Measure Theory

Felix Chen

Contents

§1 Measure spaces

§1.1 The definition of measure and its properties

The concept of "measure" is frequently used in our everyday life: length, area, weight and even propbability. They all share a similarly: the measure of a whole is equal to the sum of the measure of each part.

In the language of mathematics, let $\mathscr{E}$ be a collection of sets, and there's a function $\mu : \mathscr{E} \to [0, \infty]$ which stands for the measure.

countable additivity: Let $A_1, A_2, \cdots \in \mathscr{E}$ be pairwise disjoint sets, and $\sum_{i=1}^{\infty} A_i \in \mathscr{E}$, then

$$\mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 1.1 (Measure)**.** Suppose $\emptyset \in \mathscr{E}$, if a non-negative function

$$\mu : \mathscr{E} \to [0, \infty]$$

satisfies countable additivity, and $\mu(\emptyset) = 0$, then we say μ is a **measure** on $\mathscr{E}$.

If $\mu(A) < \infty$ for all $A \in \mathscr{E}$, we say μ is finite. (In practice we'll just simplify this to $\mu(X) < \infty$)

If $\exists A_1, A_2, \cdots \in \mathscr{E}$ are pairwise disjoint sets, s.t.

$$X = \sum_{n=1}^{\infty} A_n, \quad \mu(A_n) < \infty, \forall n.$$

Then we say μ is σ-finite.

There's a weaker version of countable additivity, that is **finite additivity**: If $A_1, \ldots, A_n \in \mathscr{E}$, pairwise disjoint, and $\sum A_i \in \mathscr{E}$,

$$\mu\left(\sum_{i=1}^{n} A_i\right) = \sum_{1=i}^{n} \mu(A_i),$$

then we say μ is finite additive.

Subtractivity: $\mu(B - A) = \mu(B) - \mu(A)$, where $A, B, B - A \in \mathscr{E}$, and $\mu(A) < \infty$.

Proposition 1.2
Measure satisfies finite additivity and subtractivity.

Example 1.3 (Counting measure)
Let $\mu(A) = \#A$, $\forall A \in \mathscr{T}_X$. Then μ is a measure.

Example 1.4 (Point measure)
Let $(X, \mathscr{F})$ be a measurable space, define $\delta_x(A) = \mathbf{I}_A(x)$. Then we can define a measure

$$\mu(A) = \sum_{i=1}^{n} p_i \delta_{x_i}(A)$$

Example 1.5 (Length)
Let $\mathscr{E} = \mathscr{Q}_{\mathbb{R}} = \{(a, b] : a, b \in \mathbb{R}\}$, $a \leq b$, then $\mu((a, b]) = b - a$ gives a measure.

Another classical example is the so-called "coin space":
Let $X = \{x = (x_1, x_2, \dots) : x_i \in 0, 1, \forall n\}$.

$$C_{i_1,\dots,i_n} := \{x : x_1 = i_1, \dots, x_n = i_n\},$$

Let

$$\mathscr{Q} = \{\emptyset, X\} \cup \{C_{i_1,\dots,i_n} : n \in \mathbb{N}, i_1, \dots, i_n \in \{0, 1\}\}$$

be a semi-ring. Then $\mu(C_{i_1,\dots,i_n}) = \frac{1}{2^n}$ gives a measure.

We need to check the countable additivity, but actually this can be realized as a compact space and the C's are open sets, so in fact we only need to check finite additivity. (Or we can prove this explicitly)

Another more complex example: finite markov chain.

Proposition 1.6
Let $X = \mathbb{R}$, $\mathscr{E} = \mathscr{R}_{\mathbb{R}}$. $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing, right continuous, then $\mu((a, b]) = F(b) - F(a)$ gives a measure on $\mathscr{E}$.

Proof. First $\mu(\emptyset) = 0$, suppose

$$\sum_{i=1}^{\infty} (a_i, b_i] = (a, b].$$

Since every partial sum has measure at most $F(b_{n+1}) - F(a_1) < F(b) - F(a)$,
$\implies \sum_{i=1}^{n} \mu((a_i, b_i]) \leq \mu((a, b])$.

