Mathematical Analysis II

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Remark 0.1 — Notes on multi-dimensional Riemann integrals:

For functions $f : \mathbb{R} \to \mathbb{R}$, recall that

$$\int_a^b f \, \mathrm{d}x = \lim_{\delta_i \to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

But in higher dimensional spaces, it's not so easy to find the suitable partition of the integral region. In fact, this requires the differential theory of multi-dimensional functions first.

As for the improper integrals, for an unbounded region D, we can simlilarly define it to be

$$\int_D f \, \mathrm{d}x = \lim_{n \to \infty} \int_{D_n} f \, \mathrm{d}x,$$

where D_n can be any shape, so the limit is actually stronger than its one-dimensional counterpart. In other words, when we partition D into small cuboids, there's an issue of the summation order.

This means the integral must be "absolutely" convergent, since by Riemann rearrangement theorem, conditional convergent sequence can be rearranged so that it becomes *divergent*.

Here we state again that if $f = g, a.e$, we regard them as the same function.

Definition 0.2 ($\mathcal{L}^p$ space)**.** Define the $\mathcal{L}^p$ space to be

$$\mathcal{L}^p(E) = \left\{ f \mid \left(\int_E |f|^p \right)^{\frac{1}{p}} < +\infty \right\}$$

Simliarly, it's a complete normal vector space.

In this course we mainly discuss about $\mathcal{L}^1$ instead of general $\mathcal{L}^p$.

Theorem 0.3

The following function spaces are dense in $\mathcal{L}^1$ space:

- Simple functions;
- Step functions;
- Continuous functions with compact support, denoted by $C_0(E)$ or $C_c(E)$.
- Smooth functions with compact support, denoted by $C_0^\infty(E)$.

Proof. • **Simple functions**:

Density is equivalent to:

$$\forall \varepsilon > 0, \exists \text{simple function } g, s.t. \|f - g\| < \varepsilon.$$

f integrable $\implies f_+, f_-$ measurable, so there exists simple functions φ_+^n and φ_-^n s.t.

$$\varphi_+^n \nearrow f_+, \varphi_-^n \nearrow f_- \overset{\text{Beppo-Levi}}{\implies} \int \varphi_+^n \nearrow \int f_+ < \infty, \int \varphi_-^n \nearrow \int f_- < \infty$$

This implies $\int (f_\pm - \varphi_\pm^n) \to 0$.

• **Step functions**:

Let $g = \sum_{k=1}^N a_k \chi_{E_k}$, we only need to consider the case $g = \chi_{E_k}$, where E_k is a measurable set with finite measure.

Take cuboids I_j s.t. $E_k \subset \bigcup_{j=1}^\infty I_j$, and $m(E_k) + \varepsilon > \sum_{j=1}^{+\infty} |I_j|$.

Let $h = \chi_{\bigcup_{j=1}^\infty I_j}$, then

$$\begin{aligned} \int |h - g| &= \left| E_k \Delta \left(\bigcup_{j=1}^N I_j \right) \right| \\ &< \varepsilon + \sum_{j>N} |I_j| \end{aligned}$$

Let N be sufficiently large, we conclude that $\int |f - g| \to 0$.

• **1-dimensional continuous functions**:

Let $l = \begin{cases} 0, & x \in (-\infty, a] \cup [b, +\infty) \\ 1, & x \in [a + \varepsilon, b - \varepsilon] \\ linear/smooth, & otherwise \end{cases}$

Then l is a continuous/smooth function s.t. $\|\chi_{[a,b]} - l\| < 2\varepsilon$.

• **Multi-dimensional continuous functions**:

Let $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a cuboid. Let $l_1, \ldots, l_n$ be continuous/smooth functions on (a_i, b_i) defined earlier. We have

$$\|l_1(x_1) \cdots l_n(x_n) - \chi_I\| < C(n)\varepsilon,$$

where $C(n)$ is a constant depending on n. □

Proposition 0.4 (Integrals are invariance under translation and scaling)

Let $f \in \mathcal{L}^1(\mathbb{R}^n)$, for $h \in \mathbb{R}^n$, define $\tau_h(f)(x) = f(x+h)$, then $\tau_h(f) \in \mathcal{L}^1$, and $\|\tau_h(f)\| = \|f\|$.

