# Geometry II

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## §0.1 Gauss map and Weingarten map

The strange definition of those curvatures don't come from nothing, in this section we'll cover this topic and give a geometric interpretation.

**Definition 0.1** (Gauss map)**.** Let $\Sigma$ be a regular surface in $\mathbb{E}^3$, denote its normal vector at $x$ by $\vec{n}(x)$. Then this map $\mathcal{G} : \Sigma \to \mathbb{S}^2$ by $x \mapsto \vec{n}(x)$ is called the **Gaussian map**.

In terms of a parametrized surface $\phi : U \to \mathbb{E}^3$, we can compute that

$$\mathcal{G} : U \to \mathbb{S}^2 : \quad \vec{n}(u) = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}$$

But each vector has a normal plane, namely $\vec{n}^\perp$, and this derives the **Weingarten map**:

**Definition 0.2** (Weingarten map)**.** For all $u \in U$, define $W : \vec{n}(u)^\perp \to \vec{n}(u)^\perp$: $\vec{v} \mapsto W(\vec{v})$, where

$$W(\vec{v}) = -\left.\frac{\mathrm{d}(\mathcal{G} \circ \gamma)}{\mathrm{d}u}\right|_{u=0}, \quad \gamma := \phi(u(r)) \text{ is a curve on the surface.}$$

> **Remark 0.3 —** In the language of modern differetial manifolds, Weingarten map is just the tangent map of Gauss map with a negative sign.

Since $\vec{n}^\perp$ has a basis $\phi_s, \phi_t$, we can compute the matrix of Weingarten map:

$$(\phi_s, \phi_t)W = (-\vec{n}_s, -\vec{n}_t).$$

Note that $-\vec{n}_s \cdot \phi_s = \vec{n} \cdot \phi_{ss} = L$, so if we take the inner product of $(\phi_s, \phi_t)$ on both sides, we get

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} W = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \implies W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Since $W$ is clearly a geometric quantity, so its trace and determinant are also geometric:

$$\operatorname{tr} W = \frac{GL - 2FM + EN}{EG - F^2} = 2H, \quad \det W = \frac{LN - M^2}{EG - F^2} = K,$$

which gives the average curvature and Gauss curvature.

Moreover, the principal curvatures are the eigenvalues of $W$, and principal directions are just the eigenspaces of $W$.

Let $\vec{v} = (\phi_s, \phi_t)X$, then its normal section has curvature

$$\kappa_n = \frac{X^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} X}{X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} X}.$$

When $\|\vec{v}\| = 1$, we can change a parameter s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = I_2$, in this case we can observe that when $\kappa_n$ attains its extremum, $\vec{v}$ is precisely the eigenvector of $W$, i.e. lies on the principal directions.

**Definition 0.4** (Curvature line)**.** A curve is called a **curvature line** if its tangent vector is the same as principal directions everywhere.

**Example 0.5**

Every curve on a sphere is curvature line.

Around a point where the principal curvatures are different, there exists a orthogonal grid of curvature lines.

**Example 0.6**

monkey saddle surface, "prong singularity"

In the case when the $s$-curve and $t$-curve are precisely the curvature lines, then we say this is a **curvature grid parameter**, and here we have $g = E\,\mathrm{d}s^2 + G\,\mathrm{d}t^2$ and $h = L\,\mathrm{d}s^2 + N\,\mathrm{d}t^2$.

**Remark 0.7** — The geometric interpretation of Gauss curvature: For $u \in D \subset U$,

$$|K(u)| = \lim_{\text{“}D \to u\text{”}} \frac{Area_{\mathbb{S}^2}(\mathcal{G}(D))}{Area_{\mathbb{E}^3}(\phi(D))}$$

while $sgn(K(u))$ is the orientation of $\mathcal{G}$ at point $u$.

**Example 0.8**

Consider the Gauss map of a torus, the "outer" part and the "inner" part of the torus maps to $\mathbb{S}^2$ bijectively. If we compute

$$\int_{T^2} K \,\mathrm{d}Area_E = \int_{\mathbb{S}^2} (1 + (-1))\,\mathrm{d}Area_S = 0 = 2\pi\chi(T^2),$$

as Gauss-Bonnet formula implies.

