

Linear Algebra II

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§0.1 Normal maps

Recall that we say two matrices A, B are orthogonally similar, if there exist $P \in O(n)$ s.t. $B = P^{-1}AP$. Again, we want to find the “simplest” matrix in each orthogonal equivalent class.

Let $T \in L(V)$ be a linear map, if there exists an orthonormal basis of V s.t. $[T]_{\mathcal{B}}$ is diagonal, then we say T is orthogonally (or unitarily) diagonalizable.

Definition 0.1.1 (Normal maps). Let V be an inner product space, $T \in L(V)$. If $TT^* = T^*T$, then we say T is a **normal map**.

It turns out that these concepts has close relations:

Theorem 0.1.2

Let V be a finite dimensional inner product space,

- If $F = \mathbb{R}$, then T orthogonally diagonalizable $\iff T$ self-adjoint;
- If $F = \mathbb{C}$, then T unitarily diagonalizable $\iff T$ normal.

Lemma 0.1.3

Let $F = \mathbb{C}$, then T normal \iff there exists self-adjoint commutative maps T_1, T_2 s.t. $T = T_1 + iT_2$.

Proof. If $T = T_1 + iT_2$, then $T^* = T_1 - iT_2$, so $T^*T = TT^*$ since T_1, T_2 commutative.

On the other hand, let $T_1 = \frac{T+T^*}{2}$, $T_2 = \frac{T-T^*}{2i}$. We can check that T_1, T_2 self-adjoint and are commutative. \square

Proof of Theorem 0.1.2. For the “ \implies ” part, let \mathcal{B} be an orthonormal basis such that $[T]_{\mathcal{B}} = \text{diag}\{c_1, \dots, c_n\}$. Then we have

$$[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^* = \text{diag}\{\bar{c}_1, \dots, \bar{c}_n\}.$$

If $F = \mathbb{R}$, then $T^* = T$, i.e. T self-adjoint.

If $F = \mathbb{C}$, clearly $TT^* = T^*T$, so T is normal.

As for the other part, we need a lemma first.

Lemma 0.1.4

Let V be a f.d. inner product space, $T \in L(V)$. If $W \subset V$ is a T -invariant space, then W^\perp is T^* -invariant.

Proof of the lemma. For all $\alpha \in W^\perp$,

$$0 = \langle \alpha, T\beta \rangle = \langle T^*\alpha, \beta \rangle, \quad \forall \beta \in W.$$

Thus $T^*\alpha \in W^\perp$. □

Corollary 0.1.5

If T is self-adjoint, $W \subset V$ is T -invariant will imply W^\perp is also T -invariant, so T is semisimple.

Lemma 0.1.6

Let V be a f.d. inner product space, $T \in L(V)$ is self-adjoint. We must have $f_T \in \mathbb{R}[x]$, and it can be decomposed to products of polynomials of degree 1.

In particular, $\sigma(T) \subset \mathbb{R}$.

Proof. Let $f_T = \prod_{j=1}^n (x - c_j)$, $c_j \in \mathbb{C}$.

Let \mathcal{B} be an orthonormal basis of V , then $A := [T]_{\mathcal{B}}$ is Hermite. Let X be a nonzero vector s.t. $AX = c_j X$, then

$$c_j X^* X = X^* AX = (AX)^* X = \bar{c}_j X^* X.$$

So $c_j \in \mathbb{R}$, and we're done. □

Lemma 0.1.7

If T is a self-adjoint map, then all the eigenspaces of T are pairwise orthogonal.

Proof. Let $c_1, c_2 \in \mathbb{R}$ be two eigenvalues of T . Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$.

$$c_1 \langle \alpha, \beta \rangle = \langle c_1 \alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T\beta \rangle = \bar{c}_2 \langle \alpha, \beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

Since $c_1 \neq c_2$, we must have $\alpha \perp \beta$, as desired. □

Returning back to [Theorem 0.1.2](#), when T is self-adjoint, let $\sigma(T) = \{c_1, \dots, c_r\}$.

