Mathematical Analysis II

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Proof. By the inverse function theorem, let $F(x, y) := \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ with

$$(x, y) \mapsto (x, f(x, y))$$

So $F(x^*, y^*) = (x^*, 0)$, and

$$\mathrm{d}F(x^*, y^*) = \begin{pmatrix} I_n & 0 \\ \mathrm{d}_x f(x^*, y^*) & \mathrm{d}_y f(x^*, y^*) \end{pmatrix}.$$

Since $\mathrm{d}_y f(x^*, y^*)$ is inversible, $\mathrm{d}F(x^*, y^*)$ is inversible as well. Hence there exists neighborhoods of (x^*, y^*) and $(x^*, 0)$, say $\widetilde{\Omega}$ and $\widetilde{\Omega}_1$, such that F is a C^1 homeomorphism $\widetilde{\Omega} \to \widetilde{\Omega}_1$.

We can find $U \ni x^*, V \ni y^*$ s.t. $U \times V \subset \widetilde{\Omega}$. Let T be the C^1 map s.t.

$$F^{-1}(x, z) = (x, T(x, z)).$$

Let $\phi(x) = T(x, 0)$, we have

$$F(x, \phi(x)) = F(x, T(x, 0)) = (x, 0) \implies f(x, \phi(x)) = 0.$$

Since F is a bijection, clearly $f(x, y) = 0 \implies y = \phi(x)$. By taking the differentiation of $f(x, \phi(x)) = 0$,

$$(\mathrm{d}_x f, \mathrm{d}_y f) \cdot \begin{pmatrix} I_n \\ \mathrm{d}\phi(x) \end{pmatrix} = 0 \implies \mathrm{d}_x f(x, \phi(x)) + \mathrm{d}_y f(x, \phi(x)) \cdot \mathrm{d}\phi(x) = 0.$$

□

§0.1 Submanifolds in Euclid space

The implicit function theorem actually gives an example of manifolds: the preimage of $f(x, y) = 0$ is an n-dimensional manifold in $\mathbb{R}^{n+p}$.

Definition 0.1.1 (Manifolds)**.** Let $M \subset \mathbb{R}^n$ be a nonempty set. If $\exists d \geq 0$, $\forall x \in M$ exists open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^d$, and a differential homeomorphism $\Phi : U \to V$, such that

$$\Phi(U \cap M) = V,$$

we say M is a **d-dimensional differential manifold**. Denote $\dim M = d$, and $n - d$ is called the **codimension** of M.

Remark 0.1.2 — There might be different maps $\phi_1 : U_1 \to V_1, \phi_2 : U_2 \to V_2$, when $U_1 \cap U_2 \cap M \neq \emptyset$, we must have $\phi_2 \circ \phi_1^{-1}$ is a differential map from V_1 to V_2. In fact when M isn't a subset of $\mathbb{R}^n$, this is the original definition of differential manifolds.

Corollary 0.1.3 (Regular value theorem)

Let $f : \Omega \to \mathbb{R}^p$ be a smooth map, where $\Omega \subset \mathbb{R}^n$ is an open set, $n \geq p$. For all $c \in \mathbb{R}^p$, we call the **fibre** of c to be its preiamge:

$$f^{-1}(c) = \{x \in \mathbb{R}^n \mid f(x) = c\}.$$

If $\forall x \in f^{-1}(c)$, $\operatorname{rank} \mathrm{d}f(x) = p$, then $f^{-1}(c)$ is a manifold with **codimension** p.

Example 0.1.4

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ with $x \mapsto |x|^2 - 1$, then $S^{n-1} = f^{-1}(0)$.

Since $\mathrm{d}f = (2x_1, 2x_2, \ldots, 2x_n)$, clearly $\operatorname{rank} \mathrm{d}f = 1$ for all $x \in S^{n-1}$, so S^{n-1} is a manifold with codimension 1.

Example 0.1.5

Consider a surface in $\mathbb{R}^4 = \mathbb{C}^2$:

$$T^2 = \{(z_1, z_2) \mid |z_1| = 1, |z_2| = 1\}.$$

Let $f(x, y, z, w) = x^2 + y^2 - 1, g(x, y, z, w) = z^2 + w^2 - 1$, then $T^2 = \begin{pmatrix} f \\ g \end{pmatrix}^{-1} (0)$.

The differentiation is

$$\mathrm{d}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & 2w \end{pmatrix},$$

so T^2 is a manifold with codimension 2.

Definition 0.1.6. Let $M \subset \mathbb{R}^n$ be a manifold. If $\dim M = 1$, we say M is a curve; if $\dim M = 2$, M is a surface; and if $\dim M = n - 1$, we say M is a hyperplane.

Lemma 0.1.7

Let $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, if $\forall x_0 \in f^{-1}(0)$, $\mathrm{d}f(x_0) \neq 0$, then $f^{-1}(0)$ is a smooth hyperplane, it is called the hyperplane globally determined by an equation.

