The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Some Correlation Basics: Properties, Motivation, Terminology

Topic 6

Exam Focus

This topic focuses on the role correlation plays as an input for quantifying risk in multiple areas of finance. We will explain how correlation changes the value and risk of structured products such as credit default swaps (CDSs), collateralized debt obligations (CDOs), multi-asset correlation options, and correlation swaps. For the exam, understand how correlation risk is related to market risk, systemic risk, credit risk, and concentration ratios, and be familiar with how changes in correlation impact implied volatility, the value of structured products, and default probabilities. Also, be prepared to discuss how the misunderstanding of correlation contributed to the financial crisis of 2007 to 2009.

Financial Correlation Risk

LO 6.1: Describe financial correlation risk and the areas in which it appears in finance.

Correlation risk measures the risk of financial loss resulting from adverse changes in correlations between financial or nonfinancial assets. An example of financial correlation risk is the negative correlation between interest rates and commodity prices. If interest rates rise, losses occur in commodity investments. Another example of this risk occurred during the 2012 Greek crisis. The positive correlation between Mexican bonds and Greek bonds caused losses for investors of Mexican bonds.

The financial crisis beginning in 2007 illustrated how financial correlation risk can impact global markets. During this time period, correlations across global markets became highly correlated. Assets that previously had very low or negative correlations suddenly become very highly positively correlated and fell in value together.

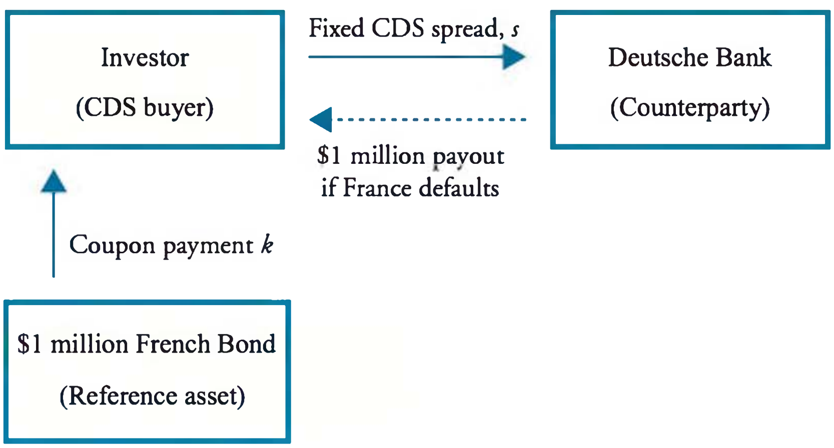
Nonfinancial assets can also be impacted by correlation risk. For example, the correlation of sovereign debt levels and currency values can result in financial losses for exporters. In 2012, U.S. exporters experienced losses due to the devaluation of the euro. Similarly, a low gross domestic product (GDP) for the United States has major adverse impacts on Asian and European exporters who rely heavily on the U.S. market. Another nonfinancial example is related to political events, such as uprisings in the Middle East that cause airline travel to decrease due to rising oil prices.

Financial correlations can be categorized as static or dynamic. **Static financial correlations** do not change and measure the relationship between assets for a specific time period. Examples of static correlation measures are value at risk (VaR), correlation copulas for collateralized debt obligations (CDOs), and the binomial default correlation model. **Dynamic financial correlations** measure the comovement of assets over time. Examples of dynamic financial correlations are pairs trading, deterministic correlation approaches, and stochastic correlation processes.

Structured products are becoming an increasing area of concern regarding correlation risk. The following example demonstrates the role correlation risk plays in **credit default swaps** (CDS). A CDS transfers credit risk from the investor (CDS buyer) to a counterparty (CDS seller).

Suppose an investor purchases $1 million of French bonds and is concerned about France defaulting. The investor (CDS buyer) can transfer the default risk to a counterparty (CDS seller). Figure 1 illustrates the process for an investor transferring credit default risk by purchasing a CDS from Deutsche Bank (a large European bank).

Figure 1: CDS Buyer Hedging Risk in Foreign Bonds



Assume the recovery rate is zero with no accrued interest in the event of default. The investor (CDS buyer) is protected if France defaults because the investor receives a $1 million payment from Deutsche Bank. The fixed CDS spread is valued based on the default probability of the reference asset (French Bond) and the joint default correlation of Deutsche Bank and France. A paper loss occurs if the correlation risk between Deutsche Bank and France increases because the value of the CDS will decrease. If Deutsche Bank and France default (worst case scenario), the investor loses the entire $1 million investment.

If there is positive correlation risk between Deutsche Bank and France, the investor has **wrong-way risk** (WWR). The higher the correlation risk, the lower the CDS spread, *s*. The increasing correlation risk increases the probability that both the French bond (reference asset) and Deutsche Bank (counterparty) default.

The dependencies between the CDS spread, *s*, and correlation risk may be *nonmonotonous*. This means that the CDS spread may sometimes increase and sometimes decrease if correlation risk increases. For example, for a correlation of –1 to –0.4, the CDS spread may increase slightly. This is due to the fact that a high negative correlation implies either France or Deutsche Bank will default, but not both. If France defaults, the $1 million is recovered from Deutsche Bank. If Deutsche Bank defaults, the investor loses the value of the CDS spread and the investor will need to repurchase a CDS spread to hedge the position. The new CDS spread cost will most likely increase in the event that Deutsche Bank defaults or if the credit quality of France decreases.

There are many areas in finance that have financial correlations. Five common finance areas where correlations play an important role are (1) investments, (2) trading, (3) risk management, (4) global markets, and (5) regulation.

Correlations in Financial Investments

In 1952, Harry Markowitz provided the foundation of modern investment theory by demonstrating the role that correlation plays in reducing risk. The portfolio return is simply the weighted average of the individual returns where the weights are the percentage of investment in each asset. The following equation defines the average return (i.e., **mean**) for a portfolio, μ_P, comprised of assets *X* and *Y*. Asset *X* has a weight of w_X and an average return of μ_X, and asset *Y* has a weight of w_Y and an average return of μ_Y.

$$\mu_P = w_X\mu_X + w_Y\mu_Y$$

The **standard deviation** of a portfolio is determined by the variances of each asset, the weights of each asset, and the covariance between assets. The risk or standard deviation (i.e., volatility) for a two-asset portfolio is calculated as follows:

$$\sigma_P = \sqrt{w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_X w_Y \operatorname{cov}_{XY}}$$

Let us review how variances, covariance, and correlation are calculated using the following example. Suppose an analyst gathers historical prices for two assets, *X* and *Y*, and calculates their average returns as illustrated in Figure 2.

Figure 2: Prices and Returns for Assets *X* and *Y*

Year	*X*	*Y*	*Return X*	*Return Y*
2009	90	150		
2010	120	180	0.3333	0.2000
2011	105	340	(0.1250)	0.8889
2012	170	320	0.6190	(0.0588)
2013	150	360	(0.1176)	0.1250
2014	270	310	0.8000	(0.1389)
		Average Return	0.3019	0.2032

The calculations for determining the standard deviations, variances, covariance, and correlation for assets *X* and *Y* are illustrated in Figure 3.

Figure 3: Variances and Covariance for Assets *X* and *Y*

Year	*Return X*	*Return Y*	$X_t - \mu_X$	$Y_t - \mu_Y$	$(X_t - \mu_X)^2$	$(Y_i - \mu_Y)^2$	$(X_t - \mu_X) \times (Y_t - \mu_Y)$
2010	0.3333	0.2000	0.0314	(0.0032)	0.0010	0.0000	(0.0001)
2011	(0.1250)	0.8889	(0.4269)	0.6857	0.1823	0.4701	(0.2927)
2012	0.6190	(0.0588)	0.3171	(0.2621)	0.1006	0.0687	(0.0831)
2013	(0.1176)	0.1250	(0.4196)	(0.0782)	0.1761	0.0061	0.0328
2014	0.8000	(0.1389)	0.4981	(0.3421)	0.2481	0.1170	(0.1704)
Mean	**0.3019**	**0.2032**			0.7079	0.6620	(0.5135)
				Variance	**0.1770**	**0.1655**	**(0.1284)**
				Standard Deviation	**0.4207**	**0.4068**	
				Correlation	**(0.7501)**		

Notice that the sixth and seventh columns of Figure 3 are used to calculate the variance of *X* and *Y*, respectively. The deviation from each respective mean is squared to calculate the variance for each asset: $(X_t - \mu_X)^2$ for *X* and $(Y_t - \mu_Y)^2$ for *Y*. The sum of the deviations is then divided by four (i.e., the number of observations minus one for degrees of freedom). For example, the asset *X* variance is calculated by taking 0.7079 and dividing by 4 (i.e., n – 1) to get 0.1770.

Covariance is a measure of how two assets move together over time. The last column of Figure 3 illustrates that the calculation for covariance is similar to the calculation for variance. However, instead of squaring each deviation from the mean, the last column multiplies the deviations from the mean for each respective asset together. This not only captures the magnitude of movement but also the direction of movement. Thus, when asset returns are moving in opposite directions for the same time period, the product of their deviations is negative. The following equation defines the calculation for covariance. The sum of the products of the deviations from the means is –0.5135 in the last column of Figure 3. Covariance is calculated as –0.1284 by dividing –0.5135 by 4 (i.e., n – 1).

$$\text{cov}_{XY} = \frac{\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y)}{n-1}$$

In finance, the **correlation coefficient** is often used to standardize the comovement or covariance between assets. The following equation defines the correlation for two assets, *X* and *Y*, by dividing covariance, cov_{XY}, by the product of the asset standard deviations, $\sigma_X\sigma_Y$.

$$\rho_{XY} = \frac{\text{cov}_{XY}}{\sigma_X\sigma_Y}$$

The correlation in this example is –0.7501, which is calculated as:

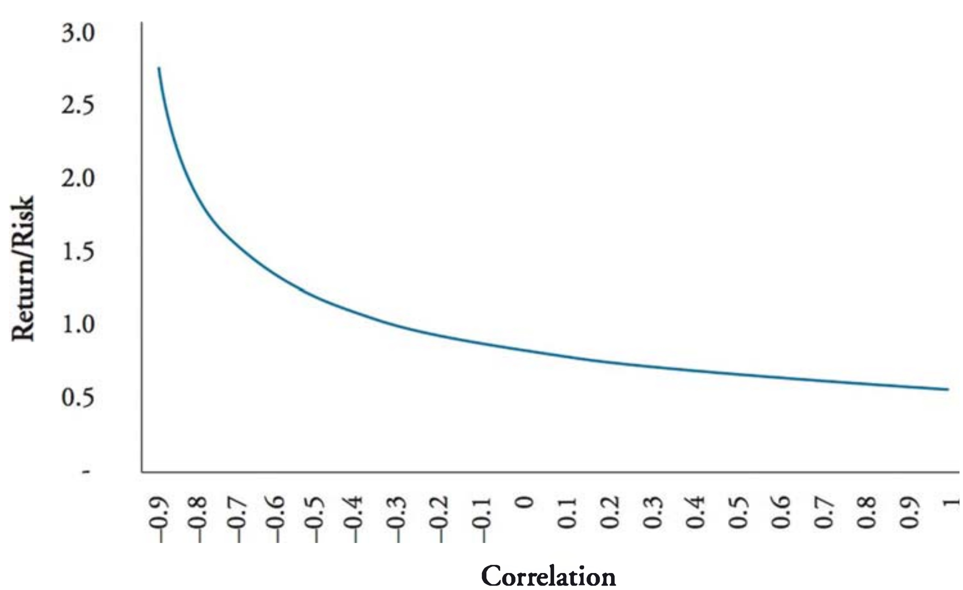
$$-0.1284 / (0.4207 \times 0.4068) = -0.7501$$

In his research, Markowitz emphasized the importance of focusing on risk-adjusted returns. The return/risk ratio measures the average return for a portfolio, μ_P, by the risk of the portfolio, σ_P. Figure 2 provided the average return for X and Y as 0.3019 and 0.2032, respectively. If we assume the portfolio is equally weighted, the average return for the portfolio is 0.2526, the correlation between assets X and Y is –0.7501, and the standard deviations for X and Y are 0.4207 and 0.4068, respectively. The standard deviation for an equally-weighted portfolio is determined using the following expression:

$$\sqrt{\left(0.5^2 \times 0.4207^2\right)+\left(0.5^2 \times 0.4068^2\right)+(2 \times 0.5 \times 0.5 \times -0.1284)}$$
$$=\sqrt{0.02142}=0.1464$$

The return/risk ratio of this equally-weighted two-asset portfolio is 1.725 (calculated as 0.2526 divided by 0.1464). Figure 4 illustrates the relationship of the return/risk ratio and correlation. The lower the correlation between the two assets, the higher the return/risk ratio. A very high negative correlation (e.g., –0.9) results in a return/risk ratio greater than 250%. A very high positive correlation (e.g., +0.9) results in a return/risk ratio near 50%.

Figure 4: Relationship of Return/Risk Ratio and Correlation



Correlation in Trading with Multi-Asset Options

Correlation trading strategies involve trading assets that have prices determined by the comovement of one or more assets over time. **Correlation options** have prices that are very sensitive to the correlation between two assets and are often referred to as *multi-asset options*.

