# Measure Theory

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Hence $(L_p/\sim, \|\cdot\|_p)$ is a normed vector space.

When $p = \infty$, define

$$\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(|f| > a) = 0\}, \quad L_\infty := \{f : \|f\|_\infty < \infty\}.$$

We call the functions in $L_\infty$ **essentially bounded**.

Let $\mu(X) < \infty$, then $f \in L_\infty \implies f \in L_p$, and $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$: For all $0 < a < \|f\|_\infty$,

$$a^p \mu(|f| > a) \le \int_X |f|^p \mathbf{I}_{|f|>a} \,\mathrm{d}\mu \le \int_X |f|^p \,\mathrm{d}\mu \le \|f\|_\infty^p \mu(X),$$

So taking the exponent $\frac{1}{p}$,

$$a \leftarrow a\mu(|f| > a)^{\frac{1}{p}} \le \|f\|_p \le \|f\|_\infty$$

But when $\mu(X) = \infty$, let $f \equiv 1$, then $f \in L_\infty$ but $f \notin L_p$.

**Theorem 0.0.1**

Let $f, g \in L_\infty$,

$$\|fg\| \le \|f\|\|g\|_\infty,$$
$$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

*Proof.*

$$\int_X |fg| \,\mathrm{d}\mu \le \int_X |f|\|g\|_\infty \,\mathrm{d}\mu = \|f\|\|g\|_\infty.$$

Since $|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty, a.e.$, we get the second inequality. □

Similarly we get $(L_\infty, \|\cdot\|_\infty)$ is a normed vector space.

The norm can deduce a *distance*:

$$\rho(f, g) := \|f - g\|.$$

**Theorem 0.0.2** ($L_p$ space is complete)
Let $1 \le p \le \infty$. If $\{f_n\} \subset L_p$ satisfying $\lim_{n,m\to\infty}\|f_n - f_m\|_p = 0$, then there exist $f \in L_p$ s.t. $\lim_{n\to\infty}\|f - f_n\|_p = 0$.

*Proof.* Take $n_1 < n_2 < \cdots$ such that

$$\|f_m - f_n\|_p \le \frac{1}{2^k}, \quad \forall n, m \ge n_k.$$

Let $g =\uparrow \lim_{k\to\infty} g_k$, where

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \in L_p, \quad g_k \ge 0.$$

Since

$$\|g_k\|_p \le \|f_{n_1}\|_p + \sum_{i=1}^{k}\|f_{n_{i+1}} - f_{n_i}\|_p \le \|f_{n_1}\|_p + 1.$$

$$\implies \|g\|_p =\uparrow \lim_{k\to\infty}\|g_k\|_p \le \|f_{n_1}\|_p + 1.$$

Here we use the monotone convergence theorem. We can check the above also holds for $p = \infty$.

Therefore $g \in L_p \implies g < \infty, a.e.$. We have

$$f := f_{n_1} + \sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i}) = \lim_{k\to\infty} f_k, a.e.$$

the series is absolutely convergent, so $f$ exists a.e. and $|f| \le g, a.e.$.

Lastly we can check: when $p = \infty$,

$$\|f_n - f\|_\infty \le \|f_n - f_{n_k}\|_\infty + \|f_{n_k} - f\|_\infty,$$

where the both term approach to 0 as $n \to \infty$.

When $p < \infty$, by Fatou's lemma,

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} |f_n - f_{n_k}|^p \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_X |f_n - f_{n_k}|^p \,\mathrm{d}\mu \le \varepsilon.$$

□

**Remark 0.0.3 —** Using the same technique we can prove that if $f_n$ is Cauchy in measure, then $f_n$ converge to some $f$ in measure:

Let $A_i = \{|f_{n_{i+1}} - f_{n_i}| > 2^{-i}\}$ s.t. $\mu(A_i) < 2^{-i}$.

Define $f = f_{n_1} + \sum_{i\ge1}(f_{n_{i+1}} - f_{n_i})$ on the set $\bigcup_{k\ge1}\bigcap_{i\ge k} A_i^c$.

This theorem implies that $(L_p, \|\cdot\|_p)$ is a Banach space. So we can try to define an *inner product* on $L_p$ space:

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

We can check $\langle\cdot,\cdot\rangle$ is bilinear only if $p = 2$, so $L_2$ is actually a Hilbert space.

