Measure Theory

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§1 Introduction

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§1.1 Starting from probablistics

Definition 1.1 (σ-algebra)**.** Let $\mathscr{F}$ be a family of subsets of a set Ω, if

- $\Omega \in \mathscr{F}$;
- If $A \in \mathscr{F}$, $A^c \in \mathscr{F}$;
- If $A_1, A_2, \cdots \in \mathscr{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.(Countable union)

then we call $\mathscr{F}$ a σ-algebra.

Some intros about probablistics (left out because I haven't studied probablistics yet;)

§1.2 What is measure theory?

It's an abstract theory, different from probablistics and real analysis. In this course we study a general set X, focus on mathematical thinking and skills, from the simple to construct the complex.

Measure theory studies the intrinsic structure of mathematical objects, and the map between different measure spaces.

§2 Measure spaces and measurable maps

§2.1 Sets and set operations

Definition 2.1. A non-empty set X is our space(universal set), its elements (points) are denoted by lower case letters $x, y, \ldots$.

Some notations:

$$x \in A, x \notin A, x \in A^c, A \subset B, A \cup B, AB = A \cap B,$$

$$B \backslash A (B - A \text{ when } A \subset B), A \Delta B.$$

A family of sets $\{A_t, t \in T\}$.

$$\bigcup_{t \in T} A_t := \{x : \exists t \in T, s.t. x \in A_t\}, \quad \bigcap_{t \in T} A_t := \{x : x \in A_t, \forall t \in T\}.$$

Sometimes we write the union of disjoint sets as sums to emphasize the disjoint property.

Monotone sequence of sets:

$$A_n \uparrow : A_n \subset A_{n+1}, \forall n; \quad A_n \downarrow : A_n \supseteq A_{n+1}, \forall n.$$

Next we define the limits of sets:

Definition 2.2. For monotone sequences:

$$\lim_{n \to \infty} A_n := \bigcup_{n=1}^{\infty} A_n \text{ or } \bigcap_{n=1}^{\infty} A_n.$$

For general sequence of sets:

$$\limsup A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n; \quad \liminf A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n.$$

A clearer intepretation of limsup and liminf:

limsup is the set of elements which occurs infinitely many times in A_n, and liminf is the elements which doesn't occur in only finitely many A_n's.

§2.2 Families of sets

Definition 2.3. A family of sets is denoted by script letters $\mathscr{A}, \mathscr{B}, \ldots$.

- A family is a **π-system** if $\mathscr{P} \neq \emptyset$ and it's closed under intersections, e.g. $\{(-\infty, a] : a \in \mathbb{R}\}$.
- **Semi-rings**: $\mathscr{Q}$ is a π-system, and for all $A \subset B$, then there exists finitely many pairwise disjoint sets $C_1, \ldots, C_n \in \mathscr{Q}$ s.t.

$$B \backslash A = \bigcup_{k=1}^{n} C_k = \sum_{k=1}^{n} C_k.$$

  e.g. $\mathscr{Q} = \{(a, b] : a, b \in \mathbb{R}\}$.

Remark 2.4 — The condition $A \subset B$ can be removed.

- **Rings**: $\mathscr{R}$ is nonempty, and it's closed under union and substraction.
  e.g. $\mathscr{R} = \{\bigcup_{k=1}^{n}(a_k, b_k] : a_k, b_k \in \mathbb{R}\}$.
- **Algebras (fields)**: $\mathscr{A}$ is a π-system that contains X, and is closed under completion.

Proposition 2.5

Semi-rings are π-systems, rings are semi-rings, algebras are rings.

Proof. By definition we only need to check rings are π-systems: $A \cap B = A \backslash (A \backslash B)$.
For algebras, $A \cup B = (A^c \cap B^c)^c, A \backslash B = A \cap B^c$, so they are rings. □

Remark 2.6 — Rings are semi-rings with unions, Algebras are rings with universal set X.

