Measure Theory

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Contents

Theorem 0.0.1 (Fauto's Lemma)
Let $\{f_n\}$ be non-negative measurable functions almost everywhere, then

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Proof. Let $g_k := \inf_{n\ge k} f_n \uparrow g := \liminf_{n\to\infty} f_n$. By monotone convergence theorem,

$$\int_X g \,\mathrm{d}\mu = \lim_{k\to\infty} \int_X g_k \,\mathrm{d}\mu \le \lim_{k\to\infty} \inf_{n\ge k} \int_X f_n \,\mathrm{d}\mu = \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

□

Corollary 0.0.2
If there exists integrable g s.t. $f_n \ge g$, then $\int_X \liminf_{n\to\infty} f_n$ exists and

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Theorem 0.0.3 (Lebesgue)
Let $f_n \to f, a.e.$ or $f_n \xrightarrow{\mu} f$, if there exists non-negative integrable function g s.t. $|f_n| \le g, \forall n$, then

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Proof. When $f_n \to f, a.e.$, by Fatou's lemma,

$$\int_X f \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} f_n \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Since $|f_n| \leq g$,

$$\int_X f \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} f_n \,\mathrm{d}\mu \geq \limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu,$$

which gives the desired.

When $f_n \xrightarrow{\mu} f$, for all subsequence $\{n_k\}$, exists a subsequence $\{n'\}$ s.t. $f_{n'} \to f, a.e.$. Thus $\int_X f_{n'} \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$, hence $\int_X f_n \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$.(Why?) □

Corollary 0.0.4

Let f_n be random variable on $(\Omega_n, \mathscr{F}_n, P_n)$, $f_n \xrightarrow{d} f$, then we have

$$\lim_{n\to\infty} \int_{X_n} f_n \,\mathrm{d}P_n = \int_X f \,\mathrm{d}P.$$

Proposition 0.0.5 (Transformation formula of integrals)

Let $g : (X, \mathscr{F}, \mu) \to (Y, \mathscr{S})$ be a measurable map. For all measurable f on $(Y, \mathscr{S})$, then

$$\int_Y f \,\mathrm{d}\mu \circ g^{-1} = \int_X f \circ g \,\mathrm{d}\mu$$

if one of them exists.

Proof. By the typical method, we only need to prove for indicator function f. □

Remark 0.0.6 — μ and $\mu \circ g^{-1}$ are the same measure in different spaces.

§0.1 Expectations

Let ξ be a r.v. on $(\Omega, \mathscr{F}, P)$,

Definition 0.1.1 (Expectations)**.** If $\int_\Omega \xi \,\mathrm{d}P$ exists, then we call it the **expectation** of ξ, denoted by $E(\xi)$ or $E\xi$.

Consider the distribution $\mu_\xi = P \circ \xi^{-1}$, $F_\xi(x) = P(\xi \leq x)$.
Let $f = \mathrm{id} : \mathbb{R} \to \mathbb{R}$, then $E(\xi) = E(\mu_\xi)$:

$$\int_{\mathbb{R}} x \,\mathrm{d}F_\xi(x) = \int_{\mathbb{R}} f \,\mathrm{d}\mu_\xi = \int_{\mathbb{R}} f \,\mathrm{d}P \circ \xi^{-1} = \int_{\mathbb{R}} f \circ \xi \,\mathrm{d}P = \int_\Omega \xi \,\mathrm{d}P = E(\xi).$$

Let f be a measurable function on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, then $f(\xi)$ is a measurable function on $(\Omega, \mathscr{F})$, and

$$Ef(\xi) = \int_{\mathbb{R}} f \,\mathrm{d}F_\xi.$$

Let $\eta = f \circ \xi$, by the transformation formula,

$$\begin{aligned} Ef(\xi) &= \int_\Omega \eta(\omega) \, \mathrm{d}P(\omega) \\ &= \int_{\overline{\mathbb{R}}} y \, \mathrm{d}P \circ \eta^{-1}(y) = \int_{\overline{\mathbb{R}}} y \, \mathrm{d}\mu_\eta(y) = \int_{\overline{\mathbb{R}}} y \, \mathrm{d}\mu_\xi \circ f^{-1}(y) \\ &= \int_{\mathbb{R}} f(x) \, \mathrm{d}\mu_\xi(x) = \int_{\mathbb{R}} f \, \mathrm{d}F_\xi. \end{aligned}$$