For the reversed inequality, first we prove that for intervals

$$\bigcup_{i=1}^{n} (c_i, d_i] \supseteq (a, b] \implies \sum_{i=1}^{n} \mu((c_i, d_i]) \geq \mu((a, b]).$$

This can be easily proved by induction, WLOG $b_{n+1} = \max_i b_i$.

Our idea is to extend each $(a_i, b_i]$ a little bit to apply above inequality.

For all $\varepsilon > 0$, take $\delta_i > 0$ s.t.

$$\tilde{b}_i := b_i + \delta_i, \quad F(\tilde{b}_i) - F(b_i) \leq \frac{\varepsilon}{2}.$$

Hence for all $\delta > 0$, $\bigcup_{i=1}^{\infty}(a_i, \tilde{b}_i) \supseteq [a+\delta, b]$, by compactness exists a finite open cover.

$$F(b) - F(a+\delta) \leq \sum_{i=1}^{n}\Big(F(\tilde{b}_i) - F(a_i)\Big) \leq \varepsilon + \sum_{i=1}^{\infty}(F(b) - F(a)).$$

Let $\varepsilon, \delta \to 0$ to conclude. $\square$

Definition 1.7 (Measure space)**.** A triple $(X, \mathscr{F}, \mu)$ is called a **measure space**, if $(X, \mathscr{F})$ is a measurable space and μ is a measure on $\mathscr{F}$.

If $N \in \mathscr{F}$ s.t. $\mu(N) = 0$, we say N is a **null set**.

A probability space is a measure space $(X, \mathscr{F}, P)$ with $P(X) = 1$.

Example 1.8 (Discrete measure)

If X is countable, $p : X \to [0, \infty]$, $\mu(A) := \sum_{x \in A} p(x)$ is a measure.

There are other important properties which we think a sensible measure would have:

- Monotonicity: If $A, B \in \mathscr{E}$, $A \subset B$, then $\mu(A) \leq \mu(B)$.
- Countable subadditivity: $A_1, A_2, \cdots \in \mathscr{E}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

- Lower continuity: $A_1, A_2, \cdots \in \mathscr{E}$ and $A_n \uparrow A \in \mathscr{E}$.

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

- Similarly there's upper continuity (which requires $\mu(A_1) < \infty$).

Theorem 1.9

The measure on a semi-ring has all the above properties.

Proof. In fact,

- Finite additivity $\implies$ monotonicity, subtractivity;
- Countable additivity $\implies$ subadditivity, upper and lower continuity.

Here we only prove the subadditivity, since others are trivial.
Let $A_1, A_2, \cdots \in \mathscr{Q}$, and $\bigcup_{i=1}^{\infty} A_i \in \mathscr{Q}$.

$$B_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in r(\mathscr{Q}) \implies B_n = \sum_{k=1}^{k_n} C_{n,k}, \quad C_{n,k} \in \mathscr{Q}.$$

$$A_n \backslash B_n \in r(\mathscr{Q}) \implies A_n \backslash B_n = \sum_{l=1}^{l_n} D_{n,l}, \quad D_{n,l} \in \mathscr{Q}.$$

Thus by countable additivity,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} \mu(C_{n,k})\right) \\ &\leq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} \mu(C_{n,k}) + \sum_{l=1}^{l_n} \mu(D_{n,l})\right) = \sum_{n=1}^{\infty} \mu(A_n).\end{aligned}$$

Using similar technique we can deduce the upper and lower continuity. □

Theorem 1.10

Let μ be a set function on a ring with finite additivity, then $1 \iff 2 \iff 3 \implies 4 \implies 5$.

- μ is countablely additive;
- μ is countablely subadditive;
- μ is lower continuous;
- μ is upper continuous;
- μ is continuous at $\emptyset$.

§1.2 Outer measure

Once we construct a measure on a semi-ring, we want to extend it to a σ-algebra. Since we can't directly do this, we shall relax some of our restrictions, say reduce countable additivity to subadditivity.