## §0.2 Fundamental equation of surfaces

Like the Fundamental theorem and Frenet frame in curve theory, we want to develop a theorem for describing surfaces using only fundamental forms.

Given a parameter on a surface, there's a natural frame $(\phi_s, \phi_t, \vec{n})$. If we take the derivative of the frame, we'll get

$$(\phi_s, \phi_t, \vec{n})_{st} = (\phi_s, \phi_t, \vec{n})_{ts}.$$

Taking the inner product with $(\phi_s, \phi_t, \vec{n})^T$ and apply the product rule:

$$\left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_s\right)_t - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_t (\phi_s, \phi_t, \vec{n})_s = \left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_t\right)_s - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_s (\phi_s, \phi_t, \vec{n})_t$$

This equation will give us some relations between the fundamental quantities. In literature these relations are known as Gauss equation and Codazzi equations.

Gauss equation can be written as:

$$(\phi_s \cdot \phi_{ts})_t - (\phi_s \cdot \phi_{tt})_s = \phi_{st} \cdot \phi_{st} - \phi_{ss} \cdot \phi_{tt}.$$

Codazzi equations are related to $\vec{n}$ and more complicated.

From Gauss equation we can deduce a famous theorem:

**Theorem 0.9** (Gauss' Theorema Egregium)
The Gauss curvature $K$ is determined by the first fundamental form.

*Proof.* Note that $(\phi_s \cdot \phi_{ts})_t = \frac{1}{2}E_{tt}$, and $(\phi_s \cdot \phi_{tt})_s = (F_t - \frac{1}{2}G_s)_s = F_{ts} - \frac{1}{2}G_{ss}$.

Suppose $\phi_{ss} = x\phi_s + y\phi_t + L\vec{n}$, then

$$\frac{1}{2}E_s = \phi_s \cdot \phi_{ss} = Ex + Fy, \quad F_s - \frac{1}{2}G_t = \phi_t \cdot \phi_{ss} = Fx + Gy$$

So $x, y$ is determined by $E, F, G$.

Similarly, we get

$$\begin{aligned}\phi_{ss} &= *\phi_s + *\phi_t + L\vec{n}\\ \phi_{st} &= *\phi_s + *\phi_t + M\vec{n}\\ \phi_{tt} &= *\phi_s + *\phi_t + N\vec{n}\end{aligned}$$

where $*$ are determined by $E, F, G$.

By Gauss equation, we get $* = -(LN - M^2) + *$, and $*$ is determined by $E, F, G$ and their partial derivatives. □

**Remark 0.10** — The computation looks messy, but in modern mathematics, we have a systematic notation which is more simplified.

**Definition 0.11** (Isometries)**.** Let $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. If a map $\psi : \widetilde{U} \to U$ satisfies $\psi^*(g) = \widetilde{g}$, then it's called an **isometry**.

Let $\mathcal{F} = (\phi_s, \phi_t, \vec{n})$. Suppose $\mathcal{F}_s = \mathcal{F}A$, and $\mathcal{F}_t = \mathcal{F}B$. Taking the second derivative we get $\mathcal{F}_{st} = \mathcal{F}(BA + A_t)$, $\mathcal{F}_{ts} = \mathcal{F}(AB + B_s)$. Thus we have

$$A_t - B_s = AB - BA.$$

But we want to express things in terms of $E, F, G$, so we can compute the dot product of $\mathcal{F}^T$ :

$$P := \mathcal{F}^T \cdot \mathcal{F} = \begin{pmatrix}E & F & \\ F & G & \\ & & 1\end{pmatrix}, \quad P^{-1} = \frac{1}{EG - F^2}\begin{pmatrix}G & -F & \\ -F & E & \\ & & 1\end{pmatrix}$$

Substituting into $\mathcal{F}_s = \mathcal{F}A$ we get

$$PA = \begin{pmatrix} \frac{E_s}{2} & \frac{E_t}{2} & -L \\ F_s - \frac{E_t}{2} & \frac{G_s}{2} & -M \\ L & M & 0 \end{pmatrix}$$

$$\mathcal{F}^T \mathcal{F}_{st} = (\mathcal{F}^T \mathcal{F}_s)_t - \mathcal{F}_t^T \mathcal{F}_s = (PA)_t - B^T PA.$$
$$\implies (PA)_t - (PB)_s = (PB)^T P^{-1}(PA) - (PA)^T P^{-1}(PB).$$

Gauss equation corresponds to the $(1,2)$ entry of the matrix equation:

$$\frac{1}{2}(E_{tt} - 2F_{st} + G_{ss}) = -(LN - M^2) + \frac{p}{4(EG - F^2)},$$

where $p$ is a polynomial in fundamental quantities and their derivatives.

