

# MEAM 620 Project 1

Modeling and Control of a Quad Rotor

January 29, 2018

## 1 Introduction

In this project you will learn about quad rotor dynamics and develop algorithms to control it. A thorough understanding of the theory and a clean implementation of the controller is crucial because you will use this code base in subsequent projects. The code will be written in Matlab and first tested in a simulator. During the lab session you will use it to fly a real, physical quad rotor with it!

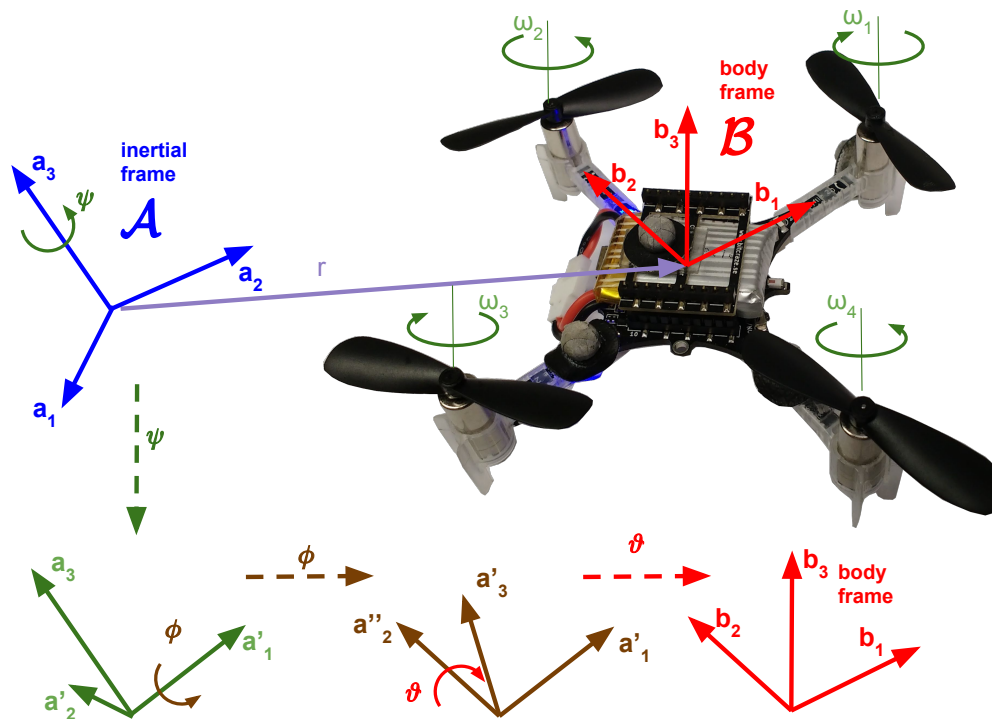


Figure 1: The CrazyFlie 2.0 robot that will be used in the lab and exercises. An Euler Z-X-Y transformation takes the inertial frame  $\mathcal{A}$  to the body-fixed frame  $\mathcal{B}$ . First a rotation by  $\psi$  around the  $a_3$  axis is performed, then a roll by  $\phi$  around the (rotated!)  $a_1$  axis, and finally a pitch by  $\theta$  around the (now twice rotated!)  $a_2$  axis. A translation  $r$  then produces the coordinate system  $\mathcal{B}$ , coinciding with the center of mass  $C$  of the robot, and aligned along the arms.

## 2 Modeling

### 2.1 Coordinate Systems and Reference frames

The coordinate systems and free body diagram for the quadrotor are shown in Fig. 1. The inertial frame,  $\mathcal{A}$ , is defined by the triad  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  with  $\mathbf{a}_3$  pointing upward. The body frame,  $\mathcal{B}$ , is attached to the center of mass of the quadrotor with  $\mathbf{b}_1$  coinciding with the preferred forward direction and  $\mathbf{b}_3$  being perpendicular to the plane of the rotors pointing vertically up during perfect hover (see Fig. 1). These vectors are parallel to the principal axes.