A quick review of the common notation for options is helpful. Assume the price of asset one and two are noted as S_1 and S_2, respectively, and that the strike price, K, for a call option is the predetermined price an asset can be purchased. Likewise, the strike price, K, for a put option is the predetermined price an asset can be sold for.

The correlation between the two assets S_1 and S_2 is an important factor in determining the price of correlation options. Figure 5 lists a number of multi-asset correlation strategies along with their payoffs. For all of these strategies, a lower correlation results in a higher option price. A low correlation is expected to result in one asset price going higher while the other is lower. Thus, there is a better chance of a higher payout.

Figure 5: Payoffs for Multi-Asset Correlation Strategies

Correlation strategies	Payoff
Option on higher of two stocks	$\max(S_1, S_2)$
Call option on maximum of two stocks	$\max[0, \max(S_1, S_2) - K]$
Exchange option	$\max(0, S_2 - S_1)$
Spread call option	$\max(0, S_2 - S_1 - K)$
Dual-strike call option	$\max(0, S_1 - K_1, S_2 - K_2)$
Portfolio of basket options	$\left[\sum_{i=1}^{n} n_i \times S_i - K, 0\right]$, where n_i = weight of asset i

Another correlation strategy that is not listed in Figure 5 is a correlation option on the worse of two stocks where the payoff is the minimum of the two stock prices. This is the only correlation option where a lower correlation is not desirable because it reduces the correlation option price.

We can better understand the role correlation plays by taking a closer look at the valuation of the exchange option. The exchange option has a payoff of $\max(0, S_2 - S_1)$. The buyer of the option has the right to receive asset 2 and give away asset 1 when the option matures. The standard deviation of the exchange option, σ_E, is the implied volatility of S_2 / S_1, which is defined as:

$$\sigma_E = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\,\text{cov}_{XY}}$$

Implied volatility is an important determinant of the option's price. Thus, the exchange option price is highly sensitive to the covariance or correlation between the two assets. The price of the exchange option is close to zero when the correlation is close to 1 because the two asset prices move together, and the spread between them does not change. The price of the exchange option increases as the correlation between the two assets decreases because the spread between the two assets is more likely to be greater.

Quanto Option

The quanto option is another investment strategy using correlation options. It protects a domestic investor from foreign currency risk. However, the financial institution selling the quanto call does not know how deep in the money the call will be or what the exchange rate will be when the option is exercised to convert foreign currency to domestic currency. Lower correlations between currencies result in higher prices for quanto options.

Example: Quanto option

Suppose a U.S. investor buys a quanto call to invest in the Nikkei index and protect potential gains by setting a fixed currency exchange rate (USD/JPY). How does the correlation between the call on the Nikkei index and the exchange rate impact the price of the quanto option?

Answer:

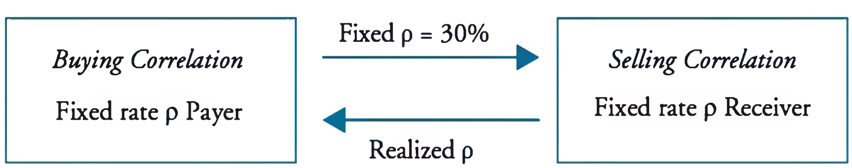
The U.S. investor buys a quanto call on the Nikkei index that has a fixed exchange rate for converting yen to dollars. If the correlation coefficient is positive (negative) between the Nikkei index and the yen relative to the dollar, an increasing Nikkei index results in an increasing (decreasing) value of the yen. Thus, the lower the correlation, the higher the price for the quanto option. If the Nikkei index increases and the yen decreases, the financial institution will need more yen to convert the profits in yen from the Nikkei investment into dollars.

Correlation Swap

LO 6.3: Describe the structure, uses, and payoffs of a correlation swap.

A correlation swap is used to trade a fixed correlation between two or more assets with the correlation that actually occurs. The correlation that will actually occur is unknown and is referred to as the *realized* or *stochastic correlation*. Figure 6 illustrates how a correlation swap is structured. In this example, the party buying a correlation swap pays a fixed correlation rate of 30%, and the entity selling a correlation receives the fixed correlation of 30%.

Figure 6: Correlation Swap with a Fixed Correlation Rate



The present value of the correlation swap increases for the correlation buyer if the realized correlation increases. The following equation calculates the realized correlation that actually occurs over the time period of the swap for a portfolio of *n* assets, where $\rho_{i,j}$ is the correlation coefficient:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j}$$

The payoff for the investor buying the correlation swap is calculated as follows:

notional amount × ($\rho_{realized} - \rho_{fixed}$)

Example: Correlation swap

Suppose a correlation swap buyer pays a fixed correlation rate of 0.2 with a notional value of $1 million for one year for a portfolio of three assets. The realized pairwise correlations of the daily log returns [$\ln(S_t / S_{t-1})$] at maturity for the three assets are $\rho_{2,1} = 0.6$, $\rho_{3,1} = 0.2$, and $\rho_{3,2} = 0.04$. (Note that for all pairs i > j.) What is the correlation swap buyer's payoff?

Answer:

The realized correlation is calculated as:

$$\rho_{realized} = \frac{2}{3^2 - 3} \times (0.6 + 0.2 + 0.04) = 0.28$$

The payoff for the correlation swap buyer is then calculated as:

$1,000,000 × (0.28 – 0.20) = $80,000

Another example of buying correlation is to buy call options on a stock index (such as the Standard & Poor's 500 Index) and sell call options on individual stocks held within the index. If correlation increases between stocks within the index, this causes the implied volatility of call options to increase. The increase in price for the index call options is expected to be greater than the increase in price for individual stocks that have a short call position.

An investor can also buy correlation by paying fixed in a variance swap on an index and receiving fixed on individual securities within the index. An increase in correlation for securities within the index causes the variance to increase. An increase in variance causes the present value of the position to increase for the fixed variance swap payer (i.e., variance swap buyer).

Risk Management

LO 6.4: Estimate the impact of different correlations between assets in the trading book on the VaR capital charge.

The primary goal of risk management is to mitigate financial risk in the form of market risk, credit risk, and operational risk. A common risk management tool used to measure market risk is **value at risk** (VaR). VaR for a portfolio measures the potential loss in value for a specific time period for a given confidence level. The formula for calculating VaR using the **variance-covariance method** (a.k.a. delta-normal method) is shown as follows:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x}$$

In this equation, σ_P is the daily volatility of the portfolio, α is the *z*-value from the standard normal distribution for a specific confidence level, and *x* is the number of trading days. The volatility of the portfolio, σ_P, includes a measurement of correlation for assets within the portfolio defined as:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v}$$

where:
β_h = horizontal β vector of investment amount
C = covariance matrix of returns
β_v = vertical β vector of investment amount

Example: Computing VaR with the variance-covariance method

Assume you have a two-asset portfolio with \$8 million in asset A and \$4 million in asset B. The portfolio correlation is 0.6, and the daily standard deviation of returns for assets A and B are 1.5% and 2%, respectively. What is the 10-day VaR of this portfolio at a 99% confidence level (i.e., α = 2.33)?

Answer:

The first step in solving for the 10-day VaR requires constructing the covariance matrix.

$$\text{cov}_{11} = \sigma_1^2 = 0.015^2 = 0.000225$$
$$\text{cov}_{22} = \sigma_2^2 = 0.02^2 = 0.0004$$
$$\text{cov}_{12} = \rho_{12} \times \sigma_1 \times \sigma_2 = 0.6 \times 0.015 \times 0.02 = 0.00018$$

Thus, the covariance matrix, C, can be represented as:

$$\begin{pmatrix} \text{cov}_{11} & \text{cov}_{12} \\ \text{cov}_{21} & \text{cov}_{22} \end{pmatrix} = \begin{pmatrix} 0.000225 & 0.00018 \\ 0.00018 & 0.0004 \end{pmatrix}$$

Next, the standard deviation of the portfolio, σ_P, is determined by first solving for $\beta_h \times C$, then solving for $(\beta_h \times C) \times \beta_v$, and finally taking the square root of the second step.

Step 1: Compute $\beta_h \times C$:

$$\begin{bmatrix} 8 & 4 \end{bmatrix} \begin{pmatrix} 0.000225 & 0.00018 \\ 0.00018 & 0.0004 \end{pmatrix}$$
$$= \begin{bmatrix} (8 \times 0.000225) + (4 \times 0.00018) & (8 \times 0.00018) + (4 \times 0.0004) \end{bmatrix}$$
$$= \begin{bmatrix} 0.00252 & 0.00304 \end{bmatrix}$$

Step 2: Compute ($\beta_h \times C$) × β_v:

$$\begin{bmatrix} 0.00252 & 0.00304 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$
$$= (0.00252 \times 8) + (0.00304 \times 4) = 0.03232$$

Step 3: Compute σ_P:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v} = \sqrt{0.03232} = 0.1798 \text{ or } 17.98\%$$

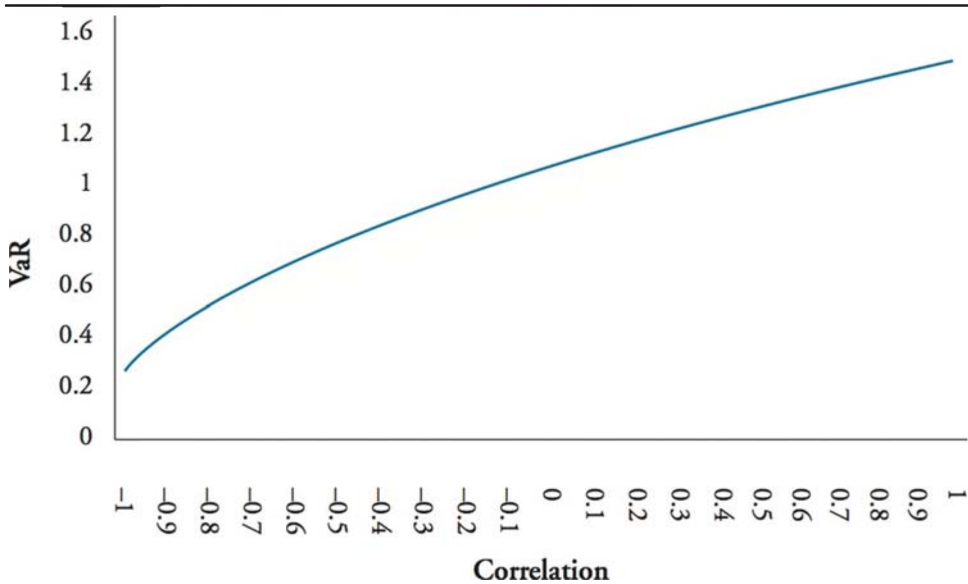
The 10-day portfolio VaR (in millions) at the 99% confidence level is then computed as:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x} = 0.1798 \times 2.33 \times \sqrt{10} = 1.3248$$

This suggests that the loss will only exceed $1,324,800 once every 100 10-day periods. This is approximately once every 1,000 trading days or once every four years assuming there are 250 trading days in a year.

Figure 7 illustrates the relationship between correlation and VaR for the previous two-asset portfolio example. The VaR for the portfolio increases as the correlation between the two assets increases.

Figure 7: Relationship Between VaR and Correlation for Two-Asset Portfolio



The Basel Committee on Banking Supervision (BCBS) requires banks to hold capital based on the VaR for their portfolios. The BCBS requires banks to hold capital for assets in the trading book of at least three times greater than 10-day VaR. The trading book includes assets that are marked-to-market, such as stocks, futures, options, and swaps. The bank in the previous example would be required by the Basel Committee to hold capital of:

$1,324,800 × 3 = $3,974,400

Correlations During the Recent Financial Crisis

LO 6.2: Explain how correlation contributed to the global financial crisis of 2007 to 2009.

The correlations of assets within and across different sectors and geographical regions were a major contributing factor for the financial crisis of 2007 to 2009. The economic environment, risk attitude, new derivative products, and new copula correlation models all contributed to the crisis.

Investors became more risk averse shortly after the internet bubble that began in the 1990s. The economy and risk environment was recovering with low credit spreads, low interest rates, and low volatility. The overly optimistic housing market led individuals to take on more debt on overvalued properties. New structured products known as collateralized debt obligations (CDOs), constant-proportion debt obligations (CPDOs), and credit default swaps (CDSs) helped encourage more speculation in real estate investments. Rating agencies, risk managers, and regulators overlooked the amount of leverage individuals and financial institutions were taking on. All of these contributing factors helped set the stage for the financial crisis that would be set off initially by defaults in the subprime mortgage market.

Risk managers, financial institutions, and investors did not understand how to properly measure correlation. Risk managers used the newly developed copula correlation model for measuring correlation in structured products. It is common for CDOs to contain up to 125 assets. The copula correlation model was designed to measure $[n \times (n - 1) / 2]$ assets in structured products. Thus, risk managers of CDOs needed to estimate and manage 7,750 correlations (i.e., $125 \times 124 / 2$).

CDOs are separated into several tranches based on the degree of default risk. The riskiest tranche is called the equity tranche, and investors in this tranche are typically exposed to the first 3% of defaults. The next tranche is referred to as the mezzanine tranche where investors are typically exposed to the next 4% of defaults (above 3% to 7%). The copula correlation model was trusted to monitor the default correlations across different tranches. A number of large hedge funds were short the CDO equity tranche and long the CDO mezzanine tranche. In other words, potential losses from the equity tranche were thought to be hedged with gains from the mezzanine tranche. Unfortunately, huge losses lead to bankruptcy filings by several large hedge funds because the correlation properties across tranches were not correctly understood.

