Mathematical Analysis II

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§1 Introduction

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Contents of this course: Real analysis

§1.1 Recap

Definition 1.1 (Measurable space)**.** Let X be a set and $\mathcal{A}$ be a σ-algebra, we say $(X, \mathcal{A})$ is a measurable space if

- $\emptyset \in \mathcal{A}$;
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- If $A_k \in \mathcal{A}$, then $\bigcup_{k=1}^{+\infty} \in \mathcal{A}$.

Outer measure m^*:

- $m^*(A) \geq 0$;
- $m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k)$;
- $m^*(A) \leq m^*(B)$ when $A \subset B$.

Caratheodry condition:

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c), \quad \forall T \subset X.$$

We define the measurable sets to be all the sets E satisfying above condition.

This implies the Lebesgue measure space $(\mathbb{R}^n, \mathscr{U}, m)$. It is a complete measure space, i.e. null sets are measurable.

Proposition 1.2

Properties of measurable sets:

- Let E be a measurable set, there exists a G_δ set G and a F_σ set F such that
$$E = G \backslash Z_1 = F \cup Z_2.$$
where Z_1, Z_2 are null sets.
- (Fatou's Lemma)
Measurable sets $E_k \nearrow E \implies \lim_{k\to\infty} m(E_k) = m(E)$ and
$$m\left(\liminf_{k\to\infty} E_k\right) \le \liminf_{k\to\infty} m(E_k).$$

Definition 1.3 (Measurable function)**.** Let f be a map from measurable space $(X, \mathcal{A})$ to $(Y, \mathcal{B})$. We say f is measurable if
$$\forall B \in \mathcal{B}, \quad f^{-1}(B) \in \mathcal{A}.$$

In the case of real functions, this is equivalent to
$$\forall t \in \mathbb{R}, \quad \{x; f(x) > t\} \text{ is a measurable set.}$$

Proposition 1.4

Let f be a non-negative measurable function, $\exists \varphi_k \nearrow f$, where φ_k are simple functions.

For a general measurable function f, decompose it to $f = f_+ - f_-$.

Theorem 1.5 (Egorov)

Let E be a measurable set and $m(E) < \infty$, $f_n \to f, a.e.$, Then $\forall \varepsilon > 0$, there exists a closed set F_ε s.t. $m(E \backslash F_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on F_ε.

Theorem 1.6 (Lusin)

Let E be a measurable set and $m(E) < \infty$. Then $\forall \varepsilon > 0, \exists F_\varepsilon$ such that $f\big|_{F_\varepsilon}$ is continuous.

Convergence patterns:

- Converge almost everywhere: $f_n \to f, a.e.$
- Converge almost uniformly: $f_n \to f, a.u.$
- Converge in measure: $f_n \xrightarrow{m} f$

§2 Lebesgue integrals

§2.1 Recap: Definition of Lebesgue integrals

- Simple functions: $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, define
$$\int f = \sum_{k=1}^{N} a_k m(E_k).$$
- $f : E \to \mathbb{R}^n$, where $m(E) < \infty$, f bounded. These functions form the set $\mathcal{L}_0$. Then $\exists \varphi_k \to f$, φ_k simple, define
$$\int f = \lim_{k\to\infty} \int \varphi_k.$$
- Non-negative function:
$$\int f = \sup \left\{ \int g \,\middle|\, 0 \le g \le f, g \in \mathcal{L}_0 \right\}.$$
- General functions:
$$\int f = \int f_+ - \int f_-.$$
Integrable $\iff \int f_+, \int f_- < +\infty$.

Relations between Riemann integrals and Lebesgue integrals:

- f is Riemann integrable on $[a, b]$ iff f bounded and the discontinuous points form a null set.
- If f is Riemann integrable on $[a, b]$, then two types of integral yield the same result.

§2.2 Dominated convergence theorem

The critical question of this section: If a sequence of functions f_n converges to f (almost everywhere), when does their integrals $\int f_n$ converge to $\int f$?

We'll discuss this issue under various conditions and reach the famous Dominated Convergence Theorem.

