Measure Theory

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Contents

Theorem 0.1

Let μ be a set function on a ring with finite additivity, then $1 \iff 2 \iff 3 \implies 4 \implies 5$.

- μ is countablely additive;
- μ is countablely subadditive;
- μ is lower continuous;
- μ is upper continuous;
- μ is continuous at $\emptyset$.

§0.1 Outer measure

Once we construct a measure on a semi-ring, we want to extend it to a σ-algebra. Since we can't directly do this, we shall relax some of our restrictions, say reduce countable additivity to subadditivity.

Definition 0.2 (Outer measure)**.** Let $\tau : \mathscr{T} \to [0, \infty]$ satisfying:

- $\tau(\emptyset) = 0$;
- If $A \subset B \subset X$, then $\tau(A) \leq \tau(B)$;
- (Countable subadditivity) $\forall A_1, A_2, \cdots \in \mathscr{T}$, we have

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \tau(A_n).$$

We call τ an **outer measure** on X.

It's easier to extend a measure on semi-ring to an outer measure:

Theorem 0.3

Let μ be a non-negative set function on a collection $\mathscr{E}$, where $\emptyset \in \mathscr{E}$ and $\mu(\emptyset) = 0$. Let

$$\tau(A) := \inf \left\{ \sum_{n=1}^{\infty} \mu(B_n) : B_n \in \mathscr{E}, \bigcup_{n=1}^{\infty} B_n \supseteq A \right\}, \quad \forall A \in \mathscr{T}.$$

By convention, $\inf \emptyset = \infty$. (μ *need not be a measure!*)

Then τ is called the outer measure generated by μ.

Proof. Clearly $\tau(\emptyset) = 0$, and $\tau(A) \le \tau(B)$ for $A \subset B$.

$$\bigcup_{n=1}^{\infty} B_n \supseteq B \implies \bigcup_{n=1}^{\infty} B_n \supseteq A.$$

For all $A_1, A_2, \dots \in \mathscr{T}$, WLOG $\tau(A_n) < \infty$. Take $B_{n,k}$ s.t. $\bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n$, such that

$$\sum_{k=1}^{\infty} \mu(B_{n,k}) < \tau(A_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{n,k} \supseteq A_n,$$

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(B_{n,k}) + \varepsilon \le \sum_{n=1}^{\infty} \tau(A_n) + \varepsilon.$$

$\square$

Example 0.4

Let $\mathscr{E} = \{X, \emptyset\}$, $\mu(X) = 1$, $\mu(\emptyset) = 0$. Then $\tau(A) = 1$, $\forall A \neq \emptyset$.

Example 0.5

Let $X = \{a, b, c\}$, $\mathscr{E} = \{\emptyset, \{a\}, \{a, b\}, \{c\}\}$. $\mu(A) = \#A$ for $A \in \mathscr{E}$.

Here something strange happens: $\tau(\{b\}) = 2$ instead of 1, and $\tau(\{b, c\}) = 3$ instead of 2.

In the above example, we found the set $\{b\}$ somehow behaves badly: if we divide $\{a, b\}$ to $\{a\} + \{b\}$, the outer measure is not the sum of two smaller measure.

Hence we want to get rid of this kind of inconsistency to get a proper measure:

Definition 0.6 (Measurable sets)**.** Let τ be an outer measure, if a set A satisfies *Caratheodory condition*:

$$\tau(D) = \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T},$$

we say A is **measurable**.

Remark 0.7 — Inorder to prove A measurable, we only need to check

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c), \quad \forall D \in \mathscr{T}.$$

Let $\mathscr{F}_\tau$ be the collection of all the τ measurable sets,

Definition 0.8 (Complete measure space)**.** Let $(X, \mathscr{F}, \mu)$ be a measure space, if for all null set A, and $\forall B \subset A, B \in \mathscr{F} \implies \mu(B) = 0$, we say $(X, \mathscr{F}, \mu)$ is **complete**.

Theorem 0.9 (Caratheodory's theorem)

Let τ be an outer measure, then $\mathscr{F} := \mathscr{F}_\tau$ is a σ-algebra, and $(X, \mathscr{F}, \tau)$ is a complete measure space.

Proof. First we prove $\mathscr{F}$ is an algebra:

Note $\emptyset \in \mathscr{F}$, and $\mathscr{F}$ is closed under completements.

