Measure Theory

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Remark 0.0.1 — If μ, ν are σ-finite measures, $\nu \ll \mu$, then

$$\int_X \mathbf{I}_A \,\mathrm{d}\nu = \int_X \mathbf{I}_A \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \implies \int_X f \,\mathrm{d}\nu = \int_X f \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.$$

§0.1 The dual space of L_p

Let $(X, \mathscr{F}, \mu)$ be a measure space, $1 < p < \infty$.

Recall that $f_n \xrightarrow{(w)L_p} f$ is defined as

$$\lim_{n\to\infty} \int_X f_n g \,\mathrm{d}\mu = \int_X fg \,\mathrm{d}\mu, \quad \forall g \in L_q.$$

By Holder's inequality,

$$\left| \int_X fg \,\mathrm{d}\mu \right| \le \|g\|_q \|f\|_p, \quad \forall f \in L_p, g \in L_q.$$

Thus given any $g \in L_q$, we can induce a **funtional** on L_p, moreover it's linear and bounded.

Definition 0.1.1. We say a funtional $\Phi : L_p \to \mathbb{R}$ is bounded linear if:

$$|\Phi(f)| \le C\|f\|_p, \quad \Phi(f_1 + cf_2) = \Phi(f_1) + c\Phi(f_2).$$

We can easily see that Φ is continuous:

$$\|f_n - f\|_p \to 0 \implies |\Phi(f_n) - \Phi(f)| \to 0.$$

Let $\|\Phi\| := \inf C = \sup_{\|f\|_p = 1} |\Phi(f)|$.

For all $A \in \mathscr{F}$, $\Phi_A := \Phi(f\mathbf{I}_A)$ is also a linear and bounded functional. It's clear that $\|\Phi_A\| \le \|\Phi\|$.

Let Φ_g denote the functional induced by $g \in L_q$:

$$\Phi_g : f \mapsto \int_X fg \,\mathrm{d}\mu, \quad |\Phi_g(f)| \le \|g\|_q \|f\|_p.$$

Moreover, take $f = |g|^{q-1}\mathrm{sgn}(g)$, we found that $\|\Phi_g\| = \|g\|_q$. We check it here:

$$\int_X |f|^p \,\mathrm{d}\mu = \int_X |g|^{p(q-1)} \,\mathrm{d}\mu = \int_X |g|^q \,\mathrm{d}\mu,$$

so $f \in L_p$, $\|f\|_p = \|g\|_q^{\frac{q}{p}} = \|g\|_q^{q-1}$. Thus the equality of Holder's inequality holds.

In fact L_q contains all the bounded linear functionals of L_p:

Theorem 0.1.2
The dual space of L_p is L_q, i.e. $L_p^* = L_q$.

The critical part is to use a signed measure φ to determine g:

$$\varphi(A) = \int_A g \,\mathrm{d}\mu = \int_X \mathbf{I}_A g \,\mathrm{d}\mu = \Phi(\mathbf{I}_A), \quad A \in \mathscr{F}.$$

We're faced with two main problems:

- $\mathbf{I}_A$ may not be in L_p.
- μ may not be σ-finite, so the derivative may not be unique.

To solve these problem, we'll start from finite measure, and proceed by finite $\to$ σ-finite $\to$ arbitary.

Proposition 0.1.3
If μ is a finite measure, then $L_p^* = L_q$.

Proof. For any bounded linear functional Φ, let $\varphi(A) = \Phi(\mathbf{I}_A)$,

$$|\varphi(A)| \le C\|\mathbf{I}_A\|_p = C\mu(A)^{\frac{1}{p}},$$

so φ is finite and $\varphi \ll \mu$.

Clearly $\varphi(\emptyset) = 0$, and $\varphi(A+B) = \varphi(A) + \varphi(B)$.

For countable additivity, let $A = \sum_{n=1}^\infty A_n$, $B_N = \sum_{n=N+1}^\infty A_n$, since $\mu(A)$ finite,

$$\left|\varphi(A) - \sum_{n=1}^N \varphi(A_n)\right| = |\varphi(B_N)| \le C\mu(B_N)^{\frac{1}{p}} \to 0.$$

By $\varphi \ll \mu$, let $g = \frac{\mathrm{d}\varphi}{\mathrm{d}\mu}$. We have $|g| < \infty, a.e.$ and $g \in L^1$, so

$$\Phi(\mathbf{I}_A) = \varphi(A) = \int_A g \,\mathrm{d}\mu = \int_X \mathbf{I}_A g \,\mathrm{d}\mu, \quad \forall A \in \mathscr{F}.$$

By the linearity of Φ, we know for simple functions the above equation holds.

