Mathematical Analysis II

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Contents

Theorem 0.0.1 (Parametrization of manifolds)

Let Ω be an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^{n+p}$ is a smooth map. If $\forall x^* \in \Omega$, $\operatorname{rank} \mathrm{d}f(x^*) = n$, then there exists an open set U, $x^* \in U$ s.t. $f(U) \subset \mathbb{R}^{n+p}$ is an n-dimensional manifold.

Proof. Let x_i be a coordinate in $\mathbb{R}^{n+p}$.

WLOG $(\frac{\partial f_i}{\partial x_j})_{1\le i,j\le n}$ is non-degenerate, let $F = (f_1, \dots, f_n)$, $G = (f_{n+1}, \dots, f_{n+p})$ and apply inverse function theorem on F, there exists open neighborhoods $U \ni x, V \ni F(x) =: y$, s.t. $F : U \to V$ is a smooth homeomorphism.

$$\begin{tikzcd} U \subset \Omega \arrow[r, "F"] \arrow[d, "f"] & V \subset \mathbb{R}^n \arrow[dl, "\phi"] \\ \mathbb{R}^{n+p} & \end{tikzcd}$$

So $f(x) = (F(x), G(x)) = (y, GF^{-1}(y))$. Let

$$\phi : V \to \mathbb{R}^n, \quad y \mapsto (y, GF^{-1}(y)).$$

We can see that ϕ is a homeomorphism $V \to f(U)$. (Indeed it's a bijection) So by definition we know $f(U)$ is a manifold. $\square$

Example 0.0.2

Let

$$\phi(\theta,r)=\begin{cases} x=\left(1+r\cos\dfrac{\theta}{2}\right)\cos\theta \\ y=\left(1+r\cos\dfrac{\theta}{2}\right)\sin\theta \\ z=r\sin\dfrac{\theta}{2} \end{cases},\quad I=[0,2\pi]\times(-1,1).$$

Then $M=\phi(I)$ is a Mobius strip, which is a two dimensional smooth manifold in $\mathbb{R}^3$, as $\mathrm{d}\phi$ has rank 2 everywhere.

Besides, there doesn't exist a function $f:\mathbb{R}^3\to\mathbb{R}$ s.t. $M=f^{-1}(0)$. Basically this is because M is not orientable, but ∇f and $-\nabla f$ are "normal" directions of M, which makes it orientable. Below we give a sketch:

Proof. Let $v(\theta)=\cos\frac{\theta}{2}e_2(\theta)-\sin\frac{\theta}{2}e_1(\theta)$, where $e_2(\theta)=(0,0,1), e_1(\theta)=(\cos\theta,\sin\theta,0)$.

Note that $e_1\perp e_2$, consider the curve $\beta:[0,2\pi]\to\mathbb{R}^3$

$$\theta\mapsto(\cos\theta,\sin\theta,0)+\varepsilon v(\theta).$$

Let ε be sufficiently small, when $\varepsilon\neq 0$ we can check β and M do not intersect. We can take ε s.t. $f(\beta(0))>0$ as $\mathrm{d}f\neq 0$. (ε can be negative)

Since $\beta(0)=(1,0,\varepsilon),\beta(2\pi)=(1,0,-\varepsilon)$, when $f(\beta(0))>0$, we must have $f(\beta(2\pi))<0$. By continuity, $\exists\theta_0$ s.t. $f(\beta(\theta_0))=0$, which means $\beta(\theta_0)\in M$, contradiction! $\square$

Midterm exam....qaq

Proposition 0.0.3

Let $\Omega\subset\mathbb{R}^n$, and $f:\Omega\to\mathbb{R}^m$ is a smooth map. Let $S\subset\mathbb{R}^m$ be a differential manifold, if for all $x\in f^{-1}(S)$, we have $\operatorname{rank}\mathrm{d}f(x)=m$, then $f^{-1}(S)$ is a differential manifold with codimension same as S.

Proof. For any $x\in S$, let Φ be the homeomorphism from an open neighborhood of x to $\mathbb{R}^m$.

