Geometry II

Felix Chen

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Theorem 0.0.1

$\pi_1(S^1) \cong \mathbb{Z}$, where the generating element is id.

Proof. Consider the map $p : \mathbb{R} \to S^1$, with $t \mapsto e^{2\pi i t}$.

Given any path $\gamma : [0, 1] \to S^1$, we can find a unique path $\tilde{\gamma} : [0, 1] \to \mathbb{R}$, s.t. $\tilde{\gamma}(0) \in \mathbb{Z}$ is any given base point. We denote this map by Φ, $\gamma \mapsto \tilde{\gamma}(1)$, where we require $\tilde{\gamma}(0) = 0$.

We can prove that $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$, and Φ only depends on the homotopy class of γ, so Φ induces a homomorphism of $\pi_1(S^1) \to \mathbb{Z}$.

Remark 0.0.2 — Since every homotopy $[0, 1] \times [0, 1] \to S^1$ can be lifted uniquely, and the endpoints of each path form a path in $\mathbb{R}$, but it's always contained in $\mathbb{Z}$, hence it must be constant.

Note that

- Φ is surjective since $s \mapsto e^{2\pi i m s}$ is mapped to m under Φ;
- Φ is injective since $\ker \Phi = \{1\}$: if $\tilde{\gamma}(1) = 0$, then $\tilde{\gamma} \simeq const$, so $\gamma = p \circ \tilde{\gamma} \simeq const$.

So Φ is an isomorphism, $\pi_1(S^1) \cong \mathbb{Z}$. □

Next we'll prove Van Kampen theorem (**??**). In fact we only need to prove that:

Claim 0.0.3. The map

$$j'_\sharp * j''_\sharp : \pi_1(U', x_0) * \pi_1(U'', x_0) \to \pi_1(X, x_0)$$

is a surjective homomorphism, and its kernel is the normal closure generated by $i'_\sharp(\delta)i''_\sharp(\bar{\delta})$.

CLearly it's a group homomorphism.

For any $\gamma \in \pi_1(X, x_0)$, it can be decompose to $a_1 b_1 a_2 \cdots a_k b_k$, where $a_i \subset U', b_i \subset U''$, let the partition points be $p_1, \ldots, p_k, q_1, \ldots, q_k \in W$, and denote s_i, t_i the path from x_0 to p_i, q_i. So we have

$$\gamma = \underbrace{a_1 \bar{s}_1}_{\in \pi_1(U', x_0)} \cdot \underbrace{s_1 b_1 \bar{t}_1}_{\in \pi_1(U'', x_0)} \cdots$$

Thus $j'_\sharp * j''_\sharp$ is indeed surjective.

At last we'll study its kernel, let $\gamma \in \ker j'_\sharp * j''_\sharp$. Since $\gamma \simeq \{x_0\}$, say the homotopy is $H : [0,1]\times[0,1] \to U' \cup U''$.

We can partition $[0,1]\times[0,1]$ to many small cells such that each cell's image is completely contained in either U' or U''.

TODO

Using the "word processing" method, since we've showed that $\gamma = \alpha_1\beta_1\cdots$ where $\alpha_i \subset U', \beta_i \subset U''$. So actually we're saying that

$$\gamma = i'_\sharp(\alpha_1) i''_\sharp(\beta_1)\cdots$$

if we some $\delta \subset U' \cap U''$, then the conjugate of $i'_\sharp(\delta)i''_\sharp(\delta)^{-1}$ can change $\cdots i'_\sharp(\delta)\cdots$ to $\cdots i''_\sharp(\delta)\cdots$.

Thus if γ is in the kernel, it can indeed be written as a product of conjugates of $i'_\sharp(\delta)i''_\sharp(\delta)^{-1}$.

Remark 0.0.4 — A more frequently used version is that W is a strong deformation kernel of some open neighborhood in X.

Example 0.0.5

For any finite representation of a group

$$G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle,$$

G can be realized as the fundamental groups of a space: Let X be a CW-complex with a single 0-cell, n 1-cells corresponding to x_i, and m 2-cells corresponding to R_i.

Remark 0.0.6 — The path connected condition of W can't be removed, e.g. two segments can fuse to S^1.

Example 0.0.7

Let $f : S \to S$ be a homeomorphism, where S is a closed surface. Consider the *mapping torus*:

$$M_f = S \times [0,1]/\sim,$$

where $(0,0) \sim (f(x), 1)$.

Let $Y = S \times \{0\} \cup (\{x_0\} \times [0,1])$, U' is an open neighborhood of Y, $U'' = M_f \backslash Y$.

