Measure Theory

Felix Chen

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Example 0.1

In probability, let $\mathscr{E}_1, \mathscr{E}_2$ be collections of sets. We say they're independent if

$$P(AB) = P(A)P(B), \quad \forall A \in \mathscr{E}_1, B \in \mathscr{E}_2.$$

By the previous theorem we can derive $\lambda(\mathscr{E}_1), \lambda(\mathscr{E}_2)$ are independent.

If $A_1, A_2, \dots$ satisfy

$$P(A_{i_1} \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

we say they are independent.

Let $\{1, 2, \dots\} = I + J$, then the σ-algebra generated by

$$\mathscr{E}_1 = \{A_\alpha \mid \alpha \in I\}, \quad \mathscr{E}_2 = \{A_\alpha \mid \alpha \in J\}$$

are independent.

Theorem 0.2 (Measure extension theorem)

Let μ be a measure on a semi-ring $\mathscr{Q}$, τ is the outer meaasure generated by μ. We have

$$\sigma(\mathscr{Q}) \in \mathscr{F}_\tau, \quad \tau|_{\mathscr{Q}} = \mu.$$

Remark 0.3 — Any measure on a semi-ring $\mathscr{Q}$ can extend to the $\sigma(\mathscr{Q})$, and if μ is σ-finite, the extension is unique.

Proof. For any $A \in \mathscr{Q}$, let $B_1 = A$, $B_n = \emptyset, n \geq 2$. Then $\tau(A) \leq \sum \mu(B_n) = \mu(A)$.

On the other hand, if $A_1, A_2, \dots \in \mathscr{Q}$ s.t. $\bigcup_{n=1}^\infty A_n \supseteq A$, then

$$\mu(A) = \mu\left(\bigcup_{n=1}^\infty \mu(AA_n)\right) \leq \sum_{n=1}^\infty \mu(AA_n) \leq \sum_{n=1}^\infty \mu(A_n).$$

Thus $\tau(A) = \mu(A)$, where we used the fact that μ is countable subadditive.

Next we prove $A \in \mathscr{F}_\tau$. We need to show that

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

WLOG $\tau(D) < \infty$. Take $B_1, B_2, \cdots \in \mathscr{Q}$ s.t.

$$\bigcup_{n=1}^{\infty} B_n \supseteq D, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(D) + \varepsilon.$$

Denote $\hat{D} := B_n \in \mathscr{Q}$ for a fixed n. Suppose $\hat{D} \cap A^c = \hat{D} \backslash A = \sum_{i=1}^{n} C_i$.

$$\mu(\hat{D}) = \mu(\hat{D} \cap A) + \sum_{i=1}^{n} \mu(C_i) \geq \tau(\hat{D} \cap A) + \tau(\hat{D} \cap A^c).$$

Apply this inequality to each B_n,

$$\tau(D) + \varepsilon > \sum_{n=1}^{\infty} (\tau(B_n \cap A) + \tau(B_n \cap A^c)) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

this implies $\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c) \implies A \in \mathscr{F}_\tau$.

At last by Caratheodory's theorem, τ is a measure on $\mathscr{F}_\tau \supseteq \sigma(\mathscr{Q})$. □

Theorem 0.4 (Equi-measure hull)

Let τ be the outer measure generated by μ,

- $\forall A \in \mathscr{F}_\tau$, $\exists B \in \sigma(\mathscr{Q})$ s.t. $B \supseteq A$ and $\tau(A) = \tau(B)$;
- If μ is σ-finite, then $\tau(B \backslash A) = 0$.

Remark 0.5 — This theroem states that $\mathscr{F}_\tau$ is just $\sigma(\mathscr{Q})$ appended with null sets.

Proof. If $\tau(A) = \infty$, $B = X$ suffices.

By definition, there exists $B_n = \bigcup_{k=1}^{k_n} B_{n,k} \supseteq A$ s.t. $\tau(B_n) < \tau(A) + \frac{1}{n}$. Let $B = \bigcap_{n=1}^{\infty} B_n$, we must have $\tau(B) = \tau(A)$.

Now for the second part, let $X = \sum_{n=1}^{\infty} A_n$, $A_n \in \mathscr{Q}$, $\mu(A_n) < \infty$.