Simlilarly, define $D_\delta f(x) = f(\delta x)$, then $D_\delta f \in \mathcal{L}^1$, $\|D_\delta f\| = \delta^{-n} \|f\|$.

Theorem 0.5 (Translation and scaling are continuous)
For $h \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$,

$$\lim_{h\to 0}\|\tau_h f - f\| = 0, \quad \lim_{\delta\to 1}\|D_\delta f - f\| = 0.$$

Proof. $\forall \varepsilon > 0$, $\exists$ step function g such that $\|g - f\| < \frac{\varepsilon}{3}$.

$$\begin{aligned}\|\tau_h f - f\| &= \|\tau_h(f-g) - (f-g) + (\tau_h g - g)\| \\ &= \|\tau_h(f-g)\| + \|f-g\| + \|\tau_h g - g\| \\ &= \|\tau_h g - g\| + \frac{2}{3}\varepsilon.\end{aligned}$$

Suppose $g = \sum_{k=1}^N a_k \chi_{I_k}$, it's sufficient to prove the case $g = \chi_I$:

$$\lim_{h\to 0}\|\tau_h g - g\| = \lim_{h\to 0}\|I\Delta(I+h)\| = 0.$$

Similarly D_δ is continuous. □

§1 Fubini's theorem

This theorem provides a way to compute nulti-dimensional integrals.

Let $f(x,y) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$. We wonder if the following equation holds:

$$\int f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{d_1}}\left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y?$$

In fact, this formula somehow says the same thing as the area of a rectangle is equal to its width and length, and this multiplication is commutative.

Theorem 1.1 (Fubini's Theorem)
Let $f(x,y) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, and f is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

1. $f(x,y)$ as a function of y is integrable on $\mathbb{R}^{d_2}$ for $x \in \mathbb{R}^{d_1}\backslash Z$ with $m(Z) = 0$.
2. Let $g(x) = \int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y$, for $x \in \mathbb{R}^{d_1}\backslash Z$, where Z is a null set. We have g is integrable on $\mathbb{R}^{d_1}$.
3. $$\int_{\mathbb{R}^{d_1+d_2}} f(x,y) = \int_{\mathbb{R}^{d_1}}\left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

Proof. Let $\mathscr{F}$ be the space consisting of all the integrable functions that satisfy Fubini's theorem.

Lemma 1.2
$\mathscr{F}$ is a vector space. Furthermore, for non-negative monotone sequence $f_n \in \mathscr{F}$, if $\lim f_n$ is integrable, then $\lim f_n \in \mathscr{F}$ as well.

Proof of the lemma. First notice that $f \in \mathscr{F} \implies cf \in \mathscr{F}$.

If $f, g \in \mathscr{F}$, consider $f + g$:

By our conditions, there exists $X_f, X_g \subset \mathbb{R}^{d_1}$, s.t. $f(x,y)$ integrable on $\mathbb{R}^{d_2}$, $\forall x \notin X_f$, and $g(x,y)$ integrable on $\mathbb{R}^{d_2}$, $\forall x \notin X_g$.

This implies $f(x,y) + g(x,y)$ integrable on $\mathbb{R}^{d_2}$ for $x \notin X_f \cup X_g$, which proves (1).

$$\int_{\mathbb{R}^{d_2}} f(x,y) + g(x,y) \,\mathrm{d}y = \int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y + \int_{\mathbb{R}^{d_2}} g(x,y) \,\mathrm{d}y.$$

So the LHS is integrable on $\mathbb{R}^{d_1}$ (this is (2)), taking the integral we get

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y \right) \mathrm{d}x + \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} g(x,y) \,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y) + g(x,y) \,\mathrm{d}y \right) \mathrm{d}x.$$

Therefore $\mathscr{F}$ is a vector space.

For a monotone non-negative sequence f_n, $\exists X_n \subset \mathbb{R}^{d_1}$ s.t. f_n is integrable with respect to y for $x \notin X_n$.

