Mathematical Analysis II

Felix Chen

Contents

Fubini's theorem is also useful when computing integrals.

Example 0.1 (Gauss integral)

Recall the Gauss integral:

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Here we give a different proof:

$$\begin{aligned}\int e^{-x^2}\,\mathrm{d}x \int e^{-y^2}\,\mathrm{d}y &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{+\infty} e^{-r^2}\,\mathrm{d}r^2 \cdot \pi = \pi.\end{aligned}$$

§1 Lebesgue differentiation

The most important theorem in calculus is no doubt the Fundamental theorem of Calculus (which is also called Newton-Lebniz formula). Since we generalized the integrals, there must be a generalized version of this theorem:

Theorem 1.1 (Lebesgue differentiation theorem, part 1)

If f is integrable on $\mathbb{R}^d$, for any ball $B \subset \mathbb{R}^d$, we have

$$\lim_{x\in B, |B|\to 0} \frac{1}{m(B)} \int_B f(y)\,\mathrm{d}y = f(x), a.e.$$

This theorem clearly holds for continuous points of f.

Our basic idea is to take a continuous g, such that $\|g-f\|_{\mathcal{L}^1} < \varepsilon$.

and to prove

$$\left\{x : \limsup_{x\in B, |B|\to 0} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,\mathrm{d}y \geq \varepsilon_0\right\}$$

is a null set.

Now we estimate

$$\begin{aligned}\frac{1}{m(B)}\int_B |f(y)-f(x)|\,\mathrm{d}y &\le \frac{1}{m(B)}\int_B (|f(y)-g(y)|+|g(y)-g(x)|+|g(x)-f(x)|)\,\mathrm{d}y\\ &= |f(x)-g(x)|+\varepsilon+\frac{1}{m(B)}\int_B |f(y)-g(y)|\,\mathrm{d}y\end{aligned}$$

We find that the last term is pretty hard to deal with, so we'll introduce some new tools:

Definition 1.2 (Hardy-Littlewood maximal function)**.** Let f be an integrable function on $\mathbb{R}^d$. Define

$$Mf(x)=\sup_{x\in B}\frac{1}{m(B)}\int_B |f(y)|\,\mathrm{d}y.$$

to be the **maximal function** of f.

Theorem 1.3 (Hardy-Littlewood)

The maximal function Mf satisfies:

- Mf is measurable;
- For x almost everywhere, $|f(x)|\le Mf(x)<+\infty$.
- There exists a constant C s.t.

$$|\{x : Mf>\alpha\}|\le \frac{C}{\alpha}\|f\|_{\mathcal{L}^1}.$$

Proof. First we prove $\{Mf>\alpha\}$ is measurable. If $Mf(x_0)>\alpha$, then exists an open ball $B\ni x_0$,

$$\int_B |f(y)|\,\mathrm{d}y>\alpha m(B).$$

This implies that $B\subset\{Mf>\alpha\}\implies\{Mf>\alpha\}$ is an open set.

For the second part, we'll prove for $\forall\varepsilon_0>0, N>0$,

$$m(\{x : Mf(x)+\varepsilon_0<|f(x)|\le N\})=0.$$

Otherwise denote the above set as E, for $\forall 0<\lambda<1$, $\exists B$ s.t. $|E\cap B|>\lambda|B|$.

Thus for $x\in E$,

$$\begin{aligned}Mf(x)&\ge\frac{1}{m(B)}\int_B |f(y)|\,\mathrm{d}y\\ &\ge\frac{1}{m(B)}\int_{E\cap B}|f(y)|\,\mathrm{d}y\\ &\ge\frac{1}{m(B)}\int_{E\cap B}\varepsilon_0+Mf(y)\,\mathrm{d}y\\ &=\frac{m(E\cap B)}{m(B)}\varepsilon_0+\frac{1}{|B|}\int_{E\cap B}Mf(y)\,\mathrm{d}y.\end{aligned}$$

Taking the integral with respect to x:

$$\left(1-\frac{|E\cap B|}{|B|}\right)\int_{E\cap B}Mf\ge\frac{|E\cap B|^2}{|B|}\varepsilon_0.$$

This implies $(1-\lambda)N \geq \lambda\varepsilon_0$, which is impossible as $\lambda \to 1$.

