Mathematical Analysis II

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§1 Introduction

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Contents of this course: Real analysis

§1.1 Recap

Definition 1.1 (Measurable space)**.** Let X be a set and $\mathcal{A}$ be a σ-algebra, we say $(X, \mathcal{A})$ is a measurable space if

- $\emptyset \in \mathcal{A}$;
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- If $A_k \in \mathcal{A}$, then $\bigcup_{k=1}^{+\infty} \in \mathcal{A}$.

Outer measure m^*:

- $m^*(A) \geq 0$;
- $m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k)$;
- $m^*(A) \leq m^*(B)$ when $A \subset B$.

Caratheodry condition:

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c), \quad \forall T \subset X.$$

We define the measurable sets to be all the sets E satisfying above condition.

This implies the Lebesgue measure space $(\mathbb{R}^n, \mathscr{U}, m)$. It is a complete measure space, i.e. null sets are measurable.

Proposition 1.2

Properties of measurable sets:

- Let E be a measurable set, there exists a G_δ set G and a F_σ set F such that

$$E = G \backslash Z_1 = F \cup Z_2.$$

where Z_1, Z_2 are null sets.

- (Fatou's Lemma)

Measurable sets $E_k \nearrow E \implies \lim_{k\to\infty} m(E_k) = m(E)$ and

$$m\left(\liminf_{k\to\infty} E_k\right) \leq \liminf_{k\to\infty} m(E_k).$$

Definition 1.3 (Measurable function). Let f be a map from measurable space $(X, \mathcal{A})$ to $(Y, \mathcal{B})$. We say f is measurable if

$$\forall B \in \mathcal{B}, \quad f^{-1}(B) \in \mathcal{A}.$$

In the case of real functions, this is equivalent to

$$\forall t \in \mathbb{R}, \quad \{x; f(x) > t\} \text{ is a measurable set.}$$

Proposition 1.4

Let f be a non-negative measurable function, $\exists \varphi_k \nearrow f$, where φ_k are simple functions.

For a general measurable function f, decompose it to $f = f_+ - f_-$.

Theorem 1.5 (Egorov)

Let E be a measurable set and $m(E) < \infty$, $f_n \to f, a.e.$, Then $\forall \varepsilon > 0$, there exists a closed set F_ε s.t. $m(E \backslash F_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on F_ε.

Theorem 1.6 (Lusin)

Let E be a measurable set and $m(E) < \infty$. Then $\forall \varepsilon > 0, \exists F_\varepsilon$ such that $f\big|_{F_\varepsilon}$ is continuous.

Convergence patterns:

- Converge **almost everywhere**: $f_n \to f, a.e.$
- Converge **almost uniformly**: $f_n \to f, a.u.$
- Converge **in measure**: $f_n \xrightarrow{m} f$

§2 Lebesgue integrals

§2.1 Recap: Definition of Lebesgue integrals

- Simple functions: $f = \sum_{k=1}^N a_k \chi_{E_k}$, define
$$\int f = \sum_{k=1}^N a_k m(E_k).$$
- $f : E \to \mathbb{R}^n$, where $m(E) < \infty$, f bounded. These functions form the set $\mathcal{L}_0$. Then $\exists \varphi_k \to f$, φ_k simple, define
$$\int f = \lim_{k\to\infty} \int \varphi_k.$$
- Non-negative function:
$$\int f = \sup \left\{ \int g \;\middle|\; 0 \le g \le f, g \in \mathcal{L}_0 \right\}.$$
- General functions:
$$\int f = \int f_+ - \int f_-.$$
Integrable $\iff$ $\int f_+, \int f_- < +\infty$.

Relations between Riemann integrals and Lebesgue integrals:

- f is Riemann integrable on $[a, b]$ iff f bounded and the discontinuous points form a null set.
- If f is Riemann integrable on $[a, b]$, then two types of integral yield the same result.

§2.2 Dominated convergence theorem

The critical question of this section: If a sequence of functions f_n converges to f (almost everywhere), when does their integrals $\int f_n$ converge to $\int f$?