Example 0.1.8

In $\mathbb{R}^3$, f, g are smooth functions. If for all $x \in \mathbb{R}^3$ with $f(x) = g(x) = 0$ we have $\nabla f, \nabla g$ are linearly independent, then $\{f = g = 0\}$ is a smooth curve.

Theorem 0.1.9 (Parametrization of manifolds)
Let Ω be an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^{n+p}$ is a smooth map. If $\forall x^* \in \Omega$, $\operatorname{rank} \mathrm{d}f(x^*) = n$, then there exists an open set U, $x^* \in U$ s.t. $f(U) \subset \mathbb{R}^{n+p}$ is an n-dimensional manifold.

Proof. Let x_i be a coordinate in $\mathbb{R}^{n+p}$.

WLOG $(\frac{\partial f_i}{\partial x_j})_{1\le i,j\le n}$ is non-degenerate, let $F = (f_1, \dots, f_n)$, $G = (f_{n+1}, \dots, f_{n+p})$ and apply inverse function theorem on F, there exists open neighborhoods $U \ni x, V \ni F(x) =: y$, s.t. $F : U \to V$ is a smooth homeomorphism.

$$\begin{tikzcd} U \subset \Omega \arrow[r, "F"] \arrow[d, "f"] & V \subset \mathbb{R}^n \arrow[dl, "\phi"] \\ \mathbb{R}^{n+p} & \end{tikzcd}$$

So $f(x) = (F(x), G(x)) = (y, GF^{-1}(y))$. Let

$$\phi : V \to \mathbb{R}^n, \quad y \mapsto (y, GF^{-1}(y)).$$

We can see that ϕ is a homeomorphism $V \to f(U)$. (Indeed it's a bijection) So by definition we know $f(U)$ is a manifold. $\square$

Example 0.1.10
Let

$$\phi(\theta, r) = \begin{cases} x = \left(1 + r\cos\dfrac{\theta}{2}\right)\cos\theta \\ y = \left(1 + r\cos\dfrac{\theta}{2}\right)\sin\theta \\ z = r\sin\dfrac{\theta}{2} \end{cases}, \quad I = [0, 2\pi] \times (-1, 1).$$

Then $M = \phi(I)$ is a Mobius strip, which is a two dimensional smooth manifold in $\mathbb{R}^3$, as $\mathrm{d}\phi$ has rank 2 everywhere.

Besides, there doesn't exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ s.t. $M = f^{-1}(0)$. Basically this is because M is not orientable, but ∇f and $-\nabla f$ are "normal" directions of M, which makes it orientable. Below we give a sketch:

Proof. Let $v(\theta) = \cos\frac{\theta}{2} e_2(\theta) - \sin\frac{\theta}{2} e_1(\theta)$, where $e_2(\theta) = (0, 0, 1), e_1(\theta) = (\cos\theta, \sin\theta, 0)$.

Note that $e_1 \perp e_2$, consider the curve $\beta : [0, 2\pi] \to \mathbb{R}^3$

$$\theta \mapsto (\cos\theta, \sin\theta, 0) + \varepsilon v(\theta).$$

Let ε be sufficiently small, when $\varepsilon \neq 0$ we can check β and M do not intersect. We can take ε s.t. $f(\beta(0)) > 0$ as $\mathrm{d}f \neq 0$. (ε can be negative)

Since $\beta(0) = (1, 0, \varepsilon), \beta(2\pi) = (1, 0, -\varepsilon)$, when $f(\beta(0)) > 0$, we must have $f(\beta(2\pi)) < 0$. By continuity, $\exists\theta_0$ s.t. $f(\beta(\theta_0)) = 0$, which means $\beta(\theta_0) \in M$, contradiction! $\square$

Midterm exam....qaq

Proposition 0.1.11

Let $\Omega \subset \mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^m$ is a smooth map. Let $S \subset \mathbb{R}^m$ be a differential manifold, if for all $x \in f^{-1}(S)$, we have $\operatorname{rank} \mathrm{d}f(x) = m$, then $f^{-1}(S)$ is a differential manifold with codimension same as S.

Proof. For any $x \in S$, let Φ be the homeomorphism from an open neighborhood of x to $\mathbb{R}^m$.

Suppose $\dim S = d$, let
$$F(x) = ((\Phi \circ f)_{d+1}, \ldots, (\Phi \circ f)_m).$$
Note that $\mathrm{d}(\Phi \circ f)$ is an $m \times n$ matrix, and its rank is m. Since
$$\mathrm{d}(\Phi \circ f) = \begin{pmatrix} \mathrm{d}(\Phi \circ f)_1 \\ \vdots \\ \mathrm{d}(\Phi \circ f)_d \\ \mathrm{d}F \end{pmatrix}$$
Thus $\mathrm{d}F$ is a $(m-d) \times n$ matrix with rank $m-d$. So $F^{-1}(0) = f^{-1}(S)$ is a manifold with dimension $n - (m-d)$. □

§0.2 Tangent space

Since the differentiation relies on the choice of coordinates, if we want to study the manifold more geometrically, we should look at the tangent lines or tangent planes of the manifold.

