Measure Theory

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Definition 0.1 (Generalize real numbers)**.** Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Similarly we can assign an order to $\overline{\mathbb{R}}$.

For the calculations, we assign 0 to $0 \cdot \pm\infty$, and $\infty - \infty$, $\frac{\infty}{\infty}$ is undefined.

For all $a \in \overline{\mathbb{R}}$, define $a^+ = \max\{a, 0\}, a^- = \max\{-a, 0\}$, so $a = a^+ - a^-$.
Define the Borel σ-algebra on $\overline{\mathbb{R}}$:

$$\mathscr{B}_{\overline{\mathbb{R}}} := \sigma(\mathscr{B}_{\mathbb{R}} \cup \{+\infty, -\infty\}).$$

For any set A, $A \in \mathscr{B}_{\overline{\mathbb{R}}} \iff A = B \cup C$, where $B \in \mathscr{B}_{\mathbb{R}}, C \subset \{+\infty, -\infty\}$.

Definition 0.2 (Measurable functions)**.** We say a function f is **measurable** if

$$f : (X, \mathscr{F}) \to (\overline{\mathbb{R}}, \mathscr{B}_{\overline{\mathbb{R}}}).$$

A **random variable (r.v.)** is a measurable map to $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.

Measurable functions are in fact random variables that can take $\pm\infty$ as its value.

Theorem 0.3

Let $(X, \mathscr{F})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$ if and only if

$$\{f \leq a\} \in \mathscr{F}, \quad \forall a \in \mathbb{R}.$$

Proof. Just note that these sets can generate $\mathscr{B}_{\overline{\mathbb{R}}}$.
Let $\mathscr{E} = \{[-\infty, a] : \forall a \in \mathbb{R}\}$. Then

$$f \text{ measurable} \iff \sigma(f) = f^{-1}\mathscr{B}_{\overline{\mathbb{R}}} = f^{-1}\sigma(\mathscr{E}) \subset \mathscr{F} \iff \sigma(f^{-1}\mathscr{E}) \subset \mathscr{F}.$$

$\square$

Example 0.4

The contant functions are measurable; the indicator functions of a measurable set are measurable $\implies$ *step functions* are measurable.

We say a function f is **Borel function** if it's a measurable function from Borel measurable space to itself.

Corollary 0.5

If f, g are measurable functions, then $\{f = a\}, \{f > g\}, \ldots$ are measurable sets.

Theorem 0.6

The arithmetic of measurable funtions are also measurable functions (if they are well-defined).

Proof. Here we only proof $f + g$ is measurable for f, g measurable. For all $a \in \mathbb{R}$, decompose $\{f + g < a\}$ to $A_1 \cup A_2 \cup A_3$:

$$A_1 := \{f = -\infty, g < +\infty\} \cup \{g = -\infty, f < +\infty\} \in \mathscr{F};$$

$$A_2 := \{f = +\infty, g > -\infty\} \cup \{g = +\infty, f > -\infty\} \in \mathscr{F};$$

$$A_3 := \{f < a - g\} \cap \{f, g \in \mathbb{R}\} = \left(\bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g < a - r\}) \right) \cap \{f, g \in \mathbb{R}\} \in \mathscr{F}.$$

$\square$

Remark 0.7 — All the measurable functions (or random variables) constitute a vector space.

Theorem 0.8

The limit inferior and limit superior of measurable functions are measurable.

Proof. If $f_1, f_2, \ldots$ are measurable, then $\inf f_n$ is measurable:

$$\left\{ \inf_{n \geq 1} f_n \geq a \right\} = \bigcap_{n=1}^{\infty} \{f_n \geq a\}.$$

Remark 0.9 — In particular, f measurable $\implies$ f^+, f^- measurable.

Hence

$$\liminf_{n \to \infty} f_n = \lim_{N \to \infty} \inf_{n \geq N} f_n = \sup_{N \geq 1} \inf_{n \geq N} f_n,$$

which is clearly measurable. $\square$

Remark 0.10 — The inferior or superior of **countable** many measurable functions are measurable as well.

Definition 0.11 (Simple functions)**.** Let $(X, \mathscr{F})$ be a measurable space. A **measurable partition** of X is a collection of subsets $\{A_1, \dots, A_n\}$ with $\sum_{i=1}^n A_i = X$, and $A_i \in \mathscr{F}$.