Now for the last part, since $\{Mf > \alpha\}$ is open, $\forall x \in \{Mf > \alpha\}$, $\exists B$ s.t.

$$\int_B |f(y)| \, \mathrm{d}y > \alpha m(B).$$

Hence for disjoint balls B_{i_k},

$$\|f\|_{\mathcal{L}^1} \geq \sum_{l=1}^k \int_{B_{i_l}} |f(y)| \, \mathrm{d}y > \alpha \sum_{l=1}^k |B_{i_l}|.$$

If we could select B_{i_l}'s such that their measure achieves say 1% of E, then we're done.

Lemma 1.4

Let $B_1, \dots, B_n$ be open balls in $\mathbb{R}^d$. There exists $i_1, \dots, i_k$ such that B_{i_j}'s are pairwise disjoint, and

$$\bigcup_{i=1}^n B_i \subset \bigcup_{j=1}^k 3B_{i_j}.$$

Here $3B$ means to multiply the radius of the ball by 3.

Proof of lemma. Trivial, just take the largest ball first and using greedy algorithm. □

Remark 1.5 — For countable many balls, the conculusion holds with 3 replaced by 5.

In particular, for all compact sets $K \subset \{Mf > \alpha\}$, there exists a finite open cover $B_1, B_2, \dots, B_n$ of K. By lemma we can select B_{i_j}'s satisfying

$$\sum_{i=1}^k m(B_{i_j}) \geq \frac{1}{3^d} m\left(\bigcup_{i=1}^n B_i\right) \geq \frac{1}{3^d} m(K).$$

Combining with our previous discussion we get $\|f\|_{\mathcal{L}^1} \geq \frac{\alpha}{3^d} m(K)$.

□

Returning to the proof of Theorem 1.1, we can assmue g is continuous with compact support,

$$\frac{1}{m(B)} \int_B |f(y) - g(y)| \, \mathrm{d}y \leq M(f-g)(x)$$

Hence by taking B sufficiently small s.t. $|g(y) - g(x)| \leq \varepsilon_0$ for all $x, y \in B$,

$$\frac{1}{m(B)} \int_B f(y) \, \mathrm{d}y \geq 3\varepsilon_0$$
$$\Longleftarrow \ |f(x) - g(x)| + M(f-g)(x) \geq 2\varepsilon_0.$$

But

$$m\{|f(x) - g(x)| \geq \varepsilon_0\} + m\{M(f-g) > \varepsilon_0\} \leq \frac{\|f-g\|_{\mathcal{L}^1}}{\varepsilon_0} + \frac{3^d}{\varepsilon_0}\|f-g\|_{\mathcal{L}^1} \leq \frac{3^d+1}{\varepsilon_0}\varepsilon.$$

This completes the proof.

Definition 1.6 (Lebesgue points)**.** Let $|f(x)| < \infty$, f is *locally integrable.* If x satisfies

$$\lim_{|B|\to 0, B\ni x} \frac{1}{|B|}\int_B |f(y)-f(x)|\,\mathrm{d}y = 0,$$

we say x is a **Lebesgue point** of f.

Remark 1.7 — Here "locally integrable" means for all bounded measurable sets E, $f\chi_E \in \mathcal{L}^1$. This is denoted by $f \in \mathcal{L}^1_{loc}$.

Let E be a measurable set, χ_E locally integrable, If point x is called a **density point** of E if it's a Lebesgue point of χ_E.

Theorem 1.8

Let E be a measurable set, then almost all the points in E are density points of E, almost all the points outside of E are not density points of E.

Proof. This is a direct corollary of Theorem 1.1. □

The differentiation theorem has some applications in convolution:

$$\begin{aligned}\frac{1}{|B|}\int_B f(y)\,\mathrm{d}y &= c_d^{-1}\varepsilon^{-d}\int_{B(x,\varepsilon)} f(y)\,\mathrm{d}y \\ &= \int f(x-y)\cdot c_d^{-1}\varepsilon^{-d}\chi_{B(0,\varepsilon)}(y)\,\mathrm{d}y \\ &= f * K_\varepsilon.\end{aligned}$$

where $K_\varepsilon = c_d^{-1}\varepsilon^{-d}\chi_{B(0,\varepsilon)}$, c_d is the measure of a unit sphere in $\mathbb{R}^d$.

