# Mathematical Analysis II

Felix Chen

## Contents



**Proposition 0.0.1**

Let $\Omega \subset \mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^m$ is a smooth map. Let $S \subset \mathbb{R}^m$ be a differential manifold, if for all $x \in f^{-1}(S)$, we have $\operatorname{rank} \mathrm{d}f(x) = m$, then $f^{-1}(S)$ is a differential manifold with codimension same as $S$.

*Proof.* For any $x \in S$, let $\Phi$ be the homeomorphism from an open neighborhood of $x$ to $\mathbb{R}^m$.

Suppose $\dim S = d$, let

$$F(x) = ((\Phi \circ f)_{d+1}, \dots, (\Phi \circ f)_m).$$

Note that $\mathrm{d}(\Phi \circ f)$ is an $m \times n$ matrix, and its rank is $m$. Since

$$\mathrm{d}(\Phi \circ f) = \begin{pmatrix} \mathrm{d}(\Phi \circ f)_1 \\ \vdots \\ \mathrm{d}(\Phi \circ f)_d \\ \mathrm{d}F \end{pmatrix}$$

Thus $\mathrm{d}F$ is a $(m-d) \times n$ matrix with rank $m-d$. So $F^{-1}(0) = f^{-1}(S)$ is a manifold with dimension $n - (m-d)$. □

## §0.1 Tangent space

Since the differentiation relies on the choice of coordinates, if we want to study the manifold more geometrically, we should look at the tangent lines or tangent planes of the manifold.

**Definition 0.1.1** (Tangent vectors)**.** Let $M$ be a differential manifold. Let $p \in M$, for all parametrized curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we say the vector $\gamma'(0) \in \mathbb{R}^n$ is the **tangent vector** of $\gamma$ at point $p$.

Let $T_pM$ denote the **tangent space** at $p$, which is defined as

$$T_pM = \{\gamma'(0) \in \mathbb{R}^n \mid \gamma(0) = p\}.$$

It's clear that $T_pM$ should be a vector space of dimension $\dim M$, next we'll prove this fact.

**Proposition 0.1.2** (Push forward of tangent spaces under differential homeomorphism)
Let $\Phi : U \to V$ be a differential homeomorphism, $M \subset U$ be a manifold, then

$$T_{\Phi(p)}\Phi(M) = (\mathrm{d}\Phi)\big|_p \cdot T_p M.$$

*Proof.* Let $\gamma$ be a parametrized curve on $M$ with $\gamma(0) = p$. Note that $\Phi \circ \gamma$ is a curve on $\Phi(M)$ passing through $\Phi(p)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \circ \gamma(t)\Big|_{t=0} = \mathrm{d}\Phi(p) \cdot \gamma'(0).$$

Thus $\mathrm{d}\Phi(p) \cdot T_p M \subset T_{\Phi(p)}\Phi(M)$.

Now we do the same thing for $\Phi^{-1}$, we can get the desired equality. □

Now we can easily calculate the tangent space: since $M$ is locally homeomorphic to $\mathbb{R}^d$, and obviously $T_{\Phi(p)}(\mathbb{R}^d \times \{0\}) = \mathbb{R}^d \times \{0\}$, by above proposition, $T_p M = (\mathrm{d}\Phi) \cdot T_{\Phi(p)}(\mathbb{R}^d \times \{0\})$ is a vector space of dimension $d$.

**Theorem 0.1.3**
Let $M$ be a manifold, $T_p M$ is a vector space of dimension $\dim M$.

**Proposition 0.1.4**
Let $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a smooth map, $\operatorname{rank} \mathrm{d}f = n$. Let $M = f^{-1}(f(p))$, then $T_p M = \ker \mathrm{d}f(p)$.

*Proof.* Let

$$F(x, y) = (x, f(x, y)), x \in \mathbb{R}^d, y \in \mathbb{R}^n.$$

$F$ is a homeomorphism, so $T_p M = (\mathrm{d}F^{-1})T_{F(p)}F(M)$.