Correlation played a key role in the bond market for U.S. automobile makers and the CDO market just prior to the financial crisis. A junk bond rating typically leads to major price decreases as pension funds, insurance companies, and other large financial institutions sell their holdings and are not willing to hold non-investment grade bonds. Bonds within specific credit quality levels typically are more highly correlated. Bonds across credit quality levels typically have lower correlations.

Rating agencies downgraded General Motors and Ford to junk bond status in May of 2005. Following the change in bond ratings for Ford and General Motors, the equity tranche spread increased dramatically. This caused losses for hedge funds that were short the equity tranche (i.e., sold credit protection). At the same time, the correlations decreased for CDOs of investment grade bonds. The lower correlations in the mezzanine tranche led to losses for hedge funds that were long the mezzanine tranche.

The CDO market, comprised primarily of residential mortgages, increased from $64 billion in 2003 to $455 billion in 2006. Liberal lending policies combined with overvalued real estate created the perfect storm in the subprime mortgage market. Housing prices became stagnate in 2006 leading to the first string of mortgage defaults. In 2007, the real estate market collapsed as the number of mortgage defaults increased. The CDO market, which was linked closely to mortgages, collapsed as well. This led to a global crisis as stock and commodities markets collapsed around the world. As a result, correlations in stock markets increased as the U.S. stock market crashed. Default correlations in CDO markets and bond markets also increased as the value of real estate and financial stability of individuals and institutions was highly questionable.

The CDO equity tranche spread typically decreases when default correlations increase. A lower equity tranche spread typically leads to an increase in value of the equity tranche. Unfortunately, the probability of default in the subprime market increased so dramatically in 2007 that it lowered the value of all CDO tranches. Thus, the default correlations across CDO tranches increased. The default rates also increased dramatically for all residential mortgages. Even the highest quality CDO tranches with AAA ratings lost 20% of their value as they were no longer protected from the lower tranches. The losses were even greater for many institutions with excess leverage in the senior tranches that were thought to be safe havens. The leverage in the CDO market caused risk exposures for investors to be 10 to 20 times higher than the investments.

In addition to the rapid growth in the CDO market, the credit default swap (CDS) market grew from $8 trillion to $60 trillion during the 2004 to 2007 time period. As mentioned earlier, CDSs are used to hedge default risk. CDSs are similar to insurance products as the risk exposure in the debt market is transferred to a broader market. The CDS seller must be financially stable enough to protect against losses. The recent financial crisis revealed that American International Group (AIG) was overextended, selling $500 billion in CDSs with little reinsurance. Also, Lehman Brothers had leverage 30.7 times greater than equity in September 2008 leading to its bankruptcy. However, the leverage was much higher considering the large number of derivatives transactions that were also held with 8,000 different counterparties.

Regulators are in the process of developing Basel III in response to the financial crisis. New standards for liquidity and leverage ratios for financial institutions are also being implemented. New correlation models are being developed and implemented such as the Gaussian copula, credit value adjustment (CVA) for correlations in derivatives transactions, and wrong-way risk (WWR) correlation. These new models hope to address correlated defaults in multi-asset portfolios.

The Role of Correlation Risk in Other Types of Risk

LO 6.5: Explain the role of correlation risk in market risk and credit risk.

LO 6.6: Relate correlation risk to systemic and concentration risk.

A major concern for risk managers is the relationship between correlation risk and other types of risk such as market, credit, systemic, and concentration risk. Examples of major factors contributing to market risk are interest rate risk, currency risk, equity price risk, and commodity risk. As discussed earlier, risk managers typically measure *market risk* in terms of VaR. Because the covariance matrix of assets is an important input of VaR, correlation risk is extremely important. Another important risk management tool used to quantify market risk is **expected shortfall** (ES). Expected shortfall measures the impact of market risk for extreme events or tail risk. Given that correlation risk refers to the risk that the correlation between assets changes over time, the concern is how the covariance matrix used for calculating VaR or ES changes over time due to changes in market risk.

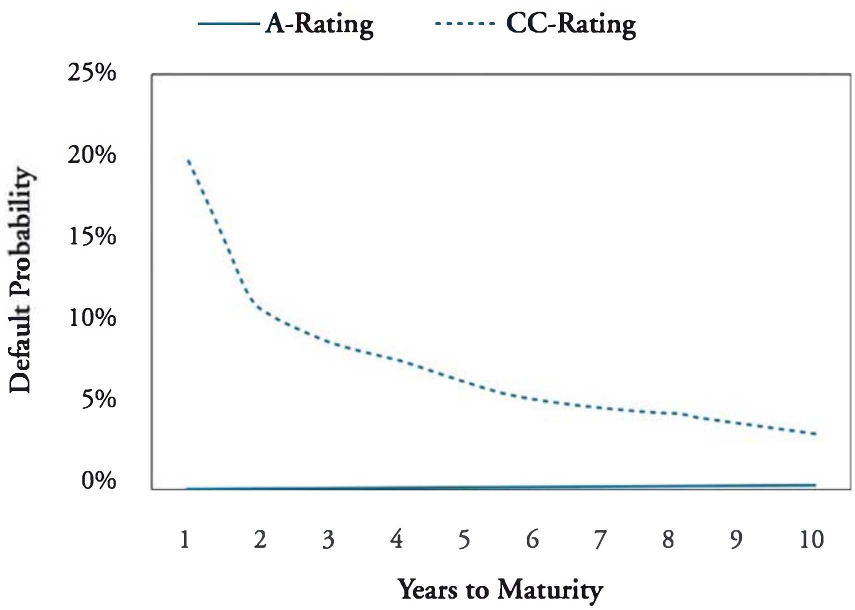
Risk managers are also concerned with measuring *credit risk* with respect to migration risk and default risk. **Migration risk** is the risk that the quality of a debtor decreases following the lowering of quality ratings. Lower debt quality ratings imply higher default probabilities. When a debt rating decreases, the present value of the underlying asset decreases, which creates a paper loss. As discussed previously, correlation risk between a reference asset and counterparty (CDS seller) is an important concern for investors. A higher correlation increases the probability of total loss of an investment.

Financial institutions such as mortgage companies and banks provide a variety of loans to individuals and entities. **Default correlation** is of critical importance to financial institutions in quantifying the degree that defaults occur at the same time. A lower default correlation is associated with greater diversification of credit risk. Empirical studies have examined historical default correlations across and within industries. Most default correlations across industries are positive with the exception of the energy sector. The energy sector has little or no correlation with other sectors and is, therefore, more resistant to recessions.

Historical data suggests that default correlations are higher within industries. This finding implies that systematic factors impacting the overall market and credit risk have much more influence in defaults than individual or company-specific factors. For example, if Chrysler defaults, then Ford and General Motors are more likely to default and have losses rather than benefit from increased market share. Thus, commercial banks limit exposures within a specific industry. The key point is that creditors benefit by diversifying exposure across industries to lower the default correlations of debtors.

Risk managers can also use a term structure of defaults to analyze credit risk. Rating agencies such as Moody's provide default probabilities based on bond ratings and time to maturity as illustrated in Figure 8.

Figure 8: Default Term Structure for A- and CC-Rated Bonds



Notice in Figure 8 that the default term structure increases slightly with time to maturity for most investment grade bonds (solid line). This is expected because bonds are more likely to default as many market or company factors can change over a longer time period. Conversely, for non-investment grade bonds (dashed line), the probability of default is higher in the immediate time horizon. If the company survives the near-term distressed situation, the probability of default decreases over time.

Lehman Brothers filed for bankruptcy in September of 2008. This bankruptcy event was an important signal of the severity of the financial crisis and the level of systemic risk. **Systemic risk** refers to the potential risk of a collapse of the entire financial system. It is interesting to examine the extent of the stock market crash that began in October 2007. From October 2007 to March 2009, the Dow Jones Industrial Average fell over 50% and only 11 stocks increased in the entire Standard & Poor's 500 Index (S&P 500). The decrease in value of 489 stocks in the S&P 500 during this time period reflected how a systemic financial crisis impacts the economy with decreasing disposable income for individuals, decreasing GDP, and increasing unemployment.

The sectors represented in the 11 increasing stocks were consumer stables (Family Dollar, Ross Stores, and Walmart), educational (Apollo Group and DeVry Inc.), pharmaceuticals (Edward Lifesciences and Gilead Pharmaceuticals), agricultural (CF Industries), entertainment (Netflix), energy (Southwestern Energy), and automotive (AutoZone). The consumer staples and pharmaceutical sector are often recession resistant as individuals continue to need basic necessities such as food, household supplies, and medications. The educational sector is also resilient as more unemployed workers go back to school for education and career changes.

Studies examined the relationship between the correlations of stocks in the U.S. stock market and the overall market during the 2007 crisis. From August of 2008 to March of 2009, there was a freefall in the U.S. equity market. During this same time period, correlations of stocks with each other increased dramatically from a pre-crisis average correlation level of 27% to over 50%. Thus, when diversification was needed most during the financial crisis, almost all stocks become more highly correlated and, therefore, less diversified. The severity of correlation risk is even greater during a systemic crisis when one considers the higher correlations of U.S. equities with bonds and international equities.

Concentration risk is the financial loss that arises from the exposure to multiple counterparties for a specific group. Concentration risk is measured by the **concentration ratio.** A lower (higher) concentration ratio reflects that the creditor has more (less) diversified default risk. For example, the concentration ratio for a creditor with 100 loans of equal size to different entities is 0.01 (= 1 / 100). If a creditor has one loan to one entity, the concentration ratio for the creditor is 1.0 (= 1 / 1). Loans can be further analyzed by grouping them into different sectors. If loan defaults are more highly correlated within sectors, when one loan defaults within a specific sector, it is more likely that another loan within the same sector will also default. The following examples illustrate the relationship between concentration risk and correlation risk.

Example: Concentration ratio for bank X and one loan to company A

Suppose commercial bank X makes a $5 million loan to company A, which has a 5% default probability. What is the concentration ratio and expected loss (EL) for commercial bank X under the worst case scenario? Assume loss given default (LGD) is 100%.

Answer:

Commercial bank X has a concentration ratio of 1.0 because there is only one loan. The worst case scenario is that company A defaults resulting in a total loss of loan value. Given that there is a 5% probability that company A defaults, EL for commercial bank X is $250,000 (= 0.05 × 5,000,000).

Example: Concentration ratio for bank Y and two loans to companies A and B

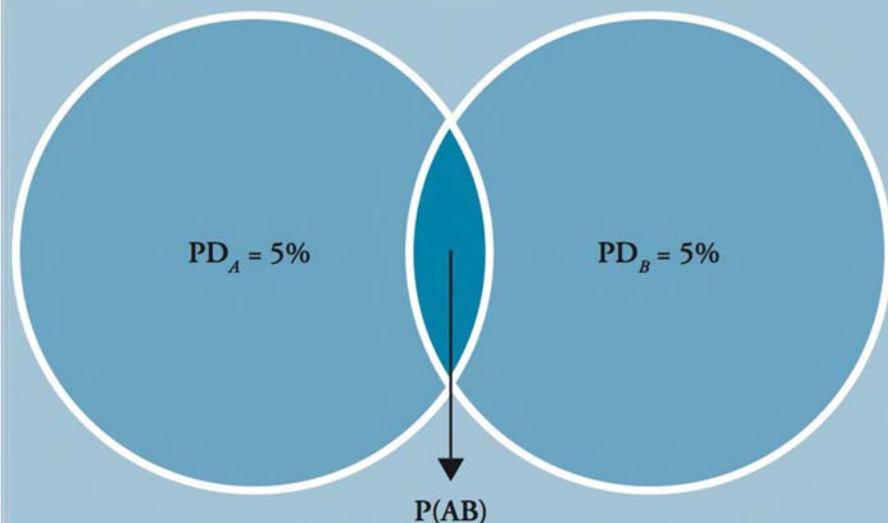
Suppose commercial bank Y makes a $2,500,000 loan to company A and a $2,500,000 loan to company B. Assuming companies A and B each have a 5% default probability, what is the concentration ratio and expected loss (EL) for commercial bank Y under the worst case scenario? Assume default correlation between companies is 1.0 and loss given default (LGD) is 100%.

Answer:

Commercial bank Y has a concentration ratio of 0.5 (calculated as 1 / 2). The expected loss for commercial bank Y depends on the default correlation of companies A and B. Note that changes in the concentration ratio are directly related to changes in the default correlations. A decrease in the concentration ratio results in a decrease in the default correlation. The default of companies A and B can be expressed as two binomial events with a value of 1 in default and 0 if not in default.

Figure 9 illustrates the joint probability that both companies A and B are in default, P(AB).