Example 0.1.2

Possion distribution: $P(\xi = i) = \frac{\lambda^i}{i!} e^{-\lambda} =: p_i$. Its expectation is

$$\int_{\mathbb{R}} x \, \mathrm{d}\mu = \int_{\mathbb{N}} x \, \mathrm{d}\mu = \sum_{i=0}^{\infty} i p_i.$$

For continuous distribution, the density function p is actually a non-negative, integrable function, and $\int_{\mathbb{R}} p(x) \, \mathrm{d}x = 1$. So $\mu(B) = \int_B p(x) \, \mathrm{d}x$ is a probability measure.

Since $\mu_\xi\big|_{\mathscr{P}_{\mathbb{R}}} = \mu\big|_{\mathscr{P}_{\mathbb{R}}}$, $\mu_\xi = \mu$. By typical method, we can prove

$$Ef(\xi) = \int_{\mathbb{R}} f \, \mathrm{d}\mu_\xi = \int_{\mathbb{R}} f(x) p(x) \, \mathrm{d}x.$$

§0.2 L_p spaces

Definition 0.2.1 (L_p spaces)**.** Let $1 \le p < \infty$. Define

$$\|f\|_p := \left(\int_X |f|^p \right)^{\frac{1}{p}}, \quad L_p(X, \mathscr{F}, \mu) := \{f : \|f\|_p < \infty\}.$$

Sometimes we'll simplify the notation as $L_p(\mu), L_p(\mathscr{F})$ or just L_p.

- $f \in L_1$ iff f integrable, let $\|f\| := \|f\|_1$.
- $f \in L_p \iff f^p \in L_1 \implies f$ is finite a.e..

In fact, L_p is a normed vector space under the norm $\|\cdot\|_p$:

Lemma 0.2.2

Let $1 \le p < \infty$, let $C_p = 2^{p-1}$, then

$$|a+b|^p \le C_p(|a|^p + |b|^p), \quad a, b \in \mathbb{R}.$$

Proof. It's a single-variable inequality, it's obvious by taking the derivative. □

Thus by taking integral on both sides,

$$\int_X |f+g|^p \, \mathrm{d}\mu \le C_p \left(\int_X |f|^p \, \mathrm{d}\mu + \int_X |g|^p \, \mathrm{d}\mu \right).$$

So L_p space is a vector space.

Lemma 0.2.3 (Holder's inequality)
Let $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\|fg\| \le \|f\|_p \|g\|_q, \quad \forall f \in L_p, g \text{ measurable}.$$

Proof. WLOG $\|f\|_p > 0$, $0 < \|g\|_q < \infty$. Let

$$a = \left(\frac{|f|}{\|f\|_p}\right)^p = \frac{|f|^p}{\int_X |f|^p \,\mathrm{d}\mu}, \quad b = \left(\frac{|g|}{\|g\|_q}\right)^q = \frac{|g|^q}{\int_X |g|^q \,\mathrm{d}\mu}.$$

By weighted AM-GM,

$$\int_X \frac{|fg|}{\|f\|_p \|g\|_q} \,\mathrm{d}\mu \le \int_X \left(\frac{a}{p} + \frac{b}{q}\right) \mathrm{d}\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

The equality holds iff $a = b$, i.e. $\exists \alpha, \beta \ge 0$ not all zero s.t. $\alpha |f|^p = \beta |g|^q, a.e.$. □

Theorem 0.2.4 (Minkowski's inequality)
Let $1 \le p < \infty$, then

$$\|f + g\|_p \le \|f\|_p + \|g\|_p, \quad \forall f, g \in L_p.$$

The equality holds iff $(1) : p = 1, fg \ge 0$; $(2) p > 1, \exists \alpha, \beta \ge 0, s.t. \alpha f = \beta g, a.e.$.