Suppose $\dim S=d$, let

$$F(x)=((\Phi\circ f)_{d+1},\dots,(\Phi\circ f)_m).$$

Note that $\mathrm{d}(\Phi\circ f)$ is an $m\times n$ matrix, and its rank is m. Since

$$\mathrm{d}(\Phi\circ f)=\begin{pmatrix}\mathrm{d}(\Phi\circ f)_1\\ \vdots\\ \mathrm{d}(\Phi\circ f)_d\\ \mathrm{d}F\end{pmatrix}$$

Thus $\mathrm{d}F$ is a $(m-d)\times n$ matrix with rank $m-d$. So $F^{-1}(0)=f^{-1}(S)$ is a manifold with dimension $n-(m-d)$. $\square$

§0.1 Tangent space

Since the differentiation relies on the choice of coordinates, if we want to study the manifold more geometrically, we should look at the tangent lines or tangent planes of the manifold.

Definition 0.1.1 (Tangent vectors)**.** Let M be a differential manifold. Let $p \in M$, for all parametrized curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we say the vector $\gamma'(0) \in \mathbb{R}^n$ is the **tangent vector** of γ at point p.

Let T_pM denote the **tangent space** at p, which is defined as

$$T_pM = \{\gamma'(0) \in \mathbb{R}^n \mid \gamma(0) = p\}.$$

It's clear that T_pM should be a vector space of dimension $\dim M$, next we'll prove this fact.

Proposition 0.1.2 (Push forward of tangent spaces under differential homeomorphism)
Let $\Phi : U \to V$ be a differential homeomorphism, $M \subset U$ be a manifold, then

$$T_{\Phi(p)}\Phi(M) = (\mathrm{d}\Phi)\big|_p \cdot T_pM.$$

Proof. Let γ be a parametrized curve on M with $\gamma(0) = p$. Note that $\Phi \circ \gamma$ is a curve on $\Phi(M)$ passing through $\Phi(p)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \circ \gamma(t)\Big|_{t=0} = \mathrm{d}\Phi(p) \cdot \gamma'(0).$$

Thus $\mathrm{d}\Phi(p) \cdot T_pM \subset T_{\Phi(p)}\Phi(M)$.

Now we do the same thing for Φ^{-1}, we can get the desired equality. □

Now we can easily calculate the tangent space: since M is locally homeomorphic to $\mathbb{R}^d$, and obviously $T_{\Phi(p)}(\mathbb{R}^d \times \{0\}) = \mathbb{R}^d \times \{0\}$, by above proposition, $T_pM = (\mathrm{d}\Phi) \cdot T_{\Phi(p)}(\mathbb{R}^d \times \{0\})$ is a vector space of dimension d.

Theorem 0.1.3
Let M be a manifold, T_pM is a vector space of dimension $\dim M$.

Proposition 0.1.4
Let $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a smooth map, $\operatorname{rank} \mathrm{d}f = n$. Let $M = f^{-1}(f(p))$, then $T_pM = \ker \mathrm{d}f(p)$.

Proof. Let

$$F(x, y) = (x, f(x, y)), x \in \mathbb{R}^d, y \in \mathbb{R}^n.$$

F is a homeomorphism, so $T_pM = (\mathrm{d}F^{-1})T_{F(p)}F(M)$.

Note that $F(M) = \{(x, p) \mid \exists y, f(x, y) = f(p)\}$, it must be a vector space of dimension d, so $T_{F(p)}F(M) = \mathbb{R}^d \times \{0\}$,

$$T_pM = (\mathrm{d}F^{-1})T_{F(p)}F(M) = \ker \mathrm{d}f(p).$$

□

Example 0.1.5

Let M be a manifold determined by $f : \mathbb{R}^n \to \mathbb{R}$,

$$T_pM = \ker \mathrm{d}f = \{v \in \mathbb{R}^n \mid \mathrm{d}f(p)v = 0\}.$$

Here $\mathrm{d}f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) = \nabla f$. So $v \in T_pM \iff \nabla f \cdot v = 0$, the dot means the inner product. In this case the vector ∇f is called **normal direction vector**.

§0.2 Smooth maps between manifolds

Next we'll briefly introduce the differential maps between manifolds. Since the differentiation on manifolds is hard to define, so what we do is actually regarding manifolds as $\mathbb{R}^d$ locally and define the differentiablity using the maps between Eucild spaces.

Definition 0.2.1. Let M, N be manifolds in $\mathbb{R}^m, \mathbb{R}^n$, respectively. $f : M \to N$ is a map, if $\forall p \in M$, there exists $p \in U \subset \mathbb{R}^m, V \subset \mathbb{R}^d$, $\Phi : U \to V$ s.t.

$$f_\Phi = f \circ \Phi^{-1}$$

is a smooth map from V to N. We say f is a smooth map from M to N.