Figure 9: Joint Probability of Default for Companies A and B



The following equation computes the joint probability that both companies A and B are in default at the same time:

$$P(AB) = \rho_{AB}\sqrt{PD_A(1-PD_A)\times PD_B(1-PD_B)} + PD_A \times PD_B$$

where:
ρ_{AB} = default correlation coefficient for *A* and *B*
$\sqrt{PD_A(1-PD_A)}$ = standard deviation of the binomial event *A*

The default probability of company A is 5%. Thus, the standard deviation for company A is:

$$\sqrt{0.05(1-0.05)} = 0.2179$$

Company B also has a default probability of 5% and, therefore, will also have a standard deviation of 0.2179. We can now calculate the expected loss under the worst case scenario where both companies A and B are in default. Assuming that the default correlation between A and B is 1.0, the joint probability of default is:

$$P(AB) = 1.0\sqrt{0.05(0.95) \times 0.05(0.95)} + 0.05 \times 0.05$$
$$= 1.0\sqrt{0.00226} + 0.0025 = 0.05$$

If the default correlation between companies A and B is 1.0, the expected loss for commercial bank Y is \$250,000 (0.05 × \$5,000,000). Notice that when the default correlation is 1.0, this is the same as making a \$5 million loan to one company.

Now, let's assume that the default correlation between companies A and B is 0.5. What is the expected loss for commercial bank Y? The joint probability of default for A and B, assuming a default correlation of 0.5, is:

$$P(AB) = 0.5\sqrt{0.00226} + 0.0025 = 0.02625$$

Thus, the expected loss for the worst case scenario for commercial bank Y is:

EL = 0.02625 × \$5,000,000 = \$131,250

If we assume the default correlation coefficient is 0, the joint probability of default is 0.0025 and the expected loss for commercial bank Y is only \$12,500. Thus, a lower default correlation results in a lower expected loss under the worst case scenario.

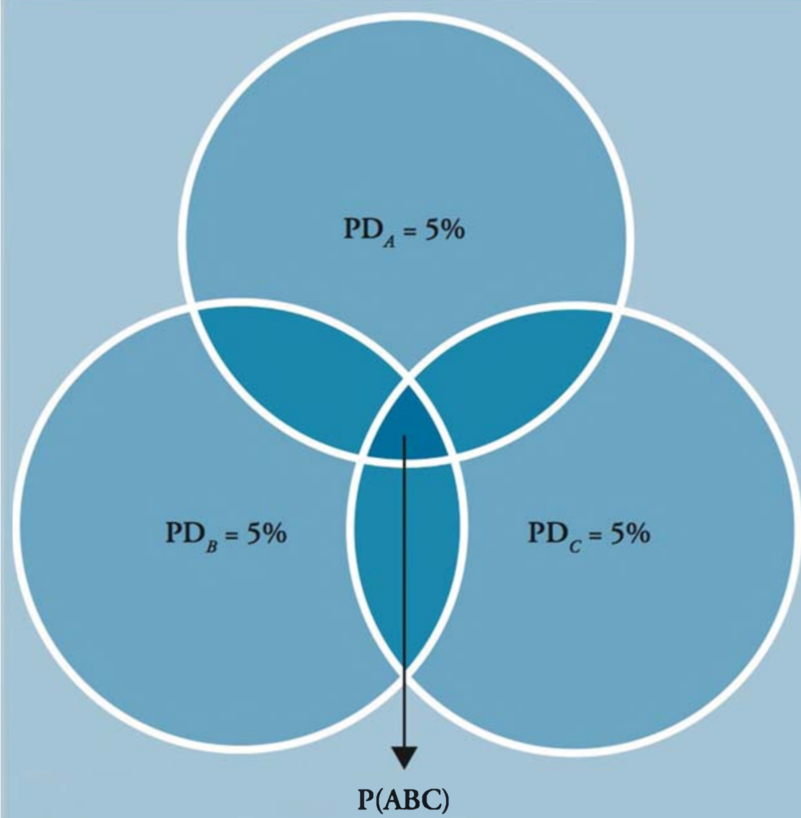
Example: Concentration ratio for bank Z and three loans to companies A, B, and C

Now we can examine what happens to the joint probability of default (i.e., the worst case scenario) if the concentration ratio is reduced further. Suppose that commercial bank Z makes three \$1,666,667 loans to companies A, B, and C. Also assume the default probability for each company is 5%. What is the concentration ratio for commercial bank Z, and how will the joint probability be impacted?

Answer:

Commercial bank Z has a concentration ratio of 0.333 (calculated as 1 / 3). Figure 10 illustrates the joint probability of all three loans defaulting at the same time, P(ABC) (i.e., the small area in the center of Figure 10 where all three default probabilities overlap). Note that as the concentration ratio decreases, the joint probability also decreases.

Figure 10: Joint Probability of Default for Companies A, B, and C



Professor Note: The assigned reading did not cover the calculation of the joint probability for three binomial events occurring. The focus here is on understanding that as the concentration ratio decreases, the probability of the worst case scenario also decreases. Both a lower concentration ratio and lower correlation coefficient reduce the joint probability of default.

Key Concepts

LO 6.1

Correlation risk measures the risk of financial loss resulting from adverse changes in correlations between financial or nonfinancial assets. For example, financial correlation risk can result from the negative correlation between interest rates and commodity prices. For almost all correlation option strategies, a lower correlation results in a higher option price.

LO 6.2

In May of 2005, several large hedge funds had losses on both sides of a hedged position short the collateralized debt obligation (CDO) equity tranche spread and long the CDO mezzanine tranche. The decrease in default correlations in the mezzanine tranche led to losses in the mezzanine tranche.

American International Group (AIG) and Lehman Brothers were highly leveraged in credit default swaps (CDSs) during the recent financial crisis. Their financial troubles revealed the impact of increasing default correlations with tremendous leverage.

LO 6.3

A correlation swap is used to trade a fixed correlation between two assets with the realized correlation. The payoff for the investor buying the correlation swap is:

$$\text{notional amount} \times (\rho_{\text{realized}} - \rho_{\text{fixed}})$$

where:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j}$$

LO 6.4

Value at risk (VaR) for a portfolio measures the potential loss in value for a specific time period for a given confidence level:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x}$$

The VaR for a portfolio increases as the correlation between assets increase. The Basel Committee on Banking Supervision requires banks to hold capital for assets in the trading book of at least three times greater than 10-day VaR (i.e., VaR capital charge = 3 × 10-day VaR).

LO 6.5

The covariance matrix of assets is an important input for value at risk (VaR) and expected shortfall (ES). These risk management tools are sensitive to changes in correlation.

A lower default correlation is associated with greater diversification of credit risk. Creditors benefit by diversifying exposure across industries to lower the default correlations of debtors. The default term structure increases with time to maturity for most investment grade bonds. The probability of default is higher in the immediate time horizon for non-investment grade bonds.

LO 6.6

Systemic risk refers to the potential risk of a collapse of the entire financial system. The severity of correlation risk is even greater during a systemic crisis considering the higher correlations of U.S. equities with bonds and international equities.

Changes in the concentration risk, which is measured by the concentration ratio, are directly related to changes in default correlations. A lower concentration ratio and lower correlation coefficient both reduce the joint probability of default.

Concept Checkers

1. Suppose an individual buys a correlation swap with a fixed correlation of 0.2 and a notional value of $1 million for one year. The realized pairwise correlations of the daily log returns at maturity for three assets are $\rho_{2,1} = 0.7$, $\rho_{3,1} = 0.2$, and $\rho_{3,2} = 0.3$. What is the correlation swap buyer's payoff at maturity?
   A. $100,000.
   B. $200,000.
   C. $300,000.
   D. $400,000.

2. Suppose a financial institution has a two-asset portfolio with $7 million in asset A and $5 million in asset B. The portfolio correlation is 0.4, and the daily standard deviation of returns for asset A and B are 2% and 1%, respectively. What is the 10-day value at risk (VaR) of this portfolio at a 99% confidence level ($\alpha = 2.33$)?
   A. $1.226 million.
   B. $1.670 million.
   C. $2.810 million.
   D. $3.243 million.

3. In May of 2005, several large hedge funds had speculative positions in the collateralized debt obligations (CDOs) tranches. These hedge funds were forced into bankruptcy due to the lack of understanding of correlations across tranches. Which of the following statements best describes the positions held by hedge funds at this time and the role of changing correlations? Hedge funds held a:
   A. long equity tranche and short mezzanine tranche when the correlations in both tranches decreased.
   B. short equity tranche and long mezzanine tranche when the correlations in both tranches increased.
   C. short senior tranche and long mezzanine tranche when the correlation in the mezzanine tranche increased.
   D. long mezzanine tranche and short equity tranche when the correlation in the mezzanine tranche decreased.

4. Suppose a creditor makes a $4 million loan to company X and a $4 million loan to company Y. Based on historical information of companies in this industry, companies X and Y each have a 7% default probability and a default correlation coefficient of 0.6. The expected loss for this creditor under the worst case scenario assuming loss given default is 100% is closest to:
   A. $280,150.
   B. $351,680.
   C. $439,600.
   D. $560,430.

5. The relationship of correlation risk to credit risk is an important area of concern for risk managers. Which of the following statements regarding default probabilities and default correlations is incorrect?
   A. Creditors benefit by diversifying exposure across industries to lower the default correlations of debtors.
   B. The default term structure increases with time to maturity for most investment grade bonds.
   C. The probability of default is higher in the long-term time horizon for non-investment grade bonds.
   D. Changes in the concentration ratio are directly related to changes in default correlations.

Concept Checker Answers

1. B First, calculate the realized correlation as follows:

$$\rho_{\text{realized}} = \frac{2}{3^2 - 3} \times (0.7 + 0.2 + 0.3) = 0.4$$

The payoff for the correlation buyer is then calculated as:

$\$1{,}000{,}000 \times (0.4 - 0.2) = \$200{,}000$

2. A The first step in solving for the 10-day VaR requires calculating the covariance matrix.

$$\begin{aligned} \text{cov}_{11} &= \sigma_1^2 = 0.02^2 = 0.0004 \\ \text{cov}_{22} &= \sigma_2^2 = 0.01^2 = 0.0001 \\ \text{cov}_{12} &= \rho_{12} \times \sigma_1 \times \sigma_2 = 0.4 \times 0.02 \times 0.01 = 0.00008 \end{aligned}$$

Thus, the covariance matrix, C, can be represented as:

$$\begin{pmatrix} 0.0004 & 0.00008 \\ 0.00008 & 0.0001 \end{pmatrix}$$

Next, the standard deviation of the portfolio, σ_P, is determined as follows:

Step 1: Compute $\beta_h \times C$:

$$\begin{aligned} &\begin{bmatrix} 7 & 5 \end{bmatrix} \begin{pmatrix} 0.0004 & 0.00008 \\ 0.00008 & 0.0001 \end{pmatrix} \\ &= \begin{bmatrix} (7 \times 0.0004) + (5 \times 0.00008) & (7 \times 0.00008) + (5 \times 0.0001) \end{bmatrix} \\ &= \begin{bmatrix} 0.0032 & 0.00106 \end{bmatrix} \end{aligned}$$

Step 2: Compute $(\beta_h \times C) \times \beta_v$:

$$\begin{aligned} &\begin{bmatrix} 0.0032 & 0.00106 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \\ &= (0.0032 \times 7) + (0.00106 \times 5) = 0.0277 \end{aligned}$$

Step 3: Compute σ_P:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v} = \sqrt{0.0277} = 0.1664 \text{ or } 16.64\%$$

The 10-day portfolio VaR (in millions) at the 99% confidence level is then computed as:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x} = 0.1664 \times 2.33 \times \sqrt{10} = \$1.226 \text{ million}$$

3. D A number of large hedge funds were short on the CDO equity tranche and long on the CDO mezzanine tranche. Following the change in bond ratings for Ford and General Motors, the equity tranche spread increased dramatically. This caused losses on the short equity tranche position. At the same time, the correlation decreased for CDOs in the mezzanine tranche, which led to losses in the mezzanine tranche.

4. **B** The worst case scenario is the joint probability that both loans default at the same time. The joint probability of default is computed as:

$$P(AB) = 0.6\sqrt{0.07(0.93)\times 0.07(0.93)} + 0.07\times 0.07$$
$$= 0.6\sqrt{0.00424} + 0.0049 = 0.04396$$

Thus, the expected loss for the worst case scenario for the creditor is:

EL = 0.04396 × \$8,000,000 = \$351,680

5. **C** The probability of default is higher in the *immediate* time horizon for non-investment grade bonds. The probability of default decreases over time if the company survives the near-term distressed situation.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Empirical Properties of Correlation: How Do Correlations Behave in the Real World?

Topic 7

Exam Focus

This topic examines how equity correlations and correlation volatility change during different economic states. It also discusses how to use a standard regression model to estimate the mean reversion rate and autocorrelation. For the exam, be able to calculate the mean reversion rate and be prepared to discuss and contrast the nature of correlations and correlation volatility for equity, bond, and default correlations. Also, be prepared to discuss the best fit distribution for these three types of correlation distributions.

Correlations During Different Economic States

LO 7.1: Describe how equity correlations and correlation volatilities behave throughout various economic states.

The recent financial crisis of 2007 provided new information on how correlation changes during different economic states. From 1972 to 2012, an empirical investigation on correlations of the 30 common stocks of the Dow Jones Industrial Average (Dow) was conducted. The correlation statistic was used to create a 30 × 30 correlation matrix for each stock in the Dow every month. This required 900 correlation calculations (30 × 30 = 900). There were 490 months in the study, so 441,000 monthly correlations were computed (900 × 490 = 441,000). However, the correlations of each stock with itself were eliminated from the study resulting in a total of 426,300 monthly correlations (441,000 – 30 × 490 = 426,300).