By differentiation theorem, $\lim_{\varepsilon\to 0} f * K_\varepsilon = f(x), a.e.$. In the homework we proved that there doesn't exist a function I s.t. $f * I = f$ for all $f \in \mathcal{L}^1$, but the functions K_ε is approximating this "convolution identity".

Definition 1.9. In general, if $\int K_\varepsilon = 1$, $|K_\varepsilon| \le A\min\{\varepsilon^{-d}, \varepsilon|x|^{-d-1}\}$ for some constant A, we say K_ε is an **approximation to the identity**.

"convolution kernel"

Let φ be a smooth function whose support is in $\{|x| \le 1\}$, and $\int\varphi = 1$. The function $K_\varepsilon := \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$ is called the Friedrichs smoothing kernel.

Theorem 1.10

If K_ε is an approximation to the identity, f integrable,

$$\lim_{\varepsilon\to 0}\|f * K_\varepsilon - f\|_{\mathcal{L}^1} = 0.$$

Proof.

$$
\begin{aligned}
|(f * K_\varepsilon)(x) - f(x)| &= \left| \int f(x-y) K_\varepsilon(y)\,\mathrm{d}y - f(x) \right| \\
&\le \int |f(x-y) - f(x)||K_\varepsilon(y)|\,\mathrm{d}y \\
&\le \int_{|y|\le R} |f(x-y) - f(x)| A\varepsilon^{-d}\,\mathrm{d}y + \int_{|y|>R} |f(x-y)-f(x)| A\varepsilon |y|^{-d-1}\,\mathrm{d}y.
\end{aligned}
$$

Taking the integral over $\mathbb{R}^d$:

$$
\begin{aligned}
&\|K_\varepsilon * f - f\|_{\mathcal{L}^1} \\
&\le A\varepsilon^{-d} \int \int_{|y|\le R} |f(x-y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x + A\varepsilon \int \int_{|y|>R} |f(x-y)-f(x)||y|^{-d-1}\,\mathrm{d}y\,\mathrm{d}x \\
&\le A\varepsilon^{-d} \int \int_{|y|\le R} |\tau_{-y} f(x) - f(x)|\,\mathrm{d}y\,\mathrm{d}x + A\varepsilon \int_{|y|>R} |y|^{-d-1} \int |\tau_{-y} f(x)| + |f(x)|\,\mathrm{d}x\,\mathrm{d}y \\
&\le A\varepsilon^{-d} \int_{|y|\le R} \|\tau_{-y} f - f\|_{\mathcal{L}^1}\,\mathrm{d}y + A\varepsilon \int_{|y|>R} |y|^{-d-1} 2\|f\|_{\mathcal{L}^1}\,\mathrm{d}y.
\end{aligned}
$$

By the continuity of translation, $\forall \varepsilon_0$, let R be sufficiently small we have

$$
\|\tau_{-y} f - f\|_{\mathcal{L}^1} < \varepsilon_0, \forall |y| < R.
$$

$$
\|K_\varepsilon * f - f\|_{\mathcal{L}^1} \le A\varepsilon^{-d} R^d c_d \varepsilon_0 + \varepsilon R^{-1} C
$$

where C is a constant.

Take suitable $\varepsilon, \varepsilon_0$ s.t. $R = \varepsilon \varepsilon_0^{-\frac{1}{2d}}$, then $LHS \le C_1 \varepsilon_0^{\frac{1}{2}} + C_2 \varepsilon_0^{\frac{1}{2d}} \to 0$. $\square$

Theorem 1.11

Let K_ε be an approximation to the identity, f integrable,

$$
\lim_{\varepsilon \to 0} f * K_\varepsilon = f(x)
$$

holds for Lebesgue points x of f.

Proof. WLOG $x = 0$, let

$$
\omega(r) = \frac{1}{r^d} \int_{B(0,r)} |f(y) - f(0)|\,\mathrm{d}y,
$$

we have $\lim_{r\to 0} \omega(r) = 0$, and ω is continuous.

$$
\omega(r) \le \frac{\|f\|_{\mathcal{L}^1}}{r^d} + |f(0)|,
$$

Thus ω is bounded.