More formally, the coordinates of a vector  $\mathbf{x}$  that is expressed in  $\mathcal{A}$  as  $\mathbf{x} = \sum_i^{\mathcal{A}} x_i \mathbf{a}_i$  and in  $\mathcal{B}$  as  $\mathbf{x} = \sum_i^{\mathcal{B}} x_i \mathbf{b}_i$  are transformed into each other by the rotation matrix  ${}^{\mathcal{A}}R_{\mathcal{B}}$  and translation vector  ${}^{\mathcal{A}}\mathbf{T}_{\mathcal{B}}$ :

$${}^{\mathcal{A}}\mathbf{x} = {}^{\mathcal{A}}R_{\mathcal{B}} {}^{\mathcal{B}}\mathbf{x} + {}^{\mathcal{A}}\mathbf{T}_{\mathcal{B}}. \quad (1)$$

To express the rotational motion of the moving frame  $\mathcal{B}$ , it is useful to introduce the angular velocity vector  $\boldsymbol{\omega}$  that describes how the basis vectors  $\mathbf{b}_i$  move:

$$\frac{d}{dt} \mathbf{b}_i = \boldsymbol{\omega} \times \mathbf{b}_i. \quad (2)$$

Note that this equation is coordinate free, meaning  $\boldsymbol{\omega}$  is not yet expressed explicitly in any particular coordinate system. We will denote the components of angular velocity of the robot in the body frame by  $p$ ,  $q$ , and  $r$ :

$$\boldsymbol{\omega} = p\mathbf{b}_1 + q\mathbf{b}_2 + r\mathbf{b}_3. \quad (3)$$

The heading (yaw) angle of the robot plays a special role since we can choose it freely without directly affecting the robots dynamics. For this reason we use  $Z - X - Y$  Euler angles to describe the transform from  $\mathcal{A}$  to  $\mathcal{B}$ : first a yaw rotation by  $\psi$  around the  $\mathbf{a}_3$  axis is performed, then a roll by  $\phi$  around the (rotated!)  $\mathbf{a}_1$  axis, and finally a pitch by  $\theta$  around the (now twice rotated!)  $\mathbf{a}_2$  axis<sup>1</sup> From the Euler angles one can compute the rotation matrix as follows:

$${}^{\mathcal{A}}R_{\mathcal{B}} = \begin{bmatrix} \cos(\psi) \cos(\theta) - \sin(\phi) \sin(\psi) \sin(\theta) & -\cos(\phi) \sin(\psi) & \cos(\psi) \sin(\theta) + \cos(\theta) \sin(\phi) \sin(\psi) \\ \cos(\theta) \sin(\psi) + \cos(\psi) \sin(\phi) \sin(\theta) & \cos(\phi) \cos(\psi) & \sin(\psi) \sin(\theta) - \cos(\psi) \cos(\theta) \sin(\phi) \\ -\cos(\phi) \sin(\theta) & \sin(\phi) & \cos(\phi) \cos(\theta) \end{bmatrix}. \quad (4)$$

The angular velocity and Euler angle velocities are related by:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\cos(\phi) \sin(\theta) \\ 0 & 1 & \sin(\phi) \\ \sin(\theta) & 0 & \cos(\phi) \cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (5)$$

### 2.2 Motor Model

This section describes how we model the motors. Each rotor has an angular speed  $\omega_i$  and produces a vertical force  $F_i$  according to

$$F_i = k_F \omega_i^2. \quad (6)$$

Experimentation with a fixed rotor at steady-state shows that  $k_F \approx 6.11 \times 10^{-8} \text{ N/rpm}^2$ . The rotors also produce a moment according to

$$M_i = k_M \omega_i^2. \quad (7)$$

<sup>1</sup>In aerospace engineering, the direction of rotation for yaw and pitch are opposite to our definition, which follows the conventional right-hand rule.

The constant,  $k_M$ , is determined to be about  $1.5 \times 10^{-9} \text{ Nm/rpm}^2$  by matching the performance of the simulation to the real system.

Data obtain from system identification experiments suggest that the rotor speed is related to the commanded speed by a first-order differential equation

$$\dot{\omega}_i = k_m(\omega_i^{\text{des}} - \omega_i).$$

This motor gain,  $k_m$ , is found to be about  $20 \text{ s}^{-1}$  by matching the performance of the simulation to the real system. The desired angular velocities,  $\omega_i^{\text{des}}$ , are limited to a minimum and maximum value determined through experimentation.

However, as a first approximation, we can assume the motor controllers to be perfect and the time constant  $k_m$  associated with the motor response to be arbitrarily small. In other words, we can assume the actual motor velocities  $\omega_i$  are equal to the commanded motor velocities,  $\omega_i^{\text{des}}$ .

Notes:

1. In simulation, you will be able to directly command torque and thrust. The thrust limit is available as a parameter inside the simulator, although you don't need it for this project.
2. In the lab, we will provide the code to translate torque and thrust to motor commands. You will however encounter saturation of your motors and may have to tweak your gains.

### 2.3 Rigid Body Dynamics: Newton's Equations of Motion for the Center of Mass

We will describe the motion of the center of mass (CoM) in the inertial ("world") coordinate frame  $\mathcal{A}$ . This makes sense because we will want to specify our waypoints (where the robot should fly), trajectories and controller targets (where the robot should be at this moment) in the inertial frame.