Definition 2.7. Some other families that start from taking limits:

- **Monotone class**: If $A_1, \cdots \in \mathscr{U}$ and A_n monotone, then $\lim_{n\to\infty} A_n \in \mathscr{U}$.
- **λ-system**:
$$X \in \mathscr{L}; \quad A_1, A_2, \cdots \in \mathscr{L}, A_n \uparrow A \implies A \in \mathscr{L};$$
$$A, B \in \mathscr{L}, A \supseteq B \implies A \backslash B \in \mathscr{L}.$$
  notes: $A_n \in \mathscr{L} \iff B_n = A_n^c \in \mathscr{L}$.
- **σ-algebra**:
$$X \in \mathscr{F}; \quad A \in \mathscr{F} \implies A^c \in \mathscr{F};$$
$$A_1, A_2, \cdots \in \mathscr{F} \implies \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}.$$

Proposition 2.8

σ-algebra = algebra & monotone class = λ-system & π-system.

Definition 2.9. **σ-rings**: $\mathscr{R}$ nonempty, $A, B \in \mathscr{R} \implies A \backslash B \in \mathscr{R}$;
$$A_1, A_2, \cdots \in \mathscr{R} \implies \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}.$$

Note: We only need to verify the case when A_n's are disjoint.

Definition 2.10 (Measurable space)**.** Let $\mathscr{F}$ be a σ-algebra on a set X, we say $(X, \mathscr{F})$ is a **measurable space**.

Proposition 2.11
Let $(X, \mathscr{F})$ be a measurable space, A is a subset of X. Then $(A, A \cap \mathscr{F})$ is also a measurable space.

The smallest σ-algebra is $\{\emptyset, X\}$, the largest σ-algebra is the power set $\mathscr{T} = \mathcal{P}(X)$.

In some cases, $\mathscr{T}$ is too large, for example, when $X = \mathbb{R}$, we can't assign a "measure" to every subset that fits our common sense.

§2.3 Generation of σ-algebras

Let $\mathscr{E}$ be a nonempty collection of sets.

Definition 2.12 (Generate rings)**.** We say $\mathscr{G}$ is the ring (algebra, etc.) generated by $\mathscr{E}$, if

- $\mathscr{G} \supseteq \mathscr{E}$;
- For any ring $\mathscr{G}'$, $\mathscr{G}' \supseteq \mathscr{E} \implies \mathscr{G}' \supseteq \mathscr{G}$

Proposition 2.13
The ring (or whatever) generated by $\mathscr{E}$ always exists.

Proof. Let $\mathbf{A}$ be the set consisting of the rings containing $\mathscr{E}$, then $\bigcap_{\mathscr{G} \in \mathbf{A}} \mathscr{G}$ is the desired ring. □

Denote $r(\mathscr{E}), m(\mathscr{E}), p(\mathscr{E}), l(\mathscr{E}), \sigma(\mathscr{E})$ the ring/monotone class/π-system/λ-system/σ-algebra generated by $\mathscr{E}$.

Theorem 2.14
Let $\mathscr{A}$ be an algebra, then $\sigma(\mathscr{A}) = m(\mathscr{A})$.

Proof. Clearly $\sigma(\mathscr{A}) \supseteq m(\mathscr{A})$.

On the other hand, we only need to prove $m(\mathscr{A})$ is a σ-algebra.

Since $\mathscr{A}$ is an algebra, so $X \in \mathscr{A} \subset m(\mathscr{A})$.

For the completion:

Let $\mathscr{G} := \{A : A^c \in m(\mathscr{A})\}$, we want to prove $\mathscr{G} \supseteq m(\mathscr{A})$.

Clearly $\mathscr{A} \subset \mathscr{G}$; If $A_1, A_2, \cdots \in \mathscr{G}$, $A_n \uparrow A$, then

$$A_n^c \in m(\mathscr{A}) \implies A^c = \downarrow \lim_n A_n^c \in m(\mathscr{A}).$$

Similarly if $A_n \downarrow A$, we can also deduce $A^c \in m(\mathscr{A})$.

So $\mathscr{G}$ is a monotone class containing $\mathscr{A}$, hence it must contain $m(\mathscr{A}) \implies \forall A \in m(\mathscr{A})$, $A^c \in m(\mathscr{A})$.

