Mathematical Analysis II

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§0.1 Lebesgue Differentiation theorem for monotone functions

Recall that Fundamental theorem of Calculus has a second part, Given any differentiable function $F(x)$, if $F'(x)$ Riemann integrable, then

$$F(b) - F(a) = \int_a^b F'(x)\,\mathrm{d}x.$$

When F has nice properties (smooth), then it has a generalization to higher dimensions (Stokes' theorem).

But in Lebesgue integration theory, the derivative of a function is hard to define. Hence we'll find an alternative for $F'(x)$.

Example 0.1

Consider Heaviside function $H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$

Then H is differentiable almost everywhere, but $\int_{-1}^1 H'(t)\,\mathrm{d}t = 0 \neq H(1) - H(-1)$.

Example 0.2

Consider Cantor-Lebesgue function F, similarly we have $F'(x) = 0, a.e.$, but $\int_0^1 F'(x)\,\mathrm{d}x = 0 \neq F(1) - F(0)$.

Definition 0.3 (Dini derivatives)**.** Let $f(x)$ be a measurable function, define

$$D^+(f)(x) = \limsup_{h>0,h\to 0} \frac{f(x+h)-f(x)}{h}, \quad D^-(f)(x) = \limsup_{h<0,h\to 0} \frac{f(x+h)-f(x)}{h}.$$

$$D_+(f)(x) = \liminf_{h>0,h\to 0} \frac{f(x+h)-f(x)}{h}, \quad D_-(f)(x) = \liminf_{h<0,h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Theorem 0.4 (Lebesgue Differentiation theorem for increasing functions)
Let f be an increasing function on $[a,b]$, then $F'(x)$ exists almost everywhere, and

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

Sketch of the proof. The outline of the proof is as follows:
Step 1: Decompose $F = F_c + J$, where F_c is continuous, J is a jump function.
Step 2: Prove F_c increasing and $J' = 0, a.e.$.
Step 3: Prove $D^+(F) < +\infty, a.e.$, $D^+(F) \le D_-(F), a.e.$, and $D^-(F) \le D_+(F), a.e.$. □

We proceed step by step.

Step 1 Denote $F(x+0) = \lim_{h\to 0^+} F(x+h), F(x-0) = \lim_{h\to 0^-} F(x+h)$.
Since F increasing, let $\{x_n\}$ be all the discontinuous points of F. Define:

$$j_n(x) = \begin{cases} 0, & x < x_n \\ \beta_n, & x = x_n \\ \alpha_n, & x > x_n \end{cases}$$

where $\alpha_n = F(x_n+0) - F(x_n-0), \beta_n = F(x_n) - F(x_n-0)$.
Hence the jump function

$$J_F(x) = \sum_{n=1}^{+\infty} j_n(x) \le \sum_{n=1}^{+\infty} \alpha_n = \sum_{n=1}^{+\infty} (F(x_n+0) - F(x_n-0)) \le F(b) - F(a)$$

is well-defined and increasing.

Lemma 0.5
$F - J_F$ is continuous and increasing.

Proof. First note that

$$\lim_{h\to 0^+} (F(x+h) - J_F(x+h)) = F(x+0) - \lim_{h\to 0^+} J_F(x+h) = F(x-0) - \lim_{h\to 0^+} J_F(x-h)$$

This can be derived from the definition of J_F: If F is continuous at x, the equality is obvious;
If $x = x_n$ for some n,

$$\lim_{h\to 0^+} J_F(x+h) = \sum_{x_k \le x_n} \alpha_k + \lim_{h\to 0^+} \sum_{x_n < x_k \le x_n + h} j_k(x+h) = \sum_{x_k \le x_n} \alpha_k$$

$$\lim_{h\to 0^+} J_F(x-h) = \lim_{h\to 0^+} \sum_{x_k < x_n - h} \alpha_k + \lim_{j\to 0^+} \sum_{x_k = x_n - h} \beta_k = \sum_{x_k < x_n} \alpha_k$$

Note that $\alpha_n = F(x_n+0) - F(x_n-0)$, thus $F - J_F$ is continuous.
Secondly,

$$F(x) - J_F(x) \le F(y) - J_F(y), \quad \forall a \le x \le y \le b.$$

Because

$$J_F(y) - J_F(x) = \sum_{x < x_j < y} \alpha_j + \sum_{x_k = y} \beta_k - \sum_{x_k = x} \beta_k \le \sum_{x < x_j < y} \alpha_j + F(y) - F(y-0) \le F(y) - F(x).$$

which means $F - J_F$ is increasing. □

Step 2

Proposition 0.6

The jump function $J(x)$ is differentiable almost everywhere, and $J'(x) = 0, a.e.$.

