Geometry II

Felix Chen

Contents

Example 0.1

Consider the Gauss map of a torus, the "outer" part and the "inner" part of the torus maps to $\mathbb{S}^2$ bijectively. If we compute

$$\int_{T^2} K \,\mathrm{d}Area_E = \int_{\mathbb{S}^2} (1 + (-1))\,\mathrm{d}Area_S = 0 = 2\pi\chi(T^2),$$

as Gauss-Bonnet formula implies.

§0.1 Fundamental equation of surfaces

Like the Fundamental theorem and Frenet frame in curve theory, we want to develop a theorem for describing surfaces using only fundamental forms.

Given a parameter on a surface, there's a natural frame $(\phi_s, \phi_t, \vec{n})$. If we take the derivative of the frame, we'll get

$$(\phi_s, \phi_t, \vec{n})_{st} = (\phi_s, \phi_t, \vec{n})_{ts}.$$

Taking the inner product with $(\phi_s, \phi_t, \vec{n})^T$ and apply the product rule:

$$\left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_s\right)_t - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_t (\phi_s, \phi_t, \vec{n})_s = \left(\begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}(\phi_s, \phi_t, \vec{n})_t\right)_s - \begin{pmatrix}\phi_s\\ \phi_t\\ \vec{n}\end{pmatrix}_s (\phi_s, \phi_t, \vec{n})_t$$

This equation will give us some relations between the fundamental quantities. In literature these relations are known as Gauss equation and Codazzi equations.

Gauss equation can be written as:

$$(\phi_s \cdot \phi_{ts})_t - (\phi_s \cdot \phi_{tt})_s = \phi_{st} \cdot \phi_{st} - \phi_{ss} \cdot \phi_{tt}.$$

Codazzi equations are related to $\vec{n}$ and more complicated.

From Gauss equation we can deduce a famous theorem:

Theorem 0.2 (Gauss' Theorema Egregium)

The Gauss curvature K is determined by the first fundamental form.

Proof. Note that $(\phi_s \cdot \phi_{ts})_t = \frac{1}{2}E_{tt}$, and $(\phi_s \cdot \phi_{tt})_s = (F_t - \frac{1}{2}G_s)_s = F_{ts} - \frac{1}{2}G_{ss}$.

Suppose $\phi_{ss} = x\phi_s + y\phi_t + L\vec{n}$, then

$$\frac{1}{2}E_s = \phi_s \cdot \phi_{ss} = Ex + Fy, \quad F_s - \frac{1}{2}G_t = \phi_t \cdot \phi_{ss} = Fx + Gy$$

So x, y is determined by E, F, G.

Similarly, we get

$$\begin{aligned} \phi_{ss} &= *\phi_s + *\phi_t + L\vec{n} \\ \phi_{st} &= *\phi_s + *\phi_t + M\vec{n} \\ \phi_{tt} &= *\phi_s + *\phi_t + N\vec{n} \end{aligned}$$

where $*$ are determined by E, F, G.

By Gauss equation, we get $* = -(LN - M^2) + *$, and $*$ is determined by E, F, G and their partial derivatives. $\square$

> **Remark 0.3** — The computation looks messy, but in modern mathematics, we have a systematic notation which is more simplified.

Definition 0.4 (Isometries)**.** Let $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. If a map $\psi : \widetilde{U} \to U$ satisfies $\psi^*(g) = \widetilde{g}$, then it's called an **isometry**.