A **simple function** is defined as

$$f = \sum_{i=1}^n a_i \mathbf{I}_{A_i}.$$

where $\{A_1, \dots, A_n\}$ is a measurable partition of X, and $a_i \in \mathbb{R}$.

It's clear that simple functions are measurable.

Theorem 0.12

Let f be a measurable function, there exists simple functions $f_1, \dots$ s.t. $f_n \to f$.

- If $f \geq 0$, we have $0 \leq f_n \leq f$;
- If f is bounded, we have $f_n \rightrightarrows f$.

Proof. This is a standard truncation. For $f \geq 0$, let

$$f_n = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbf{I}_{\{k \leq 2^n f \leq k+1\}} + n\mathbf{I}_{f \geq n}.$$

It's clear that $f_n \geq 0$, $f_n \uparrow$, and $f_n(x) \to f(x)$:

$$0 \leq f(x) - f_n(x) \leq \frac{1}{2^n}, \quad f(x) < n;$$

$$n = f_n(x) \leq f(x), \quad f(x) \geq n.$$

Therefore if f is bounded, when $n > \max f(x)$ we have $|f_n(x) - f(x)| < \frac{1}{2^n}$ for all $x \in X$.

For general measurable functions, just decompose f to $f^+ - f^-$. □

Theorem 0.13

Let $g : (X, \mathscr{F}) \to (Y, \mathscr{S})$. Let h be a map $X \to \mathbb{R}$.

Then $h : (X, g^{-1}\mathscr{S})$ iff $h = f \circ g$, where $f : (Y, \mathscr{S}) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.

Remark 0.14 — For $\overline{\mathbb{R}}$ or $[a, b]$, this theorem also holds.

Proof. There's a typical method for proving something related to measurable functions:

We'll prove the statement for $h \in \mathcal{H}_i$ in order:

- $\mathcal{H}_1$: indicator functions $h = \mathbf{I}_A$, $\forall A \in g^{-1}\mathscr{S}$;
- $\mathcal{H}_2$: non-negative simple functions;
- $\mathcal{H}_3$: non-negative measurable functions;
- $\mathcal{H}_4$: measurable functions.

When $h \in \mathcal{H}_1$, suppose $h = \mathbf{I}_A$, then

$$A = g^{-1}B, B \in \mathscr{S} \implies f = \mathbf{I}_B \text{ suffices.}$$

When $h = \sum_{i=1}^n a_i \mathbf{I}_{A_i} \in \mathcal{H}_2$, since $A_i \in g^{-1}\mathscr{S}$,

$$\exists B_i \in \mathscr{S} \quad s.t. \quad A_i = \{h = a_i\} = g^{-1}B_i.$$

Thus $f = \sum_{i=1}^n a_i \mathbf{I}_{B_i}$ is the desired function.

When $h \in \mathcal{H}_3$, $\exists h_1, h_2, \ldots \uparrow h$.

Assume $h_n = f_n \circ g$, let

$$f(y) := \begin{cases} \lim\limits_{n\to\infty} f_n(y), & \text{if it exists;} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 0.15 — Here we still need to prove f is measurable.

Hence for any $x \in X$,

$$h(x) = \lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} f_n(g(x)) = f(g(x)),$$

as f_n's limit must exist at $y = g(x)$.

So for general h, let $h = h^+ - h^-$ and we're done. NOTE: We need to assert that $\infty - \infty$ doesn't occur. □

Remark 0.16 — This is the typical method we'll use frequently in measure theory: to start from simple functions and extend to general functions.

§1 Measure spaces

§1.1 The definition of measure and its properties

The concept of "measure" is frequently used in our everyday life: length, area, weight and even propbability. They all share a similarly: the measure of a whole is equal to the sum of the measure of each part.

In the language of mathematics, let $\mathscr{E}$ be a collection of sets, and there's a function $\mu : \mathscr{E} \to [0, \infty]$ which stands for the measure.

countable additivity: Let $A_1, A_2, \dots \in \mathscr{E}$ be pairwise disjoint sets, and $\sum_{i=1}^\infty A_i \in \mathscr{E}$, then

$$\mu\left(\sum_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

Definition 1.1 (Measure)**.** Suppose $\emptyset \in \mathscr{E}$, if a non-negative function

$$\mu : \mathscr{E} \to [0, \infty]$$

satisfies countable additivity, and $\mu(\emptyset) = 0$, then we say μ is a **measure** on $\mathscr{E}$.