For measurable sets A_1, A_2,

$$\begin{aligned}
\tau(D) &= \tau(D \cap A_1) + \tau(D \cap A_1^c) \\
&= \tau(D \cap A_1 \cap A_2) + \tau(D \cap (A_1) \cap A_2^c) + \tau(D \cap A_1^c) \\
&= \tau(D \cap (A_1 \cap A_2)) + \tau(D \cap (A_1 \cap A_2)^c).
\end{aligned}$$

So $A_1 \cap A_2$ is measurable.

Secondly, we prove $\mathscr{F}$ is a σ-algebra.

Let $A_1, A_2, \cdots \in \mathscr{F}$,

$$B_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in \mathscr{F},$$

Then B_i pairwise disjoint and $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i$. Let $B_f = \bigcup_{i=1}^\infty B_i$.

It's sufficient to prove

$$\tau(D) \geq \tau(D \cap B_f) + \tau(D \cap B_f^c).$$

Let $D_n = \sum_{i=1}^n B_i \cap D$, $D_f = D \cap B_f$, $D_\infty = D \backslash D_f$.

Since B_i are measurable,

$$\tau(D) = \tau(D_n) + \tau(D \backslash D_n) \geq \tau(D_n) + \tau(D_\infty) = \sum_{i=1}^n \tau(D \cap B_i) + \tau(D_\infty).$$

Now we take $n \to \infty$,

$$\tau(D) \geq \sum_{i=1}^\infty \tau(D \cap B_i) + \tau(D_\infty) \geq \tau\left(D \cap \sum_{i=1}^\infty B_i\right) + \tau(D_\infty).$$

Where the last step follows from countable subadditivity.

This implies B_f measurable $\implies \mathscr{F}$ is a σ-algebra.

Next we prove $\tau|\mathscr{F}$ is a measure: Just let $D = \sum_{i=1}^\infty B_i$ in the previous equation.

Last we prove $(X, \mathscr{F}, \tau)$ is complete:

If $\tau(A) = 0$, $\tau(D) \geq \tau(D \cap A^c) = \tau(D \cap A) + \tau(D \cap A^c)$. Thus $A \in \mathscr{F}$. $\square$

§0.2 Measure extension

Definition 0.10 (Measure extension)**.** Let μ, ν be measures on $\mathscr{E}$ and $\overline{\mathscr{E}}$, and $\mathscr{E} \subset \overline{\mathscr{E}}$. If

$$\nu(A) = \mu(A), \quad \forall A \in \mathscr{E},$$

we say ν is a extension of μ on $\overline{\mathscr{E}}$.

If we start from a measure μ on $\mathscr{E}$, ideally, μ can generate an outer measure τ, and we can take $\mathscr{F}_\tau$ to construct a mesaure space.

However, things could go wrong:

Example 0.11

Let $X = \{a, b, c\}, \mathscr{E} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ with

$$\mu(\emptyset) = 0, \mu(\{a, b\}) = 1, \mu(\{b, c\}) = 1, \mu(X) = 2.$$

Then μ is a measure on $\mathscr{E}$, and the outer measure

$$\tau(\emptyset) = 0.$$

Observe that $\mathscr{F}_\tau = \{\emptyset, X\}$, so in this case $\tau|_{\mathscr{F}}$ is the trivial measure.

Example 0.12

Let $X = \mathbb{R}$, $\mathscr{E} = \{(a, b] : a < b, a, b \in \mathbb{R}\}$. Let $\mu(\emptyset) = 0$, and $\mu(A) = \infty$ for $A \neq \emptyset$.

Then μ can be extend to the Borel σ-algebra on $\mathbb{R}$ with $\mu_\alpha = \sum_{q \in \mathbb{Q}} \alpha \delta_q$, $\forall \alpha \geq 0$. So the extension is not unique.

Therefore in order to get a "proper" extension, we must put some requirements on both the starting collection and the set function μ.

Proposition 0.13

Let $\mathscr{P}$ be a π system. If two measures μ, ν on $\sigma(\mathscr{P})$ satisfying

$$\mu|_{\mathscr{P}} = \nu|_{\mathscr{P}}, \quad \mu|_{\mathscr{P}} \text{ is } \sigma\text{-finite},$$

Then $\mu = \nu$.

Proof. Let $A_1, A_2, \cdots \in \mathscr{P}$ s.t. $X = \sum_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$.