For $f \in L_p$ non-negative, we can take simple $f_n \uparrow f$, so $\int f_n^p \,\mathrm{d}\mu \uparrow \int f^p \,\mathrm{d}\mu \implies f_n \xrightarrow{L_p} f$.

By the continuity of Φ, $\Phi(f_n) \to \Phi(f)$.

For the integral part, let $X^+ = \{g \ge 0\}, X^- = \{g < 0\}$. Then $f_n^\pm := f_n \mathbf{I}_{X^\pm}$ non-negative simple, and $f_n^\pm \xrightarrow{L_p} f^\pm := f\mathbf{I}_{X^\pm}$.

Now we can use Levi's theorem to get

$$\int_X f_n^\pm g \,\mathrm{d}\mu \to \int_X f^\pm g \,\mathrm{d}\mu.$$

Note since LHS is $\Phi(f_n^{\pm})$, RHS must be $\Phi(f^{\pm}) \in \mathbb{R}$, so we can safely apply $f = f^+ + f^-$. At last f non-negative $\implies$ f measurable is easy, so we've proven

$$\Phi(f) = \int_X fg \, \mathrm{d}\mu, \quad \forall f \in L_p.$$

Next we'll prove $g \in L_q$. Let $A_n = \{|g| \leq n\}$, let $g_n := g\mathbf{I}_{A_n}$, clearly $g_n \in L_q$ as the base measure is finite.

Since $\Phi_{g_n} = \Phi_{A_n}$, so

$$\|g_n\|_q = \|\Phi_{A_n}\| \leq \|\Phi\|.$$

Now $|g_n| \uparrow |g|, a.e.$, by Levi $\|g_n\|_q \to \|g\|_q$, so $\|g\|_q < \infty$. □

Proposition 0.1.4

When μ is σ-finite, $L_p^* = L_q$.

Proof. Let $X = \sum_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$.

There exists g_n on X_n s.t. $\Phi_{X_n} = \Phi_{g_n}$. Let $g = \sum_{n=1}^{\infty} g_n \mathbf{I}_{X_n}$.

For $f \in L_p$, $\sum_{n=1}^{N} f\mathbf{I}_{X_n} \xrightarrow{L_p} f$, we have

$$\Phi(f) \leftarrow \Phi\left(\sum_{n=1}^{N} f\mathbf{I}_{X_n}\right) = \sum_{n=1}^{N} \Phi_{X_n}(f) = \sum_{n=1}^{N} \int_{X_n} fg \, \mathrm{d}\mu.$$

Similarly, let $A^+ = \{fg \geq 0\}, A^- = \{fg < 0\}$, $f^{\pm} = f\mathbf{I}_{A^{\pm}}$, we know the integral converges.

$g \in L_q$ is also the same as before.TODO

$$\|g\|_q = \lim_{N\to\infty} \left\| \sum_{n=1}^{N} g_n \mathbf{I}_{X_n} \right\| \leq \|\Phi_g\| = \|\Phi\|.$$

□

Proposition 0.1.5

μ is an arbitary measure.

Proof. If $\mu(A) < \infty$, consider $\Phi_A : f \mapsto \Phi(f\mathbf{I}_A)$, we can get g_A.

If $A \subset B$, $\mu(B) < \infty$, then $g_B \mathbf{I}_A = g_A, a.e.$, $\|\Phi_A\| \leq \|\Phi_B\|$.

We can take $A_n \uparrow, \mu(A_n) < \infty$ s.t.

$$\sup_n \|\Phi_{A_n}\| = \sup\{\|\Phi_A\| : \mu(A) < \infty\}.$$

Remark 0.1.6 — Here we're using A_n to replace $X_1 + \ldots X_n$ in the previous proof.

