Measure Theory

Felix Chen

Contents

0.1 Properties of integrals . . . 1
0.2 Convergence theorems . . . 3
0.3 Expectations . . . 5
0.4 L_p spaces . . . 6
0.5 Convergence in L_p space . . . 9

§0.1 Properties of integrals

Theorem 0.1.1 (Linearity of integrals)
Let f, g be functions whose integral exists.

- $\forall a \in \mathbb{R}$, the integral of af exists, and $\int_X (af)\,\mathrm{d}\mu = a\int_X f\,\mathrm{d}\mu$;
- If $\int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu$ exists, then $f+g$ *a.e.* exists, its integral exists and

$$\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu.$$

Proof. The first one is trivial by definition.
As for the second,

1. First we prove $f+g$ *a.e.* exists. If $|f| < \infty, a.e.$, we're done.
   If $\mu(f=\infty) > 0$, then $\int_X f\,\mathrm{d}\mu = \infty$. This means $\int_X g\,\mathrm{d}\mu \neq -\infty$, so $\mu(g=-\infty) = 0$. Thus $f+g$ *a.e.* exists. Similarly we can deal with the case $\mu(f=-\infty) > 0$.
2. Next we prove the equality. $f+g = (f^+ + g^+) - (f^- + g^-)$. Let $\varphi = f^+ + g^+, \psi = f^- + g^-$. Our goal is
   $$\int_X (\varphi - \psi)\,\mathrm{d}\mu = \int_X \varphi\,\mathrm{d}\mu - \int_X \psi\,\mathrm{d}\mu.$$
   Since $f+g$ *a.e.* exists, so $\varphi - \psi$ exists almost everywhere. If $\int_X \varphi\,\mathrm{d}\mu = \int_X \psi\,\mathrm{d}\mu = \infty$, then the integral of f, g must be $+\infty$ and $-\infty$, which contradicts with our condition. So both sides of above equation exist.
   Since $\max\{\varphi, \psi\} = \psi + (\varphi-\psi)^+ = \varphi + (\varphi - \psi)^-$, by the linearity of non-negative integrals,
   $$\int_X \psi\,\mathrm{d}\mu + \int_X (\varphi-\psi)^+\,\mathrm{d}\mu = \int_X \varphi\,\mathrm{d}\mu + \int_X (\varphi-\psi)^-\,\mathrm{d}\mu.$$
   which rearranges to the desired equality.

Note: we need to verify that we didn't add infinity to the equation in the last step. $\square$

Proposition 0.1.2

Let f, g be integrable functions, If $\int_A f \,\mathrm{d}\mu \geq \int_A g \,\mathrm{d}\mu, \forall A \in \mathscr{F}$, then $f \geq g, a.e.$.

Proof. Let $B = \{f < g\}$, then $(g-f)\mathbf{I}_B \geq 0$,

$$\int_B (g-f)\,\mathrm{d}\mu = \int_B (g-f)\mathbf{I}_B \,\mathrm{d}\mu \geq 0.$$

By the linearity of integrals we get $(g-f)\mathbf{I}_B = 0, a.e.$, i.e. $\mu(B) = 0$. ☐

Proposition 0.1.3

If μ is σ-finite, the integral of f, g exists, the conclusion of previous proposition also holds.

Proof. Let $X = \sum_n X_n$, $\mu(X_n) < \infty$. By looking at X_n, we may assume $\mu(X) < \infty$.

Since $\{f < g\} = \{-\infty \neq f < g\} + \{f = -\infty < g\}$.

Let $B_{M,n} = \{|f| \leq M, f + \frac{1}{n} < g\}$. By condition,

$$\int_{B_{M,n}} f \,\mathrm{d}\mu \geq \int_{B_{M,n}} g \,\mathrm{d}\mu \geq \int_{B_{M,n}} f \,\mathrm{d}\mu + \frac{1}{n}\mu(B_{M,n}).$$

Since $\int_{B_{M,n}} f \,\mathrm{d}\mu \leq M\mu(X)$ is finite, we get $\mu(B_{M,n}) = 0$. This implies $\{-\infty \neq f < g\} = \bigcup B_{M,n}$ is null.