The average correlation values were compared for three states of the U.S. economy based on gross domestic product (GDP) growth rates. The state of the economy was defined as an expansionary period when GDP was greater than 3.5%, a normal economic period when GDP was between 0% and 3.5%, and a recession when there were two consecutive quarters of negative growth rates. Based on these definitions, from 1972 to 2012 there were six recessions, five expansionary periods, and five normal periods.

The average monthly correlation and correlation volatilities were then compared for each state of the economy. Correlation levels during a recession, normal period, and expansionary period were 37.0%, 32.7%, and 27.5%, respectively. Thus, as expected, correlations were highest during recessions when common stocks in equity markets tend to go down together. The low correlation levels during an expansionary period suggest common stock valuations are determined more on industry and company-specific information rather than macroeconomic factors.

The correlation volatilities during a recession, normal period, and expansionary period were 80.5%, 83.4%, and 71.2%, respectively. These results may seem a little surprising at first as one may expect volatilities are highest during a recession. However, there is perhaps slightly more uncertainty in a normal economy regarding the overall direction of the stock market. In other words, investors expect stocks to go down during a recession and up during an expansionary period, but they are less certain of direction during normal times, which results in higher correlation volatility.

Professor Note: The main lesson from this portion of the study is that risk managers should be cognizant of high correlation and correlation volatility levels during recessions and times of extreme economic distress when calibrating risk management models.

Mean Reversion and Autocorrelation

LO 7.2: Calculate a mean reversion rate using standard regression and calculate the corresponding autocorrelation.

Mean reversion implies that over time, variables or returns regress back to the mean or average return. Empirical studies reveal evidence that bond values, interest rates, credit spreads, stock returns, volatility, and other variables are mean reverting. For example, during a recession, demand for capital is low. Therefore, interest rates are lowered to encourage investment in the economy. Then, as the economy picks up, demand for capital increases and, at some point, interest rates will rise. If interest rates are too high, demand for capital decreases and interest rates decrease and approach the long-run average. The level of interest rates is also a function of monetary and fiscal policy and not just supply and demand levels of capital.

Mean reversion is statistically defined as a negative relationship between the change in a variable over time, $S_t - S_{t-1}$, and the variable in the previous period, S_{t-1}:

$$\frac{\partial(S_t - S_{t-1})}{\partial S_{t-1}}$$

In this equation, S_t is the value of the variable at time period *t*, S_{t-1} is the value of the variable in the previous period, and ∂ is a partial derivative coefficient. Mean reversion exists when S_{t-1} increases (decreases) by a small amount causing $S_t - S_{t-1}$ to decrease (increase) by a small amount. For example, if S_{t-1} increases and is high at time period $t - 1$, then mean reversion causes the next value at S_t to reverse and decrease toward the long-run average or mean value. The **mean reversion rate** is the degree of the attraction back to the mean and is also referred to as the speed or gravity of mean reversion. The mean reversion rate, *a*, is expressed as follows:

$$S_t - S_{t-1} = a(\mu - S_{t-1})\Delta t + \sigma_S \varepsilon \sqrt{\Delta t}$$

If we are only concerned with measuring mean reversion, we can ignore the last term, $\sigma_S \varepsilon \sqrt{\Delta t}$, which is the stochastic part of the equation requiring random samples from a distribution over time. By ignoring the last term and assuming $\Delta t = 1$, the mean reversion rate equation simplifies to:

$$S_t - S_{t-1} = a(\mu - S_{t-1})$$

Example: Calculating mean reversion

Suppose mean reversion exists for a variable with a value of 50 at time period $t - 1$. The long-run mean value, μ, is 80. What are the expected changes in value of the variable over the next period, $S_t - S_{t-1}$, if the mean reversion rate, a, is 0, 0.5, or 1.0?

Answer:

If the mean reversion rate is 0, there is no mean reversion and there is no expected change. If the mean reversion rate is 0.5, there is a 50% mean reversion and the expected change is 15 [i.e., 0.5 × (80 – 50)]. If the mean reversion rate is 1.0, there is 100% mean reversion and the expected change is 30 [i.e., 1.0 × (80 – 50)]. Thus, a stronger or faster mean reversion is expected with a higher mean reversion rate.

Standard regression analysis is one method used to estimate the mean reversion rate, *a*. We can think of the mean reversion rate equation in terms of a standard regression equation (i.e., $Y = \alpha + \beta X$) by applying the distributive property to reformulate the right side of the equation:

$$S_t - S_{t-1} = a\mu - aS_{t-1}$$

Thinking of this equation in terms of a standard regression implies the following terms in the regression equation:

$$S_t - S_{t-1} = Y;\ a\mu = \alpha;\ \text{and} - aS_{t-1} = \beta X$$

A regression is run where $S_t - S_{t-1}$ (i.e., the *Y* variable) is regressed with respect to S_{t-1} (i.e., the *X* variable). Thus, the β coefficient of the regression is equal to the negative of the mean reversion rate, *a*.

From the 1972 to 2012 study, the data resulted in the following regression equation:

$$Y = 0.27 - 0.78X$$

The beta coefficient of –0.78 implies a mean reversion rate of 78%. This is a relatively high mean reversion rate. Thus, if there is a large decrease (increase) from the mean correlation for one month, the following month is expected to have a large increase (decrease) in correlation.

Example: Calculating expected correlation

Suppose that in October 2012, the average monthly correlation for all Dow stocks was 30% and the long-run correlation mean of Dow stocks was 35%. A risk manager runs a regression, and the regression output estimates the following regression relationship: Y = 0.273 – 0.78X. What is the expected correlation for November 2012 given the mean reversion rate estimated in the regression analysis? (Solve for S_t in the mean reversion rate equation.)

Answer:

There is a 5% difference from the October 2012 and long-run mean correlation (35% – 30% = 5%). The β coefficient in the regression relationship implies a mean reversion rate of 78%. The November 2012 correlation is expected to revert 78% of the difference back toward the mean. Thus, the expected correlation for November 2012 is 33.9%:

$$S_t = a(\mu - S_{t-1}) + S_{t-1}$$

$$S_t = 0.78(35\% - 30\%) + 0.3 = 0.339$$

Autocorrelation measures the degree that a current variable value is correlated to past values. Autocorrelation is often calculated using an *autoregressive conditional heteroskedasticity (ARCH) model* or a *generalized autoregressive conditional heteroskedasticity (GARCH) model*. An alternative approach to measuring autocorrelation is running a regression equation. In fact, autocorrelation has the exact opposite properties of mean reversion.

Mean reversion measures the tendency to pull away from the current value back to the long-run mean. Autocorrelation instead measures the persistence to pull toward more recent historical values. The mean reversion rate in the previous example was 78% for Dow stocks. Thus, the autocorrelation for a one-period lag is 22% for the same sample. The sum of the mean reversion rate and the one-period autocorrelation rate will always equal one (i.e., 78% + 22% = 100%).

Autocorrelation for a one-period lag is statistically defined as:

$$AC(\rho_t, \rho_{t-1}) = \frac{\text{cov}(\rho_t, \rho_{t-1})}{\sigma(\rho_t) \times \sigma(\rho_{t-1})}$$

The term $AC(\rho_t, \rho_{t-1})$ represents the autocorrelation of the correlation from time period t and the correlation from time period $t - 1$. For this example, the ρ_t term can represent the correlation matrix for Dow stocks on day t, and the ρ_{t-1} term can represent the correlation matrix for Dow stocks on day $t - 1$. The covariance between the correlation measures, $\text{cov}(\rho_t, \rho_{t-1})$, is calculated the same way covariance is calculated for equity returns.

This autocorrelation equation was used to calculate the one-period lag autocorrelation of Dow stocks for the 1972 to 2012 time period, and the result was 22%, which is identical to subtracting the mean reversion rate from one. The study also used this equation to test autocorrelations for 1- to 10-day lag periods for Dow stocks. The highest autocorrelation of 26% was found using a two-day lag, which compares the time period t correlation with the $t - 2$ correlation (two months prior). The autocorrelation for longer lags decreased gradually to approximately 10% using a 10-day lag. It is common for autocorrelations to decay with longer time period lags.

Professor Note: The autocorrelation equation is exactly the same as the correlation coefficient. Correlation values for time period t *and* t – 1 *are used to determine the autocorrelation between the two correlations.*

Best-Fit Distributions for Correlations

LO 7.3: Identify the best-fit distribution for equity, bond, and default correlations.

Seventy-seven percent of the correlations between stocks listed on the Dow from 1972 to 2012 were positive. Three distribution fitting tests were used to determine the best fit for equity correlations. Based on the results of the Kolmogorov-Smirnov, Anderson-Darling, and chi-squared distribution fitting tests, the **Johnson SB distribution** (which has two shape parameters, one location parameter, and one scale parameter) provided the best fit for equity correlations. The Johnson SB distribution best fit was also robust with respect to testing different economic states for the time period in question. The normal, lognormal, and beta distributions provided a poor fit for equity correlations.

There were three mild recessions and three severe recessions from 1972 to 2012. The time periods for the mild recessions occurred in 1980, 1990 to 1991, and 2001. More severe recessions occurred from 1973 to 1974 and from 1981 to 1982. Both of these severe recessions were caused by huge increases in oil prices. The most severe recession for this time period occurred from 2007 to 2009 following the global financial crisis. The percentage change in correlation volatility prior to a recession was negative in every case except for the 1990 to 1991 recession. This is consistent with the findings discussed earlier where correlation volatility is low during expansionary periods that often occur prior to a recession.

An empirical investigation of 7,645 bond correlations found average correlations for bonds of 42%. Correlation volatility for bond correlations was 64%. Bond correlations were also found to exhibit properties of mean reversion, but the mean reversion rate was only 26%. The best fit distribution for bond correlations was found to be the **generalized extreme value** (GEV) **distribution.** However, the normal distribution is also a good fit for bond correlations.

A study of 4,655 default probability correlations revealed an average default correlation of 30%. Correlation volatility for default probability correlations was 88%. The mean reversion rate for default probability correlations was 30%, which is closer to the 26% for bond correlations. However, the default probability correlation distribution was similar to equity distributions in that the Johnson SB distribution is the best fit for both distributions. Figure 1 summarizes the findings of the empirical correlation analysis.

Figure 1: Empirical Findings for Equity, Bond, and Default Correlations

Correlation Type	*Average Correlation*	*Correlation Volatility*	*Reversion Rate*	*Best Fit Distribution*
Equity	35%	80%	78%	Johnson SB
Bond	42%	64%	26%	Generalized Extreme Value
Default Probability	30%	88%	30%	Johnson SB

Key Concepts

LO 7.1

Risk managers should be cognizant that historical correlation levels for common stocks in the Dow are highest during recessions. Correlation volatility for Dow stocks is high during recessions but highest during normal economic periods.

LO 7.2

When a regression is run where $S_t - S_{t-1}$ (the Y variable) is regressed with respect to S_{t-1} (the X variable), the β coefficient of the regression is equal to the negative mean reversion rate, a.

Equity correlations show high mean reversion rates (78%) and low autocorrelations (22%). These two rates must sum to 100%. Bond correlations and default probability correlations show much lower mean reversion rates and higher autocorrelation rates.

LO 7.3

Equity correlation distributions and default probability correlation distributions are best fit with the Johnson SB distribution. Bond correlation distributions are best fit with the generalized extreme value distribution, but the normal distribution is also a good fit.

Concept Checkers

1. Suppose a risk manager examines the correlations and correlation volatility of stocks in the Dow Jones Industrial Average (Dow) for the period beginning in 1972 and ending in 2012. Expansionary periods are defined as periods where the U.S. gross domestic product (GDP) growth rate is greater than 3.5%, periods are normal when the GDP growth rates are between 0 and 3.5%, and recessions are periods with two consecutive negative GDP growth rates. Which of the following statements characterizes correlation and correlation volatilities for this sample? The risk manager will most likely find that:
   A. correlations and correlation volatility are highest for recessions.
   B. correlations and correlation volatility are highest for expansionary periods.
   C. correlations are highest for normal periods, and correlation volatility is highest for recessions.
   D. correlations are highest for recessions, and correlation volatility is highest for normal periods.

2. Suppose mean reversion exists for a variable with a value of 30 at time period $t - 1$. Assume that the long-run mean value for this variable is 40 and ignore the stochastic term included in most regressions of financial data. What is the expected change in value of the variable for the next period if the mean reversion rate is 0.4?
   A. –10.
   B. –4.
   C. 4.
   D. 10.

3. A risk manager uses the past 480 months of correlation data from the Dow Jones Industrial Average (Dow) to estimate the long-run mean correlation of common stocks and the mean reversion rate. Based on historical data, the long-run mean correlation of Dow stocks was 32%, and the regression output estimates the following regression relationship: $Y = 0.24 - 0.75X$. Suppose that in April 2014, the average monthly correlation for all Dow stocks was 36%. What is the expected correlation for May 2014 assuming the mean reversion rate estimated in the regression analysis?
   A. 32%.
   B. 33%.
   C. 35%.
   D. 37%.

