Linear Algebra II

Felix Chen

Contents

§1 Introduction

Teacher: An Jinpeng
Homepage: https://www.math.pku.edu.cn/teachers/anjp/algebra

§1.1 recap

Direct sums of vector spaces Given a field F, let $V_1, \dots, V_k$ be vector spaces over F. The set

$$V_1 \times \cdots \times V_k = \{(v_1, \dots, v_k) \mid v_i \in V_i\}$$

forms a vector space by the operations

$$(v_1, \dots, v_k) + (w_1, \dots, w_k) = (v_1 + w_1, \dots, v_k + w_k)$$

and

$$c \cdot (v_1, \dots, v_k) = (cv_1, \dots, cv_k).$$

We call this vector space the external direct sum of $V_1, \dots, V_k$, denoted by $\bigoplus_{i=1}^k V_i$.

Obviously $(U \oplus V) \oplus W \simeq U \oplus (V \oplus W)$.

For every i, we have an injective linear map:

$$\begin{aligned} \tau_i : & V_i \to \bigoplus_{j=1}^k V_j \\ & v_i \mapsto (0, \dots, v_i, \dots, 0) \end{aligned}$$

Lemma 1.1

If $\mathcal{B}_i$ are the bases of V_i, then $\bigcup_{i=1}^k \tau_i(\mathcal{B}_i)$ is a basis for $\bigoplus_{i=1}^k V_i$.

In paricular,

$$\dim \bigoplus_{i=1}^k V_i = \sum_{i=1}^k \dim V_i.$$

Proof. Spanning part:

For any $(v_1, \ldots, v_k) \in \bigoplus_{i=1}^k V_i$,

$$v_i \in V_i = \operatorname{span}(\mathcal{B}_i) \implies \tau_i(v_i) \in \operatorname{span}\left(\tau_i(\mathcal{B})_i\right) \implies (v_1, \ldots, v_k) \in \operatorname{span}\left(\bigcup_{i=1}^k \tau_i(\mathcal{B}_i)\right)$$

Linearly independent part:

If $\bigcup_{i=1}^k \tau_i(\mathcal{B}_i)$ is linearly dependent, i.e. exists $e_{ij} \in \mathcal{B}_i$ satisfying $\exists c_{ij} \in F$,

$$\sum_{i,j} c_{ij}\tau_i(e_{ij}) = 0.$$

This expands to

$$\left(\sum_{j=1}^{m_1} c_{1j}e_{1j}, \ldots, \sum_{j=1}^{m_k} c_{kj}e_{kj}\right) = 0.$$

but e_{1j} are linear independent, which implies $c_{1j} = 0$. □

Remark 1.2 — Let V be a vector space over F, and $V_1, \ldots, V_k$ are subspaces of V.

Consider a linear map $\Phi : V_1 \oplus \cdots \oplus V_k \to V$ by $(v_1, \ldots, v_k) \mapsto v_1 + \cdots + v_k$.

Then $\operatorname{Im}(\Phi) = V_1 + \cdots + V_k$. If Φ is injective, i.e. $V_1, \ldots, V_k$ are independent, we say $V_1 + \cdots + V_k$ the internal direct sum of $V_1, \ldots, V_k$.

In this case Φ gives an isomorphism of external and internal sums:

$$\Phi : \bigoplus_{i=1}^k V_i \xrightarrow{\sim} \sum_{i=1}^k V_i.$$

Lemma 1.3

The following statements are equivalent:

1. $V_1, \ldots, V_k$ are independent;
2. For $v_i \in V_i, (i = 1, \ldots, k)$, if $\sum_{i=1}^k v_i = 0$, then $v_i = 0$.
3. For any $1 \le i \le k$, $V_i \cap (V_1 + \cdots + V_{i-1}) = \{0\}$.
4. Given arbitary bases $\mathcal{B}_i$ of V_i, they are disjoint and their union is a basis of $\bigoplus_{i=1}^k V_i$.
5. If $\dim V < +\infty$, they are also equivalent to:

$$\dim \sum_{i=1}^k V_i = \sum_{i=1}^k \dim V_i.$$

Proof. It's easy but verbose so I leave it out. □

Example 1.4

Let char $F \neq 2$, $V = F^{n\times n}$, $V_1 = \{A \in V \mid A^t = A\}$, $V_2 = \{A \in V \mid A^t = -A\}$.

Note that $V_1 \cap V_2 = \{0\}$, and $V_1 + V_2 = V$, hence $V_1 \oplus V_2 = V$ is an internal direct sum.

§2 Eigenvectors and eigenvalues

Example: google page rank?

Definition 2.1 (Diagonizable maps)**.** Let V be a vector space over F, $T \in L(V)$ is a linear map from V to itself. If the matrix of T under a certain basis is diagonal, we say T is diagonizable.

In this case the linear map T can be simply described as a diagonal matrix, thus we'll study under what condition is T diagonizable.

Definition 2.2 (Eigenvalue)**.** Let $T : V \to V$ be a linear map, for $c \in F$, let

$$V_c = \{v \in V \mid Tv = cv\} = \ker(T - c \cdot \mathrm{id}_V).$$

If $V_c \neq \{0\}$, we call c an eigenvalue of T, and V_c the eigenspace of T with respect to c. the vectors in V_c are called eigenvectors.

All the eigenvalues of T are called the spectrum of T, denoted by $\sigma(T)$.

Proposition 2.3

Let $\mathcal{B}$ be a basis of V, then $[T]_{\mathcal{B}}$ is diagonizable $\iff$ vectors in $\mathcal{B}$ are all eigenvectors.

Proof. Let $\mathcal{B} = \{e_1, \ldots, e_k\}$, $A = [T]_{\mathcal{B}}$.

$$Te_j = \sum_{i=1}^{k} A_{ij} e_i.$$

So A is diagonal $\iff$ $A_{ij} = 0$ when $i \neq j$,
$\iff \exists c_j \in F, Te_j = c_j e_j$,
$\iff$ all the vectors e_j are eigenvectors. □

Example 2.4

Let $V = F^{n\times n}$, then V_{sym} is the eigenspace of 1, and $V_{antisym}$ is the eigenspace of -1.

Lemma 2.5

Let T be a linear operator, then

$$\sigma(T) = \{c \in F \mid \det(c \cdot id_V - T) = 0\}.$$

Proof. $V_c = \ker(c \cdot \mathrm{id}_V - T)$,

$$c \in \sigma(T) \iff V_c \neq \{0\} \iff \det(c \cdot \mathrm{id}_V - T) = 0.$$

□