

# Mathematical Analysis II

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### Theorem 0.1 (Fubini's Theorem)

Let  $f(x, y) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}$ , and  $f$  is integrable on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ .

1.  $f(x, y)$  as a function of  $y$  is integrable on  $\mathbb{R}^{d_2}$  for  $x \in \mathbb{R}^{d_1} \setminus Z$  with  $m(Z) = 0$ .
2. Let  $g(x) = \int_{\mathbb{R}^{d_2}} f(x, y) dy$ , for  $x \in \mathbb{R}^{d_1} \setminus Z$ , where  $Z$  is a null set. We have  $g$  is integrable on  $\mathbb{R}^{d_1}$ .
- 3.

$$\int_{\mathbb{R}^{d_1+d_2}} f(x, y) = \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx.$$

*Proof.* Let  $\mathcal{F}$  be the space consisting of all the integrable functions that satisfy Fubini's theorem.

### Lemma 0.2

$\mathcal{F}$  is a vector space. Furthermore, for non-negative monotone sequence  $f_n \in \mathcal{F}$ , if  $\lim f_n$  is integrable, then  $\lim f_n \in \mathcal{F}$  as well.

*Proof of the lemma.* First notice that  $f \in \mathcal{F} \implies cf \in \mathcal{F}$ .

If  $f, g \in \mathcal{F}$ , consider  $f + g$ :

By our conditions, there exists  $X_f, X_g \subset \mathbb{R}^{d_1}$ , s.t.  $f(x, y)$  integrable on  $\mathbb{R}^{d_2}$ ,  $\forall x \notin X_f$ , and  $g(x, y)$  integrable on  $\mathbb{R}^{d_2}$ ,  $\forall x \notin X_g$ .

This implies  $f(x, y) + g(x, y)$  integrable on  $\mathbb{R}^{d_2}$  for  $x \notin X_f \cup X_g$ , which proves (1).

$$\int_{\mathbb{R}^{d_2}} f(x, y) + g(x, y) dy = \int_{\mathbb{R}^{d_2}} f(x, y) dy + \int_{\mathbb{R}^{d_2}} g(x, y) dy.$$

So the LHS is integrable on  $\mathbb{R}^{d_1}$  (this is (2)), taking the integral we get

$$\int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx + \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} g(x, y) dy \right) dx = \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) + g(x, y) dy \right) dx.$$

Therefore  $\mathcal{F}$  is a vector space.

For a monotone non-negative sequence  $f_n$ ,  $\exists X_n \subset \mathbb{R}^{d_1}$  s.t.  $f_n$  is integrable with respect to  $y$  for  $x \notin X_n$ .

Similarly, when  $x \notin \bigcup_{n=1}^{\infty} X_n$ , as a function of  $y$ , by Beppo-Levi (or Dominated convergence),

$$\int_{\mathbb{R}^{d_2}} f(x, y) dy = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{d_2}} f_n(x, y) dy.$$

This equation holds when  $\int f(x, y) dy$  is finite, so we need to prove it is finite almost everywhere. For  $x \notin \bigcup X_n$ , we have:

$$\begin{aligned} \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f_n(x, y) dy \right) dx &\rightarrow \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx \\ \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f_n(x, y) dy \right) dx &= \int_{\mathbb{R}^{d_1+d_2}} f_n \rightarrow \int_{\mathbb{R}^{d_1+d_2}} f \end{aligned}$$

Compare these relations we deduce

$$\int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx = \int_{\mathbb{R}^{d_1+d_2}} f < +\infty.$$

so  $\int_{\mathbb{R}^{d_2}} f(x, y) dy$  is finite almost everywhere. This gives (1), and (2), (3) follows immediatedly.  $\square$

Back to the proof of the original theorem, we want to prove  $\mathcal{F} = \mathcal{L}^1$ .

We prove the indicator function of following sets are in  $\mathcal{F}$  :

- Cuboids;
- Finite open sets;
- $G_\delta$  sets;
- Null sets;
- General measurable sets.

Let  $I$  be a cuboid,  $I = I_x \times I_y$ , so  $\chi_I = \chi_{I_x} \chi_{I_y}$ .

$$\int \chi_I = |I| = |I_x| |I_y| = \int \chi_{I_x} |I_y| dx = \int \int (\chi_{I_x} \chi_{I_y} dy) dx.$$

Let  $O$  be a finite open set,  $O = \bigcup_{n=1}^{\infty} I_n$ , where  $I_n$  are pairwise disjoint cuboids.

$$\chi_O = \lim_{n \rightarrow \infty} \chi_{\bigcup_{k=1}^n I_k} \in \mathcal{F},$$

as it's an incesing sequence.