4. A risk manager uses the past 480 months of correlation data from the Dow Jones Industrial Average (Dow) to estimate the long-run mean correlation of common stocks and the mean reversion rate. Based on this historical data, the long-run mean correlation of Dow stocks was 34%, and the regression output estimates the following regression relationship: $Y = 0.262 - 0.77X$. Suppose that in April 2014, the average monthly correlation for all Dow stocks was 33%. What is the estimated one-period autocorrelation for this time period based on the mean reversion rate estimated in the regression analysis?
   A. 23%.
   B. 26%.
   C. 30%.
   D. 33%.

5. In estimating correlation matrices, risk managers often assume an underlying distribution for the correlations. Which of the following statements most accurately describes the best fit distributions for equity correlation distributions, bond correlation distributions, and default probability correlation distributions? The best fit distribution for the equity, bond, and default probability correlation distributions, respectively are:
   A. lognormal, generalized extreme value, and normal.
   B. Johnson SB, generalized extreme value, and Johnson SB.
   C. beta, normal, and beta.
   D. Johnson SB, normal, and beta.

Concept Checker Answers

1. D Findings of an empirical study of monthly correlations of Dow stocks from 1972 to 2012 revealed the highest correlation levels for recessions and the highest correlation volatilities for normal periods. The correlation volatilities during a recession and normal period were 80.5% and 83.4%, respectively.

2. C The mean reversion rate, *a*, indicates the speed of the change or reversion back to the mean. If the mean reversion rate is 0.4 and the difference between the last variable and long-run mean is 10 (= 40 – 30), the expected change for the next period is 4 (i.e., 0.4 × 10 = 4).

3. B There is a –4% difference from the long-run mean correlation and April 2014 correlation (32% – 36% = –4%). The inverse of the β coefficient in the regression relationship implies a mean reversion rate of 75%. Thus, the expected correlation for May 2014 is 33.0%:

$$S_t = a(\mu - S_{t-1}) + S_{t-1}$$

$$S_t = 0.75(32\% - 36\%) + 0.36 = 0.33$$

4. A The autocorrelation for a one-period lag is 23% for the same sample. The sum of the mean reversion rate (77% given the beta coefficient of –0.77) and the one-period autocorrelation rate will always equal 100%.

5. B Equity correlation distributions and default probability correlation distributions are best fit with the Johnson SB distribution. Bond correlation distributions are best fit with the generalized extreme value distribution.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Statistical Correlation Models—Can We Apply Them to Finance?

Topic 8

Exam Focus

This topic addresses the limitations of financial models and popular statistical correlation measures such as the Pearson correlation measure, the Spearman rank correlation, and the Kendall τ. For the exam, understand that the major limitation of the Pearson correlation coefficient is that most financial variables have nonlinear relationships. Also, be able to discuss the limitations of ordinal correlation measures, such as Spearman's rank correlation and Kendall's τ. These nonparametric measures do not require assumptions about the underlying joint distributions of variables; however, applications of ordinal risk measures are limited to ordinal variables where only the rankings are important instead of actual numerical values.

Limitations of Financial Models

LO 8.1: Evaluate the limitations of financial modeling with respect to the model itself, calibration of the model, and the model's output.

Financial models are important tools to help individuals and institutions better understand the complexity of the financial world. Financial models always deal with uncertainty and are, therefore, only approximations of a very complex pricing system that is influenced by numerous dynamic factors. There are many different types of markets trading a variety of assets and financial products such as equities, bonds, structured products, derivatives, real estate, and exchange-traded funds. Data from multiple sources is then gathered to calibrate financial models.

Due to the complexity of the global financial system, it is important to recognize the limitations of financial models. Limitations arise in financial models as a result of inaccurate inputs, erroneous assumptions regarding asset variable distributions, and mathematical inconsistencies. Almost all financial models require *market valuations* as inputs. Unfortunately, these values are often determined by investors who do not always behave rationally. Therefore, asset values are sometimes random and may exhibit unexpected changes.

Financial models also require assumptions regarding the *underlying distribution* of the asset returns. Value at risk (VaR) models are used to estimate market risk, and these models often assume that asset returns follow a normal distribution. However, empirical studies actually find higher kurtosis in return distributions, which suggest a distribution with fatter tails than the normal distribution.

Another example of a shortcoming of financial models is illustrated with the Black-Scholes-Merton (BSM) option pricing model. The BSM option pricing model assumes strike prices have constant volatility. However, numerous empirical studies find higher volatility for out-of-the money options and a volatility skew in equity markets. Thus, option traders and risk managers often use a volatility smile (discussed in Topic 16) with higher volatilities for out-of-the money call and put options.

Financial models at times may fail to accurately measure risk due to *mathematical inconsistencies*. For example, regarding barrier options, when applying the BSM option pricing model to up-and-out calls and puts and down-and-out calls and puts, there are rare cases where the inputs make the model insensitive to changes in implied volatility and option maturity. This can occur when the knock-out strike price is equal to the strike price, and the interest rate equals the underlying asset return. Risk managers and traders need to be aware of the possibility of mathematical inconsistencies causing model risk that leads to incorrect pricing and the inability to properly hedge risk.

Limitations in the Calibration of Financial Models

Financial models calibrate parameter inputs to reflect current market values. These parameters are then used in financial models to estimate market values with limited or no pricing information. The choice of time period used to calibrate the parameter inputs for the model can have a big impact on the results. For example, during the 2007 to 2009 financial crisis, risk managers used volatility and correlation estimates from pre-crisis periods. This resulted in significantly underestimating the risk for value at risk (VaR), credit value at risk (CVaR), and collateralized debt obligation (CDO) models.

All financial models should be tested using scenarios of extreme economic conditions. This process is referred to as **stress testing**. For example, VaR estimates are calculated in the event of a systemic financial crisis or severe recession. In 2012, the Federal Reserve, under the guidelines of Basel III, required all financial institutions to use stress tests.

Limitations of Financial Model Outputs

Limitations of financial models became evident during the recent global financial crisis. Traders and risk managers used new copula correlation models to estimate values in collateralized debt obligation (CDO) models. The values of these structured products were linked to mortgages in a collapsing real estate market.

The copula correlation models failed for two reasons. First, the copula correlation models assumed a negative correlation between the equity and senior tranches of CDOs. However, during the crisis, the correlations for both tranches significantly increased causing losses for both. Second, the copula correlation models were calibrated using volatility and correlation estimates with data from time periods that had low risk, and correlations changed significantly during the crisis.

A major lesson learned from the global financial crisis is that copula models cannot be blindly trusted. There should always be an element of human judgment in assessing the risk associated with any financial model. This is especially true for extreme market conditions.

Statistical Correlation Measures

LO 8.2: Assess the Pearson correlation approach, Spearman's rank correlation, and Kendall's τ, and evaluate their limitations and usefulness in finance.

Pearson Correlation

The Pearson correlation coefficient is commonly used to measure the *linear relationship* between two variables. The Pearson correlation is defined by dividing covariance (cov_{XY}) by the product of the two assets' standard deviations ($\sigma_X \sigma_Y$).

$$\rho_{XY} = \frac{cov_{XY}}{\sigma_X \sigma_Y}$$

Covariance is a measure of how two assets move with each other over time. The Pearson correlation coefficient standardizes covariance by dividing it by the standard deviations of each asset. This is very convenient because the correlation coefficient is always between –1 and 1.

Covariance is calculated by finding the product of each asset's deviation from their respective mean return for each period. The products of the deviations for each period are then added together and divided by the number of observations less one for degrees of freedom.

$$cov_{XY} = \frac{\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y)}{n-1}$$

There is a second methodology that is used for calculating the Pearson correlation coefficient if the data is drawn from random processes with unknown outcomes (e.g., rolling a die). The following equation defines covariance with expectation values. If *E(X)* and *E(Y)* are the expected values of variables *X* and *Y*, respectively, then the expected product of deviations from these expected values is computed as follows:

$$E\{[X - E(X)][Y - E(Y)]\} \text{ or } E(XY) - E(X)E(Y)$$

When using random sets of data, the correlation coefficient can be rewritten as:

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2} \times \sqrt{E(Y^2) - (E(Y))^2}}$$

Because many financial variables have nonlinear relationships, the Pearson correlation coefficient is only an approximation of the nonlinear relationship between financial variables. Thus, when applying the Pearson correlation coefficient in financial models, risk managers and investors need to be aware of the following five limitations:

1. The Pearson correlation coefficient measures the linear relationship between two variables, but financial relationships are often nonlinear.

2. A Pearson correlation of zero does not imply independence between the two variables. It simply means there is not a linear relationship between the variables. For example, the parabola relationship defined as $Y = X^2$ has a correlation coefficient of zero. There is, however, an obvious nonlinear relationship between variables Y and X.

3. When the joint distribution between variables is not elliptical, linear correlation measures do not have meaningful interpretations. Examples of common elliptical joint distributions are the multivariate normal distribution and the multivariate Student's *t*-distribution.

4. The Pearson correlation coefficient requires that the variance calculations of the variables X and Y are finite. In cases where kurtosis is very high, such as the Student's *t*-distribution, the variance could be infinite, so the Pearson correlation coefficient would be undefined.

5. The Pearson correlation coefficient is not meaningful if the data is transformed. For example, the correlation coefficient between two variables X and Y will be different than the correlation coefficient between ln(X) and ln(Y).

Spearman's Rank Correlation

Ordinal measures are based on the order of elements in data sets. Two examples of ordinal correlation measures are the Spearman rank correlation and the Kendall τ. The Spearman rank correlation is a nonparametric approach because no knowledge of the joint distribution of the variables is necessary. The calculation is based on the relationship of the ranked variables. The following equation defines the Spearman rank correlation coefficient where n is the number of observations for each variable, and d_i is the difference between the ranking for period i.

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$$

The Spearman rank correlation coefficient is determined in three steps:

Step 1: Order the set pairs of variables X and Y with respect to the set X.
Step 2: Determine the ranks of X_i and Y_i for each time period i.
Step 3: Calculate the difference of the variable rankings and square the difference.

Example: Spearman's rank correlation

Calculate the Spearman rank correlation for the returns of stocks *X* and *Y* provided in Figure 1.

Figure 1: Returns for Stocks *X* and *Y*

Year	*X*	*Y*
2010	25.0%	–20.0%
2011	60.0%	40.0%
2012	–20.0%	10.0%
2013	40.0%	20.0%
2014	–10.0%	30.0%
Average	19.0%	16.0%

Answer:

The calculations for determining the Spearman rank correlation coefficient are shown in Figure 2. The first step involves ranking the returns for stock *X* from lowest to highest in the second column. The first column denotes the respective year for each return. The returns for stock *Y* are then listed for each respective year. The fourth and fifth columns rank the returns for variables *X* and *Y*. The differences between the rankings for each year are listed in column six. Lastly, the sum of squared differences in rankings is determined in column 7.

Figure 2: Ranking Returns for Stocks *X* and *Y*

Year	*X*	*Y*	*X Rank*	*Y Rank*	d_i	d_i^2
2012	–20.0%	10.0%	1	2	-1	1
2014	–10.0%	30.0%	2	4	-2	4
2010	25.0%	–20.0%	3	1	2	4
2013	40.0%	20.0%	4	3	1	1
2011	60.0%	40.0%	5	5	0	0
					Sum =	10

The Spearman rank correlation coefficient can then be determined as 0.5:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 10}{5(25-1)} = 0.5$$

Kendall's τ

Kendall's τ is another ordinal correlation measure that is becoming more widely applied in financial models for ordinal variables such as credit ratings. Kendall's τ is also a nonparametric measure that does not require any assumptions regarding the joint probability distributions of variables. Both Spearman's rank correlation coefficient and Kendall's τ are similar to the Pearson correlation coefficient for ranked variables because perfectly correlated variables will have a coefficient of 1. The Kendall τ will be 1 if variable *Y* always increases with an increase in variable *X*. The numerical amount of the increase does not matter for two variables to be perfectly correlated. Therefore, for most cases, the Kendall τ and the Spearman rank correlation coefficients will be different.

The mathematical definition of Kendall's τ is provided as follows:

$$\tau = \frac{n_c - n_d}{n(n-1)/2}$$

In this equation, the number of concordant pairs is represented as n_c, and the number of discordant pairs is represented as n_d. A concordant pair of observations is when the rankings of two pairs are in agreement:

$$X_t < Y_t \text{ and } X_{t^*} < Y_{t^*} \text{ or } X_t > Y_t \text{ and } X_{t^*} > Y_{t^*} \text{ and } t \neq t^*$$

A discordant pair of observations is when the rankings of two pairs are not in agreement:

$$X_t < Y_t \text{ and } X_{t^*} > Y_{t^*} \text{ or } X_t > Y_t \text{ and } X_{t^*} < Y_{t^*} \text{ and } t \neq t^*$$

A pair of rankings is neither concordant nor discordant if the rankings are equal:

$$X_t = Y_t \text{ or } X_{t^*} = Y_{t^*}$$

The denominator in the Kendall τ equation computes the total number of pair combinations. For example, if there are six pairs of observations, there will be 15 combinations of pairs:

$$[n \times (n - 1)] / 2 = (6 \times 5) / 2 = 15$$

Example: Kendall's τ

Calculate the Kendall τ correlation coefficient for the stock returns of *X* and *Y* listed in Figure 3.

