# Measure Theory

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## §1 Introduction


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### §1.1 Starting from probablistics

**Definition 1.1** ($\sigma$-algebra)**.** Let $\mathscr{F}$ be a family of subsets of a set $\Omega$, if

- $\Omega \in \mathscr{F}$;
- If $A \in \mathscr{F}$, $A^c \in \mathscr{F}$;
- If $A_1, A_2, \cdots \in \mathscr{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.(Countable union)

then we call $\mathscr{F}$ a $\sigma$-algebra.

Some intros about probablistics (left out because I haven't studied probablistics yet;)

### §1.2 What is measure theory?

It's an abstract theory, different from probablistics and real analysis. In this course we study a general set $X$, focus on mathematical thinking and skills, from the simple to construct the complex.

Measure theory studies the intrinsic structure of mathematical objects, and the map between different measure spaces.

# §2 Measure spaces and measurable maps

## §2.1 Sets and set operations

**Definition 2.1.** A non-empty set $X$ is our space(universal set), its elements (points) are denoted by lower case letters $x, y, \ldots$.

Some notations:

$$x \in A, x \notin A, x \in A^c, A \subset B, A \cup B, AB = A \cap B,$$

$$B \backslash A (B - A \text{ when } A \subset B), A \Delta B.$$

A family of sets $\{A_t, t \in T\}$.

$$\bigcup_{t \in T} A_t := \{x : \exists t \in T, s.t. x \in A_t\}, \quad \bigcap_{t \in T} A_t := \{x : x \in A_t, \forall t \in T\}.$$

Sometimes we write the union of disjoint sets as sums to emphasize the disjoint property.

Monotone sequence of sets:

$$A_n \uparrow : A_n \subset A_{n+1}, \forall n; \quad A_n \downarrow : A_n \supseteq A_{n+1}, \forall n.$$

Next we define the limits of sets:

**Definition 2.2.** For monotone sequences:

$$\lim_{n \to \infty} A_n := \bigcup_{n=1}^{\infty} A_n \text{ or } \bigcap_{n=1}^{\infty} A_n.$$

For general sequence of sets:

$$\limsup A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n; \quad \liminf A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n.$$

A clearer intepretation of limsup and liminf:

limsup is the set of elements which occurs infinitely many times in $A_n$, and liminf is the elements which doesn't occur in only finitely many $A_n$'s.

## §2.2 Families of sets

**Definition 2.3.** A family of sets is denoted by script letters $\mathscr{A}, \mathscr{B}, \ldots$.

- A family is a $\pi$-family if $\mathscr{P} \neq \emptyset$ and it's closed under intersections, e.g. $\{(-\infty, a] : a \in \mathbb{R}\}$.
- Semi-rings: $\mathscr{Q}$ is a $\pi$-family, and for all $A \subset B$, then there exists finitely many pairwise disjoint sets $C_1, \ldots, C_n \in \mathscr{Q}$ s.t.

$$B \backslash A = \bigcup_{k=1}^{n} C_k = \sum_{k=1}^{n} C_k.$$

  e.g. $\mathscr{Q} = \{(a, b] : a, b \in \mathbb{R}\}$.

**Remark 2.4** — The condition $A \subset B$ can be removed.

- Rings: $\mathscr{R}$ is nonempty, and it's closed under union and substraction.
  e.g. $\mathscr{R} = \{\bigcup_{k=1}^{n}(a_k, b_k] : a_k, b_k \in \mathbb{R}\}$.
- Algebras (fields): $\mathscr{A}$ is a $\pi$-family that contains $X$, and is closed under completion.

**Proposition 2.5**
Semi-rings are $\pi$-families, rings are semi-rings, algebras are rings.

*Proof.* By definition we only need to check rings are $\pi$-families: $A \cap B = A \backslash (A \backslash B)$.
For algebras, $A \cup B = (A^c \cap B^c)^c, A \backslash B = A \cap B^c$, so they are rings. □

**Remark 2.6** — Rings are semi-rings with unions, Algebras are rings with universal set $X$.

**Definition 2.7.** Some other families that start from taking limits:

- Monotone families: If $A_1, \cdots \in \mathscr{U}$ and $A_n$ monotone, then $\lim_{n\to\infty} A_n = \mathscr{U}$.
- $\lambda$-families:
$$X \in \mathscr{L}; \quad A_1, A_2, \cdots \in \mathscr{L}, A_n \uparrow A \implies A \in \mathscr{L};$$
$$A, B \in \mathscr{L}, A \supseteq B \implies A \backslash B \in \mathscr{L}.$$
  notes: $A_n \in \mathscr{L} \iff B_n = A_n^c \in \mathscr{L}$.
- $\sigma$-algebra:
$$X \in \mathscr{F}; \quad A \in \mathscr{F} \implies A^c \in \mathscr{F};$$
$$A_1, A_2, \cdots \in \mathscr{F} \implies \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}.$$

**Proposition 2.8**
$\sigma$-algebra = algebra & monotone family;
$\sigma$-algebra = $\lambda$-family & $\pi$-family.

**Definition 2.9.** $\sigma$-rings: $\mathscr{R}$ nonempty, $A, B \in \mathscr{R} \implies A \backslash B \in \mathscr{R}$;
$$A_1, A_2, \cdots \in \mathscr{R} \implies \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}.$$

Note: We only need to verify the case when $A_n$'s are disjoint.

**Definition 2.10** (Measurable space)**.** Let $\mathscr{F}$ be a $\sigma$-algebra on a set $X$, we say $(X, \mathscr{F})$ is a measurable space.

**Proposition 2.11**

Let $(X, \mathscr{F})$ be a measure space, $A$ is a subset of $X$. Then $(A, A \cap \mathscr{F})$ is also a measurable space.

The smallest $\sigma$-algebra is $\{\emptyset, X\}$, the largest $\sigma$-algebra is the power set $\mathscr{T} = \mathcal{P}(X)$.

In some cases, $\mathscr{T}$ is too large, for example, when $X = \mathbb{R}$, we can't assign a "measure" to every subset that fits our common sense.