Geometry II

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§1 Introduction

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This course covers the topic of elementary *differential geometry* and *fundamental groups in algebraic topology*.

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§1.1 Intro

Definition 1.1 (Manifold)**.** Let M be an open subset of $\mathbb{R}^n$, we call M an **m-dimensional regular manifold** of $\mathbb{R}^n$, if $\forall p \in M$, exists an open neighborhood $W \subset \mathbb{R}^n$ such that there exists open set $U \subset \mathbb{R}^m$ and homeomorphism $\phi : U \to M \cap W$, satisfying the Jabobi matrix of ϕ is injective everywhere, i.e.

$$\mathrm{D}\phi(x) = \begin{pmatrix} \frac{\partial \phi_1}{\partial u_1} & \cdots & \frac{\partial \phi_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial u_1} & \cdots & \frac{\partial \phi_n}{\partial u_m} \end{pmatrix}$$

has rank m for all $x \in U$.

When $n = 3$, we say M is a curve for $m = 1$, and a surface for $m = 2$.

Remark 1.2 — The term "C^r regular manifold" means ϕ is a C^r function.

Example 1.3

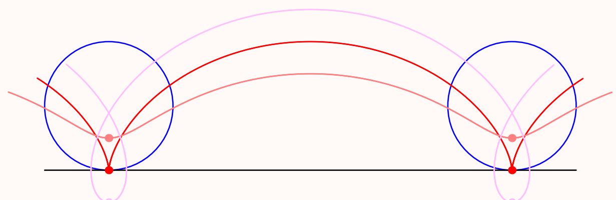
Some quadratic surfaces like cylinders, biparted hyperboloids and saddle surfaces are all regular 2-manifolds, but a cone is not a regular manifold.

Example 1.4

The curve $\phi(r) = (\cos(2\pi r), \sin(2\pi r), r)$ is a 1-manifold in $\mathbb{R}^3$. This curve is called a *helix*.

Example 1.5 (Cycloid)

A cycloid is the locus of a point on a circle while the circle "rolls" along a line. When the point lies inside resp. outside the circle, the curve is called curtate cycloid resp. procolate cycloid.



The cycloid is not a manifold because it has singularity where it touches the line, and procolate cycloids are also not manifolds as they have self-intersections.

Remark 1.6 — The regular manifolds we talk about are also called "embedded manifolds", the ones with self-intersections can be discribed as "immersed manifolds", such as the curves in the previous example. The immersed manifolds are complex and hence beyond the scope of this class.

However, it turns out that the curves or surfaces with self-intersections also have some properties, so we need to find a way to describe them. This induces:

Definition 1.7 (Regular parametrized curve)**.** Let $\gamma : J \to \mathbb{R}^3$ be a function, where J is an open interval. if for every point $p \in J$, there exists open neighborhood J' s.t. $\gamma\big|_{J'}$ is a regular 1-manifold, then we say $\gamma(J)$ is a **regular parametrized curve**, and $\gamma\big|_{J'}$ is called its **regular parametrization**.

Likewise, we have:

Definition 1.8 (Regular parametrizd surface)**.** Let $\phi : U \to \mathbb{R}^3$ be a function, where $U \in \mathbb{R}^2$ is an open set. If for every point $p \in U$, there exists open neighborhood U' s.t. $\phi\big|_{U'}$ is a regular 2-manifold, we say $\phi(U)$ is a **regular parametrized surface**.

§1.2 Prerequisites

Vector calculus Let $\vec{v}(t)$ be a 3-dimensional vector function, its derivative resp. integration is the vector formed by taking derivative resp. integration of each component, and the derivative satisfies Leibniz's rule with respect to both dot product and cross product.

Multi-variable calculus If $f : \mathbb{R}^m \to \mathbb{R}^n$ is C^2, then the partial derivative can change order with each other:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f.$$

Integration of surfaces has two types:

1. $\iint f \, \mathrm{d}x \, \mathrm{d}y$, multiple integrals.
2. $\iint f \, \mathrm{d}x \wedge \mathrm{d}y$, integrals with orientation.

Remark 1.9 — On how to construct regular manifolds:

Let $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be a smooth function, for a fixed $y \in \mathbb{R}^{n-m}$, if $\forall x \in f^{-1}(y)$, $Df(x)$ has rank $n - m$, then $M := f^{-1}(y)$ is an m-dimensional regular manifold.

In fact this is known as "Regular Value Theorem" in literarture, and y is called a regular value of f. This leads to a branch in mathematics, namely *differential topology*.

Remark 1.10 — On real/complex analysis: Holomorphic (which is the complex version of differentiable) is way stronger than smooth condition.

§2 Theory of space curve

In this section we mainly discuss the regular parametrized curves $\gamma : J \to \mathbb{E}^3$.

Our goal is to find some identities to describe the "shape" of the curves. Since the curve is 1-dimensional manifold in 3 dimensional space, somehow we should find 3 identities to describe it, including length and another two concerning how it "bends".

