Geometry II

Felix Chen

Contents

Proposition 0.1

Torsion can be represented in general parameter:

$$\mathrm{Tors}_\gamma(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

Remark 0.2 — The torsion can be negative (while curvature is always non-negative), and it is only defined at the points where the curvature is nonzero.

Note that the vectors $\vec{t}, \vec{n}, \vec{b}$ form a right-handed orthonormal basis in $\mathbb{R}^3$, and it's called the curve $\gamma(s)$'s **Frenet frame**.

The plane containing $\vec{n}$ and $\vec{b}$ is called **normal plane**, the plane containing $\vec{t}$ and $\vec{n}$ is called **osculating plane**, and the last plane which contains $\vec{t}, \vec{b}$ is called **rectifying plane**.

The Frenet frame is not a fixed frame, it's moving with the point along the curve. So we can compute its derivative (with respect to s, the arc length parameter):

$$(\vec{t}', \vec{n}', \vec{b}') = (\vec{t}, \vec{n}, \vec{b}) \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Example 0.3
When γ lies on the surface of a sphere, assmue $\kappa > 0$ on $\gamma\big|_J$, then

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = c^2.$$

where c is the radius of the sphere.

Proof. Let $\vec{u} = \gamma(s) - p$, then $\vec{u}\cdot\vec{u} = c^2$. To get a relation of κ and τ, we only need to represent $\vec{u}$ in terms of $\vec{t}, \vec{n}$ and $\vec{b}$.

Taking derivative WRT s:

$$0 = 2\vec{u}' \cdot \vec{u} = 2\vec{t} \cdot \vec{u}.$$

and then by taking the second and third derivative,

$$0 = \vec{t}' \cdot \vec{u} + \vec{t}^2 = \kappa\vec{n} \cdot \vec{u} + 1.$$

We get $\vec{u} \cdot \vec{n} = -\frac{1}{\kappa}$.

$$(\kappa\vec{n})' = \kappa'\vec{n} + \kappa(-\kappa\vec{t} + \tau\vec{b}),$$

so the third derivative should be

$$0 = \kappa\vec{n} \cdot \vec{t} + \kappa'\vec{n} \cdot \vec{u} + \kappa(-\kappa\vec{t} + \tau\vec{b}) \cdot \vec{u} = -\frac{\kappa'}{\kappa} + \kappa\tau\vec{u} \cdot \vec{b},$$

hence $\vec{u} \cdot \vec{b} = \frac{1}{\tau}(\frac{1}{\kappa})'$.

At last we just plugged everything into $\vec{u}^2 = c^2$ to conclude. □

Note: the inverse statement does not hold, e.g. helix (which has constant curvature).

This example shows that Frenet frame is a powerful tool for handling the local properties of a curve. In fact, we could totally "determine" a curve near a point given the curvature and torsion.

Example 0.4
We can expand the curve $(\gamma(s), \vec{t}, \vec{u}, \vec{b})$ around $s = 0$:

$$\begin{cases} x(s) = x(0) + s - \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ y(s) = y(0) + \dfrac{\kappa(0)}{2}s^2 + \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ z(s) = z(0) + \dfrac{\kappa'(0)\tau(0)}{6}s^3 + o(s^3) \end{cases}$$

Remark 0.5 — By Frenet's formula, we can expand it to higher degrees, but the expansion need not converge to the original curve (similar reason as Taylor's formula). Also we can expand the curve with any parameter instead of arclength.

§0.1 Fundamental theorem of curve theory

Theorem 0.6 (Fundamental theorem of curve theory)
Let $\kappa, \tau : J \to \mathbb{R}$ be smooth functions, $\kappa(s) > 0$ on J. There exists a curve with arc length parameter $\gamma : J \to \mathbb{E}^3$, such that $\mathrm{Curv}_\gamma = \kappa$, $\mathrm{Tors}_\gamma = \tau$ holds on J.

Moreover, if $\tilde{\gamma}$ also satisfies above conditions, then exists $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ perserving orientation and distance s.t. $\tilde{\gamma} = \sigma \circ \gamma$.

