

Mathematical Analysis II

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§1 Introduction

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Contents of this course: Real analysis

§1.1 Recap

Definition 1.1 (Measurable space). Let X be a set and \mathcal{A} be a σ -algebra, we say (X, \mathcal{A}) is a measurable space if

- $\emptyset \in \mathcal{A}$;
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- If $A_k \in \mathcal{A}$, then $\bigcup_{k=1}^{+\infty} A_k \in \mathcal{A}$.

Outer measure m^* :

- $m^*(A) \geq 0$;
- $m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k)$;
- $m^*(A) \leq m^*(B)$ when $A \subset B$.

Caratheodry condition:

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c), \quad \forall T \subset X.$$

We define the measurable sets to be all the sets E satisfying above condition.

This implies the Lebesgue measure space $(\mathbb{R}^n, \mathcal{U}, m)$. It is a complete measure space, i.e. null sets are measurable.

Proposition 1.2

Properties of measurable sets:

- Let E be a measurable set, there exists a G_δ set G and a F_σ set F such that

$$E = G \setminus Z_1 = F \cup Z_2.$$

where Z_1, Z_2 are null sets.

- (Fatou's Lemma)

Measurable sets $E_k \nearrow E \implies \lim_{k \rightarrow \infty} m(E_k) = m(E)$ and

$$m\left(\liminf_{k \rightarrow \infty} E_k\right) \leq \liminf_{k \rightarrow \infty} m(E_k).$$

Definition 1.3 (Measurable function). Let f be a map from measurable space (X, \mathcal{A}) to (Y, \mathcal{B}) . We say f is measurable if

$$\forall B \in \mathcal{B}, \quad f^{-1}(B) \in \mathcal{A}.$$

In the case of real functions, this is equivalent to

$$\forall t \in \mathbb{R}, \quad \{x; f(x) > t\} \text{ is a measurable set.}$$

Proposition 1.4

Let f be a non-negative measurable function, $\exists \varphi_k \nearrow f$, where φ_k are simple functions.

For a general measurable function f , decompose it to $f = f_+ - f_-$.

Theorem 1.5 (Egorov)

Let E be a measurable set and $m(E) < \infty$, $f_n \rightarrow f, a.e.$, Then $\forall \varepsilon > 0$, there exists a closed set F_ε s.t. $m(E \setminus F_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on F_ε .

Theorem 1.6 (Lusin)

Let E be a measurable set and $m(E) < \infty$. Then $\forall \varepsilon > 0, \exists F_\varepsilon$ such that $f|_{F_\varepsilon}$ is continuous.

Convergence patterns:

- Converge **almost everywhere**: $f_n \rightarrow f, a.e.$
- Converge **almost uniformly**: $f_n \rightarrow f, a.u.$
- Converge **in measure**: $f_n \xrightarrow{m} f$

§2 Lebesgue integrals

§2.1 Recap: Definition of Lebesgue integrals

- Simple functions: $f = \sum_{k=1}^N a_k \chi_{E_k}$, define

$$\int f = \sum_{k=1}^N a_k m(E_k).$$

- $f : E \rightarrow \mathbb{R}^n$, where $m(E) < \infty$, f bounded. These functions form the set \mathcal{L}_0 . Then $\exists \varphi_k \rightarrow f$, φ_k simple, define

$$\int f = \lim_{k \rightarrow \infty} \int \varphi_k.$$

- Non-negative function:

$$\int f = \sup \left\{ \int g \mid 0 \leq g \leq f, g \in \mathcal{L}_0 \right\}.$$

- General functions:

$$\int f = \int f_+ - \int f_-.$$

$$\text{Integrable} \iff \int f_+, \int f_- < +\infty.$$

Relations between Riemann integrals and Lebesgue integrals:

- f is Riemann integrable on $[a, b]$ iff f bounded and the discontinuous points form a null set.
- If f is Riemann integrable on $[a, b]$, then two types of integral yield the same result.

§2.2 Dominated convergence theorem

The critical question of this section: If a sequence of functions f_n converges to f (almost everywhere), when does their integrals $\int f_n$ converge to $\int f$?

We'll discuss this issue under various conditions and reach the famous Dominated Convergence Theorem.

Theorem 2.1

Let E be a measurable set with finite measure. Measurable functions $f_n \rightarrow f, a.e.$ on E . Furthermore, f_n is uniformly bounded almost everywhere ($|f_n| < M, a.e.$). Then we have

$$\int_E |f_n - f| \rightarrow 0 \implies \lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Proof. By Egorov's Theorem, $\forall \varepsilon > 0$, there exists $F_\varepsilon \subset E$ s.t. $f_n \rightarrow f$ uniformly on F_ε , and $m(E \setminus F_\varepsilon) < \varepsilon$.