If $\mu(A) < \infty$ for all $A \in \mathscr{E}$, we say μ is finite. (In practice we'll just simplify this to $\mu(X) < \infty$)
If $\exists A_1, A_2, \dots \in \mathscr{E}$ are pairwise disjoint sets, s.t.

$$X = \sum_{n=1}^{\infty} A_n, \quad \mu(A_n) < \infty, \forall n.$$

Then we say μ is σ-finite.

There's a weaker version of countable additivity, that is **finite additivity**: If $A_1, \dots, A_n \in \mathscr{E}$, pairwise disjoint, and $\sum A_i \in \mathscr{E}$,

$$\mu\left(\sum_{i=1}^{n} A_i\right) = \sum_{1=i}^{n} \mu(A_i),$$

then we say μ is finite additive.

Subtractivity: $\mu(B - A) = \mu(B) - \mu(A)$, where $A, B, B - A \in \mathscr{E}$, and $\mu(A) < \infty$.

Proposition 1.2
Measure satisfies finite additivity and subtractivity.

Example 1.3 (Counting measure)
Let $\mu(A) = \#A$, $\forall A \in \mathscr{T}_X$. Then μ is a measure.

Example 1.4 (Point measure)
Let $(X, \mathscr{F})$ be a measurable space, define $\delta_x(A) = \mathbf{I}_A(x)$. Then we can define a measure

$$\mu(A) = \sum_{i=1}^{n} p_i \delta_{x_i}(A)$$

Example 1.5 (Length)
Let $\mathscr{E} = \mathscr{Q}_{\mathbb{R}} = \{(a, b] : a, b \in \mathbb{R}\}$, $a \leq b$, then $\mu((a, b]) = b - a$ gives a measure.

Another classical example is the so-called "coin space":
Let $X = \{x = (x_1, x_2, \dots) : x_i \in 0, 1, \forall n\}$.

$$C_{i_1, \dots, i_n} := \{x : x_1 = i_1, \dots, x_n = i_n\},$$

Let

$$\mathscr{Q} = \{\emptyset, X\} \cup \{C_{i_1, \dots, i_n} : n \in \mathbb{N}, i_1, \dots, i_n \in \{0, 1\}\}$$

be a semi-ring. Then $\mu(C_{i_1, \dots, i_n}) = \frac{1}{2^n}$ gives a measure.

We need to check the countable additivity, but actually this can be realized as a compact space and the C's are open sets, so in fact we only need to check finite additivity. (Or we can prove this explicitly)

Another more complex example: finite markov chain.

Proposition 1.6

Let $X = \mathbb{R}$, $\mathscr{E} = \mathscr{R}_{\mathbb{R}}$. $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing, right continuous, then $\mu((a,b]) = F(b) - F(a)$ gives a measure on $\mathscr{E}$.

Proof. First $\mu(\emptyset) = 0$, suppose

$$\sum_{i=1}^{\infty}(a_i, b_i] = (a, b].$$

Since every partial sum has measure at most $F(b_{n+1}) - F(a_1) < F(b) - F(a)$,
$\implies \sum_{i=1}^{n} \mu((a_i, b_i]) \le \mu((a,b])$.

For the reversed inequality, first we prove that for intervals

$$\bigcup_{i=1}^{n}(c_i, d_i] \supseteq (a, b] \implies \sum_{i=1}^{n} \mu((c_i, d_i]) \ge \mu((a, b]).$$

This can be easily proved by induction, WLOG $b_{n+1} = \max_i b_i$.

Our idea is to extend each $(a_i, b_i]$ a little bit to apply above inequality.

