Mathematical Analysis II

Felix Chen

Contents

Example 0.0.1

Consider the spherical coordinates $x = r\sin\theta\sin\varphi$, $y = r\sin\theta\cos\varphi$, $z = r\cos\theta$.

Let $F : (r, \theta, \varphi) \mapsto (x, y, z)$.

$$J_F = \begin{pmatrix} \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

So $\det(J_F) = r^2\sin\theta$. Thus

$$\int_{\mathbb{R}^3} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{(0,2\pi)^2}\int_0^{+\infty} f(r, \theta, \phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Theorem 0.0.2 (Clairaut-Schwarz)

Given an open set $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. Assume $\frac{\partial f}{\partial x_i}(x)$, $\frac{\partial f}{\partial x_j}(x)$, $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})(x)$ exists and are continuous, then $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})(x)$ exists and

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\right)(x).$$

Proof. WLOG $n = 2$. We'll just expand and compute:

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{s\to 0}\frac{1}{s}\left(\frac{\partial f}{\partial x}(x_0, y_0 + s) - \frac{\partial f}{\partial x}(x_0, y_0)\right) \\ &= \lim_{s\to 0}\frac{1}{s}\lim_{t\to 0}\frac{1}{t}(f(x_0 + t, y_0 + s) - f(x_0, y_0 + s) - f(x_0 + t, y_0) + f(x_0, y_0)).\end{aligned}$$

Since

$$(f(x_0 + t, y_0 + s) - f(x_0, y_0 + s) - f(x_0 + t, y_0) + f(x_0, y_0)) = \int_0^s\int_0^t \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0 + \tilde{t}, y_0 + \tilde{s})\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{s}.$$

So by Fubini's theorem, $\square$

Notation: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiple index, where $\alpha_i \geq 0$ are integers. define

$$\partial^\alpha f = \left(\frac{\partial f}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial f}{\partial x_n}\right)^{\alpha_n} f.$$

or we can write

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

Theorem 0.0.3 (Multi-dimensional Taylor expansion)
Let $\Omega \subset \mathbb{R}^n$ be a convex open set. Let $f \in C^{k+1}(\Omega)$, for all $x, y \in \Omega$, then $\exists \theta \in (0,1]$ s.t.

$$f(y) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x)}{\alpha!}(y-x)^\alpha + \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(x+\theta(y-x))}{\alpha!}(y-x)^\alpha.$$

where $(y-x)^\alpha = \prod_{i=1}^n (x_i - y_i)^{\alpha_i}$, $\alpha! = \prod_{i=1}^n \alpha_i!$.

Proof. Let $g(t) = f(ty + (1-t)x)$, $g \in C^{k+1}((-1,1))$. By Taylor expansion, there exists $\theta \in [0,1]$,

$$g(1) = \sum_{l=0}^k \frac{g^{(l)}(0)}{l!} + \frac{g^{(k+1)}(\theta)}{(k+1)!}.$$

so it's just a differential formula of composite function, which can be easily proved by induction, and I don't bother to write it down. □

§0.1 Implicit function theorem

As usual let $C^k(\Omega)$ denote the k times continuously differentiable functions on Ω.

Definition 0.1.1 (Differential homeomorphisms). Let $U, V \subset \mathbb{R}^n$, if there exists a bijection $f : U \to V$, such that f, f^{-1} are smooth, then we say U and V are **smoothly homeomorphic**.
Denoted by $C^\infty(U,V)$ the smooth homeomorphisms from U to V.

Example 0.1.2
Let $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^3$, then f is a smooth bijection, but f^{-1} is not differentiable at 0.

Recall that in $\mathbb{R}$ we have the following results:

- If f is strictly increasing and continuous, then f^{-1} is continuous.
- If f is strictly increasing and C^1, $f' \neq 0$, then $f^{-1} \in C^1$.