Claim 0.1.8. $V = \bigoplus_{i=1}^r V_{c_i}$, i.e. T is diagonalizable.

Let $W = \bigoplus_{i=1}^r V_{c_i}$, if $W^\perp \neq \{0\}$, then W^\perp is T -invariant.

When $F = \mathbb{C}$, then T_{W^\perp} has eigenvectors; when $F = \mathbb{R}$, then T_{W^\perp} is self-adjoint, so it must have a eigenvector (by lemma).

Since V_{c_i} are pairwise orthogonal, so we can actually take an orthonormal basis of V_{c_i} to get an orthonormal basis of V . Hence T is orthogonally diagonalizable.

Now for the case when T is normal, let T_1, T_2 be self-adjoint maps s.t. $T = T_1 + iT_2$. Since T_1, T_2 commute, the proof is nearly identical to the simultaneously diagonalizable property.

Let $V = \bigoplus_{i=1}^r V_{c_i}$ be the eigenspace decomposition of T_1 . Note that V_{c_i} are also T_2 -invariant.

Since $(T_2)_{V_{c_i}}$ self-adjoint, $(T_2)_{V_{c_i}}$ is unitarily diagonalizable. Therefore we can concatenate those basis to get a basis of V , and T_1, T_2 are both diagonal under this basis. \square

There's another proof of " \Leftarrow " part of the theorem:

Proposition 0.1.9

Let V be an inner product space, $T \in L(V)$ normal. Let $W \subset V$ be a T -invariant space, then W^\perp is T -invariant, and W is T^* -invariant.

Proof. Take an orthonormal basis of W, W^\perp , so $A := [T]_B$ normal.

Since W is T -invariant, $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Note that:

$$AA^* = \begin{pmatrix} BB^* + CC^* & * \\ * & * \end{pmatrix}, \quad A^*A = \begin{pmatrix} B^*B & * \\ * & * \end{pmatrix}.$$

As A normal, $BB^* + CC^* = B^*B$, by looking at the trace of both sides, we get $\text{tr}(CC^*) = 0 \implies C = 0$, the conclusion follows. \square

Corollary 0.1.10

Let $A \in \mathbb{C}^{n \times n}$ be an upper triangular matrix, then A normal $\iff A$ diagonal.

Proposition 0.1.11

Let T be a normal map, then the eigenspaces of T are pairwise orthogonal.

Proof. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$, since $\text{span}\{\beta\}$ is a T -invariant space, so $T^*\beta \in \text{span}\{\beta\}$,

$$\langle T^*\beta, \beta \rangle = \langle \beta, T\beta \rangle = \bar{c}_2 \langle \beta, \beta \rangle.$$

Thus $T^*\beta = \bar{c}_2\beta$.

$$c_1 \langle \alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

But $c_1 \neq c_2$, we have $\alpha \perp \beta$. \square

When $F = \mathbb{C}$: Let $W = \bigoplus_{i=1}^r V_{c_i}$. Since W^\perp is T -invariant, so when $W^\perp \neq \{0\}$, T must have eigenvalues in W^\perp , contradiction!

Now we've proved that V_{c_i} are pairwise orthogonal, so T is unitarily diagonalizable.

Proposition 0.1.12

Let V be a complex inner product space, $T \in L(V)$ normal,

- T self-adjoint $\iff \sigma(T) \subset \mathbb{R}$;
- T anti self-adjoint $\iff \sigma(T) \subset i\mathbb{R}$;
- T unitary $\iff \sigma(T) \subset \{z : |z| = 1\}$.

Proof. Take an orthonormal basis s.t. $[T]_{\mathcal{B}}$ diagonal. The rest is trivial. \square

§1 Bilinear forms

Let V be a finite dimensional vector space, $\dim V = n$.