Note that $F(M) = \{(x, p) \mid \exists y, f(x, y) = f(p)\}$, it must be a vector space of dimension $d$, so $T_{F(p)}F(M) = \mathbb{R}^d \times \{0\}$,

$$T_p M = (\mathrm{d}F^{-1})T_{F(p)}F(M) = \ker \mathrm{d}f(p).$$

□

**Example 0.1.5**
Let $M$ be a manifold determined by $f : \mathbb{R}^n \to \mathbb{R}$,

$$T_p M = \ker \mathrm{d}f = \{v \in \mathbb{R}^n \mid \mathrm{d}f(p)v = 0\}.$$

Here $\mathrm{d}f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) = \nabla f$. So $v \in T_p M \iff \nabla f \cdot v = 0$, the dot means the inner product. In this case the vector $\nabla f$ is called **normal direction vector**.

## §0.2 Smooth maps between manifolds

Next we'll briefly introduce the differential maps between manifolds. Since the differentiation on manifolds is hard to define, so what we do is actually regarding manifolds as $\mathbb{R}^d$ locally and define the differentiablity using the maps between Eucild spaces.

**Definition 0.2.1.** Let $M, N$ be manifolds in $\mathbb{R}^m, \mathbb{R}^n$, respectively. $f : M \to N$ is a map, if $\forall p \in M$, there exists $p \in U \subset \mathbb{R}^m, V \subset \mathbb{R}^d$, $\Phi : U \to V$ s.t.

$$f_\Phi = f \circ \Phi^{-1}$$

is a smooth map from $V$ to $N$. We say $f$ is a smooth map from $M$ to $N$.

We need to check this definition is well-defined: if there's another homeomorphism $\Phi'$, $f \circ \Phi' = (f \circ \Phi^{-1}) \circ (\Phi \circ \Phi'^{-1})$ is indeed a smooth map.

**Lemma 0.2.2** (Smooth maps are locally restrictions of smooth maps in Eucild spaces)
Let $f : M \to N$ be a map, then $f$ is smooth $\iff \forall p \in M, \exists p \in U \subset \mathbb{R}^m$ and a smooth map $F : U \to \mathbb{R}^n$ s.t.

$$f\big|_{U \cap M} = F\big|_{U \cap M}.$$

*Proof.* Let $\tau$ denote the embedding from $M \cap U$ to $U$. Since $f \circ \Phi^{-1} = F \circ \tau \circ \Phi^{-1}$, so $F$ smooth $\implies f_\Phi$ smooth $\implies f$ smooth.

$$\begin{tikzcd}
V \subset \mathbb{R}^d \arrow[dr, "f \circ \Phi^{-1}"'] & M \cap U \arrow[l, "\Phi"'] \arrow[r, "\tau"] \arrow[d, "f"] & U \arrow[dl, "F"] \\
 & N \subset \mathbb{R}^n &
\end{tikzcd}$$

TODO: fix this

On the other hand, let $\widetilde{\tau}$ be the projection from $U$ to $V$, then $F = f \circ \Phi^{-1} \circ \widetilde{\tau} \circ \Phi$ satisfies the desired condition. $\square$

**Example 0.2.3**
Let $A$ be an orthogonal map in $\mathbb{R}^3$, then $A$ can be restricted to $S^2 \to S^2$.

**Definition 0.2.4** (Tangent map)**.** Let $f : M \to N$ be a map between manifolds, $v \in T_pM$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a parametrized curve with $\gamma(0) = p, \gamma'(0) = v$, then $f(\gamma(t))$ is a curve on $N$.

$$\mathrm{d}f(p)(v) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\Big|_{t=0} \in T_{f(p)}N.$$

Thus $\mathrm{d}f(p) : T_pM \to T_{f(p)}N$ is a map between tangent spaces.

In fact, if $f = F\big|_M$, then $\mathrm{d}f(p)(v) = \mathrm{d}F(p) \cdot v$.

**Definition 0.2.5** (Tangent bundle)**.** Let $M$ be a manifold, $\forall p \in M$, there's a tangent space $T_pM$. Define the **tangent bundle** of $M$ to be

$$TM = \bigsqcup_{p \in M} T_pM.$$

If $X$ is a map $M \to TM$: $p \mapsto X(p)$, with $X(p) \in T_pM$, then it's called a **tangent vector field**.

In other words, a tangent vector field is just to assign a tangent vector to every point in $M$.

**Proposition 0.2.6**

Let $M \subset \mathbb{R}^n$ be a manifold, all its tangent vector field form a $C^\infty$ module $T(M, TM)$, i.e. $\forall f \in C^\infty(M)$, $X, Y$ are smooth vector fields, then $fX, X+Y$ are both smooth vector fields.

