Geometry II

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Example 0.1

The conical and cylindrical surfaces have Gaussian curvature 0.

For a general ruled surface, we can prove that $K \le 0$ everywhere.

Example 0.2

Minimal surfaces(like soap bubbles) have $H = 0$ and $K \le 0$ everywhere.

Example 0.3 (Dupin canonical form)

Let $\phi : U \to \mathbb{E}^3$ be a regular surface, then at the neighborhood of any point, there exists a parameter s.t. $\phi(s,t) = (s, t, \kappa_1 s^2 + \kappa_2 t^2) + o(|s|^2 + |t|^2)$, where κ_1, κ_2 are principal curvatures of ϕ.

In this case we can talk about concepts like "elliptic point", "parabolic point" and "hyperbolic point".

Next we'll going to switch to a more intrinsic view to study the meaning of those definitions again.

If we look at a curve γ on a surface ϕ, let r be the arc length parameter, then $\|\gamma'\| = 1$, $\|\gamma''\| = \kappa(r)$, note that γ'' can be decompose with respect to the normal vector and tangent plane:

$$\gamma'' = \kappa_n \vec{n} + \kappa_g \vec{n} \times \gamma'.$$

Here κ_n is called **normal curvature**, and κ_g is called **geodesic curvature** of γ WRT ϕ.

Moreover we have *Euler's formula*:

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

If we compute the normal curvature in terms of $u = (s,t)$:

$$\gamma' = \phi_s s' + \phi_t t'$$

$$\gamma'' = (s', t') \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} + \phi_s s'' + \phi_t t''$$

Hence

$$\kappa_n = \gamma'' \cdot \vec{n} = (s', t') \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} = L(s')^2 + 2Ms't' + N(t')^2.$$

This is the formula under the arc length parameter.

Remark 0.4 — The general formula of κ_n:

$$\kappa_n = \frac{Ls'^2 + 2Ms't' + Nt'^2}{Es'^2 + 2Fs't' + Gt'^2}.$$

The normal plane of γ intersects the surface ϕ, the section curve is called a **normal section**.

Oberserve that: if $\|\gamma'\| = 1$, and the tangent vector is $\vec{t}$, then $\kappa_n(r)$ is the curvature of the normal section at u in the plane spanned by $\vec{n}, \vec{t}$.

Hence κ_n can be viewed as a quadratic form $\vec{n}^\perp \to \mathbb{R}$ which sends a vector $\vec{t}$ to the curvature of the normal section with tangent vector $\vec{t}$.

Furthermore, the principal directions are the "eigen-directions" of κ_n, which are the directions where the curvature of normal section attains its extremum.

Example 0.5

Consider the helix and the cylinder

$$\gamma(t) = (\cos t, \sin t, at), \quad S : x^2 + y^2 = 1.$$

It's easy to verify that $\kappa = \kappa_n = \frac{1}{1+a^2}$ as γ'' is always perpendicular to z-axis.

Note that $\kappa_g = 0$ everywhere, curves satisfying $\kappa_g = 0$ are called **geodesic line**.

§0.1 Gauss map and Weingarten map

The strange definition of those curvatures don't come from nothing, in this section we'll cover this topic and give a geometric interpretation.

Definition 0.6 (Gauss map)**.** Let Σ be a regular surface in $\mathbb{E}^3$, denote its normal vector at x by $\vec{n}(x)$. Then this map $\mathcal{G} : \Sigma \to \mathbb{S}^2$ by $x \mapsto \vec{n}(x)$ is called the **Gaussian map**.

In terms of a parametrized surface $\phi : U \to \mathbb{E}^3$, we can compute that

$$\mathcal{G} : U \to \mathbb{S}^2 : \quad \vec{n}(u) = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}$$

But each vector has a normal plane, namely $\vec{n}^\perp$, and this derives the **Weingarten map**:

Definition 0.7 (Weingarten map)**.** For all $u \in U$, define $W : \vec{n}(u)^\perp \to \vec{n}(u)^\perp$: $\vec{v} \mapsto W(\vec{v})$, where

$$W(\vec{v}) = -\left.\frac{\mathrm{d}(\mathcal{G} \circ \gamma)}{\mathrm{d}u}\right|_{u=0}, \quad \gamma := \phi(u(r)) \text{ is a curve on the surface.}$$

Remark 0.8 — In the language of modern differetial manifolds, Weingarten map is just the tangent map of Gauss map with a negative sign.

Since $\vec{n}^{\perp}$ has a basis ϕ_s, ϕ_t, we can compute the matrix of Weingarten map:

$$(\phi_s, \phi_t)W = (-\vec{n}_s, -\vec{n}_t).$$

Note that $-\vec{n}_s \cdot \phi_s = \vec{n} \cdot \phi_{ss} = L$, so if we take the inner product of (ϕ_s, ϕ_t) on both sides, we get

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} W = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \implies W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Since W is clearly a geometric quantity, so its trace and determinant are also geometric:

$$\operatorname{tr} W = \frac{GL - 2FM + EN}{EG - F^2} = 2H, \quad \det W = \frac{LN - M^2}{EG - F^2} = K,$$

which gives the average curvature and Gauss curvature.