Proof. When $p = 1$, it follows by $|f + g| \le |f| + |g|$.

When $p \ge 1$, let $q = \frac{p}{p-1}$, by Holder's inequality,

$$|f + g|^p \le |f||f + g|^{p-1} + |g||f + g|^{p-1},$$

$$\implies \|f + g\|_p^p \le (\|f\|_p + \|g\|_p) \cdot \||f + g|^{p-1}\|_q.$$

Note that

$$\||f + g|^{p-1}\|_q = \left(\int_X |f + g|^p \,\mathrm{d}\mu\right)^{\frac{1}{q}} = \|f + g\|_p^{\frac{p}{q}}.$$

Since $f + g \in L_p$, we can divide both sides by $\|f + g\|_p^{\frac{p}{q}}$ to get the result. □

In L_p space, we view two functions $f = g, a.e.$ as the same function, i.e. the original function space modding the equivalence relation out.

Hence $(L_p / \sim, \|\cdot\|_p)$ is a normed vector space.

When $p = \infty$, define

$$\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(|f| > a) = 0\}, \quad L_\infty := \{f : \|f\|_\infty < \infty\}.$$

We call the functions in L_∞ **essentially bounded**.

Let $\mu(X) < \infty$, then $f \in L_\infty \implies f \in L_p$, and $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$: For all $0 < a < \|f\|_\infty$,

$$a^p \mu(|f| > a) \le \int_X |f|^p \mathbf{I}_{|f| > a} \,\mathrm{d}\mu \le \int_X |f|^p \,\mathrm{d}\mu \le \|f\|_\infty^p \mu(X),$$

So taking the exponent $\frac{1}{p}$,

$$a \leftarrow a\mu(|f| > a)^{\frac{1}{p}} \le \|f\|_p \le \|f\|_\infty$$

But when $\mu(X) = \infty$, let $f \equiv 1$, then $f \in L_\infty$ but $f \notin L_p$.

Theorem 0.2.5
Let $f, g \in L_\infty$,

$$\|fg\| \leq \|f\| \|g\|_\infty,$$
$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Proof.

$$\int_X |fg| \, \mathrm{d}\mu \leq \int_X |f| \|g\|_\infty \, \mathrm{d}\mu = \|f\| \|g\|_\infty.$$

Since $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty, a.e.$, we get the second inequality. □

Similarly we get $(L_\infty, \|\cdot\|_\infty)$ is a normed vector space.

The norm can deduce a *distance*:

$$\rho(f, g) := \|f - g\|.$$

Theorem 0.2.6 (L_p space is complete)
Let $1 \leq p \leq \infty$. If $\{f_n\} \subset L_p$ satisfying $\lim_{n,m \to \infty} \|f_n - f_m\|_p = 0$, then there exist $f \in L_p$ s.t. $\lim_{n \to \infty} \|f - f_n\|_p = 0$.

Proof. Take $n_1 < n_2 < \cdots$ such that

$$\|f_m - f_n\|_p \leq \frac{1}{2^k}, \quad \forall n, m \geq n_k.$$

Let $g = \uparrow \lim_{k \to \infty} g_k$, where

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \in L_p, \quad g_k \geq 0.$$

Since

$$\|g_k\|_p \leq \|f_{n_1}\|_p + \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \leq \|f_{n_1}\|_p + 1.$$
$$\implies \|g\|_p = \uparrow \lim_{k \to \infty} \|g_k\|_p \leq \|f_{n_1}\|_p + 1.$$

Here we use the monotone convergence theorem. We can check the above also holds for $p = \infty$.

Therefore $g \in L_p \implies g < \infty, a.e.$. We have

$$f := f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i}) = \lim_{k \to \infty} f_k, a.e.$$

the series is absolutely convergent, so f exists a.e. and $|f| \leq g, a.e.$.