Newton's equation of motion for the robot's CoM  $\mathbf{r}$  is determined by the robot's mass  $m$ , the gravitational force  $\mathbf{F}_g = m\mathbf{g}$ , and the sum of the motor's individual forces  $\mathbf{F}_i$ :

$$m\ddot{\mathbf{r}} = \mathbf{F}_g + \sum_i \mathbf{F}_i. \quad (8)$$

In the coordinates of the inertial frame  $\mathcal{A}$  this reads:

$$m^{\mathcal{A}}\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + {}^{\mathcal{A}}R_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}. \quad (9)$$

where  ${}^{\mathcal{A}}R_{\mathcal{B}}$  is the rotation matrix from Eq (1). If the robot is tilted,  ${}^{\mathcal{A}}R_{\mathcal{B}}$  (cf Eqn (4)), will mix the propeller forces into the  $x$  and  $y$  plane and so generate horizontal acceleration of the robot. In other words, we can accomplish movement in all three directions by manipulating the attitude of the vehicle and its thrust.

Equation (9) further suggests that rather than using the forces  $F_i$  as control inputs, we should define our first input  $u_1$  as the sum

$$u_1 = \sum_{i=1}^4 F_i. \quad (10)$$

## 2.4 Euler's Equations of Motion for the Attitude

Since from Eq (9) we know that attitude control is necessary to generate horizontal motion, in this section we will examine how the motors affect the orientation of the vehicle.

In the inertial frame, the rate at which the robot's angular momentum  $\mathbf{L} = I\boldsymbol{\omega}$  changes is determined by the total moment  $\mathbf{M}$  generated by the propellers:

$$\dot{\mathbf{L}} = \mathbf{M} . \quad (11)$$

While this equation looks clean and simple in the inertial frame, it is actually hard to work with because the inertial tensor  $I$  changes with the attitude of the robot, and is thus time dependent and non diagonal! For control purposes, this equation is best expressed in the body frame that is aligned with the principal axis, where the inertia tensor  $I$  is constant, and diagonal. However, since now the basis vectors  $\mathbf{b}_i$  are time-dependent, the equation for the time derivative of the angular momentum in the body frame becomes:

$$\dot{\mathbf{L}} = \sum_i {}^{\mathcal{B}}\dot{L}_i \mathbf{b}_i + \sum_i {}^{\mathcal{B}}L_i \dot{\mathbf{b}}_i = \sum_i {}^{\mathcal{B}}\dot{L}_i \mathbf{b}_i + \sum_i \boldsymbol{\omega} \times {}^{\mathcal{B}}L_i \mathbf{b}_i = \sum_i {}^{\mathcal{B}}\dot{L}_i \mathbf{b}_i + \boldsymbol{\omega} \times \mathbf{L} . \quad (12)$$

Combining (11) and (12), and using the fact that  $I$  is constant in  $\mathcal{B}$  yield's Euler's Equation:

$$I^{\mathcal{B}}\dot{\boldsymbol{\omega}} = {}^{\mathcal{B}}\mathbf{M} - {}^{\mathcal{B}}\boldsymbol{\omega} \times I^{\mathcal{B}}\boldsymbol{\omega} , \quad (13)$$

where depending on context  $I$  means alternatingly a tensor or a matrix expressed in the body frame.

How do the motors come into play? First, by generating a force  $F_i$  that is a distance  $l$  away from the CoM, each motor can exert a torque that is in the  $\mathbf{b}_1, \mathbf{b}_2$  plane. In addition, each rotor produces a moment  $M_i$  perpendicular to the plane of rotation of the blade. Rotors 1 and 3 rotate clockwise in the  $-\mathbf{b}_3$  direction while 2 and 4 rotate counter clockwise in the  $\mathbf{b}_3$  direction. Since the moment produced on the quadrotor is opposite to the direction of rotation of the blades,  $M_1$  and  $M_3$  act in the  $\mathbf{b}_3$  direction while  $M_2$  and  $M_4$  act in the  $-\mathbf{b}_3$  direction. In contrast to this, the forces  $F_i$  are all in the positive  $\mathbf{b}_3$  direction due to the fact that the pitch is reversed on two of the propellers, see Fig 1.

Using this geometric intuition, and expressing the angular velocity in the body frame by  $p, q,$  and  $r$  as in Eq (3), the Euler equation (13) becomes:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix} . \quad (14)$$

Note that the total net torque along the yaw ( $\mathbf{b}_3$ ) axis of the robot is simply the signed sum of the motor's torques  $M_i$  (why does the distance  $l$  to the center of the robot play no role?).