For  $G_\delta = \bigcap_{n=1}^{\infty} O_n$ ,  $\chi_{O_n} \searrow \chi_{G_\delta} \implies \chi_{G_\delta} \in \mathcal{F}$ .

For null set  $E$ , if  $\chi_E \in \mathcal{F}$ ,  $\forall A \subset E$ ,

$$0 = \int \chi_E = \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} \chi_E dy \right) dx.$$

hence  $\int_{\mathbb{R}^{d_2}} \chi_E dy = 0$ , for  $x, a.e. \implies \int_{\mathbb{R}^{d_2}} \chi_A dy = 0$  for  $x, a.e..$

Taking the integral with respect to  $x$ , we have  $\chi_A \in \mathcal{F}$ .

Therefore if  $E$  is a null set, by taking its equi-measure hull we deduce  $\chi_E \in \mathcal{F}$ .

Finally, for a general measurable set  $E$ , let  $O$  be its equi-measure hull, and  $E = O \setminus A$ . since  $\mathcal{F}$  is a vector space,  $\chi_E \in \mathcal{F}$ .

The rest is trival now: Because all the simple functions are in  $\mathcal{F}$ , and any measurable functions can be expressed as limits of increasing simple functions, so  $\mathcal{F} = \mathcal{L}^1(\mathbb{R}^{d_1+d_2})$ .  $\square$

**Theorem 0.3** (Tonelli's theorem)

Let  $f$  be a non-negative measurable function on  $\mathbb{R}^d$ .

- $f(x, y)$  is measurable on  $\mathbb{R}^{d_2}$  for  $x$  almost everywhere;
- $\int_{\mathbb{R}^{d_2}} f(x, y) dy$  as a function of  $x$  is measurable;
- The integral satisfies:

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx.$$

*Proof.* Consider the truncation function  $f(x, y)\chi_{|x|+|y|<k}\chi_{f<k}$ . □

**Proposition 0.4**

Let  $E$  be a measurable set on  $\mathbb{R}^d$ . For  $x$  almost everywhere,  $E^x = \{y \mid (x, y) \in E\}$  is measurable on  $\mathbb{R}^{d_2}$ .

As a function of  $x$ ,  $m(E^x)$  satisfies

$$m(E) = \int_{\mathbb{R}^{d_1}} m(E^x).$$

*Proof.* Consider  $f = \chi_E$  and use Tonelli's theorem. □

**§0.1 Applications of Fubini's theorem**

**Definition 0.5** (Product measure). Let  $(X, \mathcal{F}, m)$  and  $(Y, \mathcal{G}, m)$  be measure spaces, define a measure on  $X \times Y$ : The measure  $m$  induces an outer measure on  $X \times Y$ , and complete it to a normal measure by using Caratheodory conditions. This measure is called the **product measure** on  $X \times Y$ .

**Theorem 0.6**

Let  $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ ,  $E_1, E_2$  are subsets of  $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$ , respectively.

- If  $E_1, E_2$  are measurable, then  $E$  is measurable as well, and  $m(E) = m(E_1)m(E_2)$ .
- If  $E$  is measurable, then  $E_1, E_2$  are measurable, and  $m(E) = m(E_1)m(E_2)$ , unless one of  $E_1, E_2$  is null set, which means  $E$  is null as well.

*Proof.* First it's easy to note that

$$m^*(E) \leq m^*(E_1)m^*(E_2).$$

So we directly conclude that if one of  $E_1, E_2$  is null set,  $E$  must be null.

Thus we may assume below that  $E_1, E_2$  have finite nonzero measure. By taking the equimeasure hull of  $E_1, E_2$  (denoted by  $F_1, F_2$ ), let  $Z_1 = F_1 \setminus E_1, Z_2 = F_2 \setminus E_2$ , we have

$$(F_1 \times F_2) \setminus (Z_1 \times F_2 \cup F_1 \times Z_2) \subset E \subset F_1 \times F_2,$$

so  $E$  is measurable.