Therefore we can compute

$$\begin{aligned}|K_\varepsilon * f(x) - f(x)| &\le \int |K_\varepsilon(y)||f(x-y)-f(x)|\,\mathrm{d}y \\ &\le \int_{B(0,r)} |K_\varepsilon(y)||f(x-y)-f(x)|\,\mathrm{d}y + \sum_{k\ge 0}\int_{2^k r\le |y|<2^{k+1}r} |K_\varepsilon(y)||f(x-y)-f(x)|\,\mathrm{d}y \\ &\le A\varepsilon^{-d}\cdot r^d\omega(r) + \sum_{k\ge 0}\int_{2^k r\le |y|<2^{k+1}r} A\varepsilon |y|^{-d-1}|f(x-y)-f(x)|\,\mathrm{d}y \\ &\le A\varepsilon^{-d}r^d\omega(r) + \sum_{k\ge 0} A\varepsilon 2^{-k(d+1)}r^{-d-1}\cdot 2^{(k+1)d}r^d\omega(2^{k+1}r) \\ &= A\varepsilon^{-d}r^d\omega(r) + A2^d\varepsilon r^{-1}\sum_{k\ge 0}2^{-k}\omega(2^{k+1}r).\end{aligned}$$

Let $r=\varepsilon$, since $\omega(r)$ is continuous and bounded, we're done. □

§1.1 Lebesgue Differentiation theorem part 2

Recall that Fundamental theorem of Calculus has a second part, Given any differentiable function $F(x)$, if $F'(x)$ Riemann integrable, then

$$F(b)-F(a) = \int_a^b F'(x)\,\mathrm{d}x.$$

When F has nice properties (smooth), then it has a generalization to higher dimensions (Stokes' theorem).

But in Lebesgue integration theory, the derivative of a function is hard to define. Hence we'll find an alternative for $F'(x)$.

Example 1.12

Consider Heaviside function $H(t) = \begin{cases}1, & t>0\\ 0, & t\le 0\end{cases}$

Then H is differentiable almost everywhere, but $\int_{-1}^1 H'(t)\,\mathrm{d}t = 0 \ne H(1)-H(-1)$.

Example 1.13

Consider Cantor-Lebesgue function F, similarly we have $F'(x)=0, a.e.$, but $\int_0^1 F'(x)\,\mathrm{d}x = 0 \ne F(1)-F(0)$.

Definition 1.14 (Dini derivatives)**.** Let $f(x)$ be a measurable function, define

$$D^+(f)(x) = \limsup_{h>0,h\to 0}\frac{f(x+h)-f(x)}{h},\quad D^-(f)(x) = \limsup_{h<0,h\to 0}\frac{f(x+h)-f(x)}{h}.$$

$$D_+(f)(x) = \liminf_{h>0,h\to 0}\frac{f(x+h)-f(x)}{h},\quad D_-(f)(x) = \liminf_{h<0,h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Theorem 1.15 (Lebesgue Differentiation theorem for increasing functions)
Let f be an increasing function on $[a, b]$, then $F'(x)$ exists almost everywhere, and

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

Sketch of the proof. The outline of the proof is as follows:
Step 1: Decompose $F = F_c + J$, where F_c is continuous, J is a jump function.
Step 2: Prove F_c increasing and $J' = 0, a.e.$.
Step 3: Prove $D^+(F) < +\infty, a.e.$, $D^+(F) \leq D_-(F), a.e.$, and $D^-(F) \leq D_+(F), a.e.$. □

We proceed step by step.

Step 1 Denote $F(x+0) = \lim_{h\to 0^+} F(x+h), F(x-0) = \lim_{h\to 0^-} F(x+h)$.
Since F increasing, let $\{x_n\}$ be all the discontinuous points of F. Define:

$$j_n(x) = \begin{cases} 0, & x < x_n \\ \beta_n, & x = x_n \\ \alpha_n, & x > x_n \end{cases}$$

where $\alpha_n = F(x_n+0) - F(x_n-0), \beta_n = F(x_n) - F(x_n-0)$.
Hence the jump function

$$J_F(x) = \sum_{n=1}^{+\infty} j_n(x) \leq \sum_{n=1}^{+\infty} \alpha_n = \sum_{n=1}^{+\infty} (F(x_n+0) - F(x_n-0)) \leq F(b) - F(a)$$

is well-defined and increasing.