Definition 1.11 (Outer measure)**.** Let $\tau : \mathscr{T} \to [0, \infty]$ satisfying:

- $\tau(\emptyset) = 0$;
- If $A \subset B \subset X$, then $\tau(A) \leq \tau(B)$;
- (Countable subadditivity) $\forall A_1, A_2, \cdots \in \mathscr{T}$, we have

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \tau(A_n).$$

We call τ an **outer measure** on X.

It's easier to extend a measure on semi-ring to an outer measure:

Theorem 1.12

Let μ be a non-negative set function on a collection $\mathscr{E}$, where $\emptyset \in \mathscr{E}$ and $\mu(\emptyset) = 0$. Let

$$\tau(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(B_n) : B_n \in \mathscr{E}, \bigcup_{n=1}^{\infty} B_n \supseteq A\right\}, \quad \forall A \in \mathscr{T}.$$

By convention, $\inf \emptyset = \infty$. (μ *need not be a measure!*)

Then τ is called the outer measure generated by μ.

Proof. Clearly $\tau(\emptyset) = 0$, and $\tau(A) \le \tau(B)$ for $A \subset B$.

$$\bigcup_{n=1}^{\infty} B_n \supseteq B \implies \bigcup_{n=1}^{\infty} B_n \supseteq A.$$

For all $A_1, A_2, \dots \in \mathscr{T}$, WLOG $\tau(A_n) < \infty$. Take $B_{n,k}$ s.t. $\bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n$, such that

$$\sum_{k=1}^{\infty} \mu(B_{n,k}) < \tau(A_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n,$$

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(B_{n,k}) + \varepsilon \le \sum_{n=1}^{\infty} \tau(A_n) + \varepsilon.$$

$\square$

Example 1.13

Let $\mathscr{E} = \{X, \emptyset\}$, $\mu(X) = 1$, $\mu(\emptyset) = 0$. Then $\tau(A) = 1$, $\forall A \ne \emptyset$.

Example 1.14

Let $X = \{a, b, c\}$, $\mathscr{E} = \{\emptyset, \{a\}, \{a, b\}, \{c\}\}$. $\mu(A) = \#A$ for $A \in \mathscr{E}$.

Here something strange happens: $\tau(\{b\}) = 2$ instead of 1, and $\tau(\{b, c\}) = 3$ instead of 2.

In the above example, we found the set $\{b\}$ somehow behaves badly: if we divide $\{a, b\}$ to $\{a\} + \{b\}$, the outer measure is not the sum of two smaller measure.

Hence we want to get rid of this kind of inconsistency to get a proper measure:

Definition 1.15 (Measurable sets)**.** Let τ be an outer measure, if a set A satisfies *Caratheodory condition*:

$$\tau(D) = \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T},$$

we say A is **measurable**.

Remark 1.16 — Inorder to prove A measurable, we only need to check

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T}.$$

Let $\mathscr{F}_\tau$ be the collection of all the τ measurable sets,

Definition 1.17 (Complete measure space)**.** Let $(X, \mathscr{F}, \mu)$ be a measure space, if for all null set A, and $\forall B \subset A, B \in \mathscr{F} \implies \mu(B) = 0$, we say $(X, \mathscr{F}, \mu)$ is **complete**.

Theorem 1.18 (Caratheodory's theorem)
Let τ be an outer measure, then $\mathscr{F} := \mathscr{F}_\tau$ is a σ-algebra, and $(X, \mathscr{F}, \tau)$ is a complete measure space.

Proof. First we prove $\mathscr{F}$ is an algebra:
Note $\emptyset \in \mathscr{F}$, and $\mathscr{F}$ is closed under completements.
For measurable sets A_1, A_2,

$$\begin{aligned}\tau(D) &= \tau(D \cap A_1) + \tau(D \cap A_1^c) \\ &= \tau(D \cap A_1 \cap A_2) + \tau(D \cap (A_1) \cap A_2^c) + \tau(D \cap A_1^c) \\ &= \tau(D \cap (A_1 \cap A_2)) + \tau(D \cap (A_1 \cap A_2)^c).\end{aligned}$$

