Linear Algebra II

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This means we always have $W \oplus W^\perp = V$.

The orthogonal completion is similar to the annihiltor we studied last semester, in fact, when we view $\langle \cdot, \beta \rangle$ as a function $f_\beta \in V^*$, $f_\beta \in S^0 \iff \beta \in S^\perp$. (Note that the inner product is linear with respect to only the first entry)

This process induces a map $\phi : V \to V^*$ by $\beta \mapsto f_\beta$. It's clear that ϕ is conjugate-linear. So ϕ is a linear map between *real* vector space $V \to V^*$, i.e. $\phi \in \operatorname{Hom}_\mathbb{R}(V, V^*)$. thus $\ker \phi = \{0\}$ implies ϕ is an isomorphism on $\mathbb{R}$, so ϕ is a bijection, $\phi(S^\perp) = S^0$.

For $E \subset V^*$, then $E^0 \subset V$, this corresponds to $\phi(S)^0 = S^\perp$. Indeed, $\alpha \in \phi(S)^0 \iff \forall \beta \in S, \langle \alpha, \beta \rangle = 0 \iff \alpha \in S^\perp$. Hence

$$\dim_\mathbb{C} W^\perp = 2 \dim_\mathbb{R} \phi(W^\perp) = 2 \dim_\mathbb{R} W^0 = \dim_\mathbb{C} W^0.$$

The above proposition can be derived directly by $\dim W + \dim W^0 = \dim V$.

We can also get $W = (W^0)^0 = \phi(W^\perp)^0 = (W^\perp)^\perp$.

Definition 0.0.1 (Orthogonal projection). Since $V = W \oplus W^\perp$, for all $\alpha \in V$, there exists unique $\beta \in W, \gamma \in W^\perp$ s.t. $\alpha = \beta + \gamma$. Let $p_W : V \to W$ be the map $\alpha \mapsto \beta$, this is called the **orthogonal projection** from V to W.

§0.1 Adjoint maps

Let $\{\alpha_1, \ldots, \alpha_m\}$ be an orthonormal basis of W, then $p_W(\beta) = \sum_{j=1}^m \langle \beta, \alpha_j \rangle \alpha_j$. So p_W is a linear map. Moreover $p_W + p_{W^\perp} = \operatorname{id}_V$, $p_W^2 = p_W$. By our geometry intuition, $p_W \beta = \arg\min_\alpha \|\alpha - \beta\|$, this fact is useful in funtional analysis.

Recall that for $T \in L(V)$, $T^t \in L(V^*)$, then what's the map $\phi^{-1} \circ T^t \circ \phi$? Unluckily it's not T, but another map denoted by T^*, the **adjoint map** of T. Keep in mind that T^* depends on the inner product.

$$
\begin{array}{ccc}
V^* & \xrightarrow{\ T^t\ } & V^* \\
{\scriptstyle \phi}\big\uparrow & & {\scriptstyle \phi}\big\uparrow \\
V & \xrightarrow{\ T^*\ } & V
\end{array}
$$

Since $T^t \circ \phi = \phi \circ T^* \iff \langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle, \forall \alpha, \beta \in V$, so T^* can be described as the map satisfying this relation.

> **Proposition 0.1.1**
>
> When $\mathcal{B}$ is an orthonormal basis, we have $[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^*$.

Proof. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$, then $\phi(\mathcal{B})$ is the dual basis of $\mathcal{B}$. i.e. $\phi(\alpha_j)(\alpha_k) = \delta_{jk}$.
Hence $[T^t]_{\phi(\mathcal{B})} = [T]_{\mathcal{B}}^t$. Let $[T^*]_{\mathcal{B}} = A$, then

$$T^*\alpha_k = \sum_{j=1}^{n} A_{jk}\alpha_j \implies \phi(T^*\alpha_k) = \sum_{j=1}^{n} \overline{A_{jk}}\phi(\alpha_j).$$

So $[T^t]_{\phi(\mathcal{B})} = \overline{A}$, which completes the proof. $\qquad\square$

> **Proposition 0.1.2**
>
> $\ker(T^*) = \mathrm{Im}(T)^{\perp}$, $\mathrm{Im}(T^*) = \ker(T)^{\perp}$. $(cT + U)^* = \bar{c}T^* + U^*$, $(TU)^* = U^*T^*$, $T^{**} = T$.
> This means the map $T \mapsto T^*$ is a conjugate anti-automorphism of $L(V)$, and it's an involution.