Theorem 0.1.3
Let Φ be an differential homeomorphism $U \to V$, $f \in C^k(V)$. Then $f \circ \Phi =: \Phi^* f \in C^k(\Omega)$, this is called the **pullback** of f by Φ.

Proof. We proceed by induction on k. When $k = 0$, this is just the continuity of composite functions.

Assume $k = n$ holds, then for $k = n + 1$,

$$\frac{\partial \Phi^* f}{\partial x_j} = \frac{\partial f(\Phi(x))}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(\Phi(x)) \cdot \frac{\partial \Phi_i(x)}{\partial x_j}.$$

Since $f \in C^{n+1} \implies \frac{\partial f}{\partial y_i} \in C^n$, and $\frac{\partial \Phi_i}{\partial x_j}$ is smooth, so $\frac{\partial \Phi^* f}{\partial x_j} \in C^n$. $\square$

Note that the condition $f' \neq 0$ grants that f is indeed a bijection locally. In higher dimensional spaces, the derivatives are more complex, so let's look at some simple cases first.

Lemma 0.1.4

Let $U, V \subset \mathbb{R}^d$ be open regions. Let $f : U \to V$ be a C^1 bijection, and $J(f)$ is non-degenerate (i.e. $\det J(f) \neq 0$). Then $f^{-1} : V \to U$ is continuously differentiable.

Proof. Let $x_0 \in U$, $y_0 = f(x_0) = V$,

$$f(x_0 + v) = y_0 + A \cdot v + o(|v|).$$

Let $E(\delta)$ be a function which satisfies

$$f^{-1}(y_0 + \delta) = x_0 + A^{-1}\delta + E(\delta).$$

this can be derived from taking f^{-1} on both sides of the above equation.

$$\begin{aligned} y_0 + \delta = f(x_0 + A^{-1}\delta + E(\delta)) &= y_0 + \delta + AE(\delta) + o(|A^{-1}\delta + E(\delta)|) \\ \implies AE(\delta) &+ o(A^{-1}\delta + E(\delta)) = 0. \end{aligned}$$

From this we can calculate

$$\begin{aligned} \frac{|E(\delta)|}{|\delta|} = \frac{|o(A^{-2}\delta + A^{-1}E(\delta))|}{|\delta|} &\leq \frac{|o(A^{-2}\delta + A^{-1}E(\delta))|}{|A^{-2}\delta + A^{-1}E(\delta)|} \cdot \frac{|A^{-2}\delta + A^{-1}E(\delta)|}{|\delta|} \\ &\leq o(1)\left(C + C\frac{|E(\delta)|}{|\delta|}\right). \end{aligned}$$

Hence $\lim_{|\delta| \to 0} \frac{|E(\delta)|}{|\delta|} = 0$. $\square$

In this case we are given f^{-1} exists, but generally we need to prove this existence.

Theorem 0.1.5 (Inverse function theorem)

Let $f : \Omega \to \mathbb{R}^d$ be a C^1 map, and $\mathrm{d}f(x_0)$ is non-degenerate, then f is a C^1 differential homeomorphism in some neighborhood of x_0.

This is to say, $\exists U \ni x_0, V \ni f(x_0)$ s.t. f is a bijection from U to V and $f^{-1} : V \to U$ is a C^1 map.

Proof. WLOG $x_0 = 0$, $f(x_0) = 0$, also we can apply a linear transformation such that $\mathrm{d}f(x_0) = I$.

There exists $\delta > 0$, s.t.

$$|f(v) - v| < \varepsilon_0 |v|, \quad \|J(f)(v) - I\| < \varepsilon_0, \quad \forall |v| < \delta.$$

note that

$$\begin{aligned} f(v) - f(u) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tv + (1-t)u) \,\mathrm{d}t &= \int_0^1 \mathrm{d}f(tv + (1-t)u) \cdot (v - u) \,\mathrm{d}t \\ &= \int_0^1 (Jf(tv + (1-t)u) - I) \cdot (v - u) \,\mathrm{d}t + (u - v). \end{aligned}$$

but when $|u|, |v| < \delta$, $|f(v) - f(u) - (v - u)| < \varepsilon_0 |v - u|$.