When $0 < p < 1$, let

$$\|f\|_p := \int_X |f|^p \,\mathrm{d}\mu, \quad L_p = \{f : \|f\|_p < \infty\}.$$

**Lemma 0.0.4**

Let $0 < p < 1$, $C_p = 1$, then

$$|a+b|^p \le C_p(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}.$$

So $L_p$ is a vector space.

**Theorem 0.0.5** (Minkowski)

Let $0 < p < 1$ then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

**Remark 0.0.6** — When $\|f\|_p = (\int_X |f|^p \,\mathrm{d}\mu)^{\frac{1}{p}}$, $0 < p < 1$. then it won't satisfy Minkowski's inequality.

Thus $L_p$ is only a metric space but not a normed vector space. Using the same method we can prove $L_p$ is a complete metric space.

## §0.1 Convergence in $L_p$ space

**Definition 0.1.1.** Let $0 < p \le \infty$, $f, f_1, f_2, \cdots \in L_p$. When $\|f_n - f\|_p \to 0$, then we write $f_n \xrightarrow{L_p} f$, called **average converge of order $p$**.

**Theorem 0.1.2**

Let $0 < p < \infty$, $f, f_1, \cdots \in L_p$,

- If $f_n \xrightarrow{L_p} f$, then $f_n \xrightarrow{\mu} f$, and $\|f_n\|_p \to \|f\|_p$.
- If $f_n \to f, a.e.$ or in measure, then $\|f_n\|_p \to \|f\|_p \iff f_n \xrightarrow{L_p} f$.

*Proof.* When $f_n \xrightarrow{L_p} f$, let $A := \{|f_n - f| > \varepsilon\}$,

$$\mu(A) \le \frac{1}{\varepsilon^p} \int_X |f_n - f|^p \mathbf{I}_A \,\mathrm{d}\mu \le \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \to 0.$$

and obviously $\|f_n\|_p \to \|f\|_p$

On the other hand, when $f_n \to f, a.e.$ and $\|f_n\|_p \to \|f\|_p$, From $|a+b|^p \le C_p(|a|^p + |b|^p)$,

$$g_n := C_p(|f_n|^p + |f|^p) - |f_n - f|^p \ge 0.$$

$g_n \to 2C_p|f|^p, a.e.$, so

$$\int_X 2C_p|f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = 2C_p \int_X |f|^p \,\mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f|^p \,\mathrm{d}\mu.$$

When $f_n \to f$ in measure, for any subsequence there exist its subsequence $f_{n'} \to f, a.e.$, so $\|f_{n'} - f\|_p \to 0$, hence $\|f_n - f\|_p \to 0$. □

**Remark 0.1.3** — This theorem implies for any $L_p$ function $f$, we can take simple functions $f_1, f_2, \cdots \to f$ and $|f_n| \uparrow |f|$, so $f_n \xrightarrow{L_p} f$.

**Definition 0.1.4** (Weak convergence)**.** Let $1 < p < \infty$, and $f_1, f_2 \cdots \in L_p$. If

$$\lim_{n\to\infty} \int_X f_n g \, \mathrm{d}\mu = \int_X fg \, \mathrm{d}\mu, \quad \forall g \in L_q.$$

Then we say $f_n$ **weak convergent** to $f$, denoted by $f_n \xrightarrow{(w)L_p} f$.

When $p = 1$ and $(X, \mathscr{F}, \mu)$ is a $\sigma$-finite measure space, and the condition also holds, we say $\{f_n\}$ weak convergent to $f$ in $L_1$.

**Corollary 0.1.5**

Let $1 \le p < \infty$, then

$$f_n \xrightarrow{L_p} f \implies f_n \xrightarrow{(w)L_p} f.$$

*Proof.* By Holder's inequality,

$$\left| \int_X (f_n - f) g \, \mathrm{d}\mu \right| \le \|f_n - f\|_p \|g\|_q \to 0.$$

□

If $\sup_{t\in T} \|f_t\|_p =: M < \infty$, then we say $\{f_t, t \in T\}$ is **bounded in $L_p$**.

**Theorem 0.1.6**

Let $1 < p < \infty$, $\{f_n\} \subset L_p$, there exists $M$ s.t. $\|f_n\|_p \le M$, $\forall n$. If $f_n \to f$, *a.e.* or in measure, then $f \in L_p$ and $f_n \to f$ weakly.

