Linear Algebra II

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Proposition 0.1

T is diagonalizable $\iff$ $\exists f \in M_T$ s.t. f is the product of different polynomials of degree 1.

Before we prove this proposition, let us take a look at the properties of annihilating polynomials.

Since $F[x]$ is a PID, M_T must be generated by one element, namely p_T, the *minmimal polynomial* of T. Thus we can WLOG assmue $f = p_T$ in the above proposition.

Speaking of polynomials and linear maps, one thing that pops into our mind is the characteristic polynomial f_T. In fact there is strong relations between p_T and f_T:

Theorem 0.2 (Cayley-Hamilton)

The characteristic polynomial of a linear operator T is its annihilating polynomial, i.e. $f_T(T) = 0$.

This theorem is also true when T is a matrix on a module. To prove it more generally, we introduce the concept of modules.

Definition 0.3 (Modules over commutative rings)**.** Let R be a commutative ring. A set M is called a **module** over R or an **R-module** if:

- There is a binary operation (addition) $M \times M \to M : (\alpha, \beta) \mapsto \alpha + \beta$ such that M becomes a commutative group under this operation.
- There is an operation (scaling) $R \times M \to M : (r, \alpha) \mapsto r\alpha$ with assosiativity and distribution over addition (both left and right). We also require $1_R\alpha = \alpha$ for all $\alpha \in M$.

Example 0.4

A commutative group automatically has a structure of $\mathbb{Z}$-module. (view the group operation as addition in definition of modules)

Example 0.5

Let $R = F[x]$, T a linear operator on V. Define $R \times V \to V : (f, \alpha) \mapsto f\alpha := f(T)\alpha$. We can check V becomes a module over R.

We can also define matrices on a commutative ring R, with addition and multiplication identical to the usual matrices. So the determinant and characteristic polynomial make sense as well.

Note that each $m \times n$ matrix represents a homomorphism $R^m \to R^n$.

Proof of Theorem 0.2. Take a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of V. Let $A = [T]_{\mathcal{B}}$. If we view V as a R-module ($R = F[x]$),

$$(\alpha_1, \dots, \alpha_n)A = (T\alpha_1, \dots, T\alpha_n) = (x\alpha_1, \dots, x\alpha_n) = (\alpha_1, \dots, \alpha_n) \cdot xI_n.$$

This implies $(\alpha_1, \dots, \alpha_n)(xI_n - A) = (0, \dots, 0)$.

Claim 0.6. If $f \in F[x]$ s.t. $\exists B \in R^{n\times n}$ s.t. $(xI_n - A)B = fI_n$, then $f(T) = 0$.

Proof of the claim.

$$0 = (\alpha_1, \dots, \alpha_n)(xI_n - A)B = (\alpha_1, \dots, \alpha_n) \cdot fI_n = (f(T)\alpha_1, \dots, f(T)\alpha_n).$$

Since $\alpha_1, \dots, \alpha_n$ is a basis, $f(T)$ must equal to 0. □

Now it's sufficient to prove f_T satisfies the condition in the claim. This follows from letting $B = A^{\mathrm{adj}}$, the adjoint matrix of A. □

Remark 0.7 — In fact this proof is derived from the proof of Nakayama's lemma, which is an important result in commutative algebra.

As a corollary, $p_T \mid f_T$.

Proof of Proposition 0.1. First we prove a lemma:

Lemma 0.8

Let $T_1, \dots, T_k \in L(V)$, $\dim V < \infty$. Then

$$\dim\ker(T_1T_2\dots T_n) \le \sum_{i=1}^{k} \dim\ker(T_i).$$

Proof of the lemma. By induction we only need to prove the case $k = 2$.

Note that $\ker(T_1T_2) = \ker(T_2) + \ker(T_1|_{\operatorname{im} T_2})$. So

$$\dim\ker(T_1T_2) = \dim\ker(T_2) + \dim\ker(T_1|_{\operatorname{im} T_2}) \le \dim\ker(T_2) + \dim\ker(T_1).$$

□

If T is diagonalizable, suppose the matrix of T is $diag\{c_1, \dots, c_r\}$, then $g = \prod_{i=1}^{r}(x - c_i)$ is an annihilating polynomial of T.

Conversely, if $\prod_{i=1}^{r}(T - c_iI) = 0$, by lemma

$$n = \ker\left(\prod_{i=1}^{r}(T - c_iI)\right) \le \sum_{i=1}^{r} \ker(T - c_iI) = \sum_{i=1}^{r} \dim V_{c_i}.$$

This forces $V = \bigoplus_{i=1}^{r} V_{c_i}$, which completes the proof. □

§0.1 Invariant subspaces

There may not exist a subspace W' s.t. $W \oplus W' = V$, so we can instead study the quotient space.