Figure 3: Ranked Returns for Stocks *X* and *Y*

Year	*X*	*Y*	*X Rank*	*Y Rank*
2012	–20.0%	10.0%	1	2
2014	–10.0%	30.0%	2	4
2010	25.0%	–20.0%	3	1
2013	40.0%	20.0%	4	3
2011	60.0%	40.0%	5	5

Answer:

Begin by comparing the rankings of *X* and *Y* stock returns in columns four and five of Figure 3. There are five pairs of observations, so there will be ten combinations. Figure 4 summarizes the pairs of rankings based on the stock returns for *X* and *Y*. There are two concordant pairs, four discordant pairs, and four pairs that are neither concordant nor discordant.

Figure 4: Categorizing Pairs of Stock *X* and *Y* Returns

Concordant Pairs	*Discordant Pairs*	*Neither*
{(1,2),(2,4)}	{(1,2),(3,1)}	{(1,2),(5,5)}
{(3,1),(4,3)}	{(1,2),(4,3)}	{(2,4),(5,5)}
	{(2,4),(3,1)}	{(3,1),(5,5)}
	{(2,4),(4,3)}	{(4,3),(5,5)}

Kendall's τ can then be determined as –0.2:

$$\tau = \frac{n_c - n_d}{n(n-1)/2} = \frac{2-4}{5(5-1)/2} = -0.2$$

Thus, the relationship between the stock returns of *X* and *Y* is slightly negative based on the Kendall τ correlation coefficient.

Limitations of Ordinal Risk Measures

Ordinal correlation measures based on ranking (i.e., Spearman's rank correlation and Kendall's τ) are implemented in copula correlation models to analyze the dependence of market prices and counterparty risk. Because ordinal numbers simply show the rank of observations, problems arise when ordinal measures are used for cardinal observations, which show the quantity, number, or value of observations.

Example: Impact of outliers on ordinal measures

Suppose we triple the returns of *X* in the previous example to show the impact of outliers. If outliers are important sources of information and financial variables are cardinal, what are the implications for ordinal correlation measures?

Answer:

Notice from Figure 5 that Spearman's rank correlation and Kendall's τ do not change with an increased probability of outliers. Thus, ordinal correlation measures are less sensitive to outliers, which are extremely important in VaR and stress test models during extreme economic conditions. Numerical values are not important for ordinal correlation measures where only the rankings matter. Thus, since outliers do not change the rankings, *ordinal measures underestimate risk by ignoring the impact of outliers.*

Figure 5: Ranking Returns with Outliers

Year	*3X*	*Y*	*3X Rank*	*Y Rank*	d_i	d_i^2
2012	–60.0%	10.0%	1	2	–1	1
2014	–30.0%	30.0%	2	4	–2	4
2010	75.0%	–20.0%	3	1	2	4
2013	120.0%	20.0%	4	3	1	1
2011	180.0%	40.0%	5	5	0	0
					Sum =	10

Another limitation of Kendall's τ occurs when there are a large number of pairs that are neither concordant nor discordant. In other words, the Kendall τ calculation can be distorted when there are only a few concordant and discordant pairs. For example, there were 4 out of 10 pairs that were neither concordant nor discordant in Figure 4. Thus, the Kendall τ calculation was based on only 6 out of 10, or 60%, of the observations.

Key Concepts

LO 8.1

Limitations of financial models arise due to inaccurate input values, erroneous underlying distribution assumptions, and mathematical inconsistencies.

Copula correlation models failed during the 2007–2009 financial crisis due to assumptions of a negative correlation between the equity and senior tranches in a collateralized debt obligation (CDO) structure and the calibration of correlation estimates with pre-crisis data.

LO 8.2

A major limitation of the Pearson correlation coefficient is that it measures linear relationships when most financial variables are nonlinear.

The Spearman rank correlation coefficient, where n is the number of observations for each variable and d_i is the difference between the ranking for period i, is computed as follows:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$$

The Kendall τ correlation coefficient, where the number of concordant pairs is represented as n_c and the number of discordant pairs is represented as n_d, is computed as follows:

$$\tau = \frac{n_c - n_d}{n(n-1)/2}$$

Spearman's rank and Kendall's τ correlation coefficients should not be used with cardinal financial variables because ordinal measures underestimate risk by ignoring the impact of outliers.

Concept Checkers

1. Kirk Rozenboom, FRM, uses the Black-Scholes-Merton (BSM) model to value options. Following the financial crisis of 2007–2009, he is more aware of the limitations of the BSM option pricing model. Which of the following statements best characterizes a major limitation of the BSM option pricing model?
   A. The BSM model assumes strike prices have nonconstant volatility.
   B. Option traders often use a volatility smile with lower volatilities for out-of-the-money call and put options when applying the BSM model.
   C. For up-and-out calls and puts, the BSM model is insensitive to changes in implied volatility when the knock-out strike price is equal to the strike price and the interest rate equals the underlying asset return.
   D. For down-and-out calls and puts, the BSM model is insensitive to changes in option maturity when the knock-out strike price is greater than the strike price and the interest rate is greater than the underlying asset return.

2. New copula correlation models were used by traders and risk managers during the 2007–2009 global financial crisis. This led to miscalculations in the underlying risk for structured products such as collateralized debt obligation (CDO) models. Which of the following statements least likely explains the failure of these new copula correlation models during the financial crisis?
   A. The copula correlation models assumed a negative correlation between the equity and senior tranches of CDOs.
   B. Correlations for equity tranches of CDOs increased during the financial crisis.
   C. The correlation copula models were calibrated with data from time periods that had low risk.
   D. Correlations for senior tranches of CDOs decreased during the financial crisis.

3. A risk manager gathers five years of historical returns to calculate the Spearman rank correlation coefficient for stocks X and Y. The stock returns for X and Y from 2010 to 2014 are as follows:

Year	*X*	*Y*
2010	5.0%	–10.0%
2011	50.0%	–5.0%
2012	–10.0%	20.0%
2013	–20.0%	40.0%
2014	30.0%	15.0%

   What is the Spearman rank correlation coefficient for the stock returns of X and Y?
   A. –0.7.
   B. –0.3.
   C. 0.3.
   D. 0.7.

4. A risk manager gathers five years of historical returns to calculate the Kendall τ correlation coefficient for stocks *X* and *Y*. The stock returns for *X* and *Y* from 2010 to 2014 are as follows:

Year	*X*	*Y*
2010	5.0%	–10.0%
2011	50.0%	–5.0%
2012	–10.0%	20.0%
2013	–20.0%	40.0%
2014	30.0%	15.0%

What is the Kendall τ correlation coefficient for the stock returns of *X* and *Y*?
A. –0.3.
B. –0.2.
C. 0.4.
D. 0.7.

5. A risk manager is using a copula correlation model to perform stress tests of financial risk during systemic economic crises. If the risk manager is concerned about extreme outliers, which of the following correlation coefficient measures should be used?
A. Kendall's τ correlation.
B. Ordinal correlation.
C. Pearson correlation.
D. Spearman's rank correlation.

Concept Checker Answers

1. **C** For up-and-out calls and puts and for down-and-out calls and puts, the BSM option pricing model is insensitive to changes in implied volatility when the knock-out strike price is equal to the strike price and the interest rate equals the underlying asset return. The BSM model assumes strike prices have a *constant* volatility, and option traders often use a volatility smile with *higher* volatilities for out-of-the-money call and put options.

2. **D** During the crisis, the correlations for both the equity and senior tranches of CDOs significantly increased causing losses in value for both.

3. **A** The following table illustrates the calculations used to determine the sum of squared ranking deviations:

Year	*X*	*Y*	*X Rank*	*Y Rank*	d_i	d_i^2
2013	–20.0%	40.0%	1	5	–4	16
2012	–10.0%	20.0%	2	4	–2	4
2010	5.0%	–10.0%	3	1	2	4
2014	30.0%	15.0%	4	3	1	1
2011	50.0%	–5.0%	5	2	3	9
					Sum =	34

Thus, the Spearman rank correlation coefficient is –0.7:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 34}{5(25-1)} = -0.7$$

4. B The following table provides the ranking of pairs with respect to *X*.

Year	*X*	*Y*	*X Rank*	*Y Rank*
2013	–20.0%	40.0%	1	5
2012	–10.0%	20.0%	2	4
2010	5.0%	–10.0%	3	1
2014	30.0%	15.0%	4	3
2011	50.0%	–5.0%	5	2

There are four concordant pairs and six discordant pairs shown as follows:

Concordant Pairs	*Discordant Pairs*
{(1,5),(2,4)}	{(1,5),(3,1)}
{(3,1),(4,3)}	{(1,5),(4,3)}
{(3,1),(5,2)}	{(1,5),(5,2)}
{(4,3),(5,2)}	{(2,4),(3,1)}
	{(2,4),(4,3)}
	{(2,4),(5,2)}

Thus, the Kendall τ correlation coefficient is –0.2:

$$\tau = \frac{n_c - n_d}{n(n-1)/2} = \frac{4-6}{5(5-1)/2} = -0.2$$

5. C The Pearson correlation coefficient is preferred to ordinal measures when outliers are a concern. Spearman's rank correlation and Kendall's τ are ordinal correlation coefficients that should not be used with cardinal financial variables because they underestimate risk by ignoring the impact of outliers.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Financial Correlation Modeling—Bottom-Up Approaches

Topic 9

Exam Focus

A copula is a joint multivariate distribution that describes how variables from marginal distributions come together. Copulas provide an alternative measure of dependence between random variables that is not subject to the same limitations as correlation in applications such as risk measurement. For the exam, understand how a correlation copula is created by mapping two or more unknown distributions to a known distribution that has well-defined properties. Also, know how the Gaussian copula is used to estimate joint probabilities of default for specific time periods and the default time for multiple assets. The material in this topic is relatively complex, so your focus here should be on gaining a general understanding of how a copula function is applied.

Copula Functions

LO 9.1: Explain the purpose of copula functions and the translation of the copula equation.

A **correlation copula** is created by converting two or more unknown distributions that may have unique shapes and mapping them to a known distribution with well-defined properties, such as the normal distribution. A copula creates a joint probability distribution between two or more variables while maintaining their individual marginal distributions. This is accomplished by mapping multiple distributions to a single multivariate distribution. For example, the following expression defines a **copula function**, *C*, that transforms an *n*-dimensional function on the interval [0,1] to a one-dimensional function.

$$C:[0,1]^n \rightarrow [0,1]$$

Suppose $G_i(u_i) \in [0,1]$ is a univariate, uniform distribution with $u_i = u_1,\ldots, u_n$, and $i \in N$ (i.e., i is an element of set N). A copula function, *C*, can then be defined as follows:

$$C\left[G_1(u_1),\ldots,G_n(u_n)\right] = F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n));\rho_F\right]$$

In this equation, $G_i(u_i)$ are the marginal distributions, F_n is the joint cumulative distribution function, F_1^{-1} is the inverse function of F_n, and ρ_F is the correlation matrix structure of the joint cumulative function F_n.

This copula function is translated as follows. Suppose there are n marginal distributions, $G_1(u_1)$ to $G_n(u_n)$. A copula function exists that maps the marginal distributions of $G_1(u_1)$ to $G_n(u_n)$ via $F_1^{-1}G_i(u_i)$ and allows for the joining of the separate values $F_1^{-1}G_i(u_i)$ to a single n-variate function $F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n))\right]$ that has a correlation matrix of ρ_F. Thus, this equation defines the process where unknown marginal distributions are mapped to a well-known distribution, such as the standard multivariate normal distribution.

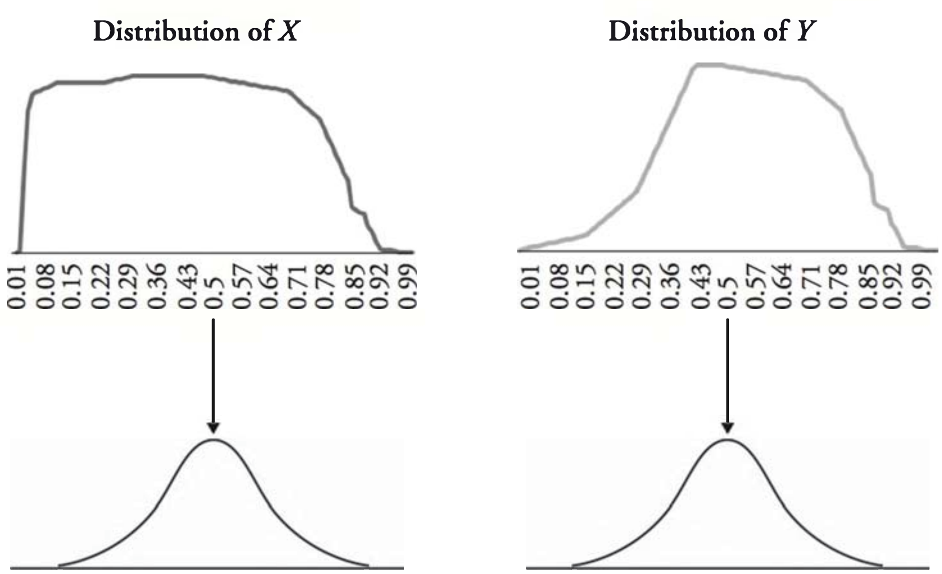
Gaussian Copula

LO 9.2: Describe the Gaussian copula and explain how to use it to derive the joint probability of default of two assets.