*Proof.* First $\|f\|_p \le M$:

$$\int_X |f|^p \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \, \mathrm{d}\mu \le M^p.$$

Next we prove the weak convergence: For all $g \in L_q$, recall the bounded convergence theorem in probability, we can view $M$ as a bound of $f_n$, and $\|g\|_q$ as $P$.

Let $B = \{|f_n - f| \le \hat{\varepsilon}\}$, consider

$$a := \int_B (f_n - f) g \, \mathrm{d}\mu, \quad b := \int_{B^c} (f_n - f) g \, \mathrm{d}\mu.$$

Note that

$$|a| \le \hat{\varepsilon} \int_X |g| \, \mathrm{d}\mu.$$

But $\int_X |g| \, \mathrm{d}\mu$ might be infinity, so let $A_k := \{\frac{1}{k} \le |g|^q \le k\}$, we have

$$\int_{A_k} |g| \, \mathrm{d}\mu \le k^{\frac{1}{q}} \mu(A_k) < \infty.$$

$(\frac{1}{k}\mu(A_k) < \int_{A_k} |g|^q \,\mathrm{d}\mu < \infty$ since $g \in L_q)$.

Now we can proceed:

$$a := \int_{A_k B} (f_n - f) g \,\mathrm{d}\mu, \quad b := \int A_k^c \cup B^c (f_n - f) g \,\mathrm{d}\mu.$$

Now $|a| \le \hat{\varepsilon} k^{\frac{1}{q}} \mu(A_k) < \varepsilon$.

$$\left| \int_X (f_n - f) g \mathbf{I}_{A_k^c \cup B^c} \,\mathrm{d}\mu \right| \le \|f_n - f\|_p \|g \mathbf{I}_{A_k^c \cup B^c}\|_q \le 2M \left( \int_{A_k^c} |g|^q \,\mathrm{d}\mu + \int_{A_k \setminus B} |g|^q \,\mathrm{d}\mu \right).$$

By LDC(Dominated convergence), $A_k^c \to \{g = 0, \infty\}$, so $\int_{A_k^c} |g|^q \,\mathrm{d}\mu < \varepsilon$.

Since $\mu(A_k) < \infty$, $f_n \to f, a.e. \implies f_n \xrightarrow{\mu} f$. By the continuity of integrals, $\mu(A_k \setminus B) \le \mu(B^c) < \delta \implies \int_{A_k \setminus B} |g|^q \,\mathrm{d}\mu < \varepsilon$.

Now we can conclude: $\forall \varepsilon > 0$, first choose $k$ large, then $\hat{\varepsilon}$ small, we get

$$\int_X (f_n - f) g \,\mathrm{d}\mu \le \varepsilon + 4M\varepsilon \implies f_n \xrightarrow{(w)L_p} f.$$

$\square$

**Remark 0.1.7 —** The proof is a little complicated, we divide the entire integral to three part, and estimate them respectively.

When $p = 1$, $f_n$ bounded in $L_p$ *cannot* imply weak convergence.

**Example 0.1.8**

Let $X = \mathbb{N}$, $\mu(\{k\}) = 1, \forall k$, clearly it's $\sigma$-finite.

Let $f_n(k) = \mathbf{I}_{k=n}$, then $\|f_n\| = \sum_k \mu(k)|f_n(k)| = 1$, and $f_n \to 0, a.e.$.

But let $g = 1 \in L_\infty$, $\int_X (f_n - f) g \,\mathrm{d}\mu = 1 \not\to 0$.

**Proposition 0.1.9**

Let $f_1, f_2, \cdots \in L_1$, then:

$$\|f_n\| \to \|f\| \& f_n \to f, a.e. \implies f_n \xrightarrow{L_1} f \implies f_n \xrightarrow{(w)L_1} f \implies \int_A f_n \,\mathrm{d}\mu \to \int_A f \,\mathrm{d}\mu, \forall A.$$

*Proof.* For the last part let $g = \mathbf{I}_A$, the rest is trivial. $\square$

## §0.2 Integrals in probability space

We can also consider $L_p$ space in probability space $(\Omega, \mathscr{F}, P)$.