Lastly we can check: when $p = \infty$,

$$\|f_n - f\|_\infty \leq \|f_n - f_{n_k}\|_\infty + \|f_{n_k} - f\|_\infty,$$

where the both term approach to 0 as $n \to \infty$.

When $p < \infty$, by Fatou's lemma,

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p \, \mathrm{d}\mu = \int_X \lim_{k \to \infty} |f_n - f_{n_k}|^p \, \mathrm{d}\mu \leq \liminf_{k \to \infty} \int_X |f_n - f_{n_k}|^p \, \mathrm{d}\mu \leq \varepsilon.$$

□

Remark 0.2.7 — Using the same technique we can prove that if f_n is Cauchy in measure, then f_n converge to some f in measure:
Let $A_i = \{|f_{n_{i+1}} - f_{n_i}| > 2^{-i}\}$ s.t. $\mu(A_i) < 2^{-i}$.
Define $f = f_{n_1} + \sum_{i\geq 1}(f_{n_{i+1}} - f_{n_i})$ on the set $\bigcup_{k\geq 1}\bigcap_{i\geq k} A_i^c$.

This theorem implies that $(L_p, \|\cdot\|_p)$ is a Banach space. So we can try to define an *inner product* on L_p space:

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

We can check $\langle\cdot,\cdot\rangle$ is bilinear only if $p = 2$, so L_2 is actually a Hilbert space.

When $0 < p < 1$, let

$$\|f\|_p := \int_X |f|^p \,\mathrm{d}\mu, \quad L_p = \{f : \|f\|_p < \infty\}.$$

Lemma 0.2.8

Let $0 < p < 1$, $C_p = 1$, then

$$|a+b|^p \leq C_p(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}.$$

So L_p is a vector space.

Theorem 0.2.9 (Minkowski)

Let $0 < p < 1$ then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark 0.2.10 — When $\|f\|_p = (\int_X |f|^p \,\mathrm{d}\mu)^{\frac{1}{p}}$, $0 < p < 1$. then it won't satisfy Minkowski's inequality.

Thus L_p is only a metric space but not a normed vector space. Using the same method we can prove L_p is a complete metric space.

§0.3 Convergence in L_p space

Definition 0.3.1. Let $0 < p \leq \infty$, $f, f_1, f_2, \cdots \in L_p$. When $\|f_n - f\|_p \to 0$, then we write $f_n \xrightarrow{L_p} f$, called **average converge of order p**.

Theorem 0.3.2

Let $0 < p < \infty$, $f, f_1, \cdots \in L_p$,

- If $f_n \xrightarrow{L_p} f$, then $f_n \xrightarrow{\mu} f$, and $\|f_n\|_p \to \|f\|_p$.
- If $f_n \to f, a.e.$ or in measure, then $\|f_n\|_p \to \|f\|_p \iff f_n \xrightarrow{L_p} f$.

Proof. When $f_n \xrightarrow{L_p} f$, let $A := \{|f_n - f| > \varepsilon\}$,

$$\mu(A) \le \frac{1}{\varepsilon^p}\int_X |f_n - f|^p \mathbf{I}_A \,\mathrm{d}\mu \le \frac{1}{\varepsilon^p}\|f_n - f\|_p^p \to 0.$$

and obviously $\|f_n\|_p \to \|f\|_p$

On the other hand, when $f_n \to f, a.e.$ and $\|f_n\|_p \to \|f\|_p$, From $|a+b|^p \le C_p(|a|^p + |b|^p)$,

$$g_n := C_p(|f_n|^p + |f|^p) - |f_n - f|^p \ge 0.$$

$g_n \to 2C_p|f|^p, a.e.$, so

$$\int_X 2C_p|f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty}\int_X g_n \,\mathrm{d}\mu = 2C_p\int_X |f|^p \,\mathrm{d}\mu - \limsup_{n\to\infty}\int_X |f_n - f|^p \,\mathrm{d}\mu.$$

When $f_n \to f$ in measure, for any subsequence there exist its subsequence $f_{n'} \to f, a.e.$, so $\|f_{n'} - f\|_p \to 0$, hence $\|f_n - f\|_p \to 0$. □

Remark 0.3.3 — This theorem implies for any L_p function f, we can take simple functions $f_1, f_2, \cdots \to f$ and $|f_n| \uparrow |f|$, so $f_n \xrightarrow{L_p} f$.