Fix n, let $B = A_n$, we want to prove that

$$\mu(B \cap A) = \nu(B \cap A), \quad \forall A \in \sigma(\mathscr{P}).$$

Let $B \in \mathscr{P}$ with $\mu(B) < \infty$,

$$\mathscr{L} := \{A \in \sigma(\mathscr{P}) : \mu(A \cap B) = \nu(A \cap B)\}.$$

We'll prove $\mathscr{L}$ is a λ system, so that $\mathscr{L} \supseteq \sigma(\mathscr{P})$.

Suppose $A_1, A_2 \in \mathscr{L}$ and $A_1 \supseteq A_2$, by $\mu(B) < \infty$,

$$\mu((A_1 - A_2)B) = \mu(A_1 B) - \mu(A_2 B) = \nu(A_1 B - A_2 B) = \nu((A_1 - A_2)B).$$

So $A_1 - A_2 \in \mathscr{L}$.

Let $A_1, A_2, \cdots \in \mathscr{L}$ and $A_n \uparrow A$, then

$$\mu(AB) = \lim_{n\to\infty} \mu(A_n B) = \lim_{n\to\infty} \nu(A_n B) = \nu(AB).$$

Which implies $A \in \mathscr{L}$.

Hence $\sigma(\mathscr{P}) \subset \mathscr{L}$, i.e.

$$\mu(A \cap A_n) = \nu(A \cap A_n), \quad \forall A \in \sigma(\mathscr{P}).$$

Therefore

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A_n) = \sum_{n=1}^{\infty} \nu(A \cap A_n) = \nu(A), \quad \forall A \in \sigma(\mathscr{P}).$$

□

Example 0.14

In probability, let $\mathscr{E}_1, \mathscr{E}_2$ be collections of sets. We say they're independent if

$$P(AB) = P(A)P(B), \quad \forall A \in \mathscr{E}_1, B \in \mathscr{E}_2.$$

By the previous theorem we can derive $\lambda(\mathscr{E}_1), \lambda(\mathscr{E}_2)$ are independent.

If $A_1, A_2, \ldots$ satisfy

$$P(A_{i_1} \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

we say they are independent.

Let $\{1, 2, \ldots\} = I + J$, then the σ-algebra generated by

$$\mathscr{E}_1 = \{A_\alpha \mid \alpha \in I\}, \quad \mathscr{E}_2 = \{A_\alpha \mid \alpha \in J\}$$

are independent.

Theorem 0.15 (Measure extension theorem)

Let μ be a measure on a semi-ring $\mathscr{Q}$, τ is the outer meaasure generated by μ. We have

$$\sigma(\mathscr{Q}) \in \mathscr{F}_\tau, \quad \tau|_{\mathscr{Q}} = \mu.$$

Remark 0.16 — Any measure on a semi-ring $\mathscr{Q}$ can extend to the $\sigma(\mathscr{Q})$, and if μ is σ-finite, the extension is unique.

Proof. For any $A \in \mathscr{Q}$, let $B_1 = A$, $B_n = \emptyset, n \geq 2$. Then $\tau(A) \leq \sum \mu(B_n) = \mu(A)$.

On the other hand, if $A_1, A_2, \cdots \in \mathscr{Q}$ s.t. $\bigcup_{n=1}^{\infty} A_n \supseteq A$, then

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} \mu(AA_n)\right) \leq \sum_{n=1}^{\infty} \mu(AA_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Thus $\tau(A) = \mu(A)$, where we used the fact that μ is countable subadditive.

Next we prove $A \in \mathscr{F}_\tau$. We need to show that

$$\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

WLOG $\tau(D) < \infty$. Take $B_1, B_2, \dots \in \mathscr{Q}$ s.t.

$$\bigcup_{n=1}^{\infty} B_n \supseteq D, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(D) + \varepsilon.$$

Denote $\hat{D} := B_n \in \mathscr{Q}$ for a fixed n. Suppose $\hat{D} \cap A^c = \hat{D} \backslash A = \sum_{i=1}^n C_i$.

$$\mu(\hat{D}) = \mu(\hat{D} \cap A) + \sum_{i=1}^{n} \mu(C_i) \geq \tau(\hat{D} \cap A) + \tau(\hat{D} \cap A^c).$$

Apply this inequality to each B_n,

$$\tau(D) + \varepsilon > \sum_{n=1}^{\infty} (\tau(B_n \cap A) + \tau(B_n \cap A^c)) \geq \tau(D \cap A) + \tau(D \cap A^c).$$

this implies $\tau(D) \geq \tau(D \cap A) + \tau(D \cap A^c) \implies A \in \mathscr{F}_\tau$.

