Geometry II

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Here we explain the above computation a little.

The vector

$$\sum_\gamma \Gamma^\gamma_{\_\alpha\beta} v_\gamma =: \nabla_\beta \vec{v}_\alpha$$

is called the covariant derivative of $\vec{v}_\alpha$. It's projection of the derivative of $\vec{v}_\alpha$ onto the tagent space.

$$\frac{\partial}{\partial u^\beta}\frac{\partial}{\partial u^\gamma}(v_\alpha) = \frac{\partial}{\partial u^\gamma}\frac{\partial}{\partial u^\beta}(v_\alpha)$$

$$\implies -\nabla_\beta\nabla_\gamma v_\alpha + \nabla_\gamma\nabla_\beta v_\alpha = h_{\alpha\beta}\nabla_\gamma\vec{n} - h_{\alpha\gamma}\nabla_\gamma\vec{n}.$$

So the covariant derivative is not commutative, and the "curvature" or the second fundamental form basically measures this discommutation.

Now if we look at

$$\frac{\partial g_{\delta\alpha}}{\partial u^\beta} = \frac{\partial v_\delta \cdot v_\alpha}{\partial u^\beta} = \frac{\partial v_\delta}{\partial u^\beta}v_\alpha + v_\delta\frac{\partial v_\alpha}{\partial u^\beta} = \sum_\gamma g_{\alpha\gamma}\Gamma^\gamma_{\_\delta\beta} + \sum_\gamma g_{\delta\gamma}\Gamma^\gamma_{\_\alpha\beta},$$

similarly, by symmetry, computing

$$\frac{\partial g_{\alpha\beta}}{\partial u^\delta}, \quad \frac{\partial g_{\delta\beta}}{\partial u^\alpha},$$

will yield

$$\Gamma^\gamma_{\_\alpha\beta} = \sum_\delta \frac{g^{\gamma\delta}}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right).$$

In fact this is more intuitive in Einstein summation notation.

§0.1   Fundamental theorem of surface theory

> **Theorem 0.1.1** (Fundamental theorem of surface theory)
>
> Let $D \subset \mathbb{R}^2$, $u = (u^1, u^2)$ is the coordinate. Let $g_{\alpha\beta}, h_{\alpha\beta} : D \to \mathbb{R}$ be C^3 functions, and the matrix $(g_{\alpha\beta})$ is symmetrical anf positive definite, $(h_{\alpha\beta})$ is symmetrical.
>
>   Let $g^{\alpha\beta}$ be the inverse matrix of $g_{\alpha\beta}$, and $R_{\delta\alpha\beta\gamma}$ is as above. If these functions satisfies Gauss equation and Codazzi equation, then:
>
>   For all $p \in D$, exists a neighborhood $U = U(p) \subset D$ and a regular surface $\phi : U \to \mathbb{E}^3$, such that $g_{\alpha\beta}, h_{\alpha\beta}$ are the first and second fundamental quantities of ϕ.
>
>   Furthermore, if $\widetilde{\phi} : U \to \mathbb{E}^3$ also satisfies above conditions, then $\widetilde{\phi} = \sigma \circ \phi$, where σ is an isometry of $\mathbb{E}^3$.

Basically we need to solve a partial differential equation, and we need to consider how to construct this equation.

Proof. Let $\phi : U \to \mathbb{E}^3$, $v_\alpha, \vec{n} : D \to V(\mathbb{E}^3)$ be unknown functions satisfying

$$
\begin{cases}
v_\alpha = \dfrac{\partial \phi}{\partial u^\alpha} \\[2mm]
\dfrac{\partial v_\alpha}{\partial u^\beta} = \displaystyle\sum_\gamma \Gamma^\gamma_{\_\alpha\beta} v_\gamma + h_{\alpha\beta} \vec{n} \\[2mm]
\dfrac{\partial \vec{n}}{\partial u^\beta} = -\displaystyle\sum_\gamma h^\gamma_{\_\beta} v_\gamma
\end{cases}
$$

This is a linear homogeneous PDE of degree 1, and it actually has a unique solution.

Consider the initial-value problem in the neighborhood of a given point $p \in D$.

We hope to prove that

- The above PDE initial-value problem has a unique solution under the Gauss-Codazzi equations;

- If initially (i.e. at p) we have

$$
\vec{n} = \frac{v_1 \times v_2}{\|v_1 \times v_2\|},
$$

  then it holds for all $p' \in U(p)$.

For the second statement, we can compute $\frac{\partial}{\partial u^\beta}(\vec{n} \cdot v_\alpha) = 0$, so they are constant.