Let $C_M = \{g > -M\}$, similarly,

$$-\infty \cdot \mu(C_M) = \int_{C_M} f \,\mathrm{d}\mu \geq \int_{C_M} g \,\mathrm{d}\mu = -M\mu(C_M).$$

Hence $\mu(C_M) = 0$, $\{-\infty = f < g\} = \bigcup C_M$ is null. ☐

Remark 0.1.4 — When $\geq$ is replaced by $=$, the conclusion holds as well. This proposition tells us that the integrals of f totally determines f. (In calculus, taking the derivative of integrals gives original functions)

Theorem 0.1.5 (Absolute continuity of integrals)

Let f be an integrable function, $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall A \in \mathscr{F}$,

$$\mu(A) < \delta \implies \int_A |f| \,\mathrm{d}\mu < \varepsilon.$$

Proof. Take non-negative simple functions $g_n \uparrow |f|$. Since $\int |f| \,\mathrm{d}\mu < \infty$, $\exists N$ s.t.

$$\int_X (|f| - g_N)\,\mathrm{d}\mu = \int_X |f| \,\mathrm{d}\mu - \int_X g_N \,\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Let $M = \max_{x \in X} g_N(x)$, $\delta = \frac{\varepsilon}{2M}$, so

$$\int_A |f| \,\mathrm{d}\mu < \frac{\varepsilon}{2} + \int_A g_N \,\mathrm{d}\mu = \frac{\varepsilon}{2} + M\mu(A) < \varepsilon.$$

☐

Example 0.1.6

Fundamental theorem of Calculus, Lebesgue version: Let g be a measurable function, then g is absolutely continuous iff $\exists f : [a, b] \to \mathbb{R}$ Lebesgue integrable, s.t.

$$g(x) - g(a) = \int_a^x f(z)\,\mathrm{d}z.$$

The absolute continuity can be implied by the absolute continuity of integrals.

§0.2 Convergence theorems

Levi, Fatou, Lebesgue.

In this section we mainly discuss the commutativity of integrals and limits, i.e. if $f_n \to f$, we care when does the following holds:

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Theorem 0.2.1 (Monotone convergence theorem, Levi's theorem)

Let $f_n \uparrow f, a.e.$ be non-negative functions, then

$$\int_X f_n \,\mathrm{d}\mu \uparrow \int_X f \,\mathrm{d}\mu.$$

Proof. By removing countable null sets, we may assume $0 \leq f_n(x) \uparrow f$.

Take non-negative simple functions $f_{n,k} \uparrow f_n$. Let $g_k = \max_{1\leq n\leq k} f_{n,k}$ be simple functions.

$$g_k = \max_{1\leq n\leq k} f_{n,k} \leq \max_{1\leq n\leq k+1} f_{n,k+1} = g_{k+1}.$$

So $g_k \uparrow$, say $g_k \to g$ for some function g. Clearly $g \leq f$ as $g_k \leq f_k$, $\forall k$.

Note as $k \to \infty$, $g_k \geq f_{n,k} \implies g \geq f_n, \forall n$. so $g = f$.

By definition of integrals,

$$\int_X f \,\mathrm{d}\mu = \lim_{k\to\infty} \int_X g_n \,\mathrm{d}\mu,$$

and

$$\int_X g_n \,\mathrm{d}\mu \leq \int_X f_n \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu.$$

So the conclusion follows. □

Corollary 0.2.2

Let f_n be functions whose integrals exist, if

$$f_n \uparrow f, a.e. \quad \int_X f_1^- \,\mathrm{d}\mu < \infty, \quad \text{or} \quad f_n \downarrow f, a.e. \quad \int_X f_1^+ \,\mathrm{d}\mu < \infty,$$

then the integral of f exists, and $\int_X f_n \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$.