Proof. The Dini derivatives of $J(x)$ exist and are non-negative (since J is increasing).

$$\overline{D}(J)(x) = \max\{D^+(J)(x), D^-(J)(x)\}.$$

Let $E_\varepsilon = \{\overline{D}(J)(x) > \varepsilon > 0\}$. We'll prove E_ε is null for all ε. If $x \in E_\varepsilon$, $\exists h$ s.t.

$$\frac{J(x+h) - J(x)}{h} > \varepsilon \implies J(x+h) - J(x-h) > \varepsilon h.$$

Let $N \in \mathbb{N}$ s.t. $\sum_{n>N} \alpha_n < \frac{\varepsilon\delta}{10}$. Define $J_N(x) = \sum_{n>N} j_n(x)$.

$$E_{\varepsilon,N} = \{\overline{D}(J_N)(x) > \varepsilon\}, \quad E_\varepsilon \subset E_{\varepsilon,N} \cup \{x_1, \ldots, x_N\},$$

Since for $x \neq x_i$,

$$\begin{aligned}\overline{D}(J)(x) &= \limsup_{h\to 0} \frac{J(x+h) - J(x)}{h} \\ &= \limsup_{h\to 0} \left(\frac{J_N(x+h) - J_N(x)}{h} + \frac{1}{h}\sum_{n=1}^{N}(j_n(x+h) - j_n(x)) \right) = \overline{D}(J_N)(x).\end{aligned}$$

Next we need to control the measure of $E_{\varepsilon,N}$.

For all $y \in E_{\varepsilon,N}$, there exists sufficiently small h s.t. $J_N(y+h) - J_N(y) > h\varepsilon$. So the intervals $(y-h, y+h)$ is a covering of $E_{\varepsilon,N}$, and it can be controled using the value of J_N. Therefore we hope to find some *disjoint* intervals which cover certain ratio of $E_{\varepsilon,N}$.

Lemma 0.7

Let $\mathcal{B}$ be a collection of balls with bounded radius in $\mathbb{R}^d$. There exists countably many disjoint balls B_i s.t.

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Proof. Let $r(B)$ denote the radius of B. Take B_1 s.t. $r(B_1) > \frac{1}{2}\sup_{b\in\mathcal{B}} r(B)$.

The rest is the same as before. ☐

By lemma, there exists countably many disjoint intervals $(x_i + h_i, x_i - h_i)$ s.t.

$$\begin{aligned}m^*(E_{\varepsilon,N}) &\leq 5\sum_{i=1}^{\infty} 2h_i \\ &\leq 10\sum_{i=1}^{\infty} \varepsilon^{-1}(J_N(x_i+h_i) - J_N(x_i - h_i)) \\ &\leq 10\varepsilon^{-1}(J_N(b) - J_N(a)) < \delta.\end{aligned}$$

Hence $m^*(E_\varepsilon) \leq m^*(E_{\varepsilon,N}) < \delta \implies m^*(E_\varepsilon) = 0$, which gives $\overline{D}(J) = 0, a.e.$. ☐

Step 3 First we prove $D^+(F) < \infty, a.e.$.

Let $E_\gamma = \{x : D^+(F)(x) > \gamma\}$.

When $h \in [\frac{1}{n+1}, \frac{1}{n}]$:

$$\frac{F(x+h) - F(x)}{h} \leq \frac{n+1}{n} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}},$$
$$\geq \frac{n}{n+1} \frac{F(x+\frac{1}{n+1}) - F(x)}{\frac{1}{n+1}}.$$

Thus

$$D^+(F)(x) = \limsup_{h\to 0} \frac{F(x+h) - F(x)}{h} = \limsup_{n\to\infty} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}$$

is measurable.