Conversely, if  $E$  is measurable, consider the measurable function  $\chi_E$ , by definition  $\chi_E = \chi_{E_1}\chi_{E_2}$ , hence by Tonelli's theorem, for  $x$  almost everywhere,  $\chi_{E_1}(x)\chi_{E_2}$  is measurable on  $\mathbb{R}^{d_2} \implies E_2$  is measurable.

Therefore we have the equation

$$m(E) = \int_{\mathbb{R}^d} \chi_E = \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} \chi_{E_1}\chi_{E_2} \right) = m(E_1)m(E_2).$$

This proves the theorem.  $\square$

### Corollary 0.7

Let  $f(x)$  be a measurable function on  $\mathbb{R}^{d_1}$ , we have  $g(x, y) = f(x)$  is measurable on  $\mathbb{R}^{d_2}$ .

*Proof.* It's sufficient to prove that  $\{(x, y) | f(x) > t\}$  is measurable in  $\mathbb{R}^d$ . This follows from the fact that

$$\{(x, y) | f(x) > t\} = \{x | f(x) > t\} \times \mathbb{R}^{d_2},$$

and the previous theorem.  $\square$

### Proposition 0.8

Let  $L$  be a linear map  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $E \subset \mathbb{R}^d$  a measurable set, then  $L(E)$  is measurable, and

$$m(L(E)) = |\det L| m(E).$$

*Proof.* In fact we only need to prove it for cuboids  $E$  and elementary linear transformation  $L$ .

Now we only need to look at the case where  $L = \begin{pmatrix} 1 & c & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$  since the other cases are

trivial or similar to this case.

Thus by Fubini's theorem, WLOG  $E$  is the unit cube,

$$m(L(E)) = \int \chi_{L(E)} = \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} \chi_{L(E)} dx_1 \right) = \int_{\mathbb{R}^{d-1}} \chi_{E'} \cdot 1 = 1 = |\det L| m(E),$$

where  $E' = \{(x_2, \dots, x_n) | 0 \leq x_i \leq 1\}$ .  $\square$

From this transformation formula we deduce the integral version:

Let  $f$  be an integrable function on  $\mathbb{R}^d$ , then  $f(L(x))$  is also integrable, and

$$\int f(L(x)) = \frac{1}{|\det L|} \int f(x).$$

Here we require  $L \in \text{GL}(n)$ , since if  $\det L = 0$ , the function  $f(L(x))$  need not be measurable.

At last we take a look at Fubini's theorem with the convolution product.

**Definition 0.9** (Convolution). Let  $f, g$  be smooth functions with compact support, define their **convolution** to be

$$f * g = \int f(x - y)g(y) \, dy.$$

Then  $f * g$  is also a smooth function with compact support.

In fact we can generalize this definition for  $f, g \in L^1$ .

First note that  $f(x - y), g(y)$  are measurable functions on  $\mathbb{R}^{2d}$ , by Tonelli's theorem,

$$\int_{\mathbb{R}^{2d}} |f(x - y)| |g(y)| \, dx \, dy = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x - y)| |g(y)| \, dx \right) dy = \|f\|_{L^1} \|g\|_{L^1} < +\infty.$$

This shows that  $f(x - y)g(y)$  is integrable on  $\mathbb{R}^{2d}$ . Hence by Fubini's theorem  $f(x - y)g(y)$  is integrable as a function of  $y$ , and  $f * g$  is integrable on  $\mathbb{R}^d$ .

Moreover we have

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$

The equality holds when both  $f$  and  $g$  are non-negative.

Fubini's theorem is also useful when computing integrals.

**Example 0.10** (Gauss integral)

Recall the Gauss integral:

$$\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.$$

Here we give a different proof:

$$\begin{aligned} \int e^{-x^2} \, dx \int e^{-y^2} \, dy &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy \\ &= \int_0^{+\infty} e^{-r^2} \, dr^2 \cdot \pi = \pi. \end{aligned}$$

## §1 Lebesgue differentiation

The most important theorem in calculus is no doubt the Fundamental theorem of Calculus (which is also called Newton-Lebniz formula). Since we generalized the integrals, there must be a generalized version of this theorem:

**Theorem 1.1** (Lebesgue differentiation theorem, part 1)

If  $f$  is integrable on  $\mathbb{R}^d$ , for any ball  $B \subset \mathbb{R}^d$ , we have

$$\lim_{x \in B, |B| \rightarrow 0} \frac{1}{m(B)} \int_B f(y) \, dy = f(x), a.e.$$

This theorem clearly holds for continuous points of  $f$ .