Theorem 2.1

Let E be a measurable set with finite measure. Measurable functions $f_n \to f, a.e.$ on E. Furthermore, f_n is uniformly bounded almost everywhere ($|f_n| < M, a.e.$). Then we have
$$\int_E |f_n - f| \to 0 \implies \lim_{m\to\infty} \int_E f_n = \int_E f.$$

Proof. By Egorov's Theorem, $\forall \varepsilon > 0$, there exists $F_\varepsilon \subset E$ s.t. $f_n \to f$ uniformly on F_ε, and $m(E \backslash F_\varepsilon) < \varepsilon$.

Hence
$$\begin{aligned}\int_E |f_n - f| &= \int_{F_\varepsilon} |f_n - f| + \int_{E\backslash F_\varepsilon} |f_n - f| \\ &\le \varepsilon_0 m(E) + 2M\varepsilon,\end{aligned}$$
which proves the result. □

Lemma 2.2 (Fatou's Lemma)

If $f_n \geq 0$, then
$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Proof. For any $g \in \mathcal{L}_0$, $0 \leq g \leq \liminf_{n\to\infty} f_n$, we need to prove $\int g \leq \liminf \int f_n$.

Let $g_k = \min\{f_k, g\}$, assmue g is uniformly bounded so that $g_k \in \mathcal{L}_0$.

We'll prove $g_k \to g$: Assmue by contradiction that $\exists \varepsilon_0 > 0, \exists x_0$ s.t.
$$g(x_0) - g_{k'}(x_0) > \varepsilon_0.$$
then $g(x_0) - f_{k'}(x_0) > \varepsilon_0$, which contradicts with $g \leq \liminf_{n\to\infty} f_n$.

Thus for sufficiently large k, $g_k(x) \leq g(x) \leq g_k(x) + \varepsilon_0, \implies g_k \to g$.

Therefore by Theorem 2.1 (note $g_k \in \mathcal{L}_0$),
$$\begin{aligned}\int g &= \lim_{k\to\infty} \int g_k \\ &\leq \liminf_{k\to\infty} \int f_k,\end{aligned}$$
and we're done. □

Remark 2.3 — This is nearly indentical to the measure version of Fatou's Lemma (Proposition 1.2). It shows some similarities between measure and integrals.

Theorem 2.4 (Beppo-Levi)

If non-negative functions $f_n \nearrow f$, we have
$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof.
$$f_n \leq f \implies \lim_{n\to\infty} \int f_n \leq \int f.$$
By Fatou's Lemma (2.2),
$$\begin{aligned}\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n &= \lim_{n\to\infty} \int f_n, \\ \implies \int f \leq \lim_{n\to\infty} \int f_n.\end{aligned}$$
Combining the two inequalities we get the desired equality. □

Corollary 2.5

Let f_n be non-negative functions, then
$$\int \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \int f_k(x).$$

Proposition 2.6

Let f be an integrable function, $\forall \varepsilon > 0$, we have:

- There exists a set B with finite measure s.t.

$$\int_{B^c} |f| < \varepsilon.$$

- (Absolute continuity of integrals) $\exists \delta > 0$ s.t. $\forall E$, if $m(E) < \delta$,

$$\int_E |f| < \varepsilon.$$

  This is equivalent to

$$\lim_{m(E) \to 0} \int_E |f| = 0.$$

Proof. Let $f_N(x) = |f(x)|$ when $|x| \leq N, |f(x)| \leq N$, and $f_N(x) = 0$ otherwise. Then $f_N \nearrow |f|$, so by Beppo-Levi (Theorem 2.4), we get

$$\lim_{N \to \infty} \int f_N = \int |f|.$$

Let $B = \{x \mid |x| \leq N, |f(x)| \leq N\}$, when N gets sufficiently large, we must have $\int_{B^c} |f| < \varepsilon$.

For the second part, when N sufficiently large we have $\int (|f| - f_N) < \frac{\varepsilon}{2}$, so

$$\begin{aligned} \int_E |f| &= \int_E f_N + \int_E (|f| - f_N) \\ &\leq N \cdot m(E) + \frac{\varepsilon}{2}. \end{aligned}$$

Let $\delta = \frac{\varepsilon}{2N}$ to finish. $\square$