Geometry II

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Contents

Proof. If x is an interior point, $x \in U$ and U homeomorphic to $\mathbb{R}^n$, then $U \backslash \{x\}$ can deform to a $n-1$ dimensional sphere, thus $\pi_1(U\backslash\{x\}) \neq \{1\}$.

But if x is a boundary point, then $\pi_1(U\backslash\{x\}) = \{1\}$, contradiction! □

Proof. Assume by contradiction that there exists $0 \in U$ s.t. $f(0) \in \mathbb{R}^n$ has no open neighborhood lying completely in $f(U)$.

We can construct a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\|x - g(f(x))\| \leq 1, \quad x \in B(0,1); \quad g(f(x)) \neq 0.$$

Then by Bronwer fixed point theorem on $x \mapsto x - g(f(x))$ we get a contradiction.

The construction is as below:

Since $f(\partial B(0,1))$ must be at least say 10ε away from 0, and $B(f(0), \varepsilon)$ has a point outside of the image of f, so we have a map $P : B(f(0), \varepsilon)\backslash\{p\} \to \partial B(f(0), \varepsilon)$.

Then consider $g = f^{-1} \circ P$, since f^{-1} may not exist on every point, so we need Tietze extension theorem to get an extension h. In $B(f(0), 2\varepsilon)$, we'll change h a little (i.e. take a polynomial approximation) to ensure $g(f(x)) \in B(0,1)$. □

§0.1 Covering spaces

Except van Kampen's theorem, there's another way to compute fundamental groups.

Definition 0.1.1 (Covering maps)**.** Let $p : \widetilde{X} \to X$ be a continuous map. If

- p is surjective;
- For any $x \in X$, there exists an open neighborhood $U = U(x) \subset X$, such that $p^{-1}(U)$ is a union of disjoint open sets $\{U_\alpha\}$, and p is homeomorphism from U_α onto U for each α.

Then we say p is a **covering map**, and $\widetilde{X}$ is a **covering space** of X. $p^{-1}(x)$ is called a **fiber**.

> **Remark 0.1.2** — Often we'll require $\widetilde{X}, X$ are path connected to ensure the relations with fundamental groups. In this case $\#p^{-1}(x)$ is constant.

Definition 0.1.3. We say two covering is **isomorphic** if exists homeomorphism $\tau : \widetilde{X} \to \widetilde{X}'$ s.t. $p' \circ \tau = p$. Two covering is **equivalent** if $p' \circ \widetilde{\sigma} = \sigma \circ p$. The difference is shown in the diagram.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tau"] \arrow[dr, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\sigma}"] \arrow[d, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ X \arrow[r, "\sigma"'] & X' \end{tikzcd}$$

Example 0.1.4

The map $x \mapsto e^{ix}$ is a covering map from $\mathbb{R}$ to S^1. Also $\mathbb{R}^2$ is a covering space of T^2, since T^2 can be represented as $\mathbb{R}^2/\mathbb{Z}^2$.

Example 0.1.5

The surface $2T^2$ can be viewed as an octagon with edges fused together, (an octagon with each angle $45°$) which can be realized in hyperbolic plane $\mathbb{H}^2$.

In fact, $\mathbb{H}^2$ is always the covering space of kT^2 when $k \geq 2$, and kP^2 when $k \geq 3$.

From the examples we can see that covering spaces are the "expanded" spaces of original spaces, i.e. the structures are "flattened" in covering spaces, so that we can study the structure of original spaces more easily.

An important application is that we can "lift" the maps to covering spaces.

Theorem 0.1.6 (Map lifting theorem)

Let $p : \widetilde{X} \to X$ be a covering map, X is path connected. Let A be a path connected space, $f : A \to X$ has a **lifting** $\widetilde{f} : A \to \widetilde{X}$ s.t. $\widetilde{f}(a) \in p^{-1}(f(a)), \forall a \in A$ if and only if there exists a homomorphism Φ s.t. $f_\sharp = p_\sharp \circ \Phi$:

$$\begin{tikzcd} & \pi_1(\widetilde{X}, e_0) \arrow[d, "p_\sharp"] \\ \pi_1(A, a_0) \arrow[ur, dashed, "\Phi"] \arrow[r, "f_\sharp"'] & \pi_1(X, x_0) \end{tikzcd}$$

This is equivalent to $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{X}, e_0))$.