**Theorem 0.2.1**

Let $0 < s < t < \infty$. Then $L_t \subset L_s$. If $s \ge 1$, we have $\|f\|_s \le \|f\|_t$, with equality $f$ constant.

*Proof.* When $f \in L_t$, let $p = \frac{t}{s}, q = \frac{t}{t-s}$.

$$\int_\Omega |f|^s \cdot 1 \, \mathrm{d}P \leq \||f|^s\|_p \|1\|_q = (E|f|^{sp})^{\frac{1}{p}} = (E|f|^t)^{\frac{1}{p}}.$$

So $f \in L_s \implies L_t \subset L_s$. When $s \geq 1$,

$$\|f\|_s^s \leq (\|f\|_t)^{\frac{t}{p}} = \|f\|_t^s \implies \|f\|_s \leq \|f\|_t.$$

□

From this we know $L_\infty \subset L_p$, and $\|f\|_p \uparrow \|f\|_\infty$.

> **Remark 0.2.2** — This theorem does not hold for general space. Let $X = \mathbb{N}$, $\mu(\{n\}) = 1$, $f(n) = \frac{1}{n}$, then $f \in L_2 \backslash L_1$.

The expectation $Ef^k$ is called **$k$-order moment** of random variable $f$.

**Definition 0.2.3** (Uniformly integrable)**.** Let $\{f_t, t \in T\}$ be r.v.'s, if $\forall \varepsilon > 0, \exists \lambda > 0$, such that

$$E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} < \varepsilon, \quad \forall t \in T,$$

then we say $\{f_t, t \in T\}$ **uniformly integrable**.

If $\forall \varepsilon > 0, \exists \delta > 0$ s.t $\forall A \in \mathscr{F}$,

$$P(A) < \delta \implies E|f_t|\mathbf{I}_A < \varepsilon, \forall t \in T,$$

we say $\{f_t\}$ is uniformly absolutely continuous, which is abbreviated as absolutely continuous.

**Theorem 0.2.4**

Uniformly integrable $\iff$ absolute continuity and bounded in $L_1$.

*Proof.* Firstly when $\{f_t\}$ uniformly integrable, $\forall A \in \mathscr{F}, \lambda > 0$,

$$\begin{aligned} E|f_t|\mathbf{I}_A &= E|f_t|\mathbf{I}_{A\cap\{|f_t|\leq\lambda\}} + E|f_t|\mathbf{I}_{A\cap\{|f_t|>\lambda\}} \\ &\leq \lambda P(A) + E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \end{aligned}$$

Let $A = X$ we know $E|f_t| \leq \lambda + \frac{\varepsilon}{2}, \forall t \in T$. Now let $\delta = \frac{\varepsilon}{2\lambda}$ we get AC property.

On the other hand,

$$\lambda P(|f_t| > \lambda) \leq E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \leq E|f_t| \leq M, \forall t \in T.$$

So when $\lambda > \frac{M}{\delta}$, $P(|f_t| > \lambda) < \delta$, hence $E|f_t|\mathbf{I}_{\{|f_t|>\lambda\}} \leq \varepsilon$, $\forall t \in T$. □

**Theorem 0.2.5**

Let $0 < p < \infty$, and $f_n \to f$ in probability. TFAE:

(1) $\{|f_n|^p\}$ uniformly integrable;

(2) $f_n \xrightarrow{L_p} f$;

(3) $f \in L_p$ and $\|f_n\|_p \to \|f\|_p$.

*Proof.* (1) $\implies$ (2): Take subsequence $f_{n'} \to f, a.s.$,

$$E|f|^p \le \liminf_{n\to\infty} E|f_n|^p < \infty,$$

since $\{|f_n|^p\}$ is bounded in $L_1$. This means $f \in L_p$.

Let $A_n = \{|f_n - f| > \varepsilon\}$, now we compute

$$E|f_n - f|^p \le \varepsilon^p + E|f_n - f|^p \mathbf{I}_{A_n} \le \varepsilon^p + C_p E|f_n|^p \mathbf{I}_{A_n} + C_p E|f|^p \mathbf{I}_{A_n}$$

Since $P(A_n) \to 0$ and $\{|f_n|^p\}$ absolutely continuous (also note $E|f|^p \mathbf{I}_{A_n} \to 0$), RHS converges to 0. Therefore $f_n \xrightarrow{L_p} f$.

