# Linear Algebra II

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**Proposition 0.0.1**

$\ker(T^*) = \mathrm{Im}(T)^\perp$, $\mathrm{Im}(T^*) = \ker(T)^\perp$. $(cT+U)^* = \bar{c}T^* + U^*$, $(TU)^* = U^*T^*$, $T^{**} = T$.

This means the map $T \mapsto T^*$ is a conjugate anti-automorphism of $L(V)$, and it's an involution.

If $T^* = T$, then we say $T$ is **self-adjoint**, and if $T^* = -T$, we say $T$ is **anti self-adjoint**.

Let $F = \mathbb{C}$, $T$ is self-adjoint iff $iT$ is anti self-adjoint. Like a function can be written as a sum of odd and even functions, $\forall T \in L(V)$, there exists unique self-adjoint $T_1, T_2$ s.t. $T = T_1 + iT_2$. In fact, $T_1 = \frac{T+T^*}{2}, T_2 = \frac{T-T^*}{2i}$.

Let $\mathcal{B}$ be an orthonormal basis, obviously $T$ self-adjoint $\iff$ $[T]_\mathcal{B}$ Hermite.

**Example 0.0.2**

Let $W \subset V$, $p_W$ be the orthogonal projection. then $p_W$ is self-adjoint as we can choose an orthonormal basis $\mathcal{B}$, such that $[p_W]_\mathcal{B} = \mathrm{diag}\{I_k, 0\}$, where $k = \dim W$.

Let $V, W$ be inner product spaces, we'll study the linear maps $T : V \to W$ which preserves the inner products, i.e.

$$\langle \alpha, \beta \rangle_V = \langle T\alpha, T\beta \rangle_W .$$

If $T$ is an isomorphism, then we say $T$ is the isomorphism between inner product spaces.

**Proposition 0.0.3**

$T$ preserves inner product $\iff$ $T$ is an isometry, i.e. preserves length.

In particular, isometry is always injective implies that inner product presering maps are always injective.

*Proof.* Trivial by polarization identity. $\square$

**Proposition 0.0.4**

Let $V, W$ be finite dimensional inner product spaces, $\dim V = \dim W$, $T \in \mathrm{Hom}(V, W)$, the followings are equivalent:

(1) $T$ preserves inner product;

(2) $T$ is an isomorphism between inner product spaces;

(3) $T$ maps all the orthonormal bases in $V$ to orthonormal bases in $W$ ;

(4) $T$ maps *one* orthonormal basis in $V$ to a orthonormal basis in $W$.

*Proof.* It's clear that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4), since $T$ injective $\implies$ $T$ is an isomorphism of vector space.

As for (4) $\implies$ (1), just expand everything using this orthonormal basis. □

**Corollary 0.0.5**

Inner product spaces with same dimensions are always isomorphic as inner product spaces.

Recall the positive definite matrices we defined earlier, we can also define *positive definite maps*: Let $T$ be a *self-adjoint map*, if

$$\forall \alpha \in V \backslash \{0\}, \quad \langle T\alpha, \alpha \rangle > 0,$$

then we say $T$ is positive definite.

The reason why we require $T$ self-adjoint is that,

$$\langle T\alpha, \alpha \rangle = \langle \alpha, T\alpha \rangle = \overline{\langle T\alpha, \alpha \rangle} \implies \langle T\alpha, \alpha \rangle \in \mathbb{R}.$$

so we can talk about "positive" safely.

## §0.1 Orthogonal maps and Unitary maps

**Definition 0.1.1** (Orthogonal maps)**.** Let $V$ be a real inner product space, the automorphisms of $V$ (as inner product space) are called **orthogonal maps**, denoted the set by $\mathrm{O}(V)$.

When $V$ is a complex inner product space, we use **unitary maps** and $\mathrm{U}(V)$ instead.