Claim 0.7. Let $H = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

The ODE about $F : J \to \mathrm{Mat}_{3\times 3}(\mathbb{R})$:

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}s}(s) = F(s)H(s) \\ F(s_0) = F_0 \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \end{cases}$$

always has unique solution. Moreover if $F(s_0) \in \mathrm{SO}(3)$, then $F(s) \in \mathrm{SO}(3)$ always holds.

Proof of the theorem. Since this claim requires some knowledge in ODE, which is beyond the scope of this course, we'll directly use it without proving.

WLOG $0 \in J$, let $s_0 = 0$ and $F(0) = I_3$.

Let $\mathcal{F} = (\vec{t}, \vec{n}, \vec{b}) := (\vec{e}_1, \vec{e}_2, \vec{e}_3)F(s)$ be a frame of $\mathbb{R}^3$.

Now we construct γ to be

$$\gamma(s_1) := \int_0^{s_1} \vec{t}\,\mathrm{d}s.$$

It's sufficient to prove that $\mathrm{Curv}_\gamma = \kappa$ and $\mathrm{Tors}_\gamma = \tau$.

Since $\mathcal{F}(0) = (e_1, e_2, e_3)$ is orthonormal frame, $\mathcal{F}(s)$ is orthonormal for all s.

Thus $|\vec{t}| = 1$, s is the arc length parameter. Some computation yields $\mathcal{F}$ is Frenet frame of γ. Compare its Frenet matrix to H, we get the desired result.

On the other hand, if $\tilde{\gamma}$ is as stated, take its Frenet frame $\tilde{\mathcal{F}}(s)$.

Let σ be the map which maps $\mathcal{F}(0)$ to $\tilde{\mathcal{F}}(0)$, $\gamma(0)$ to $\tilde{\gamma}(0)$. Then the Frenet frame of $\sigma \circ \gamma$ and $\tilde{\gamma}$ are the solution of the same ODE $\implies \sigma \circ \gamma = \tilde{\gamma}$ for all $s \in J$. □

Remark 0.8 — Here we give a proof of $F \in \mathrm{SO}(3)$:

Proof. Note that

$$\left(FF^T\right)' = F'F^T + F(F')^T = F\left(H + H^T\right)F^T = 0.$$

thus $FF^T = I$ as it holds at $s_0 \implies F \in \mathrm{O}(3)$.

Beisdes, it's easy to see that $\det(F)$ doesn't change sign, so $F \in \mathrm{SO}(3)$. □

In the words of tangent spaces or Lie groups, we can say that $T_I\mathrm{SO}(3) = \{X \mid X + X^T = 0\}$, and $T_F\mathrm{SO}(3) = \{FX \mid X + X^T = 0\}$.

Remark 0.9 — The above ODE cannot be solved explicitly, so here we introduce a method called "successive approximation". (For more details, see my notes of Analysis I)

Let $F_0(s) = F_0$, $F_1(s) = F_0 + \int_{s_0}^s F_0(t)H(t)\,\mathrm{d}t$, and define

$$F_{j+1}(s) = F_0 + \int_{s_0}^s F_j(t)H(t)\,\mathrm{d}t.$$

We can compute

$$|F_1(s) - F_0(s)| = \left|\int_{s_0}^s F_0(t)H(t)\,\mathrm{d}t\right| \le \int_{s_0}^s |F_0(t)H(t)|\,\mathrm{d}t \le M(s-s_0)\cdot|F_0|.$$

$$|F_{j+1}(s) - F_j(s)| = \left|\int_{s_0}^s (F_j(t) - F_{j-1}(t))H(t)\,\mathrm{d}t\right| \le M^{j+1}\frac{(s-s_0)^{j+1}}{(j+1)!}|F_0|.$$

Therefore F_j must uniformly converge to some function F on some small interval $[s_0-\delta, s_0+\delta]$.

With some effort we can check F is differentiable and satisfies the ODE. Furthermore, F can extend to the entire interval J, and it's the *unique* solution.