Let $\mathcal{F} = (\phi_s, \phi_t, \vec{n})$. Suppose $\mathcal{F}_s = \mathcal{F}A$, and $\mathcal{F}_t = \mathcal{F}B$. Taking the second derivative we get $\mathcal{F}_{st} = \mathcal{F}(BA + A_t)$, $\mathcal{F}_{ts} = \mathcal{F}(AB + B_s)$. Thus we have

$$A_t - B_s = AB - BA.$$

But we want to express things in terms of E, F, G, so we can compute the dot product of $\mathcal{F}^T$:

$$P := \mathcal{F}^T \cdot \mathcal{F} = \begin{pmatrix} E & F & \\ F & G & \\ & & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F & \\ -F & E & \\ & & 1 \end{pmatrix}$$

Substituting into $\mathcal{F}_s = \mathcal{F}A$ we get

$$PA = \begin{pmatrix} \frac{E_s}{2} & \frac{E_t}{2} & -L \\ F_s - \frac{E_t}{2} & \frac{G_s}{2} & -M \\ L & M & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{F}^T \mathcal{F}_{st} &= (\mathcal{F}^T \mathcal{F}_s)_t - \mathcal{F}_t^T \mathcal{F}_s = (PA)_t - B^T PA. \\ \implies (PA)_t - (PB)_s &= (PB)^T P^{-1}(PA) - (PA)^T P^{-1}(PB). \end{aligned}$$

Gauss equation corresponds to the $(1, 2)$ entry of the matrix equation:

$$\frac{1}{2}(E_{tt} - 2F_{st} + G_{ss}) = -(LN - M^2) + \frac{p}{4(EG - F^2)},$$

where p is a polynomial in fundamental quantities and their derivatives.

These equations still looks complicated, so we'll "package" them:

$$-(LN - M^2) = R_{1212}.$$

Let

$$A = \begin{pmatrix} \Gamma^1_{\text{-}11} & \Gamma^1_{\text{-}12} & h^1_{\text{-}1} \\ \Gamma^2_{\text{-}11} & \Gamma^2_{\text{-}12} & h^2_{\text{-}1} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{\text{-}12} & \Gamma^1_{\text{-}22} & h^1_{\text{-}2} \\ \Gamma^2_{\text{-}12} & \Gamma^2_{\text{-}22} & h^2_{\text{-}2} \\ h_{12} & h_{22} & 0 \end{pmatrix}$$

Here the Γ's are called Christoffel notations.

Codazzi equations correspond to the $(1,3),(2,3)$ enties:

$$\begin{aligned} L_t - M_s &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \\ M_t - N_s &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{21}) - N\Gamma^2_{21}. \end{aligned}$$

Remark 0.5 — The index notations for computation can be defined as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij}), \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = (g^{ij}),$$

and h is defined similarly. If we use Einstein summation notation, we can write $g_{ij}g^{jk} = \delta^k_i$.

Let $\vec{v}_1 := \phi_s, \vec{v}_2 = \phi_t$, and

$$\frac{\partial \vec{v}_\alpha}{\partial \vec{u}^\beta} = \sum_\gamma \Gamma^\gamma_{\text{-}\alpha\beta}\vec{v}_\gamma + h_{\alpha\beta}\vec{n}, \quad \frac{\partial \vec{n}}{\partial \vec{u}^\beta} = -\sum_\gamma h^\gamma_{\text{-}\beta}\vec{v}_\gamma.$$

Here the upper index is defined as:

$$h^\gamma_{\text{-}\beta} := \sum_\delta g^{\gamma\delta}h_{\delta\beta}.$$

From this we can write Γ out explicitly:

$$\Gamma^\gamma_{\text{-}\alpha\beta} = \sum_\delta \frac{g^{\gamma\delta}}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right)$$

This is called *Christoffel notations.*

$$R^\delta_{\text{-}\alpha\beta\gamma} := \frac{\partial \Gamma^\delta_{\text{-}\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\delta_{\text{-}\alpha\gamma}}{\partial u^\beta} + \sum_\eta(\Gamma^\eta_{\text{-}\alpha\beta}\Gamma^\delta_{\text{-}\eta\gamma} - \Gamma^\eta_{\text{-}\alpha\gamma}\Gamma^\delta_{\text{-}\eta\beta}).$$

This is called *Riemann symbols.* Another type is defined as:

$$R_{\delta\alpha\beta\gamma} = \sum_\eta g_{\delta\eta}R^\eta_{\text{-}\alpha\beta\gamma}.$$

In surface theory, only R_{1212} is nontrivial.