As for (3) $\implies$ (1), we'll prove a lemma:

**Lemma 0.2.6**

If $f_n \xrightarrow{P} f$, then $\forall 0 < p < \infty$,

$$|f_n|^p \mathbf{I}_{\{|f_n| \le \lambda\}} \xrightarrow{P} |f|^p \mathbf{I}_{\{|f| \le \lambda\}}, \quad \forall \lambda \in C(F_{|f|}).$$

By lemma and bounded convergence theorem, their expectation also converges. Note that $\|f_n\|_p \to \|f\|_p$, so

$$E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda\}} \to E|f|^p \mathbf{I}_{\{|f| > \lambda\}},$$

thus $\forall \varepsilon > 0, \exists \lambda_0 \in C(F_{|f|})$, s.t. $E|f|^p \mathbf{I}_{\{|f| > \lambda_0\}} < \frac{\varepsilon}{2}$, thus

$$\exists N, \quad E|f_n|^p \mathbf{I}_{\{|f_n| > \lambda_0\}} < \varepsilon, \quad \forall n > N.$$

Now we can take $\lambda > \lambda_0$ such that $\max_{n \le N} E|f_n|^p \mathbf{I}_{\{|f_n|^p > \lambda\}} < \varepsilon$, and we're done.

*Proof of the lemma.* Since $|f_n| \to |f|$ in probability, WLOG $f_n, f \ge 0$. Define

$$A_n := (\{f_n \le \lambda\} \Delta \{f \le \lambda\}) \cap \{|f_n^p - f^p| > \varepsilon\}$$
$$B_n := \{f_n, f \le \lambda, |f_n^p - f^p| > \varepsilon\}.$$

Since $x^p$ is uniformly continuous in $[0, \lambda]$, $B_n \subset \{|f_n - f| > \kappa_{\varepsilon,\lambda}\}$, $P(B_n) \to 0$.

Also $P(A_n) \to 0$ as

$$A_n \subset \{\lambda - \delta < f \le \lambda + \delta\} \cup \{|f_n - f| > \delta\},$$

and $F_{|f|}$ continuous at $\lambda$. □

□

# §1 Signed measure

## §1.1 Definitions

Let $(X, \mathscr{F}, \mu)$ be a measure space, consider

$$\varphi(A) := \int_A f \, \mathrm{d}\mu, \quad \forall A \in \mathscr{F}.$$

If the integral of $f$ exists, then $\varphi$ has countable additivity. Also note $\varphi(\emptyset) = 0$, so $\varphi$ looks like a measure, except it can take negative values.

In fact, denote $X^+ = \{f \ge 0\}, X^- = \{f < 0\}$, then $\varphi(A) = \varphi(AX^+) + \varphi(AX^-)$.

**Definition 1.1.1** (Signed measure)**.** If a set function $\varphi : \mathscr{F} \to \overline{\mathbb{R}}$ which satisfies countable additivity and $\varphi(\emptyset) = 0$, then we call $\varphi$ a **signed measure**.

If $|\varphi(A)| < \infty, \forall A \in \mathscr{F}$, then $\varphi$ is **finite**; Similarly we define **$\sigma$-finite**.

Since $\int_A f \, \mathrm{d}\mu$ can't reach both $\pm\infty$(otherwise the integral doesn't exist), so

**Proposition 1.1.2**
Let $\varphi$ be a signed measure, then:

$$\varphi(A) < \infty, \quad \forall A \in \mathscr{F}, \quad \textit{or} \quad \varphi(A) > -\infty, \quad \forall A \in \mathscr{F}.$$

*Proof.* Assume that $\varphi(A) = \infty, \varphi(B) = -\infty$, then:

$$\varphi(A \cup B) = \varphi(A) + \varphi(A \backslash B) = +\infty,$$

and similarly $\varphi(A \cup B) = -\infty$, contradiction! □

**Remark 1.1.3** — From now on we may assmue $\varphi(A) > -\infty$.

**Proposition 1.1.4**
If $A \supseteq B$, and $|\varphi(A)| < \infty$, then $|\varphi(B)| < \infty$.

*Proof.* Trivial, same as above proposition. □

**Proposition 1.1.5**
Let $A_1, A_2, \ldots$ be pairwise disjoint sets, and $|\varphi(\sum_{n=1}^{\infty} A_n)| < \infty$, then

$$\sum_{n=1}^{\infty} |\varphi(A_n)| < \infty.$$

*Proof.* Let $I = \{n : \varphi(A_n) > 0\}$, $J = \{n : \varphi(A_n) < 0\}$,

$$B = \sum_{n \in I} A_n, \quad C = \sum_{n \in J} A_n,$$

since $B, C \subset \sum_{n=1}^{\infty} A_n$, thus $\varphi(B), \varphi(C) \in \mathbb{R}$.

