Linear Algebra II

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Contents

Let $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}$, and $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}$. In the language of groups, $\mathrm{SO}(n)$ has only 2 coset in $\mathrm{O}(n)$, while the structure of the cosets of $\mathrm{SU}(n)$ in $\mathrm{U}(n)$ look like S^1.

Example 0.0.1

Let's look at some low dimensional orthogonal groups. $\mathrm{O}(1) = \{1, -1\}$, $\mathrm{SO}(1) = \{1\} = \mathrm{SU}(1)$, $\mathrm{U}(1) = \{z \mid |z| = 1\}$.

The group $\mathrm{SO}(2)$ is the rotations of $\mathbb{R}^2$:

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$

While $\mathrm{O}(2)$ also consisting of reflections.

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}.$$

In fact these groups are *lie groups*, which means they have the structure of differential manifolds. It's clear that $\mathrm{U}(1) \simeq \mathrm{SO}(2) \simeq S^1$, and we can see $\mathrm{SU}(2) \simeq S^3$.

Theorem 0.0.2 (QR-decomposition)

Any invertible matrix A can be uniquely decomposed to $Q \cdot R$, where $Q \in \mathrm{O}(n)$, R is an upper triangular matrix with positive diagonal entries. When $F = \mathbb{C}$, $\mathrm{O}(n)$ is replaced by $\mathrm{U}(n)$.

Proof. This is essentially Schmidt orthogonalozation. □

Corollary 0.0.3 (Ivasawa decomposition, KAN decomposition)

For all $A \in \mathrm{GL}_n(\mathbb{R})$, it has a unique decomposition $A = A_k A_a A_n$, $A_k \in \mathrm{O}(n)$, A_a is diagonal, A_n is upper triangular matrix with diagonal entries 1. It also holds for $\mathbb{C}$.

Let $\mathcal{B}, \mathcal{B}'$ be orthonormal bases of V, $T \in L(V)$. We know that $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P$ for some $P \in \mathrm{GL}(V)$. By orthogonality, P must be an orthogonal matrix, wich means $P^t = P^{-1}$.

Definition 0.0.4. Let $A, B \in \mathbb{R}^{n\times n}$, we say they are **orthogonally similar** if $A = P^{-1}BP$ for some $P \in \mathrm{O}(n)$. The name is changed to **unitarily similar** for complex matrices.

Theorem 0.0.5 (Schur triangularization theorem)
Let $F = \mathbb{C}, T \in L(V)$. There exists an orthonormal basis $\mathcal{B}$, such that $[T]_{\mathcal{B}}$ is upper triangular.

Proof. Recall that T is triangulable (which is always true in $\mathbb{C}$) iff there exists a T-invariant flag $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = V$. We can take an orthonormal basis s.t. $W_k = \mathrm{span}\{\alpha_1, \ldots, \alpha_k\}$.
Obviously T is upper triangular under this basis. $\square$

§0.1 Normal maps

Recall that we say two matrices A, B are orthogonally similar, if there exist $P \in \mathrm{O}(n)$ s.t. $B = P^{-1}AP$. Again, we want to find the "simpliest" matrix in each orthogonal equivalent class.
Let $T \in L(V)$ be a linear map, if there exists an orthonomal basis of V s.t. $[T]_{\mathcal{B}}$ is diagonal, then we say T is orthogonally (or unitarily) diagonalizable.

Definition 0.1.1 (Normal maps)**.** Let V be an inner product space, $T \in L(V)$. If $TT^* = T^*T$, then we say T is a **nomal map**.

It turns out that these concepts has close relations:

Theorem 0.1.2
Let V be a finite dimentional inner product space,

- If $F = \mathbb{R}$, then T orthogonally diagonalizable $\iff$ T self-adjoint;
- If $F = \mathbb{C}$, then T unitarily diagonalizable $\iff$ T normal.

Lemma 0.1.3
Let $F = \mathbb{C}$, then T normal $\iff$ there exists self-adjoint commutative maps T_1, T_2 s.t. $T = T_1 + iT_2$.