These equations still looks complicated, so we'll "package" them:

$$-(LN - M^2) = R_{1212}.$$

Let

$$A = \begin{pmatrix} \Gamma^1_{-11} & \Gamma^1_{-12} & h^1_{-1} \\ \Gamma^2_{-11} & \Gamma^2_{-12} & h^2_{-1} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{-12} & \Gamma^1_{-22} & h^1_{-2} \\ \Gamma^2_{-12} & \Gamma^2_{-22} & h^2_{-2} \\ h_{12} & h_{22} & 0 \end{pmatrix}$$

Here the $\Gamma$'s are called Christoffel notations.

Codazzi equations correspond to the $(1,3),(2,3)$ enties:

$$L_t - M_s = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}$$
$$M_t - N_s = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{21}) - N\Gamma^2_{21}.$$

**Remark 0.12 —** The index notations for computation can be defined as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij}), \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = (g^{ij}),$$

and $h$ is defined similarly. If we use Einstein summation notation, we can write $g_{ij}g^{jk} = \delta_i^k$.

Let $\vec{v}_1 := \phi_s, \vec{v}_2 = \phi_t$, and

$$\frac{\partial \vec{v}_\alpha}{\partial \vec{u}^\beta} = \sum_\gamma \Gamma^\gamma_{-\alpha\beta} \vec{v}_\gamma + h_{\alpha\beta}\vec{n}, \quad \frac{\partial \vec{n}}{\partial \vec{u}^\beta} = -\sum_\gamma h^\gamma_{-\beta} \vec{v}_\gamma.$$

Here the upper index is defined as:

$$h^\gamma_{-\beta} := \sum_\delta g^{\gamma\delta} h_{\delta\beta}.$$

From this we can write $\Gamma$ out explicitly:

$$\Gamma^\gamma_{-\alpha\beta} = \sum_\delta \frac{g^{\gamma\delta}}{2} \left( \frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta} \right)$$

This is called *Christoffel notations.*

$$R^{\delta}_{\text{-}\alpha\beta\gamma} := \frac{\partial \Gamma^{\delta}_{\text{-}\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\text{-}\alpha\gamma}}{\partial u^{\beta}} + \sum_{\eta} (\Gamma^{\eta}_{\text{-}\alpha\beta}\Gamma^{\delta}_{\text{-}\eta\gamma} - \Gamma^{\eta}_{\text{-}\alpha\gamma}\Gamma^{\delta}_{\text{-}\eta\beta}).$$

This is called *Riemann symbols*. Another type is defined as:

$$R_{\delta\alpha\beta\gamma} = \sum_{\eta} g_{\delta\eta} R^{\eta}_{\text{-}\alpha\beta\gamma}.$$

In surface theory, only $R_{1212}$ is nontrivial.

Using these notations, we can write the equations as:

- Gauss:
$$R_{\delta\alpha\beta\gamma} = -(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}).$$

- Codazzi:
$$\frac{\partial h_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial h_{\alpha\gamma}}{\partial u^{\beta}} = \sum_{\delta} (h_{\beta\delta}\Gamma^{\delta}_{\text{-}\alpha\gamma} - h_{\gamma\delta}\Gamma^{\delta}_{\text{-}\alpha\beta}).$$

Here we explain the above computation a little.
The vector

$$\sum_{\gamma} \Gamma^{\gamma}_{\text{-}\alpha\beta} v_{\gamma} =: \nabla_{\beta}\vec{v}_{\alpha}$$

is called the covariant derivative of $\vec{v}_{\alpha}$. It's projection of the derivative of $\vec{v}_{\alpha}$ onto the tagent space.