Similarly, when $x \notin \bigcup_{n=1}^{\infty} X_n$, as a function of y, by Beppo-Levi (or Dominated convergence),

$$\int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y = \lim_{n \to \infty} \int_{\mathbb{R}^{d_2}} f_n(x,y) \,\mathrm{d}y.$$

This equation holds when $\int f(x,y) \,\mathrm{d}y$ is finite, so we need to prove it is finite almost everywhere.

For $x \notin \bigcup X_n$, we have:

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f_n(x,y) \,\mathrm{d}y \right) \mathrm{d}x \to \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y \right) \mathrm{d}x$$

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f_n(x,y) \,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1+d_2}} f_n \to \int_{\mathbb{R}^{d_1+d_2}} f$$

Compare these relations we deduce

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^{d_1+d_2}} f < +\infty.$$

so $\int_{\mathbb{R}^{d_2}} f(x,y) \,\mathrm{d}y$ is finite almost everywhere. This gives (1), and (2), (3) follows immediatedly. $\square$

Back to the proof of the original theorem, we want to prove $\mathscr{F} = \mathcal{L}^1$.

We prove the indicator function of following sets are in $\mathscr{F}$:

- Cuboids;
- Finite open sets;
- G_δ sets;
- Null sets;
- General measurable sets.

Let I be a cuboid, $I = I_x \times I_y$, so $\chi_I = \chi_{I_x} \chi_{I_y}$.

$$\int \chi_I = |I| = |I_x||I_y| = \int \chi_{I_x} |I_y| \,\mathrm{d}x = \int \int (\chi_{I_x} \chi_{I_y} \,\mathrm{d}y) \,\mathrm{d}x.$$

Let O be a finite open set, $O = \bigcup_{n=1}^{\infty} I_n$, where I_n are pairwise disjoint cuboids.

$$\chi_O = \lim_{n\to\infty} \chi_{\bigcup_{k=1}^{n} I_k} \in \mathscr{F},$$

as it's an inceasing sequence.

For $G_\delta = \bigcap_{n=1}^{\infty} O_n$, $\chi_{O_n} \searrow \chi_{G_\delta}$. $\implies \chi_{G_\delta} \in \mathscr{F}$.

For null set E, if $\chi_E \in \mathscr{F}$, $\forall A \subset E$,

$$0 = \int \chi_E = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} \chi_E \,\mathrm{d}y \right) \mathrm{d}x.$$

hence $\int_{\mathbb{R}^{d_2}} \chi_E \,\mathrm{d}y = 0$, for $x, a.e.$ $\implies \int_{\mathbb{R}^{d_2}} \chi_A \,\mathrm{d}y = 0$ for $x, a.e.$.

Taking the integral with respect to x, we have $\chi_A \in \mathscr{F}$.

Therefore if E is a null set, by taking its equi-measure hull we deduce $\chi_E \in \mathscr{F}$.

Finally, for a general measurable set E, let O be its equi-measure hull, and $E = O \backslash A$. since $\mathscr{F}$ is a vector space, $\chi_E \in \mathscr{F}$.

The rest is trival now: Because all the simple functions are in $\mathscr{F}$, and any measurable functions can be expressed as limits of increasing simple functions, so $\mathscr{F} = \mathcal{L}^1(\mathbb{R}^{d_1+d_2})$. □

Theorem 1.3 (Tonelli's theorem)

Let f be a non-negative measurable function on $\mathbb{R}^d$.

- $f(x,y)$ is measurable on $\mathbb{R}^{d_2}$ for x almost everywhere;
- $\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y$ as a function of x is measurable;
- The integral satisfies:
$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Proof. Consider the truncation function $f(x,y)\chi_{|x|+|y|<k}\chi_{f<k}$. □

Proposition 1.4

Let E be a measurable set on $\mathbb{R}^d$. For x almost everywhere, $E^x = \{y \mid (x,y) \in E\}$ is measurable on $\mathbb{R}^{d_2}$.

As a function of x, $m(E^x)$ satisfies

$$m(E) = \int_{\mathbb{R}^{d_1}} m(E^x).$$

Proof. Consider $f = \chi_E$ and use Tonelli's theorem. □