**Proposition 0.1.2**

Let $V$ be an inner product space,

$$T \in \mathrm{O}(V) \iff T^* = T^{-1}.$$

*Proof.*

$$T \in \mathrm{O}(V) \iff \langle \alpha, \beta \rangle = \langle T\alpha, T\beta \rangle = \langle \alpha, T^*T\beta \rangle, \quad \forall \alpha, \beta \in V.$$

This also holds for $\mathrm{U}(V)$. □

**Proposition 0.1.3**

Let $A \in \mathbb{R}^{n\times n}$, TFAE:

- $A^t A = I_n$ ;
- The column (row) vectors of $A$ form an orthonormal basis of $\mathbb{R}^n$.

*Proof.* Since $A$ maps the standard basis to the column vectors of $A$, so the conclusion follows immediately (use $A^t$ to get the row vectors). □

Let $\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} \mid A^t A = I_n\}$, and $\mathrm{U}(n) = \{A \in \mathbb{C}^{n\times n} \mid A^* A = I_n\}$. We can see that $A^t A = I_n \implies \det(A) = \pm 1$, and $A^* A = I_n \implies |\det(A)| = 1$.

Let $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}$, and $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}$. In the language of groups, $\mathrm{SO}(n)$ has only 2 coset in $\mathrm{O}(n)$, while the structure of the cosets of $\mathrm{SU}(n)$ in $\mathrm{U}(n)$ look like $S^1$.

**Example 0.1.4**

Let's look at some low dimensional orthogonal groups. $\mathrm{O}(1) = \{1, -1\}$, $\mathrm{SO}(1) = \{1\} = \mathrm{SU}(1)$, $\mathrm{U}(1) = \{z \mid |z| = 1\}$.

The group $\mathrm{SO}(2)$ is the rotations of $\mathbb{R}^2$:

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$

While $\mathrm{O}(2)$ also consisting of reflections.

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}.$$

In fact these groups are *lie groups*, which means they have the structure of differential manifolds. It's clear that $\mathrm{U}(1) \simeq \mathrm{SO}(2) \simeq S^1$, and we can see $\mathrm{SU}(2) \simeq S^3$.

**Theorem 0.1.5** (QR-decomposition)

Any invertible matrix $A$ can be uniquely decomposed to $Q \cdot R$, where $Q \in \mathrm{O}(n)$, $R$ is an upper triangular matrix with positive diagonal entries. When $F = \mathbb{C}$, $\mathrm{O}(n)$ is replaced by $\mathrm{U}(n)$.

*Proof.* This is essentially Schmidt orthogonalozation. □

**Corollary 0.1.6** (Ivasawa decomposition, KAN decomposition)

For all $A \in \mathrm{GL}_n(\mathbb{R})$, it has a unique decomposition $A = A_k A_a A_n$, $A_k \in \mathrm{O}(n)$, $A_a$ is diagonal, $A_n$ is upper triangular matrix with diagonal entries 1. It also holds for $\mathbb{C}$.

Let $\mathcal{B}, \mathcal{B}'$ be orthonormal bases of $V$, $T \in L(V)$. We know that $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$ for some $P \in \mathrm{GL}(V)$. By orthogonality, $P$ must be an orthogonal matrix, wich means $P^t = P^{-1}$.

**Definition 0.1.7.** Let $A, B \in \mathbb{R}^{n\times n}$, we say they are **orthogonally similar** if $A = P^{-1}BP$ for some $P \in \mathrm{O}(n)$. The name is changed to **unitarily similar** for complex matrices.

**Theorem 0.1.8** (Schur triangularization theorem)

Let $F = \mathbb{C}, T \in L(V)$. There exists an orthonormal basis $\mathcal{B}$, such that $[T]_{\mathcal{B}}$ is upper triangular.

*Proof.* Recall that $T$ is triangulable (which is always true in $\mathbb{C}$) iff there exists a $T$-invariant flag $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = V$. We can take an orthonormal basis s.t. $W_k = \operatorname{span}\{\alpha_1, \ldots, \alpha_k\}$.

Obviously $T$ is upper triangular under this basis. □

## §0.2 Normal maps

Recall that we say two matrices $A, B$ are orthogonally similar, if there exist $P \in \mathrm{O}(n)$ s.t. $B = P^{-1}AP$. Again, we want to find the "simpliest" matrix in each orthogonal equivalent class.