If $T^* = T$, then we say T is **self-adjoint**, and if $T^* = -T$, we say T is **anti self-adjoint**.
Let $F = \mathbb{C}$, T is self-adjoint iff iT is anti self-adjoint. Like a function can be written as a sum of odd and even functions, $\forall T \in L(V)$, there exists unique self-adjoint T_1, T_2 s.t. $T = T_1 + iT_2$. In fact, $T_1 = \frac{T+T^*}{2}, T_2 = \frac{T-T^*}{2i}$.
Let $\mathcal{B}$ be an orthonormal basis, obviously T self-adjoint $\iff [T]_{\mathcal{B}}$ Hermite.

> **Example 0.1.3**
>
> Let $W \subset V$, p_W be the orthogonal projection. then p_W is self-adjoint as we can choose an orthonormal basis $\mathcal{B}$, such that $[p_W]_{\mathcal{B}} = \mathrm{diag}\{I_k, 0\}$, where $k = \dim W$.

Let V, W be inner product spaces, we'll study the linear maps $T : V \to W$ which preserves the inner products, i.e.
$$\langle \alpha, \beta \rangle_V = \langle T\alpha, T\beta \rangle_W.$$
If T is an isomorphism, then we say T is the isomorphism between inner product spaces.

> **Proposition 0.1.4**
>
> T preserves inner product $\iff T$ is an isometry, i.e. preserves length.
> In particular, isometry is always injective implies that inner product presering maps are always injective.

Proof. Trivial by polarization identity. $\qquad\square$

> **Proposition 0.1.5**
>
> Let V, W be finite dimensional inner product spaces, $\dim V = \dim W$, $T \in \operatorname{Hom}(V, W)$, the followings are equivalent:
>
> (1) T preserves inner product;
>
> (2) T is an isomorphism between inner product spaces;
>
> (3) T maps all the orthonormal bases in V to orthonormal bases in W ;
>
> (4) T maps *one* orthonormal basis in V to a orthonormal basis in W.

Proof. It's clear that $(1) \implies (2) \implies (3) \implies (4)$, since T injective $\implies T$ is an isomorphism of vector space.

   As for $(4) \implies (1)$, just expand everything using this orthonormal basis.   □

> **Corollary 0.1.6**
>
> Inner product spaces with same dimensions are always isomorphic as inner product spaces.

   Recall the positive definite matrices we defined earlier, we can also define *positive definite maps*: Let T be a *self-adjoint map*, if

$$\forall \alpha \in V \backslash \{0\}, \quad \langle T\alpha, \alpha \rangle > 0,$$

then we say T is positive definite.

   The reason why we require T self-adjoint is that,

$$\langle T\alpha, \alpha \rangle = \langle \alpha, T\alpha \rangle = \overline{\langle T\alpha, \alpha \rangle} \implies \langle T\alpha, \alpha \rangle \in \mathbb{R}.$$

so we can talk about "positive" safely.

§0.2 Orthogonal maps and Unitary maps

Definition 0.2.1 (Orthogonal maps)**.** Let V be a real inner product space, the automorphisms of V (as inner product space) are called **orthogonal maps**, denoted the set by $\operatorname{O}(V)$.

   When V is a complex inner product space, we use **unitary maps** and $\operatorname{U}(V)$ instead.

