Geometry II

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We say $\phi(s, t_0)$ is an s-curve and $\phi(s_0, t)$ is a t-curve. If s-curve and t-curve are orthogonal at every point $u \in U$, i.e. $F = 0$, or the matrix is diagonal, we say ϕ is an **orthogonal parametrization**, and s, t are **orthogonal parameters**.

Moreover, if $E = G, F = 0$ for all $u \in U$, (the matrix is a scalar) then we call ϕ an **isothermal parametrization**, and s, t are **isothermal parameters**. (Sometimes also called **comformal parameters**)

Example 0.1

The longtitude and latitude on a sphere are orthogonal parameters, but not isothermal parameters; While the stereographical projection is an isothermal parametrization of the sphere.

Remark 0.2 — The word "isothermal" is connected to thermology in a rather complicated way. The word "conformal" provides a more intuitive comprehension.

Remark 0.3 — A fun fact: Isothermal parameters always exist locally on regular parametrized surfaces. This only holds for 2-dimensional manifold.

§0.1 Linear algebra review

Let V be a vector space over $\mathbb{F}$.

Symmetrical bilinear form vs. quadratic form

A symmetrical bilinear form is a linear map $B : V \times V \to \mathbb{F}$ with $B(v, w) = B(w, v)$. A quadratic form is a map $Q : V \to \mathbb{F}$ with $Q(v) = B(v, v)$ for some symmetrical bilinear form B.

By taking a basis of V, we can use the matrix to express them:

$$B(v, w) = vAw^T, \quad Q(v) = vAv^T.$$

where A is a symmetrical matrix. When we change the basis, the matrix A differs by a congruent transformation.

We could also write $B \in V^* \otimes V^*$ for a bilinear form B. All the symmetrical bilinear forms constitute a subspace of $V^* \otimes V^*$ of dimension $\frac{n(n+1)}{2}$. This is denoted by $\mathrm{Sym}^2(V)$.

Remark 0.4 — The subspace of anti-symmetric matrices is denoted by $\mathrm{Alt}^2(V)$, and $\mathrm{Alt}^2(V) \oplus \mathrm{Sym}^2(V) = V^* \otimes V^*$.

§0.2 Tangent spaces

A surface $\phi : U \to \mathbb{E}^3$ has a tangent space at every point $\phi(u)$, which is just the space (in this case, a plane) spanned by $\phi_s(u)$ and $\phi_t(u)$. We can prove that this tangent space is independent to the parameters. Furthermore, we can equip it with the inner product in $\mathbb{E}^3$, the matrix of this product is exactly $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

Remark 0.5 — In modern differential manifold theory, there's an intrinsic definition of tangent spaces, but this definition is too abstract.

Here we present one of these intrinsic definitions.

Definition 0.6 (Tangent vectors)**.** Define an equivalence relation on smooth curves in $\phi(U)$:

Let $\gamma(r) = \phi(s(r), t(r))$ be a smooth curve $(-\epsilon, \epsilon) \to \phi(U)$. Two curves γ_1, γ_2 are equivalent iff $s_1'(0) = s_2'(0)$ and $t_1'(0) = t_2'(0)$.

Each equivalence class is a "tangent vector" at point $\phi(s_0, t_0)$.

§0.3 The second fundamental form

Since the first fundamental form is not sufficient to describe all the properties of the surface (it can only describe the curves lying on it and the area), we thereby introduce the second fundamental form.

Definition 0.7 (The second fundamental form)**.** Let $\phi : U \to \mathbb{E}^3$ be a regular parametrized surface. The normal vector at the point $u = (s, t)$ is defined as

$$\vec{n} = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}.$$

Since the cross product cares about orientation, so the normal vector is only fixed under orientation-preserving reparametrization.

Now we expand ϕ to the second derivative:

$$\phi(s + \Delta s, t + \Delta t) = \phi(s, t) + (\phi_s, \phi_t) \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + \frac{1}{2}(\Delta s, \Delta t) \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} + o(|\Delta s|^2 + |\Delta t|^2).$$

Hence we define

$$L = \phi_{ss} \cdot \vec{n}, \quad M = \phi_{st} \cdot \vec{n}, \quad N = \phi_{tt} \cdot \vec{n}.$$

The second fundamental form is defined as $h = L\,\mathrm{d}s^2 + M\,\mathrm{d}s\,\mathrm{d}t + N\,\mathrm{d}t^2$.