At last by Caratheodory's theorem, τ is a measure on $\mathscr{F}_\tau \supseteq \sigma(\mathscr{Q})$. □

Theorem 0.17 (Equi-measure hull)

Let τ be the outer measure generated by μ,

- $\forall A \in \mathscr{F}_\tau$, $\exists B \in \sigma(\mathscr{Q})$ s.t. $B \supseteq A$ and $\tau(A) = \tau(B)$;
- If μ is σ-finite, then $\tau(B \backslash A) = 0$.

Remark 0.18 — This theroem states that $\mathscr{F}_\tau$ is just $\sigma(\mathscr{Q})$ appended with null sets.

Proof. If $\tau(A) = \infty$, $B = X$ suffices.

By definition, there exists $B_n = \bigcup_{k=1}^{k_n} B_{n,k} \supseteq A$ s.t. $\tau(B_n) < \tau(A) + \frac{1}{n}$. Let $B = \bigcap_{n=1}^{\infty} B_n$, we must have $\tau(B) = \tau(A)$.

Now for the second part, let $X = \sum_{n=1}^{\infty} A_n$, $A_n \in \mathscr{Q}$, $\mu(A_n) < \infty$.

Since $A = \sum_{n=1}^{\infty} AA_n$, we have

$$AA_n \in \mathscr{F}_\tau, \quad \tau(AA_n) \leq \tau(A_n) = \mu(A_n) < \infty.$$

Let $B_n \in \sigma(\mathscr{Q})$ s.t. $B_n \supseteq AA_n$ and $\tau(B_n) = \tau(AA_n) < \infty$. Let $B := \bigcup_{n=1}^{\infty} B_n$ we have

$$\tau(B - A) = \tau\left(\bigcup_{n=1}^{\infty} (B_n - AA_n)\right) \leq \sum_{n=1}^{\infty} \tau(B_n - AA_n) = 0.$$

□

Let $\mathscr{R}, \mathscr{A}, \mathscr{F}$ be the ring, algebra, σ-algebra generated by $\mathscr{Q}$, respectively. The outer measure τ restricts to a measure on each of these collections, denoted by μ_1, μ_2, μ_3. Each μ_i can generate an outer measure τ_i, but actually they're all the same as our original τ, since τ_i are "build up" from τ, intuitively τ_i cannot be any better than τ. (The proof says exactly the same thing, so I'll omit it)

Proposition 0.19

Let μ be a measure on an algebra $\mathscr{A}$. τ is the outer measure generated by μ, for all $A \in \sigma(\mathscr{A})$, if $\tau(A) < \infty$, then $\forall \varepsilon > 0$, $\exists B \in \mathscr{A}$ s.t. $\tau(A \Delta B) < \varepsilon$.

Remark 0.20 — In practice we often replace τ with a σ-finite measure μ on $\sigma(\mathscr{A})$. (Here σ-finite is on $\mathscr{A}$)

Proof. Choose $B_1, B_2, \dots \in \mathscr{A}$ s.t.

$$\hat{B} := \bigcup_{n=1}^{\infty} B_n \supseteq A, \quad \sum_{n=1}^{\infty} \mu(B_n) < \tau(A) + \frac{\varepsilon}{2}.$$

Let N be a sufficiently large number, $B := \bigcup_{n=1}^{N} B_n \in \mathscr{A}$,

$$\tau(A \backslash B) \leq \tau\left(\bigcup_{n=N+1}^{\infty} B_n\right) \leq \sum_{n=N+1}^{\infty} \tau(B_n) \leq \frac{\varepsilon}{2}.$$

As $\tau(B\backslash A) \leq \tau(\hat{B}\backslash A) < \frac{\varepsilon}{2}$, $\tau(A \Delta B) < \varepsilon$. ☐

Example 0.21

Consider the Bernoulli test, recall $C_{i_1,\dots,i_n}$ we defined earlier. A measure(probability) μ is defined on the semi-ring $\{C_{i_1,\dots,i_n}\} \cup \{\emptyset, X\}$, then it can extend uniquely to the σ-algebra generated by it. This is how the probability of Bernoulli test comes from.