Remark 0.2.3 — Counter example when $\int_X f_1^+ \,\mathrm{d}\mu = \infty$: let $X = \mathbb{R}$,

$$f_n = \mathbf{I}_{[n,\infty)} \downarrow f = 0, \quad \int_X f_n \,\mathrm{d}\mu = \infty, \quad \int_X f \,\mathrm{d}\mu = 0.$$

Corollary 0.2.4

If the integral of f exists, then for any measure partition $\{A_n\}$,

$$\int_X f \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{A_n} f \,\mathrm{d}\mu.$$

If $f \geq 0$, then $\nu : A \mapsto \int_A f \,\mathrm{d}\mu$ is a measure on $\mathscr{F}$. If we don't require $f \geq 0$, ν will become a signed measure which we'll cover later.

Theorem 0.2.5 (Fauto's Lemma)

Let $\{f_n\}$ be non-negative measurable functions almost everywhere, then

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Proof. Let $g_k := \inf_{n\geq k} f_n \uparrow g := \liminf_{n\to\infty} f_n$. By monotone convergence theorem,

$$\int_X g \,\mathrm{d}\mu = \lim_{k\to\infty} \int_X g_k \,\mathrm{d}\mu \leq \lim_{k\to\infty} \inf_{n\geq k} \int_X f_n \,\mathrm{d}\mu = \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

□

Corollary 0.2.6

If there exists integrable g s.t. $f_n \geq g$, then $\int_X \liminf_{n\to\infty} f_n$ exists and

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Theorem 0.2.7 (Lebesgue)

Let $f_n \to f, a.e.$ or $f_n \xrightarrow{\mu} f$, if there exists non-negative integrable function g s.t. $|f_n| \leq g, \forall n$, then

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Proof. When $f_n \to f, a.e.$, by Fatou's lemma,

$$\int_X f \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Since $|f_n| \le g$,

$$\int_X f \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} f_n \,\mathrm{d}\mu \ge \limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu,$$

which gives the desired.

When $f_n \xrightarrow{\mu} f$, for all subsequence $\{n_k\}$, exists a subsequence $\{n'\}$ s.t. $f_{n'} \to f, a.e.$. Thus $\int_X f_{n'} \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$, hence $\int_X f_n \,\mathrm{d}\mu \to \int_X f \,\mathrm{d}\mu$.(Why?) □

Corollary 0.2.8

Let f_n be random variable on $(\Omega_n, \mathscr{F}_n, P_n)$, $f_n \xrightarrow{d} f$, then we have

$$\lim_{n\to\infty} \int_{X_n} f_n \,\mathrm{d}P_n = \int_X f \,\mathrm{d}P.$$

Proposition 0.2.9 (Transformation formula of integrals)

Let $g : (X, \mathscr{F}, \mu) \to (Y, \mathscr{S})$ be a measurable map. For all measurable f on $(Y, \mathscr{S})$, then

$$\int_Y f \,\mathrm{d}\mu \circ g^{-1} = \int_X f \circ g \,\mathrm{d}\mu$$

if one of them exists.

Proof. By the typical method, we only need to prove for indicator function f. □

Remark 0.2.10 — μ and $\mu \circ g^{-1}$ are the same measure in different spaces.

§0.3 Expectations

Let ξ be a r.v. on $(\Omega, \mathscr{F}, P)$,

Definition 0.3.1 (Expectations)**.** If $\int_\Omega \xi \,\mathrm{d}P$ exists, then we call it the **expectation** of ξ, denoted by $E(\xi)$ or $E\xi$.

Consider the distribution $\mu_\xi = P \circ \xi^{-1}$, $F_\xi(x) = P(\xi \le x)$.
Let $f = \mathrm{id} : \mathbb{R} \to \mathbb{R}$, then $E(\xi) = E(\mu_\xi)$:

$$\int_{\mathbb{R}} x \,\mathrm{d}F_\xi(x) = \int_{\mathbb{R}} f \,\mathrm{d}\mu_\xi = \int_{\mathbb{R}} f \,\mathrm{d}P \circ \xi^{-1} = \int_{\mathbb{R}} f \circ \xi \,\mathrm{d}P = \int_\Omega \xi \,\mathrm{d}P = E(\xi).$$