**Proposition 0.2.7**

Let $M \subset \mathbb{R}^n$ be a smooth manifold, we have

$$TM = \{(x, v) \mid x \in M, v \in T_x M\}$$

is a smooth manifold in $\mathbb{R}^{2n}$, and $\dim TM = 2 \dim M$.

*Proof.* There exists a local homeomorphism $\phi : V \to U \subset \mathbb{R}^n$ s.t. $V \subset \mathbb{R}^d$, $\phi(V) = M \cap U$.

Define map $T\phi : V \times \mathbb{R}^n \to U \times \mathbb{R}^n$, $(x, v) \mapsto (\phi(x), \mathrm{d}\phi(x) \cdot v)$. Since $T\phi$ is injective($\phi$ is homeomorphism), and

$$\mathrm{d}T\phi = \begin{pmatrix} \mathrm{d}\phi & 0 \\ \mathrm{d}(\mathrm{d}\phi)(v) & \mathrm{d}\phi \end{pmatrix}$$

is non-degenerate, so $T\phi$ is a bijection and hence differential homeomorphism.

Since the tangent space of $V$ is just $\mathbb{R}^d$, so $T(U \cap M)$ is the image of $T\phi$ restricted on $V \times \mathbb{R}^d$. (Note that $\mathrm{d}\phi(x) \cdot v \in T_{\phi(x)} M$) Thus $TM$ is a manifold in $\mathbb{R}^{2n}$ with dimension $2d$. $\square$

**Definition 0.2.8** (Tangent maps)**.** Earlier we know that $\mathrm{d}f(p)$ is a map $T_p M \to T_{f(p)} N$, combined with tangent bundle we can write $\mathrm{d}f : TM \to TN$, this map is called the **tangent map** or the **differentiation** of $f$.

If we have a vector field $X$ and a smooth function $f : M \to \mathbb{R}^n$, consider

$$X(f)(p) = \mathrm{d}f(X)(p) := \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\Big|_{t=0}, \quad \gamma(0) = p, \gamma'(0) = X(p).$$

So $X$ induces a smooth map $C^\infty(M) \to C^\infty(M)$.

Now we can generalize a well known result to manifolds:

**Proposition 0.2.9**

Let $M \subset \mathbb{R}^n$ be a smooth manifold, $f \in C^\infty(M)$. If $f$ achieves a local extremum at $p \in M$, we must have $\mathrm{d}f(p) = 0$.

*Proof.* It suffices to prove $\mathrm{d}f(p)(v) = 0$, $\forall v \in T_p M$. Take $\gamma$ s.t. $\gamma(0) = p, \gamma'(0) = v$, then $f(\gamma(t))$ achieves its extremum at $t = 0$, so $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\big|_{t=0} = 0 = \mathrm{d}f(p)(v)$. $\square$

## §0.3 Conditional extremum problem

Consider a function $f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ and some constraint conditions

$$\begin{cases} g_1(x_1, \ldots, x_n) = 0 \\ \qquad \vdots \\ g_m(x_1, \ldots, x_n) = 0 \end{cases}$$

We want to compute the extremum of $f$ under these conditions.

Well, you probably heard of *Lagrange multipliers*, i.e. let

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x) - \sum_{j=1}^{m} \lambda_j g_j(x).$$

But here we'll provide a different point of view. Let $M$ be the manifold in $\mathbb{R}^n$ under those conditions, Suppose $p \in M$ is a local extremum of $f$, then $T_pM \subset \ker \mathrm{d}f(p)$.

Also recall that $T_pM = \ker \mathrm{d}g(p) = \bigcap_{j=1}^{m} \ker \mathrm{d}g_j(p)$. This means that, $\exists \lambda_1, \ldots, \lambda_m$ s.t.

$$\mathrm{d}f(p) = \sum_{j=1}^{m} \lambda_j \, \mathrm{d}g_j(p).$$

Surprisingly, we get the same result of Lagrange multipliers! Hence what we've done is to give a geometrical comprehension of Lagrange multipliers.

**Example 0.3.1**

Let $g : \mathbb{R}^n \to \mathbb{R}$ be the constraint function, then $f$ can achieve its extremum only if $\mathrm{d}f = \lambda \, \mathrm{d}g$.