Our basic idea is to take a continuous  $g$ , such that  $\|g - f\|_{L^1} < \varepsilon$ .

and to prove

$$\left\{ x : \limsup_{x \in B, |B| \rightarrow 0} \frac{1}{m(B)} \int_B |f(y) - f(x)| \, dy \geq \varepsilon_0 \right\}$$

is a null set.

Now we estimate

$$\begin{aligned} \frac{1}{m(B)} \int_B |f(y) - f(x)| \, dy &\leq \frac{1}{m(B)} \int_B (|f(y) - g(y)| + |g(y) - g(x)| + |g(x) - f(x)|) \, dy \\ &= |f(x) - g(x)| + \varepsilon + \frac{1}{m(B)} \int_B |f(y) - g(y)| \, dy \end{aligned}$$

We find that the last term is pretty hard to deal with, so we'll introduce some new tools:

**Definition 1.2** (Hardy-Littlewood maximal function). Let  $f$  be an integrable function on  $\mathbb{R}^d$ . Define

$$Mf(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, dy.$$

to be the **maximal function** of  $f$ .

**Theorem 1.3** (Hardy-Littlewood)

The maximal function  $Mf$  satisfies:

- $Mf$  is measurable;
- For  $x$  almost everywhere,  $|f(x)| \leq Mf(x) < +\infty$ .
- There exists a constant  $C$  s.t.

$$|\{x : Mf > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{\mathcal{L}^1}.$$

*Proof.* First we prove  $\{Mf > \alpha\}$  is measurable. If  $Mf(x_0) > \alpha$ , then exists an open ball  $B \ni x_0$ ,

$$\int_B |f(y)| \, dy > \alpha m(B).$$

This implies that  $B \subset \{Mf > \alpha\} \implies \{Mf > \alpha\}$  is an open set.

For the second part, we'll prove for  $\forall \varepsilon_0 > 0, N > 0$ ,

$$m(\{x : Mf(x) + \varepsilon_0 < |f(x)| \leq N\}) = 0.$$

Otherwise denote the above set as  $E$ , for  $\forall 0 < \lambda < 1, \exists B$  s.t.  $|E \cap B| > \lambda|B|$ .

Thus for  $x \in E$ ,

$$\begin{aligned} Mf(x) &\geq \frac{1}{m(B)} \int_B |f(y)| \, dy \\ &\geq \frac{1}{m(B)} \int_{E \cap B} |f(y)| \, dy \\ &\geq \frac{1}{m(B)} \int_{E \cap B} \varepsilon_0 + Mf(y) \, dy \\ &= \frac{m(E \cap B)}{m(B)} \varepsilon_0 + \frac{1}{|B|} \int_{E \cap B} Mf(y) \, dy. \end{aligned}$$

Taking the integral with respect to  $x$ :

$$\left(1 - \frac{|E \cap B|}{|B|}\right) \int_{E \cap B} Mf \geq \frac{|E \cap B|^2}{|B|} \varepsilon_0.$$

This implies  $(1 - \lambda)N \geq \lambda\varepsilon_0$ , which is impossible as  $\lambda \rightarrow 1$ .

Now for the last part, since  $\{Mf > \alpha\}$  is open,  $\forall x \in \{Mf > \alpha\}$ ,  $\exists B$  s.t.

$$\int_B |f(y)| \, dy > \alpha m(B).$$

Hence for disjoint balls  $B_{i_k}$ ,

$$\|f\|_{\mathcal{L}^1} \geq \sum_{l=1}^k \int_{B_{i_l}} |f(y)| \, dy > \alpha \sum_{l=1}^k |B_{i_l}|.$$

If we could select  $B_{i_l}$ 's such that their measure achieves say 1% of  $E$ , then we're done.

**Lemma 1.4**

Let  $B_1, \dots, B_n$  be open balls in  $\mathbb{R}^d$ . There exists  $i_1, \dots, i_k$  such that  $B_{i_j}$ 's are pairwise disjoint, and

$$\bigcup_{i=1}^n B_i \subset \bigcup_{j=1}^k 3B_{i_j}.$$

Here  $3B$  means to multiply the radius of the ball by 3.

□