Geometry II

Felix Chen

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It's easy to prove that isometric = conformal + area-perserving. These three properties induce Riemann geometry, complex geometry and sympletic geometry, respectively (in two dimensional).

§0.0.1 Isometries

Firstly by Gauss' Theorema Egregium, Isometries perserves Gaussian curvature.

Example 0.0.1

Let $S_{a,b} : \frac{x^2}{a} + \frac{y^2}{b} = 2z$ be a saddle surface. Let $(x, y, z) = (as, bt, \frac{as^2+bt^2}{2})$ be a parametrization.

We can compute the fundamental forms:

$$g = a^2(1 + s^2)\,\mathrm{d}s^2 + 2abst\,\mathrm{d}s\,\mathrm{d}t + b^2(1 + t^2)\,\mathrm{d}t^2,$$

$$h = \frac{a\,\mathrm{d}s^2 + b\,\mathrm{d}t^2}{\sqrt{1 + s^2 + t^2}}.$$

So $K = \frac{1}{ab(1+s^2+t^2)^2}$. In fact the Gaussian curvature of some different surfaces, say $S_{2,3}$ and $S_{1,6}$ are the same.

But there is not an isometry between them:

If $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then τ fixes the circles centered at $(0, 0)$ as their Gaussian curvature are the same. Then $\tau_* = \mathrm{d}\tau : T_{(0,0)}\mathbb{R}^2 \to T_{(0,0)}\mathbb{R}^2$ can only be rotation or reflection. (If τ_* is not orthogonal, it will map small circles to ellipse)

While $g(0, 0) = a^2\,\mathrm{d}s^2 + b^2\,\mathrm{d}s^2$, which has eigenvalue a^2 and b^2, and they're fixed under τ_*, so $S_{2,3}$ isn't isometric to $S_{1,6}$.

Remark 0.0.2 — Given $E, F, G : D \to \mathbb{R}$ s.t. $g = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ positive definite, is there a surface $D \to \mathbb{E}^3$ can have g as its first fundamental form locally?

When we require E, F, G to be C^ω(analytic), the answer is "yes", but if we only require C^∞, it's still an open problem.

Even though we don't know the situation in 3 dimensional space, we can study the case in higher dimensions:

Theorem 0.0.3

It's always possible to construct $\phi : D \to \mathbb{E}^4$ to have E, F, G as its first fundamental form.

Surfaces with Gaussian curvature 0 everywhere are called **developable surfaces**. Developable surfaces can only be cylinder, cone, tangent surface of a curve and their concatenation.

Example 0.0.4 (Pseudosphere)

Let $\phi(x, y) = (\frac{\cos x}{y}, \frac{\sin x}{y}, \cosh^{-1}(y) - \frac{\sqrt{y^2-1}}{y})$, where $(x, y) \in (-\pi, \pi) \times [1, +\infty)$.

It's obtained by rotating a *tractrix* around its asymptote. We can calculate its Gaussian curvature, whichis a constant -1. This is where the name comes from.

Recall that hyperbolic plane also has constant curvature -1, in fact they are locally isometric. In 1901, Hilbert proved a theorem that there exists an isometry $\mathbb{H}^2 \to \mathbb{E}^3$.

At last we'll prove an interesting fact:

Proposition 0.0.5 (The local existence of isothermal parameters)

Let $\phi : U \to \mathbb{E}^3$, for all $\hat{u} \in U$, there exists a neighborhood $\widetilde{U}$ and a reparametrization $u = u(\widetilde{u})$, such that

$$g(\widetilde{u}) = \rho^2(\widetilde{u})(\widetilde{E}\,\mathrm{d}\widetilde{s}^2 + \widetilde{G}\,\mathrm{d}\widetilde{t}^2).$$

Remark 0.0.6 — Note that the right hand side is clearly conformal to regions in $\mathbb{E}^2$, so this in fact implies that any surfaces is locally conformal to $\mathbb{E}^2$.