§2.1 Arc length

Definition 2.1 (Arc length)**.** Let $\gamma : J \to \mathbb{E}^3$ be a regular parametrized curve. In an interval $[a, b] \subset J$, we define its length to be

$$Length_\gamma([a, b]) := \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

where $\gamma'(t) \in V(\mathbb{E}^3) = \mathbb{R}^3$.

Proposition 2.2

Arc length is a **geometry quantity**, i.e. fixed under reparametrization.

Proof. For an arbitary regular reparametrization $t = t(\tilde{t})$, $\tilde{\gamma}(\tilde{t}) = \gamma(t)$, by Chain rule we get

$$\begin{aligned} Length_\gamma([a, b]) &= \int_a^b |\gamma'(t)| \, \mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))| \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} \, \mathrm{d}\tilde{t} \\ &= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}'(\tilde{t})| \, \mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{a}, \tilde{b}]). \end{aligned}$$

However, here we used the fact that $\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}$ is positive, so when the reparametrization reverses the orientation, we need to take extra care of it.

$$\begin{aligned} Length_\gamma([a,b]) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\gamma'(t(\tilde{t}))|\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{b}}^{\tilde{a}} |\tilde{\gamma}'(\tilde{t})|\,\mathrm{d}\tilde{t} = Length_{\tilde{\gamma}}([\tilde{b},\tilde{a}]). \end{aligned}$$

□

The arc length induces a parametrization for regular curves, namely the **arc length parameter** $\gamma(s)$, with $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\| = 1$ everywhere.

§2.2 Curvature

Definition 2.3 (Curvature)**.** Let $\gamma(s)$ be a regular curve with arc length parameter, define its **curvature** to be

$$\mathrm{Curv}_\gamma(s) = \kappa(s) := \|\gamma''(s)\|.$$

Since it is deduced from arc length (which is a geometry quantity), it must be a geometry quantity as well.

Remark 2.4 — Sometimes $\gamma''(s)$ is called the curvature vector. It's parallel to the normal vector and can be interpreted as centripedal force.

Example 2.5

For a straight line, its curvature is always 0.

For a circle with radius R, $\gamma(s) = (R\cos(\frac{s}{R}), R\sin(\frac{s}{R}))$, so $\mathrm{Curv}_\gamma(s) = \frac{1}{R}$.

Proposition 2.6

When the parameter is a general parameter $\gamma(t)$, the curvature is equal to:

$$\mathrm{Curv}_\gamma(t) = \frac{\|\gamma''(t) \times \gamma'(t)\|}{\|\gamma'(t)\|^3}.$$

Example 2.7

Let $\Gamma : x^2 + k^2y^2 = 1$, calculate curvature of Γ at point (x, y).

Solution. First we take a parametrization for Γ: $(x, y) = (\cos t, \frac{1}{k}\sin t)$.

Then compute the derivatives: $(x', y') = (-\sin t, \frac{1}{k}\cos t) = (-ky, \frac{1}{k}x)$, $(x'', y'') = (-\cos t, -\frac{1}{k}\sin t) = (-x, -y)$.

$$\mathrm{Curv}_\Gamma = \frac{|ky^2 + \frac{1}{k}x^2|}{(k^2y^2 + \frac{1}{k^2}x^2)^{\frac{3}{2}}} = \frac{1}{k(\frac{1}{k^2}x^2 + k^2y^2)^{\frac{3}{2}}}.$$

When $(x, y) = (1, 0)$, $\mathrm{Curv} = k^2$; when $(x, y) = (0, \frac{1}{k})$, $\mathrm{Curv} = \frac{1}{k}$. □

Remark 2.8 — Osculating circle: A circle tangent to the curve with the same curvature as the curve at the tangent point. Specifically, its radius is equal to $\frac{1}{\mathrm{Curv}}$.

This is useful in engineering to indicate the curvature of a curve.

§2.3 Torsion and Frenet frame

Definition 2.9 (Torsion)**.** Let $\gamma(s)$ be a curve with arc length parameter.

Let $\vec{t} := \gamma'(s), \vec{n} := \frac{\gamma''(s)}{\|\gamma''(s)\|}$ be the tangent vector and normal vector.

Let $\vec{b} = \vec{t} \times \vec{n}$ be the **binormal vector**. Define the **torsion** to be

$$\mathrm{Tors}_\gamma(s) = \tau(s) := -\vec{b}' \cdot \vec{n}.$$

In fact $\vec{b}'$ is parallel to $\vec{n}$:

$$\vec{b}' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}' = \vec{t} \times \vec{n}' \perp \vec{t},$$

and $\|\vec{b}\| = 1$, so $\vec{b} \perp \vec{b}'$, so $\vec{b}' \parallel \vec{n}$.

The torsion's geometric meaning is less intuitive than the previous ones. It describes how much the curve is moving "out" the plane it currently lies in.