Observe that $U' \simeq S \vee circle$, and $U'' \simeq (S\backslash disk) \times (\varepsilon, 1-\varepsilon) \simeq S\backslash disk$.

$$\pi_1(M_f) \cong \pi_1(X) * \langle t \rangle / (g \sim t f_\sharp(g) t^{-1}) \cong \pi_1(S) \rtimes_{f_\sharp} \langle t \rangle$$

Seifent-vanKampen: if $i'_\sharp, i''_\sharp$ are both injective, then $j'_\sharp, j''_\sharp$ are also injective.

Next we'll see some applications of fundamental groups:

- Bronwer fixed point theorem: A continuous map $f : D^n \to D^n$ must have a fixed point.
- Invariance of the boundary: If $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times [0, +\infty)$, s.t. $x \in \mathbb{R}^{n-1} \times \{0\}$, then x doesn't have a neighborhood homeomorphic to $\mathbb{R}^n$.
- Invariance of regions: If $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is also open, i.e. $f : U \to f(U)$ is a homeomorphism.

Here we can only prove the case $n = 2$, since the complete proof need homotopy groups of rank n (i.e. π_n), but here we only introduced π_1. However, the idea is nearly identical.

Proof. Assmue by contradiction that f has no fixed points, let $g(x) = \frac{x-f(x)}{\|x-f(x)\|}$, then $g : D^n \to S^{n-1}$ is a deformation. Thus $g_\sharp : \pi_1(D^2) \to \pi_1(S^1)$ surjective, but $\pi_1(D^2) = \{1\}, \pi_1(S^1) = \mathbb{Z}$, contradiction! □

Proof. If x is an interior point, $x \in U$ and U homeomorphic to $\mathbb{R}^n$, then $U \backslash \{x\}$ can deform to a $n-1$ dimensional sphere, thus $\pi_1(U\backslash\{x\}) \neq \{1\}$.

But if x is a boundary point, then $\pi_1(U\backslash\{x\}) = \{1\}$, contradiction! □

Proof. Assume by contradiction that there exists $0 \in U$ s.t. $f(0) \in \mathbb{R}^n$ has no open neighborhood lying completely in $f(U)$.

We can construct a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\|x - g(f(x))\| \leq 1, \quad x \in B(0,1); \quad g(f(x)) \neq 0.$$

Then by Bronwer fixed point theorem on $x \mapsto x - g(f(x))$ we get a contradiction.

The construction is as below:

Since $f(\partial B(0,1))$ must be at least say 10ε away from 0, and $B(f(0),\varepsilon)$ has a point outside of the image of f, so we have a map $P : B(f(0),\varepsilon)\backslash\{p\} \to \partial B(f(0),\varepsilon)$.

Then consider $g = f^{-1} \circ P$, since f^{-1} may not exist on every point, so we need Tietze extension theorem to get an extension h. In $B(f(0), 2\varepsilon)$, we'll change h a little (i.e. take a polynomial approximation) to ensure $g(f(x)) \in B(0,1)$. □

§0.1 Covering spaces

Except van Kampen's theorem, there's another way to compute fundamental groups.

Definition 0.1.1 (Covering maps)**.** Let $p : \widetilde{X} \to X$ be a continuous map. If

- p is surjective;
- For any $x \in X$, there exists an open neighborhood $U = U(x) \subset X$, such that $p^{-1}(U)$ is a union of disjoint open sets $\{U_\alpha\}$, and p is homeomorphism from U_α onto U for each α.

Then we say p is a **covering map**, and $\widetilde{X}$ is a **covering space** of X. $p^{-1}(x)$ is called a **fiber**.

Remark 0.1.2 — Often we'll require $\widetilde{X}, X$ are path connected to ensure the relations with fundamental groups. In this case $\#p^{-1}(x)$ is constant.

Definition 0.1.3. We say two covering is **isomorphic** if exists homeomorphism $\tau : \widetilde{X} \to \widetilde{X}'$ s.t. $p' \circ \tau = p$. Two covering is **equivalent** if $p' \circ \widetilde{\sigma} = \sigma \circ p$. The difference is shown in the diagram.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tau"] \arrow[dr, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\sigma}"] \arrow[d, "p"'] & \widetilde{X}' \arrow[d, "p'"] \\ X \arrow[r, "\sigma"'] & X' \end{tikzcd}$$

Example 0.1.4

The map $x \mapsto e^{ix}$ is a covering map from $\mathbb{R}$ to S^1. Also $\mathbb{R}^2$ is a covering space of T^2, since T^2 can be represented as $\mathbb{R}^2/\mathbb{Z}^2$.

Example 0.1.5

The surface $2T^2$ can be viewed as an octagon with edges fused together, (an octagon with each angle $45°$) which can be realized in hyperbolic plane $\mathbb{H}^2$.

In fact, $\mathbb{H}^2$ is always the covering space of kT^2 when $k \geq 2$, and kP^2 when $k \geq 3$.

