Linear Algebra II

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Contents

Here we'll present multiple proofs to emphasize some intermediate result.

Proposition 0.0.1

Let T be a normal map, if $W \subset V$ is T-invariant, then T_W is also normal.

Proof. First note that $W, W^\perp$ are T^*-invariant. For $\alpha, \beta \in W$, we have

$$\langle (T_W)^*\alpha, \beta\rangle = \langle \alpha, T_W\beta\rangle = \langle \alpha, T\beta\rangle = \langle T^*\alpha, \beta\rangle .$$

Thus $(T_W)^* = T_W^*$. The conclusion follows. □

Proposition 0.0.2

Let T be a normal map, there exists an orthogonal decomposition $V = \bigoplus_{i=1}^k V_i$, such that each V_i is T-invariant, and T_{V_i} simple.

Proof. Note that if W is T-invariant, then $W^\perp$ is also T-invariant. By induction and the previous proposition this is trivial. □

Therefore to prove **??**, we only need to prove the case when T is simple.

Proof of **??**. WLOG $\dim V > 1$.

Since T simple $\implies f_T \in \mathbb{R}[x]$ prime, thus $\deg f_T = 2$, $\dim V = 2$ and $f_T = (x-c)(x-\overline{c})$.

Take any orthonormal basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, let $r = |c|$, $A = r^{-1}[T]_{\mathcal{B}}$. Clearly A normal and $\sigma(A) = \{r^{-1}c, r^{-1}\overline{c}\}$, so A is unitarily similar to $\operatorname{diag}(r^{-1}c, r^{-1}\overline{c})$, A is unitary.

Moreover A is a real matrix so A orthogonal, and $\det A = 1$, thus $A = Q_\theta, \theta \in [0, 2\pi]$.

At last by T has no eigenvector, and we can change α_2 to $-\alpha_2$, so we can require $\theta \in (0, \pi)$. □

Proposition 0.0.3

Let $T \in L(V)$, then $\ker(T)^\perp = \operatorname{im}(T^*), \operatorname{im}(T)^\perp = \ker(T^*)$.

Proof. Trivial, just some computation. □

Proposition 0.0.4
Let $T \in L(V)$, $\sigma(T^*) = \overline{\sigma(T)}$,

$$\forall c \in \sigma(T), \quad \dim\ker(T - cI) = \dim\ker(T^* - \overline{c}I).$$

Proof. By the previous proposition,

$$\dim\ker(T - cI) = n - \dim\mathrm{im}(T^* - \overline{c}I) = \dim\ker(T^* - \overline{c}I)$$

which also implies $\sigma(T) = \overline{\sigma(T^*)}$. □

Proposition 0.0.5
If T normal, then $\ker(T - cI) = \ker(T^* - \overline{c}I)$.

Proof. Let $W = \ker(T - cI)$, T_W^* is just $(c\,\mathrm{id}_W)^* = \overline{c}\,\mathrm{id}_W$. Thus $W \subset \ker(T^* 0 \overline{c}I)$, by dimensional reasons they must be equal. □

Proposition 0.0.6
Let T be a normal map, $f, g \in F[x]$ coprime $\implies \ker(f(T)) \perp \ker(g(T))$.

Proof. Since $g(T)^* = \overline{g}(T^*)$, $g(T)$ is normal, thus $\ker(g(T))^\perp = \mathrm{im}(g(T))$.

Let $W = \ker(f(T))$, let $a, b \in F[x]$ s.t. $af + bg = 1$, so $a(T)f(T) + b(T)g(T) = \mathrm{id}_V$. Restrict this equation to W, we get $b(T)_W g(T)_W = \mathrm{id}_W$, hence $W \subset \mathrm{im}(g(T))$. □

Proposition 0.0.7
Let T be a normal map,

- The primary decomposition of T are orthogonal decomposition;
- The cyclic decomposition of T can be orthogonal.

Proof. The first one is trivial by previous proposition.

For cyclic decomposition, we proceed by induction on $\dim V$.

