Mathematical Analysis II

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Since $AA^T = \begin{pmatrix} GG^T & 0 \\ 0 & I_{n-d} \end{pmatrix}$, $|\det A| = \sqrt{\det GG^T}$, we say GG^T is the **Gram matrix** of G.

Another example is the length of a curve. Recall that we have the formula

$$L(\gamma) = \int_0^1 |\gamma'(t)| \, \mathrm{d}t.$$

The length of a curve is essentially the "volume" of a 1-dimensional manifold, so the idea of higher dimensional manifold is the same.

Definition 0.0.1. Let M be a manifold in $\mathbb{R}^n$. Let $\Phi : V \subset \mathbb{R}^d \to U \subset M$ be a smooth homeomorphism, $\operatorname{rank} \Phi = d$. We can split U to many small regions and use the paraloids to approximate the volume of each regoin.

Thus we define:

$$m(U) = \int_V \sqrt{\det(\mathrm{d}\Phi(x)^T \, \mathrm{d}\Phi(x))} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_d.$$

The above differential form inside the integral is called the **volume form**:

$$\mathrm{d}\sigma(y) = \sqrt{\det(\mathrm{d}\Phi^T \, \mathrm{d}\Phi)} \, \mathrm{d}y.$$

Moreover, with this measure we can define the integral of general measurable function f (measurable means locally measurable on $\mathbb{R}^d$):

$$\int_U f \, \mathrm{d}\sigma = \int_V f(\Phi(x)) \sqrt{\det(\mathrm{d}\Phi^T \, \mathrm{d}\Phi)} \, \mathrm{d}x.$$

Just like the length of a curve, the volume defined above is also a geometric quantity, i.e. independent of coordinates.

Example 0.0.2

Let $d = 1$, $\gamma : (-1, 1) \to \mathbb{R}^n$, $\gamma'(0) \neq 0$. For fixed $-1 < a < b < 1$ and a function f on γ, let C_a^b denote the curve between $\gamma(a), \gamma(b)$,

$$\int_{C_a^b} f \, \mathrm{d}\sigma = \int_a^b f(\gamma(t)) |\gamma'(t)| \, \mathrm{d}t$$

is called the **curve integral of the first type**.

Example 0.0.3

Let $d = n - 1$, $f : \mathbb{R}^{n-1} \to \mathbb{R}$, the graph of f is a hyper-surface $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}^{n-1}\}$. It has a parametrization $\Phi(x) = (x, f(x))$, so

$$\mathrm{d}\Phi = \begin{pmatrix} I_{n-1} \\ \nabla f \end{pmatrix} \implies \mathrm{d}\Phi^T \, \mathrm{d}\Phi = I_{n-1} + \nabla f^T \cdot \nabla f.$$

Hence $\det(\mathrm{d}\Phi^T \, \mathrm{d}\Phi) = 1 + |\nabla f|^2$. (This can be obtained by looking at the eigenvectors)

Therefore for φ on $\mathbb{R}^n$, we have

$$\int_{\Gamma_f} \varphi \, \mathrm{d}\sigma = \int_{\mathbb{R}^{n-1}} \varphi(x, f(x)) \sqrt{1 + |\nabla f|^2} \, \mathrm{d}x.$$

Next we'll compute the surface area of unit sphere S^{n-1}.

Let c_n denote the volume of unit sphere in $\mathbb{R}^n$,

$$c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.$$

We claim in advance that the surface area of unit sphere $\omega_{n-1} = nc_n$. Here we use the sphere coordinates:

$$x_1 = r \prod_{k=1}^{n-1} \sin\theta_k, \quad x_i = r \sin\theta_{n-1} \cdots \sin\theta_i \cos\theta_{i-1}, 2 \le i \le n.$$

Let $F_n(r, \theta_1, \ldots, \theta_{n-1}) = (x_1, \ldots, x_n)$.

$$\mathrm{d}F_n = \begin{pmatrix} \sin\theta_{n-1} \, \mathrm{d}F_{n-1} & \cos\theta_{n-1} F_{n-1}^T \\ (\cos\theta_{n-1}, 0, \ldots, 0) & -r \sin\theta_{n-1} \end{pmatrix}.$$