We need to check this definition is well-defined: if there's another homeomorphism Φ', $f \circ \Phi' = (f \circ \Phi^{-1}) \circ (\Phi \circ \Phi'^{-1})$ is indeed a smooth map.

Lemma 0.2.2 (Smooth maps are locally restrictions of smooth maps in Eucild spaces)

Let $f : M \to N$ be a map, then f is smooth $\iff \forall p \in M, \exists p \in U \subset \mathbb{R}^m$ and a smooth map $F : U \to \mathbb{R}^n$ s.t.

$$f\big|_{U\cap M} = F\big|_{U\cap M}.$$

Proof. Let τ denote the embedding from $M \cap U$ to U. Since $f \circ \Phi^{-1} = F \circ \tau \circ \Phi^{-1}$, so F smooth $\implies f_\Phi$ smooth $\implies f$ smooth.

$$\begin{tikzcd}
V \subset \mathbb{R}^d \arrow[dr, "f\circ\Phi^{-1}"'] & M \cap U \arrow[l, "\Phi"'] \arrow[r, "\tau"] \arrow[d, "f"] & U \arrow[dl, "F"] \\
& N \subset \mathbb{R}^n &
\end{tikzcd}$$

TODO: fix this

On the other hand, let $\widetilde{\tau}$ be the projection from U to V, then $F = f \circ \Phi^{-1} \circ \widetilde{\tau} \circ \Phi$ satisfies the desired condition. $\square$

Example 0.2.3

Let A be an orthogonal map in $\mathbb{R}^3$, then A can be restricted to $S^2 \to S^2$.

Definition 0.2.4 (Tangent map)**.** Let $f : M \to N$ be a map between manifolds, $v \in T_pM$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a parametrized curve with $\gamma(0) = p, \gamma'(0) = v$, then $f(\gamma(t))$ is a curve on N.

$$\mathrm{d}f(p)(v) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\Big|_{t=0} \in T_{f(p)}N.$$

Thus $\mathrm{d}f(p) : T_pM \to T_{f(p)}N$ is a map between tangent spaces.

In fact, if $f = F\big|_M$, then $\mathrm{d}f(p)(v) = \mathrm{d}F(p) \cdot v$.

Definition 0.2.5 (Tangent bundle)**.** Let M be a manifold, $\forall p \in M$, there's a tangent space T_pM. Define the **tangent bundle** of M to be

$$TM = \bigsqcup_{p \in M} T_pM.$$

If X is a map $M \to TM$: $p \mapsto X(p)$, with $X(p) \in T_pM$, then it's called a **tangent vector field**.

In other words, a tangent vector field is just to assign a tangent vector to every point in M.

Proposition 0.2.6

Let $M \subset \mathbb{R}^n$ be a manifold, all its tangent vector field form a C^∞ module $T(M, TM)$, i.e. $\forall f \in C^\infty(M)$, X, Y are smooth vector fields, then $fX, X+Y$ are both smooth vector fields.

Proposition 0.2.7

Let $M \subset \mathbb{R}^n$ be a smooth manifold, we have

$$TM = \{(x, v) \mid x \in M, v \in T_xM\}$$

is a smooth manifold in $\mathbb{R}^{2n}$, and $\dim TM = 2\dim M$.

Proof. There exists a local homeomorphism $\phi : V \to U \subset \mathbb{R}^n$ s.t. $V \subset \mathbb{R}^d$, $\phi(V) = M \cap U$.

Define map $T\phi : V \times \mathbb{R}^n \to U \times \mathbb{R}^n$, $(x, v) \mapsto (\phi(x), \mathrm{d}\phi(x) \cdot v)$. Since $T\phi$ is injective(ϕ is homeomorphism), and

$$\mathrm{d}T\phi = \begin{pmatrix} \mathrm{d}\phi & 0 \\ \mathrm{d}(\mathrm{d}\phi)(v) & \mathrm{d}\phi \end{pmatrix}$$

is non-degenerate, so $T\phi$ is a bijection and hence differential homeomorphism.