Let $W \subset V$ be a T-invariant subspace. Define $T_W = T|_W \in L(W)$, $T_{V/W} \in L(V/W)$: $T_{V/W}(\alpha + W) = T(\alpha) + W$. It's clear that $T_{V/W}$ is well-defined.

However, this decomposition loses some information about T, i.e. they can't determine T completely. For example when $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, the matrix B will not be carried to T_W and $T_{V/W}$ as their matrices are A, C respectively.

Since $\det T = \det T_W \det T_{V/W}$, $f_T = f_{T_W} \cdot f_{T_{V/W}}$. The minimal polynomials satisfy

$$lcm(p_{T_W}, p_{T_{V/W}}) \mid p_T, \quad p_T \mid p_{T_W} p_{T_{V/W}}.$$

This follows by the definition of $T_W, T_{V/W}$, readers can check it manually. Hint: The image of $p_{T_{V/W}}(T)$ is in W. So by Proposition 0.1, T is diagonalizable $\iff$ $T_W, T_{V/W}$ are both diagonalizable.

Definition 0.9 (Simultaneous diagonalization)**.** Let $\mathcal{F} \subset L(V)$, if there exists $\mathcal{B}$ s.t. $\forall T \in \mathcal{F}$, $[T]_{\mathcal{B}}$ is diagonal matrix, then we say $\mathcal{F}$ can be simultaneously diagonalized.

Proposition 0.10

Let $\mathcal{F} \subset L(V)$, TFAE:

- $\mathcal{F}$ can be simultaneously diagonalized;
- Any element in $\mathcal{F}$ is diagonalizable, and any two elements commute with each other.

Proof. It's obvious the fisrt statement implies the second.

On the other hand, we proceed by induction on the dimension of the space V.

Assume $\dim V = n \geq 2$, WLOG $T \in \mathcal{F}$ is not a scalar matrix.

Let $\sigma(T) = \{c_1, \ldots, c_r\}, V = \bigoplus_{i=1}^r V_{c_i}$, where $r \geq 2$, $V_{c_i} \neq V$. Since T commutes with other elements in $\mathcal{F}$, so $V_{c_i} = \ker(T - c_i \operatorname{id}_V)$ is invariant under all the maps in $\mathcal{F}$.

Hence we can restrict $\mathcal{F}$ to V_{c_i} and apply induction hypothesis, i.e. for any $U \in \mathcal{F}$, $U|_{V_{c_i}}$ can be simultaneously diagonalized.

Therefore $\exists \mathcal{B}_i$ s.t. $[U|_{V_{c_i}}]_{\mathcal{B}_i}$ is diagonal $\implies$ $[U]_{\mathcal{B}}$ is diagonal, where $\mathcal{B} = \bigcup \mathcal{B}_i$. $\square$

Definition 0.11 (Triangulable matrix)**.** Let $T \in L(V)$. If $[T]_{\mathcal{B}}$ is an upper triangular matrix for some basis $\mathcal{B}$, we say T is **triangulable**.

Proposition 0.12

Let $\dim V = n$, for $T \in L(V)$, TFAE:

1. T is triangulable;
2. f_T(or p_T) can be decomposed to product of polynomials of degree 1.
3. There exists a sequence of T-invariant subspaces $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = V$.

   This kind of sequence is called a **flag**. (Flag itself does not require T-invariant)

Remark 0.13 — In particular, when the base field is *algebraically closed*, the above statements always holds.

Proof. It's obvious that (1) $\implies$ (2).

For (3) $\implies$ (4): We proceed by induction, for W_1 just take the space spanned by one of the eigenvectors of T.

Assume that we have constructed W_j for $0 \leq j \leq i$. Instead of finding an invariant subspace of dimension $i+1$, we'll find an invariant subspace of dimension 1 in V/W_i.

Let Q denote the quotient map $V \to V/W_i$. Consider the map $T_{V/W_i} : \alpha + W_i \mapsto T(\alpha) + W_i$. We have

$$T_{V/W_i} \circ Q = Q \circ T.$$

Since $p_{T_{V/W_i}} \mid p_T \implies p_{T_{V/W_i}}$ is product of polynomials of degree 1, T_{V/W_i} must have an eigenvector. Let L denote the subspace spanned by this vector, and $W_{i+1} = Q^{-1}(L)$.

