Mathematical Analysis II

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§0.1 Stolkes' formula

Intuitively, Stolkes' formula states that: Let D be a region, $\mathrm{d}\omega$ be a differential form, then

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

Here ∂D means the "boundary" of D.

Of course we need some "regularity" requirements of D and ω, and it's the generalization of Newton-Lebniz formula into higher dimensions.

Definition 0.1.1 (Bounded regions with boundary)**.** Let $\Omega \subset \mathbb{R}^n$ be a compact set, we say it's a **bounded region with boundary** if $\forall x \in \partial\Omega$, there exists open sets $U, V \subset \mathbb{R}^n$, $x \in U$ and a continuous homeomorphism $\Phi : U \to V$, such that

$$\Phi(U \cap \Omega) = V \cap \{x_n \geq 0\}, \quad \Phi(U \cap \partial\Omega) = V \cap \{x_n = 0\}.$$

If Φ is also C^1, we say $x \in \partial\Omega$ is a **regular point**, otherwise a **singular point**.

Lemma 0.1.2

Let Ω be a bounded region with boundary, for all regular $p \in \partial\Omega$, there exists a unique unit vector $\nu(p) \in \mathbb{R}^n$, and $\varepsilon > 0$, s.t.

$$\nu(p) \perp T_p\partial\Omega, |\nu(p)| = 1, p - t\nu(p) \in \Omega, \forall 0 < t < \varepsilon.$$

We call $\nu(p)$ the **outward unit normal vector** of p.

Proof. By the definition of regular points, we may assume that:

$$\Omega \cap V = \{x \in V \mid f(x) \geq 0\}, \quad \partial\Omega \cap V = \{x \in V \mid f(x) = 0\}.$$

Where f is a C^1 function.

Since ∇f is nonzero, the tangent space $T_p\partial\Omega = \{v \mid v \cdot \nabla f = 0\}$.

Let $\nu(p) = -\frac{\nabla f}{|\nabla f|}$, then it's obvious $\nu(p)$ points outside of Ω. □

Now for a cuboid I and a C^1 function ϕ,

$$\begin{aligned}\int_I \frac{\mathrm{d}\phi}{\mathrm{d}x_n}\,\mathrm{d}x &= \int_{I_{n-1}} \phi(x_1,\dots,x_{n-1},b_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1} - \int_{I_{n-1}} \phi(x_1,\dots,x_{n-1},a_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\\ &= \int_{\partial I} \phi\cdot\nu_n\,\mathrm{d}\sigma.\end{aligned}$$

Where σ is the measure on the boundary, ν is the outward unit normal vector.

Lemma 0.1.3

Let K be a compact set in $\mathbb{R}^n$, $U \supset K$ is open, there exists a smooth function f such that $\operatorname{supp} f \subset U$, and $f\big|_K > 0$.

Proof. Let $\rho(x)$ be a smooth function s.t. $\rho(x) = 1$ for $|x| \le 1$ and $\rho(x) = 0$ for $|x| \ge 2$. Let

$$g(x) = \int_{|y|\le 2} f(x-\delta y)\rho(y)\,\mathrm{d}y.$$

Then g is a smooth non-negative function. □

Theorem 0.1.4 (Unit decomposition on compact sets)

Let K be a compact set, $\{U_1,\dots,U_k\}$ is an open covering of K. There exists smooth functions $f_1,\dots,f_k$ s.t.

$$1 = f_1(x) + f_2(x) + \cdots + f_k(x), \quad \operatorname{supp} f_i(x) \subset U_i.$$

Proof. For $1 \le i \le k$, $\delta > 0$, define

$$K_i^\delta = \{x \in U_i \mid d(x, U_i^c) > \delta\}.$$

Note that $\{\bigcup_{i=1}^k K_i^{\frac{1}{m}}\}_{m=1}^\infty$ is also an open covering of K, thus there exists N s.t.

$$K \subset \bigcup_{i=1}^k K_i^{\frac{1}{N}}.$$

Hence by lemma we have g_i s.t. $\operatorname{supp} g_i \subset U_i$ and $g_i > 0$ on the closure of $K_i^{\frac{1}{N}}$. Similarly we have a smooth function g s.t. $g(x) = 0$ on K, and $g > 0$ outside of the closure of $\bigcup_{i=1}^k K_i^{\frac{1}{N}}$.