Hence

$$\begin{aligned} \int_E |f_n - f| &= \int_{F_\varepsilon} |f_n - f| + \int_{E \setminus F_\varepsilon} |f_n - f| \\ &\leq \varepsilon_0 m(E) + 2M\varepsilon, \end{aligned}$$

which proves the result. \square

Lemma 2.2 (Fatou's Lemma)

If $f_n \geq 0$, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Proof. For any $g \in \mathcal{L}_0$, $0 \leq g \leq \liminf_{n \rightarrow \infty} f_n$, we need to prove $\int g \leq \liminf_{n \rightarrow \infty} \int f_n$.

Let $g_k = \min\{f_k, g\}$, assume g is uniformly bounded so that $g_k \in \mathcal{L}_0$.

We'll prove $g_k \rightarrow g$: Assume by contradiction that $\exists \varepsilon_0 > 0, \exists x_0$ s.t.

$$g(x_0) - g_{k'}(x_0) > \varepsilon_0.$$

then $g(x_0) - f_{k'}(x_0) > \varepsilon_0$, which contradicts with $g \leq \liminf_{n \rightarrow \infty} f_n$.

Thus for sufficiently large k , $g_k(x) \leq g(x) \leq g_k(x) + \varepsilon_0, \implies g_k \rightarrow g$.

Therefore by [Theorem 2.1](#) (note $g_k \in \mathcal{L}_0$),

$$\begin{aligned} \int g &= \lim_{k \rightarrow \infty} \int g_k \\ &\leq \liminf_{k \rightarrow \infty} \int f_k, \end{aligned}$$

and we're done. □

Remark 2.3 — This is nearly identical to the measure version of Fatou's Lemma ([Proposition 1.2](#)). It shows some similarities between measure and integrals.

Theorem 2.4 (Beppo-Levi)

If non-negative functions $f_n \nearrow f$, we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof.

$$f_n \leq f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f.$$

By Fatou's Lemma ([2.2](#)),

$$\begin{aligned} \int \liminf_{n \rightarrow \infty} f_n &\leq \liminf_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int f_n, \\ \implies \int f &\leq \lim_{n \rightarrow \infty} \int f_n. \end{aligned}$$

Combining the two inequalities we get the desired equality. □

Corollary 2.5

Let f_n be non-negative functions, then

$$\int \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \int f_k(x).$$

Proposition 2.6

Let f be an integrable function, $\forall \varepsilon > 0$, we have:

- There exists a set B with finite measure s.t.

$$\int_{B^c} |f| < \varepsilon.$$

- (**Absolute continuity** of integrals) $\exists \delta > 0$ s.t. $\forall E$, if $m(E) < \delta$,

$$\int_E |f| < \varepsilon.$$

This is equivalent to

$$\lim_{m(E) \rightarrow 0} \int_E |f| = 0.$$

Proof. Let $f_N(x) = |f(x)|$ when $|x| \leq N$, $|f(x)| \leq N$, and $f_N(x) = 0$ otherwise. Then $f_N \nearrow |f|$, so by Beppo-Levi ([Theorem 2.4](#)), we get

$$\lim_{N \rightarrow \infty} \int f_N = \int |f|.$$

Let $B = \{x \mid |x| \leq N, |f(x)| \leq N\}$, when N gets sufficiently large, we must have $\int_{B^c} |f| < \varepsilon$.

For the second part, when N is sufficiently large we have $\int (|f| - f_N) < \frac{\varepsilon}{2}$, so

$$\begin{aligned} \int_E |f| &= \int_E f_N + \int_E (|f| - f_N) \\ &\leq N \cdot m(E) + \frac{\varepsilon}{2}. \end{aligned}$$

Let $\delta = \frac{\varepsilon}{2N}$ to finish. □

Now we take a look at what we get so far:

- If bounded functions $f_n \in \mathcal{L}_0$, $f_n \rightarrow f$, then $\int f_n \rightarrow \int f$.
- If f_n is non-negative, then $\int \liminf f_n \leq \liminf \int f_n$. (Fatou)

This corresponds to: $m(\liminf E_n) \leq \liminf m(E_n)$.

- If $f_n \nearrow f$, then $\int f_n \nearrow \int f$. (Beppo-Levi)

This corresponds to: $E_n \subset E_{n+1} \implies m(\bigcup E_n) = \lim m(E_n)$.