We can rewrite this as:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix} . \quad (15)$$

where  $\gamma = \frac{k_M}{k_F}$  is the relationship between lift and drag given by Equations (6-7). Accordingly, we will define our second set of inputs to be the vector of moments  $\mathbf{u}_2$  given by:

$$\mathbf{u}_2 = \begin{bmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} . \quad (16)$$

Dropping the reference frame superscripts for brevity, we can now write the equations of motion for center of mass and orientation in compact form:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} \quad (17)$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \mathbf{u}_2 - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad (18)$$

where (17) is in inertial coordinates, and (18) is in body coordinates. The inputs  $u_1$  and  $\mathbf{u}_2$  are related to the motor forces  $F_i$  via the linear system of equations formed by (10) and (16). We are thus facing a system governed by two coupled second order differential equations which we will exploit in section 3 to design controllers.

### 3 Robot Controllers

#### 3.1 Trajectory Generation

Our ultimate goal is to make the robot follow a trajectory. For the time let's assume we are given a trajectory generator that for any given time  $t$  produces a trajectory target  $\mathbf{z}^{\text{des}}(t)$  consisting of target position  $\mathbf{r}_T(t)$  and target yaw  $\psi_T(t)$ :

$$\mathbf{z}^{\text{des}}(t) = \begin{bmatrix} \mathbf{r}_T(t) \\ \psi_T(t) \end{bmatrix} \quad (19)$$

and its first and second derivatives  $\dot{\mathbf{z}}^{\text{des}}$  and  $\ddot{\mathbf{z}}^{\text{des}}$ . To hover for example, the trajectory generator would produce a constant  $\mathbf{r}(t) = \mathbf{r}_0$  and e.g. a fixed  $\psi(t) = \psi_0$ , with all derivatives being zero.

The controller's job will then be to produce the correct torque  $\mathbf{u}_2$  and thrust  $u_1$  to bring the robot to the desired state specified by the trajectory generator.

#### 3.2 Linear Backstepping Controller

For this controller we will make the following assumptions:

1. The robot is near the hover points, meaning the roll angle  $\phi$  and pitch angle  $\theta$  are small enough to allow linearization of trigonometric functions, i.e.  $\cos(\phi) = 1$ ,  $\sin(\phi) = \phi$ ,  $\cos(\theta) = 1$ ,  $\sin(\theta) = \theta$ . This will allow us to linearize the rotation matrix in Eq (17).
2. The robot's angular velocity is small enough for the cross product term between angular momentum and velocity in (18) to be negligible. This is usually a good assumption for almost any quad rotor.
3. The attitude of the quad rotor can be controlled at a much smaller time scale than the position. This "backstepping" approach to controller design allows a decoupling of position and attitude control loops. In practice this is generally warranted since the attitude controller usually runs almost an order of magnitude faster than the position controller.

Making the backstepping approximation is equivalent to assuming that  $R$  in Eq (17) can be commanded instantaneously. This further implies that it is possible to directly command the acceleration  $\mathbf{r}^{\text{des}}$ . Figure 2 shows how trajectory generator, position, and attitude controller play together.

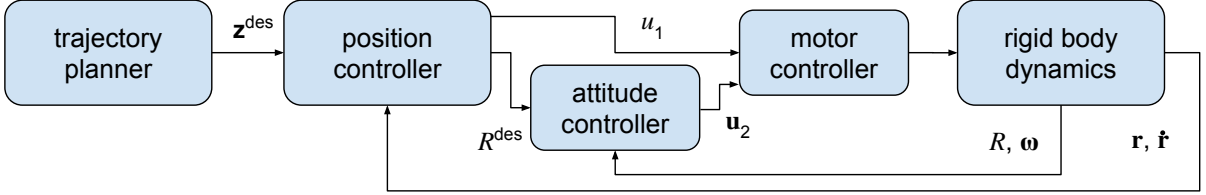


Figure 2: The position and attitude control loops.

Define the position error in terms of components by:

$$e_i = (r_i - r_{i,T}).$$

In order to guarantee that this error goes exponentially to zero, we require

$$(\ddot{r}_i^{\text{des}} - \ddot{r}_{i,T}) + k_{d,i}(\dot{r}_i - \dot{r}_{i,T}) + k_{p,i}(r_i - r_{i,T}) = 0. \quad (20)$$

In Eq (20),  $r_{i,T}$  and its derivatives are given by the trajectory generator, and  $r_i$  and  $\dot{r}_i$  are provided by the state estimation system, allowing for the commanded acceleration  $\ddot{r}_i^{\text{des}}$  to be calculated:

$$\ddot{r}_i^{\text{des}} = \ddot{r}_{i,T} - k_{d,i}(\dot{r}_i - \dot{r}_{i,T}) - k_{p,i}(r_i - r_{i,T}). \quad (21)$$