We'll discuss this issue under various conditions and reach the famous Dominated Convergence Theorem.

Theorem 2.1

Let E be a measurable set with finite measure. Measurable functions $f_n \to f, a.e.$ on E. Furthermore, f_n is uniformly bounded almost everywhere ($|f_n| < M, a.e.$). Then we have
$$\int_E |f_n - f| \to 0 \implies \lim_{m\to\infty} \int_E f_n = \int_E f.$$

Proof. By Egorov's Theorem, $\forall \varepsilon > 0$, there exists $F_\varepsilon \subset E$ s.t. $f_n \to f$ uniformly on F_ε, and $m(E \backslash F_\varepsilon) < \varepsilon$.

Hence
$$\begin{aligned}\int_E |f_n - f| &= \int_{F_\varepsilon} |f_n - f| + \int_{E\backslash F_\varepsilon} |f_n - f| \\ &\le \varepsilon_0 m(E) + 2M\varepsilon,\end{aligned}$$
which proves the result. □

Lemma 2.2 (Fatou's Lemma)
If $f_n \geq 0$, then
$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Proof. For any $g \in \mathcal{L}_0$, $0 \leq g \leq \liminf_{n\to\infty} f_n$, we need to prove $\int g \leq \liminf \int f_n$.
Let $g_k = \min\{f_k, g\}$, assmue g is uniformly bounded so that $g_k \in \mathcal{L}_0$.
We'll prove $g_k \to g$: Assmue by contradiction that $\exists \varepsilon_0 > 0, \exists x_0$ s.t.
$$g(x_0) - g_{k'}(x_0) > \varepsilon_0.$$
then $g(x_0) - f_{k'}(x_0) > \varepsilon_0$, which contradicts with $g \leq \liminf_{n\to\infty} f_n$.
Thus for sufficiently large k, $g_k(x) \leq g(x) \leq g_k(x) + \varepsilon_0, \implies g_k \to g$.
Therefore by Theorem 2.1 (note $g_k \in \mathcal{L}_0$),
$$\begin{aligned}\int g &= \lim_{k\to\infty} \int g_k \\ &\leq \liminf_{k\to\infty} \int f_k,\end{aligned}$$
and we're done. □

Remark 2.3 — This is nearly indentical to the measure version of Fatou's Lemma (Proposition 1.2). It shows some similarities between measure and integrals.

Theorem 2.4 (Beppo-Levi)
If non-negative functions $f_n \nearrow f$, we have
$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof.
$$f_n \leq f \implies \lim_{n\to\infty} \int f_n \leq \int f.$$
By Fatou's Lemma (2.2),
$$\begin{aligned}\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n &= \lim_{n\to\infty} \int f_n, \\ \implies \int f \leq \lim_{n\to\infty} \int f_n.\end{aligned}$$
Combining the two inequalities we get the desired equality. □

Corollary 2.5
Let f_n be non-negative functions, then
$$\int \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \int f_k(x).$$

Proposition 2.6

Let f be an integrable function, $\forall \varepsilon > 0$, we have:

- There exists a set B with finite measure s.t.
$$\int_{B^c} |f| < \varepsilon.$$
- (**Absolute continuity** of integrals) $\exists \delta > 0$ s.t. $\forall E$, if $m(E) < \delta$,
$$\int_E |f| < \varepsilon.$$
This is equivalent to
$$\lim_{m(E)\to 0} \int_E |f| = 0.$$

Proof. Let $f_N(x) = |f(x)|$ when $|x| \le N, |f(x)| \le N$, and $f_N(x) = 0$ otherwise. Then $f_N \nearrow |f|$, so by Beppo-Levi (Theorem 2.4), we get
$$\lim_{N\to\infty} \int f_N = \int |f|.$$
Let $B = \{x \mid |x| \le N, |f(x)| \le N\}$, when N gets sufficiently large, we must have $\int_{B^c} |f| < \varepsilon$.