For the intersection:

- $\forall A \in \mathscr{A}, B \in m(\mathscr{A}), AB \in m(\mathscr{A})$: If $B \in \mathscr{A}$, this clearly holds;

  Moreover, such B's constitude a monotone class:

> **Claim 2.15.** Let $\mathscr{M}$ be a monotone class, then $\forall C \in \mathscr{M}$, $\mathscr{G}_C = \{D : CD \in \mathscr{M}\}$ is a monotone class.

If $D_1, D_2, \cdots \to D$ satisfy $C \cap D_i \in m(\mathscr{A})$, then $D \cap C = \lim_n D_i \cap C \in \mathscr{M}$.
Therefore such B's constitude a monotone class $\mathscr{G}_A$ containing $\mathscr{A} \implies \mathscr{G}_A \supseteq m(\mathscr{A})$.

- All the A's which satisfies the first condition constitude a monotone class:
  Let $\mathscr{G}_B = \{A : AB \in m(\mathscr{A})\}$, then $\mathscr{G} = \bigcup_{B \in m(\mathscr{A})} \mathscr{G}_B$ is a monotone class containing $\mathscr{A}$.
  Hence $\mathscr{G} \supseteq m(\mathscr{A}) \implies \forall A \in m(\mathscr{A})$, $\forall B \in m(\mathscr{A})$, we have $AB \in m(\mathscr{A})$.

□

Theorem 2.16 (λ-π theorem)
Let $\mathscr{P}$ be a π-system, then $\sigma(\mathscr{P}) = l(\mathscr{P})$.

Proof. Obviously $\sigma(\mathscr{P}) \supseteq l(\mathscr{P})$.
We only need to check that $l(\mathscr{P})$ is a π-system, i.e. closed under intersection.

> **Claim 2.17.** If $\mathscr{L}$ is a λ-system, then $\forall C \in \mathscr{L}$, $\mathscr{G}_C$ is a λ-system, where
> $$\mathscr{G}_C := \{D : CD \in \mathscr{L}\}.$$

Proof of the claim. First of all, $X \in \mathscr{G}_C$ as $CX = C \in \mathscr{G}_C$.
Second, if $D_1, D_2 \in \mathscr{G}_C$,

$$CD_1, CD_2 \in \mathscr{L} \implies C(D_1 - D_2) = CD_1 - CD_2 \in \mathscr{L} \implies D_1 - D_2 \in \mathscr{G}_C.$$

Lastly, if $D_n \in \mathscr{G}_C$, $D_n \to D$,

$$CD_n \in \mathscr{L} \implies CD = \lim_n CD_n \in \mathscr{L} \implies D \in \mathscr{G}_C$$

□

The rest is similar to the previous theorem:

- $\forall A \in \mathscr{P}, B \in l(\mathscr{P}), AB \in l(\mathscr{P})$: If $B \in \mathscr{P}$ this clearly holds;
  By the claim, $\mathscr{G}_A = \{B : AB \in l(\mathscr{P})\}$ is a λ-system, so $\mathscr{G}_A \supseteq l(\mathscr{P})$.
- For $B \in l(\mathscr{P})$, let
  $$\mathscr{G}_B = \{A : AB \in l(\mathscr{P})\}.$$
  By our claim, $\mathscr{G}_B$'s are λ-systems. So $\mathscr{G} = \bigcap_{B \in l(\mathscr{P})} \mathscr{G}_B$ is a λ-system.
  Moreover $\mathscr{G} \supseteq \mathscr{P}$ (This is proved above), so $\mathscr{G} \supseteq l(\mathscr{P})$.
  This means $\forall A, B \in l(\mathscr{P}), AB \in l(\mathscr{P})$.

□

Remark 2.18 — These two proofs are very similar. Note how we make use of the conditions.

Let X be a topological space, $\mathscr{O}$ is the collection of all the open sets.

Let $\mathscr{B}_X := \sigma(\mathscr{O})$ be the **Borel σ-algebra** on the space X, $B \in \mathscr{B}_X$ are called **Borel sets**, and $(X, \mathscr{B}_X)$ is called the **topological measurable space**.