So we can compute its determinant (expand using the last row), note that the first column of $\mathrm{d}F_{n-1}$ is $r^{-1} F_{n-1}^T$,

$$\begin{aligned} \det \mathrm{d}F_n =& - r \sin\theta_{n-1} (\sin\theta_{n-1})^{n-1} \det(\mathrm{d}F_{n-1}) \\ &+ (-1)^{n-1} (\cos\theta_{n-1})^2 (-1)^{n-2} r (\sin\theta_{n-1})^{n-2} \det(\mathrm{d}F_{n-1}) \\ =& - r (\sin\theta_{n-1})^{n-2} \det(\mathrm{d}F_{n-1}). \end{aligned}$$

Hence $\mathrm{d}x = r^{n-1} \sin^{n-2}\theta_{n-1} \cdots \sin\theta_2 \, \mathrm{d}r \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1} = r^{n-1} \, \mathrm{d}r \, \mathrm{d}\omega$.

Denote F_n^S to be the function F_n restricted to S^{n-1}. Then $\mathrm{d}F_n = (r^{-1} F_n^T, \mathrm{d}F_n^S)$. We can compute that the Gram determinant of $\mathrm{d}F_n^S$ is just $\det \mathrm{d}F_n$ with $r = 1$.

The rest is some integrals with gamma function and beta function, which is left out.

§0.1 Stolkes' formula

Intuitively, Stolkes' formula states that: Let D be a region, $\mathrm{d}\omega$ be a differential form, then

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

Here ∂D means the "boundary" of D.

Of course we need some "regularity" requirements of D and ω, and it's the generalization of Newton-Lebniz formula into higher dimensions.

Definition 0.1.1 (Bounded regions with boundary)**.** Let $\Omega \subset \mathbb{R}^n$ be a compact set, we say it's a **bounded region with boundary** if $\forall x \in \partial\Omega$, there exists open sets $U, V \subset \mathbb{R}^n$, $x \in U$ and a continuous homeomorphism $\Phi : U \to V$, such that

$$\Phi(U \cap \Omega) = V \cap \{x_n \geq 0\}, \quad \Phi(U \cap \partial\Omega) = V \cap \{x_n = 0\}.$$

If Φ is also C^1, we say $x \in \partial\Omega$ is a **regular point**, otherwise a **singular point**.

Lemma 0.1.2

Let Ω be a bounded region with boundary, for all regular $p \in \partial\Omega$, there exists a unique unit vector $\nu(p) \in \mathbb{R}^n$, and $\varepsilon > 0$, s.t.

$$\nu(p) \perp T_p\partial\Omega, |\nu(p)| = 1, p - t\nu(p) \in \Omega, \forall 0 < t < \varepsilon.$$

We call $\nu(p)$ the **outward unit normal vector** of p.

Proof. By the definition of regular points, we may assume that:

$$\Omega \cap V = \{x \in V \mid f(x) \geq 0\}, \quad \partial\Omega \cap V = \{x \in V \mid f(x) = 0\}.$$

Where f is a C^1 function.

Since ∇f is nonzero, the tangent space $T_p\partial\Omega = \{v \mid v \cdot \nabla f = 0\}$.

Let $\nu(p) = -\frac{\nabla f}{|\nabla f|}$, then it's obvious $\nu(p)$ points outside of Ω. □

Now for a cuboid I and a C^1 function ϕ,

$$\begin{aligned}\int_I \frac{\mathrm{d}\phi}{\mathrm{d}x_n}\,\mathrm{d}x &= \int_{I_{n-1}} \phi(x_1, \ldots, x_{n-1}, b_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} - \int_{I_{n-1}} \phi(x_1, \ldots, x_{n-1}, a_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \\ &= \int_{\partial I} \phi \cdot \nu_n\,\mathrm{d}\sigma.\end{aligned}$$

Where σ is the measure on the boundary, ν is the outward unit normal vector.

Lemma 0.1.3

Let K be a compact set in $\mathbb{R}^n$, $U \supset K$ is open, there exists a smooth function f such that $\operatorname{supp} f \subset U$, and $f\big|_K > 0$.