> **Proposition 0.2.2**
>
> Let V be an inner product space,
>
> $$T \in \operatorname{O}(V) \iff T^* = T^{-1}.$$

Proof.
$$T \in \operatorname{O}(V) \iff \langle \alpha, \beta \rangle = \langle T\alpha, T\beta \rangle = \langle \alpha, T^*T\beta \rangle, \quad \forall \alpha, \beta \in V.$$

This also holds for $\operatorname{U}(V)$.   □

> **Proposition 0.2.3**
>
> Let $A \in \mathbb{R}^{n \times n}$, TFAE:
>
> - $A^t A = I_n$;
>
> - The column (row) vectors of A form an orthonormal basis of $\mathbb{R}^n$.

Proof. Since A maps the standard basis to the column vectors of A, so the conclusion follows immediately (use A^t to get the row vectors). $\qquad\square$

Let $\mathrm{O}(n) = \{A \in \mathbb{R}^{n \times n} \mid A^t A = I_n\}$, and $\mathrm{U}(n) = \{A \in \mathbb{C}^{n \times n} \mid A^* A = I_n\}$. We can see that $A^t A = I_n \implies \det(A) = \pm 1$, and $A^* A = I_n \implies |\det(A)| = 1$.

Let $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}$, and $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}$. In the language of groups, $\mathrm{SO}(n)$ has only 2 coset in $\mathrm{O}(n)$, while the structure of the cosets of $\mathrm{SU}(n)$ in $\mathrm{U}(n)$ look like S^1.

> **Example 0.2.4**
>
> Let's look at some low dimensional orthogonal groups. $\mathrm{O}(1) = \{1, -1\}$, $\mathrm{SO}(1) = \{1\} = \mathrm{SU}(1)$, $\mathrm{U}(1) = \{z \mid |z| = 1\}$.
>
> The group $\mathrm{SO}(2)$ is the rotations of $\mathbb{R}^2$:
>
> $$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$
>
> While $\mathrm{O}(2)$ also consisting of reflections.
>
> $$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}.$$
>
> In fact these groups are *lie groups*, which means they have the structure of differential manifolds. It's clear that $\mathrm{U}(1) \simeq \mathrm{SO}(2) \simeq S^1$, and we can see $\mathrm{SU}(2) \simeq S^3$.

> **Theorem 0.2.5** (QR-decomposition)
>
> Any invertible matrix A can be uniquely decomposed to $Q \cdot R$, where $Q \in \mathrm{O}(n)$, R is an upper triangular matrix with positive diagonal entries. When $F = \mathbb{C}$, $\mathrm{O}(n)$ is replaced by $\mathrm{U}(n)$.

Proof. This is essentially Schmidt orthogonalozation. $\qquad\square$

> **Corollary 0.2.6** (Ivasawa decomposition, KAN decomposition)
>
> For all $A \in \mathrm{GL}_n(\mathbb{R})$, it has a unique decomposition $A = A_k A_a A_n$, $A_k \in \mathrm{O}(n)$, A_a is diagonal, A_n is upper triangular matrix with diagonal entries 1. It also holds for $\mathbb{C}$.

Let $\mathcal{B}, \mathcal{B}'$ be orthonormal bases of V, $T \in L(V)$. We know that $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$ for some $P \in \mathrm{GL}(V)$. By orthogonality, P must be an orthogonal matrix, wich means $P^t = P^{-1}$.

Definition 0.2.7. Let $A, B \in \mathbb{R}^{n \times n}$, we say they are **orthogonally similar** if $A = P^{-1}BP$ for some $P \in \mathrm{O}(n)$. The name is changed to **unitarily similar** for complex matrices.

> **Theorem 0.2.8** (Schur triangularization theorem)
>
> Let $F = \mathbb{C}, T \in L(V)$. There exists an orthonormal basis $\mathcal{B}$, such that $[T]_\mathcal{B}$ is upper triangular.

Proof. Recall that T is triangulable (which is always true in $\mathbb{C}$) iff there exists a T-invariant flag $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = V$. We can take an orthonormal basis s.t. $W_k = \mathrm{span}\{\alpha_1, \ldots, \alpha_k\}$. Obviously T is upper triangular under this basis. $\square$

§0.3 Normal maps

Recall that we say two matrices A, B are orthogonally similar, if there exist $P \in \mathrm{O}(n)$ s.t. $B = P^{-1}AP$. Again, we want to find the "simplest" matrix in each orthogonal equivalent class.