Theorem 2.19

Let $\mathscr{Q}$ be a semi-ring, then

$$r(\mathscr{Q}) = \mathscr{G} := \bigcup_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} A_k : A_1, \dots, A_n \in \mathscr{Q} \text{ and pairwise disjoint} \right\}.$$

Remark 2.20 — If $\mathscr{R}$ is a ring, then $\mathscr{A} = a(\mathscr{R}) = \mathscr{R} \cup \{A^c : A \in \mathscr{R}\}$ can also be written out explicitly, while $\sigma(\mathscr{A})$ usually cannot be expressed explicitly.

Proof. Since $r(\mathscr{Q})$ is closed under finite unions, so $r(\mathscr{Q}) \supseteq \mathscr{G}$.

Reversely, $\mathscr{G}$ is nonempty. In fact we only need to prove it's closed under subtraction, as

$$A \cap B = A \backslash (A \backslash B) \in \mathscr{G}, A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B) \in \mathscr{G}.$$

Suppose $A = \sum_{i=1}^{n} A_i, B = \sum_{j=1}^{m} B_j$.

Then $A_i \backslash B_1$ can be split to several disjoint sets C_k in $\mathscr{Q}$. Continue this process, each C_k can be split again into smaller set. When all of the B_j's are removed, we end up with many tiny sets which are in $\mathscr{Q}$ and pairwise disjoint. (This process can be formalized using induction)

Therefore $A \backslash B \in \mathscr{G}$, the conclusion follows. $\square$

§2.4 Measurable maps and measurable functions

For a map $f : X \to Y$, we say the **preimage** of $B \subset Y$ is $f^{-1}(B) := \{x : f(x) \in B\}$.

Some properties of preimage:

$$f^{-1}(\emptyset) = \emptyset, \quad f^{-1}(Y) = X;$$
$$B_1 \subset B_2 \implies f^{-1}(B_1) \subset f^{-1}(B_2), \quad (f^{-1}(B))^c = f^{-1}(B^c);$$
$$f^{-1}\left(\bigcup_{t \in T} A_t\right) = \bigcup_{t \in T} f^{-1}(A_t), \quad f^{-1}\left(\bigcap_{t \in T} A_t\right) = \bigcap_{t \in T} f^{-1}(A_t).$$

Proposition 2.21

Let $\mathscr{T}$ be a σ-algebra on Y, then $f^{-1}(\mathscr{T})$ is also a σ-algebra on X.

Furthermore, for $\mathscr{E}$ on Y,

$$\sigma(f^{-1}(\mathscr{E})) = f^{-1}(\sigma(\mathscr{E})).$$

Proof. $f^{-1}(\mathscr{E}) \subset f^{-1}(\sigma(\mathscr{E})) \implies f^{-1}(\sigma(E)) \supseteq \sigma(f^{-1}(\mathscr{E}))$.

Again, let

$$\mathscr{G} := \{B \subset Y : f^{-1}(B) \in \sigma(f^{-1}(\mathscr{E}))\}.$$

We need to prove $\mathscr{G}$ is a σ-algebra. This can be checked easily by previous properties, so I leave them out. Hence $\mathscr{G} \supseteq \mathscr{E} \implies \mathscr{G} \supseteq \sigma(\mathscr{E}) \implies f^{-1}(\sigma(\mathscr{E})) \subset \sigma(f^{-1}(\mathscr{E}))$. $\square$

Definition 2.22 (Measurable maps)**.** Let $(X, \mathscr{F})$ and $(Y, \mathscr{S})$, and $f : X \to Y$ a map. We say f is **measurable** if $f^{-1}(\mathscr{S}) \subset \mathscr{F}$, i.e. the preimage of measurable sets are also measurable, denoted by

$$f : (X, \mathscr{F}) \to (Y, \mathscr{S}) \quad \text{or} \quad (X, \mathscr{F}) \xrightarrow{f} (Y, \mathscr{S}) \quad \text{or} \quad f \in \mathscr{F}.$$

Clearly the composition of measurable maps is measurable as well.

