Geometry II

Felix Chen

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Example 0.0.1

We can't grant that the global function exists. For example, let $D = \{x^2 + y^2 \in [a^2, b^2]\}$, and M be a helicoid.

Since there's a natural map $\phi : D \backslash ([a, b] \times \{0\}) \to M$ (projection), let g, h be the fundamental forms of ϕ, by the symmetry we can extend g, h to entire D.

It's clear that there exists local solutions but the global solutions does't exist. (In theory of differential forms, this is similar to closed forms may not be exact)

But if the region D is *simply connected*, the global solution always exist.

§0.1 Isometric, conformal and area-perserving maps

Let $U, \widetilde{U} \subset \mathbb{R}^2$, and $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. Let $f : \widetilde{U} \to U$ be a map between two surfaces.

Earlier we introduced isometric maps (isometry), i.e. $f^*(g) = \widetilde{g}$. Since the length depends only on the first fundamental form, the isometry perserves the length, angles and areas on surfaces.

The **conformal** maps perserves the angles on the surfaces, and it's easy to imply this is equivalent to $f^*(g) = \lambda \widetilde{g}$ for some $\lambda \in \mathbb{R}$.

As the name suggests, the **area-perserving** maps perserves the areas on two surfaces, which is saying $\det f^*(g) = \det \widetilde{g}$.

It's easy to prove that isometric = conformal + area-perserving. These three properties induce Riemann geometry, complex geometry and sympletic geometry, respectively (in two dimensional).

§0.1.1 Isometries

Firstly by Gauss' Theorema Egregium, Isometries perserves Gaussian curvature.

Example 0.1.1

Let $S_{a,b} : \frac{x^2}{a} + \frac{y^2}{b} = 2z$ be a saddle surface. Let $(x, y, z) = (as, bt, \frac{as^2+bt^2}{2})$ be a parametrization.

We can compute the fundamental forms:

$$g = a^2(1+s^2)\,\mathrm{d}s^2 + 2abst\,\mathrm{d}s\,\mathrm{d}t + b^2(1+t^2)\,\mathrm{d}t^2,$$

$$h = \frac{a\,\mathrm{d}s^2 + b\,\mathrm{d}t^2}{\sqrt{1+s^2+t^2}}.$$

So $K = \frac{1}{ab(1+s^2+t^2)^2}$. In fact the Gaussian curvature of some different surfaces, say $S_{2,3}$ and $S_{1,6}$ are the same.

But there is not an isometry between them:

If $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then τ fixes the circles centered at $(0,0)$ as their Gaussian curvature are the same. Then $\tau_* = \mathrm{d}\tau : T_{(0,0)}\mathbb{R}^2 \to T_{(0,0)}\mathbb{R}^2$ can only be rotation or reflection. (If τ_* is not orthogonal, it will map small circles to ellipse)

While $g(0,0) = a^2\,\mathrm{d}s^2 + b^2\,\mathrm{d}s^2$, which has eigenvalue a^2 and b^2, and they're fixed under τ_*, so $S_{2,3}$ isn't isometric to $S_{1,6}$.

Remark 0.1.2 — Given $E, F, G : D \to \mathbb{R}$ s.t. $g = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ positive definite, is there a surface $D \to \mathbb{E}^3$ can have g as its first fundamental form locally?

When we require E, F, G to be C^ω(analytic), the answer is “yes”, but if we only require C^∞, it's still an open problem.

Even though we don't know the situation in 3 dimensional space, we can study the case in higher dimensions:

Theorem 0.1.3

It's always possible to construct $\phi : D \to \mathbb{E}^4$ to have E, F, G as its first fundamental form.

Surfaces with Gaussian curvature 0 everywhere are called **developable surfaces**. Developable surfaces can only be cylinder, cone, tangent surface of a curve and their concatenation.

Example 0.1.4 (Pseudosphere)

Let $\phi(x, y) = (\frac{\cos x}{y}, \frac{\sin x}{y}, \cosh^{-1}(y) - \frac{\sqrt{y^2-1}}{y})$, where $(x, y) \in (-\pi, \pi) \times [1, +\infty)$.

It's obtained by rotating a *tractrix* around its asymptote. We can calculate its Gaussian curvature, whichis a constant -1. This is where the name comes from.

Recall that hyperbolic plane also has constant curvature -1, in fact they are locally isometric. In 1901, Hilbert proved a theorem that there exists an isometry $\mathbb{H}^2 \to \mathbb{E}^3$.

At last we'll prove an interesting fact:

Proposition 0.1.5 (The local existence of isothermal parameters)
Let $\phi : U \to \mathbb{E}^3$, for all $\hat{u} \in U$, there exists a neighborhood $\widetilde{U}$ and a reparametrization $u = u(\widetilde{u})$, such that
$$g(\widetilde{u}) = \rho^2(\widetilde{u})(\widetilde{E}\,\mathrm{d}\widetilde{s}^2 + \widetilde{G}\,\mathrm{d}\widetilde{t}^2).$$

Remark 0.1.6 — Note that the right hand side is clearly conformal to regions in $\mathbb{E}^2$, so this in fact implies that any surfaces is locally conformal to $\mathbb{E}^2$.