For the PDE part, if we want a C^2 solution of some linear PDE of degree 1:

$$
\frac{\partial y^j}{\partial x^\alpha} = f^j_\alpha(x^1, \ldots, x^n, y^1, \ldots, y^m)
$$

There's a necessary condition that the partial derivatives are commutative, i.e.

$$
\frac{\partial}{\partial x^\beta} \frac{\partial y^j}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta}.
$$

This expands to

$$
\frac{\partial f^j_\alpha}{\partial x^\beta} + \sum_k f^k_\beta \frac{\partial f^j_\alpha}{\partial y^k} = \frac{\partial f^j_\beta}{\partial x^\alpha} + \sum_k f^k_\alpha \frac{\partial f^j_\beta}{\partial y^k}.
$$

In fact this is also the sufficient condition of the existence local solution.

> **Remark 0.1.2 —** The proof is beyond the scope of this course, but the basic idea is to build the y^j's dimension by dimension (from curve to surfaces to 3d manifolds ...). The 1d part can be constructed using solutions of ODE, and the compatibility follows by our condition.
>
> In the language of differential forms, let $y = (y^1, \ldots, y^m)$, we are given $\mathrm{d}y$, since the condition says $\mathrm{d}(\mathrm{d}y) = 0$, i.e. $\mathrm{d}y$ is a *closed form*, so we always have local solution of y.

Returning to our original problem, this condition is actually what we used to deduce the Gauss-Codazzi equations, so our PDE must have a unique solution on a neighborhood of p. $\qquad\square$

> **Example 0.1.3**
>
> We can't grant that the global function exists. For example, let $D = \{x^2 + y^2 \in [a^2, b^2]\}$, and M be a helicoid.
>
> Since there's a natural map $\phi : D \backslash ([a, b] \times \{0\}) \to M$ (projection), let g, h be the fundamental forms of ϕ, by the symmetry we can extend g, h to entire D.
>
> It's clear that there exists local solutions but the global solutions does't exist. (In theory of differential forms, this is similar to closed forms may not be exact)
>
> But if the region D is *simply connected*, the global solution always exist.

§0.2 Isometric, conformal and area-perserving maps

Let $U, \widetilde{U} \subset \mathbb{R}^2$, and $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. Let $f : \widetilde{U} \to U$ be a map between two surfaces.

Earlier we introduced isometric maps (isometry), i.e. $f^*(g) = \widetilde{g}$. Since the length depends only on the first fundamental form, the isometry perserves the length, angles and areas on surfaces.

The **conformal** maps perserves the angles on the surfaces, and it's easy to imply this is equivalent to $f^*(g) = \lambda \widetilde{g}$ for some $\lambda \in \mathbb{R}$.

As the name suggests, the **area-perserving** maps perserves the areas on two surfaces, which is saying $\det f^*(g) = \det \widetilde{g}$.

It's easy to prove that isometric = conformal + area-perserving. These three properties induce Riemann geometry, complex geometry and sympletic geometry, respectively (in two dimensional).

§0.2.1 Isometries

Firstly by Gauss' Theorema Egregium, Isometries perserves Gaussian curvature.

> **Example 0.2.1**
>
> Let $S_{a,b} : \frac{x^2}{a} + \frac{y^2}{b} = 2z$ be a saddle surface. Let $(x, y, z) = (as, bt, \frac{as^2 + bt^2}{2})$ be a parametrization.
>
> We can compute the fundamental forms:
> $$g = a^2(1 + s^2)\,\mathrm{d}s^2 + 2abst\,\mathrm{d}s\,\mathrm{d}t + b^2(1 + t^2)\,\mathrm{d}t^2,$$
>
> $$h = \frac{a\,\mathrm{d}s^2 + b\,\mathrm{d}t^2}{\sqrt{1 + s^2 + t^2}}.$$
>
> So $K = \frac{1}{ab(1 + s^2 + t^2)^2}$. In fact the Gaussian curvature of some different surfaces, say $S_{2,3}$ and $S_{1,6}$ are the same.
>
> But there is not an isometry between them:
>
> If $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then τ fixes the circles centered at $(0, 0)$ as their Gaussian curvature are the same. Then $\tau_* = \mathrm{d}\tau : T_{(0,0)}\mathbb{R}^2 \to T_{(0,0)}\mathbb{R}^2$ can only be rotation or reflection. (If τ_* is not orthogonal, it will map small circles to ellipse)
>
> While $g(0, 0) = a^2\,\mathrm{d}s^2 + b^2\,\mathrm{d}s^2$, which has eigenvalue a^2 and b^2, and they're fixed under τ_*, so $S_{2,3}$ isn't isometric to $S_{1,6}$.