For example, let $f(x) = d(x, z)^2$, $\mathrm{d}f(x) = 2(x_1 - z_1, \ldots, x_n - z_n)$, so $\mathrm{d}f = \lambda \, \mathrm{d}g$ means the vector $\mathrm{d}f(p)$ is orthogonal to the tangent plane of $M = \{g = 0\}$.

**Proposition 0.3.2** (Hadamard's inequality)

Let $v_1, \ldots, v_n \in \mathbb{R}^n$, then

$$|\det(v_1, \ldots, v_n)| \leq |v_1| \cdots |v_n|.$$

*Proof.* Let $f : \mathbb{R}^{n^2} \to \mathbb{R}$, $(v_1, \ldots, v_n) \mapsto \det(v_1, \ldots, v_n)$ with constraint $|v_i| = 1$. Let $v_{ij} \in \mathbb{R}$,

$$g_i(V) = -1 + \sum_{j=1}^{n} v_{ij}^2.$$

The manifold determined by $g_i$ is $M = (S^{n-1})^n$. The extremum point of $f$ in $M$ must satisfy:

$$\frac{\partial f}{\partial v_{i_0 j}} - \lambda_{i_0} \frac{\partial g_{i_0}}{\partial v_{i_0 j}} = 0.$$

This implies $v_{i_0 j}^* = 2\lambda_{i_0} v_{i_0 j}$, where $v_{i_0 j}^*$ is the *cofactors* of $v_{i_0 j}$.

This means that $\sum_{j=1}^{n} v_{i_0 j} v_{kj} = 0$, so $V$ must be an orthogonal matrix, so $|f| \leq 1$. $\square$

## §0.4 Convex functions

**Definition 0.4.1** (Hesse matrix)**.** Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function, we call the Jacobi matrix of $\nabla f$ to be the **Hesse matrix** of $f$. (Also called Hessian matrix)

$$H_f(p) = \nabla^2 f(p) = \left( \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f(p) \right)_{i,j}.$$

Since the partial derivatives commute, so $H_f$ is a symmetrical matrix, hence diagonalizable.

**Proposition 0.4.2**

Let $f \in C^2(\Omega)$, let $x_0$ be a minimum of $f$, then $\nabla f(x_0) = 0$, and $H_f(x_0)$ is semi positive definite.

*Proof.* By Taylor's expansion,

$$f(x) = f(x_0) + \mathrm{d}f(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T H_f(x_0)(x - x_0) + o(|x - x_0|^2).$$

If $H_f(x_0)$ has a negative eigenvalue $-\lambda$, with eigenvector $v$, then $f(x_0 + tv) = f(x_0) - \frac{1}{2}\lambda t^2 |v|^2 + o(|tv|^2)$, which contradicts with the minimality of $x_0$. □

**Proposition 0.4.3**

If $\nabla f(x_0) = 0$, $H_f(x_0)$ is positive definite, then $x_0$ is a local minimum of $f$.

*Proof.* Same as previous one. □

**Definition 0.4.4** (Convex functions)**.** If $f$ and $\Omega$ satisfies:

$$\forall x, y \in \Omega, tx + (1-t)y \in \Omega, \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

we say $\Omega$ is a **convex set** and $f$ a **convex function**.

**Theorem 0.4.5** (Jensen's inequality)

Let $f$ be a convex function on $\Omega$. Real numbers $t_i \geq 0, \sum_{i=1}^N t_i = 1$, for $x_i \in \Omega$,

$$f\left(\sum_{i=1}^N t_i x_i\right) \leq \sum_{i=1}^N t_i f(x_i).$$

**Example 0.4.6** (Convex functions)

Linear functions $f(x) = Ax + b$ are convex.

The norm function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is convex. Also let $A$ be an $n \times n$ positive definite matrix, then $f(x) = x^T Ax$ is convex.

Just like the one dimensional case, convex functions have nice properties.

**Theorem 0.4.7**

Let $f$ be a convex function on an open convex set $\Omega$, then $f$ is continuous, and Lipschitz continuous in any compact set, i.e.

$$|f(x) - f(y)| \leq M|x - y|, \quad x, y \in U$$

where $U$ is a compact set.