Let $\alpha_1 \in V$ s.t. $p_{\alpha_1} = p_r$, then $(R\alpha_1)^\perp$ are T-invariant, use induction hypo on it and we're done. □

Remark 0.0.8 — This means the primary cyclic decomposition of T can also be orthogonal.

This gives the second proof of **??**:

Proof. WLOG T normal and primary cyclic, then p_T is primary, and T normal $\implies$ T semisimple, so p_T has no multiple factors, thus p_T prime, which proves the result. □

Next we present the third proof:

Proposition 0.0.9

If $A, B \in \mathbb{R}^{n\times n}$ are unitarily similar, then they are orthogonally similar.

Lemma 0.0.10 (QS decomposition)

For any unitary matrix U, $U = QS$ where Q real orthogonal, S unitary and symmetrical. Moreover $\exists f \in \mathbb{C}[x]$ s.t. $S = f(U^tU)$.

Proof. Let $\sigma(U^tU) = \{c_1, \ldots, c_k\}$. We can take a polynomial $f \in \mathbb{C}[x]$ s.t. $f(c_i)^2 = c_i$.

Since U is unitary, $|c_i| = 1 \implies |f(c_i)| = 1$.

Let $S = f(U^tU)$, we claim that S unitary and $S^2 = U^tU$.

Let $U^tU = P\operatorname{diag}(c_1, \ldots, c_k)P^{-1}$, where P is unitary, then $S = P\operatorname{diag}(f(c_1), \ldots, f(c_k))P^{-1}$ is unitary, and clearly $S^2 = U^tU$.

Let $Q = US^{-1}$, then Q unitary. Since S symmetrical, $S^{-1} = S^* \implies \overline{S^{-1}} = S^t = S$,

$$\overline{Q}Q^{-1} = \overline{U}SSU^{-1} = \overline{U}U^tUU^{-1} = I_n.$$

Hence $\overline{Q} = Q$, Q is real orthogonal. $\square$

Return to the original proposition. Let A, B be real matrices unitarily similar, let $B = UAU^{-1}$, taking the conjuate we get

$$UAU^{-1} = \overline{U}AU^t \implies U^tUA = AU^tU.$$

Let $U = QS$, then $AS = SA$. We have

$$B = UAU^{-1} = QSAS^{-1}Q^{-1} = QAQ^{-1}.$$

Therefore A, B are orthogonally similar. $\square$

Corollary 0.0.11

Let A, B be normal matrices, TFAE:

(1) A, B are unitarily similar (or orthogonally similar);

(2) A, B are similar;

(3) $f_A = f_B$.

Proof. We only need to prove (3) $\implies$ (1).

When $F = \mathbb{C}$, A, B are unitarily similar to diagonal matrices D_1, D_2. Since $f_A = f_B$, D_1, D_2 only differ by a permutation, hence unitarily similar.

When $F = \mathbb{R}$, by the previous proposition and proof for $\mathbb{C}$, we get the result. $\square$

The third proof of **??** is to factorize $f_T \in \mathbb{R}[x]$ and use the above corollary.

At last we prove another property of normal maps:

Proposition 0.0.12
Let A be a normal matrix, then A^* is a complex polynomial of A.

Proof. Use the spectral decomposition. $\square$

§1 Bilinear forms

In this section we study the bilinear forms on generic fields. Let $M^2(V)$ denote all the bilinear forms on V.

For $f \in M^2(V)$, Let $(f(\alpha_i, \alpha_j))_{ij}$ be the matrix of f under basis $\{\alpha_i\}$. (Note that this differs by a transpose with previous section)

Obviously $M^2(V) \to F^{n\times n}$ by $f \mapsto [f]_{\mathcal{B}}$ is a linear isomorphism.

Proposition 1.0.1
Let $\mathcal{B}, \mathcal{B}'$ be two basis, P is the transformation matrix between them, for all $f \in M^2(V)$ we have $[f]_{\mathcal{B}'} = P^t[f]_{\mathcal{B}}P$.

Proof. Trivial. $\square$

If $A = P^tBP$ for some $P \in \mathrm{GL}(V)$, we say A, B are **congruent**.