For the second part, when N is sufficiently large we have $\int (|f| - f_N) < \frac{\varepsilon}{2}$, so
$$\begin{aligned}\int_E |f| &= \int_E f_N + \int_E (|f| - f_N) \\ &\le N \cdot m(E) + \frac{\varepsilon}{2}.\end{aligned}$$
Let $\delta = \frac{\varepsilon}{2N}$ to finish. □

Now we take a look at what we get so far:

- If bounded functions $f_n \in \mathcal{L}_0$, $f_n \to f$, then $\int f_n \to \int f$.
- If f_n is non-negative, then $\int \liminf f_n \le \liminf \int f_n$. (Fatou)
  This corresponds to: $m(\liminf E_n) \le \liminf m(E_n)$.
- If $f_n \nearrow f$, then $\int f_n \nearrow \int f$. (Beppo-Levi)
  This corresponds to: $E_n \subset E_{n+1} \implies m(\bigcup E_n) = \lim m(E_n)$.

Finally we come to the famous Lebesgue dominated convergence theorem:

Theorem 2.7 (Lebesgue Dominated Convergence Theorem)

Functions $f_n \to f, a.e.$, if there exists a function g s.t. $|f_n| \le g, a.e.$, then we have:
$$\int |f - f_n| \to 0. \left(\lim_{n\to\infty} \int f_n = \int f\right)$$

Proof. By Fatou's lemma (2.2), $2g - |f_n - f|$ is non-negative,

$$\int \liminf(2g - |f_n - f|) \leq \liminf \int (2g - |f_n - f|)$$

$$\implies 0 \leq \liminf \left(- \int |f_n - f| \right)$$

$\implies \limsup \int |f_n - f| \leq 0$, hence it must equal to 0. □

Example 2.8

Non-examples of lebesgue dominated convergence theorem:

- Let $f_n = \chi_{[n,n+1]}$, $g = 1$, note that g is not integrable, so $\int f_n = 1$ while $f_n \to 0$.
- $f_n = \frac{1}{n}\chi_{[0,n]}$, $f_n \to 0$, $\int f_n = 1 \not\to 0$. Since $g(x) = \min\{\frac{1}{x}, 1\}$, which isn't integrable.
- $f_n = n\chi_{(0,\frac{1}{n})}$, $f_n \to 0$, $\int f_n = 1 \not\to 0$. Here $g(x) = \frac{1}{x}\chi_{[0,1]}$ is not integrable.

Example 2.9

Suppose that

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

holds for any measurable set E. Then

$$\liminf_{n\to\infty} f_n \leq f \leq \limsup_{n\to\infty} f_n, a.e..$$

Proof. We only need to prove the case when $f = 0$.

For $\forall \varepsilon > 0$, define

$$E_n^\varepsilon = \{x : f_n(x) < -\varepsilon\}.$$

Note that

$$\liminf E_n^\varepsilon \subset \{x : \limsup f_n \leq -\varepsilon\} \subset \liminf E_n^{\frac{\varepsilon}{2}}.$$

Because when $\limsup f_n(x) \leq -\varepsilon$, $\exists N$ such that $\sup_{n>N} f_n(x) < -\frac{\varepsilon}{2}$

$$\implies f_n(x) < -\frac{\varepsilon}{2}, \forall n > N$$

This implies $x \in E_n^{\frac{\varepsilon}{2}}, \forall n > N$, so $x \in \liminf E_n^{\frac{\varepsilon}{2}}$.

We proceed with the proof, by using the condition ($E = \bigcap_{k\geq N} E_k^\varepsilon$),

$$0 = \lim_{n\to\infty} \int_{\bigcap_{k\geq N} E_k^\varepsilon} f_n.$$

Since $x \in \bigcap_{k\geq N} E_k^\varepsilon \implies f_k(x) < -\varepsilon$, we deduce

$$0 = \lim_{n\to\infty} \int_{\bigcap_{k\geq N} E_k^\varepsilon} f_n \leq (-\varepsilon) \cdot m \left(\bigcap_{k\geq N} E_k^\varepsilon \right)$$

Hence $E = \bigcap_{k\geq N} E_k^\varepsilon$ is a null set. □

§2.3 Integrable function space $\mathcal{L}^1(E)$

Definition 2.10 ($\mathcal{L}^1$ space)**.** Denoted by $\mathcal{L}^1(E)$ the space consisting of all the integrable functions on E.