So $A_1 \cap A_2$ is measurable.
Secondly, we prove $\mathscr{F}$ is a σ-algebra.
Let $A_1, A_2, \cdots \in \mathscr{F}$,

$$B_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in \mathscr{F},$$

Then B_i pairwise disjoint and $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i$. Let $B_f = \bigcup_{i=1}^\infty B_i$.
It's sufficient to prove

$$\tau(D) \geq \tau(D \cap B_f) + \tau(D \cap B_f^c).$$

Let $D_n = \sum_{i=1}^n B_i \cap D$, $D_f = D \cap B_f$, $D_\infty = D \backslash D_f$.
Since B_i are measurable,

$$\tau(D) = \tau(D_n) + \tau(D \backslash D_n) \geq \tau(D_n) + \tau(D_\infty) = \sum_{i=1}^n \tau(D \cap B_i) + \tau(D_\infty).$$

Now we take $n \to \infty$,

$$\tau(D) \geq \sum_{i=1}^\infty \tau(D \cap B_i) + \tau(D_\infty) \geq \tau\left(D \cap \sum_{i=1}^\infty B_i\right) + \tau(D_\infty).$$

Where the last step follows from countable subadditivity.
This implies B_f measurable $\implies \mathscr{F}$ is a σ-algebra.
Next we prove $\tau|\mathscr{F}$ is a measure: Just let $D = \sum_{i=1}^\infty B_i$ in the previous equation.
Last we prove $(X, \mathscr{F}, \tau)$ is complete:
If $\tau(A) = 0$, $\tau(D) \geq \tau(D \cap A^c) = \tau(D \cap A) + \tau(D \cap A^c)$. Thus $A \in \mathscr{F}$. $\square$

§1.3 Measure extension

Definition 1.19 (Measure extension)**.** Let μ, ν be measures on $\mathscr{E}$ and $\overline{\mathscr{E}}$, and $\mathscr{E} \subset \overline{\mathscr{E}}$. If

$$\nu(A) = \mu(A), \quad \forall A \in \mathscr{E},$$

we say ν is a extension of μ on $\overline{\mathscr{E}}$.

If we start from a measure μ on $\mathscr{E}$, ideally, μ can generate an outer measure τ, and we can take $\mathscr{F}_\tau$ to construct a mesaure space.
However, things could go wrong:

Example 1.20
Let $X = \{a, b, c\}, \mathscr{E} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ with

$$\mu(\emptyset) = 0, \mu(\{a, b\}) = 1, \mu(\{b, c\}) = 1, \mu(X) = 2.$$

Then μ is a measure on $\mathscr{E}$, and the outer measure

$$\tau(\emptyset) = 0.$$

Observe that $\mathscr{F}_\tau = \{\emptyset, X\}$, so in this case $\tau|_{\mathscr{F}}$ is the trivial measure.

Example 1.21
Let $X = \mathbb{R}$, $\mathscr{E} = \{(a, b] : a < b, a, b \in \mathbb{R}\}$. Let $\mu(\emptyset) = 0$, and $\mu(A) = \infty$ for $A \neq \emptyset$.
Then μ can be extend to the Borel σ-algebra on $\mathbb{R}$ with $\mu_\alpha = \sum_{q \in \mathbb{Q}} \alpha \delta_q$, $\forall \alpha \geq 0$. So the extension is not unique.

Therefore in order to get a "proper" extension, we must put some requirements on both the starting collection and the set function μ.

Proposition 1.22
Let $\mathscr{P}$ be a π system. If two measures μ, ν on $\sigma(\mathscr{P})$ satisfying

$$\mu|_{\mathscr{P}} = \nu|_{\mathscr{P}}, \quad \mu|_{\mathscr{P}} \text{ is } \sigma\text{-finite},$$

Then $\mu = \nu$.

Proof. Let $A_1, A_2, \cdots \in \mathscr{P}$ s.t. $X = \sum_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$.
Fix n, let $B = A_n$, we want to prove that

$$\mu(B \cap A) = \nu(B \cap A), \quad \forall A \in \sigma(\mathscr{P}).$$

Let $B \in \mathscr{P}$ with $\mu(B) < \infty$,

$$\mathscr{L} := \{A \in \sigma(\mathscr{P}) : \mu(A \cap B) = \nu(A \cap B)\}.$$

We'll prove $\mathscr{L}$ is a λ system, so that $\mathscr{L} \supseteq \sigma(\mathscr{P})$.