Lemma 0.8 (Riesz sunrise lemma)

Let $G(x)$ be a continuous function on $\mathbb{R}$. Define

$$E = \{x : \exists h > 0, s.t.\ G(x+h) > G(x)\}.$$

Then E is open and $E = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where (a_i, b_i) are disjoint finite intervals s.t. $G(a_i) = G(b_i)$.

When G is defined on finite interval $[a, b]$, we also have $G(a) \leq G(b_1)$.

Proof. Note that E is open since G is continuous.

Take a maximum open interval $(a, b) \subset E$, i.e. $a, b \notin E$, so $G(a) \geq G(b)$.

Since $b \notin E, G(x) \leq G(b), \forall x > b$. If $G(a) > G(b)$, Let $G(a+\varepsilon) > G(b)$, as $a+\varepsilon \in E$, exists $h > 0$ s.t. $G(a+\varepsilon+h) > G(a+\varepsilon)$.

But G has a maximum on $[a+\varepsilon, b]$, say $G(c)$, we must have $c \neq a+\varepsilon, b$. This leads to a contradiction. □

Remark 0.9 — This lemma provides a better estimation than previous covering lemmas, since it directly claims that E can be broken into disjoint intervals.

For $x \in E_\gamma$, $\exists h > 0$ s.t. $F(x+h) - F(x) > \gamma h$, by Lemma 0.8 on $F(x) - \gamma x$,

$$m(E_\gamma) \leq \sum_{k=1}^{\infty}(b_k - a_k) \leq \gamma^{-1}\sum_{k=1}^{\infty}(F(b_k) - F(a_k)) \leq \gamma^{-1}(F(b) - F(a)).$$

Therefore when $\gamma \to \infty$, $m(E_\gamma) \to 0$.

The last part is $D^+(F) \leq D_-(F), a.e.$.

Similarly it's sufficient to prove the following set is null for all rational numbers $r < R$:

$$E_{r,R} = \{D^+(F)(x) > R, D_-(F)(x) < r\}.$$

Since $D^+(F)$ is measurable, $E_{r,R}$ is measurable. If $m(E_{r,R}) > 0$, we can restrict it to a smaller interval $[c, d] \subset [a, b]$ such that $d - c < \frac{R}{r} m(E_{r,R})$.

Let $G(x) = F(-x) + rx$, by Lemma 0.8 on $[-d, -c]$,

$$\{s : \exists h > 0, G(x+h) > G(x)\} = \bigcup_k (-b_k, -a_k).$$

Note that $-E_{r,R}$ is contained in the above set, and $G(-b_k) \leq G(-a_k) \iff F(b_k) - F(a_k) \leq r(b_k - a_k)$,

We use Lemma 0.8 again on each (a_k, b_k) and $F(x) - Rx$,

$$E_{r,R} \cap (a_k, b_k) \subset \{x : \exists h > 0, F(x+h) - F(x) \geq Rh\} = \bigcup_{l=1}^{\infty} (a_{k,l}, b_{k,l}).$$

Hence

$$\begin{aligned} m(E_{r,R}) &\leq \sum_{k,l=1}^{\infty} (b_{k,l} - a_{k,l}) \\ &\leq R^{-1} \sum_{k,l=1}^{\infty} (F(b_{k,l}) - F(a_{k,l})) \leq R^{-1} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) \\ &\leq R^{-1} r \sum_{k=1}^{\infty} (b_k - a_k) \leq R^{-1} r (d-c), \end{aligned}$$

which gives a contradiction! So $m(E_{r,R}) = 0$ for all rationals $r < R$. Therefore we're done by

$$m(\{D^+(F) > D_-(F)\}) \leq \sum_{r,R} m(E_{r,R}) = 0$$

Now we can complete the proof of Theorem 0.4. Here we state the theorem again:

Let F be an increasing function on $[a, b]$, then F is differentiable almost everywhere, and

