Geometry II

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Contents

0.1 Fundamental groups 2
0.2 Covering spaces 5

Note that X may be disconnected, so the fundamental group is dependent of the base point x_0. If $\gamma = \langle c \rangle$ is a homotopy class of paths from x_0 to x_1, then γ induces a group homomorphism:

$$\gamma_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1) : \langle a \rangle \mapsto \langle \bar{c}ac \rangle .$$

It's easy to see $\gamma_\sharp$ is an isomorphism.

Hence $\pi_1(X, x_0)$ only depends on the path connected components of x_0. Thus if X is path connected, and X, Y are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, or sometimes we can leave the base point out, just write $\pi_1(X) \cong \pi_1(Y)$.

Remark 0.0.1 — If $x_0 = x_1$, then $\gamma \mapsto \gamma_\#$ gives a homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_1(X, x_0))$.

Example 0.0.2

If $X \simeq \{pt\}$, then $\pi_1(X) \cong \{1\}$. In this case, X is called a **contractible space**. Note that the inverse is not true, e.g. $X = S^n$ for $n \geq 2$. Path connected spaces with trivial fundamental group are called **simple connected** spaces.

Some classic contractible space includes convex sets in $\mathbb{R}^n$, trees in graph theory and cones $CX = X \times [0,1]/X \times \{1\}$.

Some more complex contractible examples including "a house with two rooms", the equitorial inclusion $S^\infty = \bigcup_{n=0}^\infty S^n$ with limit topology, i.e. the largest topology s.t. $S^n \to S^\infty$ continuous.

There are several concepts:

- Retraction: $f : X \to A$, $A \subset X$, $f\big|_A = \mathrm{id}_A$.
- Deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$, where $i : A \to X$ is the inclusion.
- Strong deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$ *rel* A.

The set A is called (strong) deformation kernel of f.

Example 0.0.3 (Differences between deformation and strong deformation)

Let X be the follwing space:

$$([0,1] \times \{0\}) \cup ([0,1]_{\mathbb{Q}} \times [0,1])$$

We know $X \simeq \{pt\}$, but $\{q\} \times [0,1]$ is deformation kernel but not strong deformation kernel.

§0.1 Fundamental groups

After introducing the fundamental groups, a natural question arises: how to compute the fundamental group of a given space? We first state the main result of this section:

Theorem 0.1.1 (Van Kampen)
Let $X = U' \cup U''$ be a topology space such that U', U'' are open and $W = U' \cap U''$ path connected, then for $x_0 \in W$, we have

$$\pi_1(X, x_0) \cong \pi_1(U', x_0) * \pi_1(U'', x_0)/N,$$

where N is the smallest normal subgroup generated by

$$i'_\sharp(\delta) i''_\sharp(\delta^{-1}) : \delta \in \pi_1(W, x_0),$$

$$\begin{tikzcd} & U' \arrow[dr, "j'"] & \\ W \arrow[ur, "i'"] \arrow[dr, "i''"'] & & X \\ & U'' \arrow[ur, "j''"'] & \end{tikzcd}$$

and $*$ means free product.

Note that this theorem is useless when both U', U'' have trivial fundamental groups. Thus we need to find a space with nontrivial fundamental group first. One of the simplest example is S^1 :

$$\pi_1(S^n, x_0) \cong \begin{cases} \{1\}, & n \geq 2, \\ \mathbb{Z}, & n = 1. \end{cases}$$

Let $X \vee Y := X \sqcup Y/(x_0 = y_0)$, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. Thus $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{k}) = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_k$, the free group of rank k.

Example 0.1.2
Since nT^2 is formed by $2n$ loops(borders of the polynomial representation) fused with a disk. Note that $W = U' \cap U'' \cong S^1$, so

$$\pi_1(nT^2) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle .$$

Similarly,

$$\pi_1(mP^2) = \left\langle c_1, \ldots, c_m \mid c_1^2 \cdots c_m^2 = 1 \right\rangle .$$

Example 0.1.3
For product spaces, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

In fact the isomorphism can be written down with i_x, i_y, p_x, p_y, i.e. $p_{x\sharp} \times p_{y\sharp}$ and $(i_{x\sharp}, i_{y\sharp})$.