Let $T \in L(V)$ be a linear map, if there exists an orthonomal basis of V s.t. $[T]_\mathcal{B}$ is diagonal, then we say T is orthogonally (or unitarily) diagonalizable.

Definition 0.3.1 (Normal maps)**.** Let V be an inner product space, $T \in L(V)$. If $TT^* = T^*T$, then we say T is a **nomal map**.

It turns out that these concepts has close relations:

> **Theorem 0.3.2**
>
> Let V be a finite dimentional inner product space,
>
> - If $F = \mathbb{R}$, then T orthogonally diagonalizable $\iff$ T self-adjoint;
>
> - If $F = \mathbb{C}$, then T unitarily diagonalizable $\iff$ T normal.

> **Lemma 0.3.3**
>
> Let $F = \mathbb{C}$, then T normal $\iff$ there exists self-adjoint commutative maps T_1, T_2 s.t. $T = T_1 + iT_2$.

Proof. If $T = T_1 + iT_2$, then $T^* = T_1 - iT_2$, so $T^*T = TT^*$ since T_1, T_2 commutative.

On the other hand, let $T_1 = \frac{T+T^*}{2}$, $T_2 = \frac{T-T^*}{2i}$. We can check that T_1, T_2 self-adjoint and are commutative. $\square$

Proof of Theorem 0.3.2. For the " $\implies$ " part, let $\mathcal{B}$ be an orthonomal basis such that $[T]_\mathcal{B} = \mathrm{diag}\{c_1, \ldots, c_n\}$. Then we have

$$[T^*]_\mathcal{B} = [T]_\mathcal{B}^* = \mathrm{diag}\{\bar{c}_1, \ldots, \bar{c}_n\}.$$

If $F = \mathbb{R}$, then $T^* = T$, i.e. T self-adjoint.

If $F = \mathbb{C}$, clearly $TT^* = T^*T$, so T is normal.

As for the other part, we need a lemma first.

> **Lemma 0.3.4**
>
> Let V be a f.d. inner product space, $T \in L(V)$. If $W \subset V$ is a T-invariant space, then $W^\perp$ is T^*-invariant.

Proof of the lemma. For all $\alpha \in W^\perp$,

$$0 = \langle \alpha, T\beta \rangle = \langle T^*\alpha, \beta \rangle, \quad \forall \beta \in W.$$

Thus $T^*\alpha \in W^\perp$. $\qquad \square$

> **Corollary 0.3.5**
>
> If T is self-adjoint, $W \subset V$ is T-invariant will imply $W^\perp$ is also T-invariant, so T is semisimple.

> **Lemma 0.3.6**
>
> Let V be a f.d. inner product space, $T \in L(V)$ is self-adjoint. We must have $f_T \in \mathbb{R}[x]$, and it can be decomposed to products of polynomials of degree 1.
> In particular, $\sigma(T) \subset \mathbb{R}$.

Proof. Let $f_T = \prod_{j=1}^n (x - c_j), c_j \in \mathbb{C}$.
Let $\mathcal{B}$ be an orthonomal basis of V, then $A := [T]_\mathcal{B}$ is Hermite. Let X be a nonzero vector s.t. $AX = c_j X$, then
$$c_j X^* X = X^* A X = (AX)^* X = \bar{c}_j X^* X.$$

So $c_j \in \mathbb{R}$, and we're done. $\qquad \square$

> **Lemma 0.3.7**
>
> If T is a self-adjoint map, then all the eigenspaces of T are pairwise orthogonal.

Proof. Let $c_1, c_2 \in \mathbb{R}$ be two eigenvalues of T. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$.

$$c_1 \langle \alpha, \beta \rangle = \langle c_1 \alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T\beta \rangle = \bar{c}_2 \langle \alpha, \beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

Since $c_1 \neq c_2$, we must have $\alpha \perp \beta$, as desired. $\qquad \square$

Returning back to Theorem 0.3.2, when T is self-adjoint, let $\sigma(T) = \{c_1, \ldots, c_r\}$.

| **Claim 0.3.8.** $V = \bigoplus_{i=1}^r V_{c_i}$, i.e. T is diagonalizable.