Proof. If $T = T_1 + iT_2$, then $T^* = T_1 - iT_2$, so $T^*T = TT^*$ since T_1, T_2 commutative.
On the other hand, let $T_1 = \frac{T+T^*}{2}$, $T_2 = \frac{T-T^*}{2i}$. We can check that T_1, T_2 self-adjoint and are commutative. $\square$

Proof of Theorem 0.1.2. For the " $\Longrightarrow$ " part, let $\mathcal{B}$ be an orthonomal basis such that $[T]_{\mathcal{B}} = \mathrm{diag}\{c_1, \ldots, c_n\}$. Then we have

$$[T^*]_{\mathcal{B}} = [T]^*_{\mathcal{B}} = \mathrm{diag}\{\bar{c}_1, \ldots, \bar{c}_n\}.$$

If $F = \mathbb{R}$, then $T^* = T$, i.e. T self-adjoint.
If $F = \mathbb{C}$, clearly $TT^* = T^*T$, so T is normal.
As for the other part, we need a lemma first.

Lemma 0.1.4

Let V be a f.d. inner product space, $T \in L(V)$. If $W \subset V$ is a T-invariant space, then $W^\perp$ is T^*-invariant.

Proof of the lemma. For all $\alpha \in W^\perp$,

$$0 = \langle \alpha, T\beta \rangle = \langle T^*\alpha, \beta \rangle, \quad \forall \beta \in W.$$

Thus $T^*\alpha \in W^\perp$. □

Corollary 0.1.5

If T is self-adjoint, $W \subset V$ is T-invariant will imply $W^\perp$ is also T-invariant, so T is semisimple.

Lemma 0.1.6

Let V be a f.d. inner product space, $T \in L(V)$ is self-adjoint. We must have $f_T \in \mathbb{R}[x]$, and it can be decomposed to products of polynomials of degree 1.

In particular, $\sigma(T) \subset \mathbb{R}$.

Proof. Let $f_T = \prod_{j=1}^{n}(x - c_j), c_j \in \mathbb{C}$.

Let $\mathcal{B}$ be an orthonomal basis of V, then $A := [T]_\mathcal{B}$ is Hermite. Let X be a nonzero vector s.t. $AX = c_jX$, then

$$c_jX^*X = X^*AX = (AX)^*X = \bar{c}_jX^*X.$$

So $c_j \in \mathbb{R}$, and we're done. □

Lemma 0.1.7

If T is a self-adjoint map, then all the eigenspaces of T are pairwise orthogonal.

Proof. Let $c_1, c_2 \in \mathbb{R}$ be two eigenvalues of T. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$.

$$c_1 \langle \alpha, \beta \rangle = \langle c_1\alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T\beta \rangle = \bar{c}_2 \langle \alpha, \beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

Since $c_1 \neq c_2$, we must have $\alpha \perp \beta$, as desired. □

Returning back to Theorem 0.1.2, when T is self-adjoint, let $\sigma(T) = \{c_1, \ldots, c_r\}$.

Claim 0.1.8. $V = \bigoplus_{i=1}^r V_{c_i}$, i.e. T is diagonalizable.

Let $W = \bigoplus_{i=1}^r V_{c_i}$, if $W^\perp \neq \{0\}$, then $W^\perp$ is T-invariant.

When $F = \mathbb{C}$, then $T_{W^\perp}$ has eigenvectors; when $F = \mathbb{R}$, then $T_{W^\perp}$ is self-adjoint, so it must have a eigenvector (by lemma).

Since V_{c_i} are pairwise orthogonal, so we can actually take an orthonomal basis of V_{c_i} to get an orthonomal basis of V. Hence T is orthogonally diagonalizable.

Now for the case when T is normal, let T_1, T_2 be self-adjoint maps s.t. $T = T_1 + iT_2$. Since T_1, T_2 commute, the proof is nearly identical to the simutaneously diagonalizable property.

Let $V = \bigoplus_{i=1}^r V_{c_i}$ be the eigenspace decomposition of T_1. Note that V_{c_i} are also T_2-invariant.