A **Gaussian copula** maps the marginal distribution of each variable to the standard normal distribution which, by definition, has a mean of zero and a standard deviation of one. The key property of a copula correlation model is preserving the original marginal distributions while defining a correlation between them. The mapping of each variable to the new distribution is done on percentile-to-percentile basis.

Figure 1 illustrates that the variables of two unknown distributions X and Y have unique marginal distributions. The observations of the unknown distributions are mapped to the standard normal distribution on a percentile-to-percentile basis to create a Gaussian copula.

Figure 1: Mapping a Gaussian Copula to the Standard Normal Distribution



For example, the 5th percentile observation for marginal distribution X is mapped to the 5th percentile point on the univariate standard normal distribution. When the 5th percentile is mapped, it will have a value of –1.645. This is repeated for each observation on a percentile-to-percentile basis. Likewise, every observation on the marginal distribution of *Y* is mapped to the corresponding percentile on the univariate standard normal distribution. The new joint distribution is now a multivariate standard normal distribution.

Now a correlation structure can be defined between the two variables *X* and *Y*. The unique marginal distributions of *X* and *Y* are not well-behaved structures, and therefore, it is difficult to define a relationship between the two variables. However, the standard normal distribution is a well-behaved distribution. Therefore, a copula is a way to indirectly define a correlation relationship between two variables when it is not possible to directly define a correlation.

A Gaussian copula, C_G, is defined in the following expression for an *n*-variate example. The joint standard multivariate normal distribution is denoted as M_n. The inverse of the univariate standard normal distribution is denoted as N_1^{-1}. The notation ρ_M denotes the n × n correlation matrix for the joint standard multivariate normal distribution M_n.

$$C_G\left[G_1(u_1),\ldots,G_n(u_n)\right]=M_n\left[N_1^{-1}(G_1(u_1)),\ldots,N_n^{-1}(G_n(u_n));\rho_M\right]$$

In finance, the Gaussian copula is a common approach for measuring default risk. The approach can be transformed to define the **Gaussian default time copula**, C_{GD}, in the following expression:

$$C_{GD}\left[Q_i(t),\ldots,Q_n(t)\right]=M_n\left[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t));\rho_M\right]$$

Marginal distributions of cumulative default probabilities, *Q(t)*, for assets *i* = 1 to *n* for fixed time periods *t* are mapped to the single *n*-variate standard normal distribution M_n with a correlation structure of ρ_M. The term $N_1^{-1}(Q_1(t))$ maps each individual cumulative default probability for asset *i* for time period *t* on a percentile-to-percentile basis to the standard normal distribution.

Example: Applying a Gaussian copula

Suppose a risk manager owns two non-investment grade assets. Figure 2 lists the default probabilities for the next five years for companies B and C that have B and C credit ratings, respectively. How can a Gaussian copula be constructed to estimate the joint default probability, Q, of these two companies in the next year, assuming a one-year Gaussian default correlation of 0.4?

Figure 2: Default Probabilities of Companies B and C

Time, t	*B Default Probability*	*C Default Probability*
1	0.065	0.238
2	0.081	0.152
3	0.072	0.113
4	0.064	0.092
5	0.059	0.072

Professor's Note: Non-investment grade companies have a higher probability of default in the near term during the company crisis state. If the company survives past the near term crisis, the probability of default will go down over time.

Answer:

In this example, there are only two companies, B and C. Thus, a bivariate standard normal distribution, M_2, with a default correlation coefficient of ρ can be applied. With two companies, only a single correlation coefficient is required, and not a correlation matrix of $\rho_{M.}$

$$C_{GD}\left[Q_B(t), Q_C(t)\right] = M_2\left[N^{-1}(Q_B(t)), N^{-1}(Q_C(t)); \rho\right]$$

Figure 3 illustrates the percentile-to-percentile mapping of cumulative default probabilities for each company to the standard normal distribution.

Figure 3: Mapping Cumulative Default Probabilities to Standard Normal Distribution

Time, t	*B Default Probability*	$Q_B(t)$	$N^{-1}(Q_B(t))$	*C Default Probability*	$Q_C(t)$	$N^{-1}(Q_C(t))$
1	0.065	0.065	–1.513	0.238	0.238	–0.712
2	0.081	0.146	–1.053	0.152	0.390	–0.279
3	0.072	0.218	–0.779	0.113	0.503	0.008
4	0.064	0.282	–0.577	0.092	0.595	0.241
5	0.059	0.341	–0.409	0.072	0.667	0.432

Columns 3 and 6 represent the cumulative default probabilities $Q_B(t)$ and $Q_C(t)$ for companies B and C, respectively. The values in columns 4 and 7 map the respective cumulative default probabilities, $Q_B(t)$ and $Q_C(t)$, to the standard normal distribution via $N^{-1}(Q(t))$. The values for the standard normal distribution are determined using the Microsoft Excel® function =NORMSINV(Q(t)) or the MATLAB® function =NORMINV(Q(t)). This process was illustrated graphically in Figure 1.

The joint probability of both Company B and Company C defaulting within one year is calculated as:

$$Q\left(t_B \leq 1 \cap t_C \leq 1\right) \equiv M\left(X_B \leq -1.513 \cap X_C \leq -0.712, \rho = 0.4\right) = 3.4\%$$

Professor's Note: You will not be asked to calculate the percentiles for mapping to the standard normal distribution because it requires the use of Microsoft Excel® or MATLAB®. In addition, you will not be asked to calculate the joint probability of default for a bivariate normal distribution due to its complexity.

Correlated Default Time

LO 9.3: Summarize the process of finding the default time of an asset correlated to all other assets in a portfolio using the Gaussian copula.

When a Gaussian copula is used to derive the default time relationship for more than two assets, a **Cholesky decomposition** is used to derive a sample $M_n(\bullet)$ from a multivariate copula $M_n(\bullet) \in [0,1]$. The default correlations of the sample are determined by the default correlation matrix ρ_M for the *n*-variate standard normal distribution, M_n.

The first step is to equate the sample $M_n(\bullet)$ to the cumulative individual default probability, *Q*, for asset *i* at time τ using the following equation. This is accomplished using Microsoft Excel® or a Newton-Raphson search procedure.

$$M_n(\bullet) = Q_i(\tau_i)$$

Next, the random samples are repeatedly drawn from the *n*-variate standard normal distribution $M_n(\bullet)$ to determine the expected default time using the Gaussian copula.

Random samples are drawn to estimate the default times, because there is no closed form solution for this equation.

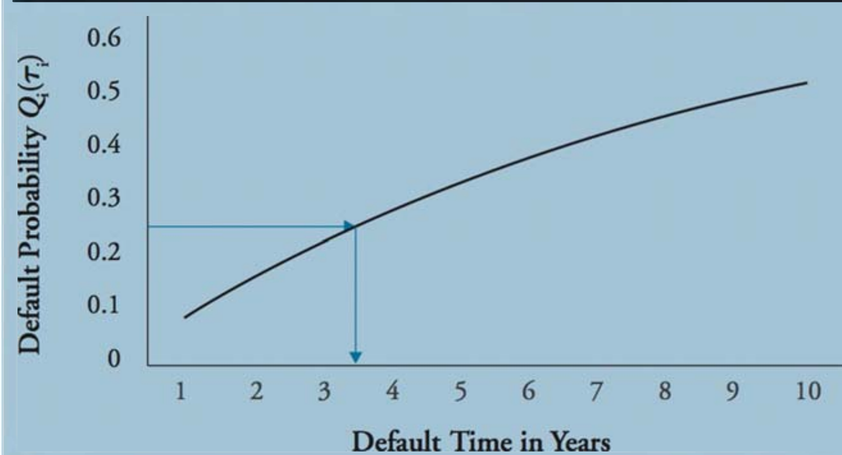
Example: Estimating default time

Illustrate how a risk manager estimates the expected default time of asset *i* using an *n*-variate Gaussian copula.

Answer:

Suppose a risk manager draws a 25% cumulative default probability for asset *i* from a random *n*-variate standard normal distribution, $M_n(\bullet)$. The *n*-variate standard normal distribution includes a default correlation matrix, ρ_M, that has the default correlations of asset *i* with all *n* assets. Figure 4 illustrates how to equate this 25% with the market determined cumulative individual default probability $Q_i(\tau_i)$. Suppose the first random sample equates to a default time τ of 3.5 years. This process is then repeated 100,000 times to estimate the default time of asset *i*.

Figure 4: Mapping Default Time for a Random Sample



Key Concepts

LO 9.1

The general equation for correlation copula, *C*, is defined as:

$$C[G_1(u_1),\ldots,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n));\rho_F\right]$$

The notation for this copula equation is translated as: $G_i(u_i)$ are marginal distributions, F_n is the joint cumulative distribution function, F_1^{-1} is the inverse function of F_n, and ρ_F is the correlation matrix structure of the joint cumulative function F_n.

LO 9.2

The Gaussian default time copula is defined as:

$$C_{GD}[Q_i(t),\ldots,Q_n(t)] = M_n\left[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t));\rho_M\right]$$

Marginal distributions of cumulative default probabilities, $Q(t)$, for assets $i = 1$ to n for fixed time periods t are mapped to the single n-variate standard normal distribution, M_n, with a correlation structure of ρ_M.

The Gaussian copula for the bivariate standard normal distribution, M_2, for two assets with a default correlation coefficient of ρ is defined as:

$$C_{GD}[Q_B(t),Q_C(t)] = M_2\left[N^{-1}(Q_B(t)),N^{-1}(Q_C(t));\rho\right]$$

LO 9.3

Random samples are drawn from an n-variate standard normal distribution sample, $M_n(\bullet)$, to estimate expected default times using the Gaussian copula:

$$M_n(\bullet) = Q_i(\tau_i)$$

Concept Checkers

1. Suppose a risk manager creates a copula function, *C*, defined by the equation:

$$C[G_1(u_1),...,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),...,F_n^{-1}(G_n(u_n));\rho_F\right]$$

Which of the following statements does not accurately describe this copula function?

A. $G_i(u_i)$ are standard normal univariate distributions.
B. F_n is the joint cumulative distribution function.
C. F_1^{-1} is the inverse function of F_n that is used in the mapping process.
D. ρ_F is the correlation matrix structure of the joint cumulative function F_n.

2. Which of the following statements best describes a Gaussian copula?

A. A major disadvantage of a Gaussian copula model is the transformation of the original marginal distributions in order to define the correlation matrix.
B. The mapping of each variable to the new distribution is done by defining a mathematical relationship between marginal and unknown distributions.
C. A Gaussian copula maps the marginal distribution of each variable to the standard normal distribution.
D. A Gaussian copula is seldom used in financial models because ordinal numbers are required.

3. A Gaussian copula is constructed to estimate the joint default probability of two assets within a one-year time period. Which of the following statements regarding this type of copula is incorrect?

A. This copula requires that the respective cumulative default probabilities are mapped to a bivariate standard normal distribution.
B. This copula defines the relationship between the variables using a default correlation matrix, ρ_M.
C. The term $N_1^{-1}(Q_1(t))$ maps each individual cumulative default probability for asset *i* for time period *t* on a percentile-to-percentile basis.
D. This copula is a common approach used in finance to estimate joint default probabilities.

4. A risk manager is trying to estimate the default time for asset *i* based on the default correlation copula of asset *i* to *n* assets. Which of the following equations best defines the process that the risk manager should use to generate and map random samples to estimate the default time?

A. $C_{GD}[Q_B(t),Q_C(t)] = M_2\left[N^{-1}(Q_B(t)),N^{-1}(Q_C(t));\rho\right]$

B. $C[G_1(u_1),...,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),...,F_n^{-1}(G_n(u_n));\rho_F\right]$

C. $C_{GD}[Q_i(t),...,Q_n(t)] = M_n\left[N_1^{-1}(Q_1(t)),...,N_n^{-1}(Q_n(t));\rho_M\right]$

D. $M_n(\bullet) = Q_i(\tau_i)$

5. Suppose a risk manager owns two non-investment grade assets and has determined their individual default probabilities for the next five years. Which of the following equations best defines how a Gaussian copula is constructed by the risk manager to estimate the joint probability of these two companies defaulting within the next year, assuming a Gaussian default correlation of 0.35?

A. $C_{GD}\left[Q_B(t),Q_C(t)\right]=M_2\left[N^{-1}(Q_B(t)),N^{-1}(Q_C(t));\rho\right]$

B. $C\left[G_1(u_1),\ldots,G_n(u_n)\right]=F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n));\rho_F\right]$

C. $C_{GD}\left[Q_i(t),\ldots,Q_n(t)\right]=M_n\left[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t));\rho_M\right]$

D. $M_n(\bullet)=Q_i(\tau_i)$

Concept Checker Answers

1. A $G_i(u_i)$ are marginal distributions that do not have well-known distribution properties.

2. C Observations of the unknown marginal distributions are mapped to the standard normal distribution on a percentile-to-percentile basis to create a Gaussian copula.

3. B Because there are only two companies, only a single correlation coefficient is required and not a correlation matrix, ρ_M.

4. D The equation $\mathrm{M_n}(\bullet) = \mathrm{Q_i}(\tau_i)$ is used to repeatedly generate random drawings from the *n*-variate standard normal distribution to determine the expected default time using the Gaussian copula.

5. A Because there are only two assets, the risk manager should use this equation to define the bivariate standard normal distribution, M_2, with a single default correlation coefficient of ρ.