Let $G(x) = g_1(x) + \cdots + g_k(x) + g(x) > 0$ on $\bigcup_{i=1}^k U_i$, then we can define $f_i(x) = \frac{g_i(x)}{G(x)}$ which satisfy the condition. □

Theorem 0.1.5

Let Φ be a C^1 homeomorphism from a cuboid I to Ω, then Ω satisfies Stolkes' formula:

$\forall \phi \in C^1(\mathring{D}) \cap C(\overline{D})$, we have

$$\int_D \nabla\phi\,\mathrm{d}x = \int_{\partial D} \phi\nu\,\mathrm{d}\sigma.$$

Proof. Since $\Omega = \Phi(I)$, let y be the coordinates on I, $x = \Phi(y)$,

$$\int_{\Phi(I)} \nabla\varphi \,\mathrm{d}x = \int_I \nabla\varphi(\Phi(y))(\mathrm{d}\Phi)^{-1} J_\Phi \,\mathrm{d}y.$$

Let $A = \mathrm{d}\Phi$, WLOG $J_\Phi > 0$. Using the index notation and Einstein summation,

$$A_{kj}A^{ji} = A^{kj}A_{ji} = \delta_{ki}.$$

Thus

$$\partial_{y_j}\varphi A^{ji}|A| = \partial_{y_j}(\varphi A^{ji}|A|) - \varphi\partial_{y_j}(A^{ji}|A|)$$

Since $|A| = A_{kl}A^{kl}|A|$, $A_{kl} = \frac{\partial\Phi_k}{\partial y_l}$.

$$\begin{aligned}\partial_{y_j}(A^{ji}|A|) &= |A|\partial_{y_j}A^{ji} + A^{ji}\partial_{y_j}|A| \\ &= |A|\partial_{y_j}A^{ji} + A^{ji}|A|\partial_{y_j}A_{kl}A^{kl} \\ &= |A|(\partial_{y_j}A^{ji} + \partial_{y_l}A_{kj}A^{kl}) \\ &= |A|(\partial_{y_j}A^{ji} - \partial_{y_j}A^{ji}) = 0.\end{aligned}$$

Hence by our previous work,

$$\int_I \partial_{y_j}(\varphi A^{ji}|A|)\,\mathrm{d}y = \int_{\partial I} \varphi A^{ji}|A|\nu_j \,\mathrm{d}\sigma.$$

Putting this together for all i's, note that $\widetilde{\nu} = \frac{\nabla\Phi_n^{-1}}{|\nabla\Phi_n^{-1}|}$,

TODO $\square$

Let $(\phi_1, \ldots, \phi_n)$ be an element in the tangent boundle TM, it can represent a vector field

$$X = (\phi_1, \ldots, \phi_n) = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}.$$

Here $X \in TM, X(p) \in T_pM$.

We define the **divergence** of X to be

$$\operatorname{div}(X) = \sum_{i=1}^n \frac{\partial\phi_i}{\partial x_i}.$$

The Stolke's formula can be presented as divergence theorem:

Theorem 0.1.6 (Divergence theorem)
Let X be a vector field,

$$\int_D \operatorname{div}(X)\,\mathrm{d}x = \int_{\partial D} X\cdot\nu\,\mathrm{d}\sigma.$$

Another commonly-used operator is the **Laplace operator**:

$$\Delta = \operatorname{div}\cdot\nabla, \quad \Delta\phi = \operatorname{div}(\nabla\phi) = \operatorname{tr}(H_\phi) = \sum_{i=1}^n \frac{\partial^2\phi}{\partial x_i^2}.$$

When $n = 2$, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$,

$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma.$$

Since ∂D is a curve $\gamma(t)$, so $\mathrm{d}\sigma = |\gamma'(t)| \, \mathrm{d}t$. Let $\gamma(t) = (x(t), y(t))$, then $\nu(t) = \frac{(y'(t), -x'(t))}{|\gamma'(t)|}$. Here we must take $\gamma(t)$ to be *counterclockwise* to ensure ν points outside of D.

Thus we get

$$\int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma = \int_\gamma \frac{Py'(t) - Qx'(t)}{|\gamma'(t)|} |\gamma'(t)| \, \mathrm{d}t = \int_{\partial D} (P \, \mathrm{d}y - Q \, \mathrm{d}x).$$

This result is known as *Green's formula*.

This leads to the curve integrals of the second type: let $\gamma(t) \in \mathbb{R}^d$, X a vector field, we call the integral

$$\int_\gamma \sum_{i=1}^d X^i \, \mathrm{d}x_i = \int_\gamma X \cdot \mathrm{d}\gamma(t).$$

the **curve integral of the second type**.