A bilinear form will induce two linear maps $V \to V^*$, namely L_f, R_f:

$$L_f(\alpha)(\beta) = R_f(\beta)(\alpha) = f(\alpha, \beta).$$

Proposition 1.0.2
For any basis $\mathcal{B}$, we have $\operatorname{rank} L_f = \operatorname{rank} R_f = \operatorname{rank}[f]_{\mathcal{B}}$. This number is called the rank of f, denoted by $\operatorname{rank} f$.

If $\operatorname{rank} f = n$, we say f is non-degenrate, this is equivalent to L_f invertible or R_f invertible.

§1.1 Some special bilinear forms

Definition 1.1.1. For $f \in M^2(V)$,

- If $f(\alpha, \beta) = f(\beta, \alpha), \forall \alpha, \beta \in V$, then we say f is **symmetrical**.
- If $f(\alpha, \beta) = -f(\beta, \alpha), \forall \alpha, \beta \in V$, we say f is **anti-symmetrical**.
- If $f(\alpha, \alpha) = 0, \forall \alpha \in V$, we say f is **alternating**.

We denote the above functions by $S^2(V), A^2(V), \Lambda^2(V)$.

We can see that $\Lambda^2(V) \subset A^2(V)$, and they are all subspaces of $M^2(V)$.

Proposition 1.1.2

If $\operatorname{char} F \neq 2$, then $A^2(V) = \Lambda^2(V)$, and $M^2(V) = A^2(V) \oplus S^2(V)$.

Proof. Already proved in last semester. $\square$

Proposition 1.1.3

Let $\mathcal{B}$ be any basis of V,

- f symmetrical $\iff$ $[f]_{\mathcal{B}}$ symmetrical;
- f anti-symmetrical $\iff$ $[f]_{\mathcal{B}}$ anti-symmetrical;
- f alternating $\iff$ $[f]_{\mathcal{B}}$ anti-symmetrical and the diagonal entries are all zero.

Definition 1.1.4 (Quadratic forms)**.** Let $q : V \to F$ be a function, we say q is a **quadratic form** if there exists $f \in M^2(V)$ s.t.

$$q(\alpha) = f(\alpha, \alpha), \quad \forall \alpha \in V.$$

When $V = F^n$, quadratic forms are just a homogenous quadratic polynomial with n variables, i.e.

$$q(X) = X^t AX, \quad A \in F^{n\times n}, X \in F^n.$$

Let $Q(V)$ denote all the quadratic forms on V, it's an F-vector space. By definition there's a surjective linear map $M^2(V) \to Q(V)$ by $\Phi(f)(\alpha) = f(\alpha, \alpha)$.

Proposition 1.1.5

Let $\operatorname{char} F \neq 2$,

- The map $\Phi : S^2(V) \to Q(V)$ is an isomorphism.
- Let $q \in Q(V)$, if $f \in S^2(V)$ and $\Phi(f) = q$, then

$$f(\alpha, \beta) = \frac{1}{4}(q(\alpha + \beta) - q(\alpha - \beta)).$$

Proof. The first one can be proved by $\ker(\Phi) = \Lambda^2(V)$ and $M^2(V) = S^2(V) \oplus \Lambda^2(V)$.

The second one is trivial by direct computation. $\square$

From this we can define the matrix of a quadratic form q to be the matrix of the symmetrical bilinear form $\Phi^{-1}(q)$, thus $[q]_{\mathcal{B}}$ is always symmetrical.

Theorem 1.1.6

Let $f \in M^2(V)$,

- If $\operatorname{char} F \neq 2$, then $f \in S^2(V) \iff \exists \mathcal{B}$, s.t. $[f]_{\mathcal{B}}$ diagonal;
- $f \in \Lambda^2(V) \iff \exists \mathcal{B}$ s.t. $[f]_{\mathcal{B}}$ is block diagonal with each block being 0 or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

To prove this theorem, it's sufficient to prove:

Lemma 1.1.7

Let $f \in S^2(V) \cup A^2(V)$, $W \subset V$ is a subspace, let

$$W^{\perp} = \{\beta \in V \mid f(\alpha, \beta) = 0, \forall \alpha \in W\}.$$

If $f\big|_W$ is non-degenerate, then $V = W \oplus W^{\perp}$. In this case, let $\mathcal{B}_1, \mathcal{B}_2$ be basis of W, $W^{\perp}$, and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, we have

$$[f]_{\mathcal{B}} = \operatorname{diag}([f|_W]_{\mathcal{B}_1}, [f|_{W^{\perp}}]_{\mathcal{B}_2}).$$

Proof. Since $f\big|_W$ non-degenerate, $W \cap W^{\perp} = 0$. Note that

$$W^{\perp} = \bigcap_{\alpha \in W} \ker(L_f(\alpha)) = L_f(W)^0.$$

Thus $\dim W^{\perp} = n - \dim L_f(W) \geq n - \dim W$. This implies that $V = W \oplus W^{\perp}$.

For the second part, since $f(\alpha, \beta) = 0 \implies f(\beta, \alpha) = 0$, thus the matrix must obey the conclusion. □

Now by induction it's trivial when $n = 1$,

- When $f \in S^2(V)$, WLOG $f \neq 0$, $\exists \alpha$ s.t. $f(\alpha, \alpha) \neq 0$. Let $W = \operatorname{span}\{\alpha\}$, by lemma and induction hypo we're done.
- When $f \in A^2(V)$, there exists α, β s.t. $f(\alpha, \beta) = 1$. Let $W = \operatorname{span}\{\alpha, \beta\}$, similarly by lemma and induction hypo, we're done.

Corollary 1.1.8

For any $q \in Q(V)$, there exists a basis of V s.t. $[q]_{\mathcal{B}}$ diagonal.

The non-degenerate alternating bilinear forms are called **symplectic forms**.

Corollary 1.1.9

If there exists symplectic form f on V, then $\dim V = 2m$ and

$$[f]_{\mathcal{B}} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

for some basis $\mathcal{B}$.

Theorem 1.1.10

Let F be an algebraically closed field, and $\operatorname{char} F \neq 2$. Let $f \in S^2(V)$, there exists a basis $\mathcal{B}$, s.t. $[f]_{\mathcal{B}}$ diagonal and the diagonal entries can only be 0 or 1.

Proof. Use the previous result and multiply some scalars (the root of $x^2 = c$). $\square$

When $F = \mathbb{R}$, using similar technique we can prove the diagonal entries can only be $0, 1$ or -1.

§1.2 Lie algebras

There's a class I missed, so the notes may not be complete.

Definition 1.2.1 (Lie algebra)**.** Let L be a vector space over a field F. Suppose an operation (called **Lie bracket**)

$$L \times L \to L, \quad (x, y) \mapsto [x, y]$$

is given and satisfies:

- (Bilinearity)
$$\begin{cases} [ax + by, z] = a[x, z] + b[y, z], \\ [x, ay + bz] = a[x, y] + b[x, z], \end{cases} \quad \forall x, y, z \in L, a, b \in F;$$

- (Alternativity)
$$[x, x] = 0, \quad \forall x \in L;$$

- (Jacobi identity)
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \quad \forall x, y, z \in L.$$

Then L is called a **Lie algebra** over F.

The Lie algebra can be viewed as a vectorization of Lie groups, where Lie bracket is the commutator in Lie groups.

Example 1.2.2

On any F-vector space L, one can define a trivial Lie bracket by

$$[x, y] = 0, \quad \forall x, y \in L$$

Then L becomes a Lie algebra, called an **abelian Lie algebra**.

We can also define homomorphisms by $\phi([x, y]) = [\phi(x), \phi(y)]$.

Definition 1.2.3 (Representation)**.** Let L be a Lie algebra over F. A **representation** of L is a homomorphism $\phi : L \to \mathfrak{gl}(V)$, where V is some finite-dimensional F-vector space.