Let $(X, \mathscr{F}, P)$ be a probability space, $A_1, A_2, \dots \in \mathscr{F}$. We define the **tail σ-algebra** $\mathscr{T}$:

$$\mathscr{G}_n := \sigma(\{A_{n+1}, A_{n+2}, \dots\}), \quad \mathscr{T} := \bigcap_{n=1}^{\infty} \mathscr{G}_n.$$

Let $f_1, f_2, \dots$ be random variable, the tail σ-algebra generated by them is defined similarly:

$$\mathscr{G}_n := \sigma(\{f_{n+1}, f_{n+2}, \dots\}), \quad \mathscr{T} := \bigcap_{n=1}^{\infty} \mathscr{G}_n.$$

Theorem 0.22 (Kolmogorov's 0-1 law)

If $A_1, A_2, \dots \in \mathscr{F}$ are independent, then $P(A) \in \{0, 1\}$, $\forall A \in \mathscr{T}$.

Proof. Let $\mathscr{F}_n := \sigma(\{A_1, \dots, A_n\})$ and $\mathscr{G}_n$. They are clearly independent.

Note that $\mathscr{A} := \bigcup_{n=1}^{\infty} \mathscr{F}_n$ is an algebra.

Let $\mathscr{H} := \sigma(\mathscr{A}) \supseteq \mathscr{G}_n \supseteq \mathscr{T}$.

Hence $\forall A \in \mathscr{T} \subset \mathscr{H}$, $\forall \varepsilon > 0$, exists $B \in \mathscr{A}$ s.t. $P(A \Delta B) < \varepsilon$, so

$$P(A) - P(AB) \leq \varepsilon, \quad |P(A) - P(B)| \leq \varepsilon.$$

Since $B \in \mathscr{F}_n$ for some n, thus it is independent to A.

$$|P(A) - P(A)^2| \leq |P(A) - P(AB)| + |P(AB) - P(A)^2| \leq 2\varepsilon.$$

Let $\varepsilon \to 0$, we'll get $P(A) \in \{0, 1\}$. ☐

Remark 0.23 — When A_i's are replace by random variables, this theorem also holds.

Example 0.24

finite Markov chain

§0.3 The completion of measure spaces

Let $(X, \mathscr{F}, \mu)$ be a measure space, and

$$\widetilde{\mathscr{F}} := \{A \cup N : A \in \mathscr{F}, \exists B \in \mathscr{F} \ s.t. \ \mu(B) = 0, N \subset B\}.$$

Another way to define it is: $\widetilde{\mathscr{F}} := \{A \backslash N\}$, since

$$A \cup N = A + NA^c = (A \cup B) \backslash (BA^c \backslash N);$$

$$A \backslash N = A - NA = (A \backslash B) + (BA \backslash N).$$

In fact, we can do even more: $\widetilde{\mathscr{F}} := \{A \Delta N\}$.

Next we define the measure

$$\widetilde{\mu}(A \cup N) := \mu(A), \quad \forall A \cup N \in \widetilde{\mathscr{F}}$$

We need to check several things:

- $\widetilde{\mathscr{F}}$ is a σ-algebra.
- $\widetilde{\mu}$ is well-defined.
- $(X, \widetilde{\mathscr{F}}, \widetilde{\mu})$ is a complete measure space.

Remark 0.25 — The mesure $\widetilde{\mu}$ is the *minimal complete extension* of μ, i.e. if $(X, \mathscr{G}, \nu)$ is another complete extension, then

$$\nu(B) = \mu(B) = 0 \implies \forall N \subset B, N \in \mathscr{G}, \nu(N) = 0.$$

$$\mu(A) = \nu(A) \leq \nu(A \cup N) \leq \nu(A) + \nu(N) = \nu(A).$$

Thus $\mathscr{G} \supseteq \widetilde{\mathscr{F}}$ and $\nu(A) = \widetilde{\mu}(A)$ for $A \in \widetilde{\mathscr{F}}$.

Therefore we call $(X, \widetilde{\mathscr{F}}, \widetilde{\mu})$ the **completion** of $(X, \mathscr{F}, \mu)$.

Obviously $\emptyset \in \widetilde{\mathscr{F}}$; For $A \cup N \in \widetilde{\mathscr{F}}$, $(A \cup N)^c = A^c - A^c N \in \widetilde{\mathscr{F}}$.

$$\bigcup_{n=1}^{\infty} (A_n \cup N_n) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} N_n, \quad N = \bigcup_{n=1}^{\infty} N_n \subset \bigcup_{n=1}^{\infty} B_n.$$

Thus $\widetilde{\mathscr{F}}$ is a σ-algebra.

For $\widetilde{\mu}$, if $A_1 \cup N_1 = A_2 \cup N_2$,

$$\mu(A_1) = \mu(A_1 \cup B_2) \geq \mu(A_2).$$

Last we prove the countable additivity of $\widetilde{\mu}$. It's easy to check, so left out.