Clearly $\dim W_{i+1} = 1 + \dim W_i = i + 1$. It suffices to check that W_{i+1} is T-invariant:

$$T(W_{i+1}) = T(Q^{-1}(L)) = Q^{-1}(T_{V/W_i}(L)) \subset Q^{-1}(L) = W_{i+1}.$$

Now for the last part (3) $\implies$ (1):

Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$, such that $\operatorname{span}\{\alpha_1, \ldots, \alpha_i\} = W_i$. The matrix of T under $\mathcal{B}$ is clearly an upper triangular matrix. ☐

Proposition 0.14

Let F be an algebraically closed field. Suppose the elements of $\mathcal{F} \subset L(V)$ are pairwise commutative, then $\mathcal{F}$ is simultaneously triangulable.

Remark 0.15 — The inverse of this proposition is not true: Just let $\mathcal{F}$ be the set consisting of all the upper triangular matrices.

Lemma 0.16

There's a common eigenvector of $\mathcal{F}$.

Proof of lemma. WLOG $\mathcal{F}$ is finite. (In fact, $\operatorname{span} \mathcal{F} \subset L(V)$ is a finite dimensional vector space, so we can take a basis $\mathcal{F}_0$.)

Now by induction, if $T_1, \ldots, T_{k-1}$ have common eigenvector α, let $T_i\alpha = c_i\alpha$. Then

$$W = \bigcap_{i=1}^{k-1} \ker(T_i - c_i \operatorname{id}_V) \neq \{0\}$$

is a T_k-invariant space.

So any eigenvector α' of $T_k\big|_W$ is the common eigenvector. ☐

Proof of the proposition. It suffices to prove that there exists an $\mathcal{F}$-invariant flag. By the lemma, the proof is nearly identical as the proof of previous proposition. ☐

§0.2 Decomposition of linear maps

In this section we mainly study how a linear map is decomposed into irreducible maps and the structure of irreducible maps.

Recall that every vector space V is an $F[x]$-module given a linear operator T. If a subspace $W \subset V$ is a T-invariant space, then W is a submodule of V.

Hence it leads to decompose V into direct sums of submodules.

Definition 0.17. Let V, W be isomorphic vector spaces. $T \in L(V)$, $T' \in L(W)$. If there exists an isomorphism $\Phi : V \to W$ s.t. $\Phi \circ T = T' \circ \Phi$, we say T and T' are **equivalent**.

Definition 0.18 (Primary maps)**.** Let $T \in L(V)$ be a linear map. We say T is **primary** if p_T is a power of prime polynomials.

Theorem 0.19 (Primary decomposition)

Let $T \in L(V)$, $p_T = \prod_{i=1}^{k} p_i^{r_i}$, where p_i are different monic prime polynomials of degree 1.

We have

$$V = \bigoplus_{i=1}^{k} W_i, \quad W_i = \ker\left(p_i^{r_i}(T)\right),$$

with $W_i \neq \{0\}$ and $T\big|_{W_i}$ primary.

Proof. Let $f_i = \prod_{j \neq i} p_j^{r_j}$, f_i and p_i are coprime.

Note that $f_i(T) \neq 0$ and $f_i(T) p_i^{r_i}(T) = p_T(T) = 0$, thus $p_i^{r_i}(T)$ is not inversible, which implies $W_i \neq \{0\}$.

W_i independent : If there exists $\alpha_j \in W_j$ s.t. $\sum_{j=1}^{k} \alpha_j = 0$, applying f_i we get $f_i(\alpha_i) = 0$. But $p_i^{r_i}(\alpha_i) = 0 \implies \alpha_i = 0, \forall i$.

To prove $V = \sum_{i=1}^{k} W_i$, observe that

$$\gcd(f_1, \ldots, f_k) = 1 \implies \exists g_1, \ldots, g_k \quad s.t. \quad 1 = \sum_{i=1}^{k} g_i f_i \implies \alpha = \sum_{i=1}^{k} g_i(f_i \alpha), \quad \forall \alpha \in V.$$

Since $f_i \alpha \in W_i$, W_i is T-invariant $\implies g_i f_i \alpha \in W_i$.

Lastly, we'll prove that the minimal polynomial q_i of $T\big|_{W_i}$ is $p_i^{r_i}$.

Clearly $p_i^{r_i}(T\big|_{W_i}) = 0$, so $q_i \mid p_i^{r_i}$.

On the other hand, $q_1 q_2 \ldots q_k$ is an annihilating polynomial of T, hence

$$\prod_{i=1}^{k} p_i^{r_i} \mid \prod_{i=1}^{k} q_i \implies q_i = p_i^{r_i}, \quad \forall i.$$

□