Let $g_n := g_{A_n} \uparrow g$, then $g \in L_q$:

$$\|g\|_q^q = \int_X \lim_{n\to\infty} |g_n|^q \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X |g_n|^q \, \mathrm{d}\mu \leq \|\Phi\|^q.$$

Let $A = \bigcup_{n=1}^{\infty} A_n$, since $g \in L_q$, by Holder and LDC,

$$\int_X fg \,\mathrm{d}\mu \leftarrow \int_X fg_n \,\mathrm{d}\mu = \Phi_{A_n}(f) = \Phi(f\mathbf{I}_{A_n}) \to \Phi(f\mathbf{I}_A).$$

The last part is to prove $\Phi(f\mathbf{I}_{A^c}) = 0$. Otherwise let $D_n = \{|f| > \frac{1}{n}\} \cap A^c$, then $\mu(D_n) < \infty$ since

$$\mu(D_n) \le \mu\left(|f| > \frac{1}{n}\right) \le \int_X (n|f|\mathbf{I}_{D_n})^p \,\mathrm{d}\mu < \infty.$$

By LDC, $f\mathbf{I}_{D_n} \xrightarrow{L_p} f\mathbf{I}_{A^c}$, so $\Phi(f\mathbf{I}_{D_n}) \neq 0$ for some n. But $\mu(D) < \infty$, let $B_n = A_n + D$ we'll find a contradiction on $\sup_n \|\Phi_{B_n}\| > \sup_n \|\Phi_{A_n}\|$. □

When $p = 1$, we can prove for σ-fintie measure μ that $L_1^* = L_\infty$. The method is the same as above.

§0.2 Lebesgue decomposition

Let φ, ϕ be two signed measures.

If $\varphi \ll |\phi|$, then we say φ is absolute continuous with respect to ϕ, denoted by $\varphi \ll \phi$. We can see that $\varphi \ll \phi \iff |\varphi| \ll |\phi|$.

Definition 0.2.1. If $\exists N \in \mathscr{F}$ such that

$$|\varphi|(N^c) = |\phi|(N) = 0,$$

then we say φ and ϕ are **mutually singular**, denoted by $\varphi \perp \phi$.

Lemma 0.2.2

$\varphi \perp \phi$ iff there exists $N \in \mathscr{F}$ such that

$$\varphi(A \cap N^c) = \phi(A \cap N) = 0, \quad \forall A.$$

Proof. This is trivial by $|\varphi|(A) = 0 \iff \varphi(B) = 0, \forall B \subset A$. □

Two measures are mutually singular is to say their supports are disjoint.

Lemma 0.2.3

If $\varphi \ll \phi$ and $\varphi \perp \phi$, then $\varphi \equiv 0$.

Proof. Take N s.t. $|\varphi|(N^c) = |\phi|(N) = 0$, since $\varphi \ll \phi$, $|\varphi|(N) = 0$ as well, thus $|\varphi|(X) = 0$. □

Theorem 0.2.4 (Lebesgue decomposition)

Let φ, ϕ be σ-finite signed measures, there exists unique σ-finite signed measures φ_c, φ_s s.t.

$$\varphi = \varphi_c + \varphi_s, \quad \varphi_c \ll \phi, \varphi_s \perp \phi.$$

Again, we'll start from finite measures, and reach σ-finite signed measures step by step.

Proposition 0.2.5

Let φ, μ be finite measures, then the Lebesgue decomposition holds.

Proof. Since $\varphi \ll \varphi + \mu$, let $f = \frac{\mathrm{d}\varphi}{\mathrm{d}(\varphi+\mu)}$, note that $0 \le f \le 1$, $(\varphi+\mu)$-a.e. (here we use the finite condition) and $1 - f = \frac{\mathrm{d}\mu}{\mathrm{d}(\varphi+\mu)}$.

Let $N = \{f = 1\}$,

$$\varphi_c(A) = \varphi(A \cap N^c), \quad \varphi_s(A) = \varphi(A \cap N).$$

Clearly $\varphi_s(N^c) = 0$,

$$\varphi(N) = \int_N f \,\mathrm{d}(\varphi + \mu) = \int_N 1 \,\mathrm{d}(\varphi + \mu) = \varphi(N) + \mu(N)$$

so $\mu(N) = 0, \varphi_s \perp \mu$.

On the other hand, if $\mu(A) = 0$, since $1 - f > 0$,

$$0 = \mu(AN^c) = \int_{AN^c} (1 - f) \,\mathrm{d}(\varphi + \mu) \implies \varphi_c(A) \le (\varphi + \mu)(AN^c) = 0.$$

Thus $\varphi_c \ll \mu$, we're done. □

From this proof, we can see that the critical point is to find a set N, s.t. $\mu(N) = 0$ and $\varphi_c = \varphi(\cdot \cap N^c) \ll \mu$, i.e. in some sense the "largest" null set of μ.

So this can give another proof:

Proof. Let $\gamma := \sup\{\varphi(A) : A \in \mathscr{F}, \mu(A) = 0\}$.

Let $A_n \in \mathscr{F}, \mu(A_n) = 0$ and $\varphi(A_n) \to \gamma$. Let $N = \bigcup A_n$, then $\varphi(N) = \gamma, \mu(N) = 0$.

