Geometry II

Felix Chen

Contents

In fact $\mathbb{C}P^2$ is a 4-dimensional closed manifold, and it's also a 2-dimensional complex manifold. $PSL(3, \mathbb{C})$ acts transitively on $\mathbb{C}P^2$.

Example 0.0.1

We can fuse the edges of polygons to get manifolds: By fusing together opposite edges of a square, we can get torus or Klein bottle.

We'll use the word "fuse" frequently in the future, so here we'll make it clear what we mean by "fusing" things together.

Definition 0.0.2 (Quotient maps)**.** A continuous map $f : X \to Y$ is called a **quotient map**, if it's surjective, and $\forall B \subset Y$, $f^{-1}(B)$ open $\implies B$ open.

This is saying that the topology on Y is the "largest" topology (or quotient topology) while keeping f continuous.

So when we "fusing" things together, we're actually giving an equivalence relation on the original space, and the result is the quotient topology induced from the natural projection map.

Now we look at the elliptic curves again, let $U = \mathbb{C} \backslash ([\lambda_1, \lambda_2] \cup [\lambda_3, \infty])$. Let X be the path end compactification of U, then $X \simeq S^1 \times [0, 1]$.

Let X_1, X_2 be two copies of X, and fusing the corresponding circles at the end in the reversed direction, we'll get a torus without 4 points, by adding $\lambda_1, \lambda_2, \lambda_3$ back we'll get $T^2 \backslash \{pt\}$.

Remark 0.0.3 — The quotient topology may have some bad properties, like not being Hausdorff: Consider $\mathbb{R}^2 \backslash \{(0, 0)\}$ with connected vertical lines as equivalence class, then we'll get a line with 2 points at the origin, which is a typical non-Hausdorff space.

A closed surface is a connected compact 2-dimensional manifold with no edges. We have the following classification theorem:

Theorem 0.0.4

All the closed surfaces must be homeomorphic to $nT^2 (n \geq 0)$ or $mP^2 (m \geq 1)$. Here n is called the **genus** of orientable surfaces.

nT^2 can be viewed as S^2 fused with n handles (torus), and mP^2 can be viewed as S^2 fused with m crosscaps (Mobius strip).

In this course we mainly talk about surfaces with triangulation, i.e. we take it for granted that all surfaces has triangulation.

Here we'll prove part of this theorem (since the other part needs further knowledge).

> **Remark 0.0.5 —** X has a triangulation means that X is homeomorphic to finitely many n-simplex fused together at the boundary linearly, and the *link* of each vertical is a triangulation of S^{n-1}.

Proof. Observe that given a triangulation, we can get a polygon fusing presentation of the surface by adding the triangles one by one, fusing only one edge each time.

If we write down the edges of this polygon at a certain order, using letters to indicate differet edges and bars for direction, we can get something like $ab\overline{a}\overline{b}$ for a torus.

TODO: pictures!

In fact, nT^2 can be presented as $[a_1, b_1][a_2, b_2] \dots [a_n, b_n]$, where $[a, b] = ab\overline{a}\overline{b}$. Likely, mP^2 is $c_1^2 c_2^2 \dots c_m^2$ since P^2 is c^2. So our goal is to say that any given "edge words" can be reformed to one of the above standard forms.

Note that $(A) : Wa\overline{a} = W$, and $(B) : aUV\overline{a}U'V' = bVU\overline{b}V'U'$. The second operation is cut the polygon in the middle to get b, and fuse two parts together to eliminate a. There's also a reversed version: $aUVaU'V' = bV'VbUU'$. Also note that the word is cyclic, so $(C) : UV = VU$.

TODO: pictures!

This is kind of like Olympiad combinatorics problem. So we need techniques like:

- A "complexity" to measure how close we are to destination:

  vertical numbers(verticals fused together are regarded as one) and egde pair numbers
- Some labels to control different branches:

  whether it has edges with the same direction
- Some efficient "combo moves"

Observe that

- (A) will reduce vetical and edge pair by 1,
- (B) won't effect edge pairs, but may change vertical numbers,
- (C) won't change anything.

In fact we can reduce the vertical number to 1, i.e. all the verticals are fused to one point in the surface. If we have at least 2 verticals, say P and Q, and PQ is an edge. There must be another edge connecting P, Q, If those two P are different in the polygon, we can use (B) to eliminate one P vertical (by adding edge pair of QQ), and use (A) to eliminate they're the same.

TODO: pictures!!!

Repeating above process we can make the vertical number become 1.

If we have $aUbV\overline{a}U'\overline{b}V'$, we can use (B) twice to reform it to $cd\overline{c}\overline{d}W$.

TODO: pictures!!!

So we can achive nT^2 from a word with no same-direction-pairs. Techniquely we still need to prove that we can always find $a \dots b \dots \overline{a} \dots \overline{b}$ in original word, but this can be proved easily otherwise we can perform (A) to reduce edges.

Now for mP^2 :

After some fancy operations we're done. □

Remark 0.0.6 — On the existence of triangulation

§0.1 Homotopy

Definition 0.1.1 (Homotopy)**.** Given two continuous maps $f, g : X \to Y$, if there exists a continuous map

$$H : X \times [0,1] \to Y$$

such that $f = H_0, g = H_1$, where $H_t = H\big|_{X\times\{t\}}$, then we say f and g are **homotopic**, denoted by $f \simeq g$, and the map H is a **homotopy**.

Definition 0.1.2 (Relative homotopy)**.** Let $A \subset X$, $f, g : X \to Y$, and $f\big|_A = g\big|_A$. We say f and g are homotopic relative to A ($f \simeq g$ rel A), if H satisfies $H_t\big|_A = f\big|_A$.