Let $T \in L(V)$ be a linear map, if there exists an orthonomal basis of $V$ s.t. $[T]_{\mathcal{B}}$ is diagonal, then we say $T$ is orthogonally (or unitarily) diagonalizable.

**Definition 0.2.1** (Normal maps)**.** Let $V$ be an inner product space, $T \in L(V)$. If $TT^* = T^*T$, then we say $T$ is a **nomal map**.

It turns out that these concepts has close relations:

**Theorem 0.2.2**

Let $V$ be a finite dimentional inner product space,

- If $F = \mathbb{R}$, then $T$ orthogonally diagonalizable $\iff$ $T$ self-adjoint;
- If $F = \mathbb{C}$, then $T$ unitarily diagonalizable $\iff$ $T$ normal.

**Lemma 0.2.3**

Let $F = \mathbb{C}$, then $T$ normal $\iff$ there exists self-adjoint commutative maps $T_1, T_2$ s.t. $T = T_1 + iT_2$.

*Proof.* If $T = T_1 + iT_2$, then $T^* = T_1 - iT_2$, so $T^*T = TT^*$ since $T_1, T_2$ commutative.

On the other hand, let $T_1 = \frac{T+T^*}{2}$, $T_2 = \frac{T-T^*}{2i}$. We can check that $T_1, T_2$ self-adjoint and are commutative. □

*Proof of Theorem 0.2.2.* For the " $\implies$ " part, let $\mathcal{B}$ be an orthonomal basis such that $[T]_{\mathcal{B}} = \operatorname{diag}\{c_1, \ldots, c_n\}$. Then we have

$$[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^* = \operatorname{diag}\{\bar{c}_1, \ldots, \bar{c}_n\}.$$

If $F = \mathbb{R}$, then $T^* = T$, i.e. $T$ self-adjoint.

If $F = \mathbb{C}$, clearly $TT^* = T^*T$, so $T$ is normal.

As for the other part, we need a lemma first.

**Lemma 0.2.4**

Let $V$ be a f.d. inner product space, $T \in L(V)$. If $W \subset V$ is a $T$-invariant space, then $W^{\perp}$ is $T^*$-invariant.

*Proof of the lemma.* For all $\alpha \in W^{\perp}$,

$$0 = \langle \alpha, T\beta \rangle = \langle T^*\alpha, \beta \rangle, \quad \forall \beta \in W.$$

Thus $T^*\alpha \in W^{\perp}$. □

**Corollary 0.2.5**

If $T$ is self-adjoint, $W \subset V$ is $T$-invariant will imply $W^{\perp}$ is also $T$-invariant, so $T$ is semisimple.

**Lemma 0.2.6**

Let $V$ be a f.d. inner product space, $T \in L(V)$ is self-adjoint. We must have $f_T \in \mathbb{R}[x]$, and it can be decomposed to products of polynomials of degree 1.

In particular, $\sigma(T) \subset \mathbb{R}$.

*Proof.* Let $f_T = \prod_{j=1}^{n}(x - c_j), c_j \in \mathbb{C}$.

Let $\mathcal{B}$ be an orthonomal basis of $V$, then $A := [T]_{\mathcal{B}}$ is Hermite. Let $X$ be a nonzero vector s.t. $AX = c_jX$, then

$$c_jX^*X = X^*AX = (AX)^*X = \bar{c}_jX^*X.$$

So $c_j \in \mathbb{R}$, and we're done. □

**Lemma 0.2.7**

If $T$ is a self-adjoint map, then all the eigenspaces of $T$ are pairwise orthogonal.

*Proof.* Let $c_1, c_2 \in \mathbb{R}$ be two eigenvalues of $T$. Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$.

$$c_1 \langle \alpha, \beta \rangle = \langle c_1\alpha, \beta \rangle = \langle T\alpha, \beta \rangle = \langle \alpha, T\beta \rangle = \bar{c}_2 \langle \alpha, \beta \rangle = c_2 \langle \alpha, \beta \rangle.$$

Since $c_1 \neq c_2$, we must have $\alpha \perp \beta$, as desired. □

Returning back to Theorem 0.2.2, when $T$ is self-adjoint, let $\sigma(T) = \{c_1, \ldots, c_r\}$.