From the examples we can see that covering spaces are the "expanded" spaces of original spaces, i.e. the structures are "flattened" in covering spaces, so that we can study the structure of original spaces more easily.

An important application is that we can "lift" the maps to covering spaces.

Theorem 0.1.6 (Map lifting theorem)

Let $p : \widetilde{X} \to X$ be a covering map, X is path connected. Let A be a path connected space, $f : A \to X$ has a **lifting** $\widetilde{f} : A \to \widetilde{X}$ s.t. $\widetilde{f}(a) \in p^{-1}(f(a)), \forall a \in A$ if and only if there exists a homomorphism Φ s.t. $f_\sharp = p_\sharp \circ \Phi$:

$$\begin{tikzcd} & \pi_1(\widetilde{X}, e_0) \arrow[d, "p_\sharp"] \\ \pi_1(A, a_0) \arrow[ur, dashed, "\Phi"] \arrow[r, "f_\sharp"'] & \pi_1(X, x_0) \end{tikzcd}$$

This is equivalent to $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{X}, e_0))$.

Proof. If we fixed $\tilde{f}(a_0) = e_0$, then for a neighborhood V of e_0, there's a unique map $\tilde{f} : U \to V$, where U is a neighborhood of a_0. This is because p restricted on V is a homeomorphism, and f continuous implies U is open, U is called a *basic neighborhood* of a_0.

For any $b \in A$, there's a path γ from a_0 to b. Since γ is compact, it can be splited to several segments, where each segment lies inside a basic neighborhood of some point.

Therefore the lifting of γ can be uniquely determined by the lifting of one point. Hence $\tilde{f}(b)$ is also determined.

Next we'll show that this $\tilde{f}$ is well-defined and continuous. Let α, β be two paths from a_0 to b. Then $f \circ \alpha, f \circ \beta$ are two paths from x_0 to $f(b)$.

When $f_\sharp(\pi_1(A, a_0)) \leqslant p_\sharp(\pi_1(\widetilde{(X)}, e_0))$, let $w = \alpha\beta^{-1} \in \pi_1(A)$, then there exists $\varphi \in \pi_1(\widetilde{X})$ s.t. $f \circ w = p \circ \varphi$.

But there's a unique lifting for $f \circ \alpha, f \circ \beta$, so $\tilde{f}(\alpha)\tilde{f}(\beta)^{-1} = \varphi$, thus $\tilde{f}(b)$ is well-defined.

Clearly $\tilde{f}$ is continuous, so we're done. □

Remark 0.1.7 — Different base points will result in the image $p_\sharp$ and $\tilde{f}$.

Example 0.1.8

Let M be a closed surface, $M \neq S^2, \mathbb{RP}^2$. Note that M has a contractible covering space, so any map $S^n \to M$ is always homotopic to constant, where $n \geq 2$.

Now if we look at the definition of isomorphic coverings, we'll find that this is just a map lifting, where τ is a lifting of p, τ^{-1} is a lifting of p'. By map lifting theorem we get:

Corollary 0.1.9

Two covering spaces $\widetilde{X}, \widetilde{X}'$ of X are isomorphic iff $p_\sharp(\pi_1(\widetilde{X})) = p'_\sharp(\pi_1(\widetilde{X}'))$.

From this we discover that each covering of X corresponds to a subgroup of $\pi_1(X)$. In fact the inverse is also true:

Theorem 0.1.10 (Existence theorem of covering spaces)

Let X be a path connected space, then for all subgroups $G \leqslant \pi_1(X, x_0)$, there exists a covering $p : \widetilde{X} \to X$ s.t.

$$p_\sharp(\pi_1(\widetilde{X}, e_0)) = G.$$

Remark 0.1.11 — This implies that **universal coverings** always exists, i.e. the covering space $\widetilde{X}$ which has trivial fundamental group.

The proof is quite complex, so we'll put it off here.

Definition 0.1.12 (Regular covering space)**.** If $p_\sharp(\pi_1(\widetilde{X}, e_0))$ is a normal subgroup of $\pi_1(X, x_0)$, then we say it's a **regular covering** of X.

In this case the base point will not change the image of $p_\sharp$.

Proposition 0.1.13

Let $p : \widetilde{X} \to X$ be a regular covering, for any $e_1, e_2 \in p^{-1}(x_0)$, there exists a homeomorphism $f : \widetilde{X} \to \widetilde{X}$ s.t.

- f is a lifting of p;
- $f(e_1) = f(e_2)$.

We say the lifting of p with respect to itself **deck transformation**. Regular coverings have *orbit transitivity*: $\forall e_1, e_2 \in p^{-1}(x_0)$, there exists a self-homeomorphism sending e_1 to e_2.