Proof. Let $\rho(x)$ be a smooth function s.t. $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| \geq 2$. Let

$$g(x) = \int_{|y| \leq 2} f(x - \delta y)\rho(y)\,\mathrm{d}y.$$

Then g is a smooth non-negative function. □

Theorem 0.1.4 (Unit decomposition on compact sets)

Let K be a compact set, $\{U_1, \ldots, U_k\}$ is an open covering of K. There exists smooth functions $f_1, \ldots, f_k$ s.t.

$$1 = f_1(x) + f_2(x) + \cdots + f_k(x), \quad \operatorname{supp} f_i(x) \subset U_i.$$

Proof. For $1 \le i \le k$, $\delta > 0$, define

$$K_i^\delta = \{x \in U_i \mid d(x, U_i^c) > \delta\}.$$

Note that $\{\bigcup_{i=1}^k K_i^{\frac{1}{m}}\}_{m=1}^\infty$ is also an open covering of K, thus there exists N s.t.

$$K \subset \bigcup_{i=1}^k K_i^{\frac{1}{N}}.$$

Hence by lemma we have g_i s.t. $\operatorname{supp} g_i \subset U_i$ and $g_i > 0$ on the closure of $K_i^{\frac{1}{N}}$. Similarly we have a smooth function g s.t. $g(x) = 0$ on K, and $g > 0$ outside of the closure of $\bigcup_{i=1}^k K_i^{\frac{1}{N}}$.

Let $G(x) = g_1(x) + \cdots + g_k(x) + g(x) > 0$ on $\bigcup_{i=1}^k U_i$, then we can define $f_i(x) = \frac{g_i(x)}{G(x)}$ which satisfy the condition. $\square$

Theorem 0.1.5

Let Φ be a C^1 homeomorphism from a cuboid I to Ω, then Ω satisfies Stolkes' formula: $\forall \phi \in C^1(\mathring{D}) \cap C(\overline{D})$, we have

$$\int_D \nabla\phi \,\mathrm{d}x = \int_{\partial D} \phi\nu \,\mathrm{d}\sigma.$$

Proof. Since $\Omega = \Phi(I)$, let y be the coordinates on I, $x = \Phi(y)$,

$$\int_{\Phi(I)} \nabla\varphi \,\mathrm{d}x = \int_I \nabla\varphi(\Phi(y))(\mathrm{d}\Phi)^{-1} J_\Phi \,\mathrm{d}y.$$

Let $A = \mathrm{d}\Phi$, WLOG $J_\Phi > 0$. Using the index notation and Einstein summation,

$$A_{kj}A^{ji} = A^{kj}A_{ji} = \delta_{ki}.$$

Thus

$$\partial_{y_j}\varphi A^{ji}|A| = \partial_{y_j}(\varphi A^{ji}|A|) - \varphi\partial_{y_j}(A^{ji}|A|)$$

Since $|A| = A_{kl}A^{kl}|A|$, $A_{kl} = \frac{\partial\Phi_k}{\partial y_l}$.

$$\begin{aligned}
\partial_{y_j}(A^{ji}|A|) &= |A|\partial_{y_j}A^{ji} + A^{ji}\partial_{y_j}|A| \\
&= |A|\partial_{y_j}A^{ji} + A^{ji}|A|\partial_{y_j}A_{kl}A^{kl} \\
&= |A|(\partial_{y_j}A^{ji} + \partial_{y_l}A_{kj}A^{kl}) \\
&= |A|(\partial_{y_j}A^{ji} - \partial_{y_j}A^{ji}) = 0.
\end{aligned}$$

Hence by our previous work,

$$\int_I \partial_{y_j}(\varphi A^{ji}|A|)\,\mathrm{d}y = \int_{\partial I} \varphi A^{ji}|A|\nu_j \,\mathrm{d}\sigma.$$

Putting this together for all i's, note that $\widetilde{\nu} = \frac{\nabla\Phi_n^{-1}}{|\nabla\Phi_n^{-1}|}$,

TODO $\square$

Let $(\phi_1, \dots, \phi_n)$ be an element in the tangent boundle TM, it can represent a vector field

$$X = (\phi_1, \dots, \phi_n) = \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial x_i}.$$

Here $X \in TM, X(p) \in T_pM$.