**Claim 0.2.8.** $V = \bigoplus_{i=1}^{r} V_{c_i}$, i.e. $T$ is diagonalizable.

Let $W = \bigoplus_{i=1}^{r} V_{c_i}$, if $W^{\perp} \neq \{0\}$, then $W^{\perp}$ is $T$-invariant.

When $F = \mathbb{C}$, then $T_{W^{\perp}}$ has eigenvectors; when $F = \mathbb{R}$, then $T_{W^{\perp}}$ is self-adjoint, so it must have a eigenvector (by lemma).

Since $V_{c_i}$ are pairwise orthogonal, so we can actually take an orthonomal basis of $V_{c_i}$ to get an orthonomal basis of $V$. Hence $T$ is orthogonally diagonalizable.

Now for the case when $T$ is normal, let $T_1, T_2$ be self-adjoint maps s.t. $T = T_1 + iT_2$. Since $T_1, T_2$ commute, the proof is nearly identical to the simutaneously diagonalizable property.

Let $V = \bigoplus_{i=1}^{r} V_{c_i}$ be the eigenspace decomposition of $T_1$. Note that $V_{c_i}$ are also $T_2$-invariant.

Since $(T_2)_{V_{c_i}}$ self-adjoint, $(T_2)_{V_{c_i}}$ is unitarily diagonalizable. Therefore we can concatenate those basis to get a basis of $V$, and $T_1, T_2$ are both diagonal under this basis. □

There's another proof of " $\Longleftarrow$ " part of the theorem:

**Proposition 0.2.9**

Let $V$ be an inner product space, $T \in L(V)$ normal. Let $W \subset V$ be a $T$-invariant space, then $W^\perp$ is $T$-invariant, and $W$ is $T^*$-invariant.

*Proof.* Take an orthonomal basis of $W, W^\perp$, so $A := [T]_{\mathcal{B}}$ normal.

Since $W$ is $T$-invariant, $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Note that:

$$AA^* = \begin{pmatrix} BB^* + CC^* & * \\ * & * \end{pmatrix}, \quad A^*A = \begin{pmatrix} B^*B & * \\ * & * \end{pmatrix}.$$

As $A$ normal, $BB^* + CC^* = B^*B$, by looking at the trace of both sides, we get $\operatorname{tr}(CC^*) = 0 \implies C = 0$, the conclusion follows. □

**Corollary 0.2.10**

Let $A \in \mathbb{C}^{n\times n}$ be an upper triangular martix, then $A$ normal $\iff$ $A$ diagonal.

**Proposition 0.2.11**

Let $T$ be a normal map, then the eigenspaces of $T$ are pairwise orthogonal.

*Proof.* Let $\alpha \in V_{c_1}, \beta \in V_{c_2}$, since $\operatorname{span}\{\beta\}$ is a $T$-invariant space, so $T^*\beta \in \operatorname{span}\{\beta\}$,

$$\langle T^*\beta, \beta\rangle = \langle \beta, T\beta\rangle = \overline{c}_2 \langle \beta, \beta\rangle .$$

Thus $T^*\beta = \overline{c}_2\beta$.

$$c_1 \langle \alpha, \beta\rangle = \langle T\alpha, \beta\rangle = \langle \alpha, T^*\beta\rangle = c_2 \langle \alpha, \beta\rangle .$$

But $c_1 \neq c_2$, we have $\alpha \perp \beta$. □

When $F = \mathbb{C}$: Let $W = \bigoplus_{i=1}^r V_{c_i}$. Since $W^\perp$ is $T$-invariant, so when $W \perp\neq \{0\}$, $T$ must have eigenvalues in $W^\perp$, contradiction!

Now we've proved that $V_{c_i}$ are pairwise orthogonal, so $T$ is unitarily diagonalizable.