Example 1.2.4 (Adjoint representation)

Let L be a Lie algebra over F. Define a linear map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ by

$$\mathrm{ad}(x)(y) = [x, y], \quad \forall x, y \in L.$$

We claim that it is a representation, called the **adjoint representation** of L. In fact, it follows from the Jacobi identity that for any $x, y, z \in L$,

$$\begin{aligned}\mathrm{ad}([x, y])(z) &= [[x, y], z] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= \mathrm{ad}(x)([y, z]) - \mathrm{ad}(y)([x, z]) \\ &= [\mathrm{ad}(x), \mathrm{ad}(y)](z).\end{aligned}$$

Definition 1.2.5 (Subalgebra, ideal, quotient algebra)**.** Let L be a Lie algebra over F.

- If $S, T \subset L$ are subspaces, write

$$[S, T] := \mathrm{span}\{[x, y] : x \in S, y \in T\}.$$

- A subspace $K \subset L$ is called a **subalgebra** if $[K, K] \subset K$, denoted $K < L$.
- A subspace $I \subset L$ is an **ideal** if $[I, L] \subset I$, denoted $I \lhd L$.
- Let $I \lhd L$. On the quotient space L/I, we introduce the Lie bracket

$$[x + I, y + I] := [x, y] + I, \quad \forall x, y \in L.$$

  Then L/I becomes a Lie algebra, called the **quotient algebra** of L by I.

Example 1.2.6

Let $\phi : L \to L'$ be a homomorphism. Then

$$\ker \phi \lhd L, \quad \mathrm{im}(\phi) \lhd L', \quad \mathrm{im}(\phi) \cong L/\ker \phi.$$

The **center** of L is defined as

$$Z(L) := \{x \in L : [x, y] = 0, \forall y \in L\}.$$

We have $Z(L) \lhd L$ and $Z(L) = \ker \mathrm{ad}$.

Definition 1.2.7 (Direct sum)**.** Let $L_1, \ldots L_r$ be Lie algebras over F. On the (external) vector space Direct sum $L_1 \oplus \cdots \oplus L_r$ we introduce the Lie bracket

$$[(x_1, \ldots, x_r), (y_1, \ldots, y_r)] = ([x_1, y_1], \ldots, [x_r, y_r])$$

This makes $L_1 \oplus \cdots \oplus L_r$ a Lie algebra, called the **(external) Lie algebra direct sum** of $L_1, \ldots, L_r$.

Definition 1.2.8 (Linear Lie algebra)**.** Subalgebras of $\mathfrak{gl}_n(F)$ and $\mathfrak{gl}(V)$ are called **linear Lie algebras**.

We have the following deep result:

Theorem 1.2.9 (Ado-Iwasawa)
All finite-dimensional Lie algebras over F are isomorphic to linear Lie algebras.

Let us introduce some important linear Lie algebras.

Example 1.2.10 (Special linear Lie algebra)
Let
$$\mathfrak{sl}_n(F) = \{x \in \mathfrak{gl}_n(F) : \mathrm{tr}(x) = 0\}, \mathfrak{sl}(V) = \{x \in \mathfrak{sl}(V) : \mathrm{tr}(V) = 0\},$$
where V is a vector space over F. We have $\mathfrak{sl}(V) \lhd \mathfrak{gl}(V)$.

Example 1.2.11 (The Lie algebra $L(V, f)$)
Let V be a finite-dimensional F-vector space, and $f : V \times V \to F$ be a bilinear form. For $x \in \mathfrak{gl}(V)$, we say that f is **invariant under x (in the infinitesimal snese)** if
$$f(xv, w) + f(v, xw) = 0, \quad \forall v, w \in V.$$
This comes from the derivative of Lie groups: Let $L \in \mathrm{GL}(V)$, $g(0) = \mathrm{id}_V$. By taking derivatives at $t = 0$ on
$$f(g(t)v, g(t)w) = f(v, w),$$
we get $f(g'(0)v, w) + f(v, g'(0)w) = 0$.

Let $L(V, f) \subset \mathfrak{gl}(V)$ be the subspace of all $x \in \mathfrak{gl}(V)$ that leave f invariant, we claim that $L(V, f) < \mathfrak{gl}(V)$.