Let $W = \bigoplus_{i=1}^r V_{c_i}$, if $W^\perp \neq \{0\}$, then $W^\perp$ is T-invariant.
When $F = \mathbb{C}$, then $T_{W^\perp}$ has eigenvectors; when $F = \mathbb{R}$, then $T_{W^\perp}$ is self-adjoint, so it must have a eigenvector (by lemma).

Since V_{c_i} are pairwise orthogonal, so we can actually take an orthonomal basis of V_{c_i} to get an orthonomal basis of V. Hence T is orthogonally diagonalizable.
Now for the case when T is normal, let T_1, T_2 be self-adjoint maps s.t. $T = T_1 + iT_2$. Since T_1, T_2 commute, the proof is nearly identical to the simutaneously diagonalizable property.
Let $V = \bigoplus_{i=1}^r V_{c_i}$ be the eigenspace decomposition of T_1. Note that V_{c_i} are also T_2-invariant.
Since $(T_2)_{V_{c_i}}$ self-adjoint, $(T_2)_{V_{c_i}}$ is unitarily diagonalizable. Therefore we can concatenate those basis to get a basis of V, and T_1, T_2 are both diagonal under this basis. $\qquad \square$

There's another proof of " $\Longleftarrow$ " part of the theorem:

> **Proposition 0.3.9**
>
> Let V be an inner product space, $T \in L(V)$ normal. Let $W \subset V$ be a T-invariant space, then $W^\perp$ is T-invariant, and W is T^*-invariant.

Proof. Take an orthonomal basis of $W, W^\perp$, so $A := [T]_\mathcal{B}$ normal.

Since W is T-invariant, $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Note that:

$$AA^* = \begin{pmatrix} BB^* + CC^* & * \\ * & * \end{pmatrix}, \quad A^*A = \begin{pmatrix} B^*B & * \\ * & * \end{pmatrix}.$$

As A normal, $BB^* + CC^* = B^*B$, by looking at the trace of both sides, we get $\operatorname{tr}(CC^*) = 0 \implies C = 0$, the conclusion follows. $\square$

> **Corollary 0.3.10**
>
> Let $A \in \mathbb{C}^{n \times n}$ be an upper triangular martix, then A normal $\iff$ A diagonal.

> **Proposition 0.3.11**
>
> Let T be a normal map, then the eigenspaces of T are pairwise orthogonal.

Proof. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$, since $\operatorname{span}\{\beta\}$ is a T-invariant space, so $T^*\beta \in \operatorname{span}\{\beta\}$,

$$\langle T^*\beta, \beta \rangle = \langle \beta, T\beta \rangle = \bar{c}_2 \langle \beta, \beta \rangle.$$

Thus $T^*\beta = \bar{c}_2 \beta$.

$$c_1 \langle \alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

But $c_1 \neq c_2$, we have $\alpha \perp \beta$. $\square$

When $F = \mathbb{C}$: Let $W = \bigoplus_{i=1}^r V_{c_i}$. Since $W^\perp$ is T-invariant, so when $W^\perp \neq \{0\}$, T must have eigenvalues in $W^\perp$, contradiction!

Now we've proved that V_{c_i} are pairwise orthogonal, so T is unitarily diagonalizable.

> **Proposition 0.3.12**
>
> Let V be a complex inner product space, $T \in L(V)$ normal,
>
> - T self-adjoint $\iff$ $\sigma(T) \subset \mathbb{R}$;
>
> - T anti self-adjoint $\iff$ $\sigma(T) \subset i\mathbb{R}$;
>
> - T unitary $\iff$ $\sigma(T) \subset \{z : |z| = 1\}$.

Proof. Take an orthonomal basis s.t. $[T]_\mathcal{B}$ diagonal. The rest is trivial. $\square$