Proof. The critical idea is to realize $\mathbb{R}^2$ as $\mathbb{C}$. To be more precise, we'll follow the steps below:

- Find a way to express $E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ as $(a\,\mathrm{d}s + b\,\mathrm{d}t)(\overline{a}\,\mathrm{d}s + \overline{b}\,\mathrm{d}t)$, where a, b are functions with complex value.
- If there exists a complex function f s.t. $\mathrm{d}f(s+it) = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$, then $g = \frac{1}{|\rho|^2}\,\mathrm{d}f\,\mathrm{d}\overline{f}$.
- Assume further that f is *holomorphic* and non-degenerate, then $f(u) = \widetilde{x}(u) + i\widetilde{y}(u)$ is locally inversible, i.e. exists $u = u(\widetilde{x}, \widetilde{y})$, then
$$g = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x} + i\,\mathrm{d}\widetilde{y})(\mathrm{d}\widetilde{x} - i\,\mathrm{d}\widetilde{y}) = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x}^2 + \mathrm{d}\widetilde{y}^2).$$

Let $a = \sqrt{E}$, $b = \frac{-F + i\sqrt{EG - F^2}}{\sqrt{E}}$. (Note $EG - F^2 > 0$ as g is positive definite)
Next we'll choose suitable f, ρ. Consider the differential equation $T = T(s,t)$:
$$\frac{\partial T}{\partial s} = -\frac{a(s,T)}{b(s,T)}, \quad T(\hat{s}, t) = t.$$

From the relation $f(s, T(s,t)) = t - \hat{t}$ and implicit function theorem we can uniquely determine f.

Remark 0.1.7 — The detail of the solution to this equation in complex functions is beyond the scope of this class.

Such f satisfies $\mathrm{d}f = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$.
When $f(s,t) = (\widetilde{x}, \widetilde{y})$, the Jacobian determinant is
$$\widetilde{x}_s\widetilde{y}_t - \widetilde{x}_t\widetilde{y}_s = -|\rho|^2(a\overline{b} - b\overline{a}) = |\rho|^2\sqrt{EG - F^2} > 0.$$
so f must be non-degenerate. □

§0.2 A bit of manifold

First we'll introduce a few concepts before we move on.

- We say a topological space is an n-dimensional **topological manifold** if it's Hausdorff and locally homeomorphic to $\mathbb{R}^n$. Sometimes we also require manifolds to be compact / paracompact / C_2. Here paracompact means that any open covering has a locally finite subcovering.
- Manifolds with boundary: locally homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$.

- When we talk about the regularity of manifolds, we must appoint an atlas first. Let $\phi_i : U_i \to E_i \subset \mathbb{R}^n$ be homeomorphisms mentioned above, then each ϕ_i is a **chart**, and $\{(U_i, \phi_i)\}_{i\in I}$ is the **atlas**. The map
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
is called **transition functions**.

  The regularity of the manifold is actually the regularity of transition functions, such as C^r, C^∞, piecewise linear, etc.

Example 0.2.1

The sphere $\mathbb{S}^2$ and projective plane $\mathbb{R}P^2$ are 2d manifolds. But they're different since $\mathbb{R}P^2$ is not *orientable*. In fact $\mathbb{R}P^2$ can be obtained by fusing the edge of a Mobius band to a disk(keep in mind that Mobius band has only one edge!).

There are many manifolds which looks wired, but I can't draw them on the computer ;)

Example 0.2.2 (Projective curves)

Consider a quadratic equation
$$C : z^2 + w^2 = 1, \quad (z, w) \in \mathbb{C}^2.$$
What does this surface look like?

Let $Z = z + iw, W = z - iw$, the equation becomes $ZW = 1$, hence the surface is $(\zeta, \frac{1}{\zeta}), \zeta \in \mathbb{C}\backslash\{0\}$. So C is homeomorphic to $\mathbb{C}\backslash\{0\}$.

We can also discuss this in $\mathbb{C}P^2 = \mathbb{C}P^1 \cup \mathbb{C}^2$, where $\mathbb{C}P^1 = \{\infty\} \cup \mathbb{C} \cong \mathbb{S}^2$.

So in homogeneous coordinate, the equation can be written as $ZW = T^2$. The surface is consisting of $(1, 0, 0), (0, 1, 0), (\zeta, \frac{1}{\zeta}, 0)$. Thus the projective completion of C is homeomorphic to $\mathbb{S}^2$, which is $\mathbb{C}\backslash\{0\}$ appending with two points.

Example 0.2.3 (Elliptic curves)

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ pairwise different.
$$E : w^2 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3).$$
What does E looks like in $\mathbb{C}P^2$?

Observe that for $z \in \mathbb{C}\backslash\{\lambda_1, \lambda_2, \lambda_3\}$, there're 2 values for w. So the image of E is two planes($\mathbb{C}$) fused together at $\lambda_1, \lambda_2, \lambda_3$ and ∞ with some adjust.

In fact this can be realized as two cylinder fused together at their edges.

$E \cong T^2\backslash\{pt\}$ in $\mathbb{C}^2$, and T^2 in $\mathbb{C}P^2$.