Definition 0.3.4 (Weak convergence)**.** Let $1 < p < \infty$, and $f_1, f_2 \cdots \in L_p$. If

$$\lim_{n\to\infty}\int_X f_n g \,\mathrm{d}\mu = \int_X fg \,\mathrm{d}\mu, \quad \forall g \in L_q.$$

Then we say f_n **weak convergent** to f, denoted by $f_n \xrightarrow{(w)L_p} f$.

When $p = 1$ and $(X, \mathscr{F}, \mu)$ is a σ-finite measure space, and the condition also holds, we say $\{f_n\}$ weak convergent to f in L_1.

Corollary 0.3.5

Let $1 \le p < \infty$, then

$$f_n \xrightarrow{L_p} f \implies f_n \xrightarrow{(w)L_p} f.$$

Proof. By Holder's inequality,

$$\left|\int_X (f_n - f)g \,\mathrm{d}\mu\right| \le \|f_n - f\|_p\|g\|_q \to 0.$$

□

If $\sup_{t\in T}\|f_t\|_p =: M < \infty$, then we say $\{f_t, t \in T\}$ is **bounded in L_p**.

Theorem 0.3.6

Let $1 < p < \infty$, $\{f_n\} \subset L_p$, there exists M s.t. $\|f_n\|_p \le M, \forall n$. If $f_n \to f, a.e.$ or in measure, then $f \in L_p$ and $f_n \to f$ weakly.

Proof. First $\|f\|_p \le M$:

$$\int_X |f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \,\mathrm{d}\mu \le M^p.$$

Next we prove the weak convergence: For all $g \in L_q$, recall the bounded convergence theorem in probability, we can view M as a bound of f_n, and $\|g\|_q$ as P.

Let $B = \{|f_n - f| \le \hat{\varepsilon}\}$, consider

$$a := \int_B (f_n - f)g \,\mathrm{d}\mu, \quad b := \int_{B^c} (f_n - f)g \,\mathrm{d}\mu.$$

Note that

$$|a| \le \hat{\varepsilon} \int_X |g| \,\mathrm{d}\mu.$$

But $\int_X |g| \,\mathrm{d}\mu$ might be infinity, so let $A_k := \{\frac{1}{k} \le |g|^q \le k\}$, we have

$$\int_{A_k} |g| \,\mathrm{d}\mu \le k^{\frac{1}{q}} \mu(A_k) < \infty.$$

($\frac{1}{k}\mu(A_k) < \int_{A_k} |g|^q \,\mathrm{d}\mu < \infty$ since $g \in L_q$).

Now we can proceed:

$$a := \int_{A_k B} (f_n - f)g \,\mathrm{d}\mu, \quad b := \int A_k^c \cup B^c (f_n - f)g \,\mathrm{d}\mu.$$

Now $|a| \le \hat{\varepsilon} k^{\frac{1}{q}} \mu(A_k) < \varepsilon$.

$$\left| \int_X (f_n - f) g \mathbf{I}_{A_k^c \cup B^c} \,\mathrm{d}\mu \right| \le \|f_n - f\|_p \|g \mathbf{I}_{A_k^c \cup B^c}\|_q \le 2M \left(\int_{A_k^c} |g|^q \,\mathrm{d}\mu + \int_{A_k \backslash B} |g|^q \,\mathrm{d}\mu \right).$$

By LDC(Dominated convergence), $A_k^c \to \{g = 0, \infty\}$, so $\int_{A_k^c} |g|^q \,\mathrm{d}\mu < \varepsilon$.