Since $(T_2)_{V_{c_i}}$ self-adjoint, $(T_2)_{V_{c_i}}$ is unitarily diagonalizable. Therefore we can concatenate those basis to get a basis of V, and T_1, T_2 are both diagonal under this basis. $\square$

There's another proof of " $\Longleftarrow$ " part of the theorem:

Proposition 0.1.9

Let V be an inner product space, $T \in L(V)$ normal. Let $W \subset V$ be a T-invariant space, then $W^\perp$ is T-invariant, and W is T^*-invariant.

Proof. Take an orthonomal basis of $W, W^\perp$, so $A := [T]_{\mathcal{B}}$ normal.

Since W is T-invariant, $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Note that:

$$AA^* = \begin{pmatrix} BB^* + CC^* & * \\ * & * \end{pmatrix}, \quad A^*A = \begin{pmatrix} B^*B & * \\ * & * \end{pmatrix}.$$

As A normal, $BB^* + CC^* = B^*B$, by looking at the trace of both sides, we get $\operatorname{tr}(CC^*) = 0 \implies C = 0$, the conclusion follows. $\square$

Corollary 0.1.10

Let $A \in \mathbb{C}^{n\times n}$ be an upper triangular martix, then A normal $\iff$ A diagonal.

Proposition 0.1.11

Let T be a normal map, then the eigenspaces of T are pairwise orthogonal.

Proof. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$, since $\operatorname{span}\{\beta\}$ is a T-invariant space, so $T^*\beta \in \operatorname{span}\{\beta\}$,

$$\langle T^*\beta, \beta\rangle = \langle \beta, T\beta\rangle = \overline{c}_2 \langle \beta, \beta\rangle .$$

Thus $T^*\beta = \overline{c}_2\beta$.

$$c_1 \langle \alpha, \beta\rangle = \langle T\alpha, \beta\rangle = \langle \alpha, T^*\beta\rangle = c_2 \langle \alpha, \beta\rangle .$$

But $c_1 \neq c_2$, we have $\alpha \perp \beta$. $\square$

When $F = \mathbb{C}$: Let $W = \bigoplus_{i=1}^r V_{c_i}$. Since $W^\perp$ is T-invariant, so when $W \perp \neq \{0\}$, T must have eigenvalues in $W^\perp$, contradiction!

Now we've proved that V_{c_i} are pairwise orthogonal, so T is unitarily diagonalizable.

Proposition 0.1.12

Let V be a complex inner product space, $T \in L(V)$ normal,

- T self-adjoint $\iff$ $\sigma(T) \subset \mathbb{R}$;
- T anti self-adjoint $\iff$ $\sigma(T) \subset i\mathbb{R}$;
- T unitary $\iff$ $\sigma(T) \subset \{z : |z| = 1\}$.

Proof. Take an orthonomal basis s.t. $[T]_{\mathcal{B}}$ diagonal. The rest is trivial. □

§1 Bilinear forms

Let V be a finite dimensional vector space, $\dim V = n$.

Definition 1.0.1. Let $F = \mathbb{C}$, we say a function $f : V \times V \to V$ is a **semi bilinear form** if:

- $f(c_1\alpha + \beta, \gamma) = c_1 f(\alpha, \gamma) + f(\beta, \gamma)$;
- $f(\alpha, c_1\beta + \gamma) = \bar{c}_1 f(\alpha, \beta) + f(\alpha, \gamma)$.

Let $\mathrm{Form}(V)$ denote the (semi) bilinear forms on (complex) real vector space V.

For $f \in \mathrm{Form}(V)$, fix a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of V, let $[f]_{\mathcal{B}} \in F^{n\times n}$ be the matrix

$$([f]_{\mathcal{B}})_{jk} = f(\alpha_k, \alpha_j).$$

which is called **the matrix of f under $\mathcal{B}$**.