Suppose $A_1, A_2 \in \mathscr{L}$ and $A_1 \supseteq A_2$, by $\mu(B) < \infty$,

$$\mu((A_1 - A_2)B) = \mu(A_1 B) - \mu(A_2 B) = \nu(A_1 B - A_2 B) = \nu((A_1 - A_2)B).$$

So $A_1 - A_2 \in \mathscr{L}$.

Let $A_1, A_2, \cdots \in \mathscr{L}$ and $A_n \uparrow A$, then

$$\mu(AB) = \lim_{n\to\infty} \mu(A_n B) = \lim_{n\to\infty} \nu(A_n B) = \nu(AB).$$

Which implies $A \in \mathscr{L}$.

Hence $\sigma(\mathscr{P}) \subset \mathscr{L}$, i.e.

$$\mu(A \cap A_n) = \nu(A \cap A_n), \quad \forall A \in \sigma(\mathscr{P}).$$

Therefore

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A_n) = \sum_{n=1}^{\infty} \nu(A \cap A_n) = \nu(A), \quad \forall A \in \sigma(\mathscr{P}).$$

□

Example 1.23

In probability, let $\mathscr{E}_1, \mathscr{E}_2$ be collections of sets. We say they're independent if

$$P(AB) = P(A)P(B), \quad \forall A \in \mathscr{E}_1, B \in \mathscr{E}_2.$$

By the previous theorem we can derive $\lambda(\mathscr{E}_1), \lambda(\mathscr{E}_2)$ are independent.

If $A_1, A_2, \ldots$ satisfy

$$P(A_{i_1} \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

we say they are independent.

Let $\{1, 2, \ldots\} = I + J$, then the σ-algebra generated by

$$\mathscr{E}_1 = \{A_\alpha \mid \alpha \in I\}, \quad \mathscr{E}_2 = \{A_\alpha \mid \alpha \in J\}$$

are independent.

Theorem 1.24 (Measure extension theorem)

Let μ be a measure on a semi-ring $\mathscr{Q}$, τ is the outer meaasure generated by μ. We have

$$\sigma(\mathscr{Q}) \in \mathscr{F}_\tau, \quad \tau|_{\mathscr{Q}} = \mu.$$

Remark 1.25 — Any measure on a semi-ring $\mathscr{Q}$ can extend to the $\sigma(\mathscr{Q})$, and if μ is σ-finite, the extension is unique.

Proof. For any $A \in \mathscr{Q}$, let $B_1 = A$, $B_n = \emptyset, n \geq 2$. Then $\tau(A) \leq \sum \mu(B_n) = \mu(A)$.

On the other hand, if $A_1, A_2, \cdots \in \mathscr{Q}$ s.t. $\bigcup_{n=1}^{\infty} A_n \supseteq A$, then

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} \mu(AA_n)\right) \leq \sum_{n=1}^{\infty} \mu(AA_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Thus $\tau(A) = \mu(A)$, where we used the fact that μ is countable subadditive.

Next we prove $A \in \mathscr{F}_\tau$. We need to show that

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

WLOG $\tau(D) < \infty$. Take $B_1, B_2, \dots \in \mathscr{Q}$ s.t.

$$\bigcup_{n=1}^{\infty} B_n \supseteq D, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(D) + \varepsilon.$$

Denote $\hat{D} := B_n \in \mathscr{Q}$ for a fixed n. Suppose $\hat{D} \cap A^c = \hat{D} \backslash A = \sum_{i=1}^{n} C_i$.

$$\mu(\hat{D}) = \mu(\hat{D} \cap A) + \sum_{i=1}^{n} \mu(C_i) \geq \tau(\hat{D} \cap A) + \tau(\hat{D} \cap A^c).$$

Apply this inequality to each B_n,

$$\tau(D) + \varepsilon > \sum_{n=1}^{\infty} (\tau(B_n \cap A) + \tau(B_n \cap A^c)) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

this implies $\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c) \implies A \in \mathscr{F}_\tau$.