If $\mu(A) = 0, \varphi_c(A) > 0$ for some A, then $\mu(N \cup A) = 0$,

$$\varphi(N \cup A) = \varphi(N) + \varphi(A \cap N^c) > \varphi(N) = \gamma,$$

contradiction!

Hence $\varphi_c \ll \mu$. □

Proposition 0.2.6

Let φ, μ be σ-finite measures, the Lebesgue decomposition holds.

Proof. Let $\{A_n\}$ be a partition of X, $\varphi(A_n) < \infty, \mu(A_n) < \infty$.

On $(A_n, A_n \cap \mathscr{F})$, there exists Lebesgue decomposition $\varphi_{n,c}, \varphi_{n,s}$, let $\varphi_c(A) = \sum_{n=1}^{\infty} \varphi_{n,c}(A \cap A_n)$, φ_s similarly defined, we can easily check that $\varphi_c \ll \mu$ and $\varphi_s \perp \mu$. □

At last we prove the Lebesgue decomposition: Let X^+, X^- be the Hahn decomposition of φ, WLOG φ^- finite.

By previous propositions, we have $\varphi_c^{\pm}, \varphi_s^{\pm}$, since φ_s^-, φ_c^- finite, so φ_c, φ_s is well-defined. The rest is some trivial work to check they satisfy the condition.

Now it remains to check the uniqueness. Suppose $\varphi_{c,i}, \varphi_{s,i}$ are two decompositions, $i = 1, 2$.

Let N_i be sets s.t. $\mu(N_i) = |\varphi_{s,i}|(N_i^c) = 0$, let $N = N_1 \cup N_2$, we have

$$\mu(N) = 0 \implies \varphi_{c,i}(N) = 0; \quad |\varphi_{s,i}|(N^c) = 0, i = 1, 2.$$

Thus $\varphi_{c,i}(A) = \varphi_{c,i}(AN^c) = \varphi(AN^c)$, and $\varphi_{s,i}(A) = \varphi_{s,i}(AN) = \varphi(AN)$.

At last we take $\mu = |\phi|$ to finally conclude.

Example 0.2.7

Let μ be a probability on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, λ is Lebesgue measure.

If $\mu \ll \lambda$, we say μ is continuous, and $\frac{d\mu}{d\lambda}$ is the density function of μ.

If $\mu(\{x\}) > 0$, then we say x is an atom of μ,

$$D = D_\mu := \{x \in \mathbb{R} : \mu(\{x\}) > 0\},$$

then μ finite $\implies$ D countable.

If $\mu(D) = 1$, then we say μ is discrete.

If $\mu \perp \lambda$ and $D_\mu = \emptyset$, then we say μ is singular.

Then for any finite measure μ, let $\mu = \mu_c + \mu_s$ be the Lebesgue decomposition with respect to λ. Let $\mu_1 = \mu_c, \mu_2 = \mu(\cdot \cap D_\mu), \mu_3 = \mu_s - \mu_2$.

Then μ_1, μ_2, μ_3 are pairwise singular.

§0.3 Conditional expectations

Let $(X, \mathscr{F}, P)$ be a probability space. Let $\mathscr{G}$ be a sub σ-algebra of $\mathscr{F}$. Then we have another probability space $(X, \mathscr{G}, P)$.

Recall that $L_2(\mathscr{G}) \subset L_2(\mathscr{F})$ are Hilbert spaces.

Let $g \in \mathscr{G}$ be a function, $g \geq 0$, then $\int_X g \,\mathrm{d}P$ is the same in two spaces.(By Levi's theorem)

By linear algebra, for any $f \in \mathscr{F}$, there's a unique optimal approximation (or orthogonal projection) $f^* \in \mathscr{G}$ s.t.

$$\|f - f^*\|_2 = \inf_{g \in L_2(\mathscr{G})} \|f - g\|_2.$$

Therefore by orthogonality,

$$Efg = Ef^*g, \forall g \in L_2(\mathscr{G}) \iff Ef\mathbf{I}_A = Ef^*\mathbf{I}_A, \forall A \in \mathscr{G}.$$

Let $\varphi(A) = Ef\mathbf{I}_A$, $\varphi \ll P$, in fact we have $f^* = \frac{\mathrm{d}\varphi}{\mathrm{d}P}$ in $\mathscr{G}$.