Example 1.2.12
Let's consider 2 special cases of $L(V, f)$:

- Let $V = F^n$, and f be the symmetrical form given by
$$f(v, w) = v^t w, \quad \forall v, w \in F^n.$$
Then $\mathfrak{o}_n(F) := L(F^n, f)$ is called the **orthogonal Lie algebra**. Under the identification $\mathfrak{gl}(F^n) \cong \mathfrak{gl}_n(F)$, we have $\mathfrak{o}_n(F) = \{x \in \mathfrak{gl}_n(F) : x^t + x = 0\}$.

- Let $V = F^{2n}$, and f be the symplectic form given by
$$f(v, w) = v^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} w, \quad \forall v, w \in V.$$
Then $\mathfrak{sp}_{2n}(F) := L(F^{2n}, f)$ is called the **symplectic Lie algebra**.

Suppose $I \lhd L$, and we understand I and L/I, then we understand L (in the rough sense). This motivates the following:

Definition 1.2.13 (Simple Lie algebra, semisimple Lie algebra)**.** Let L be a finite-dimensional Lie algebra over F.

- L is **simple** if it's nonabelian and has no nontrivial ideals.
- L is **semisimple** if it's nonzero and has no nonzero abelian ideal.

Clearly, a simple Lie algebra is semisimple. One of our main purposes is to explain the proof of the following classification theorem:

Theorem 1.2.14
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$.

(1) L is semisimple iff it's isomorphic to the direct sum of finitely many simple Lie algebras.

(2) L is simple iff it's isomorphic to one of the following Lie algebras:

- $\mathfrak{sl}_n(\mathbb{C}), n \geq 2$;
- $\mathfrak{o}_n(\mathbb{C}), n \geq 7$;
- $\mathfrak{sp}_{2n}(\mathcal{C}), n \geq 2$;
- one of the 5 exceptional complex simple Lie algebras, denoted by $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$ respectively.

Remark 1.2.15 — It can be shown that

$$\mathfrak{o}_2(\mathbb{C}) \cong \mathbb{C}, \quad \mathfrak{o}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sp}_2(\mathbb{C}),$$

$$\mathfrak{o}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), \quad \mathfrak{o}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C}), \quad \mathfrak{o}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C}).$$

§1.3 Abelian, nilpotent and solvable Lie algebras

From now on, let us make the convention that L always denotes a finite-dimensional complex Lie algebra, and V always denoted a complex vector space.

Recall that for $x \in \mathfrak{gl}(V)$, x is said to be semisimple if it's diagonalizable; and nilpotent if $x^r = 0$ for some $r \geq 1$.

Definition 1.3.1 (ad-semisimple and ad-nilpotent)**.** x is **ad-semisimple** if $\mathrm{ad}(x) \in \mathfrak{gl}(V)$ is semisimple. Similarly define ad-nilpotent.

Proposition 1.3.2
Let $L < \mathfrak{gl}(V), x \in L$. If x is semisimple, then it's ad-semisimple. If x is nilpotent, then it's ad-nilpotent.

Remark 1.3.3 — If L is semisimple, then the converse of the proposition holds.

Theorem 1.3.4
A Lie algebra L is abelian iff it consists only of ad-semisimple elements.

For a Lie algebra L, we define two sequences of ideals

$$L = L^0 \supset L^1 \supset \cdots, \quad L = L^{(0)} \supset L^{(1)} \supset \cdots$$

by

$$L^k = [L, L^{k-1}], \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}].$$

Definition 1.3.5. L is said to be **nilpotent** if $L^k = 0$ for some k. L is said to be **solvable** if $L^{(k)} = 0$ for some k.

It's easy to see $L^k \supset L^{(k)}$, thus nilpotent Lie algebras must be solvable.

Proposition 1.3.6

Let L be a finite-dimensional Lie algebra, TFAE:

- L is semisimple;
- L has no nonzero nilpotent subalgebras;
- L has no nonzero solvable subalgebras.

Theorem 1.3.7 (Engel)

Let $L < \mathfrak{gl}(V)$ be a linear Lie algebra consisting of nilpotent transformations, then the following statement holds:

- There exists $v \in V$ s.t. $Lv = 0$.
- There exists a basis of V s.t. elements in L are all upper triangular.

Remark 1.3.8 — This implies that L is nilpotent.