Theorem 1.16
$F - J_F$ is continuous and increasing.

Proof. First note that

$$\lim_{h\to 0^+} (F(x+h) - J_F(x+h)) = F(x+0) - \lim_{h\to 0^+} J_F(x+h) = F(x-0) - \lim_{h\to 0^+} J_F(x-h)$$

This can be derived from the definition of J_F: If F is continuous at x, the equality is obvious;
If $x = x_n$ for some n,

$$\lim_{h\to 0^+} J_F(x+h) = \sum_{x_k \leq x_n} \alpha_k + \lim_{h\to 0^+} \sum_{x_n < x_k \leq x_n + h} j_k(x+h) = \sum_{x_k \leq x_n} \alpha_k$$

$$\lim_{h\to 0^+} J_F(x-h) = \lim_{h\to 0^+} \sum_{x_k < x_n - h} \alpha_k + \lim_{j\to 0^+} \sum_{x_k = x_n - h} \beta_k = \sum_{x_k < x_n} \alpha_k$$

Note that $\alpha_n = F(x_n+0) - F(x_n-0)$, thus $F - J_F$ is continuous.
Secondly,

$$F(x) - J_F(x) \leq F(y) - J_F(y), \quad \forall a \leq x \leq y \leq b.$$

Because

$$J_F(y) - J_F(x) = \sum_{x < x_j < y} \alpha_j + \sum_{x_k = y} \beta_k - \sum_{x_k = x} \beta_k \leq \sum_{x < x_j < y} \alpha_j + F(y) - F(y-0) \leq F(y) - F(x).$$

which means $F - J_F$ is increasing. □

Step 2

Proposition 1.17

The jump function $J(x)$ is differentiable almost everywhere, and $J'(x) = 0, a.e.$.

Proof. The Dini derivatives of $J(x)$ exist and are non-negative (since J is increasing).

$$\overline{D}(J)(x) = \max\{D^+(J)(x), D^-(J)(x)\}.$$

Let $E_\varepsilon = \{\overline{D}(J)(x) > \varepsilon > 0\}$. We'll prove E_ε is null for all ε. If $x \in E_\varepsilon$, $\exists h$ s.t.

$$\frac{J(x+h) - J(x)}{h} > \varepsilon \implies J(x+h) - J(x-h) > \varepsilon h.$$

Let $N \in \mathbb{N}$ s.t. $\sum_{n>N} \alpha_n < \frac{\varepsilon\delta}{10}$. Define $J_N(x) = \sum_{n>N} j_n(x)$.

$$E_{\varepsilon,N} = \{\overline{D}(J_N)(x) > \varepsilon\}, \quad E_\varepsilon \subset E_{\varepsilon,N} \cup \{x_1, \dots, x_N\},$$

Since for $x \neq x_i$,

$$\overline{D}(J)(x) = \limsup_{h \to 0} \frac{J(x+h) - J(x)}{h} = \limsup_{h \to 0} \left(\frac{J_N(x+h) - J_N(x)}{h} + \frac{1}{h} \sum_{n=1}^{N} (j_n(x+h) - j_n(x)) \right).$$

Lemma 1.18

Let $\mathcal{B}$ be a collection of balls with bounded radius in $\mathbb{R}^d$. There exists countably many disjoint balls B_i s.t.

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Proof. Let $r(B)$ denote the radius of B. Take B_1 s.t. $r(B_1) > \frac{1}{2} \sup_{b \in \mathcal{B}} r(B)$.
The rest is the same as before. $\square$

By lemma, there exists countably many disjoint intervals $(x_i + h_i, x_i - h_i)$ s.t.

$$\begin{aligned} m^*(E_{\varepsilon,N}) &\leq 5 \sum_{i=1}^{\infty} 2h_i \\ &\leq 10 \sum_{i=1}^{\infty} \varepsilon^{-1} (J_N(x_i + h_i) - J_N(x_i - h_i)) \\ &\leq 10\varepsilon^{-1} (J_N(b) - J_N(a)) < \delta. \end{aligned}$$

Hence $E_{\varepsilon,N}$ is a null set $\implies E_\varepsilon$ null, and at last $\overline{D}(J) = 0, a.e.$. $\square$

Step 3 First we prove $D^+(F) < \infty, a.e.$.
Let $E_\gamma = \{x : D^+(F)(x) > \gamma\}$.