Proof. The critical idea is to realize $\mathbb{R}^2$ as $\mathbb{C}$. To be more precise, we'll follow the steps below:

- Find a way to express $E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$ as $(a\,\mathrm{d}s + b\,\mathrm{d}t)(\overline{a}\,\mathrm{d}s + \overline{b}\,\mathrm{d}t)$, where a, b are functions with complex value.
- If there exists a complex function f s.t. $\mathrm{d}f(s + it) = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$, then $g = \frac{1}{|\rho|^2}\,\mathrm{d}f\,\mathrm{d}\overline{f}$.
- Assume further that f is *holomorphic* and non-degenerate, then $f(u) = \widetilde{x}(u) + i\widetilde{y}(u)$ is locally inversible, i.e. exists $u = u(\widetilde{x}, \widetilde{y})$, then

$$g = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x} + i\,\mathrm{d}\widetilde{y})(\mathrm{d}\widetilde{x} - i\,\mathrm{d}\widetilde{y}) = \frac{1}{|\rho|^2}(\mathrm{d}\widetilde{x}^2 + \mathrm{d}\widetilde{y}^2).$$

Let $a = \sqrt{E}$, $b = \frac{-F + i\sqrt{EG - F^2}}{\sqrt{E}}$. (Note $EG - F^2 > 0$ as g is positive definite)

Next we'll choose suitable f, ρ. Consider the differential equation $T = T(s, t)$:

$$\frac{\partial T}{\partial s} = -\frac{a(s, T)}{b(s, T)}, \quad T(\hat{s}, t) = t.$$

From the relation $f(s, T(s, t)) = t - \hat{t}$ and implicit function theorem we can uniquely determine f.

Remark 0.0.7 — The detail of the solution to this equation in complex functions is beyond the scope of this class.

Such f satisfies $\mathrm{d}f = \rho(a\,\mathrm{d}s + b\,\mathrm{d}t)$.

When $f(s,t) = (\widetilde{x}, \widetilde{y})$, the Jacobian determinant is

$$\widetilde{x}_s\widetilde{y}_t - \widetilde{x}_t\widetilde{y}_s = -|\rho|^2(a\bar{b} - b\bar{a}) = |\rho|^2\sqrt{EG - F^2} > 0.$$

so f must be non-degenerate. □

§1 Algebraic topology

§1.1 A bit of manifold

First we'll introduce a few concepts before we move on.

- We say a topological space is an n-dimensional **topological manifold** if it's Hausdorff and locally homeomorphic to $\mathbb{R}^n$. Sometimes we also require manifolds to be compact / paracompact / C_2. Here paracompact means that any open covering has a locally finite subcovering.
- Manifolds with boundary: locally homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$.
- When we talk about the regularity of manifolds, we must appoint an atlas first. Let $\phi_i : U_i \to E_i \subset \mathbb{R}^n$ be homeomorphisms mentioned above, then each ϕ_i is a **chart**, and $\{(U_i, \phi_i)\}_{i\in I}$ is the **atlas**. The map
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
is called **transition functions**.

  The regularity of the manifold is actually the regularity of transition functions, such as C^r, C^∞, piecewise linear, etc.

Example 1.1.1

The sphere $\mathbb{S}^2$ and projective plane $\mathbb{R}P^2$ are 2d manifolds. But they're different since $\mathbb{R}P^2$ is not *orientable*. In fact $\mathbb{R}P^2$ can be obtained by fusing the edge of a Mobius band to a disk(keep in mind that Mobius band has only one edge!).

There are many manifolds which looks wired, but I can't draw them on the computer ;)

Example 1.1.2 (Projective curves)

Consider a quadratic equation

$$C : z^2 + w^2 = 1, \quad (z, w) \in \mathbb{C}^2.$$

What does this surface look like?