For all $\varepsilon > 0$, take $\delta_i > 0$ s.t.

$$\tilde{b}_i := b_i + \delta_i, \quad F(\tilde{b}_i) - F(b_i) \le \frac{\varepsilon}{2}.$$

Hence for all $\delta > 0$, $\bigcup_{i=1}^{\infty}(a_i, \tilde{b}_i) \supseteq [a + \delta, b]$, by compactness exists a finite open cover.

$$F(b) - F(a + \delta) \le \sum_{i=1}^{n} \left(F(\tilde{b}_i) - F(a_i)\right) \le \varepsilon + \sum_{i=1}^{\infty}(F(b) - F(a)).$$

Let $\varepsilon, \delta \to 0$ to conclude. □

Definition 1.7 (Measure space)**.** A triple $(X, \mathscr{F}, \mu)$ is called a **measure space**, if $(X, \mathscr{F})$ is a measurable space and μ is a measure on $\mathscr{F}$.

If $N \in \mathscr{F}$ s.t. $\mu(N) = 0$, we say N is a **null set**.

A probability space is a measure space $(X, \mathscr{F}, P)$ with $P(X) = 1$.

Example 1.8 (Discrete measure)

If X is countable, $p : X \to [0, \infty]$, $\mu(A) := \sum_{x \in A} p(x)$ is a measure.

There are other important properties which we think a sensible measure would have:

- Monotonicity: If $A, B \in \mathscr{E}$, $A \subset B$, then $\mu(A) \le \mu(B)$.
- Countable subadditivity: $A_1, A_2, \dots \in \mathscr{E}$,
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i).$$
- Lower continuity: $A_1, A_2, \dots \in \mathscr{E}$ and $A_n \uparrow A \in \mathscr{E}$.
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
- Similarly there's upper continuity (which requires $\mu(A_1) < \infty$).

Theorem 1.9

The measure on a semi-ring has all the above properties.

Proof. In fact,

- Finite additivity $\implies$ monotonicity, subtractivity;
- Countable additivity $\implies$ subadditivity, upper and lower continuity.

Here we only prove the subadditivity, since others are trivial.
Let $A_1, A_2, \cdots \in \mathscr{Q}$, and $\bigcup_{i=1}^{\infty} A_i \in \mathscr{Q}$.

$$B_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in r(\mathscr{Q}) \implies B_n = \sum_{k=1}^{k_n} C_{n,k}, \quad C_{n,k} \in \mathscr{Q}.$$

$$A_n \backslash B_n \in r(\mathscr{Q}) \implies A_n \backslash B_n = \sum_{l=1}^{l_n} D_{n,l}, \quad D_{n,l} \in \mathscr{Q}.$$

Thus by countable additivity,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} \mu(C_{n,k})\right) \\ &\leq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} \mu(C_{n,k}) + \sum_{l=1}^{l_n} \mu(D_{n,l})\right) = \sum_{n=1}^{\infty} \mu(A_n).\end{aligned}$$

Using similar technique we can deduce the upper and lower continuity. □

Theorem 1.10

Let μ be a set function on a ring with finite additivity, then $1 \iff 2 \iff 3 \implies 4 \implies 5$.

- μ is countablely additive;
- μ is countablely subadditive;
- μ is lower continuous;
- μ is upper continuous;
- μ is continuous at $\emptyset$.

§1.2 Outer measure

Once we construct a measure on a semi-ring, we want to extend it to a σ-algebra. Since we can't directly do this, we shall relax some of our restrictions, say reduce countable additivity to subadditivity.

Definition 1.11 (Outer measure)**.** Let $\tau : \mathscr{T} \to [0, \infty]$ satisfying:

- $\tau(\emptyset) = 0$;

- If $A \subset B \subset X$, then $\tau(A) \le \tau(B)$;
- (Countable subadditivity) $\forall A_1, A_2, \cdots \in \mathscr{T}$, we have

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \tau(A_n).$$

We call τ an **outer measure** on X.

It's easier to extend a measure on semi-ring to an outer measure:

Theorem 1.12

Let μ be a non-negative set function on a collection $\mathscr{E}$, where $\emptyset \in \mathscr{E}$ and $\mu(\emptyset) = 0$. Let

$$\tau(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(B_n) : B_n \in \mathscr{E}, \bigcup_{n=1}^{\infty} B_n \supseteq A\right\}, \quad \forall A \in \mathscr{T}.$$

By convention, $\inf \emptyset = \infty$. (μ *need not be a measure!*)

Then τ is called the outer measure generated by μ.