Since $A = \sum_{n=1}^{\infty} AA_n$, we have

$$AA_n \in \mathscr{F}_\tau, \quad \tau(AA_n) \leq \tau(A_n) = \mu(A_n) < \infty.$$

Let $B_n \in \sigma(\mathscr{Q})$ s.t. $B_n \supseteq AA_n$ and $\tau(B_n) = \tau(AA_n) < \infty$. Let $B := \bigcup_{n=1}^{\infty} B_n$ we have

$$\tau(B - A) = \tau\left(\bigcup_{n=1}^{\infty} (B_n - AA_n)\right) \leq \sum_{n=1}^{\infty} \tau(B_n - AA_n) = 0.$$

□

Let $\mathscr{R}, \mathscr{A}, \mathscr{F}$ be the ring, algebra, σ-algebra generated by $\mathscr{Q}$, respectively. The outer measure τ restricts to a measure on each of these collections, denoted by μ_1, μ_2, μ_3. Each μ_i can generate an outer measure τ_i, but actually they're all the same as our original τ, since τ_i are "build up" from τ, intuitively τ_i cannot be any better than τ. (The proof says exactly the same thing, so I'll omit it)

Proposition 0.6

Let μ be a measure on an algebra $\mathscr{A}$. τ is the outer measure generated by μ, for all $A \in \sigma(\mathscr{A})$, if $\tau(A) < \infty$, then $\forall \varepsilon > 0$, $\exists B \in \mathscr{A}$ s.t. $\tau(A \Delta B) < \varepsilon$.

Remark 0.7 — In practice we often replace τ with a σ-finite measure μ on $\sigma(\mathscr{A})$. (Here σ-finite is on $\mathscr{A}$)

Proof. Choose $B_1, B_2, \dots \in \mathscr{A}$ s.t.

$$\hat{B} := \bigcup_{n=1}^{\infty} B_n \supseteq A, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(A) + \frac{\varepsilon}{2}.$$

Let N be a sufficiently large number, $B := \bigcup_{n=1}^{N} B_n \in \mathscr{A}$,

$$\tau(A \backslash B) \leq \tau\left(\bigcup_{n=N+1}^{\infty} B_n\right) \leq \sum_{n=N+1}^{\infty} \tau(B_n) \leq \frac{\varepsilon}{2}.$$

As $\tau(B \backslash A) \leq \tau(\hat{B} \backslash A) < \frac{\varepsilon}{2}$, $\tau(A \Delta B) < \varepsilon$. □

Example 0.8

Consider the Bernoulli test, recall $C_{i_1,\dots,i_n}$ we defined earlier. A measure(probability) μ is defined on the semi-ring $\{C_{i_1,\dots,i_n}\} \cup \{\emptyset, X\}$, then it can extend uniquely to the σ-algebra generated by it. This is how the probability of Bernoulli test comes from.

Let $(X, \mathscr{F}, P)$ be a probability space, $A_1, A_2, \dots \in \mathscr{F}$. We define the **tail σ-algebra** $\mathscr{T}$:

$$\mathscr{G}_n := \sigma(\{A_{n+1}, A_{n+2}, \dots\}), \quad \mathscr{T} := \bigcap_{n=1}^{\infty} \mathscr{G}_n.$$

Let $f_1, f_2, \dots$ be random variable, the tail σ-algebra generated by them is defined similarly:

$$\mathscr{G}_n := \sigma(\{f_{n+1}, f_{n+2}, \dots\}), \quad \mathscr{T} := \bigcap_{n=1}^{\infty} \mathscr{G}_n.$$

Theorem 0.9 (Kolmogorov's 0-1 law)

If $A_1, A_2, \dots \in \mathscr{F}$ are independent, then $P(A) \in \{0, 1\}$, $\forall A \in \mathscr{T}$.

Proof. Let $\mathscr{F}_n := \sigma(\{A_1, \dots, A_n\})$ and $\mathscr{G}_n$. They are clearly independent.

Note that $\mathscr{A} := \bigcup_{n=1}^{\infty} \mathscr{F}_n$ is an algebra.

Let $\mathscr{H} := \sigma(\mathscr{A}) \supseteq \mathscr{G}_n \supseteq \mathscr{T}$.

