Mathematical Analysis II

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Proof. Since $\Omega = \Phi(I)$, let y be the coordinates on I, $x = \Phi(y)$,

$$\int_{\Phi(I)} \nabla\varphi \, \mathrm{d}x = \int_I \nabla\varphi(\Phi(y))(\mathrm{d}\Phi)^{-1} J_\Phi \, \mathrm{d}y.$$

Let $A = \mathrm{d}\Phi$, WLOG $J_\Phi > 0$. Using the index notation and Einstein summation,

$$A_{kj}A^{ji} = A^{kj}A_{ji} = \delta_{ki}.$$

Thus

$$\partial_{y_j}\varphi A^{ji}|A| = \partial_{y_j}(\varphi A^{ji}|A|) - \varphi\partial_{y_j}(A^{ji}|A|)$$

Since $|A| = A_{kl}A^{kl}|A|$, $A_{kl} = \frac{\partial\Phi_k}{\partial y_l}$.

$$\begin{aligned}
\partial_{y_j}(A^{ji}|A|) &= |A|\partial_{y_j}A^{ji} + A^{ji}\partial_{y_j}|A| \\
&= |A|\partial_{y_j}A^{ji} + A^{ji}|A|\partial_{y_j}A_{kl}A^{kl} \\
&= |A|(\partial_{y_j}A^{ji} + \partial_{y_l}A_{kj}A^{kl}) \\
&= |A|(\partial_{y_j}A^{ji} - \partial_{y_j}A^{ji}) = 0.
\end{aligned}$$

Hence by our previous work,

$$\int_I \partial_{y_j}(\varphi A^{ji}|A|) \, \mathrm{d}y = \int_{\partial I} \varphi A^{ji}|A|\nu_j \, \mathrm{d}\sigma.$$

Putting this together for all i's, note that $\widetilde{\nu} = \frac{\nabla\Phi_n^{-1}}{|\nabla\Phi_n^{-1}|}$,

TODO $\square$

Let $(\phi_1, \ldots, \phi_n)$ be an element in the tangent boundle TM, it can represent a vector field

$$X = (\phi_1, \ldots, \phi_n) = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}.$$

Here $X \in TM, X(p) \in T_pM$.

We define the **divergence** of X to be

$$\mathrm{div}(X) = \sum_{i=1}^n \frac{\partial\phi_i}{\partial x_i}.$$

The Stolke's formula can be presented as divergence theorem:

Theorem 0.0.1 (Divergence theorem)
Let X be a vector field,
$$\int_D \operatorname{div}(X)\,\mathrm{d}x = \int_{\partial D} X\cdot\nu\,\mathrm{d}\sigma.$$

Another commonly-used operator is the **Laplace operator**:
$$\Delta = \operatorname{div}\cdot\nabla,\quad \Delta\phi = \operatorname{div}(\nabla\phi) = \operatorname{tr}(H_\phi) = \sum_{i=1}^n \frac{\partial^2\phi}{\partial x_i^2}.$$

When $n=2$, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$,
$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} X\cdot\nu\,\mathrm{d}\sigma.$$
Since ∂D is a curve $\gamma(t)$, so $\mathrm{d}\sigma = |\gamma'(t)|\,\mathrm{d}t$. Let $\gamma(t) = (x(t), y(t))$, then $\nu(t) = \frac{(y'(t), -x'(t))}{|\gamma'(t)|}$. Here we must take $\gamma(t)$ to be *counterclockwise* to ensure ν points outside of D.

Thus we get
$$\int_{\partial D} X\cdot\nu\,\mathrm{d}\sigma = \int_\gamma \frac{Py'(t) - Qx'(t)}{|\gamma'(t)|}|\gamma'(t)|\,\mathrm{d}t = \int_{\partial D}(P\,\mathrm{d}y - Q\,\mathrm{d}x).$$
This result is known as *Green's formula*.

This leads to the curve integrals of the second type: let $\gamma(t)\in\mathbb{R}^d$, X a vector field, we call the integral
$$\int_\gamma \sum_{i=1}^d X^i\,\mathrm{d}x_i = \int_\gamma X\cdot\mathrm{d}\gamma(t).$$
the **curve integral of the second type**.