When $h \in [\frac{1}{n+1}, \frac{1}{n}]$:

$$\frac{F(x+h)-F(x)}{h} \leq \frac{n+1}{n}\frac{F(x+\frac{1}{n})-F(x)}{\frac{1}{n}},$$
$$\geq \frac{n}{n+1}\frac{F(x+\frac{1}{n+1})-F(x)}{\frac{1}{n+1}}.$$

Thus

$$D^+(F)(x) = \limsup_{h\to 0}\frac{F(x+h)-F(x)}{h} = \limsup_{n\to\infty}\frac{F(x+\frac{1}{n})-F(x)}{\frac{1}{n}}$$

is measurable.

Lemma 1.19 (Sunrise lemma)
Let $G(x)$ be a continuous function on $\mathbb{R}$. Define

$$E = \{x : \exists h > 0, G(x+h) > G(x)\}.$$

Then E is open and $E = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where (a_i, b_i) are disjoint finite intervals s.t. $G(a_i) = G(b_i)$.

When G is defined on finite interval $[a, b]$, we also have $G(a) \leq G(b_1)$.

Proof. E is open since G is continuous.

Take an interval (a, b), by definition $a, b \notin E$, so $G(a) \geq G(b)$.

Since $b \notin E, G(x) \leq G(b), \forall x > b$. If $G(a) > G(b)$, Let $G(a+\varepsilon) > G(b)$, as $a + \varepsilon \in E$, exists $h > 0$ s.t. $G(a+\varepsilon+h) > G(a+\varepsilon)$.

But G has a maximum on $[a+\varepsilon, b]$, say $G(c)$, we must have $c \neq a+\varepsilon, b$. This leads to a contradiction. □

For $x \in E_\gamma$, $\exists h > 0$ s.t. $F(x+h) - F(x) > \gamma h$, by Sunrise Lemma on $F(x) - \gamma x$,

$$m(E_\gamma) \leq \sum_{k=1}^{\infty}(b_k - a_k) \leq \gamma^{-1}\sum_{k=1}^{\infty}(F(b_k) - F(a_k)) \leq \gamma^{-1}(F(b) - F(a)).$$

Therefore when $\gamma \to \infty$, $m(E_\gamma) \to 0$.

The last part is $D^+(F) \leq D_-(F), a.e.$.

Similarly let

$$E_{r,R} = \{D^+(F)(x) > R, D_-(F)(x) < r\}.$$

WLOG $E_{r,R} \subset [c, d]$, and $d - c < \frac{R}{r}m(E_{r,R})$.

Let $G(x) = F(-x) + rx$, by Sunrise Lemma on $[-d, -c]$,

$$\{s : \exists h > 0, G(x+h) > G(x)\} = \bigcup_k(-b_k, -a_k).$$

Note that $-E_{r,R}$ is contained in the above set, and $G(-b_k) \leq G(-a_k) \iff F(b_k) - F(a_k) \leq r(b_k - a_k)$,

We use Sunrise Lemma again on each (a_k, b_k) and $F(x) - Rx$,

$$E_{r,R} \cap (a_k, b_k) \subset \{x : \exists h > 0, F(x+h) - F(x) \geq Rh\} = \bigcup_{l=1}^{\infty}(a_{k,l}, b_{k,l}).$$

Hence

$$
\begin{aligned}
m(E_{r,R}) &\leq \sum_{k,l=1}^{\infty}(b_{k,l} - a_{k,l}) \\
&\leq R^{-1} \sum_{k,l=1}^{\infty}(F(b_{k,l}) - F(a_{k,l})) \leq R^{-1}\sum_{k=1}^{\infty}(F(b_k) - F(a_k)) \\
&\leq R^{-1} r \sum_{k=1}^{\infty}(b_k - a_k) \leq R^{-1} r(d-c),
\end{aligned}
$$

which gives a contradiction!