Hence $\forall A \in \mathscr{T} \subset \mathscr{H}$, $\forall \varepsilon > 0$, exists $B \in \mathscr{A}$ s.t. $P(A \Delta B) < \varepsilon$, so

$$P(A) - P(AB) \leq \varepsilon, \quad |P(A) - P(B)| \leq \varepsilon.$$

Since $B \in \mathscr{F}_n$ for some n, thus it is independent to A.

$$|P(A) - P(A)^2| \leq |P(A) - P(AB)| + |P(AB) - P(A)^2| \leq 2\varepsilon.$$

Let $\varepsilon \to 0$, we'll get $P(A) \in \{0, 1\}$. □

Remark 0.10 — When A_i's are replace by random variables, this theorem also holds.

Example 0.11

finite Markov chain

§0.1 The completion of measure spaces

Let $(X, \mathscr{F}, \mu)$ be a measure space, and

$$\widetilde{\mathscr{F}} := \{A \cup N : A \in \mathscr{F}, \exists B \in \mathscr{F} \ s.t. \ \mu(B) = 0, N \subset B\}.$$

Another way to define it is: $\widetilde{\mathscr{F}} := \{A \backslash N\}$, since

$$A \cup N = A + NA^c = (A \cup B) \backslash (BA^c \backslash N);$$

$$A \backslash N = A - NA = (A \backslash B) + (BA \backslash N).$$

In fact, we can do even more: $\widetilde{\mathscr{F}} := \{A \Delta N\}$.

Next we define the measure

$$\widetilde{\mu}(A \cup N) := \mu(A), \quad \forall A \cup N \in \widetilde{\mathscr{F}}$$

We need to check several things:

- $\widetilde{\mathscr{F}}$ is a σ-algebra.
- $\widetilde{\mu}$ is well-defined.
- $(X, \widetilde{\mathscr{F}}, \widetilde{\mu})$ is a complete measure space.

Remark 0.12 — The mesure $\widetilde{\mu}$ is the *minimal complete extension* of μ, i.e. if $(X, \mathscr{G}, \nu)$ is another complete extension, then

$$\nu(B) = \mu(B) = 0 \implies \forall N \subset B, N \in \mathscr{G}, \nu(N) = 0.$$

$$\mu(A) = \nu(A) \leq \nu(A \cup N) \leq \nu(A) + \nu(N) = \nu(A).$$

Thus $\mathscr{G} \supseteq \widetilde{\mathscr{F}}$ and $\nu(A) = \widetilde{\mu}(A)$ for $A \in \widetilde{\mathscr{F}}$.

Therefore we call $(X, \widetilde{\mathscr{F}}, \widetilde{\mu})$ the **completion** of $(X, \mathscr{F}, \mu)$.

Obviously $\emptyset \in \widetilde{\mathscr{F}}$; For $A \cup N \in \widetilde{\mathscr{F}}$, $(A \cup N)^c = A^c - A^c N \in \widetilde{\mathscr{F}}$.

$$\bigcup_{n=1}^{\infty} (A_n \cup N_n) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} N_n, \quad N = \bigcup_{n=1}^{\infty} N_n \subset \bigcup_{n=1}^{\infty} B_n.$$

Thus $\widetilde{\mathscr{F}}$ is a σ-algebra.

For $\widetilde{\mu}$, if $A_1 \cup N_1 = A_2 \cup N_2$,

$$\mu(A_1) = \mu(A_1 \cup B_2) \geq \mu(A_2).$$

Last we prove the countable additivity of $\widetilde{\mu}$. It's easy to check, so left out.

For the completeness, if $C \subset A \cup N, \mu(A) = 0$, then $C \subset A \cup B$ which is null.

Combining with the previous results we have

Theorem 0.13

Let τ be the outer measure generated by μ, a σ-finite measure on a semi-ring $\mathscr{Q}$. We have $(X\mathscr{F}_\tau, \tau)$ is the completion of $(X, \sigma(\mathscr{Q}), \tau)$.

Proof. Let $\mathscr{F} = \sigma(\mathscr{Q})$, we'll prove that $\widetilde{\mathscr{F}} = \mathscr{F}_\tau$.

Since $(X, \mathscr{F}_\tau, \tau)$ is complete, we have $\mathscr{F}_\tau \supseteq \widetilde{\mathscr{F}}$.

For all $C \in \mathscr{F}_\tau$, it suffices to prove $C = A + N$ for some $A \in \mathscr{F}, N \subset B$ with B null.