Now the attitude of the quad rotor must be controlled such that it will generate  $\ddot{r}^{\text{des}}$ . For this, the linearized version of (17) is used:

$$\ddot{r}_1^{\text{des}} = g(\theta^{\text{des}} \cos \psi_T + \phi^{\text{des}} \sin \psi_T) \quad (22a)$$

$$\ddot{r}_2^{\text{des}} = g(\theta^{\text{des}} \sin \psi_T - \phi^{\text{des}} \cos \psi_T) \quad (22b)$$

$$\ddot{r}_3^{\text{des}} = \frac{1}{m}u_1 - g. \quad (22c)$$

From the third equation we can directly read off the thrust control  $u_1$ . The first two equations can be solved for  $\theta^{\text{des}}$  and  $\phi^{\text{des}}$ , since  $\psi^{\text{des}} = \psi_T$  is given directly by the trajectory generator.

Linearizing (18), we get:

$$I \begin{bmatrix} \ddot{p} \\ \ddot{q} \\ \ddot{r} \end{bmatrix} = \mathbf{u}_2, \quad (23)$$

and by further exploiting that via Eq (5) Euler angle velocities are to linear approximation equal to angular velocities, it becomes evident that we can directly command the acceleration of the Euler angles:

$$I \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \mathbf{u}_2. \quad (24)$$

Thus the “inner loop” attitude control can also be done with a simple PD controller:

$$\mathbf{u}_2 = I \begin{bmatrix} -k_{p,\phi}(\phi - \phi^{\text{des}}) - k_{d,\phi}(p - p^{\text{des}}) \\ -k_{p,\theta}(\theta - \theta^{\text{des}}) - k_{d,\theta}(q - q^{\text{des}}) \\ -k_{p,\psi}(\psi - \psi_T) - k_{d,\psi}(r - \psi_T) \end{bmatrix}, \quad (25)$$

where the desired roll and pitch velocities  $p^{\text{des}}$  and  $q^{\text{des}}$  can be computed from the equations of motion and the specified trajectory, but in practice can be set to zero.

In summary, the controller then works as follows:

1. Use Eq (21) to compute the commanded acceleration  $\ddot{\mathbf{r}}^{\text{des}}$ .
2. Use Eq (22c) to compute  $u_1$  and the desired angles  $\theta^{\text{des}}$  and  $\phi^{\text{des}}$  from (22a) and (22b).
3. Use Eq (25) to compute  $\mathbf{u}_2$ .

### 3.3 A Geometric Nonlinear Controller

Nonlinear controllers are generally built based on geometric intuition, i.e. the quad rotor's  $\mathbf{b}_3$  axis is tilted to point in the desired direction, and thrust is applied. Such controllers are suitable for very aggressive maneuvers and will allow for faster flight and sharper turns, in particular in the simulator. Note that we have considerable freedom to choose a control algorithm so long as it results in a stable system.

The following section is based on the controller developed in [1], which in turn is a simplified version of the controller in [2]. For a thorough treatment and stability analysis please refer to [2].

It turns out that the basic layout of the controller remains as shown in Figure 2, i.e. there is an attitude and a position controller, although we no longer make the backstepping assumption.

We start out from the PD controller in Eq (21), but now write it in more compact vector form:

$$\ddot{\mathbf{r}}^{\text{des}} = \ddot{\mathbf{r}}_T - K_d(\dot{\mathbf{r}} - \dot{\mathbf{r}}_T) - K_p(\mathbf{r} - \mathbf{r}_T), \quad (26)$$

where  $K_d$  and  $K_p$  are diagonal, positive definite gain matrices.

We are again faced with the question how to compute the inputs  $u_1$ ,  $\mathbf{u}_2$  to generate  $\ddot{\mathbf{r}}^{\text{des}}$ . We first determine the input  $u_1$ . By rearranging Eq (17) one obtains:

$$m\ddot{\mathbf{r}}^{\text{des}} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = u_1 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (27)$$

Note that the lhs of (27) is just the total commanded force  $\mathbf{F}^{\text{des}}$  (including gravity):

$$\mathbf{F}^{\text{des}} = m\ddot{\mathbf{r}}^{\text{des}} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}. \quad (28)$$