Note that $\sum_{n \in I} |\varphi(A_n)| = |\varphi(B)|$, $\sum_{n \in J} \varphi(A_n) = |\varphi(C)|$, and we're done. □

## §1.2 Hahn decomposition and Jordan decomposition

Let's look at the indefinite integral again, notice that

$$\varphi(A) = \int_{A \cap \{f > 0\}} f \, \mathrm{d}\mu + \int_{A \cap \{f < 0\}} f \, \mathrm{d}\mu = \int_A f^+ \, \mathrm{d}\mu - \int_A f^- \, \mathrm{d}\mu.$$

It turns out that this property holds for any signed measure.

**Definition 1.2.1** (Hahn decomposition)**.** If a patition $\{X^+, X^-\}$ of $X$ satisfies:

$$\varphi(A) \geq 0, \forall A \subset X^+, \quad \varphi(A) \leq 0, \forall A \subset X^-,$$

then $\{X^+, X^-\}$ is called a **Hahn decomposition** of $\varphi$.

**Definition 1.2.2** (Jordan decomposition)**.** Let $\varphi^\pm = \int_A f^\pm \, d\mu$ be measures, if

$$\varphi = \varphi^+ - \varphi^-,$$

then it's called a **Jordan decomposition** of $\varphi$.

We're going to find $X^+$, or equivalently, find $\varphi^+$. Let $\varphi^*(A) := \sup\{\varphi(B) : B \subseteq A\}$.
It's clear that $\varphi^*$ is non-negative, monotone, and $\varphi^*(\emptyset) = 0$.
Consider $\mathscr{F}^- = \{A : \varphi^*(A) = 0\}$. Intuitively, this is all the subsets of $X^-$, unioned with "null sets" in $X^+$.

**Theorem 1.2.3** (Hahn decomposition)

Let $X^-$ be a set with maximum $|\varphi|$ in $\mathscr{F}^-$, (since $\varphi > -\infty$, $X^-$ must exist) and $X^+ = X \backslash X^-$ doesn't contain any set $A$ with $\varphi(A) < 0$.

Furthermore, the Hahn decomposition is unique:

$$\varphi(A) = 0, \quad \forall A \in X_1^+ \Delta X_2^+ = X_1^- \Delta X_2^-.$$

The critical part of this theorem is:

**Lemma 1.2.4**

If $\varphi(A) < 0$, then we can find $A_0 \subset A$ s.t. $\varphi^*(A_0) = 0$, $\varphi(A_0) < 0$.

To prove this lemma, we need another lemma:

**Lemma 1.2.5**

If $\varphi(A) < \infty$, then $\forall \varepsilon > 0$, $\exists A_\varepsilon \subset A$ s.t.

$$\varphi(A_\varepsilon) \geq 0, \quad \varphi^*(A \backslash A_\varepsilon) \leq \varepsilon.$$

*Proof.* Assume by contradiction that $\exists \varepsilon_0 \geq 0$ s.t. $\forall A_0 \subset A$, $\varphi(A_0) < 0$ or $\varphi^*(A \backslash A_0) > \varepsilon_0$, this means,

$$\varphi(A_0) \geq 0 \implies \varphi^*(A \backslash A_0) > \varepsilon_0.$$

This will clearly yield a contradiction:

Take any $\varphi(A_0) \geq 0$(say $A_0 = \emptyset$), then exists $A_1 \subset A \backslash A_0$ s.t. $\varphi(A_1) > \varepsilon_0$, and $\varphi(A_0 \cup A_1) \geq 0$, continuing this process we can get infinitely many pairwise disjoint sets $A_1, A_2, \ldots$, with $\varphi(A_n) > \varepsilon_0$, so $\varphi(\sum_{i=1}^\infty A_n) = \infty \implies \varphi(A) = \infty$, contradiction! ☐