Since $C^c \in \mathscr{F}_\tau$, $\exists B \in \mathscr{F}$ s.t.

$$B \supseteq C^c, \quad \tau(B \backslash C^c) = 0.$$

□

§0.2 Distributions

Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right continuous function (called a **quasi-distribution function**). Let ν be the measure on $\mathscr{Q}_\mathbb{R}$,

$$\nu : (a, b] \mapsto \max\{F(b) - F(a), 0\}.$$

Let τ be the outer measure generated by ν. We call the sets in $\mathscr{F}_\tau$ to be the Lebesgue-Stieljes measurable sets (L-S measurable), a measurable function

$$f : (\mathbb{R}, \mathscr{F}_\tau) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$$

is L-S measurable, and $\tau\big|_{\mathscr{F}_\tau}$ is the L-S measure.

In fact finite L-S measures and the quasi-distribution functions are 1-1 correspondent (ignoring the difference of a constant), since $\mathscr{B}_\mathbb{R} = \sigma(\mathscr{Q}_\mathbb{R})$, $(\mathbb{R}, \mathscr{F}_\tau, \tau)$ is the completion of $(\mathbb{R}, \mathscr{B}_\mathbb{R}, \tau)$, and $\mu_F = \tau\big|_{\mathscr{B}_\mathbb{R}}$ is the unique extension of ν.

Conversely, given a measure μ on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$, if $\mu((a, b]) < \infty$ for all $a < b$, then $\mu = \mu_F$, where

$$F = F_\mu : x \mapsto \mu((-\infty, x]), \quad x \in \mathbb{R}.$$

We say a probability measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ is a **distribution**. Let $F : \mathbb{R} \to \mathbb{R}$ be a quasi-distribution function, if F satisfies:

$$F(-\infty) := \lim_{x \to -\infty} F(x) = 0, \quad F(+\infty) := \lim_{x \to +\infty} F(x) = 1,$$

then we say F is a distribution function (d.f.).

From the previous example we know distribution and d.f. are one-to-one correspondent.

Theorem 0.14

Let $g : (X, \mathscr{F}) \to (Y, \mathscr{S})$, μ is a measure on $\mathscr{F}$. Let

$$\nu(B) := \mu(g^{-1}(B)) = \mu \circ g^{-1}(B), \quad \forall B \in \mathscr{S}.$$

Then ν is a measure on $\mathscr{S}$.

Proof. Trivial. Just check the definition one by one. □

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $f : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}_{\mathscr{R}})$. We say

$$P \circ f^{-1} : B \mapsto P(f \in B)$$

is the **distribution** of f, denoted by μ_f, i.e. $\mu_f(B) = P(f \in B)$ for Borel sets B.

If $\mu_f = \mu$, we say f obeys the distribution μ, denoted by $f \sim \mu$.

Let $F_f = F_{\mu_f}$ be the distribution function of f.

$$F_f := \mu_f((-\infty, x]) = P(f \leq x), \quad x \in \mathbb{R}.$$

We can also say f obeys F_f, denoted by $f \sim F_f$.

If $F_f = F_g$, then we say f and g is **equal in distribution**, denoted by $f \overset{d}{=} g$.

Theorem 0.15

Any d.f. is the distribution function of some random variable.

Proof. Let $\Omega = \mathbb{R}, \mathscr{F} = \mathscr{B}_{\mathbb{R}}, P = \mu_F$, and $f = \text{id}$. It's clear that the distribution function of f is precisely F. □

§0.3 The convergence of measurable functions

Let $(X, \mathscr{F}, \mu)$ be a measure space.

For any statement, if there exists null set N s.t. it holds for all $x \in N^c$, then we say this statement holds *almost everywhere*. (Often abbreviated as *a.e.*)

Definition 0.16. If a sequence of functions f_n satisfies

$$\mu\left(\lim_{n\to\infty} f_n \neq f\right) = 0,$$

(here f is finite a.e.) we say $\{f_n\}$ converges to f **almost everywhere**, denoted by $f_n \to f, a.e.$.

Definition 0.17. If $\forall \delta > 0$, $\exists A \in \mathscr{F}$ s.t. $\mu(A) < \delta$ and

$$\lim_{n\to\infty} \sup_{x \notin A} |f_n(x) - f(x)| = 0,$$

then we say $\{f_n\}$ converges to f **almost uniformly**, denoted by $f_n \to f, a.u.$.