Moreover, the principal curvatures are the eigenvalues of W, and principal directions are just the eigenspaces of W.

Let $\vec{v} = (\phi_s, \phi_t)X$, then its normal section has curvature

$$\kappa_n = \frac{X^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} X}{X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} X}.$$

When $\|\vec{v}\| = 1$, we can change a parameter s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = I_2$, in this case we can observe that when κ_n attains its extremum, $\vec{v}$ is precisely the eigenvector of W, i.e. lies on the principal directions.

Definition 0.9 (Curvature line)**.** A curve is called a **curvature line** if its tangent vector is the same as principal directions everywhere.

Example 0.10

Every curve on a sphere is curvature line.

Around a point where the principal curvatures are different, there exists a orthogonal grid of curvature lines.

Example 0.11

monkey saddle surface, "prong singularity"

In the case when the s-curve and t-curve are precisely the curvature lines, then we say this is a **curvature grid parameter**, and here we have $g = E\,\mathrm{d}s^2 + G\,\mathrm{d}t^2$ and $h = L\,\mathrm{d}s^2 + N\,\mathrm{d}t^2$.

Remark 0.12 — The geometric interpretation of Gauss curvature: For $u \in D \subset U$,

$$|K(u)| = \lim_{\text{“}D \to u\text{”}} \frac{Area_{\mathbb{S}^2}(\mathcal{G}(D))}{Area_{\mathbb{E}^3}(\phi(D))}$$

while $sgn(K(u))$ is the orientation of $\mathcal{G}$ at point u.

Example 0.13

Consider the Gauss map of a torus, the "outer" part and the "inner" part of the torus maps to $\mathbb{S}^2$ bijectively. If we compute

$$\int_{T^2} K \,\mathrm{d}Area_E = \int_{\mathbb{S}^2} (1 + (-1)) \,\mathrm{d}Area_S = 0 = 2\pi\chi(T^2),$$

as Gauss-Bonnet formula implies.

§0.2 Fundamental equation of surfaces

Like the Fundamental theorem and Frenet frame in curve theory, we want to develop a theorem for describing surfaces using only fundamental forms.

Given a parameter on a surface, there's a natural frame $(\phi_s, \phi_t, \vec{n})$. If we take the derivative of the frame, we'll get

$$(\phi_s, \phi_t, \vec{n})_{st} = (\phi_s, \phi_t, \vec{n})_{ts}.$$

Taking the inner product with $(\phi_s, \phi_t, \vec{n})^T$ and apply the product rule:

$$\left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_s\right)_t - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_t (\phi_s, \phi_t, \vec{n})_s = \left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_t\right)_s - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_s (\phi_s, \phi_t, \vec{n})_t$$

This equation will give us some relations between the fundamental quantities. In literature these relations are known as Gauss equation and Codazzi equations.

Gauss equation can be written as:

$$(\phi_s \cdot \phi_{ts})_t - (\phi_s \cdot \phi_{tt})_s = \phi_{st} \cdot \phi_{st} - \phi_{ss} \cdot \phi_{tt}.$$

Codazzi equations are related to $\vec{n}$ and more complicated.

From Gauss equation we can deduce a famous theorem:

Theorem 0.14 (Gauss' Theorema Egregium)

The Gauss curvature K is determined by the first fundamental form.

Proof. Note that $(\phi_s \cdot \phi_{ts})_t = \frac{1}{2}E_{tt}$, and $(\phi_s \cdot \phi_{tt})_s = (F_t - \frac{1}{2}G_s)_s = F_{ts} - \frac{1}{2}G_{ss}$.

Suppose $\phi_{ss} = x\phi_s + y\phi_t + L\vec{n}$, then

$$\frac{1}{2}E_s = \phi_s \cdot \phi_{ss} = Ex + Fy, \quad F_s - \frac{1}{2}G_t = \phi_t \cdot \phi_{ss} = Fx + Gy$$

So x, y is determined by E, F, G.

Similarly, we get

$$\begin{aligned}\phi_{ss} &= *\phi_s + *\phi_t + L\vec{n}\\ \phi_{st} &= *\phi_s + *\phi_t + M\vec{n}\\ \phi_{tt} &= *\phi_s + *\phi_t + N\vec{n}\end{aligned}$$

where $*$ are determined by E, F, G.

By Gauss equation, we get $* = -(LN - M^2) + *$, and $*$ is determined by E, F, G and their partial derivatives. □

> **Remark 0.15 —** The computation looks messy, but in modern mathematics, we have a systematic notation which is more simplified.