A map f is measurable is equivalent to $\sigma(f) \subset \mathscr{F}$, where

$$\sigma(f) := f^{-1}(\mathscr{S})$$

is the smallest σ-algebra which makes f measurable, called the generate σ-algebra of f.

Theorem 2.23

Let $\mathscr{E}$ be a nonempty collection on Y, then

$$f : (X, \mathscr{F}) \to (Y, \sigma(\mathscr{E})) \iff f^{-1}(\mathscr{E}) \subset \mathscr{F}.$$

Proof. Trivial. □

Definition 2.24 (Generalize real numbers)**.** Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Similarly we can assign an order to $\overline{\mathbb{R}}$.

For the calculations, we assign 0 to $0 \cdot \pm\infty$, and $\infty - \infty$, $\frac{\infty}{\infty}$ is undefined.

For all $a \in \overline{\mathbb{R}}$, define $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, so $a = a^+ - a^-$.
Define the Borel σ-algebra on $\overline{\mathbb{R}}$:

$$\mathscr{B}_{\overline{\mathbb{R}}} := \sigma(\mathscr{B}_{\mathbb{R}} \cup \{+\infty, -\infty\}).$$

For any set A, $A \in \mathscr{B}_{\overline{\mathbb{R}}} \iff A = B \cup C$, where $B \in \mathscr{B}_{\mathbb{R}}, C \subset \{+\infty, -\infty\}$.

Definition 2.25 (Measurable functions)**.** We say a function f is **measurable** if

$$f : (X, \mathscr{F}) \to (\overline{\mathbb{R}}, \mathscr{B}_{\overline{\mathbb{R}}}).$$

A **random variable (r.v.)** is a measurable map to $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.

Measurable functions are in fact random variables that can take $\pm\infty$ as its value.

Theorem 2.26

Let $(X, \mathscr{F})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$ if and only if

$$\{f \le a\} \in \mathscr{F}, \quad \forall a \in \mathbb{R}.$$

Proof. Just note that these sets can generate $\mathscr{B}_{\overline{\mathbb{R}}}$.
Let $\mathscr{E} = \{[-\infty, a] : \forall a \in \mathbb{R}\}$. Then

$$f \text{ measurable} \iff \sigma(f) = f^{-1}\mathscr{B}_{\overline{\mathbb{R}}} = f^{-1}\sigma(\mathscr{E}) \subset \mathscr{F} \iff \sigma(f^{-1}\mathscr{E}) \subset \mathscr{F}.$$

□

Example 2.27

The contant functions are measurable; the indicator functions of a measurable set are measurable $\implies$ *step functions* are measurable.

We say a function f is **Borel function** if it's a measurable function from Borel measurable space to itself.

Corollary 2.28

If f, g are measurable functions, then $\{f = a\}, \{f > g\}, \ldots$ are measurable sets.

Theorem 2.29

The arithmetic of measurable funtions are also measurable functions (if they are well-defined).

Proof. Here we only proof $f + g$ is measurable for f, g measurable. For all $a \in \mathbb{R}$, decompose $\{f + g < a\}$ to $A_1 \cup A_2 \cup A_3$:

$$A_1 := \{f = -\infty, g < +\infty\} \cup \{g = -\infty, f < +\infty\} \in \mathscr{F};$$

$$A_2 := \{f = +\infty, g > -\infty\} \cup \{g = +\infty, f > -\infty\} \in \mathscr{F};$$

$$A_3 := \{f < a - g\} \cap \{f, g \in \mathbb{R}\} = \left(\bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g < a - r\})\right) \cap \{f, g \in \mathbb{R}\} \in \mathscr{F}.$$

□

Remark 2.30 — All the measurable functions (or random variables) constitude a vector space.

Theorem 2.31

The limit inferior and limit superior of measurable functions are measurable.

Proof. If $f_1, f_2, \ldots$ are measurable, then $\inf f_n$ is measurable:

$$\left\{\inf_{n \geq 1} f_n \geq a\right\} = \bigcap_{n=1}^{\infty} \{f_n \geq a\}.$$

Remark 2.32 — In particular, f measurable $\implies$ f^+, f^- measurable.