At last by Caratheodory's theorem, τ is a measure on $\mathscr{F}_\tau \supseteq \sigma(\mathscr{Q})$. □

Theorem 1.26 (Equi-measure hull)

Let τ be the outer measure generated by μ,

- $\forall A \in \mathscr{F}_\tau$, $\exists B \in \sigma(\mathscr{Q})$ s.t. $B \supseteq A$ and $\tau(A) = \tau(B)$;
- If μ is σ-finite, then $\tau(B \backslash A) = 0$.

Remark 1.27 — This theroem states that $\mathscr{F}_\tau$ is just $\sigma(\mathscr{Q})$ appended with null sets.

Proof. If $\tau(A) = \infty$, $B = X$ suffices.

By definition, there exists $B_n = \bigcup_{k=1}^{k_n} B_{n,k} \supseteq A$ s.t. $\tau(B_n) < \tau(A) + \frac{1}{n}$. Let $B = \bigcap_{n=1}^{\infty} B_n$, we must have $\tau(B) = \tau(A)$.

Now for the second part, let $X = \sum_{n=1}^{\infty} A_n$, $A_n \in \mathscr{Q}$, $\mu(A_n) < \infty$.

Since $A = \sum_{n=1}^{\infty} AA_n$, we have

$$AA_n \in \mathscr{F}_\tau, \quad \tau(AA_n) \leq \tau(A_n) = \mu(A_n) < \infty.$$

Let $B_n \in \sigma(\mathscr{Q})$ s.t. $B_n \supseteq AA_n$ and $\tau(B_n) = \tau(AA_n) < \infty$. Let $B := \bigcup_{n=1}^{\infty} B_n$ we have

$$\tau(B - A) = \tau\left(\bigcup_{n=1}^{\infty} (B_n - AA_n)\right) \leq \sum_{n=1}^{\infty} \tau(B_n - AA_n) = 0.$$

□

Let $\mathscr{R}, \mathscr{A}, \mathscr{F}$ be the ring, algebra, σ-algebra generated by $\mathscr{Q}$, respectively. The outer measure τ restricts to a measure on each of these collections, denoted by μ_1, μ_2, μ_3. Each μ_i can generate an outer measure τ_i, but actually they're all the same as our original τ, since τ_i are "build up" from τ, intuitively τ_i cannot be any better than τ. (The proof says exactly the same thing, so I'll omit it)

Proposition 1.28

Let μ be a measure on an algebra $\mathscr{A}$. τ is the outer measure generated by μ, for all $A \in \sigma(\mathscr{A})$, if $\tau(A) < \infty$, then $\forall \varepsilon > 0$, $\exists B \in \mathscr{A}$ s.t. $\tau(A \Delta B) < \varepsilon$.

Remark 1.29 — In practice we often replace τ with a σ-finite measure μ on $\sigma(\mathscr{A})$. (Here σ-finite is on $\mathscr{A}$)

Proof. Choose $B_1, B_2, \cdots \in \mathscr{A}$ s.t.

$$\hat{B} := \bigcup_{n=1}^{\infty} B_n \supseteq A, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(A) + \frac{\varepsilon}{2}.$$

Let N be a sufficiently large number, $B := \bigcup_{n=1}^{N} B_n \in \mathscr{A}$,

$$\tau(A \backslash B) \leq \tau\left(\bigcup_{n=N+1}^{\infty} B_n\right) \leq \sum_{n=N+1}^{\infty} \tau(B_n) \leq \frac{\varepsilon}{2}.$$

As $\tau(B \backslash A) \leq \tau(\hat{B} \backslash A) < \frac{\varepsilon}{2}$, $\tau(A \Delta B) < \varepsilon$. $\square$

Example 1.30

Consider the Bernoulli test, recall $C_{i_1, \ldots, i_n}$ we defined earlier. A measure(probability) μ is defined on the semi-ring $\{C_{i_1, \ldots, i_n}\} \cup \{\emptyset, X\}$, then it can extend uniquely to the σ-algebra generated by it. This is how the probability of Bernoulli test comes from.