On the right hand side is  $u_1$ , multiplied with the quad rotors axis  $\mathbf{b}_3 = R[0, 0, 1]^T$ , expressed in the inertial frame. We obtain the input  $u_1$  by projecting  $\mathbf{F}^{\text{des}}$  onto  $\mathbf{b}_3$ :

$$u_1 = \mathbf{b}_3^T \mathbf{F}^{\text{des}}. \quad (29)$$

To compute  $\mathbf{u}_2$ , we observe that a quad rotor can only produce thrust along its  $\mathbf{b}_3$  axis. It makes sense to align  $\mathbf{b}_3$  with  $\mathbf{F}^{\text{des}}$ , and align  $\mathbf{b}_1$  to match the desired yaw  $\psi_T$ . Please refer to Fig. 3. In the following steps we find a triad  $R^{\text{des}} = [\mathbf{b}_1^{\text{des}}, \mathbf{b}_2^{\text{des}}, \mathbf{b}_3^{\text{des}}]$  that has the proper alignment, then proceed to develop a control input  $\mathbf{u}_2$  that produces the corresponding torque to drive the attitude towards  $R^{\text{des}}$ .

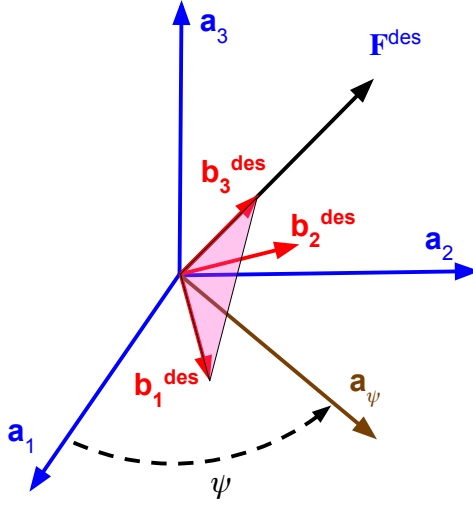


Figure 3: Geometry based attitude control. First  $\mathbf{b}_3^{\text{des}}$  is aligned along  $\mathbf{F}^{\text{des}}$ . Then  $\mathbf{b}_2^{\text{des}}$  is chosen to be *perpendicular* to the plane spanned by the desired yaw heading vector  $\mathbf{a}_\psi$  and  $\mathbf{b}_3^{\text{des}}$ . This ensures that  $\mathbf{b}_1^{\text{des}}$  and  $\mathbf{b}_3^{\text{des}}$  are *in* the plane containing  $\mathbf{a}_\psi$ .

As indicated, the  $\mathbf{b}_3^{\text{des}}$  axis should be oriented along the desired thrust:

$$\mathbf{b}_3^{\text{des}} = \frac{\mathbf{F}^{\text{des}}}{\|\mathbf{F}^{\text{des}}\|}. \quad (30)$$

Next, we want the  $\mathbf{b}_2^{\text{des}}$  axis to be perpendicular to both  $\mathbf{b}_1^{\text{des}}$  and the vector  $\mathbf{a}_\psi$  that defines the yaw direction in the plane  $(\mathbf{a}_1, \mathbf{a}_2)$  plane:

$$\mathbf{a}_\psi = \begin{bmatrix} \cos \psi_T \\ \sin \psi_T \\ 0 \end{bmatrix}. \quad (31)$$

This can be accomplished by forming the cross product between  $\mathbf{b}_3^{\text{des}}$  and  $\mathbf{a}_\psi$ :

$$\mathbf{b}_2^{\text{des}} = \frac{\mathbf{b}_3^{\text{des}} \times \mathbf{a}_\psi}{\|\mathbf{b}_3^{\text{des}} \times \mathbf{a}_\psi\|} \quad (32)$$

which guarantees that the plane formed by  $\mathbf{b}_3^{\text{des}}$  and the axis  $\mathbf{b}_1^{\text{des}}$  representing the head of the robot contains the  $\mathbf{a}_\psi$ . Finally,  $\mathbf{b}_1^{\text{des}}$  is obtained by cross product and the desired rotation matrix is:

$$R^{\text{des}} = [\mathbf{b}_2^{\text{des}} \times \mathbf{b}_3^{\text{des}}, \mathbf{b}_2^{\text{des}}, \mathbf{b}_3^{\text{des}}]. \quad (33)$$

Next, we need a measure for the error in orientation  $R^{\text{des}T}R$ . The following error vector (see [1])

$$\mathbf{e}_R = \frac{1}{2}(R^{\text{des}T}R - R^T R^{\text{des}})^\vee \quad (34)$$



is obtained from the matrices by taking the  $\vee$  operator that maps elements of  $\text{so}(3)$  to  $\mathbb{R}^3$ . This error vector is zero when  $R^{\text{des}} = R$ . It is the rotation vector that generates the error in orientation. Consequently, by applying a torque along its direction it can be decreased, suggesting the control input:

$$\mathbf{u}_2 = I(-K_R \mathbf{e}_R - K_\omega \mathbf{e}_\omega), \quad (35)$$

where  $\mathbf{e}_\omega = \boldsymbol{\omega} - \boldsymbol{\omega}^{\text{des}}$  is the error in angular velocities and  $K_R$  and  $K_\omega$  are diagonal gains matrices. The desired angular velocities  $\boldsymbol{\omega}^{\text{des}}$  can be computed from the output of the trajectory generator  $\mathbf{z}_T$  and its derivatives, but setting them to zero will work for the purpose of this project.