Hence

$$\liminf_{n \to \infty} f_n = \lim_{N \to \infty} \inf_{n \geq N} f_n = \sup_{N \geq 1} \inf_{n \geq N} f_n,$$

which is clearly measurable. □

Remark 2.33 — The inferior or superior of **countable** many measurable functions are measurable as well.

Definition 2.34 (Simple functions)**.** Let $(X, \mathscr{F})$ be a measurable space. A **measurable partition** of X is a collection of subsets $\{A_1, \dots, A_n\}$ with $\sum_{i=1}^{n} A_i = X$, and $A_i \in \mathscr{F}$.

A **simple function** is defined as

$$f = \sum_{i=1}^{n} a_i \mathbf{I}_{A_i}.$$

where $\{A_1, \dots, A_n\}$ is a measurable partition of X, and $a_i \in \mathbb{R}$.

It's clear that simple functions are measurable.

Theorem 2.35

Let f be a measurable function, there exists simple functions $f_1, \dots$ s.t. $f_n \to f$.

- If $f \geq 0$, we have $0 \leq f_n \leq f$;
- If f is bounded, we have $f_n \rightrightarrows f$.

Proof. This is a standard truncation. For $f \geq 0$, let

$$f_n = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbf{I}_{\{k \leq 2^n f \leq k+1\}} + n\mathbf{I}_{f \geq n}.$$

It's clear that $f_n \geq 0$, $f_n \uparrow$, and $f_n(x) \to f(x)$:

$$0 \leq f(x) - f_n(x) \leq \frac{1}{2^n}, \quad f(x) < n;$$

$$n = f_n(x) \leq f(x), \quad f(x) \geq n.$$

Therefore if f is bounded, when $n > \max f(x)$ we have $|f_n(x) - f(x)| < \frac{1}{2^n}$ for all $x \in X$.

For general measurable functions, just decompose f to $f^+ - f^-$. □

Theorem 2.36

Let $g : (X, \mathscr{F}) \to (Y, \mathscr{S})$. Let h be a map $X \to \mathbb{R}$.

Then $h : (X, g^{-1}\mathscr{S})$ iff $h = f \circ g$, where $f : (Y, \mathscr{S}) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.

Remark 2.37 — For $\overline{\mathbb{R}}$ or $[a, b]$, this theorem also holds.

Proof. There's a typical method for proving something related to measurable functions:

We'll prove the statement for $h \in \mathcal{H}_i$ in order:

- $\mathcal{H}_1$: indicator functions $h = \mathbf{I}_A$, $\forall A \in g^{-1}\mathscr{S}$;
- $\mathcal{H}_2$: non-negative simple functions;
- $\mathcal{H}_3$: non-negative measurable functions;

- $\mathcal{H}_4$: measurable functions.

When $h \in \mathcal{H}_1$, suppose $h = \mathbf{I}_A$, then

$$A = g^{-1}B, B \in \mathscr{S} \implies f = \mathbf{I}_B \text{ suffices.}$$

When $h = \sum_{i=1}^n a_i \mathbf{I}_{A_i} \in \mathcal{H}_2$, since $A_i \in g^{-1}\mathscr{S}$,

$$\exists B_i \in \mathscr{S} \quad s.t. \quad A_i = \{h = a_i\} = g^{-1}B_i.$$

Thus $f = \sum_{i=1}^n a_i \mathbf{I}_{B_i}$ is the desired function.

When $h \in \mathcal{H}_3$, $\exists h_1, h_2, \ldots \uparrow h$.

Assume $h_n = f_n \circ g$, let

$$f(y) := \begin{cases} \lim_{n\to\infty} f_n(y), & \text{if it exists;} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.38 — Here we still need to prove f is measurable.

Hence for any $x \in X$,

$$h(x) = \lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} f_n(g(x)) = f(g(x)),$$

as f_n's limit must exist at $y = g(x)$.

So for general h, let $h = h^+ - h^-$ and we're done. NOTE: We need to assert that $\infty - \infty$ doesn't occur. $\square$