Proof. If we fixed $\tilde{f}(a_0) = e_0$, then for a neighborhood V of e_0, there's a unique map $\tilde{f} : U \to V$, where U is a neighborhood of a_0. This is because p restricted on V is a homeomorphism, and f continuous implies U is open, U is called a *basic neighborhood* of a_0.

For any $b \in A$, there's a path γ from a_0 to b. Since γ is compact, it can be splited to several segments, where each segment lies inside a basic neighborhood of some point.

Therefore the lifting of γ can be uniquely determined by the lifting of one point. Hence $\tilde{f}(b)$ is also determined.

Next we'll show that this $\tilde{f}$ is well-defined and continuous. Let α, β be two paths from a_0 to b. Then $f \circ \alpha, f \circ \beta$ are two paths from x_0 to $f(b)$.

When $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{(X)}, e_0))$, let $w = \alpha\beta^{-1} \in \pi_1(A)$, then there exists $\varphi \in \pi_1(\widetilde{X})$ s.t. $f \circ w = p \circ \varphi$.

But there's a unique lifting for $f \circ \alpha, f \circ \beta$, so $\tilde{f}(\alpha)\tilde{f}(\beta)^{-1} = \varphi$, thus $\tilde{f}(b)$ is well-defined.

Clearly $\tilde{f}$ is continuous, so we're done. □

Remark 0.1.7 — Different base points will result in the image $p_\sharp$ and $\tilde{f}$.

Example 0.1.8

Let M be a closed surface, $M \neq S^2, \mathbb{RP}^2$. Note that M has a contractible covering space, so any map $S^n \to M$ is always homotopic to constant, where $n \geq 2$.

Now if we look at the definition of isomorphic coverings, we'll find that this is just a map lifting, where τ is a lifting of p, τ^{-1} is a lifting of p'. By map lifting theorem we get:

Corollary 0.1.9

Two covering spaces $\widetilde{X}, \widetilde{X}'$ of X are isomorphic iff $p_\sharp(\pi_1(\widetilde{X})) = p'_\sharp(\pi_1(\widetilde{X}'))$.

From this we discover that each covering of X corresponds to a subgroup of $\pi_1(X)$. In fact the inverse is also true:

Theorem 0.1.10 (Existence theorem of covering spaces)

Let X be a path connected and locally path connected space, then for all subgroups $G \leqslant \pi_1(X, x_0)$, there exists a covering $p : \widetilde{X} \to X$ s.t.

$$p_\sharp(\pi_1(\widetilde{X}, e_0)) = G.$$

Remark 0.1.11 — This implies that **universal coverings** always exists, i.e. the covering space $\widetilde{X}$ which has trivial fundamental group.

The proof is quite complex, so we'll put it off here.

Definition 0.1.12 (Regular covering space)**.** If $p_\sharp(\pi_1(\widetilde{X}, e_0))$ is a normal subgroup of $\pi_1(X, x_0)$, then we say it's a **regular covering** of X.

In this case the base point will not change the image of $p_\sharp$.

We say the lifting of p with respect to itself is a **deck transformation**. In fact, deck transformations are just automorphisms of covering spaces, and they constitute a group $Deck_X(\widetilde{X})$ or $Deck_{\widetilde{X}/X}$.

Here's another definition of regular covering: If the group action $Deck_{\widetilde{X}/X}$ onto $\widetilde{X}$ are transitive in $p^{-1}(x_0)$, then we say the covering is **regular covering**.

There should be some pictures of regular and non-regular coverings of $S^1 \vee S^1$, but I'm a bit lazy :-)

Now we'll prove this two definitions are equivalent.

Proposition 0.1.13

Let $p : \widetilde{X} \to X$ be a covering, p is regular iff $p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$ is a normal subgroup.