Theorem 0.1.4
$\pi_1(S^1) \cong \mathbb{Z}$, where the generating element is id.

Proof. Consider the map $p : \mathbb{R} \to S^1$, with $t \mapsto e^{2\pi i t}$.

Given any path $\gamma : [0,1] \to S^1$, we can find a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}$, s.t. $\tilde{\gamma}(0) \in \mathbb{Z}$ is any given base point. We denote this map by Φ, $\gamma \mapsto \tilde{\gamma}(1)$, where we require $\tilde{\gamma}(0) = 0$.

We can prove that $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$, and Φ only depends on the homotopy class of γ, so Φ induces a homomorphism of $\pi_1(S^1) \to \mathbb{Z}$.

Remark 0.1.5 — Since every homotopy $[0,1] \times [0,1] \to S^1$ can be lifted uniquely, and the endpoints of each path form a path in $\mathbb{R}$, but it's always contained in $\mathbb{Z}$, hence it must be constant.

Note that

- Φ is surjective since $s \mapsto e^{2\pi i m s}$ is mapped to m under Φ;
- Φ is injective since $\ker \Phi = \{1\}$: if $\tilde{\gamma}(1) = 0$, then $\tilde{\gamma} \simeq const$, so $\gamma = p \circ \tilde{\gamma} \simeq const$.

So Φ is an isomorphism, $\pi_1(S^1) \cong \mathbb{Z}$. □

Next we'll prove Van Kampen theorem (0.1.1). In fact we only need to prove that:

Claim 0.1.6. The map

$$j'_\sharp * j''_\sharp : \pi_1(U', x_0) * \pi_1(U'', x_0) \to \pi_1(X, x_0)$$

is a surjective homomorphism, and its kernel is the normal closure generated by $i'_\sharp(\delta) i''_\sharp(\bar{\delta})$.

CLearly it's a group homomorphism.

For any $\gamma \in \pi_1(X, x_0)$, it can be decompose to $a_1 b_1 a_2 \cdots a_k b_k$, where $a_i \subset U', b_i \subset U''$, let the partition points be $p_1, \ldots, p_k, q_1, \ldots, q_k \in W$, and denote s_i, t_i the path from x_0 to p_i, q_i. So we have

$$\gamma = \underbrace{a_1 \bar{s}_1}_{\in \pi_1(U', x_0)} \cdot \underbrace{s_1 b_1 \bar{t}_1}_{\in \pi_1(U'', x_0)} \cdots$$

Thus $j'_\sharp * j''_\sharp$ is indeed surjective.

At last we'll study its kernel, let $\gamma \in \ker j'_\sharp * j''_\sharp$. Since $\gamma \simeq \{x_0\}$, say the homotopy is $H : [0,1] \times [0,1] \to U' \cup U''$.

We can partition $[0,1] \times [0,1]$ to many small cells such that each cell's image is completely contained in either U' or U''.

TODO

Using the "word processing" method, since we've showed that $\gamma = \alpha_1 \beta_1 \cdots$ where $\alpha_i \subset U', \beta_i \subset U''$. So actually we're saying that

$$\gamma = i'_\sharp(\alpha_1) i''_\sharp(\beta_1) \cdots$$

if we some $\delta \subset U' \cap U''$, then the conjugate of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$ can change $\cdots i'_\sharp(\delta) \cdots$ to $\cdots i''_\sharp(\delta) \cdots$.

Thus if γ is in the kernel, it can indeed be written as a product of conjugates of $i'_\sharp(\delta) i''_\sharp(\delta)^{-1}$.

Remark 0.1.7 — A more frequently used version is that W is a strong deformation kernel of some open neighborhood in X.

Example 0.1.8

For any finite representation of a group

$$G = \langle x_1, \dots, x_n \mid R_1, \dots, R_n \rangle \,,$$

G can be realized as the fundamental groups of a space: Let X be a CW-complex with a single 0-cell, n 1-cells corresponding to x_i, and m 2-cells corresponding to R_i.

Remark 0.1.9 — The path connected condition of W can't be removed, e.g. two segments can fuse to S^1.

Example 0.1.10

Let $f : S \to S$ be a homeomorphism, where S is a closed surface. Consider the *mapping torus*:

$$M_f = S \times [0, 1]/ \sim,$$

where $(0, 0) \sim (f(x), 1)$.