Hence $f(u) = f(v) \implies u = v$, f is injective in $B_\delta(0)$.

As for surjectivity, it's sufficient to prove $f(\overline{B_\delta(0)})$ contains a neighborhood of $f(0) = 0$. i.e. $\forall |v| < \delta_1, \exists |u| < \delta$ s.t. $f(u) = u + o(u) = v$.

Since we don't know the non-linear term $o(u)$, we'll iterate to get a solution u: let $u_0 = v$. Define $u_{k+1} = v - (f(u_k) - u_k)$. When δ_1 is sufficiently small,

$$|u_{k+1}| \le |v| + |f(u_k) - u_k| \le |v| + \varepsilon_0 |u_k| \le \delta_1 + \varepsilon_0 \delta \le \delta.$$

Now we prove the convergency:

$$\begin{aligned} |u_{k+2} - u_{k+1}| &= |f(u_{k+1}) - u_{k+1} - f(u_k) + u_k| \\ &= |\int_0^1 (J(f)(tu_{k+1} + (1-t)u_k) - I) \,\mathrm{d}t (u_{k+1} - u_k)| \\ &\le \varepsilon_0 |u_{k+1} - u_k|. \end{aligned}$$

by contraction mapping principle we're done. □

Remark 0.1.6 — This theorem holds for any Banach space.

Corollary 0.1.7

Let $k \ge 2$ be an integer, when $f \in C^k$ in the above theorem, we can imply that $f^{-1} \in C^k(V)$.

Proof. Since

$$\mathrm{d}f^{-1}(u) = (\mathrm{d}f)^{-1}(f^{-1}(u)),$$

so $\mathrm{d}f \in C^{k-1} \implies (\mathrm{d}f)^{-1} \in C^{k-1}$. □

Theorem 0.1.8 (Implicit function theorem)

Let $f : \Omega \subset \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ be a continuously differentiable function. If $\exists (x^*, y^*) \in \Omega$ s.t. $f(x^*, y^*) = 0$, and $\mathrm{d}_y f(x^*, y^*)$ is inversible, then there exists an open neighborhood $U \subset \mathbb{R}^n$ of x^*, $V \subset \mathbb{R}^p$ of y^*, and a C^1 map $\phi : U \to V$ such that:

$$f(x, \phi(x)) = 0, \quad \mathrm{d}\phi(x) = -(\mathrm{d}_y f(x, \phi(x)))^{-1} \cdot \mathrm{d}_x f(x, \phi(x)).$$

Also if $x \in U$ and $f(x, y) = 0$, we must have $y = \phi(x)$.

Remark 0.1.9 — This is to say, if $f(x,y)=0$, $x \in U, y \in V$, then $y=\phi(x)$.
Also remember that $\mathrm{d}_y f$ is a $p \times p$ matrix, $\mathrm{d}_x f$ is a $p \times n$ matrix.

Proof. By the inverse function theorem, let $F(x,y) := \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ with

$$(x,y) \mapsto (x, f(x,y))$$

So $F(x^*,y^*) = (x^*,0)$, and

$$\mathrm{d}F(x^*,y^*) = \begin{pmatrix} I_n & 0 \\ \mathrm{d}_x f(x^*,y^*) & \mathrm{d}_y f(x^*,y^*) \end{pmatrix}.$$

Since $\mathrm{d}_y f(x^*,y^*)$ is inversible, $\mathrm{d}F(x^*,y^*)$ is inversible as well. Hence there exists neighborhoods of (x^*,y^*) and $(x^*,0)$, say $\widetilde{\Omega}$ and $\widetilde{\Omega}_1$, such that F is a C^1 homeomorphism $\widetilde{\Omega} \to \widetilde{\Omega}_1$.