Remark 0.3.1 — $\int_X f \,\mathrm{d}\mu$ only depends on $\sigma(f)$, so when $f \in \mathscr{G} \subset \mathscr{F}$, the integral is the same under both σ-algebra.

We can see that the condition L_2 is a little strong, so we can reduce it to existence of integrals.

Definition 0.3.2 (Conditional expectation)**.** Let $f \in \mathscr{F}$ whose integral exists, we say the **conditional expectation** of f under $\mathscr{G}$ is the function f^* with integral which satisfies:

$$f^* \in \mathscr{G}, \quad Ef^*\mathbf{I}_A = \int_A f \,\mathrm{d}P, \forall A \in \mathscr{G}.$$

This function is denoted by $E(f|\mathscr{G})$.

By the notation $E(f|\mathscr{G})$ we mean a family of *almost surely* equal functions which are measurable in $(X, \mathscr{G}, P)$.

The **conditional probability** of A under $\mathscr{G}$ is

$$P(A|\mathscr{G}) := E(\mathbf{I}_A|\mathscr{G}).$$

As we've said, let $\phi(A) = Ef\mathbf{I}_A$ be a signed measure, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}P} = f \in (X, \mathscr{F}), \quad \frac{\mathrm{d}\phi|_{\mathscr{G}}}{\mathrm{d}P} = f^* \in (X, \mathscr{G}).$$

All we've done is to find a approximation of f which isn't necessarily in $\mathscr{G}$

Let $g : (X, \mathscr{F}) \to (Y, \mathscr{S})$. We say the conditional expectation of f with respect to g is defined as

$$E(f|g) = E(f|\sigma(g))$$

In probability courses we learned:

$$Ef = EE(f|g),$$

now it's almost trivial since $\int_X f \,\mathrm{d}P = \int_X f^* \,\mathrm{d}P$.

Example 0.3.3

Let $\mathscr{G} = \{\emptyset, B, B^c, X\}$, where $B \in \mathscr{F}$. Then $E(f|\mathscr{G}) = \int_B f \,\mathrm{d}P P(B)^{-1}\mathbf{I}_B + \int_{B^c} f \,\mathrm{d}P P(B^c)^{-1}\mathbf{I}_{B^c}$.

We can see that the conditional expectation is indeed an "expectation".

Also, $P(A|\mathscr{G}) = P(A \cap B)P(B)^{-1}\mathbf{I}_B + P(A \cap B^c)P(B^c)^{-1}\mathbf{I}_{B^c}$, thus $P(A|B) = \frac{P(A\cap B)}{P(B)}$, which coincides with elementary probability.

Definition 0.3.4. Let $\{A_t, t \in T\}$ be a family of sets in $\mathscr{F}$, if $\forall n \geq 2, \{t_1, \ldots, t_n\} \subset T$,

$$P\left(\bigcap_{k=1}^{n} A_{t_k}\right) = \prod_{k=1}^{n} P(A_{t_k}),$$

we say $\{A_t, t \in T\}$ are independent.

Likely, we can define the independence of collections of sets, and therefore random variables.

Lemma 0.3.5

Let f be a random variable whose integral exists, if f and $\mathscr{E}$ are independent, we have

$$E(f\mathbf{I}_A) = (Ef) \cdot P(A), \quad \forall A \in \mathscr{E}$$

Next we'll study the properties of conditional expectations: Let f, g be functions whose integrals exist, $\mathscr{G}, \mathscr{G}_0$ are sub σ-algebras of $\mathscr{F}$,

(1) If $f \in \mathscr{G}$, then $E(f|\mathscr{G}) = f, a.s.$. (Trivial)

(2) If f and $\mathscr{G}$ are independent, then $E(f|\mathscr{G}) = Ef, a.s.$.

Let $f^* = Ef$, we can see

$$Ef\mathbf{I}_A = (Ef)P(A) = Ef^*\mathbf{I}_A,$$

(3) Let $\mathscr{G} \subset \mathscr{G}_0$,

$$E(E(f|\mathscr{G})|\mathscr{G}_0) = E(f|\mathscr{G}) = E(E(f|\mathscr{G}_0)|\mathscr{G}), a.s.$$

The left hand side is immediate by (1). The right hand side can be checked directly using definition.

(4) If $f \leq g, a.s.$ then $E(f|\mathscr{G}) \leq E(g|\mathscr{G}), a.s.$.

$$Ef^*\mathbf{I}_A = Ef\mathbf{I}_A \leq Eg\mathbf{I}_A = Eg^*\mathbf{I}_A, \quad \forall A \in \mathscr{G}.$$

(5) For all $a, b \in \mathbb{R}$, if $aEf + bEg$ exists, then

$$E(af + bg|\mathscr{G}) = aE(f|\mathscr{G}) + bE(g|\mathscr{G}).$$

This also can be checked using definition (let $h = af + bg$).