If $f_n \to f, a.u.$, $\forall \varepsilon > 0$, $\exists m = m_k(\varepsilon)$ s.t. when $n \geq m$, $|f_n(x) - f(x)| < \varepsilon, \forall x \in C_k$, but we could have $\sup_k m_k(\varepsilon) = \infty$, thus $f_n \rightrightarrows f$ doesn't hold. e.g. $f_n(x) = x^n, f(x) = 0, x \in [0, 1), f(1) = 1$.

Proposition 0.18

$f_n \to f, a.u. \implies f_n \to f, a.e.$.

Proof. For all n, $\exists A_n$ s.t. $\mu(A_n) < \frac{1}{n}$, and $f_n \to f$ in A_n^c. Let $A := \bigcap_n A_n$.

Then $\{f_n \not\to f\} \cup \{|f| = \infty\} \subset A$, $\mu(A) = 0$, hence $f_n \to f, a.e.$. □

Proposition 0.19

$f_n \to f, a.e.$ iff $\forall \varepsilon > 0$,

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{|f_m - f| \geq \varepsilon\}\right) = 0.$$

Note: If $f(x) - g(x)$ is not defined, we regard it as $+\infty$.

Proof. Let $A_\varepsilon := \bigcap \bigcup \{|f_m - f| > \varepsilon\}$.

$$\left\{\lim_{n\to\infty} f_n \neq f\right\} \cup \{|f| = \infty\} = \bigcup_{k=1}^{\infty} A_{\frac{1}{k}} =\uparrow \lim_{k\to\infty} A_{\frac{1}{k}}.$$

□

Proposition 0.20

$f_n \to f, a.u.$ iff $\forall \varepsilon > 0$, we have

$$\downarrow \lim_{m\to\infty} \mu\left(\bigcup_{n=m}^{\infty} \{|f_n - f| \geq \varepsilon\}\right) = 0.$$

Proof. If $f_n \to f, a.u.$, $\forall \delta, \exists A \in \mathscr{F}$ s.t. $\mu(A) < \delta$ and $f_n \rightrightarrows f, x \in A^c$.

This means for any fixed ε, $\exists m$ s.t. when $n \geq m$, $x \notin A \implies |f_n(x) - f(x)| < \varepsilon$. Thus $A \supseteq \bigcup_{n=m}^{\infty} |f_n - f| \geq \varepsilon$.

Conversely, $\forall \delta > 0$, $\exists m_k$ s.t.

$$\mu\left(\bigcup_{n=m_k}^{\infty} \{|f_n - f| \geq \frac{1}{k}\}\right) < \frac{\delta}{2^k}.$$

Denote the above set by A_k, and $A = \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) < \delta$, and $f_n(x) \rightrightarrows f(x)$ for $x \in A^c$. □

Definition 0.21. If $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu(|f_n - f| \geq \varepsilon) = 0,$$

then we say $\{f_n\}$ converges to f **in measure**, denoted by $f_n \xrightarrow{\mu} f$.

Theorem 0.22

$$f_n \to f, a.u. \implies f_n \to f, a.e., \quad f_n \xrightarrow{\mu} f.$$

If $\mu(X) < \infty$, then

$$f_n \to f, a.u. \iff f_n \to f, a.e. \implies f_n \xrightarrow{\mu} f.$$

Theorem 0.23

$f_n \to f$ in measure iff for any subsequence of $\{f_n\}$, exists its subsequence $\{f_{n'}\}$ s.t.

$$f_{n'} \to f, a.u.$$

Proof. When $f_n \to f$ in measure, let $n_0 = 0$. Take $n_k > n_{k-1}$ inductively such that

$$\mu\left(\left\{|f_n - f| \geq \frac{1}{k}\right\}\right) \leq \frac{1}{2^k}, \quad \forall n \geq n_k.$$

Then $\forall \varepsilon > 0$, $\exists \frac{1}{m} < \varepsilon$, $\{|f_{n_k} - f| \geq \varepsilon\} \subset \{|f_{n_k} - f| \geq \frac{1}{k}\}$,

$$\mu\left(\bigcup_{k=m}^{\infty} \{|f_{n_k} - f| \geq \varepsilon\}\right) \leq \mu\left(\bigcup_{k=m}^{\infty} \left\{|f_{n_k} - f| \geq \frac{1}{k}\right\}\right) \leq \frac{1}{2^{m-1}} \to 0.$$