Definition 0.16 (Isometries)**.** Let $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. If a map $\psi : \widetilde{U} \to U$ satisfies $\psi^*(g) = \widetilde{g}$, then it's called an **isometry**.

Let $\mathcal{F} = (\phi_s, \phi_t, \vec{n})$. Suppose $\mathcal{F}_s = \mathcal{F}A$, and $\mathcal{F}_t = \mathcal{F}B$. Taking the second derivative we get $\mathcal{F}_{st} = \mathcal{F}(BA + A_t)$, $\mathcal{F}_{ts} = \mathcal{F}(AB + B_s)$. Thus we have

$$A_t - B_s = AB - BA.$$

But we want to express things in terms of E, F, G, so we can compute the dot product of $\mathcal{F}^T$:

$$P := \mathcal{F}^T \cdot \mathcal{F} = \begin{pmatrix} E & F & \\ F & G & \\ & & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F & \\ -F & E & \\ & & 1 \end{pmatrix}$$

Substituting into $\mathcal{F}_s = \mathcal{F}A$ we get

$$PA = \begin{pmatrix} \frac{E_s}{2} & \frac{E_t}{2} & -L \\ F_s - \frac{E_t}{2} & \frac{G_s}{2} & -M \\ L & M & 0 \end{pmatrix}$$

$$\mathcal{F}^T \mathcal{F}_{st} = (\mathcal{F}^T \mathcal{F}_s)_t - \mathcal{F}_t^T \mathcal{F}_s = (PA)_t - B^T PA.$$
$$\implies (PA)_t - (PB)_s = (PB)^T P^{-1}(PA) - (PA)^T P^{-1}(PB).$$

Gauss equation corresponds to the $(1, 2)$ entry of the matrix equation:

$$\frac{1}{2}(E_{tt} - 2F_{st} + G_{ss}) = -(LN - M^2) + \frac{p}{4(EG - F^2)},$$

where p is a polynomial in fundamental quantities and their derivatives.

These equations still looks complicated, so we'll "package" them:

$$-(LN - M^2) = R_{1212}.$$

Let

$$A = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & h^1_{1} \\ \Gamma^2_{11} & \Gamma^2_{12} & h^2_{1} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} & h^1_{2} \\ \Gamma^2_{12} & \Gamma^2_{22} & h^2_{2} \\ h_{12} & h_{22} & 0 \end{pmatrix}$$

Here the Γ's are called Christoffel notations.

Codazzi equations correspond to the $(1, 3), (2, 3)$ enties:

$$L_t - M_s = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}$$
$$M_t - N_s = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{21}) - N\Gamma^2_{21}.$$

> **Remark 0.17 —** The index notations for computation can be defined as
>
> $$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij}), \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = (g^{ij}),$$
>
> and h is defined similarly. If we use Einstein summation notation, we can write $g_{ij}g^{jk} = \delta_i^k$.

Let $\vec{v}_1 := \phi_s, \vec{v}_2 = \phi_t$, and

$$\frac{\partial \vec{v}_\alpha}{\partial \vec{u}^\beta} = \sum_\gamma \Gamma^{\gamma}_{-\alpha\beta} \vec{v}_\gamma + h_{\alpha\beta}\vec{n}, \quad \frac{\partial \vec{n}}{\partial \vec{u}^\beta} = -\sum_\gamma h^{\gamma}_{-\beta} \vec{v}_\gamma.$$

Here the upper index is defined as:

$$h^{\gamma}_{-\beta} := \sum_\delta g^{\gamma\delta} h_{\delta\beta}.$$

From this we can write Γ out explicitly:

$$\Gamma^{\gamma}_{-\alpha\beta} = \sum_\delta \frac{g^{\gamma\delta}}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right)$$

This is called *Christoffel notations*.

$$R^{\delta}_{-\alpha\beta\gamma} := \frac{\partial \Gamma^{\delta}_{-\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^{\delta}_{-\alpha\gamma}}{\partial u^\beta} + \sum_\eta (\Gamma^{\eta}_{-\alpha\beta}\Gamma^{\delta}_{-\eta\gamma} - \Gamma^{\eta}_{-\alpha\gamma}\Gamma^{\delta}_{-\eta\beta}).$$

This is called *Riemann symbols*. Another type is defined as:

$$R_{\delta\alpha\beta\gamma} = \sum_\eta g_{\delta\eta} R^{\eta}_{-\alpha\beta\gamma}.$$

In surface theory, only R_{1212} is nontrivial.

Using these notations, we can write the equations as:

- Gauss:
$$R_{\delta\alpha\beta\gamma} = -(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}).$$

- Codazzi:
$$\frac{\partial h_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial h_{\alpha\gamma}}{\partial u^\beta} = \sum_\delta (h_{\beta\delta}\Gamma^{\delta}_{-\alpha\gamma} - h_{\gamma\delta}\Gamma^{\delta}_{-\alpha\beta}).$$