Let f be a measurable function on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, then $f(\xi)$ is a measurable function on $(\Omega, \mathscr{F})$, and

$$Ef(\xi) = \int_{\mathbb{R}} f \,\mathrm{d}F_\xi.$$

Let $\eta = f \circ \xi$, by the transformation formula,

$$\begin{aligned} Ef(\xi) &= \int_\Omega \eta(\omega)\,\mathrm{d}P(\omega) \\ &= \int_{\overline{\mathbb{R}}} y\,\mathrm{d}P\circ\eta^{-1}(y) = \int_{\overline{\mathbb{R}}} y\,\mathrm{d}\mu_\eta(y) = \int_{\overline{\mathbb{R}}} y\,\mathrm{d}\mu_\xi \circ f^{-1}(y) \\ &= \int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_\xi(x) = \int_{\mathbb{R}} f\,\mathrm{d}F_\xi. \end{aligned}$$

Example 0.3.2

Possion distribution: $P(\xi = i) = \frac{\lambda^i}{i!}e^{-\lambda} =: p_i$. Its expectation is

$$\int_{\mathbb{R}} x\,\mathrm{d}\mu = \int_{\mathbb{N}} x\,\mathrm{d}\mu = \sum_{i=0}^{\infty} ip_i.$$

For continuous distribution, the density function p is actually a non-negative, integrable function, and $\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$. So $\mu(B) = \int_B p(x)\,\mathrm{d}x$ is a probability measure.

Since $\mu_\xi\big|_{\mathscr{P}_{\mathbb{R}}} = \mu\big|_{\mathscr{P}_{\mathbb{R}}}$, $\mu_\xi = \mu$. By typical method, we can prove

$$Ef(\xi) = \int_{\mathbb{R}} f\,\mathrm{d}\mu_\xi = \int_{\mathbb{R}} f(x)p(x)\,\mathrm{d}x.$$

§0.4 L_p spaces

Definition 0.4.1 (L_p spaces)**.** Let $1 \le p < \infty$. Define

$$\|f\|_p := \left(\int_X |f|^p\right)^{\frac{1}{p}}, \quad L_p(X, \mathscr{F}, \mu) := \{f : \|f\|_p < \infty\}.$$

Sometimes we'll simplify the notation as $L_p(\mu), L_p(\mathscr{F})$ or just L_p.

- $f \in L_1$ iff f integrable, let $\|f\| := \|f\|_1$.
- $f \in L_p \iff f^p \in L_1 \implies f$ is finite a.e..

In fact, L_p is a normed vector space under the norm $\|\cdot\|_p$:

Lemma 0.4.2

Let $1 \le p < \infty$, let $C_p = 2^{p-1}$, then

$$|a+b|^p \le C_p(|a|^p + |b|^p), \quad a, b \in \mathbb{R}.$$

Proof. It's a single-variable inequality, it's obvious by taking the derivative. □

Thus by taking integral on both sides,

$$\int_X |f+g|^p\,\mathrm{d}\mu \le C_p\left(\int_X |f|^p\,\mathrm{d}\mu + \int_X |g|^p\,\mathrm{d}\mu\right).$$

So L_p space is a vector space.

Lemma 0.4.3 (Holder's inequality)
Let $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\|fg\| \le \|f\|_p \|g\|_q, \quad \forall f \in L_p, g \text{ measurable}.$$

Proof. WLOG $\|f\|_p > 0$, $0 < \|g\|_q < \infty$. Let

$$a = \left(\frac{|f|}{\|f\|_p}\right)^p = \frac{|f|^p}{\int_X |f|^p \,\mathrm{d}\mu}, \quad b = \left(\frac{|g|}{\|g\|_q}\right)^q = \frac{|g|^q}{\int_X |g|^q \,\mathrm{d}\mu}.$$

By weighted AM-GM,

$$\int_X \frac{|fg|}{\|f\|_p \|g\|_q} \,\mathrm{d}\mu \le \int_X \left(\frac{a}{p} + \frac{b}{q}\right) \mathrm{d}\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

The equality holds iff $a = b$, i.e. $\exists \alpha, \beta \ge 0$ not all zero s.t. $\alpha|f|^p = \beta|g|^q, a.e.$. □

Theorem 0.4.4 (Minkowski's inequality)
Let $1 \le p < \infty$, then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p, \quad \forall f, g \in L_p.$$

The equality holds iff $(1): p = 1, fg \ge 0$; $(2) p > 1, \exists \alpha, \beta \ge 0, s.t. \alpha f = \beta g, a.e.$.