*Proof.* WLOG $0 \in \Omega$, take an orthogonal basis $e_1, \ldots, e_n$. Let

$$x = \sum_{i=1}^{n} \lambda_i \overline{e}_i, \quad \overline{e}_i = e_i \text{ or } -e_i, \lambda_i \geq 0.$$

When $|x|$ sufficiently small, $\sum_{i=1}^{n} \lambda_i < 1$, so by Jensen's inequality,

$$f(x) \leq \sum_{i=1}^{n} \lambda_i f(\overline{e}_i) + \lambda f(0),$$

$$f(x) - f(0) \leq \sum_{i=1}^{n} \lambda_i (f(\overline{e}_i) - f(0)) \leq \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} (f(\overline{e}_i) - f(0))^2 \right)^{\frac{1}{2}} \leq |x| C,$$

since we can change the length of $e_i$, and $f$ is continuous on a straight line.

This means $f$ is continuous. For the second part, let $\lambda_0 = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i}$, since $0 = \lambda_0 x + \sum_{i=1}^{n} \lambda_0 \lambda_i (-\overline{e}_i)$, by Jensen's inequality, we'll get the desired property. □

**Proposition 0.4.8**

Let $f$ be a differentiable function on a covex set $\Omega$,
$f$ is convex $\iff f(x) \geq f(x_0) + \mathrm{d}f(x_0)(x - x_0)$.

*Proof.* If $f$ is convex, just use the definition and let $t \to 0$:

$$f(x_0) + f'(x_0) t (x - x_0) + o(t(x - x_0)) \leq t f(x) + (1 - t) f(x_0).$$

Conversely, let $z = tx + (1 - t) x_0$,

$$f(x) \geq f(z) + f'(z)(1 - t)(x - x_0), f(x_0) \geq f(z) + f'(z) t (x_0 - x).$$

Thus adding these together we get

$$t f(x) + (1 - t) f(x_0) \geq f(z).$$

□

**Theorem 0.4.9**

Let $\Omega \subset \mathbb{R}^n$ be an open convex set, $f \in C^2(\Omega)$, $f$ convex $\iff H_f(x)$ semi positive definite.

*Proof.* One direction can be proved using Taylor's expansion.

On the other hand, let $H(t) = f(x_0 + t(x - x_0)) - f(x_0) - t\,\mathrm{d}f(x_0)(x - x_0)$, then $H'(t) = \mathrm{d}f(x_0 + t(x - x_0))(x - x_0) - \mathrm{d}f(x_0)(x - x_0)$,

$$H''(t) = (x - x_0)^T H_f(p)(x_0 + t(x - x_0))(x - x_0) \geq 0.$$

So $H(t)$ is a convex function, $H(0) = 0, H'(0) = 0$. □

# §1 Integrals on surfaces

## §1.1 Measures on manifolds

To define integrals, we need to define a measure on it first.

For example, let $v_1, \ldots, v_d \in \mathbb{R}^n$ be linearly independent vectors, and unit vectors $v_{d+1}, \ldots, v_n$ complete them to a basis, satisfying $v_j \perp v_i, j > d, j > i$.

Let $A$ be a linear map s.t. $Ae_i = v_i$, then the volume of $A(E)$ is $|\det A| = \sqrt{\det(G \cdot G^T)}$, where $G = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$ is a $d \times n$ matrix.

Since $AA^T = \begin{pmatrix} GG^T & 0 \\ 0 & I_{n-d} \end{pmatrix}$, $|\det A| = \sqrt{\det GG^T}$, we say $GG^T$ is the **Gram matrix** of $G$.

Another example is the length of a curve. Recall that we have the formula

$$L(\gamma) = \int_0^1 |\gamma'(t)| \,\mathrm{d}t.$$

The length of a curve is essentially the "volume" of a 1-dimensional manifold, so the idea of higher dimensional manifold is the same.

**Definition 1.1.1.** Let $M$ be a manifold in $\mathbb{R}^n$. Let $\Phi : V \subset \mathbb{R}^d \to U \subset M$ be a smooth homeomorphism, $\operatorname{rank} \Phi = d$. We can split $U$ to many small regions and use the paraloids to approximate the volume of each regoin.

