Geometry II

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We say the lifting of p with respect to itself is a **deck transformation**. In fact, deck transformations are just automorphisms of covering spaces, and they constitute a group $Deck_X(\widetilde{X})$ or $Deck_{\widetilde{X}/X}$.

Here's another definition of regular covering: If the group action $Deck_{\widetilde{X}/X}$ onto $\widetilde{X}$ are transitive in $p^{-1}(x_0)$, then we say the covering is **regular covering**.

There should be some pictures of regular and non-regular coverings of $S^1 \vee S^1$, but I'm a bit lazy :-)

Now we'll prove this two definitions are equivalent.

Proposition 0.0.1

Let $p : \widetilde{X} \to X$ be a covering, p is regular iff $p_\sharp \pi_1(\widetilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$ is a normal subgroup.

Proof. When $p_\sharp \pi_1(\widetilde{X}, \tilde{x}_0) \lhd \pi_1(X, x_0)$, for $\tilde{x}_0, \tilde{x}'_0 \in \widetilde{X}$, we need to prove that there exists $\tau \in Deck_{\widetilde{X}/X}$ s.t. $\tau(\tilde{x}_0) = \tilde{x}'_0$.

We'll use lifting theorem on p, thus we only need to show

$$p_\sharp \pi_1(\widetilde{X}, \tilde{x}_0) \subset p_\sharp \pi_1(\widetilde{X}, \tilde{x}'_0).$$

Let γ be a path from $\tilde{x}_0$ to $\tilde{x}'_0$, and $\alpha \in \pi_1(\widetilde{X}, \tilde{x}_0)$. Note that $\alpha \simeq \gamma\overline{\gamma}\alpha\gamma\overline{\gamma}$, $\alpha' = \overline{\gamma}\alpha\gamma \in \pi_1(\widetilde{X}, \tilde{x}'_0)$. Hence

$$p_\sharp(\alpha) = p_\sharp(\gamma) p_\sharp(\alpha') p_\sharp(\overline{\gamma}) \in h p_\sharp \pi_1(\widetilde{X}, \tilde{x}'_0) h^{-1} = p_\sharp \pi_1(\widetilde{X}, \tilde{x}'_0).$$

The converse is the same. □

Now we'll prove **??**: First we'll handle the case of universal covering.

Theorem 0.0.2

Universal covering space and is unique under isomorphism for path connected and locally path connected space X. If X is also locally semi-simply connected, then universal covering exists.

Proof. If $\widetilde{x}, \widetilde{X}'$ are both universal coverings, by map lifting theorem, since $\pi_1(\widetilde{X})$ is trivial, $p : \widetilde{X} \to X$ can be lifted to $\sigma : \widetilde{X} \to \widetilde{X}'$, similarly we have σ', and it's easy to see σ and σ' are inverse maps, so they are isomorphic.

For existence part, X locally semi-simply connected means for $\forall x \in X$, there exists a neighborhood basis $\{U_i\}$ s.t. $\pi_1(U_i, x) \to \pi_1(X, x)$ is trivial.

Let $P(X, x_0)$ be all paths in X starting from x_0, and $\mathscr{X}$ is the homology equivalent classes (with fixed endpoints) of $P(X, x_0)$.

Let $p : \mathscr{X} \to X$ by $\langle a \rangle \mapsto a(1)$, and $\tilde{x}_0$ denote the constant path.

Next we'll define the topology on $\mathscr{X}$:

Let $\{U_\alpha\}$ be a topology basis of X, consider the following sets:

$$U(U_\alpha, a) = \{\langle ac \rangle \mid c \in P(U_\alpha, a(1))\}.$$

Let the topology basis on $\mathscr{X}$ be the above sets. We claim $p : \mathscr{X} \to X$ is indeed a covering. □

Example 0.0.3

A counter example of above theorem when X is not locally semi-simply connected: Hawaiian earrings (a family of tangent circles with radius $\to 0$).

§0.1 Covering spaces and group actions

Now we can view all these things from group actions.

Let X be a topological space, G is a group acting on X. We say the action is **freely discontinuous** if for all $x \in X$, there's a neighborhood U s.t. $gU \cap U \neq \emptyset$ only holds for $g = e$.