Proof. When $p = 1$, it follows by $|f+g| \le |f| + |g|$.

When $p \ge 1$, let $q = \frac{p}{p-1}$, by Holder's inequality,

$$|f+g|^p \le |f||f+g|^{p-1} + |g||f+g|^{p-1},$$
$$\implies \|f+g\|_p^p \le (\|f\|_p + \|g\|_p) \cdot \||f+g|^{p-1}\|_q.$$

Note that

$$\||f+g|^{p-1}\|_q = \left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}}.$$

Since $f + g \in L_p$, we can divide both sides by $\|f+g\|_p^{\frac{p}{q}}$ to get the result. □

In L_p space, we view two functions $f = g, a.e.$ as the same function, i.e. the original function space modding the equivalence relation out.

Hence $(L_p/\sim, \|\cdot\|_p)$ is a normed vector space.

When $p = \infty$, define

$$\|f\|_\infty := \inf\{a \in \mathbb{R} : \mu(|f| > a) = 0\}, \quad L_\infty := \{f : \|f\|_\infty < \infty\}.$$

We call the functions in L_∞ **essentially bounded**.

Let $\mu(X) < \infty$, then $f \in L_\infty \implies f \in L_p$, and $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$: For all $0 < a < \|f\|_\infty$,

$$a^p \mu(|f| > a) \le \int_X |f|^p \mathbf{I}_{|f|>a} \,\mathrm{d}\mu \le \int_X |f|^p \,\mathrm{d}\mu \le \|f\|_\infty^p \mu(X),$$

So taking the exponent $\frac{1}{p}$,

$$a \leftarrow a\mu(|f| > a)^{\frac{1}{p}} \le \|f\|_p \le \|f\|_\infty$$

But when $\mu(X) = \infty$, let $f \equiv 1$, then $f \in L_\infty$ but $f \notin L_p$.

Theorem 0.4.5

Let $f, g \in L_\infty$,

$$\|fg\| \leq \|f\|\|g\|_\infty,$$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Proof.

$$\int_X |fg| \,\mathrm{d}\mu \leq \int_X |f|\|g\|_\infty \,\mathrm{d}\mu = \|f\|\|g\|_\infty.$$

Since $|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty, a.e.$, we get the second inequality. □

Similarly we get $(L_\infty, \|\cdot\|_\infty)$ is a normed vector space.

The norm can deduce a *distance*:

$$\rho(f,g) := \|f-g\|.$$

Theorem 0.4.6 (L_p space is complete)

Let $1 \leq p \leq \infty$. If $\{f_n\} \subset L_p$ satisfying $\lim_{n,m\to\infty}\|f_n - f_m\|_p = 0$, then there exist $f \in L_p$ s.t. $\lim_{n\to\infty}\|f - f_n\|_p = 0$.

Proof. Take $n_1 < n_2 < \cdots$ such that

$$\|f_m - f_n\|_p \leq \frac{1}{2^k}, \quad \forall n, m \geq n_k.$$

Let $g =\uparrow \lim_{k\to\infty} g_k$, where

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \in L_p, \quad g_k \geq 0.$$

Since

$$\|g_k\|_p \leq \|f_{n_1}\|_p + \sum_{i=1}^{k}\|f_{n_{i+1}} - f_{n_i}\|_p \leq \|f_{n_1}\|_p + 1.$$

$$\implies \|g\|_p =\uparrow \lim_{k\to\infty}\|g_k\|_p \leq \|f_{n_1}\|_p + 1.$$

Here we use the monotone convergence theorem. We can check the above also holds for $p = \infty$.