For $\alpha = \sum_{k=1}^n x_k\alpha_k, \beta = \sum_{j=1}^n y_j\alpha_j \in V$. It's clear that

$$f(\alpha, \beta) = \sum_{j,k=1}^n x_k\bar{y}_j f(\alpha_k, \alpha_j) = \sum_{j,k=1}^n x_k\bar{y}_j([f]_{\mathcal{B}})_{jk} = [\beta]_{\mathcal{B}}^*[f]_{\mathcal{B}}[\alpha]_{\mathcal{B}}.$$

From this we know that the map $\mathrm{Form}(V) \to F^{n\times n}, f \mapsto [f]_{\mathcal{B}}$ is a linear isomorphism. Since if $[f]_{\mathcal{B}} = 0$, then $f(\alpha, \beta) = 0$ for all $\alpha, \beta \in V$. Thus it's injective. Obviously it's surjective and linear, so

$$\dim \mathrm{Form}(V) = n^2$$

Example 1.0.2

Let $A \in F^{n\times n}$. Let $f \in \mathrm{Form}(F^{n\times 1})$ be

$$f(X, Y) = Y^*AX, \quad \forall X, Y \in F^{n\times 1}.$$

Let $\mathcal{B}$ be the standard basis of F, it's clear that $[f]_{\mathcal{B}} = A$.

Proposition 1.0.3

Let $\mathcal{B}' = \{\alpha_1', \dots, \alpha_n'\}$ be another basis of V, $P \in \mathrm{GL}_n(F)$ satisfies

$$(\alpha_1', \dots, \alpha_n') = (\alpha_1, \dots, \alpha_n)P.$$

Then $[f]_{\mathcal{B}'} = P^*[f]_{\mathcal{B}}P$.

Proof. Since $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$, just plug this into the definition of $[f]_{\mathcal{B}}$, the rest is trivial. □

Definition 1.0.4. Let $f \in \mathrm{Form}(V)$.

- When $F = \mathbb{R}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = f(\beta, \alpha)$, then we say f is symmetrical (also called Hermite);
- When $F = \mathbb{C}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, we say f is Hermite.

Proposition 1.0.5

When $F = \mathbb{C}$, f Hermite $\iff f(\alpha, \alpha) \in \mathbb{R}, \forall \alpha \in V$.

Proof. For the " $\Longleftarrow$ " direction, consider $f(\alpha+\beta, \alpha+\beta) \in \mathbb{R}$. Expanding we'll get $f(\alpha, \beta) + f(\beta, \alpha) \in \mathbb{R}$, i.e.

$$f(\alpha, \beta) + f(\beta, \alpha) = \overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Replace α with $i\alpha$, we get

$$f(\alpha, \beta) - f(\beta, \alpha) = -\overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Combining two equations we get the conclusion. □

Definition 1.0.6. Let $f \in \mathrm{Form}(V)$ be an Hermite form. If $\forall \alpha \in V \backslash \{0\}$, $f(\alpha, \alpha) > 0$, we say f is **positive definite**.

Similarly we can define negative definite and semi positive definite.

Note that a positive definite Hermite form is nothing but an inner product.

§1.1 Positive define matrices

In this section we'll dig deeper into properties of positive definite matrices.

It's clear that if a matrix A is positive definite, then A is inversible, and P^*AP is also positive definite. In particular, P^*P is positive definite.

Theorem 1.1.1 (Cholesky decomposition)

Let $A \in F^{n\times n}$ be a positive definite matrix, there exists a unique upper triangular matrix R with positive diagonal entries s.t. $A = R^*R$.

Proof. Consider the inner product $f(X, Y) = Y^*AX$. Let the standard inner product on V be $f_0(X, Y) = Y^*X$.

Since inner product spaces with same dimensions are isomorphic, so there exists a matrix $R \in \mathrm{GL}_n(F)$, such that

$$R : (F^{n\times 1}, f) \to (F^{n\times 1}, f_0), \quad X \mapsto RX$$

is an isomorphism of inner product space, i.e. $f_0(RX, RY) = f(X, Y)$. This is equivalent to $A = R^*R$.

For any $P \in GL_n(F)$, P is also an isomorphism of $(F^{n\times 1}, f) \to (F^{n\times 1}, f_0)$ iff RP^{-1} preserves the inner product f_0, iff $RP^{-1} \in \mathrm{O}(n)$ or $\mathrm{U}(n)$.

By QR decomposition, $R = RP^{-1} \cdot P$, so there must be a unique P s.t. P upper triangular with positive diagonal entries. □

Corollary 1.1.2

A positive definite $\implies \det A > 0$.