Finally we come to the famous Lebesgue dominated convergence theorem:

Theorem 2.7 (Lebesgue Dominated Convergence Theorem)

Functions $f_n \rightarrow f$, a.e., if there exists a function g s.t. $|f_n| \leq g$, a.e., then we have:

$$\int |f - f_n| \rightarrow 0. \quad \left(\lim_{n \rightarrow \infty} \int f_n = \int f \right)$$

Proof. By Fatou's lemma (2.2), $2g - |f_n - f|$ is non-negative,

$$\begin{aligned} \int \liminf (2g - |f_n - f|) &\leq \liminf \int (2g - |f_n - f|) \\ &\implies 0 \leq \liminf \left(- \int |f_n - f| \right) \end{aligned}$$

$\implies \limsup \int |f_n - f| \leq 0$, hence it must equal to 0. \square

Example 2.8

Non-examples of lebesgue dominated convergence theorem:

- Let $f_n = \chi_{[n, n+1]}$, $g = 1$, note that g is not integrable, so $\int f_n = 1$ while $f_n \rightarrow 0$.
- $f_n = \frac{1}{n} \chi_{[0, n]}$, $f_n \rightarrow 0$, $\int f_n = 1 \not\rightarrow 0$. Since $g(x) = \min\{\frac{1}{x}, 1\}$, which isn't integrable.
- $f_n = n \chi_{(0, \frac{1}{n})}$, $f_n \rightarrow 0$, $\int f_n = 1 \not\rightarrow 0$. Here $g(x) = \frac{1}{x} \chi_{[0, 1]}$ is not integrable.

Example 2.9

Suppose that

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$$

holds for any measurable set E . Then

$$\liminf_{n \rightarrow \infty} f_n \leq f \leq \limsup_{n \rightarrow \infty} f_n, a.e..$$

Proof. We only need to prove the case when $f = 0$.

For $\forall \varepsilon > 0$, define

$$E_n^\varepsilon = \{x : f_n(x) < -\varepsilon\}.$$

Note that

$$\liminf E_n^\varepsilon \subset \{x : \limsup f_n \leq -\varepsilon\} \subset \liminf E_n^{\frac{\varepsilon}{2}}.$$

Because when $\limsup f_n(x) \leq -\varepsilon$, $\exists N$ such that $\sup_{n > N} f_n(x) < -\frac{\varepsilon}{2}$

$$\implies f_n(x) < -\frac{\varepsilon}{2}, \forall n > N$$

This implies $x \in E_n^{\frac{\varepsilon}{2}}, \forall n > N$, so $x \in \liminf E_n^{\frac{\varepsilon}{2}}$.

We proceed with the proof, by using the condition $(E = \bigcap_{k \geq N} E_k^\varepsilon)$,

$$0 = \lim_{n \rightarrow \infty} \int_{\bigcap_{k \geq N} E_k^\varepsilon} f_n.$$

Since $x \in \bigcap_{k \geq N} E_k^\varepsilon \implies f_k(x) < -\varepsilon$, we deduce

$$0 = \lim_{n \rightarrow \infty} \int_{\bigcap_{k \geq N} E_k^\varepsilon} f_n \leq (-\varepsilon) \cdot m\left(\bigcap_{k \geq N} E_k^\varepsilon\right)$$

Hence $E = \bigcap_{k \geq N} E_k^\varepsilon$ is a null set. \square

§2.3 Integrable function space $\mathcal{L}^1(E)$

Definition 2.10 (\mathcal{L}^1 space). Denoted by $\mathcal{L}^1(E)$ the space consisting of all the integrable functions on E .

If $f = g$, a.e., then $\int |f - g| = 0$, we regard them as equivalent elements in $\mathcal{L}^1(E)$.

Observe that $\mathcal{L}^1(E)$ is a vector space, define the norm:

$$\|f\| = \int_E |f|.$$

It's easy to check that $\mathcal{L}^1(E)$ becomes a normal vector space.

Moreover, it's also a **Banach space** (complete normal vector space).

Theorem 2.11

$\mathcal{L}^1(E)$ is a Banach space.

Proof. Let f_n be a Cauchy sequence in $\mathcal{L}^1(E)$, suppose $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k}$.

Let $f = \sum_{k=0}^{\infty} (f_{n_{k+1}} - f_{n_k})$, where $f_{n_0} = 0$. Because

$$\int_E \sum_{k=0}^{\infty} |f_{n_{k+1}} - f_{n_k}| = \sum_{k=0}^{\infty} \int_E |f_{n_{k+1}} - f_{n_k}| \leq \sum_{k=0}^{\infty} 2^{-k} < +\infty.$$

so our f is well-defined (convergent). Now we compute

$$\begin{aligned} \|f - f_m\| &= \|f_m - f_{n_l}\| + \left\| \sum_{k=l}^{\infty} (f_{n_{k+1}} - f_{n_k}) \right\| \\ &\leq \|f_m - f_{n_l}\| + \sum_{k=l}^{\infty} \|f_{n_{k+1}} - f_{n_k}\| \\ &\leq \|f_m - f_{n_l}\| + 2^{-l+1}. \end{aligned}$$

As m gets large, $\|f_m - f_{n_l}\|$ and 2^{-l+1} both converge to 0, so $f_n \rightarrow f$ in $\mathcal{L}^1(E)$. □