Proof. Clearly $\tau(\emptyset) = 0$, and $\tau(A) \le \tau(B)$ for $A \subset B$.

$$\bigcup_{n=1}^{\infty} B_n \supseteq B \implies \bigcup_{n=1}^{\infty} B_n \supseteq A.$$

For all $A_1, A_2, \cdots \in \mathscr{T}$, WLOG $\tau(A_n) < \infty$. Take $B_{n,k}$ s.t. $\bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n$, such that

$$\sum_{k=1}^{\infty} \mu(B_{n,k}) < \tau(A_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n,$$

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(B_{n,k}) + \varepsilon \le \sum_{n=1}^{\infty} \tau(A_n) + \varepsilon.$$

□

Example 1.13

Let $\mathscr{E} = \{X, \emptyset\}$, $\mu(X) = 1$, $\mu(\emptyset) = 0$. Then $\tau(A) = 1$, $\forall A \ne \emptyset$.

Example 1.14

Let $X = \{a, b, c\}$, $\mathscr{E} = \{\emptyset, \{a\}, \{a, b\}, \{c\}\}$. $\mu(A) = \#A$ for $A \in \mathscr{E}$.

Here something strange happens: $\tau(\{b\}) = 2$ instead of 1, and $\tau(\{b, c\}) = 3$ instead of 2.

In the above example, we found the set $\{b\}$ somehow behaves badly: if we divide $\{a,b\}$ to $\{a\}+\{b\}$, the outer measure is not the sum of two smaller measure.

Hence we want to get rid of this kind of inconsistency to get a proper measure:

Definition 1.15 (Measurable sets)**.** Let τ be an outer measure, if a set A satisfies *Caratheodory condition*:

$$\tau(D) = \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T},$$

we say A is **measurable**.

Remark 1.16 — Inorder to prove A measurable, we only need to check

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T}.$$

Let $\mathscr{F}_\tau$ be the collection of all the τ measurable sets,

Definition 1.17 (Complete measure space)**.** Let $(X, \mathscr{F}, \mu)$ be a measure space, if for all null set A, and $\forall B \subset A, B \in \mathscr{F} \implies \mu(B) = 0$, we say $(X, \mathscr{F}, \mu)$ is **complete**.

Theorem 1.18 (Caratheodory's theorem)

Let τ be an outer measure, then $\mathscr{F} := \mathscr{F}_\tau$ is a σ-algebra, and $(X, \mathscr{F}, \tau)$ is a complete measure space.

Proof. First we prove $\mathscr{F}$ is an algebra:

Note $\emptyset \in \mathscr{F}$, and $\mathscr{F}$ is closed under completements.

For measurable sets A_1, A_2,

$$\begin{aligned}
\tau(D) &= \tau(D \cap A_1) + \tau(D \cap A_1^c) \\
&= \tau(D \cap A_1 \cap A_2) + \tau(D \cap (A_1) \cap A_2^c) + \tau(D \cap A_1^c) \\
&= \tau(D \cap (A_1 \cap A_2)) + \tau(D \cap (A_1 \cap A_2)^c).
\end{aligned}$$

So $A_1 \cap A_2$ is measurable.

Secondly, we prove $\mathscr{F}$ is a σ-algebra.

Let $A_1, A_2, \cdots \in \mathscr{F}$,

$$B_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in \mathscr{F},$$

Then B_i pairwise disjoint and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Let $B_f = \bigcup_{i=1}^{\infty} B_i$.

It's sufficient to prove

$$\tau(D) \geq \tau(D \cap B_f) + \tau(D \cap B_f^c).$$

Let $D_n = \sum_{i=1}^{n} B_i \cap D$, $D_f = D \cap B_f$, $D_\infty = D \backslash D_f$.

Since B_i are measurable,

$$\tau(D) = \tau(D_n) + \tau(D \backslash D_n) \geq \tau(D_n) + \tau(D_\infty) = \sum_{i=1}^{n} \tau(D \cap B_i) + \tau(D_\infty).$$

Now we take $n \to \infty$,

$$\tau(D) \geq \sum_{i=1}^{\infty} \tau(D \cap B_i) + \tau(D_\infty) \geq \tau\left(D \cap \sum_{i=1}^{\infty} B_i\right) + \tau(D_\infty).$$

Where the last step follows from countable subadditivity.

This implies B_f measurable $\implies \mathscr{F}$ is a σ-algebra.