Since $\mu(A_k) < \infty$, $f_n \to f, a.e. \implies f_n \xrightarrow{\mu} f$. By the continuity of integrals, $\mu(A_k \backslash B) \le \mu(B^c) < \delta \implies \int_{A_k \backslash B} |g|^q \,\mathrm{d}\mu < \varepsilon$.

Now we can conclude: $\forall \varepsilon > 0$, first choose k large, then $\hat{\varepsilon}$ small, we get

$$\int_X (f_n - f)g \,\mathrm{d}\mu \le \varepsilon + 4M\varepsilon \implies f_n \xrightarrow{(w)L_p} f.$$

□

Remark 0.3.7 — The proof is a little complicated, we divide the entire integral to three part, and estimate them respectively.

When $p = 1$, f_n bounded in L_p *cannot* imply weak convergence.

Example 0.3.8

Let $X = \mathbb{N}$, $\mu(\{k\}) = 1, \forall k$, clearly it's σ-finite.

Let $f_n(k) = \mathbf{I}_{k=n}$, then $\|f_n\| = \sum_k \mu(k)|f_n(k)| = 1$, and $f_n \to 0, a.e.$.

But let $g = 1 \in L_\infty$, $\int_X (f_n - f)g \,\mathrm{d}\mu = 1 \not\to 0$.

Proposition 0.3.9

Let $f_1, f_2, \cdots \in L_1$, then:

$$\|f_n\| \to \|f\| \& f_n \to f, a.e. \implies f_n \xrightarrow{L_1} f \implies f_n \xrightarrow{(w)L_1} f \implies \int_A f_n \, \mathrm{d}\mu \to \int_A f \, \mathrm{d}\mu, \forall A.$$

Proof. For the last part let $g = \mathbf{I}_A$, the rest is trivial. □

§0.4 Integrals in probability space

We can also consider L_p space in probability space $(\Omega, \mathscr{F}, P)$.

Theorem 0.4.1

Let $0 < s < t < \infty$. Then $L_t \subset L_s$. If $s \geq 1$, we have $\|f\|_s \leq \|f\|_t$, with equality f constant.

Proof. When $f \in L_t$, let $p = \frac{t}{s}, q = \frac{t}{t-s}$.

$$\int_\Omega |f|^s \cdot 1 \, \mathrm{d}P \leq \||f|^s\|_p \|1\|_q = (E|f|^{sp})^{\frac{1}{p}} = (E|f|^t)^{\frac{1}{p}}.$$

So $f \in L_s \implies L_t \subset L_s$. When $s \geq 1$,

$$\|f\|_s^s \leq (\|f\|_t)^{\frac{t}{p}} = \|f\|_t^s \implies \|f\|_s \leq \|f\|_t.$$

□

From this we know $L_\infty \subset L_p$, and $\|f\|_p \uparrow \|f\|_\infty$.

Remark 0.4.2 — This theorem does not hold for general space. Let $X = \mathbb{N}$, $\mu(\{n\}) = 1$, $f(n) = \frac{1}{n}$, then $f \in L_2 \backslash L_1$.

The expectation Ef^k is called **k-order moment** of random variable f.

Definition 0.4.3 (Uniformly integrable)**.** Let $\{f_t, t \in T\}$ be r.v.'s, if $\forall \varepsilon > 0, \exists \lambda > 0$, such that

$$E|f_t|\mathbf{I}_{\{|f_t| > \lambda\}} < \varepsilon, \quad \forall t \in T,$$

then we say $\{f_t, t \in T\}$ **uniformly integrable**.

If $\forall \varepsilon > 0, \exists \delta > 0$ s.t $\forall A \in \mathscr{F}$,

$$P(A) < \delta \implies E|f_t|\mathbf{I}_A < \varepsilon, \forall t \in T,$$

we say $\{f_t\}$ is uniformly absolutely continuous, which is abbreviated as absolutely continuous.

Theorem 0.4.4

Uniformly integrable $\iff$ absolute continuity and bounded in L_1.