We can find $U \ni x^*, V \ni y^*$ s.t. $U \times V \subset \widetilde{\Omega}$. Let T be the C^1 map s.t.

$$F^{-1}(x,z) = (x, T(x,z)).$$

Let $\phi(x) = T(x,0)$, we have

$$F(x,\phi(x)) = F(x,T(x,0)) = (x,0) \implies f(x,\phi(x)) = 0.$$

Since F is a bijection, clearly $f(x,y)=0 \implies y=\phi(x)$. By taking the differentiation of $f(x,\phi(x))=0$,

$$(\mathrm{d}_x f, \mathrm{d}_y f) \cdot \begin{pmatrix} I_n \\ \mathrm{d}\phi(x) \end{pmatrix} = 0 \implies \mathrm{d}_x f(x,\phi(x)) + \mathrm{d}_y f(x,\phi(x)) \cdot \mathrm{d}\phi(x) = 0.$$

□

§0.2 Submanifolds in Euclid space

The implicit function theorem actually gives an example of manifolds: the preimage of $f(x,y)=0$ is an n-dimensional manifold in $\mathbb{R}^{n+p}$.

Definition 0.2.1 (Manifolds)**.** Let $M \subset \mathbb{R}^n$ be a nonempty set. If $\exists d \geq 0$, $\forall x \in M$ exists open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^d$, and a differential homeomorphism $\Phi : U \to V$, such that

$$\Phi(U \cap M) = V,$$

we say M is a **d-dimensional differential manifold**. Denote $\dim M = d$, and $n-d$ is called the **codimension** of M.

Remark 0.2.2 — There might be different maps $\phi_1 : U_1 \to V_1, \phi_2 : U_2 \to V_2$, when $U_1 \cap U_2 \cap M \neq \emptyset$, we must have $\phi_2 \circ \phi_1^{-1}$ is a differential map from V_1 to V_2. In fact when M isn't a subset of $\mathbb{R}^n$, this is the original definition of differential manifolds.

Corollary 0.2.3 (Regular value theorem)
Let $f : \Omega \to \mathbb{R}^p$ be a smooth map, where $\Omega \subset \mathbb{R}^n$ is an open set, $n \geq p$. For all $c \in \mathbb{R}^p$, we call the **fibre** of c to be its preiamge:

$$f^{-1}(c) = \{x \in \mathbb{R}^n \mid f(x) = c\}.$$

If $\forall x \in f^{-1}(c)$, $\operatorname{rank} \mathrm{d}f(x) = p$, then $f^{-1}(c)$ is a manifold with **codimension** p.

Example 0.2.4

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ with $x \mapsto |x|^2 - 1$, then $S^{n-1} = f^{-1}(0)$.

Since $\mathrm{d}f = (2x_1, 2x_2, \ldots, 2x_n)$, clearly $\operatorname{rank} \mathrm{d}f = 1$ for all $x \in S^{n-1}$, so S^{n-1} is a manifold with codimension 1.

Example 0.2.5

Consider a surface in $\mathbb{R}^4 = \mathbb{C}^2$:

$$T^2 = \{(z_1, z_2) \mid |z_1| = 1, |z_2| = 1\}.$$

Let $f(x, y, z, w) = x^2 + y^2 - 1, g(x, y, z, w) = z^2 + w^2 - 1$, then $T^2 = \begin{pmatrix} f \\ g \end{pmatrix}^{-1} (0)$.

The differentiation is

$$\mathrm{d} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & 2w \end{pmatrix},$$

so T^2 is a manifold with codimension 2.

Definition 0.2.6. Let $M \subset \mathbb{R}^n$ be a manifold. If $\dim M = 1$, we say M is a curve; if $\dim M = 2$, M is a surface; and if $\dim M = n - 1$, we say M is a hyperplane.

