(S^1) = \mathbb{Z}$, contradiction! $\square$