We define the **divergence** of X to be

$$\operatorname{div}(X) = \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i}.$$

The Stolke's formula can be presented as divergence theorem:

Theorem 0.1.6 (Divergence theorem)
Let X be a vector field,

$$\int_D \operatorname{div}(X) \, \mathrm{d}x = \int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma.$$

Another commonly-used operator is the **Laplace operator**:

$$\Delta = \operatorname{div} \cdot \nabla, \quad \Delta\phi = \operatorname{div}(\nabla\phi) = \operatorname{tr}(H_\phi) = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}.$$

When $n = 2$, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$,

$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma.$$

Since ∂D is a curve $\gamma(t)$, so $\mathrm{d}\sigma = |\gamma'(t)| \, \mathrm{d}t$. Let $\gamma(t) = (x(t), y(t))$, then $\nu(t) = \frac{(y'(t), -x'(t))}{|\gamma'(t)|}$. Here we must take $\gamma(t)$ to be *counterclockwise* to ensure ν points outside of D.

Thus we get

$$\int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma = \int_\gamma \frac{Py'(t) - Qx'(t)}{|\gamma'(t)|} |\gamma'(t)| \, \mathrm{d}t = \int_{\partial D} (P \, \mathrm{d}y - Q \, \mathrm{d}x).$$

This result is known as *Green's formula*.

This leads to the curve integrals of the second type: let $\gamma(t) \in \mathbb{R}^d$, X a vector field, we call the integral

$$\int_\gamma \sum_{i=1}^{d} X^i \, \mathrm{d}x_i = \int_\gamma X \cdot \mathrm{d}\gamma(t).$$

the **curve integral of the second type**.

When $n = 3$, the result is called *Gauss's formula*, we have $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$,

$$\int_D \operatorname{div}(X) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_{\partial D} X \cdot \nu \, \mathrm{d}\sigma.$$

Let $\gamma(u, v) = (x, y, z)$ be a parametrization of ∂D. We have two tangent vector γ_u, γ_v, so the normal vector is defined as $\nu = \frac{\gamma_u \times \gamma_v}{|\gamma_u \times \gamma_v|}$. Also $\mathrm{d}\sigma = |\gamma_u \times \gamma_v| \, \mathrm{d}u \, \mathrm{d}v$. After some computation we can get

$$\nu \, \mathrm{d}\sigma = (\mathrm{d}y \, \mathrm{d}z, \mathrm{d}z \, \mathrm{d}x, \mathrm{d}x \, \mathrm{d}y).$$

$$\int_D \operatorname{div}(X) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_{\partial D} (P \, \mathrm{d}y \, \mathrm{d}z + Q \, \mathrm{d}z \, \mathrm{d}x + R \, \mathrm{d}x \, \mathrm{d}y).$$

§0.2 Differential forms

Let T_p^*M denote the *dual space* of T_pM, and $\mathrm{d}x_i$ is the dual basis of $\frac{\partial}{\partial x_i}$. The linear combination of $\mathrm{d}x_i$ are called **differential forms**, and a differential form on a manifold can be written as $\sum_{i=1}^n a_i \,\mathrm{d}x_i$, where a_i are functions on M.

We can construct differential forms of higher order, the order is $1 \le k \le n$, called **k-forms**, which is a linear combination of

$$\mathrm{d}x_{i_1}\,\mathrm{d}x_{i_2}\cdots\mathrm{d}x_{i_k}, \quad i_1 < i_2 < \cdots < i_k.$$

Here the product is *wedge product*, i.e. $\mathrm{d}x_i\,\mathrm{d}x_j = -\,\mathrm{d}x_j\,\mathrm{d}x_i$. We denote the space of all k-forms by $\Lambda^k(\Omega)$.

We can define the multiplication of forms: let $\omega_1 \in \Lambda^{k_1}, \omega_2 \in \Lambda^{k_2}$, then $\omega_1 \wedge \omega_2 \in \Lambda^{k_1+k_2}$ by multiplying the coefficients and $\mathrm{d}x_i$'s respectively.

There's also an operator called **exterior differentiation** $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$, where

$$\mathrm{d}(a\,\mathrm{d}x_{i_1}\cdots\mathrm{d}x_{i_k}) = \sum_{j=1}^n \frac{\partial a}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_{i_1}\cdots\mathrm{d}x_{i_k}.$$

This operator behaves like the derivatives very much:

$$\mathrm{d}(\omega_1+\omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2, \quad \mathrm{d}(\omega_1\omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1k_2}\omega_1 \wedge \mathrm{d}\omega_2.$$