**Proposition 0.2.12**

Let $V$ be a complex inner product space, $T \in L(V)$ normal,

- $T$ self-adjoint $\iff$ $\sigma(T) \subset \mathbb{R}$;
- $T$ anti self-adjoint $\iff$ $\sigma(T) \subset i\mathbb{R}$;
- $T$ unitary $\iff$ $\sigma(T) \subset \{z : |z| = 1\}$.

*Proof.* Take an orthonomal basis s.t. $[T]_{\mathcal{B}}$ diagonal. The rest is trivial. □

# §1 Bilinear forms on inner product space

Let $V$ be a finite dimensional vector space, $\dim V = n$.

**Definition 1.0.1.** Let $F = \mathbb{C}$, we say a function $f : V \times V \to V$ is a **semi bilinear form** if:

- $f(c_1\alpha + \beta, \gamma) = c_1 f(\alpha, \gamma) + f(\beta, \gamma)$;
- $f(\alpha, c_1\beta + \gamma) = \overline{c}_1 f(\alpha, \beta) + f(\alpha, \gamma)$.

Let $\operatorname{Form}(V)$ denote the (semi) bilinear forms on (complex) real vector space $V$.

For $f \in \operatorname{Form}(V)$, fix a basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ of $V$, let $[f]_{\mathcal{B}} \in F^{n\times n}$ be the matrix

$$([f]_{\mathcal{B}})_{jk} = f(\alpha_k, \alpha_j).$$

which is called **the matrix of $f$ under $\mathcal{B}$**.

For $\alpha = \sum_{k=1}^n x_k\alpha_k, \beta = \sum_{j=1}^n y_j\alpha_j \in V$. It's clear that

$$f(\alpha, \beta) = \sum_{j,k=1}^n x_k\overline{y}_j f(\alpha_k, \alpha_j) = \sum_{j,k=1}^n x_k\overline{y}_j([f]_{\mathcal{B}})_{jk} = [\beta]_{\mathcal{B}}^*[f]_{\mathcal{B}}[\alpha]_{\mathcal{B}}.$$

From this we know that the map $\operatorname{Form}(V) \to F^{n\times n}, f \mapsto [f]_{\mathcal{B}}$ is a linear isomorphism. Since if $[f]_{\mathcal{B}} = 0$, then $f(\alpha, \beta) = 0$ for all $\alpha, \beta \in V$. Thus it's injective. Obviously it's surjective and linear, so

$$\dim \operatorname{Form}(V) = n^2$$

**Example 1.0.2**

Let $A \in F^{n\times n}$. Let $f \in \operatorname{Form}(F^{n\times 1})$ be

$$f(X, Y) = Y^*AX, \quad \forall X, Y \in F^{n\times 1}.$$

Let $\mathcal{B}$ be the standard basis of $F$, it's clear that $[f]_{\mathcal{B}} = A$.

**Proposition 1.0.3**

Let $\mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$ be another basis of $V$, $P \in \mathrm{GL}_n(F)$ satisfies

$$(\alpha_1', \ldots, \alpha_n') = (\alpha_1, \ldots, \alpha_n)P.$$

Then $[f]_{\mathcal{B}'} = P^*[f]_{\mathcal{B}}P$.

*Proof.* Since $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$, just plug this into the definition of $[f]_{\mathcal{B}}$, the rest is trivial. □

**Definition 1.0.4.** Let $f \in \operatorname{Form}(V)$.

- When $F = \mathbb{R}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = f(\beta, \alpha)$, then we say $f$ is symmetrical (also called Hermite);
- When $F = \mathbb{C}$, if $\forall \alpha, \beta \in V$ we have $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, we say $f$ is Hermite.

**Proposition 1.0.5**

When $F = \mathbb{C}$, $f$ Hermite $\iff f(\alpha, \alpha) \in \mathbb{R}$, $\forall \alpha \in V$.