For the completeness, if $C \subset A \cup N, \mu(A) = 0$, then $C \subset A \cup B$ which is null.

Combining with the previous results we have

Theorem 0.26

Let τ be the outer measure generated by μ, a σ-finite measure on a semi-ring $\mathscr{Q}$. We have $(X\mathscr{F}_\tau, \tau)$ is the completion of $(X, \sigma(\mathscr{Q}), \tau)$.

Proof. Let $\mathscr{F} = \sigma(\mathscr{Q})$, we'll prove that $\widetilde{\mathscr{F}} = \mathscr{F}_\tau$.

Since $(X, \mathscr{F}_\tau, \tau)$ is complete, we have $\mathscr{F}_\tau \supseteq \widetilde{\mathscr{F}}$.

For all $C \in \mathscr{F}_\tau$, it suffices to prove $C = A + N$ for some $A \in \mathscr{F}, N \subset B$ with B null.

Since $C^c \in \mathscr{F}_\tau$, $\exists B \in \mathscr{F}$ s.t.

$$B \supseteq C^c, \quad \tau(B \backslash C^c) = 0.$$

□

§0.4 Distributions

Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right continuous function (called a **quasi-distribution function**). Let ν be the measure on $\mathscr{Q}_\mathbb{R}$,

$$\nu : (a, b] \mapsto \max\{F(b) - F(a), 0\}.$$

Let τ be the outer measure generated by ν. We call the sets in $\mathscr{F}_\tau$ to be the Lebesgue-Stieljes measurable sets (L-S measurable), a measurable function

$$f : (\mathbb{R}, \mathscr{F}_\tau) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$$

is L-S measurable, and $\tau\big|_{\mathscr{F}_\tau}$ is the L-S measure.

In fact finite L-S measures and the quasi-distribution functions are 1-1 correspondent (ignoring the difference of a constant), since $\mathscr{B}_\mathbb{R} = \sigma(\mathscr{Q}_\mathbb{R})$, $(\mathbb{R}, \mathscr{F}_\tau, \tau)$ is the completion of $(\mathbb{R}, \mathscr{B}_\mathbb{R}, \tau)$, and $\mu_F = \tau\big|_{\mathscr{B}_\mathbb{R}}$ is the unique extension of ν.

Conversely, given a measure μ on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$, if $\mu((a, b]) < \infty$ for all $a < b$, then $\mu = \mu_F$, where

$$F = F_\mu : x \mapsto \mu((-\infty, x]), \quad x \in \mathbb{R}.$$

We say a probability measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ is a **distribution**. Let $F : \mathbb{R} \to \mathbb{R}$ be a quasi-distribution function, if F satisfies:

$$F(-\infty) := \lim_{x \to -\infty} F(x) = 0, \quad F(+\infty) := \lim_{x \to +\infty} F(x) = 1,$$

then we say F is a distribution function (d.f.).

From the previous example we know distribution and d.f. are one-to-one correspondent.

Theorem 0.27

Let $g : (X, \mathscr{F}) \to (Y, \mathscr{S})$, μ is a measure on $\mathscr{F}$. Let

$$\nu(B) := \mu(g^{-1}(B)) = \mu \circ g^{-1}(B), \quad \forall B \in \mathscr{S}.$$

Then ν is a measure on $\mathscr{S}$.

Proof. Trivial. Just check the definition one by one. □

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $f : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}_{\mathscr{R}})$. We say

$$P \circ f^{-1} : B \mapsto P(f \in B)$$

is the **distribution** of f, denoted by μ_f, i.e. $\mu_f(B) = P(f \in B)$ for Borel sets B.

If $\mu_f = \mu$, we say f obeys the distribution μ, denoted by $f \sim \mu$.

Let $F_f = F_{\mu_f}$ be the distribution function of f.

$$F_f := \mu_f((-\infty, x]) = P(f \leq x), \quad x \in \mathbb{R}.$$

We can also say f obeys F_f, denoted by $f \sim F_f$.

If $F_f = F_g$, then we say f and g is **equal in distribution**, denoted by $f \overset{d}{=} g$.

Theorem 0.28

Any d.f. is the distribution function of some random variable.

Proof. Let $\Omega = \mathbb{R}, \mathscr{F} = \mathscr{B}_{\mathbb{R}}, P = \mu_F$, and $f = \text{id}$. It's clear that the distribution function of f is precisely F. $\square$