Theorem 0.3.6
Let $f_1, f_2, \dots$ be r.v. whose integrals exist, $\mathscr{G} \subset \mathscr{F}$, then the limit theorems also holds:

- If $0 \le f_n \uparrow f, a.s.$, then
$$0 \le E(f_n|\mathscr{G}) \uparrow E(f|\mathscr{G}), a.s.;$$
- If $f_n \ge 0, a.s.$, then
$$E\left(\liminf_{n\to\infty} f_n|\mathscr{G}\right) \le \liminf_{n\to\infty} E(f_n|\mathscr{G}), a.s.;$$
- If $|f_n| \le g, a.s.$ and $g \in L_1$, $f_n \to f, a.s.$ or in measure.
$$E(f|\mathscr{G}) = \lim_{n\to\infty} E(f_n|\mathscr{G}), a.s.$$

Proof.

- Let $f_n^* = E(f_n|\mathscr{G})$, then they are a.s. increasing, let $\hat{f} = \lim_{n\to\infty} f_n^*$, then $\hat{f} \in \mathscr{G}$, and
$$E\hat{f}\mathbf{I}_A = \lim_{n\to\infty} Ef_n^*\mathbf{I}_A = Ef\mathbf{I}_A.$$
- Similarly, let
$$g_n := \inf_{m\ge n} f_m \uparrow \liminf_{n\to\infty} f_n =: f.$$
We have $g_n^* \uparrow f^*$, so
$$g_n \le f_n \implies g_n^* \le f^* \implies f^* \le \liminf_{n\to\infty} f_n^*, a.s.$$
- Lebesgue dominated theorem can be proved similarly.

□

Theorem 0.3.7
Let f, g are r.v. whose integrals exist, $g \in \mathscr{G} \subset \mathscr{F}$.
$$E(fg|\mathscr{G}) = gE(f|\mathscr{G}), a.s.$$

Proof. Fix f, we use typical method on g. When $g = \mathbf{I}_A$, $A \in \mathscr{G}$, then the conclusion holds:
$$E(f^*\mathbf{I}_A\mathbf{I}_B) = E(f^*\mathbf{I}_{AB}) = Ef\mathbf{I}_{AB} = E(f\mathbf{I}_A\mathbf{I}_B).$$
Since $AB \in \mathscr{G}$.

Now using the linearity and limit theorems we're done. Note that we need to prove on $\{f, g \ge 0\}$ and other 3 sets respectively. □

§0.4 Regular conditional distribution

Let $\{A_n\}$ be a partition of X, $\mathscr{G} = \sigma(\{A_n\})$, $P(A_n) > 0$. Thus if $B \in \mathscr{G}$ and $P(B) = 0 \implies B = \emptyset$. So the conditional expectations are uniquely determined (the only null set is the empty set).

We'll compute the conditional expectation of f under $\mathscr{G}$.
$$f^*(x) = \sum_n a_n\mathbf{I}_{A_n}(x), \quad \forall x. \quad Ef^*\mathbf{I}_{A_n} = Ef\mathbf{I}_{A_n} \implies a_n = \frac{Ef\mathbf{I}_{A_n}}{P(A_n)}.$$

Hence $\forall x \in X, A \in \mathscr{F}$,

$$p(x, A) = P(A|\mathscr{G})(x) = (\mathbf{I}_A)^*(x) = \sum_n \frac{P(A \cap A_n)}{P(A_n)} \mathbf{I}_A(x).$$

We get a function $p(x, \cdot)$, which is a probability on $\mathscr{F}$, and $p(x, \cdot) = P(\cdot|A_n)$ when $x \in A_n$.

For a fixed x,

$$(\mathbf{I}_A)^*(x) = \int_X \mathbf{I}_A(y) \, \mathrm{d}p(x, \cdot), \quad \forall A \in \mathscr{F}.$$

Now using typical method we can generalize $\mathbf{I}_A$ to any measurable function f. Since here $a.s.$ means equal at every point, so all the functions are determined.

Now what we've done is to realize conditional expectation as an integral over **conditional probabilities** $p(x, \cdot)$:

$$f^*(x) = \int_X f(y) \, \mathrm{d}p(x, \cdot) = \int_X f(y) p(x, \mathrm{d}y).$$