Geometry II

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At last we'll prove an interesting fact:

Proposition 0.0.1 (The local existence of isothermal parameters)
Let $\phi : U \to \mathbb{E}^3$, for all $\hat{u} \in U$, there exists a neighborhood $\widetilde{U}$ and a reparametrization $u = u(\widetilde{u})$, such that
$$g(\widetilde{u}) = \rho^2(\widetilde{u})(\widetilde{E}\,\mathrm{d}\widetilde{s}^2 + \widetilde{G}\,\mathrm{d}\widetilde{t}^2).$$

Remark 0.0.2 — Note that the right hand side is clearly conformal to regions in $\mathbb{E}^2$, so this in fact implies that any surfaces is locally conformal to $\mathbb{E}^2$.

Proof. The critical idea is to realize $\mathbb{R}^2$ as $\mathbb{C}$. To be more precise, we'll follow the steps below:

- Find a way to express $E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ as $(a\,\mathrm{d}s + b\,\mathrm{d}t)(\overline{a}\,\mathrm{d}s + \overline{b}\,\mathrm{d}t)$, where a, b are functions with complex value.
- If there exists a complex function f s.t. $\mathrm{d}f(s+it) = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$, then $g = \frac{1}{|\rho|^2}\,\mathrm{d}f\,\mathrm{d}\overline{f}$.
- Assume further that f is *holomorphic* and non-degenerate, then $f(u) = \widetilde{x}(u) + i\widetilde{y}(u)$ is locally inversible, i.e. exists $u = u(\widetilde{x}, \widetilde{y})$, then
$$g = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x} + i\,\mathrm{d}\widetilde{y})(\mathrm{d}\widetilde{x} - i\,\mathrm{d}\widetilde{y}) = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x}^2 + \mathrm{d}\widetilde{y}^2).$$

Let $a = \sqrt{E}$, $b = \frac{-F + i\sqrt{EG - F^2}}{\sqrt{E}}$. (Note $EG - F^2 > 0$ as g is positive definite)
Next we'll choose suitable f, ρ. Consider the differential equation $T = T(s,t)$:
$$\frac{\partial T}{\partial s} = -\frac{a(s,T)}{b(s,T)}, \quad T(\hat{s}, t) = t.$$

From the relation $f(s, T(s,t)) = t - \hat{t}$ and implicit function theorem we can uniquely determine f.

Remark 0.0.3 — The detail of the solution to this equation in complex functions is beyond the scope of this class.

Such f satisfies $\mathrm{d}f = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$.

When $f(s,t) = (\widetilde{x}, \widetilde{y})$, the Jacobian determinant is

$$\widetilde{x}_s\widetilde{y}_t - \widetilde{x}_t\widetilde{y}_s = -|\rho|^2(a\bar{b} - b\bar{a}) = |\rho|^2\sqrt{EG - F^2} > 0.$$

so f must be non-degenerate. □

§1 Algebraic topology

§1.1 A bit of manifold

First we'll introduce a few concepts before we move on.

- We say a topological space is an n-dimensional **topological manifold** if it's Hausdorff and locally homeomorphic to $\mathbb{R}^n$. Sometimes we also require manifolds to be compact / paracompact / C_2. Here paracompact means that any open covering has a locally finite subcovering.
- Manifolds with boundary: locally homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$.
- When we talk about the regularity of manifolds, we must appoint an atlas first. Let $\phi_i : U_i \to E_i \subset \mathbb{R}^n$ be homeomorphisms mentioned above, then each ϕ_i is a **chart**, and $\{(U_i, \phi_i)\}_{i\in I}$ is the **atlas**. The map
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
is called **transition functions**.

  The regularity of the manifold is actually the regularity of transition functions, such as C^r, C^∞, piecewise linear, etc.

Example 1.1.1

The sphere $\mathbb{S}^2$ and projective plane $\mathbb{R}P^2$ are 2d manifolds. But they're different since $\mathbb{R}P^2$ is not *orientable*. In fact $\mathbb{R}P^2$ can be obtained by fusing the edge of a Mobius band to a disk(keep in mind that Mobius band has only one edge!).

There are many manifolds which looks wired, but I can't draw them on the computer ;)

Example 1.1.2 (Projective curves)

Consider a quadratic equation

$$C : z^2 + w^2 = 1, \quad (z, w) \in \mathbb{C}^2.$$

What does this surface look like?

Let $Z = z + iw, W = z - iw$, the equation becomes $ZW = 1$, hence the surface is $(\zeta, \frac{1}{\zeta}), \zeta \in \mathbb{C}\backslash\{0\}$. So C is homeomorphic to $\mathbb{C}\backslash\{0\}$.

We can also discuss this in $\mathbb{C}P^2 = \mathbb{C}P^1 \cup \mathbb{C}^2$, where $\mathbb{C}P^1 = \{\infty\} \cup \mathbb{C} \cong \mathbb{S}^2$.

So in homogeneous coordinate, the equation can be written as $ZW = T^2$. The surface is consisting of $(1,0,0), (0,1,0), (\zeta, \frac{1}{\zeta}, 0)$. Thus the projective completion of C is homeomorphic to $\mathbb{S}^2$, which is $\mathbb{C}\backslash\{0\}$ appending with two points.

Example 1.1.3 (Elliptic curves)

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ pairwise different.

$$E : w^2 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3).$$

What does E looks like in $\mathbb{C}P^2$?