Proof. Firstly when $\{f_t\}$ uniformly integrable, $\forall A \in \mathscr{F}, \lambda > 0$,

$$\begin{aligned} E|f_t|\mathbf{I}_A &= E|f_t|\mathbf{I}_{A\cap\{|f_t|\le\lambda\}} + E|f_t|\mathbf{I}_{A\cap\{|f_t|>\lambda\}} \\ &\le \lambda P(A) + E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \end{aligned}$$

Let $A = X$ we know $E|f_t| \le \lambda + \frac{\varepsilon}{2}, \forall t \in T$. Now let $\delta = \frac{\varepsilon}{2\lambda}$ we get AC property.

On the other hand,

$$\lambda P(|f_t| > \lambda) \le E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \le E|f_t| \le M, \forall t \in T.$$

So when $\lambda > \frac{M}{\delta}$, $P(|f_t| > \lambda) < \delta$, hence $E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \le \varepsilon$, $\forall t \in T$. □

Theorem 0.4.5

Let $0 < p < \infty$, and $f_n \to f$ in probability. TFAE:

(1) $\{|f_n|^p\}$ uniformly integrable;

(2) $f_n \xrightarrow{L_p} f$;

(3) $f \in L_p$ and $\|f_n\|_p \to \|f\|_p$.

Proof. (1) $\implies$ (2): Take subsequence $f_{n'} \to f, a.s.$,

$$E|f|^p \le \liminf_{n\to\infty} E|f_n|^p < \infty,$$

since $\{|f_n|^p\}$ is bounded in L_1. This means $f \in L_p$.

Let $A_n = \{|f_n - f| > \varepsilon\}$, now we compute

$$E|f_n - f|^p \le \varepsilon^p + E|f_n - f|^p\mathbf{I}_{A_n} \le \varepsilon^p + C_pE|f_n|^p\mathbf{I}_{A_n} + C_pE|f|^p\mathbf{I}_{A_n}$$

Since $P(A_n) \to 0$ and $\{|f_n|^p\}$ absolutely continuous (also note $E|f|^p\mathbf{I}_{A_n} \to 0$), RHS converges to 0. Therefore $f_n \xrightarrow{L_p} f$.

As for (3) $\implies$ (1), we'll prove a lemma:

Lemma 0.4.6

If $f_n \xrightarrow{P} f$, then $\forall 0 < p < \infty$,

$$|f_n|^p\mathbf{I}_{\{|f_n|\le\lambda\}} \xrightarrow{P} |f|^p\mathbf{I}_{\{|f|\le\lambda\}}, \quad \forall\lambda \in C(F_{|f|}).$$

By lemma and bounded convergence theorem, their expectation also converges. Note that $\|f_n\|_p \to \|f\|_p$, so

$$E|f_n|^p\mathbf{I}_{\{|f_n|>\lambda\}} \to E|f|^p\mathbf{I}_{\{|f|>\lambda\}},$$

thus $\forall\varepsilon > 0, \exists\lambda_0 \in C(F_{|f|})$, s.t. $E|f|^p\mathbf{I}_{\{|f|>\lambda_0\}}$ TODO!

Proof of the lemma. Since $|f_n| \to |f|$ in probability, WLOG $f_n, f \ge 0$. Define

$$A_n := \{f_n \le \lambda\}\Delta\{f \le \lambda\} \cap \{|f_n^p - f^p| < \varepsilon\}$$

$$B_n := \{f_n, f \le \lambda, |f_n^p - f^p| < \varepsilon\}.$$

Since x^p is uniformly continuous in $[0, \lambda]$, $B_n \subset \{|f_n - f| > \kappa_{\varepsilon,\lambda}\}$, $P(B_n) \to 0$.

Also $P(A_n) \to 0$ as

$$A_n \subset \{\lambda - \delta < f \leq \lambda + \delta\} \cup \{|f_n - f| \geq \delta\},$$

and $F_{|f|}$ continuous at λ. $\square$

$\square$