Therefore $g \in L_p \implies g < \infty, a.e.$. We have

$$f := f_{n_1} + \sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i}) = \lim_{k\to\infty} f_k, a.e.$$

the series is absolutely convergent, so f exists a.e. and $|f| \leq g, a.e.$.

Lastly we can check: when $p = \infty$,

$$\|f_n - f\|_\infty \leq \|f_n - f_{n_k}\|_\infty + \|f_{n_k} - f\|_\infty,$$

where the both term approach to 0 as $n \to \infty$.

When $p < \infty$, by Fatou's lemma,

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} |f_n - f_{n_k}|^p \,\mathrm{d}\mu \leq \liminf_{k\to\infty} \int_X |f_n - f_{n_k}|^p \,\mathrm{d}\mu \leq \varepsilon.$$

□

Remark 0.4.7 — Using the same technique we can prove that if f_n is Cauchy in measure, then f_n converge to some f in measure:

Let $A_i = \{|f_{n_{i+1}} - f_{n_i}| > 2^{-i}\}$ s.t. $\mu(A_i) < 2^{-i}$.

Define $f = f_{n_1} + \sum_{i\geq 1}(f_{n_{i+1}} - f_{n_i})$ on the set $\bigcup_{k\geq 1}\bigcap_{i\geq k} A_i^c$.

This theorem implies that $(L_p, \|\cdot\|_p)$ is a Banach space. So we can try to define an *inner product* on L_p space:

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

We can check $\langle\cdot,\cdot\rangle$ is bilinear only if $p = 2$, so L_2 is actually a Hilbert space.

When $0 < p < 1$, let

$$\|f\|_p := \int_X |f|^p \,\mathrm{d}\mu, \quad L_p = \{f : \|f\|_p < \infty\}.$$

Lemma 0.4.8

Let $0 < p < 1$, $C_p = 1$, then

$$|a+b|^p \leq C_p(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}.$$

So L_p is a vector space.

Theorem 0.4.9 (Minkowski)

Let $0 < p < 1$ then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark 0.4.10 — When $\|f\|_p = (\int_X |f|^p \,\mathrm{d}\mu)^{\frac{1}{p}}$, $0 < p < 1$. then it won't satisfy Minkowski's inequality.

Thus L_p is only a metric space but not a normed vector space. Using the same method we can prove L_p is a complete metric space.

§0.5 Convergence in L_p space

Definition 0.5.1. Let $0 < p \leq \infty$, $f, f_1, f_2, \cdots \in L_p$. When $\|f_n - f\|_p \to 0$, then we write $f_n \xrightarrow{L_p} f$, called **average converge of order p**.

Theorem 0.5.2

Let $0 < p < \infty$, $f, f_1, \cdots \in L_p$,

- If $f_n \xrightarrow{L_p} f$, then $f_n \xrightarrow{\mu} f$, and $\|f_n\|_p \to \|f\|_p$.
- If $f_n \to f$, *a.e.* or in measure, then $\|f_n\|_p \to \|f\|_p \iff f_n \xrightarrow{L_p} f$.

Proof. When $f_n \xrightarrow{L_p} f$, let $A := \{|f_n - f| > \varepsilon\}$,

$$\mu(A) \leq \frac{1}{\varepsilon^p} \int_X |f_n - f|^p \mathbf{I}_A \, \mathrm{d}\mu \leq \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \to 0.$$

and obviously $\|f_n\|_p \to \|f\|_p$

On the other hand, when $f_n \to f, a.e.$ and $\|f_n\|_p \to \|f\|_p$, From $|a+b|^p \leq C_p(|a|^p + |b|^p)$,

$$g_n := C_p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0.$$

$g_n \to 2C_p|f|^p, a.e.$, so

$$\int_X 2C_p|f|^p \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu = 2C_p \int_X |f|^p \, \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f|^p \, \mathrm{d}\mu.$$

When $f_n \to f$ in measure, for any subsequence there exist its subsequence $f_{n'} \to f, a.e.$, so $\|f_{n'} - f\|_p \to 0$, hence $\|f_n - f\|_p \to 0$. $\square$