Using these notations, we can write the equations as:

- Gauss:

$$R_{\delta\alpha\beta\gamma} = -(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}).$$

- Codazzi:

$$\frac{\partial h_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial h_{\alpha\gamma}}{\partial u^\beta} = \sum_\delta(h_{\beta\delta}\Gamma^\delta_{\text{-}\alpha\gamma} - h_{\gamma\delta}\Gamma^\delta_{\text{-}\alpha\beta}).$$

Here we explain the above computation a little.

The vector

$$\sum_\gamma \Gamma^\gamma_{\_\alpha\beta} v_\gamma =: \nabla_\beta \vec{v}_\alpha$$

is called the covariant derivative of $\vec{v}_\alpha$. It's projection of the derivative of $\vec{v}_\alpha$ onto the tagent space.

$$\frac{\partial}{\partial u^\beta}\frac{\partial}{\partial u^\gamma}(v_\alpha) = \frac{\partial}{\partial u^\gamma}\frac{\partial}{\partial u^\beta}(v_\alpha)$$

$$\implies -\nabla_\beta\nabla_\gamma v_\alpha + \nabla_\gamma\nabla_\beta v_\alpha = h_{\alpha\beta}\nabla_\gamma \vec{n} - h_{\alpha\gamma}\nabla_\gamma \vec{n}.$$

So the covariant derivative is not commutative, and the "curvature" or the second fundamental form basically measures this discommutation.

Now if we look at

$$\frac{\partial g_{\delta\alpha}}{\partial u^\beta} = \frac{\partial v_\delta \cdot v_\alpha}{\partial u^\beta} = \frac{\partial v_\delta}{\partial u^\beta} v_\alpha + v_\delta \frac{\partial v_\alpha}{\partial u^\beta} = \sum_\gamma g_{\alpha\gamma}\Gamma^\gamma_{\_\delta\beta} + \sum_\gamma g_{\delta\gamma}\Gamma^\gamma_{\_\alpha\beta},$$

similarly, by symmetry, computing

$$\frac{\partial g_{\alpha\beta}}{\partial u^\delta}, \quad \frac{\partial g_{\delta\beta}}{\partial u^\alpha},$$

will yield

$$\Gamma^\gamma_{\_\alpha\beta} = \sum_\delta \frac{g^{\gamma\delta}}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right).$$

In fact this is more intuitive in Einstein summation notation.

§0.2 Fundamental theorem of surface theory

Theorem 0.6 (Fundamental theorem of surface theory)
Let $D \subset \mathbb{R}^2$, $u = (u^1, u^2)$ is the coordinate. Let $g_{\alpha\beta}, h_{\alpha\beta} : D \to \mathbb{R}$ be C^3 functions, and the matrix $(g_{\alpha\beta})$ is symmetrical anf positive definite, $(h_{\alpha\beta})$ is symmetrical.

Let $g^{\alpha\beta}$ be the inverse matrix of $g_{\alpha\beta}$, and $R_{\delta\alpha\beta\gamma}$ is as above. If these functions satisfies Gauss equation and Codazzi equation, then:

For all $p \in D$, exists a neighborhood $U = U(p) \subset D$ and a regular surface $\phi : U \to \mathbb{E}^3$, such that $g_{\alpha\beta}, h_{\alpha\beta}$ are the first and second fundamental quantities of ϕ.

Furthermore, if $\widetilde{\phi} : U \to \mathbb{E}^3$ also satisfies above conditions, then $\widetilde{\phi} = \sigma \circ \phi$, where σ is an isometry of $\mathbb{E}^3$.

Basically we need to solve a partial differential equation, and we need to consider how to construct this equation.

