

Measure Theory

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Contents

1	Introduction	1
1.1	Starting from probablistics	1
1.2	What is measure theory?	1
2	Measure spaces and measurable maps	2
2.1	Sets and set operations	2
2.2	Families of sets	2

§1 Introduction

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§1.1 Starting from probablistics

Definition 1.1 (σ -algebra). Let \mathcal{F} be a family of subsets of a set Ω , if

- $\Omega \in \mathcal{F}$;
- If $A \in \mathcal{F}$, $A^c \in \mathcal{F}$;
- If $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. (Countable union)

then we call \mathcal{F} a σ -algebra.

Some intros about probablistics (left out because I haven't studied probablistics yet;)

§1.2 What is measure theory?

It's an abstract theory, different from probablistics and real analysis. In this course we study a general set X , focus on mathematical thinking and skills, from the simple to construct the complex.

Measure theory studies the intrinsic structure of mathematical objects, and the map between different measure spaces.

§2 Measure spaces and measurable maps

§2.1 Sets and set operations

Definition 2.1. A non-empty set X is our space(universal set), its elements (points) are denoted by lower case letters x, y, \dots

Some notations:

$$x \in A, x \notin A, x \in A^c, A \subset B, A \cup B, AB = A \cap B,$$

$$B \setminus A (B - A \text{ when } A \subset B), A \Delta B.$$

A family of sets $\{A_t, t \in T\}$.

$$\bigcup_{t \in T} A_t := \{x : \exists t \in T, s.t. x \in A_t\}, \quad \bigcap_{t \in T} A_t := \{x : x \in A_t, \forall t \in T\}.$$

Sometimes we write the union of disjoint sets as sums to emphasize the disjoint property.

Monotone sequence of sets:

$$A_n \uparrow: A_n \subset A_{n+1}, \forall n; \quad A_n \downarrow: A_n \supseteq A_{n+1}, \forall n.$$

Next we define the limits of sets:

Definition 2.2. For monotone sequences:

$$\lim_{n \rightarrow \infty} A_n := \bigcup_{n=1}^{\infty} A_n \text{ or } \bigcap_{n=1}^{\infty} A_n.$$

For general sequence of sets:

$$\limsup A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n; \quad \liminf A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n.$$

A clearer interpretation of limsup and liminf:

limsup is the set of elements which occurs infinitely many times in A_n , and liminf is the elements which doesn't occur in only finitely many A_n 's.

§2.2 Families of sets

Definition 2.3. A family of sets is denoted by script letters $\mathcal{A}, \mathcal{B}, \dots$

- A family is a π -system if $\mathcal{P} \neq \emptyset$ and it's closed under intersections, e.g. $\{(-\infty, a] : a \in \mathbb{R}\}$.
- Semi-rings: \mathcal{Q} is a π -system, and for all $A \subset B$, then there exists finitely many pairwise disjoint sets $C_1, \dots, C_n \in \mathcal{Q}$ s.t.

$$B \setminus A = \bigcup_{k=1}^n C_k = \sum_{k=1}^n C_k.$$

e.g. $\mathcal{Q} = \{(a, b] : a, b \in \mathbb{R}\}$.

Remark 2.4 — The condition $A \subset B$ can be removed.

- Rings: \mathcal{R} is nonempty, and it's closed under union and subtraction.
e.g. $\mathcal{R} = \{\bigcup_{k=1}^n (a_k, b_k] : a_k, b_k \in \mathbb{R}\}$.
- Algebras (fields): \mathcal{A} is a π -system that contains X , and is closed under completion.

Proposition 2.5

Semi-rings are π -systems, rings are semi-rings, algebras are rings.

Proof. By definition we only need to check rings are π -systems: $A \cap B = A \setminus (A \setminus B)$.

For algebras, $A \cup B = (A^c \cap B^c)^c$, $A \setminus B = A \cap B^c$, so they are rings. \square

Remark 2.6 — Rings are semi-rings with unions, Algebras are rings with universal set X .

Definition 2.7. Some other families that start from taking limits:

- Monotone class: If $A_1, \dots \in \mathcal{U}$ and A_n monotone, then $\lim_{n \rightarrow \infty} A_n = \mathcal{U}$.
- λ -system:

$$X \in \mathcal{L}; \quad A_1, A_2, \dots \in \mathcal{L}, A_n \uparrow A \implies A \in \mathcal{L};$$

$$A, B \in \mathcal{L}, A \supseteq B \implies A \setminus B \in \mathcal{L}.$$

notes: $A_n \in \mathcal{L} \iff B_n = A_n^c \in \mathcal{L}$.

- σ -algebra:

$$X \in \mathcal{F}; \quad A \in \mathcal{F} \implies A^c \in \mathcal{F};$$

$$A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Proposition 2.8

σ -algebra = algebra & monotone class;

σ -algebra = λ -system & π -system.

Definition 2.9. σ -rings: \mathcal{R} nonempty, $A, B \in \mathcal{R} \implies A \setminus B \in \mathcal{R}$;

$$A_1, A_2, \dots \in \mathcal{R} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}.$$

Note: We only need to verify the case when A_n 's are disjoint.

Definition 2.10 (Measurable space). Let \mathcal{F} be a σ -algebra on a set X , we say (X, \mathcal{F}) is a measurable space.

Proposition 2.11

Let (X, \mathcal{F}) be a measure space, A is a subset of X . Then $(A, A \cap \mathcal{F})$ is also a measurable space.

The smallest σ -algebra is $\{\emptyset, X\}$, the largest σ -algebra is the power set $\mathcal{T} = \mathcal{P}(X)$.

In some cases, \mathcal{T} is too large, for example, when $X = \mathbb{R}$, we can't assign a "measure" to every subset that fits our common sense.