§1 Theory of surfaces

§1.1 The first fundamental form

Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. We denote the point in $U \subset \mathbb{R}^2$ as $u = (s,t)$. Hence the partial derivative of ϕ gives:

$$\phi_s(u) = \frac{\partial\phi}{\partial s}(u), \quad \phi_t(u) = \frac{\partial\phi}{\partial t}(u).$$

Define the first fundamental quantities

$$E(u) = \phi_s(u)\cdot\phi_s(u), \quad F(u) = \phi_s(u)\cdot\phi_t(u), \quad G(u) = \phi_t(u)\cdot\phi_t(u).$$

The first fundamental form of the surface ϕ is the real bilinear form $T_u\phi(U)\times T_u\phi(U) \to \mathbb{R}$:

$$g(u) = E(u)\,\mathrm{d}s^2 + 2F(u)\,\mathrm{d}s\,\mathrm{d}t + G(u)\,\mathrm{d}t^2.$$

We say $\phi(s,t_0)$ is an s-curve and $\phi(s_0,t)$ is a t-curve. If s-curve and t-curve are orthogonal at every point $u \in U$, i.e. $F = 0$, we say ϕ is an **orthogonal parametrization**, and s,t are **orthogonal parameters**.

Moreover, if $E = G, F = 0$ for all $u \in U$, then we call ϕ an **isothermal parametrization**, and s,t are **isothermal parameters**. (Sometimes also called **comformal parameters**)

Example 1.1

The longtitude and latitude on a sphere are orthogonal parameters, but not isothermal parameters; While the stereographical projection is an isothermal parametrization of the sphere.

Remark 1.2 — The word "isothermal" is connected to thermology in a rather complicated way. The word "conformal" provides a more intuitive comprehension.

§1.2 Linear algebra review

Let V be a vector space over $\mathbb{F}$.

Symmetrical bilinear form vs. quadratic form

A symmetrical bilinear form is a linear map $B : V \times V \to \mathbb{F}$ with $B(v,w) = B(w,v)$. A quadratic form is a map $Q : V \to \mathbb{F}$ with $Q(v) = B(v,v)$ for some symmetrical bilinear form B.

By taking a basis of V, we can use the matrix to express them:

$$B(v,w) = vAw^T, \quad Q(v) = vAv^T.$$

where A is a symmetrical matrix. When we change the basis, the matrix A differs by a congruent transformation.

We could also write $B \in V^* \otimes V^*$ for a bilinear form B. All the symmetrical bilinear forms constitude a subspace of $V^* \otimes V^*$ of dimension $\frac{n(n+1)}{2}$. This is denoted by $\mathrm{Sym}^2(V)$.

> **Remark 1.3 —** The subspace of anti-symmetric matrices is denoted by $\mathrm{Alt}^2(V)$, and $\mathrm{Alt}^2(V) \oplus \mathrm{Sym}^2(V) = V^* \otimes V^*$.

§1.3 Tangent spaces

A surface $\phi : U \to \mathbb{E}^3$ has a tangent space at every point $\phi(u)$, which is just the space (in this case, a plane) spanned by $\phi_s(u)$ and $\phi_t(u)$. We can prove that this tangent space is independent to the parameters. Furthermore, we can equip it with the inner product in $\mathbb{E}^3$, the matrix of this product is exactly $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

> **Remark 1.4 —** In modern differential manifold theory, there's an intrinsic definition of tangent spaces, but this definition is too abstract.

Here we present one of these intrinsic definitions.

Definition 1.5 (Tangent vectors)**.** Define an equivalence relation on smooth curves in $\phi(U)$:

Let $\gamma(r) = \phi(s(r), t(r))$ be a smooth curve $(-\epsilon, \epsilon) \to \phi(U)$. Two curves γ_1, γ_2 are equivalent iff $s_1'(0) = s_2'(0)$ and $t_1'(0) = t_2'(0)$.

Each equivalence class is a "tangent vector" at point $\phi(s_0, t_0)$.