To summarize, the steps to implement the controller are:

1. calculate  $\mathbf{F}^{\text{des}}$  from Eq (28), (27), and (26).
2. compute  $u_1$  from Eq (29)
3. determine  $R^{\text{des}}$  from Eq (33) and the definitions of  $\mathbf{b}_i^{\text{des}}$ .
4. find the error orientation error vector  $\mathbf{e}_R$  from Eq (34) and substitute  $\boldsymbol{\omega}$  for  $\mathbf{e}_\omega$ .
5. compute  $\mathbf{u}_2$  from Eq (35).

## 4 System Description

### 4.1 Quadrotor Platform

For this project we will be using the CrazyFlie 2.0 platform made by Bitcraze, shown in Fig. 1. The CrazyFlie has a motor to motor (diagonal) distance of 92 mm, and a mass of 30 g, including a battery. A microcontroller allows low-level control and estimation to be done onboard the robot. An onboard IMU provides feedback of angular velocities and accelerations.

### 4.2 Software and Integration

Position control and other high level commands are computed in Matlab at 100 Hz and sent to the robot via the CrazyRadio (2.4GHz). Attitude control is performed onboard using the microcontroller

### 4.3 Inertial Properties

Since  $\mathbf{b}_i$  are principal axes, the inertia matrix referenced to the center of mass along the  $\mathbf{b}_i$  reference triad,  $I$ , is a diagonal matrix. In practice, the three moments of inertia can be estimated by weighing individual components of the quadrotor and building a physically accurate model in SolidWorks. The key parameters for the rigid body dynamics for the CrazyFlie platform are as follows:

- (a) mass:  $m = 0.030$  kg;
- (b) the distance from the center of mass to the axis of a motor:  $l = 0.046$  m; and
- (c) the components of the inertia dyadic using  $\mathbf{b}_i$  in  $\text{kg m}^2$ :

$$[I_C]^{\mathbf{b}_i} = \begin{bmatrix} 1.43 \times 10^{-5} & 0 & 0 \\ 0 & 1.43 \times 10^{-5} & 0 \\ 0 & 0 & 2.89 \times 10^{-5} \end{bmatrix}.$$

Note that these constants will be made available to you as a matlab structure for both simulation and lab session.

## 5 Project Work

### 5.1 Simulator

The quadrotor simulator can be found under “projects” on the MEAM620 web page and comes with skeleton code that has interface documentation. Before implementing your own functions, you should first try running `runsim.m` in your matlab. If you see a quadrotor falling from position  $(0, 0, 0)$  then the simulator works on your computer and you may continue with other tasks. This is because the outputs of `controller.m` are all zeros, thus no thrust is generated.

When you have the basic version of your trajectory generator and controller done (see below), you will be able to see the quadrotor flying in the space with proper roll, pitch, and yaw, leaving trails of desired and actual position behind. The desired position is color-coded **blue** and the actual position is **red**. After the simulation finishes, two plots of position and velocity will be generated to give you an overview of how well your trajectory generator and controller are doing. Note that you will not be graded on these plots but by the automatic grader.

### 5.2 Tasks

You will simulate the quadrotor dynamics and control using the matlab simulator. The simulator relies on the numerical solver `ode45`, details about this solver can be easily found online. You don't need to be an expert in numerical methods, but it would be beneficial if you know the basics of how ode solvers work. Your tasks include:

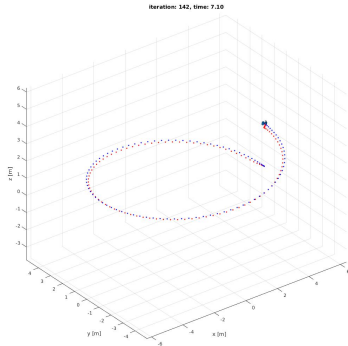


Figure 4: Helix trajectory (`circle.m`)

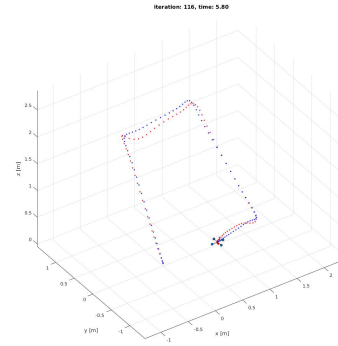


Figure 5: Diamond trajectory (`diamond.m`)