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

Let $F_n(x) = n(F(x + \frac{1}{n}) - F(x))$, where $F(x) = F(b)$ for $x > b$. Since $F_n \geq 0$, by Fatou's Lemma, (we've already proved F is differentiable almost everywhere and $F' \geq 0$)

$$\begin{aligned} \int_a^b \liminf_{n\to\infty} F_n \,\mathrm{d}x &\leq \liminf_{n\to\infty} \int_a^b F_n \,\mathrm{d}x \\ \implies \int_a^b F'(x)\,\mathrm{d}x &\leq \liminf_{n\to\infty} \int_a^b n\left(F\left(x + \frac{1}{n}\right) - F(x)\right)\mathrm{d}x \\ &= \liminf_{n\to\infty} n\left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} F(x) - \int_a^b F(x)\right)\mathrm{d}x \\ &= \liminf_{n\to\infty} \left(F(b) - n\int_a^{a+\frac{1}{n}} F(x)\,\mathrm{d}x\right) \\ &\leq F(b) - F(a) \end{aligned}$$

§0.2 Absolute continuous functions

Definition 0.10 (Absolute continuity)**.** We say a function $F(x)$ is **absolutely continuous** on interval $[a, b]$, if $\forall \varepsilon > 0, \exists \delta > 0$, such that for all disjoint intervals $(a_k, b_k), k = 1, \ldots, N$ with

$$\sum_{k=1}^{N} (b_k - a_k) < \delta,$$

we must have

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon.$$

The space consisting of all the absolutely continuous functions on $[a, b]$ is denoted by $Ac([a, b])$.

Example 0.11

A C^1 function with bounded derivative or a Lipschtiz function is absolutely continuous.

Some obvious properties of absolutely continuous function F:

- F is continuous;
- F has bounded variation, i.e. $F \in BV$.
- F is differentiable almost everywhere, since $F = F_1 - F_2$, where F_1, F_2 are increasing.

  In fact we have
  $$T_F([a, b]) = \int_a^b |F'(x)| \, \mathrm{d}x.$$
- If N is a null set, then $F(N)$ is also null. In particular F maps measurable sets to measurable sets.

Proof of the last property. Take intervals (a_k, b_k) s.t. $N \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$. Since $F(N) \subset F(\bigcup(a_k, b_k))$,

$$|F(N)| \leq \sum_{k=1}^{\infty} |F([a_k, b_k])| \leq \sum_{k=1}^{\infty} |F(\tilde{b}_k) - F(\tilde{a}_k)| < \varepsilon.$$

□

Proposition 0.12

The space $Ac([a, b]) \subset BV([a, b])$, moreover it's an algebra, and it's a separable Banach space under the norm induced from BV.

Finally we come to the full generalization of Newton-Lebniz formula:

Theorem 0.13 (Fundamental theorem of Calculus)

A function $F \in Ac([a, b]) \implies F$ is differentiable almost everywhere, F' is integrable, and

$$F(x) - F(a) = \int_a^x F'(\tilde{x}) \, \mathrm{d}\tilde{x}, \quad \forall x \in [a, b].$$

Proof. Let $\tilde{F}(x) = F(a) + \int_a^b F'(y) \, \mathrm{d}y \in Ac([a, b])$ (by the absolute continuity of integrals).

We have $F - \tilde{F} \in Ac([a, b])$ and $(F - \tilde{F})' = 0, a.e.$.

Thus it suffices to prove the following theorem:

Theorem 0.14

Let $F \in Ac([a,b])$, and $F' = 0, a.e.$, then $F(a) = F(b)$, i.e. F is constant on $[a,b]$.

To prove this, we'll need Vitali covering theorem:

Definition 0.15 (Vitali covering)**.** Let $\mathcal{B} = \{B_\alpha\}$, where B_α are closed balls in $\mathbb{R}^d$. We say $\mathcal{B}$ is a **Vitali covering** of a set E, if $\forall x \in E, \forall \eta > 0$, exists $B_\alpha \in \mathcal{B}$ s.t. $m(B_\alpha) < \eta$, $x \in B_\alpha$.

Theorem 0.16 (Vitali)

Let $E \subset \mathbb{R}^d$ with $m^*(E) < \infty$, for any Vitali covering $\mathcal{B}$ of E and $\delta > 0$, exists disjoint balls $B_1, \ldots, B_n \in \mathcal{B}$, such that

$$m^*\left(E \backslash \bigcup_{i=1}^{n} B_i\right) < \delta.$$

Proof. For all $\varepsilon > 0$, exists an open set A s.t. $E \subset A$ and $m(A) < m^*(E) + \varepsilon < +\infty$.