Remark 0.8 — Another expression of L, M, N:

$$L = -\phi_s \cdot \vec{n}_s, \quad M = -\phi_s \cdot \vec{n}_t = -\phi_t \cdot \vec{n}_s, \quad N = -\phi_t \cdot \vec{n}_t.$$

Intuitively, the second fundamental form describes how much the surface is "going out" of the tangent plane.

Since the first fundamental form g gives an inner product of the tangent space, so we can compute the "canonical form" of h with respect to g, this process will generate some geometric quantities.

Definition 0.9. Define the **average curvature** and **Gaussian curvature**:

$$H := \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}, \quad K := \frac{LN - M^2}{EG - F^2}.$$

These expressions look complicated and ugly, the reason is that we didn't choose the right parameters. Indeed, if at some point $u = (s, t)$ we have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Definition 0.10 (Principal curvatures)**.** The characteristic polynomial $\lambda^2 - 2H\lambda + K$ has two real roots, they are called the **principal curvatures** of ϕ. The **principal directions** are defined as the directions of eigenvectors of $h : T_uU \times T_uU \to \mathbb{R}$ WRT the inner product g.

Now we'll dig deeper into the geometric meaning of these formulas.

Proposition 0.11

H and K are geometric quantities.

Proof. For any reparametrization $s = s(\tilde{s}, \tilde{t}), t = t(\tilde{s}, \tilde{t})$, we have

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J \begin{pmatrix} E & F \\ F & G \end{pmatrix} J^{-1}.$$

Similarly we can verify that

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = J \begin{pmatrix} L & M \\ M & N \end{pmatrix} J^{-1}.$$

Since

$$H = \frac{1}{2}\operatorname{tr}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad K = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Thus H, K are fixed under orientation-preserving reparametrization. □

Remark 0.12 — When the reparametrization is orientation-reversing, L, M, N all differ by a sign, thus H will change while K is still fixed.

Some examples about curvatures:

Example 0.13

A sphere with radius R:

$$H = \frac{1}{R}, \quad K = \frac{1}{R^2}.$$

Note that

$$\int_{S^2} K \, \mathrm{d}Area = \frac{1}{R^2} 4\pi R^2 = 2\pi\chi(S^2).$$

This is an example of Gauss-Bonnet formula which we'll cover later.

Example 0.14

The conical and cylindrical surfaces have Gaussian curvature 0.

For a general ruled surface, we can prove that $K \leq 0$ everywhere.

Example 0.15

Minimal surfaces(like soap bubbles) have $H = 0$ and $K \leq 0$ everywhere.

Example 0.16 (Dupin canonical form)

Let $\phi : U \to \mathbb{E}^3$ be a regular surface, then at the neighborhood of any point, there exists a parameter s.t. $\phi(s,t) = (s, t, \kappa_1 s^2 + \kappa_2 t^2) + o(|s|^2 + |t|^2)$, where κ_1, κ_2 are principal curvatures of ϕ.

In this case we can talk about concepts like "elliptic point", "parabolic point" and "hyperbolic point".

Next we'll going to switch to a more intrinsic view to study the meaning of those definitions again.

If we look at a curve γ on a surface ϕ, let r be the arc length parameter, then $\|\gamma'\| = 1$, $\|\gamma''\| = \kappa(r)$, note that γ'' can be decompose with respect to the normal vector and tangent plane:

$$\gamma'' = \kappa_n \vec{n} + \kappa_g \vec{n} \times \gamma'.$$

Here κ_n is called **normal curvature**, and κ_g is called **geodesic curvature** of γ WRT ϕ.

Moreover we have *Euler's formula*:

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

If we compute the normal curvature in terms of $u = (s,t)$:

$$\gamma' = \phi_s s' + \phi_t t'$$

$$\gamma'' = (s', t') \begin{pmatrix} \phi_{ss} & \phi_{st} \\ \phi_{ts} & \phi_{tt} \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} + \phi_s s'' + \phi_t t''$$

Hence

$$\kappa_n = \gamma'' \cdot \vec{n} = (s', t') \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} s' \\ t' \end{pmatrix} = L(s')^2 + 2Ms't' + N(t')^2.$$

This is the formula under the arc length parameter.

> **Remark 0.17** — The general formula of κ_n:
> $$\kappa_n = \frac{Ls'^2 + 2Ms't' + Nt'^2}{Es'^2 + 2Fs't' + Gt'^2}.$$

The normal plane of γ intersects the surface ϕ, the section curve is called a **normal section**.