When $n = 3$, the result is called *Gauss's formula*, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$,

$$\int_D \operatorname{div}(X) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma.$$

Let $\gamma(u, v) = (x, y, z)$ be a parametrization of ∂D. We have two tangent vector γ_u, γ_v, so the normal vector is defined as $\nu = \frac{\gamma_u \times \gamma_v}{|\gamma_u \times \gamma_v|}$. Also $\mathrm{d}\sigma = |\gamma_u \times \gamma_v| \, \mathrm{d}u \, \mathrm{d}v$. After some computation we can get

$$\nu \, \mathrm{d}\sigma = (\mathrm{d}y \, \mathrm{d}z, \mathrm{d}z \, \mathrm{d}x, \mathrm{d}x \, \mathrm{d}y).$$

$$\int_D \operatorname{div}(X) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_{\partial D} (P \, \mathrm{d}y \, \mathrm{d}z + Q \, \mathrm{d}z \, \mathrm{d}x + R \, \mathrm{d}x \, \mathrm{d}y).$$

§0.2 Differential forms

Let $T_p^* M$ denote the *dual space* of $T_p M$, and $\mathrm{d}x_i$ is the dual basis of $\frac{\partial}{\partial x_i}$. The linear combination of $\mathrm{d}x_i$ are called **differential forms**, and a differential form on a manifold can be written as $\sum_{i=1}^n a_i \, \mathrm{d}x_i$, where a_i are functions on M.

We can construct differential forms of higher order, the order is $1 \le k \le n$, called **k-forms**, which is a linear combination of

$$\mathrm{d}x_{i_1} \, \mathrm{d}x_{i_2} \cdots \mathrm{d}x_{i_k}, \quad i_1 < i_2 < \cdots < i_k.$$

Here the product is *wedge product*, i.e. $\mathrm{d}x_i \, \mathrm{d}x_j = -\mathrm{d}x_j \, \mathrm{d}x_i$. We denote the space of all k-forms by $\Lambda^k(\Omega)$.

We can define the multiplication of forms: let $\omega_1 \in \Lambda^{k_1}, \omega_2 \in \Lambda^{k_2}$, then $\omega_1 \wedge \omega_2 \in \Lambda^{k_1+k_2}$ by multiplying the coefficients and $\mathrm{d}x_i$'s respectively.

There's also an operator called **exterior differentiation** $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$, where

$$\mathrm{d}(a \, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}) = \sum_{j=1}^n \frac{\partial a}{\partial x_j} \, \mathrm{d}x_j \, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}.$$

This operator behaves like the derivatives very much:

$$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2, \quad \mathrm{d}(\omega_1 \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1 k_2} \omega_1 \wedge \mathrm{d}\omega_2.$$

Note that $\omega_1 \wedge \omega_2 = (-1)^{k_1 k_2} \omega_2 \wedge \omega_1$, so when $k_1 k_2$ is even, the wedge product may not be anti-symmetrical.

If we have a coordinate transformation $\Phi : (x_1, \dots, x_n) \to (y_1, \dots, y_n)$, we have

$$\mathrm{d}y_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_i} \mathrm{d}x_i,$$

thus $\mathrm{d}y_1 \cdots \mathrm{d}y_n = J_\Phi \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$. Here J_Φ can be negative, so the differential forms already contains the information of orientation.

Theorem 0.2.1

Let ω be a differential form, $\mathrm{d}(\mathrm{d}\omega) = 0$.

Proof. Partial derivatives commute. $\square$

Definition 0.2.2. Let ω be a differential form, if $\mathrm{d}\omega = 0$, we say ω is a **closed form**, if there exists ω_1 s.t. $\mathrm{d}\omega_1 = \omega$, then ω is a **exact form**.

The theorem above tells us that exact forms must be closed, but in general closed forms may not be exact, it depends on the topology structure of Ω.

Theorem 0.2.3 (Poincare)

The closed forms on $\mathbb{R}^n$ must be exact.

Proof. Use induction, when ω is an n-form this can be proved by computation.

For a generic form $\omega = \omega_1 + \mathrm{d}x_1 \wedge \omega_2$, where ω_1, ω_2 do not contain $\mathrm{d}x_1$. We want to find ω_3 s.t. $\mathrm{d}\omega_3 = \mathrm{d}x_1 \wedge \omega_2 + \omega_4$, where ω_3, ω_4 don't contain $\mathrm{d}x_1$ as well. (The construction is direct)

Since $\omega - \mathrm{d}\omega_3 = \omega_1 - \omega_4$, and

$$\mathrm{d}(\omega - \mathrm{d}\omega_3) = \mathrm{d}\omega = 0 \implies \mathrm{d}(\omega_1 - \omega_4) = 0.$$

Since $\mathrm{d}(\omega_1 - \omega_4) = 0$ and it doesn't contain $\mathrm{d}x_1$, hence all its coefficients can't contain $\mathrm{d}x_1$. Thus we can view it as a differential form in $\mathbb{R}^{n-1}$. $\square$

Remark 0.2.4 — When Ω is simply connected, then all the closed 1-forms are exact. Also this is equivalent to the integral on any closed curves are 0.

We can rewrite Stolkes' formula using differential forms:

Theorem 0.2.5 (Stolkes' formula)

Let D be a $k+1$ dimensional orientable manifold, $\omega \in \Lambda^k(\mathbb{R}^n)$, we have

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$