Definition 1.1.3. Let $A \in F^{n\times n}$, for $1 \le k \le n$, define

$$\Delta_k(A) := \det(A^{1\le j\le k}_{1\le i\le k})$$

be the **leading principal minor**.

Theorem 1.1.4

Let $A \in F^{n\times n}$ be an Hermite matrix. Then A positive definite $\iff \Delta_k(A) > 0$, $k = 1, \dots, n$.

Lemma 1.1.5 (LU decomposition)

Let F be *any field*. For $A \in \mathrm{GL}_n(F)$, the followings are equivalent:

- $\Delta_k(A) \neq 0, k = 1, \dots, n$;
- $A = LU$, where L lower triangular, and U upper triangular with diagonal entries 1.

Proof. On one hand, Let L_k, U_k be the top-left $k \times k$ submatrix of L, U, since L, U inversible, L_k, U_k inversible. By the triangular property, $\Delta_k(A) = \det(L_k U_k) \neq 0$.

On the other hand, it's sufficient to prove:

$$\exists N \text{ strictly upper triangular}, A(N + I_n) \text{ lower triangular}$$

Let A_k be the k-th leading principal submatrix of A, and $\alpha_{k+1}, \beta_{k+1} \in F^{n\times 1}$ the $(k+1)$-th column of A, N.

Now compute the first k rows of the $(k+1)$-th column of $A(N+I)$, which is equal to $A_k\beta'_{k+1} + \alpha'_{k+1}$, where $\alpha'_{k+1}, \beta'_{k+1}$ is the first k entries of $\alpha_{k+1}, \beta_{k+1}$.

Since A_k inversible, $\exists \beta'_{k+1}$ s.t. $A_k\beta'_{k+1} + \alpha'_{k+1} = 0$.

Hence these β'_{k+1} forms a strictly upper triangular matrix N, as desired. □

Proof of the theorem. Let A be an Hermite matrix, if A positive definite, then $\det A \geq 0$.

Let A_k be the upper left $k \times k$ submatrix of A. For $X \in F^{k\times 1} \backslash \{0\}$, we have

$$X^* A_k X = \begin{pmatrix} X \\ 0 \end{pmatrix}^* A \begin{pmatrix} X \\ 0 \end{pmatrix} > 0.$$

Hence A_k positive definite, $\det A_k = \Delta_k(A) \geq 0$.

Conversely, by our lemma let $A = LU$, let $D = (U^*)^{-1}L$, $A = U^*DU$.

Hence A Hermite $\implies D$ Hermite. Moreover D is lower triangular, so D is diagonal.

Some computation yields that $A_k = U_k^* D_k U_k$. Therefore

$$\Delta_k(A) \geq 0 \implies \det(U_k^* D_k U_k) \geq 0 \implies \det D_k \geq 0.$$

From this we deduce that all the diagonal entries of D are positive, so D positive definite $\implies A$ positive definite. □

§1.2 Bilinear forms on inner product spaces

Let V be an inner product space, given a basis of V, recall that there are two linear isomorphism:

$$\mathrm{Form}(V) \to F^{n\times n}, f \mapsto [f]_{\mathcal{B}} \quad L(V) \to F^{n\times n}, T \mapsto [T]_{\mathcal{B}}$$

Hence we can define a map $\mathrm{Form}(V) \to L(V)$ by composing these two isomorphism. Denote this map by $f \mapsto T_f$. It seems like this map also depends on the choice of the basis, but in fact it's independent as long as $\mathcal{B}$ is orthonormal!

Let $\mathcal{B}'$ be another orthonormal basis, then $[T_f]_{\mathcal{B}'} = P^{-1}[T_f]_{\mathcal{B}}P$, while $[f]_{\mathcal{B}'} = P^*[f]_{\mathcal{B}}P$, but P is orthogonal (or unitary), so $P^{-1} = P^*$, i.e. T_f doesn't change under the new basis.

Since T_f do not depend on the basis, thus we wonder whether we can define this map intrinsically.

Proposition 1.2.1

For all $T \in L(V)$, T induces a (semi) bilinear form $f(\alpha, \beta) = \langle T\alpha, \beta\rangle$. We claim that this map $\mathcal{F}$ gives an isomorphism of $L(V)$ and $\mathrm{Form}(V)$.