Lemma 0.2.7

Let $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, if $\forall x_0 \in f^{-1}(0)$, $\mathrm{d}f(x_0) \neq 0$, then $f^{-1}(0)$ is a smooth hyperplane, it is called the hyperplane globally determined by an equation.

Example 0.2.8

In $\mathbb{R}^3$, f, g are smooth functions. If for all $x \in \mathbb{R}^3$ with $f(x) = g(x) = 0$ we have $\nabla f, \nabla g$ are linearly independent, then $\{f = g = 0\}$ is a smooth curve.

Theorem 0.2.9 (Parametrization of manifolds)

Let Ω be an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^{n+p}$ is a smooth map. If $\forall x^* \in \Omega$, $\operatorname{rank} \mathrm{d}f(x^*) = n$, then there exists an open set U, $x^* \in U$ s.t. $f(U) \subset \mathbb{R}^{n+p}$ is an n-dimensional manifold.

Proof. Let x_i be a coordinate in $\mathbb{R}^{n+p}$.

WLOG $(\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq n}$ is non-degenerate, let $F = (f_1, \ldots, f_n)$, $G = (f_{n+1}, \ldots, f_{n+p})$ and apply inverse function theorem on F, there exists open neighborhoods $U \ni x, V \ni F(x) =: y$, s.t. $F : U \to V$ is a smooth homeomorphism.

$$\begin{tikzcd} U \subset \Omega \arrow[r, "F"] \arrow[d, "f"] & V \subset \mathbb{R}^n \arrow[dl, "\phi"] \\ \mathbb{R}^{n+p} & \end{tikzcd}$$

So $f(x) = (F(x), G(x)) = (y, GF^{-1}(y))$. Let

$$\phi : V \to \mathbb{R}^n, \quad y \mapsto (y, GF^{-1}(y)).$$

We can see that ϕ is a homeomorphism $V \to f(U)$. (Indeed it's a bijection) So by definition we know $f(U)$ is a manifold. □

Example 0.2.10

Let

$$\phi(\theta, r) = \begin{cases} x = \left(1 + r\cos\dfrac{\theta}{2}\right)\cos\theta \\ y = \left(1 + r\cos\dfrac{\theta}{2}\right)\sin\theta \\ z = r\sin\dfrac{\theta}{2} \end{cases}, \quad I = [0, 2\pi] \times (-1, 1).$$

Then $M = \phi(I)$ is a Mobius strip, which is a two dimensional smooth manifold in $\mathbb{R}^3$, as $\mathrm{d}\phi$ has rank 2 everywhere.

Besides, there doesn't exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ s.t. $M = f^{-1}(0)$. Basically this is because M is not orientable, but ∇f and $-\nabla f$ are "normal" directions of M, which makes it orientable. Below we give a sketch:

Proof. Let $v(\theta) = \cos\frac{\theta}{2} e_2(\theta) - \sin\frac{\theta}{2} e_1(\theta)$, where $e_2(\theta) = (0, 0, 1), e_1(\theta) = (\cos\theta, \sin\theta, 0)$.

Note that $e_1 \perp e_2$, consider the curve $\beta : [0, 2\pi] \to \mathbb{R}^3$

$$\theta \mapsto (\cos\theta, \sin\theta, 0) + \varepsilon v(\theta).$$

Let ε be sufficiently small, when $\varepsilon \neq 0$ we can check β and M do not intersect. We can take ε s.t. $f(\beta(0)) > 0$ as $\mathrm{d}f \neq 0$. (ε can be negative)

Since $\beta(0) = (1, 0, \varepsilon), \beta(2\pi) = (1, 0, -\varepsilon)$, when $f(\beta(0)) > 0$, we must have $f(\beta(2\pi)) < 0$. By continuity, $\exists \theta_0$ s.t. $f(\beta(\theta_0)) = 0$, which means $\beta(\theta_0) \in M$, contradiction! □