More often we'll talk about homotopy between paths, here by path we mean a map $\gamma : [0,1] \to X$. We say two paths are homotopic if they are homotopic relative to the endpoints($i.e.\{0,1\}$)

Proposition 0.1.3

The homotopic relation is an equivalence relation.

Besides studying the homotopy of maps, we can also consider the homotopy between spaces:

Definition 0.1.4. We say two topological spaces X, Y are **homotopy equivalent** or have the same **homotopy type**, if there exists $f : X \to Y$, $g : Y \to X$, such that

$$f \circ g = \mathrm{id}_Y, \quad g \circ f = \mathrm{id}_X .$$

Example 0.1.5

The following spaces are homotopy equivalent:

Definition 0.1.6 (Fundamental groups)**.** Let $\Omega(X, x_0)$ denote all the loops starting at x_0, i.e. $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

Define the **fundamental group** of X to be:

$$\pi_1(X, x_0) = \Omega(X, x_0)/\simeq,$$

where $\simeq$ is the homotopy relative to x_0.

We define the group operation to be the *concatenation* of paths, denoted by $(a, b) \mapsto ab$, where

$$ab(t) = \begin{cases} a(2t), & t \in [0, \frac{1}{2}]; \\ b(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proposition 0.1.7

The concatenation descends to a well-defined group operation:

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0).$$

Proof. Just some trivial checking. Note that the inverse of a is just $\overline{a}(t) := a(1-t)$. $\square$

Proposition 0.1.8

An homeomorphism $f : (X, x_0) \to (Y, y_0)$ will induce a group homomorphism $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Note that X may be disconnected, so the fundamental group is dependent of the base point x_0. If $\gamma = \langle c \rangle$ is a homotopy class of paths from x_0 to x_1, then γ induces a group homomorphism:

$$\gamma_\# : \pi_1(X, x_0) \to \pi_1(X, x_1) : \langle a \rangle \mapsto \langle \bar{c}ac \rangle .$$

It's easy to see $\gamma_\#$ is an isomorphism.

Hence $\pi_1(X, x_0)$ only depends on the path connected components of x_0. Thus if X is path connected, and X, Y are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, or sometimes we can leave the base point out, just write $\pi_1(X) \cong \pi_1(Y)$.

Remark 0.1.9 — If $x_0 = x_1$, then $\gamma \mapsto \gamma_\#$ gives a homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_1(X, x_0))$.

Example 0.1.10

If $X \simeq \{pt\}$, then $\pi_1(X) \cong \{1\}$. In this case, X is called a **contractible space**. Note that the inverse is not true, e.g. $X = S^n$ for $n \geq 2$. Path connected spaces with trivial fundamental group are called **simple connected** spaces.

Some classic contractible space includes convex sets in $\mathbb{R}^n$, trees in graph theory and cones $CX = X \times [0,1]/X \times \{1\}$.

Some more complex contractible examples including "a house with two rooms", the equitorial inclusion $S^\infty = \bigcup_{n=0}^{\infty} S^n$ with limit topology, i.e. the largest topology s.t. $S^n \to S^\infty$ continuous.

There are several concepts:

- Retraction: $f : X \to A$, $A \subset X$, $f\big|_A = \mathrm{id}_A$.
- Deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$, where $i : A \to X$ is the inclusion.
- Strong deformation retraction: f as above with $i \circ f \simeq \mathrm{id}_X$ *rel* A.

The set A is called (strong) deformation kernel of f.

Example 0.1.11 (Differences between deformation and strong deformation)

Let X be the follwing space:

$$([0,1] \times \{0\}) \cup ([0,1]_{\mathbb{Q}} \times [0,1])$$

We know $X \simeq \{pt\}$, but $\{q\} \times [0,1]$ is deformation kernel but not strong deformation kernel.

After introducing the fundamental groups, a natural question arises: how to compute the fundamental group of a given space?

Theorem 0.1.12 (Van Kampen)
Let $X = U' \cup U''$ be a topology space such that U', U'' are open and $W = U' \cap U''$ path connected, then for $x_0 \in W$, we have

$$\pi_1(X, x_0) \cong \pi_1(U', x_0) * \pi_1(U'', x_0)/N,$$

where N is the smallest normal subgroup generated by

$$i'_{\#}(\delta) i''_{\#}(\delta^{-1}) : \delta \in \pi_1(W, x_0),$$

$$\begin{tikzcd} & U' \arrow[dr, "j'"] & \\ W \arrow[ur, "i'"] \arrow[dr, "i''"'] & & X \\ & U'' \arrow[ur, "j''"'] & \end{tikzcd}$$

and $*$ means free product.

Note that this theorem is useless when both U', U'' have trivial fundamental groups. Thus we need to find a space with nontrivial fundamental group first. One of the simplest example is S^1 :

$$\pi_1(S^n, x_0) \cong \begin{cases} \{1\}, & n \geq 2, \\ \mathbb{Z}, & n = 1. \end{cases}$$

Let $X \vee Y := X \sqcup Y / (x_0 = y_0)$, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. Thus $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{k}) = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_k$, the free group of rank k.

Example 0.1.13
Since nT^2 is formed by $2n$ loops(borders of the polynomial representation) fused with a disk. Note that $W = U' \cap U'' \cong S^1$, so

$$\pi_1(nT^2) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle .$$

Similarly,

$$\pi_1(mP^2) = \left\langle c_1, \ldots, c_m \mid c_1^2 \cdots c_m^2 = 1 \right\rangle .$$

Example 0.1.14
For product spaces, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

In fact the isomorphism can be written down with i_x, i_y, p_x, p_y.