Definition 1.0.1. Let $F = \mathbb{C}$, we say a function $f : V \times V \rightarrow F$ is a **semi bilinear form** if:

- $f(c_1\alpha + \beta, \gamma) = c_1f(\alpha, \gamma) + f(\beta, \gamma)$;
- $f(\alpha, c_1\beta + \gamma) = \bar{c}_1f(\alpha, \beta) + f(\alpha, \gamma)$.

Let $\text{Form}(V)$ denote the (semi) bilinear forms on (complex) real vector space V .

For $f \in \text{Form}(V)$, fix a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of V , let $[f]_{\mathcal{B}} \in F^{n \times n}$ be the matrix

$$([f]_{\mathcal{B}})_{jk} = f(\alpha_k, \alpha_j).$$

which is called **the matrix of f under \mathcal{B}** .

For $\alpha = \sum_{k=1}^n x_k \alpha_k, \beta = \sum_{j=1}^n y_j \alpha_j \in V$. It's clear that

$$f(\alpha, \beta) = \sum_{j,k=1}^n x_k \bar{y}_j f(\alpha_k, \alpha_j) = \sum_{j,k=1}^n x_k \bar{y}_j ([f]_{\mathcal{B}})_{jk} = [\beta]_{\mathcal{B}}^* [f]_{\mathcal{B}} [\alpha]_{\mathcal{B}}.$$

From this we know that the map $\text{Form}(V) \rightarrow F^{n \times n}, f \mapsto [f]_{\mathcal{B}}$ is a linear isomorphism. Since if $[f]_{\mathcal{B}} = 0$, then $f(\alpha, \beta) = 0$ for all $\alpha, \beta \in V$. Thus it's injective. Obviously it's surjective and linear, so

$$\dim \text{Form}(V) = n^2$$

Example 1.0.2

Let $A \in F^{n \times n}$. Let $f \in \text{Form}(F^{n \times 1})$ be

$$f(X, Y) = Y^* A X, \quad \forall X, Y \in F^{n \times 1}.$$

Let \mathcal{B} be the standard basis of F , it's clear that $[f]_{\mathcal{B}} = A$.

Proposition 1.0.3

Let $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_n\}$ be another basis of V , $P \in \text{GL}_n(F)$ satisfies

$$(\alpha'_1, \dots, \alpha'_n) = (\alpha_1, \dots, \alpha_n)P.$$

Then $[f]_{\mathcal{B}'} = P^* [f]_{\mathcal{B}} P$.

Proof. Since $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$, just plug this into the definition of $[f]_{\mathcal{B}}$, the rest is trivial. \square

Definition 1.0.4. Let $f \in \text{Form}(V)$.

- When $F = \mathbb{R}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = f(\beta, \alpha)$, then we say f is symmetrical (also called Hermite);
- When $F = \mathbb{C}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, we say f is Hermite.

Proposition 1.0.5

When $F = \mathbb{C}$, f Hermite $\iff f(\alpha, \alpha) \in \mathbb{R}, \forall \alpha \in V$.

Proof. For the “ \Leftarrow ” direction, consider $f(\alpha + \beta, \alpha + \beta) \in \mathbb{R}$. Expanding we’ll get $f(\alpha, \beta) + f(\beta, \alpha) \in \mathbb{R}$, i.e.

$$f(\alpha, \beta) + f(\beta, \alpha) = \overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Replace α with $i\alpha$, we get

$$f(\alpha, \beta) - f(\beta, \alpha) = -\overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Combining two equations we get the conclusion. \square

Definition 1.0.6. Let $f \in \text{Form}(V)$ be an Hermite form. If $\forall \alpha \in V \setminus \{0\}$, $f(\alpha, \alpha) > 0$, we say f is **positive definite**.

Similarly we can define negative definite and semi positive definite.

Note that a positive definite Hermite form is nothing but an inner product.

§1.1 Positive definite matrices

In this section we’ll dig deeper into properties of positive definite matrices.