Thus we define:

$$m(U) = \int_V \sqrt{\det(\mathrm{d}\Phi(x)^T \,\mathrm{d}\Phi(x))} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \cdots \mathrm{d}x_d.$$

The above differential form inside the integral is called the **volume form**:

$$\mathrm{d}\sigma(y) = \sqrt{\det(\mathrm{d}\Phi^T \,\mathrm{d}\Phi)} \,\mathrm{d}y.$$

Moreover, with this measure we can define the integral of general measurable function $f$ (measurable means locally measurable on $\mathbb{R}^d$):

$$\int_U f \,\mathrm{d}\sigma = \int_V f(\Phi(x)) \sqrt{\det(\mathrm{d}\Phi^T \,\mathrm{d}\Phi)} \,\mathrm{d}x.$$

Just like the length of a curve, the volume defined above is also a geometric quantity, i.e. independent of coordinates.

**Example 1.1.2**

Let $d = 1$, $\gamma : (-1, 1) \to \mathbb{R}^n$, $\gamma'(0) \neq 0$. For fixed $-1 < a < b < 1$ and a function $f$ on $\gamma$, let $C_a^b$ denote the curve between $\gamma(a), \gamma(b)$,

$$\int_{C_a^b} f \,\mathrm{d}\sigma = \int_a^b f(\gamma(t)) |\gamma'(t)| \,\mathrm{d}t$$

is called the **curve integral of the first type**.

**Example 1.1.3**

Let $d = n-1$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$, the graph of $f$ is a hyper-surface $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}^{n-1}\}$. It has a parametrization $\Phi(x) = (x, f(x))$, so

$$\mathrm{d}\Phi = \begin{pmatrix} I_{n-1} \\ \nabla f \end{pmatrix} \implies \mathrm{d}\Phi^T \, \mathrm{d}\Phi = I_{n-1} + \nabla f^T \cdot \nabla f.$$

Hence $\det(\mathrm{d}\Phi^T \, \mathrm{d}\Phi) = 1 + |\nabla f|^2$. (This can be obtained by looking at the eigenvectors)

Therefore for $\varphi$ on $\mathbb{R}^n$, we have

$$\int_{\Gamma_f} \varphi \, \mathrm{d}\sigma = \int_{\mathbb{R}^{n-1}} \varphi(x, f(x)) \sqrt{1 + |\nabla f|^2} \, \mathrm{d}x.$$

Next we'll compute the surface area of unit sphere $S^{n-1}$.

Let $c_n$ denote the volume of unit sphere in $\mathbb{R}^n$,

$$c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.$$

We claim in advance that the surface area of unit sphere $\omega_{n-1} = nc_n$. Here we use the sphere coordinates:

$$x_1 = r \prod_{k=1}^{n-1} \sin \theta_k, \quad x_i = r \sin \theta_{n-1} \cdots \sin \theta_i \cos \theta_{i-1}, 2 \le i \le n.$$

Let $F_n(r, \theta_1, \ldots, \theta_{n-1}) = (x_1, \ldots, x_n)$.

$$\mathrm{d}F_n = \begin{pmatrix} \sin \theta_{n-1} \, \mathrm{d}F_{n-1} & \cos \theta_{n-1} F_{n-1}^T \\ (\cos \theta_{n-1}, 0, \ldots, 0) & -r \sin \theta_{n-1} \end{pmatrix}.$$

So we can compute its determinant (expand using the last row), note that the first column of $\mathrm{d}F_{n-1}$ is $r^{-1} F_{n-1}^T$,

$$\begin{aligned} \det \mathrm{d}F_n &= - r \sin \theta_{n-1} (\sin \theta_{n-1})^{n-1} \det(\mathrm{d}F_{n-1}) \\ &\quad + (-1)^{n-1} (\cos \theta_{n-1})^2 (-1)^{n-2} r (\sin \theta_{n-1})^{n-2} \det(\mathrm{d}F_{n-1}) \\ &= - r (\sin \theta_{n-1})^{n-2} \det(\mathrm{d}F_{n-1}). \end{aligned}$$

Hence $\mathrm{d}x = r^{n-1} \sin^{n-2} \theta_{n-1} \cdots \sin \theta_2 \, \mathrm{d}r \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1} = r^{n-1} \, \mathrm{d}r \, \mathrm{d}\omega$.

Denote $F_n^S$ to be the function $F_n$ restricted to $S^{n-1}$. Then $\mathrm{d}F_n = (r^{-1} F_n^T, \mathrm{d}F_n^S)$. We can compute that the Gram determinant of $\mathrm{d}F_n^S$ is just $\det \mathrm{d}F_n$ with $r = 1$.

The rest is some integrals with gamma function and beta function, which is left out.