Remove all the balls in $\mathcal{B}$ with radius greater than 1. Each time we take a ball B_i with radius greater than $\frac{1}{2}\sup_{B \in \mathcal{B}'} r(B)$, where $\mathcal{B}'$ are the remaining balls, and remove all the balls which intersect with B_i.

If we end up with finitely many balls $B_1, \ldots, B_n$, we must have $E \subset \bigcup_{i=1}^{n} B_i$, otherwise $x \notin B_i \implies \exists B \ni x$, $B \cap B_i = \emptyset$, contradiction!

If we take out countably many balls $B_1, B_2, \cdots \subset A$, since $\sum_{i=1}^{\infty} m(B_i) \le m(A) < \infty$, there exists N s.t. $\sum_{i=N+1}^{\infty} m(B_i) < 5^{-d}\delta$.

Now we only need to prove

$$E \backslash \bigcup_{i=1}^{N} B_i \subset \bigcup_{i>N} 5B_i.$$

□

Let $E = \{x : F'(x) = 0\}$, $\forall x \in E$, $\exists \delta(x) > 0$, s.t.

$$|F(y) - F(x)| < \varepsilon|y - x|, \forall |y - x| < \delta(x).$$

Hence $[x - h, x + h], 0 < h < \delta(x)$ is a Vitali covering of E. By Theorem 0.16, there exists finitely many disjoint intervals $[x_k - h_k, x_k + h_k] = I_k$ s.t.

$$m^*\left(E \backslash \bigcup_{k=1}^{N} I_k\right) < \varepsilon.$$

Assmue $a \le a_1 < b_1 < \cdots < a_N < b_N \le b$, by absolute continuity and $|F(b_k) - F(a_k)| < \varepsilon(b_k - a_k)$,

$$F(b) - F(a) \le \sum_{k=1}^{N} |F(b_k) - F(a_k)| + \sum_{k=0}^{N} |F(a_{k+1}) - F(b_k)| \le \varepsilon(b - a) + \delta.$$

Here we complete the proof of the generalized Fundamental theorem of Calculus. □

There's another version of this thoerem which looks like Newton-Lebniz formula more:

Theorem 0.17

Let F be a differentiable function on $[a,b]$, if F' is Lebesgue integrable, then

$$F(b)-F(a)=\int_a^b F'(x)\,\mathrm{d}x.$$

We need to prove a lemma first.

Theorem 0.18

Let F be real function on $[a,b]$, if F is differentiable on E, and $|F'|\le M$ in E, then

$$m^*(F(E))\le Mm^*(E).$$

Proof. For all $\varepsilon>0$, $x\in E$, $\exists\delta>0$,

$$\left|\frac{F(x+h)-F(x)}{h}-M\right|<\varepsilon,\quad \forall|h|<\delta.$$

So $[x-h,x+h]$ is a Vitali covering of E. By Vitali's theorem (0.16), exists disjoint intervals $I_i=[x_i-h_i,x_i+h_i]$ s.t.

$$m^*\left(E\backslash\bigcup_{i=1}^\infty I_i\right)=0,\quad \sum_{i=1}^\infty 2h_i\le m^*(E)+\varepsilon.$$

But for $y\in I_i$, $|F(y)-F(x_i)|\le(M+\varepsilon)h_i$, thus $m^*(F(I_i))\le 2(M+\varepsilon)h_i=(M+\varepsilon)|I_i|$.

$$\begin{aligned}
m^*(F(E))&\le m^*(F(E\cap\bigcup_{i=1}^\infty I_i))+m^*(F(E\backslash\bigcup_{i=1}^\infty I_i))\\
&\le\sum_{i=1}^\infty m^*(F(I_i))+m^*(F(E\backslash\bigcup_{i=1}^\infty I_i))\\
&\le(M+\varepsilon)(m^*(E)+\varepsilon)+m^*(F(E\backslash\bigcup_{i=1}^\infty I_i))
\end{aligned}$$