It’s clear that if a matrix A is positive definite, then A is invertible, and P^*AP is also positive definite. In particular, P^*P is positive definite.

Theorem 1.1.1 (Cholesky decomposition)

Let $A \in F^{n \times n}$ be a positive definite matrix, there exists a unique upper triangular matrix R with positive diagonal entries s.t. $A = R^*R$.

Proof. Consider the inner product $f(X, Y) = Y^*AX$. Let the standard inner product on V be $f_0(X, Y) = Y^*X$.

Since inner product spaces with same dimensions are isomorphic, so there exists a matrix $R \in \text{GL}_n(F)$, such that

$$R : (F^{n \times 1}, f) \rightarrow (F^{n \times 1}, f_0), \quad X \mapsto RX$$

is an isomorphism of inner product space, i.e. $f_0(RX, RY) = f(X, Y)$. This is equivalent to $A = R^*R$.

For any $P \in \text{GL}_n(F)$, P is also an isomorphism of $(F^{n \times 1}, f) \rightarrow (F^{n \times 1}, f_0)$ iff RP^{-1} preserves the inner product f_0 , iff $RP^{-1} \in \text{O}(n)$ or $\text{U}(n)$.

By QR decomposition, $R = RP^{-1} \cdot P$, so there must be a unique P s.t. P upper triangular with positive diagonal entries. \square

Corollary 1.1.2

A positive definite $\implies \det A > 0$.

Definition 1.1.3. Let $A \in F^{n \times n}$, for $1 \leq k \leq n$, define

$$\Delta_k(A) := \det(A_{1 \leq i \leq k}^{1 \leq j \leq k})$$

be the **leading principal minor**.

Theorem 1.1.4

Let $A \in F^{n \times n}$ be an Hermite matrix. Then A positive definite $\iff \Delta_k(A) > 0, k = 1, \dots, n$.

Lemma 1.1.5 (LU decomposition)

Let F be any field. For $A \in \text{GL}_n(F)$, the followings are equivalent:

- $\Delta_k(A) \neq 0, k = 1, \dots, n$;
- $A = LU$, where L lower triangular, and U upper triangular with diagonal entries 1.

Proof. On one hand, Let L_k, U_k be the top-left $k \times k$ submatrix of L, U , since L, U invertible, L_k, U_k invertible. By the triangular property, $\Delta_k(A) = \det(L_k U_k) \neq 0$.

On the other hand, it's sufficient to prove:

$\exists N$ strictly upper triangular, $A(N + I_n)$ lower triangular

Let A_k be the k -th leading principal submatrix of A , and $\alpha_{k+1}, \beta_{k+1} \in F^{n \times 1}$ the $(k+1)$ -th column of A, N .

Now compute the first k rows of the $(k+1)$ -th column of $A(N + I)$, which is equal to $A_k \beta'_{k+1} + \alpha'_{k+1}$, where $\alpha'_{k+1}, \beta'_{k+1}$ is the first k entries of $\alpha_{k+1}, \beta_{k+1}$.

Since A_k invertible, $\exists \beta'_{k+1}$ s.t. $A_k \beta'_{k+1} + \alpha'_{k+1} = 0$.

Hence these β'_{k+1} forms a strictly upper triangular matrix N , as desired. \square

Proof of the theorem. Let A be an Hermite matrix, if A positive definite, then $\det A \geq 0$.

Let A_k be the upper left $k \times k$ submatrix of A . For $X \in F^{k \times 1} \setminus \{0\}$, we have

$$X^* A_k X = \begin{pmatrix} X \\ 0 \end{pmatrix}^* A \begin{pmatrix} X \\ 0 \end{pmatrix} > 0.$$

Hence A_k positive definite, $\det A_k = \Delta_k(A) \geq 0$.

Conversely, by our lemma let $A = LU$, let $D = (U^*)^{-1}L$, $A = U^* D U$.