When $n=3$, the result is called *Gauss's formula*, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$,
$$\int_D \operatorname{div}(X)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\partial D} X\cdot\nu\,\mathrm{d}\sigma.$$
Let $\gamma(u,v) = (x,y,z)$ be a parametrization of ∂D. We have two tangent vector γ_u, γ_v, so the normal vector is defined as $\nu = \frac{\gamma_u\times\gamma_v}{|\gamma_u\times\gamma_v|}$. Also $\mathrm{d}\sigma = |\gamma_u\times\gamma_v|\,\mathrm{d}u\,\mathrm{d}v$. After some computation we can get
$$\nu\,\mathrm{d}\sigma = (\mathrm{d}y\,\mathrm{d}z, \mathrm{d}z\,\mathrm{d}x, \mathrm{d}x\,\mathrm{d}y).$$
$$\int_D \operatorname{div}(X)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\partial D}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y).$$

§0.1 Differential forms

Let T_p^*M denote the *dual space* of T_pM, and $\mathrm{d}x_i$ is the dual basis of $\frac{\partial}{\partial x_i}$. The linear combination of $\mathrm{d}x_i$ are called **differential forms**, and a differential form on a manifold can be written as $\sum_{i=1}^n a_i\,\mathrm{d}x_i$, where a_i are functions on M.

We can construct differential forms of higher order, the order is $1\le k\le n$, called **k-forms**, which is a linear combination of
$$\mathrm{d}x_{i_1}\,\mathrm{d}x_{i_2}\cdots\mathrm{d}x_{i_k},\quad i_1 < i_2 < \cdots < i_k.$$

Here the product is *wedge product*, i.e. $\mathrm{d}x_i\,\mathrm{d}x_j = -\,\mathrm{d}x_j\,\mathrm{d}x_i$. We denote the space of all k-forms by $\Lambda^k(\Omega)$.

We can define the multiplication of forms: let $\omega_1 \in \Lambda^{k_1}, \omega_2 \in \Lambda^{k_2}$, then $\omega_1 \wedge \omega_2 \in \Lambda^{k_1+k_2}$ by multiplying the coefficients and $\mathrm{d}x_i$'s respectively.

There's also an operator called **exterior differentiation** $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$, where

$$\mathrm{d}(a\,\mathrm{d}x_{i_1}\cdots\mathrm{d}x_{i_k}) = \sum_{j=1}^{n} \frac{\partial a}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_{i_1}\cdots\mathrm{d}x_{i_k}.$$

This operator behaves like the derivatives very much:

$$\mathrm{d}(\omega_1+\omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2, \quad \mathrm{d}(\omega_1\omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1k_2}\omega_1 \wedge \mathrm{d}\omega_2.$$

Note that $\omega_1 \wedge \omega_2 = (-1)^{k_1k_2}\omega_2 \wedge \omega_1$, so when k_1k_2 is even, the wedge product may not be anti-symmetrical.

If we have a coordinate transformation $\Phi : (x_1, \ldots, x_n) \to (y_1, \ldots, y_n)$, we have

$$\mathrm{d}y_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_i}\,\mathrm{d}x_i,$$

thus $\mathrm{d}y_1\cdots\mathrm{d}y_n = J_\Phi\,\mathrm{d}x_1\cdots\mathrm{d}x_n$. Here J_Φ can be negative, so the differential forms already contains the information of orientation.

Theorem 0.1.1

Let ω be a differential form, $\mathrm{d}(\mathrm{d}\omega) = 0$.

Proof. Partial derivatives commute. □

Definition 0.1.2. Let ω be a differential form, if $\mathrm{d}\omega = 0$, we say ω is a **closed form**, if there exists ω_1 s.t. $\mathrm{d}\omega_1 = \omega$, then ω is a **exact form**.

The theorem above tells us that exact forms must be closed, but in general closed forms may not be exact, it depends on the topology structure of Ω.

Theorem 0.1.3 (Poincare)

The closed forms on $\mathbb{R}^n$ must be exact.

Proof. Use induction, when ω is an n-form this can be proved by computation.

For a generic form $\omega = \omega_1 + \mathrm{d}x_1 \wedge \omega_2$, where ω_1, ω_2 do not contain $\mathrm{d}x_1$. We want to find ω_3 s.t. $\mathrm{d}\omega_3 = \mathrm{d}x_1 \wedge \omega_2 + \omega_4$, where ω_3, ω_4 don't contain $\mathrm{d}x_1$ as well. (The construction is direct)

Since $\omega - \mathrm{d}\omega_3 = \omega_1 - \omega_4$, and

$$\mathrm{d}(\omega - \mathrm{d}\omega_3) = \mathrm{d}\omega = 0 \implies \mathrm{d}(\omega_1 - \omega_4) = 0.$$

Since $\mathrm{d}(\omega_1 - \omega_4) = 0$ and it doesn't contain $\mathrm{d}x_1$, hence all its coefficients can't contain $\mathrm{d}x_1$. Thus we can view it as a differential form in $\mathbb{R}^{n-1}$. □

Remark 0.1.4 — When Ω is simply connected, then all the closed 1-forms are exact. Also this is equivalent to the integral on any closed curves are 0.

We can rewrite Stolkes' formula using differential forms:

Theorem 0.1.5 (Stolkes' formula)
Let D be a $k+1$ dimensional orientable manifold, $\omega \in \Lambda^k(\mathbb{R}^n)$, we have

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$