Definition 0.2.1 (Tangent vectors)**.** Let M be a differential manifold. Let $p \in M$, for all parametrized curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we say the vector $\gamma'(0) \in \mathbb{R}^n$ is the **tangent vector** of γ at point p.

Let T_pM denote the **tangent space** at p, which is defined as
$$T_pM = \{\gamma'(0) \in \mathbb{R}^n \mid \gamma(0) = p\}.$$

It's clear that T_pM should be a vector space of dimension $\dim M$, next we'll prove this fact.

Proposition 0.2.2 (Push forward of tangent spaces under differential homeomorphism)

Let $\Phi : U \to V$ be a differential homeomorphism, $M \subset U$ be a manifold, then
$$T_{\Phi(p)}\Phi(M) = (\mathrm{d}\Phi)\big|_p \cdot T_pM.$$

Proof. Let γ be a parametrized curve on M with $\gamma(0) = p$. Note that $\Phi \circ \gamma$ is a curve on $\Phi(M)$ passing through $\Phi(p)$. Since
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \circ \gamma(t)\Big|_{t=0} = \mathrm{d}\Phi(p) \cdot \gamma'(0).$$
Thus $\mathrm{d}\Phi(p) \cdot T_pM \subset T_{\Phi(p)}\Phi(M)$.

Now we do the same thing for Φ^{-1}, we can get the desired equality. □

Now we can easily calculate the tangent space: since M is locally homeomorphic to $\mathbb{R}^d$, and obviously $T_{\Phi(p)}(\mathbb{R}^d \times \{0\}) = \mathbb{R}^d \times \{0\}$, by above proposition, $T_pM = (\mathrm{d}\Phi) \cdot T_{\Phi(p)}(\mathbb{R}^d \times \{0\})$ is a vector space of dimension d.

Theorem 0.2.3

Let M be a manifold, T_pM is a vector space of dimension $\dim M$.

Proposition 0.2.4

Let $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a smooth map, $\operatorname{rank} \mathrm{d}f = n$. Let $M = f^{-1}(f(p))$, then $T_pM = \ker \mathrm{d}f(p)$.

Proof. Let

$$F(x, y) = (x, f(x, y)), x \in \mathbb{R}^d, y \in \mathbb{R}^n.$$

F is a homeomorphism, so $T_pM = (\mathrm{d}F^{-1})T_{F(p)}F(M)$.

Note that $F(M) = \{(x, p) \mid \exists y, f(x, y) = f(p)\}$, it must be a vector space of dimension d, so $T_{F(p)}F(M) = \mathbb{R}^d \times \{0\}$,

$$T_pM = (\mathrm{d}F^{-1})T_{F(p)}F(M) = \ker \mathrm{d}f(p).$$

□

Example 0.2.5

Let M be a manifold determined by $f : \mathbb{R}^n \to \mathbb{R}$,

$$T_pM = \ker \mathrm{d}f = \{v \in \mathbb{R}^n \mid \mathrm{d}f(p)v = 0\}.$$

Here $\mathrm{d}f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) = \nabla f$. So $v \in T_pM \iff \nabla f \cdot v = 0$, the dot means the inner product. In this case the vector ∇f is called **normal direction vector**.

Next we'll briefly introduce the differential maps between manifolds. Since the differentiation on manifolds is hard to define, so what we do is actually regarding manifolds as $\mathbb{R}^d$ locally and define the differentiablity using the maps between Eucild spaces.

Definition 0.2.6. Let M, N be manifolds in $\mathbb{R}^m, \mathbb{R}^n$, respectively. $f : M \to N$ is a map, if $\forall p \in M$, there exists $p \in U \subset \mathbb{R}^m, V \subset \mathbb{R}^d$, $\Phi : U \to V$ s.t.

$$f_\Phi = f \circ \Phi^{-1}$$

is a smooth map from V to N. We say f is a smooth map from M to N.

We need to check this definition is well-defined: if there's another homeomorphism Φ', $f \circ \Phi' = (f \circ \Phi^{-1}) \circ (\Phi \circ \Phi'^{-1})$ is indeed a smooth map.

Lemma 0.2.7 (Smooth maps are locally restrictions of smooth maps in Eucild spaces)

Let $f : M \to N$ be a map, then f is smooth $\iff \forall p \in M, \exists p \in U \subset \mathbb{R}^m$ and a smooth map $F : U \to \mathbb{R}^n$ s.t.

$$f\big|_{U \cap M} = F\big|_{U \cap M}.$$

Proof. Let τ denote the embedding from $M \cap U$ to U. Since $f \circ \Phi^{-1} = F \circ \tau \circ \Phi^{-1}$, so F smooth $\implies f_\Phi$ smooth $\implies f$ smooth.

$$\begin{tikzcd}
V \subset \mathbb{R}^d \arrow[dr, "f \circ \Phi^{-1}"'] & M \cap U \arrow[l, "\Phi"'] \arrow[r, "\tau"] \arrow[d, "f"] & U \arrow[dl, "F"] \\
& N \subset \mathbb{R}^n &
\end{tikzcd}$$

On the other hand, $\square$