$$\frac{\partial}{\partial u^{\beta}}\frac{\partial}{\partial u^{\gamma}}(v_{\alpha}) = \frac{\partial}{\partial u^{\gamma}}\frac{\partial}{\partial u^{\beta}}(v_{\alpha})$$

$$\implies -\nabla_{\beta}\nabla_{\gamma}v_{\alpha} + \nabla_{\gamma}\nabla_{\beta}v_{\alpha} = h_{\alpha\beta}\nabla_{\gamma}\vec{n} - h_{\alpha\gamma}\nabla_{\gamma}\vec{n}.$$

So the covariant derivative is not commutative, and the "curvature" or the second fundamental form basically measures this discommutation.

Now if we look at

$$\frac{\partial g_{\delta\alpha}}{\partial u^{\beta}} = \frac{\partial v_{\delta}\cdot v_{\alpha}}{\partial u^{\beta}} = \frac{\partial v_{\delta}}{\partial u^{\beta}}v_{\alpha} + v_{\delta}\frac{\partial v_{\alpha}}{\partial u^{\beta}} = \sum_{\gamma} g_{\alpha\gamma}\Gamma^{\gamma}_{\text{-}\delta\beta} + \sum_{\gamma} g_{\delta\gamma}\Gamma^{\gamma}_{\text{-}\alpha\beta},$$

similarly, by symmetry, computing

$$\frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}, \quad \frac{\partial g_{\delta\beta}}{\partial u^{\alpha}},$$

will yield

$$\Gamma^{\gamma}_{\text{-}\alpha\beta} = \sum_{\delta} \frac{g^{\gamma\delta}}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}\right).$$

In fact this is more intuitive in Einstein summation notation.

## §0.3 Fundamental theorem of surface theory

**Theorem 0.13** (Fundamental theorem of surface theory)
Let $D \subset \mathbb{R}^2$, $u = (u^1, u^2)$ is the coordinate. Let $g_{\alpha\beta}, h_{\alpha\beta} : D \to \mathbb{R}$ be $C^3$ functions, and the matrix $(g_{\alpha\beta})$ is symmetrical anf positive definite, $(h_{\alpha\beta})$ is symmetrical.

Let $g^{\alpha\beta}$ be the inverse matrix of $g_{\alpha\beta}$, and $R_{\delta\alpha\beta\gamma}$ is as above. If these functions satisfies Gauss equation and Codazzi equation, then:

For all $p \in D$, exists a neighborhood $U = U(p) \subset D$ and a regular surface $\phi : U \to \mathbb{E}^3$, such that $g_{\alpha\beta}, h_{\alpha\beta}$ are the first and second fundamental quantities of $\phi$.

Furthermore, if $\widetilde{\phi} : U \to \mathbb{E}^3$ also satisfies above conditions, then $\widetilde{\phi} = \sigma \circ \phi$, where $\sigma$ is an isometry of $\mathbb{E}^3$.

Basically we need to solve a partial differential equation, and we need to consider how to construct this equation.

*Proof.* Let $\phi : U \to \mathbb{E}^3$, $v_\alpha, \vec{n} : D \to V(\mathbb{E}^3)$ be unknown functions satisfying

$$\begin{cases} v_\alpha = \dfrac{\partial \phi}{\partial u^\alpha} \\ \dfrac{\partial v_\alpha}{\partial u^\beta} = \displaystyle\sum_\gamma \Gamma^{\gamma}_{\ \alpha\beta} v_\gamma + h_{\alpha\beta}\vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = -\displaystyle\sum_\gamma h^{\gamma}_{\ \beta} v_\gamma \end{cases}$$

This is a linear homogeneous PDE of degree 1, and it actually has a unique solution.

Consider the initial-value problem in the neighborhood of a given point $p \in D$.

We hope to prove that

- The above PDE initial-value problem has a unique solution under the Gauss-Codazzi equations;
- If initially (i.e. at $p$) we have
$$\vec{n} = \frac{v_1 \times v_2}{\|v_1 \times v_2\|},$$
then it holds for all $p' \in U(p)$.

For the second statement, we can compute $\frac{\partial}{\partial u^\beta}(\vec{n} \cdot v_\alpha) = 0$, so they are constant. □