Let $Z = z + iw, W = z - iw$, the equation becomes $ZW = 1$, hence the surface is $(\zeta, \frac{1}{\zeta}), \zeta \in \mathbb{C}\backslash\{0\}$. So C is homeomorphic to $\mathbb{C}\backslash\{0\}$.

We can also discuss this in $\mathbb{C}P^2 = \mathbb{C}P^1 \cup \mathbb{C}^2$, where $\mathbb{C}P^1 = \{\infty\} \cup \mathbb{C} \cong \mathbb{S}^2$.

So in homogeneous coordinate, the equation can be written as $ZW = T^2$. The surface is consisting of $(1, 0, 0), (0, 1, 0), (\zeta, \frac{1}{\zeta}, 0)$. Thus the projective completion of C is homeomorphic to $\mathbb{S}^2$, which is $\mathbb{C}\backslash\{0\}$ appending with two points.

Example 1.1.3 (Elliptic curves)

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ pairwise different.

$$E : w^2 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3).$$

What does E looks like in $\mathbb{C}P^2$?

Observe that for $z \in \mathbb{C}\backslash\{\lambda_1, \lambda_2, \lambda_3\}$, there're 2 values for w. So the image of E is two planes($\mathbb{C}$) fused together at $\lambda_1, \lambda_2, \lambda_3$ and ∞ with some adjust.

In fact this can be realized as two cylinder fused together at their edges.

$E \cong T^2\backslash\{pt\}$ in $\mathbb{C}^2$, and T^2 in $\mathbb{C}P^2$.

In fact $\mathbb{C}P^2$ is a 4-dimensional closed manifold, and it's also a 2-dimensional complex manifold. $PSL(3, \mathbb{C})$ acts transitively on $\mathbb{C}P^2$.

Example 1.1.4

We can fuse the edges of polygons to get manifolds: By fusing together opposite edges of a square, we can get torus or Klein bottle.

We'll use the word "fuse" frequently in the future, so here we'll make it clear what we mean by "fusing" things together.

Definition 1.1.5 (Quotient maps)**.** A continuous map $f : X \to Y$ is called a **quotient map**, if it's surjective, and $\forall B \subset Y$, $f^{-1}(B)$ open $\implies$ B open.

This is saying that the topology on Y is the "largest" topology (or quotient topology) while keeping f continuous.

So when we "fusing" things together, we're actually giving an equivalence relation on the original space, and the result is the quotient topology induced from the natural projection map.

Now we look at the elliptic curves again, let $U = \mathbb{C}\backslash([\lambda_1, \lambda_2] \cup [\lambda_3, \infty])$. Let X be the path end compactification of U, then $X \simeq S^1 \times [0, 1]$.

Let X_1, X_2 be two copies of X, and fusing the corresponding circles at the end in the reversed direction, we'll get a torus without 4 points, by adding $\lambda_1, \lambda_2, \lambda_3$ back we'll get $T^2\backslash\{pt\}$.

Remark 1.1.6 — The quotient topology may have some bad properties, like not being Hausdorff: Consider $\mathbb{R}^2\backslash\{(0,0)\}$ with connected vertical lines as equivalence class, then we'll get a line with 2 points at the origin, which is a typical non-Hausdorff space.

A closed surface is a connected compact 2-dimensional manifold with no edges. We have the following classification theorem:

Theorem 1.1.7

All the closed surfaces must be homeomorphic to $nT^2(n \geq 0)$ or $mP^2(m \geq 1)$. Here n is called the **genus** of orientable surfaces.

nT^2 can be viewed as S^2 fused with n handles (torus), and mP^2 can be viewed as S^2 fused with m crosscaps (Mobius strip).

In this course we mainly talk about surfaces with triangulation, i.e. we take it for granted that all surfaces has triangulation.

Here we'll prove part of this theorem (since the other part needs further knowledge).

Proof. Observe that given a triangulation, we can get a polygon fusing expression of the surface by adding the triangles one by one, fusing only one edge each time. □