Since the tangent space of V is just $\mathbb{R}^d$, so $T(U \cap M)$ is the image of $T\phi$ restricted on $V \times \mathbb{R}^d$. (Note that $\mathrm{d}\phi(x) \cdot v \in T_{\phi(x)}M$) Thus TM is a manifold in $\mathbb{R}^{2n}$ with dimension $2d$. $\square$

Definition 0.2.8 (Tangent maps)**.** Earlier we know that $\mathrm{d}f(p)$ is a map $T_pM \to T_{f(p)}N$, combined with tangent bundle we can write $\mathrm{d}f : TM \to TN$, this map is called the **tangent map** or the **differentiation** of f.

If we have a vector field X and a smooth function $f : M \to \mathbb{R}^n$, consider

$$X(f)(p) = \mathrm{d}f(X)(p) := \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\Big|_{t=0}, \quad \gamma(0) = p, \gamma'(0) = X(p).$$

So X induces a smooth map $C^\infty(M) \to C^\infty(M)$.

Now we can generalize a well known result to manifolds:

Proposition 0.2.9

Let $M \subset \mathbb{R}^n$ be a smooth manifold, $f \in C^\infty(M)$. If f achieves a local extremum at $p \in M$, we must have $\mathrm{d}f(p) = 0$.

Proof. It suffices to prove $\mathrm{d}f(p)(v) = 0$, $\forall v \in T_pM$. Take γ s.t. $\gamma(0) = p, \gamma'(0) = v$, then $f(\gamma(t))$ achieves its extremum at $t = 0$, so $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\big|_{t=0} = 0 = \mathrm{d}f(p)(v)$. $\square$

§0.3 Conditional extremum problem

Consider a function $f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ and some constraint conditions

$$\begin{cases} g_1(x_1, \ldots, x_n) = 0 \\ \qquad \vdots \\ g_m(x_1, \ldots, x_n) = 0 \end{cases}$$

We want to compute the extremum of f under these conditions.

Well, you probably heard of *Lagrange multipliers*, i.e. let

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x) - \sum_{j=1}^{m} \lambda_j g_j(x).$$

But here we'll provide a different point of view. Let M be the manifold in $\mathbb{R}^n$ under those conditions, Suppose $p \in M$ is a local extremum of f, then $T_pM \subset \ker \mathrm{d}f(p)$.

Also recall that $T_pM = \ker \mathrm{d}g(p) = \bigcap_{j=1}^{m} \ker \mathrm{d}g_j(p)$. This means that, $\exists \lambda_1, \ldots, \lambda_m$ s.t.

$$\mathrm{d}f(p) = \sum_{j=1}^{m} \lambda_j \, \mathrm{d}g_j(p).$$

Surprisingly, we get the same result of Lagrange multipliers! Hence what we've done is to give a geometrical comprehension of Lagrange multipliers.

Example 0.3.1

Let $g : \mathbb{R}^n \to \mathbb{R}$ be the constraint function, then f can achieve its extremum only if $\mathrm{d}f = \lambda \, \mathrm{d}g$.

For example, let $f(x) = d(x, z)^2$, $\mathrm{d}f(x) = 2(x_1 - z_1, \ldots, x_n - z_n)$, so $\mathrm{d}f = \lambda \, \mathrm{d}g$ means the vector $\mathrm{d}f(p)$ is orthogonal to the tangent plane of $M = \{g = 0\}$.

Proposition 0.3.2 (Hadamaid's inequality)

Let $v_1, \ldots, v_n \in \mathbb{R}^n$, then

$$|\det(v_1, \ldots, v_n)| \le |v_1| \cdots |v_n|.$$

Proof. Let $f : \mathbb{R}^{n^2} \to \mathbb{R}$, $(v_1, \ldots, v_n) \mapsto \det(v_1, \ldots, v_n)$ with constraint $|v_i| = 1$. Let $v_{ij} \in \mathbb{R}$,

$$g_i(V) = -1 + \sum_{j=1}^{n} v_{ij}^2.$$

The manifold determined by g_i is $M = (S^{n-1})^n$. The extremum point of f in M must satisfy:

$$\frac{\partial f}{\partial v_{i_0 j}} - \lambda_{i_0} \frac{\partial g_{i_0}}{\partial v_{i_0 j}} = 0.$$

This implies $v_{i_0 j}^* = 2\lambda_{i_0} v_{i_0 j}$, where $v_{i_0 j}^*$ is the *cofactors* of $v_{i_0 j}$.

This means that $\sum_{j=1}^{n} v_{i_0 j} v_{kj} = 0$, so V must be an orthogonal matrix, so $|f| \le 1$. □