Proof. Clearly $\mathcal{F}$ is injective:

$$\langle T\alpha, \beta\rangle = 0, \forall\beta \implies T\alpha = 0,$$

thus $\ker \mathcal{F} = \{0\}$.

By dimenional reasons $\mathcal{F}$ must be an isomorphism. □

By considering $\mathcal{F}^{-1}$, we get an one-to-one map $f \mapsto T_f$ s.t.

$$f(\alpha, \beta) = \langle T_f\alpha, \beta\rangle .$$

We'll see that this definition coincide with the initial one. In fact it's sufficient to prove $[T_f]_{\mathcal{B}} = [f]_{\mathcal{B}}$, which is just a bunch of computation ;)

Remark 1.2.2 — We can construct T_f explicitly from f:

The inner product gives a conjugate linear isomorphism

$$\Phi : V \to V^*, \quad \Phi(\alpha)(\beta) = \langle \beta, \alpha\rangle = \overline{\langle \alpha, \beta\rangle}.$$

Similarly, $f \in \mathrm{Form}(V)$ gives a conjugate linear map

$$\Phi_f : V \to V^*, \quad \Phi_f(\alpha)(\beta) = \overline{f(\alpha, \beta)}.$$

Then $T = \Phi^{-1} \circ \Phi_f$ is the desired linear map:

$$\langle T\alpha, \beta\rangle = \overline{\Phi(T\alpha)(\beta)} = \overline{\Phi_f(\alpha)(\beta)} = f(\alpha, \beta).$$

Hence all the properties of linear maps can be carried over to the forms, and vice versa (using the matrix representation).

Corollary 1.2.3

Let $F = \mathbb{C}$, $T \in L(V)$, T self-adjoint iff $\langle T\alpha, \alpha\rangle \in \mathbb{R}, \forall\alpha \in V$.

Proof. T self-adjoint iff f Hermite iff $f(\alpha, \alpha) \in \mathbb{R}$. □

Corollary 1.2.4

Let $f \in \mathrm{Form}(V)$.

- If f Hermite, there exists an orthonormal basis of V s.t. $[f]_{\mathcal{B}}$ is real diagonal.
- If $F = \mathbb{C}$, there exists an orthonormal basis such that $[f]_{\mathcal{B}}$ upper triangular.

§1.3 Spectral decomposition

Theorem 1.3.1 (Spectral decomposition of normal maps)

Let $T \in L(V)$ be a self-adjoint map (or normal map in complex field), let $\sigma(T) = \{c_1, \ldots, c_k\}$, $P_i \in L(V)$ are the projection onto V_{c_i}. Then for any $f \in F[x]$, we have

$$f(T) = \sum_{i=1}^{k} f(c_i) P_i.$$

In particular, $T = \sum_{i=1}^{k} c_i P_i$.

Proof. Consider the orthogonal direct sum

$$V = \bigoplus_{i=1}^{k} V_{c_i},$$

since previously we've proven that T is orthogonally diagonalizable (or unitarily diagonalizable).

Using this decomposition, the conclusion is somewhat trivial. □

Corollary 1.3.2

Each P_i is a polynomial of T.

Proof. Take $f_i \in F[x]$ s.t. $f_i(c_i) = \delta_{ij}$. Then $f_i(T) = \sum_{j=1}^{k} f_i(c_j) P_j = P_i$. □

Using simialr technique, we can consider functions other than polynomials of T, defined by $\phi(T) = \sum_{i=1}^{k} \phi(c_i) T$. By Lagrange interpolation, we can always find a polynomial p s.t. $p(c_i) = \phi(c_i)$ for all $c_i \in \sigma(T)$.

Example 1.3.3

If T semi positive definite normal matrix, $\sigma(T) \subset [0, +\infty)$, so we can define $\sqrt{T} = \sum_{i=1}^{k} \sqrt{c_i} P_i$.

Proposition 1.3.4

T self-adjoint (normal) $\implies$ $\phi(T)$ self-adjoint (normal); $\sigma(\phi(T)) = \phi(\sigma(T))$.

Proof. Note that T and $\phi(T)$ are diagonal matrices under orthonormal basis of V_{c_i}. □