If $f = g, a.e.$, then $\int |f-g| = 0$, we regard them as equivalent elements in $\mathcal{L}^1(E)$.

Observe that $\mathcal{L}^1(E)$ is a vector space, define the norm:

$$\|f\| = \int_E |f|.$$

It's easy to check that $\mathcal{L}^1(E)$ becomes a normal vector space.

Moreover, it's also a **Banach space** (complete normal vector space).

Theorem 2.11

$\mathcal{L}^1(E)$ is a Banach space.

Proof. Let f_n be a Cauchy sequence in $\mathcal{L}^1(E)$, suppose $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k}$.

Let $f = \sum_{k=0}^{\infty}(f_{n_{k+1}} - f_{n_k})$, where $f_{n_0} = 0$. Because

$$\int_E \sum_{k=0}^{\infty} |f_{n_{k+1}} - f_{n_k}| = \sum_{k=0}^{\infty} \int_E |f_{n_{k+1}-f_{n_k}}| \le \sum_{k=0}^{\infty} 2^{-k} < +\infty.$$

so our f is well-defined (convergent). Now we compute

$$\begin{aligned} \|f - f_m\| &= \|f_m - f_{n_l}\| + \left\| \sum_{k=l}^{\infty} (f_{n_{k+1}} - f_{n_k}) \right\| \\ &\le \|f_m - f_{n_l}\| + \sum_{k=l}^{\infty} \|f_{n_{k+1}} - f_{n_k}\| \\ &\le \|f_m - f_{n_l}\| + 2^{-l+1}. \end{aligned}$$

As m gets large, $\|f_m - f_{n_l}\|$ and 2^{-l+1} both converge to 0, so $f_n \to f$ in $\mathcal{L}^1(E)$. □

Remark 2.12 — Notes on multi-dimensional Riemann integrals:

For functions $f : \mathbb{R} \to \mathbb{R}$, recall that

$$\int_a^b f \,\mathrm{d}x = \lim_{\delta_i \to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

But in higher dimensional spaces, it's not so easy to find the suitable partition of the integral region. In fact, this requires the differential theory of multi-dimensional functions first.

As for the improper integrals, for an unbounded region D, we can simlilarly define it to be

$$\int_D f \,\mathrm{d}x = \lim_{n \to \infty} \int_{D_n} f \,\mathrm{d}x,$$

where D_n can be any shape, so the limit is actually stronger than its one-dimensional counterpart. In other words, when we partition D into small cuboids, there's an issue of the summation order.

> This means the integral must be "absolutely" convergent, since by Riemann rearrangement theorem, conditional convergent sequence can be rearranged so that it becomes *divergent*.

Here we state again that if $f = g, a.e$, we regard them as the same function.

Definition 2.13 ($\mathcal{L}^p$ space)**.** Define the $\mathcal{L}^p$ space to be

$$\mathcal{L}^p(E) = \left\{ f \mid \left(\int_E |f|^p \right)^{\frac{1}{p}} < +\infty \right\}$$

Simliarly, it's a complete normal vector space.

In this course we mainly discuss about $\mathcal{L}^1$ instead of general $\mathcal{L}^p$.

Theorem 2.14

The following function spaces are dense in $\mathcal{L}^1$ space:

- Simple functions;
- Step functions;
- Continuous functions with compact support, denoted by $C_0(E)$ or $C_c(E)$.
- Smooth functions with compact support, denoted by $C_0^\infty(E)$.