Proof. Let $\phi : U \to \mathbb{E}^3$, $v_\alpha, \vec{n} : D \to V(\mathbb{E}^3)$ be unknown functions satisfying

$$\begin{cases} v_\alpha = \dfrac{\partial \phi}{\partial u^\alpha} \\ \dfrac{\partial v_\alpha}{\partial u^\beta} = \displaystyle\sum_\gamma \Gamma^\gamma_{\_\alpha\beta} v_\gamma + h_{\alpha\beta}\vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = -\displaystyle\sum_\gamma h^\gamma_{\_\beta} v_\gamma \end{cases}$$

This is a linear homogeneous PDE of degree 1, and it actually has a unique solution.

Consider the initial-value problem in the neighborhood of a given point $p \in D$.

We hope to prove that

- The above PDE initial-value problem has a unique solution under the Gauss-Codazzi equations;
- If initially (i.e. at p) we have
$$\vec{n} = \frac{v_1 \times v_2}{\|v_1 \times v_2\|},$$
then it holds for all $p' \in U(p)$.

For the second statement, we can compute $\frac{\partial}{\partial u^\beta}(\vec{n} \cdot v_\alpha) = 0$, so they are constant.

For the PDE part, if we want a C^2 solution of some linear PDE of degree 1:

$$\frac{\partial y^j}{\partial x^\alpha} = f^j_\alpha(x^1, \ldots, x^n, y^1, \ldots, y^m)$$

There's a necessary condition that the partial derivatives are commutative, i.e.

$$\frac{\partial}{\partial x^\beta}\frac{\partial y^j}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta}.$$

This expands to

$$\frac{\partial f^j_\alpha}{\partial x^\beta} + \sum_k f^k_\beta \frac{\partial f^j_\alpha}{\partial y^k} = \frac{\partial f^j_\beta}{\partial x^\alpha} + \sum_k f^k_\alpha \frac{\partial f^j_\beta}{\partial y^k}.$$

In fact this is also the sufficient condition of the existence local solution.

Remark 0.7 — The proof is beyond the scope of this course, but the basic idea is to build the y^j's dimension by dimension (from curve to surfaces to 3d manifolds ...). The 1d part can be constructed using solutions of ODE, and the compatibility follows by our condition.

In the language of differential forms, let $y = (y^1, \ldots, y^m)$, we are given $\mathrm{d}y$, since the condition says $\mathrm{d}(\mathrm{d}y) = 0$, i.e. $\mathrm{d}y$ is a *closed form*, so we always have local solution of y.

Returning to our original problem, this condition is actually what we used to deduce the Gauss-Codazzi equations, so our PDE must have a unique solution on a neighborhood of p. □

Example 0.8

We can't grant that the global function exists. For example, let $D = \{x^2 + y^2 \in [a^2, b^2]\}$, and M be a helicoid.

Since there's a natural map $\phi : D \backslash ([a, b] \times \{0\}) \to M$ (projection), let g, h be the fundamental forms of ϕ, by the symmetry we can extend g, h to entire D.

It's clear that there exists local solutions but the global solutions does't exist. (In theory of differential forms, this is similar to closed forms may not be exact)

But if the region D is *simply connected*, the global solution always exist.

§0.3 Isometric, conformal and area-perserving maps

Let $U, \widetilde{U} \subset \mathbb{R}^2$, and $\phi : U \to \mathbb{E}^3, \widetilde{\phi} : \widetilde{U} \to \mathbb{E}^3$ be two surfaces. Let $f : \widetilde{U} \to U$ be a map between two surfaces.

Earlier we introduced isometric maps (isometry), i.e. $f^*(g) = \widetilde{g}$. Since the length depends only on the first fundamental form, the isometry perserves the length, angles and areas on surfaces.

The **conformal** maps perserves the angles on the surfaces, and it's easy to imply this is equivalent to $f^*(g) = \lambda \widetilde{g}$ for some $\lambda \in \mathbb{R}$.

As the name suggests, the **area-perserving** maps perserves the areas on two surfaces, which is saying $\det f^*(g) = \det \widetilde{g}$.