#### 1. Trajectory Generator

You will first implement three trajectory generators to produce a hover, a circle, and a diamond trajectory. In order to specify the trajectory in the simulator, you will need to change the variable `trajhandle` in `runsim.m`. The handle `trajhandle` is the name of a function that takes in current time  $t$  and current quadrotor number  $qn$  and returns a struct of desired state as a function of time. The struct `desired_state` has 5 fields, which are `pos`, `vel`, `acc`, `yaw` and `yawdot`. For a proper trajectory, you need to specify `pos`, `vel` and `acc`, whereas `yaw` and `yawdot` can be left as 0.

(a) `hover.m`

Hover at whatever altitude you wish. This is just a test trajectory for you to debug your controller and will not be used by the auto grader.

(b) `circle.m`

A helix in the  $xy$  plane of radius 5 m centered about the point  $(0, 0, 0)$  starting at the point  $(5, 0, 0)$ . The  $z$  coordinate should start at 0 and end at 2.5. The helix should go counter-clockwise. See Fig. 4.

(c) `diamond.m`

A "diamond helix" with corners at  $(0, 0, 0)$ ,  $(0, \sqrt{2}, \sqrt{2})$ ,  $(0, 0, 2\sqrt{2})$ , and  $(0, -\sqrt{2}, \sqrt{2})$  when projected into the  $yz$  plane, and an  $x$  coordinate starting at 0 and ending at 1. The quadrotor should start at  $(0, 0, 0)$  and end at  $(1, 0, 0)$ . See Fig. 5.

## 2. Controller

You will then implement a controller in file `controller.m` that makes sure that your quadrotor follows the desired trajectory. This controller will be used again in the following phases. The controller takes in a cell struct `qd`, current time `t`, current quadrotor number `qn` and quadrotor parameters `params` and outputs thrust `F`, moments `M`, desired thrust, roll, pitch and yaw `trpy` and derivatives of desired roll, pitch and yaw `drpy`.

Although in this phase of the project you will only need to output thrust `F` and moments `M`, you should also set `trpy` and `drpy`. Although they will be ignored in the simulator, you will need to specify them when you later fly in the lab. The cell array `qd` contains structs for each quadrotor. This means `qd{qn}` has all the information (position, velocity, euler angles and etc.) about quadrotor number `qn`. Those information can be accessed via `qd{qn}.pos` for example.

## 3. Simulation

Lastly, you need to fill in the appropriate variables for `trajhandle` and `controlhandle` in the script `runsim.m` for simulating your result.

Hints:

1. Make sure that your controller can also handle arbitrary yaw targets, and test that it works! Beware of yaw angle wrap-around, consider using Matlab's `angdiff()` function.
2. Tune attitude and position gains separately. Which ones should you do first?

## 5.3 Submission

When you are finished you may submit your code via `turnin`. The project name for this assignment is titled "proj1phase1" so the command to submit should be

```
# turnin -c meam620 -p proj1phase1 -v *
```

Your `turnin` submission should contain:

1. A README file detailing anything we should be aware of.
2. Files `crazyflie.m`, `controller.m`, `diamond.m`, `circle.m`, as well as any other Matlab files you need to run your code. (That being said, we recommend you to only submit those 4 files)

Shortly after submitting you should receive an e-mail from `meam620@seas.upenn.edu` stating that your submission has been received. You can check on the status of your submission at `https://alliance.seas.upenn.edu/~meam620/monitor/`. Once the automated tests finish running, you should receive another e-mail containing your test results. This report will only tell you whether you passed or failed tests; it will not tell you what the test inputs were or why your code failed. Your code will be given at most 5 minutes to complete all the tests. There is no limit on the number of times you can submit your code.

Please do not attempt to determine what the automated tests are or otherwise try to game the automated test system. Any attempt to do so will be considered cheating, resulting in a 0 on this assignment and possible disciplinary action.

You will be graded on successful completion of the code and how quickly and accurately your quadrotor follows the circle and diamond paths and two other trajectories which will not be released.

## References

- [1] D. Mellinger and V. Kumar, "Minimum snap trajectory generation and control for quadrotors," in *Proc. of the IEEE Int. Conf. on Robotics and Automation*, Shanghai, China, May 2011.
- [2] T. Lee, M. Leok, and N. McClamroch, "Geometric tracking control of a quadrotor uav on  $SE(3)$ ," in *Proc. of the IEEE Conf. on Decision and Control*, 2010.