Proof. When $p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \lhd \pi_1(X, x_0)$, for $\tilde{x}_0, \tilde{x}'_0 \in \widetilde{X}$, we need to prove that there exists $\tau \in Deck_{\widetilde{X}/X}$ s.t. $\tau(\tilde{x}_0) = \tilde{x}'_0$.

We'll use lifting theorem on p, thus we only need to show

$$p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0) \subset p_\sharp\pi_1(\widetilde{X}, \tilde{x}'_0).$$

Let γ be a path from $\tilde{x}_0$ to $\tilde{x}_0'$, and $\alpha \in \pi_1(\widetilde{X}, \tilde{x}_0)$. Note that $\alpha \simeq \gamma\overline{\gamma}\alpha\gamma\overline{\gamma}$, $\alpha' = \overline{\gamma}\alpha\gamma \in \pi_1(\widetilde{X}, \tilde{x}_0')$. Hence

$$p_\sharp(\alpha) = p_\sharp(\gamma)p_\sharp(\alpha')p_\sharp(\overline{\gamma}) \in hp_\sharp\pi_1(\widetilde{X}, \tilde{x}_0')h^{-1} = p_\sharp\pi_1(\widetilde{X}, \tilde{x}_0').$$

The converse is the same. □

Now we'll prove Theorem 0.1.10: First we'll handle the case of universal covering.

Theorem 0.1.14

Universal covering space and is unique under isomorphism for path connected and locally path connected space X. If X is also locally semi-simply connected, then universal covering exists.

Proof. If $\widetilde{x}, \widetilde{X}'$ are both universal coverings, by map lifting theorem, since $\pi_1(\widetilde{X})$ is trivial, $p : \widetilde{X} \to X$ can be lifted to $\sigma : \widetilde{X} \to \widetilde{X}'$, similarly we have σ', and it's easy to see σ and σ' are inverse maps, so they are isomorphic.

For existence part, X locally semi-simply connected means for $\forall x \in X$, there exists a neighborhood basis $\{U_i\}$ s.t. $\pi_1(U_i, x) \to \pi_1(X, x)$ is trivial.

Let $P(X, x_0)$ be all paths in X starting from x_0, and $\mathscr{X}$ is the homology equivalent classes (with fixed endpoints) of $P(X, x_0)$.

Let $p : \mathscr{X} \to X$ by $\langle a \rangle \mapsto a(1)$, and $\tilde{x}_0$ denote the constant path.

Next we'll define the topology on $\mathscr{X}$:

Let $\{U_\alpha\}$ be a topology basis of X, consider the following sets:

$$U(U_\alpha, a) = \{\langle ac \rangle \mid c \in P(U_\alpha, a(1))\}.$$

Let the topology basis on $\mathscr{X}$ be the above sets. We claim $p : \mathscr{X} \to X$ is indeed a covering. □

Example 0.1.15

A counter example of above theorem when X is not locally semi-simply connected: Hawaiian earrings (a family of tangent circles with radius $\to 0$).

Now we can view all these things from group actions.

Let X be a topological space, G is a group acting on X. We say the action is **freely discontinuous** if for all $x \in X$, there's a neighborhood U s.t. $gU \cap U \neq \emptyset$ only holds for $g = e$.

Proposition 0.1.16

Let $G \curvearrowright X$ be a freely discontinuous action, then the quotient map $X \to X/G$ by $x \mapsto Gx$ is a regular covering, and the group action is just deck transformations.

Example 0.1.17

The antipodal map in S^n generates a group $\{\pm 1\}$, and the action is freely discontinuous, so $S^n \to S^n/\{\pm 1\} = \mathbb{R}P^n$ is a covering.

Let $\alpha : (x, y) \mapsto (x, y + 1)$ and $\beta : (x, y) \mapsto (x + 1, -y)$ on $\mathbb{E}^2$ generates a group action $G \curvearrowright \mathbb{E}^2$. This is also freely discontinuous, and $\mathbb{E}^2/G$ is a Klein bottle.

Let X be a topological space, G is a group acting on X. We say the action is **properly discontinuous** if for all compact set $K \subset X$, $gK \cap K \neq \emptyset$ only holds for finitely many g.