*Proof.* For the " $\Longleftarrow$ " direction, consider $f(\alpha+\beta, \alpha+\beta) \in \mathbb{R}$. Expanding we'll get $f(\alpha, \beta) + f(\beta, \alpha) \in \mathbb{R}$, i.e.

$$f(\alpha, \beta) + f(\beta, \alpha) = \overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Replace $\alpha$ with $i\alpha$, we get

$$f(\alpha, \beta) - f(\beta, \alpha) = -\overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}.$$

Combining two equations we get the conclusion. □

**Definition 1.0.6.** Let $f \in \mathrm{Form}(V)$ be an Hermite form. If $\forall \alpha \in V \backslash \{0\}$, $f(\alpha, \alpha) > 0$, we say $f$ is **positive definite**.

Similarly we can define negative definite and semi positive definite.

Note that a positive definite Hermite form is nothing but an inner product.

It's clear that if a matrix $A$ is positive definite, then $A$ is inversible, and $P^*AP$ is also positive definite. In particular, $P^*P$ is positive definite.

**Theorem 1.0.7** (Cholesky decomposition)

Let $A \in F^{n\times n}$ be a positive definite matrix, there exists a unique upper triangular matrix $R$ with positive diagonal entries s.t. $A = R^*R$.

*Proof.* Consider the inner product $f(X, Y) = Y^*AX$. Let the standard inner product on $V$ be $f_0(X, Y) = Y^*X$.

Since inner product spaces with same dimensions are isomorphic, so there exists a matrix $R \in \mathrm{GL}_n(F)$, such that

$$R : (F^{n\times 1}, f) \to (F^{n\times 1}, f_0), \quad X \mapsto RX$$

is an isomorphism of inner product space, i.e. $f_0(RX, RY) = f(X, Y)$. This is equivalent to $A = R^*R$.

For any $P \in GL_n(F)$, $P$ is also an isomorphism of $(F^{n\times 1}, f) \to (F^{n\times 1}, f_0)$ iff $RP^{-1}$ preserves the inner product $f_0$, iff $RP^{-1} \in \mathrm{O}(n)$ or $\mathrm{U}(n)$.

By QR decomposition, $R = RP^{-1} \cdot P$, so there must be a unique $P$ s.t. $P$ upper triangular with positive diagonal entries. □

**Corollary 1.0.8**

$A$ positive definite $\implies \det A > 0$.

**Definition 1.0.9.** Let $A \in F^{n\times n}$, for $1 \le k \le n$, define

$$\Delta_k(A) := \det(A^{1\le j\le k}_{1\le i\le k})$$

be the **leading principal minor**.

**Theorem 1.0.10**

Let $A \in F^{n\times n}$ be an Hermite matrix. Then $A$ positive definite $\iff$ $\Delta_k(A) > 0$, $k = 1, \dots, n$.

**Lemma 1.0.11** (LU decomposition)

Let $F$ be *any field*. For $A \in \mathrm{GL}_n(F)$, the followings are equivalent:

- $\Delta_k(A) \neq 0, k = 1, \dots, n$;
- $A = LU$, where $L$ lower triangular, and $U$ upper triangular with diagonal entries 1.

*Proof.* On one hand, Let $L_k, U_k$ be the top-left $k \times k$ submatrix of $L, U$, since $L, U$ inversible, $L_k, U_k$ inversible. By the triangular property, $\Delta_k(A) = \det(L_k U_k) \neq 0$.

On the other hand, it's sufficient to prove:

$$\exists N \text{ strictly upper triangular}, A(N + I_n) \text{ lower triangular}$$

Let $A_k$ be the $k$-th leading principal submatrix of $A$, and $\alpha_{k+1}, \beta_{k+1} \in F^{n\times 1}$ the $(k+1)$-th column of $A, N$.

Now compute the first $k$ rows of the $(k+1)$-th column of $A(N+I)$, which is equal to $A_k\beta'_{k+1} + \alpha'_{k+1}$, where $\alpha'_{k+1}, \beta'_{k+1}$ is the first $k$ entries of $\alpha_{k+1}, \beta_{k+1}$.

Since $A_k$ inversible, $\exists \beta'_{k+1}$ s.t. $A_k\beta'_{k+1} + \alpha'_{k+1} = 0$.

Hence these $\beta'_{k+1}$ forms a strictly upper triangular matrix $N$, as desired. □