Oberserve that: if $\|\gamma'\| = 1$, and the tangent vector is $\vec{t}$, then $\kappa_n(r)$ is the curvature of the normal section at u in the plane spanned by $\vec{n}, \vec{t}$.

Hence κ_n can be viewed as a quadratic form $\vec{n}^\perp \to \mathbb{R}$ which sends a vector $\vec{t}$ to the curvature of the normal section with tangent vector $\vec{t}$.

Furthermore, the principal directions are the "eigen-directions" of κ_n, which are the directions where the curvature of normal section attains its extremum.

> **Example 0.18**
> Consider the helix and the cylinder
> $$\gamma(t) = (\cos t, \sin t, at), \quad S : x^2 + y^2 = 1.$$
> It's easy to verify that $\kappa = \kappa_n = \frac{1}{1+a^2}$ as γ'' is always perpendicular to z-axis.
>
> Note that $\kappa_g = 0$ everywhere, curves satisfying $\kappa_g = 0$ are called **geodesic line**.

§0.4 Gauss map and Weingarten map

The strange definition of those curvatures don't come from nothing, in this section we'll cover this topic and give a geometric interpretation.

Definition 0.19 (Gauss map)**.** Let Σ be a regular surface in $\mathbb{E}^3$, denote its normal vector at x by $\vec{n}(x)$. Then this map $\mathcal{G} : \Sigma \to \mathbb{S}^2$ by $x \mapsto \vec{n}(x)$ is called the **Gaussian map**.

In terms of a parametrized surface $\phi : U \to \mathbb{E}^3$, we can compute that
$$\mathcal{G} : U \to \mathbb{S}^2 : \quad \vec{n}(u) = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|}$$

But each vector has a normal plane, namely $\vec{n}^\perp$, and this derives the **Weingarten map**:

Definition 0.20 (Weingarten map)**.** For all $u \in U$, define $W : \vec{n}(u)^\perp \to \vec{n}(u)^\perp$: $\vec{v} \mapsto W(\vec{v})$, where
$$W(\vec{v}) = -\left.\frac{\mathrm{d}(\mathcal{G} \circ \gamma)}{\mathrm{d}u}\right|_{u=0}, \quad \gamma := \phi(u(r)) \text{ is a curve on the surface.}$$

> **Remark 0.21** — In the language of modern differetial manifolds, Weingarten map is just the tangent map of Gauss map with a negative sign.

Since $\vec{n}^\perp$ has a basis ϕ_s, ϕ_t, we can compute the matrix of Weingarten map:
$$(\phi_s, \phi_t)W = (-\vec{n}_s, -\vec{n}_t).$$
Note that $-\vec{n}_s \cdot \phi_s = \vec{n} \cdot \phi_{ss} = L$, so if we take the inner product of (ϕ_s, ϕ_t) on both sides, we get
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} W = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \implies W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Since W is clearly a geometric quantity, so its trace and determinant are also geometric:

$$\operatorname{tr} W = \frac{GL - 2FM + EN}{EG - F^2} = 2H, \quad \det W = \frac{LN - M^2}{EG - F^2} = K,$$

which gives the average curvature and Gauss curvature.

Moreover, the principal curvatures are the eigenvalues of W, and principal directions are just the eigenspaces of W.

Let $\vec{v} = (\phi_s, \phi_t)X$, then its normal section has curvature

$$\kappa_n = \frac{X^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} X}{X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} X}.$$

When $\|\vec{v}\| = 1$, we can change a parameter s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = I_2$, in this case we can observe that when κ_n attains its extremum, $\vec{v}$ is precisely the eigenvector of W, i.e. lies on the principal directions.

Definition 0.22 (Curvature line)**.** A curve is called a **curvature line** if its tangent vector is the same as principal directions everywhere.

Example 0.23

Every curve on a sphere is curvature line.

Around a point where the principal curvatures are different, there exists a orthogonal grid of curvature lines.

Example 0.24

monkey saddle surface, "prong singularity"

In the case when the s-curve and t-curve are precisely the curvature lines, then we say this is a **curvature grid parameter**, and here we have $g = E\,\mathrm{d}s^2 + G\,\mathrm{d}t^2$ and $h = L\,\mathrm{d}s^2 + N\,\mathrm{d}t^2$.

Remark 0.25 — The geometric interpretation of Gauss curvature: For $u \in D \subset U$,

$$|K(u)| = \lim_{\text{“}D \to u\text{''}} \frac{Area_{\mathbb{S}^2}(\mathcal{G}(D))}{Area_{\mathbb{E}^3}(\phi(D))}$$