Hence A Hermite $\implies D$ Hermite. Moreover D is lower triangular, so D is diagonal.

Some computation yields that $A_k = U_k^* D_k U_k$. Therefore

$$\Delta_k(A) \geq 0 \implies \det(U_k^* D_k U_k) \geq 0 \implies \det D_k \geq 0.$$

From this we deduce that all the diagonal entries of D are positive, so D positive definite $\implies A$ positive definite. \square

§1.2 Bilinear forms on inner product spaces

Let V be an inner product space, given a basis of V , recall that there are two linear isomorphism:

$$\text{Form}(V) \rightarrow F^{n \times n}, f \mapsto [f]_{\mathcal{B}} \quad L(V) \rightarrow F^{n \times n}, T \mapsto [T]_{\mathcal{B}}$$

Hence we can define a map $\text{Form}(V) \rightarrow L(V)$ by composing these two isomorphism. Denote this map by $f \mapsto T_f$. It seems like this map also depends on the choice of the basis, but in fact it's independent as long as \mathcal{B} is orthonormal!

Let \mathcal{B}' be another orthonormal basis, then $[T_f]_{\mathcal{B}'} = P^{-1}[T_f]_{\mathcal{B}}P$, while $[f]_{\mathcal{B}'} = P^*[f]_{\mathcal{B}}P$, but P is orthogonal (or unitary), so $P^{-1} = P^*$, i.e. T_f doesn't change under the new basis.

Since T_f do not depend on the basis, thus we wonder whether we can define this map intrinsically.

Proposition 1.2.1

For all $T \in L(V)$, T induces a (semi) bilinear form $f(\alpha, \beta) = \langle T\alpha, \beta \rangle$. We claim that this map \mathcal{F} gives an isomorphism of $L(V)$ and $\text{Form}(V)$.

Proof. Clearly \mathcal{F} is injective:

$$\langle T\alpha, \beta \rangle = 0, \forall \beta \implies T\alpha = 0,$$

thus $\ker \mathcal{F} = \{0\}$.

By dimensional reasons \mathcal{F} must be an isomorphism. □

By considering \mathcal{F}^{-1} , we get an one-to-one map $f \mapsto T_f$ s.t.

$$f(\alpha, \beta) = \langle T_f \alpha, \beta \rangle.$$

We'll see that this definition coincide with the initial one. In fact it's sufficient to prove $[T_f]_{\mathcal{B}} = [f]_{\mathcal{B}}$, which is just a bunch of computation ;)

Remark 1.2.2 — We can construct T_f explicitly from f :

The inner product gives a conjugate linear isomorphism

$$\Phi : V \rightarrow V^*, \quad \Phi(\alpha)(\beta) = \langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}.$$

Similarly, $f \in \text{Form}(V)$ gives a conjugate linear map

$$\Phi_f : V \rightarrow V^*, \quad \Phi_f(\alpha)(\beta) = \overline{f(\alpha, \beta)}.$$

Then $T = \Phi^{-1} \circ \Phi_f$ is the desired linear map:

$$\langle T\alpha, \beta \rangle = \overline{\Phi(T\alpha)(\beta)} = \overline{\Phi_f(\alpha)(\beta)} = f(\alpha, \beta).$$

Hence all the properties of linear maps can be carried over to the forms, and vice versa (using the matrix representation).

Corollary 1.2.3

Let $F = \mathbb{C}$, $T \in L(V)$, T self-adjoint iff $\langle T\alpha, \alpha \rangle \in \mathbb{R}, \forall \alpha \in V$.

Proof. T self-adjoint iff f Hermite iff $f(\alpha, \alpha) \in \mathbb{R}$. □

Corollary 1.2.4

Let $f \in \text{Form}(V)$.

- If f Hermite, there exists an orthonormal basis of V s.t. $[f]_{\mathcal{B}}$ is real diagonal.
- If $F = \mathbb{C}$, there exists an orthonormal basis such that $[f]_{\mathcal{B}}$ upper triangular.