Observe that for $z \in \mathbb{C}\backslash\{\lambda_1, \lambda_2, \lambda_3\}$, there're 2 values for w. So the image of E is two planes($\mathbb{C}$) fused together at $\lambda_1, \lambda_2, \lambda_3$ and ∞ with some adjust.

In fact this can be realized as two cylinder fused together at their edges.

$E \cong T^2\backslash\{pt\}$ in $\mathbb{C}^2$, and T^2 in $\mathbb{C}P^2$.

In fact $\mathbb{C}P^2$ is a 4-dimensional closed manifold, and it's also a 2-dimensional complex manifold. $PSL(3, \mathbb{C})$ acts transitively on $\mathbb{C}P^2$.

Example 1.1.4

We can fuse the edges of polygons to get manifolds: By fusing together opposite edges of a square, we can get torus or Klein bottle.

We'll use the word "fuse" frequently in the future, so here we'll make it clear what we mean by "fusing" things together.

Definition 1.1.5 (Quotient maps)**.** A continuous map $f : X \to Y$ is called a **quotient map**, if it's surjective, and $\forall B \subset Y$, $f^{-1}(B)$ open $\implies$ B open.

This is saying that the topology on Y is the "largest" topology (or quotient topology) while keeping f continuous.

So when we "fusing" things together, we're actually giving an equivalence relation on the original space, and the result is the quotient topology induced from the natural projection map.

Now we look at the elliptic curves again, let $U = \mathbb{C}\backslash([\lambda_1, \lambda_2] \cup [\lambda_3, \infty])$. Let X be the path end compactification of U, then $X \simeq S^1 \times [0, 1]$.

Let X_1, X_2 be two copies of X, and fusing the corresponding circles at the end in the reversed direction, we'll get a torus without 4 points, by adding $\lambda_1, \lambda_2, \lambda_3$ back we'll get $T^2\backslash\{pt\}$.

Remark 1.1.6 — The quotient topology may have some bad properties, like not being Hausdorff: Consider $\mathbb{R}^2\backslash\{(0, 0)\}$ with connected vertical lines as equivalence class, then we'll get a line with 2 points at the origin, which is a typical non-Hausdorff space.

A closed surface is a connected compact 2-dimensional manifold with no edges. We have the following classification theorem:

Theorem 1.1.7

All the closed surfaces must be homeomorphic to $nT^2 (n \geq 0)$ or $mP^2 (m \geq 1)$. Here n is called the **genus** of orientable surfaces.

nT^2 can be viewed as S^2 fused with n handles (torus), and mP^2 can be viewed as S^2 fused with m crosscaps (Mobius strip).

In this course we mainly talk about surfaces with triangulation, i.e. we take it for granted that all surfaces has triangulation.

Here we'll prove part of this theorem (since the other part needs further knowledge).

> **Remark 1.1.8 —** X has a triangulation means that X is homeomorphic to finitely many n-simplex fused together at the boundary linearly, and the *link* of each vertical is a triangulation of S^{n-1}.

Proof. Observe that given a triangulation, we can get a polygon fusing presentation of the surface by adding the triangles one by one, fusing only one edge each time.

If we write down the edges of this polygon at a certain order, using letters to indicate differet edges and bars for direction, we can get something like $ab\overline{a}\overline{b}$ for a torus.

TODO: pictures!

In fact, nT^2 can be presented as $[a_1, b_1][a_2, b_2] \dots [a_n, b_n]$, where $[a, b] = ab\overline{a}\overline{b}$. Likely, mP^2 is $c_1^2 c_2^2 \dots c_m^2$ since P^2 is c^2. So our goal is to say that any given "edge words" can be reformed to one of the above standard forms.

Note that $(A) : Wa\overline{a} = W$, and $(B) : aUV\overline{a}U'V' = bVU\overline{b}V'U'$. The second operation is cut the polygon in the middle to get b, and fuse two parts together to eliminate a. There's also a reversed version: $aUVaU'V' = bV'VbUU'$. Also note that the word is cyclic, so $(C) : UV = VU$.

TODO: pictures!

This is kind of like Olympiad combinatorics problem. So we need techniques like:

- A "complexity" to measure how close we are to destination:

  vertical numbers(verticals fused together are regarded as one) and egde pair numbers

- Some labels to control different branches:

  whether it has edges with the same direction

- Some efficient "combo moves"

Observe that

- (A) will reduce vetical and edge pair by 1,
- (B) won't effect edge pairs, but may change vertical numbers,
- (C) won't change anything.

In fact we can reduce the vertical number to 1, i.e. all the verticals are fused to one point in the surface. If we have at least 2 verticals, say P and Q, and PQ is an edge. There must be another edge connecting P, Q, If those two P are different in the polygon, we can use (B) to eliminate one P vertical (by adding edge pair of QQ), and use (A) to eliminate they're the same.

TODO: pictures!!!

Repeating above process we can make the vertical number become 1.

If we have $aUbV\overline{a}U'\overline{b}V'$, we can use (B) twice to reform it to $cd\overline{c}\overline{d}W$.

TODO: pictures!!!

So we can achive nT^2 from a word with no same-direction-pairs. Techniquely we still need to prove that we can always find $a \dots b \dots \overline{a} \dots \overline{b}$ in original word, but this can be proved easily otherwise we can perform (A) to reduce edges.

Now for mP^2 :

After some fancy operations we're done. $\square$