Proof. • **Simple functions**:

Density is equivalent to:

$$\forall \varepsilon > 0, \exists \text{simple function } g, s.t. \|f - g\| < \varepsilon.$$

f integrable $\implies$ f_+, f_- measurable, so there exists simple functions φ_+^n and φ_-^n s.t.

$$\varphi_+^n \nearrow f_+, \varphi_-^n \nearrow f_- \overset{\text{Beppo-Levi}}{\Longrightarrow} \int \varphi_+^n \nearrow \int f_+ < \infty, \int \varphi_-^n \nearrow \int f_- < \infty$$

This implies $\int (f_\pm - \varphi_\pm^n) \to 0$.

• **Step functions**:

Let $g = \sum_{k=1}^N a_k \chi_{E_k}$, we only need to consider the case $g = \chi_{E_k}$, where E_k is a measurable set with finite measure.

Take cuboids I_j s.t. $E_k \subset \bigcup_{j=1}^\infty I_j$, and $m(E_k) + \varepsilon > \sum_{j=1}^{+\infty} |I_j|$.

Let $h = \chi_{\bigcup_{j=1}^\infty I_j}$, then

$$\begin{aligned} \int |h - g| &= \left| E_k \Delta \left(\bigcup_{j=1}^N I_j \right) \right| \\ &< \varepsilon + \sum_{j>N} |I_j| \end{aligned}$$

Let N be sufficiently large, we conclude that $\int |f - g| \to 0$.

• **1-dimensional continuous functions**:

Let $l = \begin{cases} 0, & x \in (-\infty, a] \cup [b, +\infty) \\ 1, & x \in [a + \varepsilon, b - \varepsilon] \\ linear/smooth, & otherwise \end{cases}$

Then l is a continuous/smooth function s.t. $\|\chi_{[a,b]} - l\| < 2\varepsilon$.

• **Multi-dimensional continuous functions**:

Let $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a cuboid. Let $l_1, \ldots, l_n$ be continuous/smooth functions on (a_i, b_i) defined earlier. We have

$$\|l_1(x_1) \cdots l_n(x_n) - \chi_I\| < C(n)\varepsilon,$$

where $C(n)$ is a constant depending on n. □

Proposition 2.15 (Integrals are invariance under translation and scaling)

Let $f \in \mathcal{L}^1(\mathbb{R}^n)$, for $h \in \mathbb{R}^n$, define $\tau_h(f)(x) = f(x+h)$, then $\tau_h(f) \in \mathcal{L}^1$, and $\|\tau_h(f)\| = \|f\|$.

Simlilarly, define $D_\delta f(x) = f(\delta x)$, then $D_\delta f \in \mathcal{L}^1$, $\|D_\delta f\| = \delta^{-n}\|f\|$.

Theorem 2.16 (Translation and scaling are continuous)

For $h \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$,

$$\lim_{h \to 0} \|\tau_h f - f\| = 0, \quad \lim_{\delta \to 1} \|D_\delta f - f\| = 0.$$

Proof. $\forall \varepsilon > 0$, $\exists$ step function g such that $\|g - f\| < \frac{\varepsilon}{3}$.

$$\begin{aligned} \|\tau_h f - f\| &= \|\tau_h(f-g) - (f-g) + (\tau_h g - g)\| \\ &= \|\tau_h(f-g)\| + \|f-g\| + \|\tau_h g - g\| \\ &= \|\tau_h g - g\| + \frac{2}{3}\varepsilon. \end{aligned}$$

Suppose $g = \sum_{k=1}^N a_k \chi_{I_k}$, it's sufficient to prove the case $g = \chi_I$:

$$\lim_{h \to 0} \|\tau_h g - g\| = \lim_{h \to 0} \|I \Delta (I + h)\| = 0.$$

Similarly D_δ is continuous. □

§3 Fubini's theorem

This theorem provides a way to compute nulti-dimensional integrals.

Let $f(x,y) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$. We wonder if the following equation holds:

$$\int f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y?$$

In fact, this formula somehow says the same thing as the area of a rectangle is equal to its width and length, and this multiplication is commutative.