§1.3 Spectral decomposition**Theorem 1.3.1** (Spectral decomposition of normal maps)

Let $T \in L(V)$ be a self-adjoint map (or normal map in complex field), let $\sigma(T) = \{c_1, \dots, c_k\}$, $P_i \in L(V)$ are the projection onto V_{c_i} . Then for any $f \in F[x]$, we have

$$f(T) = \sum_{i=1}^k f(c_i)P_i.$$

In particular, $T = \sum_{i=1}^k c_i P_i$.

Proof. Consider the orthogonal direct sum

$$V = \bigoplus_{i=1}^k V_{c_i},$$

since previously we've proven that T is orthogonally diagonalizable (or unitarily diagonalizable).

Using this decomposition, the conclusion is somewhat trivial. \square

Corollary 1.3.2

Each P_i is a polynomial of T .

Proof. Take $f_i \in F[x]$ s.t. $f_i(c_i) = \delta_{ij}$. Then $f_i(T) = \sum_{j=1}^k f_i(c_j)P_j = P_i$. \square

Using similar technique, we can consider functions other than polynomials of T , defined by $\phi(T) = \sum_{i=1}^k \phi(c_i)P_i$. By Lagrange interpolation, we can always find a polynomial p s.t. $p(c_i) = \phi(c_i)$ for all $c_i \in \sigma(T)$.

Example 1.3.3

If T semi positive definite normal matrix, $\sigma(T) \subset [0, +\infty)$, so we can define $\sqrt{T} = \sum_{i=1}^k \sqrt{c_i}P_i$.

Proposition 1.3.4

T self-adjoint (normal) $\implies \phi(T)$ self-adjoint (normal); $\sigma(\phi(T)) = \phi(\sigma(T))$.

Proof. Note that T and $\phi(T)$ are diagonal matrices under orthonormal basis of V_{c_i} . \square

Theorem 1.3.5

Let $T \in L(V)$ be semi positive definite.

- \sqrt{T} semi positive definite, and $\sqrt{T}^2 = T$.
- T positive definite $\iff \sqrt{T}$ positive definite.
- If $S \in L(V)$ semi positive definite and $S^2 = T$, then $S = \sqrt{T}$.

Proof. Since $[\sqrt{T}]_{\mathcal{B}} = \text{diag}(\sqrt{c_1}I_{d_1}, \dots, \sqrt{c_k}I_{d_k})$, the first two statements are trivial.

Let $\sigma(S) = \{s_1, \dots, s_r\}$, $V_i = \ker(S - s_i \text{id})$. Since S self-adjoint, $V = \bigoplus_{i=1}^r V_i$.

For any $\alpha \in V_i$, $T\alpha = S^2\alpha = s_i^2\alpha$, thus $V_i \subset \ker(T - s_i^2 \text{id})$. Since $s_i \geq 0$, $\sqrt{T} = S$. \square

Note that T^*T is always positive definite, so we can consider $\sqrt{T^*T}$. We call the eigen-values of $\sqrt{T^*T}$ **singular values** of T .

In some sense, $\sqrt{T^*T}$ is a semi positive approximation of T :

Lemma 1.3.6

For any $\alpha \in V$, $\|T\alpha\| = \|\sqrt{T^*T}\alpha\|$. In particular, $\ker T = \ker \sqrt{T^*T}$.

Proof. Let $N = \sqrt{T^*T}$,

$$\|N\alpha\|^2 = \langle N\alpha, N\alpha \rangle = \langle N^2\alpha, \alpha \rangle = \langle T^*T\alpha, \alpha \rangle = \langle T\alpha, T\alpha \rangle = \|T\alpha\|^2.$$

\square

Theorem 1.3.7 (Polar decomposition)

Let $T \in L(V)$,

- (1) There exists $U \in L(V)$ orthogonal or unitary, $N \in L(V)$ semi positive definite, $T = UN$.
- (2) We must have $N = \sqrt{T^*T}$.
- (3) T invertible iff N positive definite, in this case U is unique.