> **Remark 0.2.2 —** Given $E, F, G : D \to \mathbb{R}$ s.t. $g = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ positive definite, is there a surface $D \to \mathbb{E}^3$ can have g as its first fundamental form locally?

> When we require E, F, G to be C^ω(analytic), the answer is "yes", but if we only require C^∞, it's still an open problem.

Even though we don't know the situation in 3 dimensional space, we can study the case in higher dimensions:

Theorem 0.2.3

It's always possible to construct $\phi : D \to \mathbb{E}^4$ to have E, F, G as its first fundamental form.

Surfaces with Gaussian curvature 0 everywhere are called **developable surfaces**. Developable surfaces can only be cylinder, cone, tangent surface of a curve and their concatenation.

Example 0.2.4 (Pseudosphere)

Let $\phi(x, y) = (\frac{\cos x}{y}, \frac{\sin x}{y}, \cosh^{-1}(y) - \frac{\sqrt{y^2-1}}{y})$, where $(x, y) \in (-\pi, \pi) \times [1, +\infty)$.
    It's obtained by rotating a *tractrix* around its asymptote. We can calculate its Gaussian curvature, whichis a constant -1. This is where the name comes from.
    Recall that hyperbolic plane also has constant curvature -1, in fact they are locally isometric. In 1901, Hilbert proved a theorem that there exists an isometry $\mathbb{H}^2 \to \mathbb{E}^3$.

At last we'll prove an interesting fact:

Proposition 0.2.5 (The local existence of isothermal parameters)

Let $\phi : U \to \mathbb{E}^3$, for all $\hat{u} \in U$, there exists a neighborhood $\widetilde{U}$ and a reparametrization $u = u(\widetilde{u})$, such that
$$g(\widetilde{u}) = \rho^2(\widetilde{u})(\widetilde{E} \, \mathrm{d}\widetilde{s}^2 + \widetilde{G} \, \mathrm{d}\widetilde{t}^2).$$

> **Remark 0.2.6** — Note that the right hand side is clearly conformal to regions in $\mathbb{E}^2$, so this in fact implies that any surfaces is locally conformal to $\mathbb{E}^2$.

Proof. The critical idea is to realize $\mathbb{R}^2$ as $\mathbb{C}$. To be more precise, we'll follow the steps below:

- Find a way to express $E \, \mathrm{d}s^2 + 2F \, \mathrm{d}s \, \mathrm{d}t + G \, \mathrm{d}t^2$ as $(a \, \mathrm{d}s + b \, \mathrm{d}t)(\overline{a} \, \mathrm{d}s + \overline{b} \, \mathrm{d}t)$, where a, b are functions with complex value.

- If there exists a complex function f s.t. $\mathrm{d}f(s + it) = \rho(a \, \mathrm{d}s + b \, \mathrm{d}t)$, then $g = \frac{1}{|\rho|^2} \, \mathrm{d}f \, \mathrm{d}\overline{f}$.

- Assume further that f is *holomorphic* and non-degenerate, then $f(u) = \widetilde{x}(u) + i\widetilde{y}(u)$ is locally inversible, i.e. exists $u = u(\widetilde{x}, \widetilde{y})$, then

$$g = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x} + i \, \mathrm{d}\widetilde{y})(\mathrm{d}\widetilde{x} - i \, \mathrm{d}\widetilde{y}) = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x}^2 + \mathrm{d}\widetilde{y}^2).$$

Let $a = \sqrt{E}$, $b = \frac{-F + i\sqrt{EG - F^2}}{\sqrt{E}}$. (Note $EG - F^2 > 0$ as g is positive definite)
Next we'll choose suitable f, ρ. Consider the differential equation $T = T(s, t)$:

$$\frac{\partial T}{\partial s} = -\frac{a(s, T)}{b(s, T)}, \quad T(\hat{s}, t) = t.$$

From the relation $f(s, T(s,t)) = t - \hat{t}$ and implicit function theorem we can uniquely determine f. Such f satisfies $\mathrm{d}f = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$.

When $f(s,t) = (\widetilde{x}, \widetilde{y})$, the Jacobian determinant is

$$\widetilde{x}_s \widetilde{y}_t - \widetilde{x}_t \widetilde{y}_s = -|\rho|^2 (a\bar{b} - b\bar{a}) = |\rho|^2 \sqrt{EG - F^2} > 0.$$

so f must be non-degenerate. $\qquad\square$