So it suffices to prove the case when E is null. Define

$$E_n=\left\{x\in E:|F(y)-F(x)|\le(M+\varepsilon)|y-x|,\forall|y-x|<\frac{1}{n}\right\}.$$

Observe that $E_n\nearrow E$ and $F(E_n)\nearrow F(E)$. There exists disjoint intervals $J_{n,k}$ s.t.

$$E_n\subset\bigcup_{k=1}^\infty J_{n,k},\quad \sum_{k=1}^\infty|J_{n,k}|\le\min\left\{\frac{1}{n},\varepsilon\right\}.$$

Thus

$$m^*(F(E_n))\le\sum_{k=1}^\infty m^*(F(E_n\cap J_{n,k}))\le\sum_{k=1}^\infty(M+\varepsilon)|J_{n,k}|\le\varepsilon(M+\varepsilon).$$

Taking $\varepsilon\to0$ we get $F(E_n)$ is null. So $F(E)=\lim_{n\to\infty}F(E_n)$ is null, which completes the proof. $\square$

Returning to the proof of the theorem, in fact we only need to prove

$$|F(b) - F(a)| \le \int_a^b |F'(x)| \,\mathrm{d}x,$$

since this implies F is absolutely continuous. For all $\varepsilon > 0$, let

$$E_n = \{x \in [a,b] : n\varepsilon \le |F'(x)| < (n+1)\varepsilon\}.$$

By our lemma, $m^*(F(E_n)) \le (n+1)\varepsilon m(E_n) \le \varepsilon m(E_n) + \int_{E_n} |F'(x)| \,\mathrm{d}x$.

Hence

$$\begin{aligned} |F(b) - F(a)| \le m(F([a,b])) &\le \sum_{n=0}^{\infty} m^*(F(E_n)) \\ &\le \varepsilon(b-a) + \int_a^b |F'(x)| \,\mathrm{d}x. \end{aligned}$$

Theorem 0.19

A rectifiable curve $\gamma(t) = (x(t), y(t))$ with x, y absolutely continuous has length

$$L(\gamma) = \int_a^b |\gamma'(t)| \,\mathrm{d}t.$$

Proof. Since $|\gamma(t_i) - \gamma(t_{i-1})| = |\int_{t_{i-1}}^{t_i} \gamma'(t) \,\mathrm{d}t| \le \int_{t_{i-1}}^{t_i} |\gamma'(t)| \,\mathrm{d}t$, thus $L(\gamma) \le \int_a^b |\gamma'(t)| \,\mathrm{d}t$.

$\forall \varepsilon > 0$, we can take a step function (with vector values) g s.t. $\gamma' = g + h$, and $\int_a^b |h| \,\mathrm{d}x < \varepsilon$.

Define

$$G(x) = G(a) + \int_a^x g(t) \,\mathrm{d}t, \quad H(x) = H(a) + \int_a^x h(t) \,\mathrm{d}t.$$

We have $\gamma(t) = G(t) + h(t)$, and $T_\gamma([a,b]) \ge T_G([a,b]) - T_H([a,b])$.

$$\begin{aligned} L(\gamma) = T_\gamma([a,b]) &\ge \int_a^b |g| \,\mathrm{d}t - \int_a^b |h| \,\mathrm{d}t \\ &\ge \int_a^b |\gamma'(t)| \,\mathrm{d}t - 2\int_a^b |h| \,\mathrm{d}t \\ &\ge \int_a^b |\gamma'(t)| \,\mathrm{d}t - 2\varepsilon. \end{aligned}$$

which gives the opposite inequality. □

Proposition 0.20 (substitution formula)

Let $\phi : [a,b] \to [c,d]$ be strictly increasing AC function. For a function f on $[c,d]$, we have

$$\int_c^d f(y) \,\mathrm{d}y = \int_a^b f(\phi(x))\phi'(x) \,\mathrm{d}x.$$

Proof. It's equivalent to $m(\phi(E)) = \int_E \phi' \,\mathrm{d}x$. □

§1 Multi-dimensional Calculus

In this section we'll generalize the differentiation and integration theory to higher dimensions. Recall that Lebesgue integral is already defined on higher dimensions, so here we mainly study the differentiation of multi-dimensional functions.