Conversely, we assume for contradiction that $\exists \varepsilon > 0$ s.t. $\mu(\{|f_n - f| \geq \varepsilon\}) \not\to 0$.
So $\exists \delta > 0$ and subsequence $\{n_k\}$ s.t. $\mu(\{|f_{n_k} - f| \geq \varepsilon\}) > \delta$.
Hence there doesn't exist a subsequence $\{f_{n'}\}$ of $\{f_{n_k}\}$ s.t. $f_{n'} \to f, a.u.$. □

Example 0.24

Consider measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \lambda)$, the Lebesgue measure, $f_n = \mathbf{I}_{|x|>n}$, then

$$f_n \to 0, \forall x \implies f_n \to 0, a.e..$$

let $\varepsilon = 1$, it's clear that f_n doesn't converge to f in measure, hence not almost uniformly.

Example 0.25

Let $f_{k,i} = \mathbf{I}_{\frac{i-1}{k} < x \leq \frac{i}{k}}$, $i = 1, \dots, k$. It's clear that $f_{k,i} \to 0$ in measure, but not almost everywhere, and hence not almost uniformly.

§0.4 Probability space

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Here almost everywhere is renamed to almost surely.

Let F be a real function, let $C(F)$ be the continuous points of F.

Let $F, F_1, F_2, \dots$ be non-decreasing functions, if

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \forall x \in C(F),$$

then we say $\{F_n\}$ converge to F weakly, $F_n \xrightarrow{w} F$.

Let $F, F_1, F_2, \dots$ be distribution functions, $f_n \sim F_n$,

Definition 0.26. If $F_n \xrightarrow{w} F$, then we say $\{f_n\}$ converge to F in distribution, denoted by $f_n \xrightarrow{d} F$. For $f \sim F$, we can also write $f_n \xrightarrow{d} f$.

Theorem 0.27
$f_n \xrightarrow{P} f \implies f_n \xrightarrow{d} f.$

Proof.

$$\begin{aligned} P(h \le y) &\le P(h \le y, |h-g| < \varepsilon) + P(h \le y, |h-g| \ge \varepsilon) \\ &\le P(g \le y + \varepsilon) + P(|h-g| \ge \varepsilon). \end{aligned}$$

Let $F_n(x) = P_n(f \le x)$ Let $h = f_n$, $g = f$, $y = x$.

$$\limsup_{n\to\infty} F_n(x) \le F(x+\varepsilon), \quad \forall \varepsilon > 0.$$

Thus $\limsup_{n\to\infty} F_n(x) \le F(x)$. TODO ☐

Theorem 0.28 (Skorokhod)
If $f_n \xrightarrow{d} f$, then exists a probability space $(\widetilde{X}, \widetilde{\mathscr{F}}, \widetilde{P})$, with random variables $\{\tilde{f}_n\}$ and $\tilde{f}$, such that

$$\tilde{f}_n \overset{d}{=} f_n, \tilde{f} \overset{d}{=} f, \quad \tilde{f}_n \to \tilde{f}, a.s.$$

Proof. If $F_n \to F$ weakly, then $F_n^{\leftarrow} \to F^{\leftarrow}$ weakly. (Prove this yourself!)
Since $\mathbb{R} \backslash C(F_n^{\leftarrow})$ is countable, TODO ☐

If f is defined almost everywhere, we can extend it to $\tilde{f} = f \cdot \mathbf{I}_{N^c}$. So from now on when we talk about $f = g$, we mean $f = g, a.e.$.

§0.5 Review of first two sections

Here we list some concepts so that you can recall their definition and properties.
Collections of sets:

- π-system
- Semi-ring
- Ring, algebra
- σ-algebra
- Monotone class, λ-system

Measure:

- σ-finite
- Outer measure
- Caratheodory condition, measurable sets
- Measure extension, semi-ring $\to$ σ-algebra
- Complete measure space, completion

- For $\mathscr{F} = \sigma(\mathscr{A})$, $\forall F \in \mathscr{F}$, $\varepsilon > 0, \exists A \in \mathscr{A}$ s.t. $F = A \Delta N_\varepsilon$, $\mu(N_\varepsilon) \leq \varepsilon$.

Functions:

- Measurable map
- $h \in \sigma(g) \implies h = f \circ g$ for some f.
- Typical method, simple non-negative functions $\to$ measurable functions
- Almost uniformly, almost everywhere, converge in measure