*Proof of Lemma 1.2.4.* Applying above lemma repeatedly and take a limit:
Take $C_1 \subset A$ s.t. $\varphi(C_1) \geq 0$ and $\varphi^*(A \backslash C_1) \leq 1$. Let $A_1 = A \backslash C_1$, $\varphi(A_1) < 0$.
Again take

$$C_{k+1} \subset A_k, A_{k+1} = A_k \backslash C_{k+1} \implies \varphi^*(A_{k+1}) \leq \frac{1}{k+1}, \varphi(A_{k+1}) < 0.$$

Since $A_k \downarrow$, let $A_0 = \lim_{k\to\infty} A_k$, note $\varphi^*(A_k) \downarrow 0$, we must have $\varphi^*(A_0) = 0$.
Also $\varphi(\sum C_k) = \sum \varphi(C_k) \geq 0$, so $\varphi(A_0) < 0$. □

*Proof of Theorem 1.2.3.* First we prove that $\mathscr{F}^-$ is a $\sigma$-ring: $\emptyset \in \mathscr{F}^-$, if $A_1, A_2 \in \mathscr{F}^-$,

$$0 \leq \varphi^*(A_1 \backslash A_2) \leq \varphi(A_1) = 0.$$

Thus $A_1 \backslash A_2 \in \mathscr{F}^-$.
If $A_1, A_2, \cdots \in \mathscr{F}^-$ pairwise disjoint,

$$\varphi(B) = \sum_{n=1}^{\infty} \varphi(B \cap A_n) \leq 0, \quad \forall B \subset \sum_{n=1}^{\infty} A_n.$$

Hence $\sum_{n=1}^{\infty} A_n \in \mathscr{F}^-$.
Next we'll prove Hahn decomposition exists:
Let $\alpha := \inf\{\varphi(A) : A \in \mathscr{F}^-\}$, $\alpha \leq 0$.
Let $\{A_n\} \in \mathscr{F}^-$ s.t. $\varphi(A_n) \to \alpha$, then $X^- := \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}^-$.

$$\varphi(X^-) = \varphi(A_n) + \varphi(X^- \backslash A_n) \leq \varphi(A_n) + \varphi^*(X^- \backslash A_n) = \varphi(A_n) \to \alpha.$$

Therefore $-\infty < \varphi(X^-) = \alpha$.
Hence $\forall A, \varphi(AX^-) \leq \varphi^*(X^-) = 0$. By Lemma 1.2.4 we get $\forall A, \varphi(AX^+) \geq 0$, otherwise $\exists A_0 \subset A$ s.t. $\varphi^*(A_0) = 0, \varphi(A_0) < 0$. Then $\varphi(X^- \cup A_0) = \alpha + \varphi(A_0) < \alpha$, contradiction!
At last we'll prove the uniqueness:
If $X_1^\pm, X_2^\pm$ are both Hahn decompositions, then $A \in X_1^+ \cap X_2^- + X_1^- \cap X_2^+$, it's clear $\varphi(A) = 0$. □

**Theorem 1.2.6** (Jordan decomposition)
The Jordan decomposition exists and is unique:

$$\varphi = \varphi^+ - \varphi^-, \quad \varphi^+ = \varphi^*, \varphi^- = (-\varphi)^*.$$

*Proof.* Let $\varphi^\pm$ be measures with $\varphi^\pm = \pm\varphi(A \cap X^\pm)$. It's clear that this is a Jordan decomposition.
Now given any Jordan decomposition $\varphi^\pm$.
Since

$$\forall B \subset A, \varphi(B) \leq \varphi^+(B) \leq \varphi^+(A),$$

so $\varphi^* \leq \varphi^+$. But $A \cap X^+ \subset A$, so $\varphi^* \geq \varphi^+$, which proves the result.
Similarly $\varphi^- = (-\varphi)^*$, so it is unique. □

**Remark 1.2.7 —** The support of $\varphi^\pm$ are disjoint, but if $\phi \neq 0$, then the support of $\varphi^\pm + \phi$ intersects. $\varphi^\pm$ are called the **upper variation** and **lower variation**, respectively, and $|\varphi| = \varphi^+ + \varphi^-$ is called the **total variation**.