Remark 1.3.8 — This is a generalization of $z = re^{i\theta}$ in \mathbb{C} . That's where the name comes from.

Proof. If (1) holds, then

$$T^* = NU^* \implies T^*T = NU^*UN = N^2 \implies N = \sqrt{T^*T}.$$

Since T, N are semi positive definite, T invertible iff T positive definite. Now we must have $U = TN^{-1}$, which is unique.

To prove (1), when T invertible, let N, U as above, by our lemma,

$$\|U\alpha\| = \|TN^{-1}\alpha\| = \|\alpha\|$$

Thus U is orthogonal or unitary.

When T is not invertible, $\ker T = \ker N$, thus $\exists U_1 : \text{Im}(N) \rightarrow \text{Im}(T)$ s.t. $T = U_1 N$. (Just take $N\alpha \mapsto T\alpha$)

Moreover U_1 is an isomorphism of inner product space: $\|U_1 N\alpha\| = \|T\alpha\| = \|N\alpha\|$. So U_1 preserves inner product and hence injective. By dimension reasons, U_1 must be an isomorphism.

Now we can take an arbitrary isomorphism $U_2 : \text{Im}(N)^\perp \rightarrow \text{Im}(T)^\perp$, $U := U_1 \oplus U_2$ is the desired map. \square

Looking back at the singular values, consider the image of unit sphere $S \subset V$ under T , $N(S)$ is an ellipsoid:

$$N(S) = \left\{ \sum_{i=1}^n c_i x_i \alpha_i : \sum_{i=1}^n x_i^2 = 1 \right\}.$$

Since $T = UN$, U is a rotation, so $T(S)$ is also an ellipsoid, whose axes lengths are $2c_i$, where c_i are singular values of T .

Corollary 1.3.9 (Singular value decomposition, SVD)

Let $A \in F^{n \times n}$, then there exists decomposition $A = U_1 D U_2$, where D is a diagonal matrix with non-negative entries, U_1, U_2 are orthogonal or unitary matrices.

Proof. Consider the polar decomposition $A = UN$, let $N = PDP^{-1}$, where P orthogonal or unitary, D non-negative diagonal. Thus we can take $U_1 = UP, U_2 = P^{-1}$.

Note that the diagonal entries of D is precisely the singular value of A . \square

Corollary 1.3.10

Let $T \in L(V)$, then T map *some* orthogonal basis to another orthogonal basis.

Proof. Let $T = UN$ be the polar decomposition. Let $\alpha_1, \dots, \alpha_n$ be an orthonormal basis s.t. N diagonal, then

$$T\alpha_i = UN\alpha_i = c_i U\alpha_i,$$

obviously $c_i U\alpha_i$ constitute an orthogonal basis. \square

§1.4 Further on normal maps

For $\theta \in \mathbb{R}$, let Q_θ be the rotation of angle θ . The main goal of this section is to prove:

Theorem 1.4.1

Let V be a finite dimensional real inner product space, $T \in L(V)$ normal. There exists an orthonormal basis \mathcal{B} s.t.

$$[T]_{\mathcal{B}} = \text{diag}(a_1, \dots, a_l, r_1 Q_{\theta_1}, \dots, r_m Q_{\theta_m}),$$

where $a_i \in \mathbb{R}, r_j > 0, \theta_j \in (0, \pi)$.

Let's look at a corollary of this theorem first:

Corollary 1.4.2

If T orthogonal, then

$$[T]_{\mathcal{B}} = \text{diag}(I_{l_1}, -I_{l_2}, Q_{\theta_1}, \dots, Q_{\theta_m}).$$

Proof. Applying the theorem, since each block is orthogonal, $a_i = \pm 1$, $r_j = 1$. □

This gives us a comprehension of rotations in higher dimensional spaces.