Let $Y = S \times \{0\} \cup (\{x_0\} \times [0, 1])$, U' is an open neighborhood of Y, $U'' = M_f \backslash Y$.

Observe that $U' \simeq S \vee circle$, and $U'' \simeq (S\backslash disk) \times (\varepsilon, 1 - \varepsilon) \simeq S\backslash disk$.

$$\pi_1(M_f) \cong \pi_1(X) * \langle t\rangle \,/(g \sim tf_\sharp(g)t^{-1}) \cong \pi_1(S) \rtimes_{f_\sharp} \langle t\rangle$$

Seifent-vanKampen: if $i'_\sharp, i''_\sharp$ are both injective, then $j'_\sharp, j''_\sharp$ are also injective.

Next we'll see some applications of fundamental groups:

- Bronwer fixed point theorem: A continuous map $f : D^n \to D^n$ must have a fixed point.
- Invariance of the boundary: If $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$, s.t. $x \in \mathbb{R}^{n-1} \times \{0\}$, then x doesn't have a neighborhood homeomorphic to $\mathbb{R}^n$.
- Invariance of regions: If $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is also open, i.e. $f : U \to f(U)$ is a homeomorphism.

Here we can only prove the case $n = 2$, since the complete proof need homotopy groups of rank n (i.e. π_n), but here we only introduced π_1. However, the idea is nearly identical.

Proof. Assmue by contradiction that f has no fixed points, let $g(x) = \frac{x-f(x)}{\|x-f(x)\|}$, then $g : D^n \to S^{n-1}$ is a deformation. Thus $g_\sharp : \pi_1(D^2) \to \pi_1(S^1)$ surjective, but $\pi_1(D^2) = \{1\}, \pi_1(S^1) = \mathbb{Z}$, contradiction! □

Proof. If x is an interior point, $x \in U$ and U homeomorphic to $\mathbb{R}^n$, then $U\backslash\{x\}$ can deform to a $n - 1$ dimensional sphere, thus $\pi_1(U\backslash\{x\}) \neq \{1\}$.

But if x is a boundary point, then $\pi_1(U\backslash\{x\}) = \{1\}$, contradiction! □

Proof. Assume by contradiction that there exists $0 \in U$ s.t. $f(0) \in \mathbb{R}^n$ has no open neighborhood lying completely in $f(U)$.

We can construct a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\|x - g(f(x))\| \leq 1, \quad x \in B(0, 1); \quad g(f(x)) \neq 0.$$

Then by Bronwer fixed point theorem on $x \mapsto x - g(f(x))$ we get a contradiction.

The construction is as below:

Since $f(\partial B(0,1))$ must be at least say 10ε away from 0, and $B(f(0),\varepsilon)$ has a point outside of the image of f, so we have a map $P : B(f(0),\varepsilon)\backslash\{p\} \to \partial B(f(0),\varepsilon)$.

Then consider $g = f^{-1} \circ P$, since f^{-1} may not exist on every point, so we need Tietze extension theorem to get an extension h. In $B(f(0), 2\varepsilon)$, we'll change h a little (i.e. take a polynomial approximation) to ensure $g(f(x)) \in B(0,1)$. □

§0.2 Covering spaces

Except van Kampen's theorem, there's another way to compute fundamental groups.

Definition 0.2.1 (Covering maps)**.** Let $p : \widetilde{X} \to X$ be a continuous map. If

- p is surjective;
- For any $x \in X$, there exists an open neighborhood $U = U(x) \subset X$, such that $p^{-1}(U)$ is a union of disjoint open sets $\{U_\alpha\}$, and p is homeomorphism from U_α onto U for each α.

Then we say p is a **covering map**, and $\widetilde{X}$ is a **covering space** of X. $p^{-1}(x)$ is called a **fiber**.

Remark 0.2.2 — Often we'll require $\widetilde{X}, X$ are path connected to ensure the relations with fundamental groups. In this case $\#p^{-1}(x)$ is constant.

Definition 0.2.3. We say two covering is **isomorphic** if exists $\tau : \widetilde{X} \to \widetilde{X}'$ s.t. $p' \circ \tau = p$. Two covering is **equivalent** if $p' \circ \widetilde{\sigma} = \sigma \circ p$.