Next we prove $\tau|\mathscr{F}$ is a measure: Just let $D = \sum_{i=1}^{\infty} B_i$ in the previous equation.

Last we prove $(X, \mathscr{F}, \tau)$ is complete:

If $\tau(A) = 0$, $\tau(D) \geq \tau(D \cap A^c) = \tau(D \cap A) + \tau(D \cap A^c)$. Thus $A \in \mathscr{F}$. □

§1.3 Measure extension

Definition 1.19 (Measure extension)**.** Let μ, ν be measures on $\mathscr{E}$ and $\overline{\mathscr{E}}$, and $\mathscr{E} \subset \overline{\mathscr{E}}$. If

$$\nu(A) = \mu(A), \quad \forall A \in \mathscr{E},$$

we say ν is a extension of μ on $\overline{\mathscr{E}}$.

If we start from a measure μ on $\mathscr{E}$, ideally, μ can generate an outer measure τ, and we can take $\mathscr{F}_\tau$ to construct a mesaure space.

However, things could go wrong:

Example 1.20

Let $X = \{a, b, c\}, \mathscr{E} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ with

$$\mu(\emptyset) = 0, \mu(\{a, b\}) = 1, \mu(\{b, c\}) = 1, \mu(X) = 2.$$

Then μ is a measure on $\mathscr{E}$, and the outer measure

$$\tau(\emptyset) = 0.$$

Observe that $\mathscr{F}_\tau = \{\emptyset, X\}$, so in this case $\tau|_{\mathscr{F}}$ is the trivial measure.

Example 1.21

Let $X = \mathbb{R}$, $\mathscr{E} = \{(a, b] : a < b, a, b \in \mathbb{R}\}$. Let $\mu(\emptyset) = 0$, and $\mu(A) = \infty$ for $A \neq \emptyset$.

Then μ can be extend to the Borel σ-algebra on $\mathbb{R}$ with $\mu_\alpha = \sum_{q \in \mathbb{Q}} \alpha \delta_q$, $\forall \alpha \geq 0$. So the extension is not unique.

Therefore in order to get a "proper" extension, we must put some requirements on both the starting collection and the set function μ.

Proposition 1.22

Let $\mathscr{P}$ be a π system. If two measures μ, ν on $\sigma(\mathscr{P})$ satisfying

$$\mu|_{\mathscr{P}} = \nu|_{\mathscr{P}}, \quad \mu|_{\mathscr{P}} \text{ is } \sigma\text{-finite},$$

Then $\mu = \nu$.

Proof. Let $A_1, A_2, \cdots \in \mathscr{P}$ s.t. $X = \sum_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$.

Fix n, let $B = A_n$, we want to prove that

$$\mu(B \cap A) = \nu(B \cap A), \quad \forall A \in \sigma(\mathscr{P}).$$

Let $B \in \mathscr{P}$ with $\mu(B) < \infty$,

$$\mathscr{L} := \{A \in \sigma(\mathscr{P}) : \mu(A \cap B) = \nu(A \cap B)\}.$$

We'll prove $\mathscr{L}$ is a λ system, so that $\mathscr{L} \supseteq \sigma(\mathscr{P})$.

Suppose $A_1, A_2 \in \mathscr{L}$ and $A_1 \supseteq A_2$, by $\mu(B) < \infty$,

$$\mu((A_1 - A_2)B) = \mu(A_1 B) - \mu(A_2 B) = \nu(A_1 B - A_2 B) = \nu((A_1 - A_2)B).$$

So $A_1 - A_2 \in \mathscr{L}$.

Let $A_1, A_2, \cdots \in \mathscr{L}$ and $A_n \uparrow A$, then

$$\mu(AB) = \lim_{n\to\infty} \mu(A_n B) = \lim_{n\to\infty} \nu(A_n B) = \nu(AB).$$

Which implies $A \in \mathscr{L}$.

Hence $\sigma(\mathscr{P}) \subset \mathscr{L}$, i.e.

$$\mu(A \cap A_n) = \nu(A \cap A_n), \quad \forall A \in \sigma(\mathscr{P}).$$

Therefore

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A_n) = \sum_{n=1}^{\infty} \nu(A \cap A_n) = \nu(A), \quad \forall A \in \sigma(\mathscr{P}).$$

$\square$