Proposition 0.1.1

Let $G \curvearrowright X$ be a freely discontinuous action, then the quotient map $X \to X/G$ by $x \mapsto Gx$ is a regular covering, and the group action is just deck transformations.

Example 0.1.2

The antipodal map in S^n generates a group $\{\pm 1\}$, and the action is freely discontinuous, so $S^n \to S^n/\{\pm 1\} = \mathbb{R}P^n$ is a covering.

Let $\alpha : (x, y) \mapsto (x, y + 1)$ and $\beta : (x, y) \mapsto (x + 1, -y)$ on $\mathbb{E}^2$ generates a group action $G \curvearrowright \mathbb{E}^2$. This is also freely discontinuous, and $\mathbb{E}^2/G$ is a Klein bottle.

Let X be a topological space, G is a group acting on X. We say the action is **properly discontinuous** if for all compact set $K \subset X$, $gK \cap K \neq \emptyset$ only holds for finitely many g.

Usually we suppose X is a locally compact Hausdorff space.

Example 0.1.3

Let $\mathbb{Z}$ acts on $\mathbb{C}$ by $\sigma : x + iy \mapsto \lambda x + i\lambda^{-1}y$. Then it's not properly discontinuous.

Proposition 0.1.4

Let G acts on X properly discontinuously. If X is locally compact Hausdorff, then so is X/G.

Proof. For $\overline{x} \neq \overline{y} \in X/G$, take compact neighborhoods $K(x), K(y) \subset X$, since $gK(x) \cap K(y) \neq \emptyset$ only holds for finitely many g, we can "shrink" $K(x)$ and $K(y)$ so that $gK(x) \cap K(y) = \emptyset$ for all $g \in G$. Thus X/G is Hausdorff.

Clearly X/G is locally compact. □

Proposition 0.1.5

Let X be a locally compact Hausdorff space, G act on X. The action is properly discontinuous + free $\iff$ it's freely discontinuous.

Proof. Trivial. $\square$

Corollary 0.1.6

Let X be a locally compact Hausdorff space, and G acts on X properly discontinuously. If G has no torsion, then the action is freely discontinuous.

Proof. If the action is not free, there exists $g \neq \mathrm{id}$, $gx = x$. Thus $\{x\} \cap \{gx\} \neq \emptyset$ holds for any g^n.

Since g is not a torsion, this contradicts with proper discontinuity. $\square$

Example 0.1.7

Consider the action of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ on the space $UHP = \{z \in \mathbb{C} \mid Im(z) > 0\}$ by fractional linear transformation. (hyberbolic transformation)

This action is properly discontinuous, let $\Gamma(2) = \{A \in \Gamma \mid A \equiv I(\text{mod} 2)\}$, which has no torsion, thus it's freely discontinuous. Note that $[\Gamma : \Gamma(2)] = 6$, and $UHP \to UHP/\Gamma(2)$ is a covering map.

At last, we'll combined what we've learned and prove a well-known theorem:

Theorem 0.1.8

Simply connected surfaces with complete metric and constant curvature -1 are globally isometrically isomorphic to $\mathbb{H}^2$.

Remark 0.1.9 — Here the curvature is the Gauss curvature. The proof is similar for $\mathbb{S}^2(k = 1)$ and $\mathbb{E}^2(k = 0)$.

Sketch of the proof. The surface can be viewed as a manifold, whose charts are assigned the first fundamental form. The proof can be spilt to 2 parts, one for local properties and one for global properties.

If we have the local result, i.e. each point has an open neighborhood homeomorphic to an open disk in $\mathbb{H}^2$, we'll prove the theorem:

- There exists a unique well-defined locally isometric extension $f : M \to \mathbb{H}^2$. (Here we need M simply connected)

Since f is locally isometric, f is a covering map. But $\pi_1(M) = \{1\}$, by the uniqueness of universal covering, there exists an isomorphism of coverings σ s.t. $f = \mathrm{id} \circ \sigma = \sigma$. Thus f is a homeomorphism.

- Locally, we'll take a geodesic parallel parameter, i.e. the y-axis and x-curves are geodesic lines. We have $I = \mathrm{d}x^2 + G(x, y)\,\mathrm{d}y^2$, where $G(0, y) = 1$, $G_x(0, y) = 0$. $\square$