The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Empirical Approaches to Risk Metrics and Hedging

Topic 10

Exam Focus

This topic discusses how dollar value of a basis point (DV01)-style hedges can be improved. Regression-based hedges enhance DV01-style hedges by examining yield changes over time. Principal components analysis (PCA) greatly simplifies bond hedging techniques. For the exam, understand the drawbacks of a standard DV01-netural hedge, and know how to compute the face value of an offsetting position using DV01 and how to adjust this position using regression-based hedging techniques.

DV01-Neutral Hedge

LO 10.1: Explain the drawbacks to using a DV01-neutral hedge for a bond position.

A standard DV01-neutral hedge assumes that the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points. However, a one-to-one relationship does not always exist in practice. For example, if a trader hedges a T-bond which uses a nominal yield with a treasury security indexed to inflation [i.e., Treasury Inflation Protected Security (TIPS)] which uses a real yield, the hedge will likely be imprecise when changes in yield occur. In general, more dispersion surrounds the change in the nominal yield for a given change in the real yield. Empirically, the nominal yield adjusts by more than one basis point for every basis point adjustment in the real yield.

DV01-style metrics and hedges focus on how rates change relative to one another. As mentioned, the presumption that yields on nominal bonds and TIPS change by the same amount is not very realistic. To improve this DV01-neutral hedge approach, we can apply regression analysis techniques. Using a regression hedge examines the volatility of historical rate differences and adjusts the DV01 hedge accordingly, based on historical volatility.

Regression Hedge

LO 10.2: Describe a regression hedge and explain how it can improve a standard DV01-neutral hedge.

A regression hedge takes DV01-style hedges and adjusts them for projected nominal yield changes compared to projected real yield changes. Least squares regression analysis, which is used for regression-based hedges, looks at the historical relationship between real and nominal yields.

The advantage of a regression framework is that it provides an estimate of a hedged portfolio's volatility. An investor can gauge the expected gain in advance and compare it to historical volatility to determine whether the hedged portfolio is an attractive investment.

For example, assume a relative value trade is established whereby a trader sells a U.S. Treasury bond and buys a U.S. TIPS (which makes inflation-adjusted payments) to hedge the T-bond. The initial spread between these two securities represents the current views on inflation. Over time, changes in yields on nominal bonds and TIPS do not track one-for-one. To illustrate this hedge, assume the following data for yields and DV01s of a TIPS and a T-bond. Also assume that the trader is selling 100 million of the T-bond.

Bond	*Yield (%)*	*DV01*
TIPS	1.325	0.084
T-Bond	3.475	0.068

If the trade was made DV01-neutral, which assumes that the yield on the TIPS and the nominal bond will increase/decrease by the same number of basis points, the trade will not earn a profit or sustain a loss. The calculation for the amount of TIPS to purchase to hedge the short nominal bond is as follows:

$$F^R \times \frac{0.084}{100} = 100M \times \frac{0.068}{100}$$

$$F^R = 100M \times \frac{0.068}{0.084} = \$80.95 \text{ million}$$

where:
F^R = face amount of the real yield bond

To improve this hedge, the trader gathers yield data over time and plots a regression line, whereby the real yield is the independent variable and the nominal yield is the dependent variable. To compensate for the dispersion in the change in the nominal yield for a given change in the real yield, the trader would adjust the DV01-neutral hedge.

Hedge Adjustment Factor

LO 10.3: Calculate the regression hedge adjustment factor, beta.

In order to profit from a hedge, we must assume variability in the spread between the real and nominal yields over time. As mentioned, least squares regression is conducted to analyze these changes. The alpha and beta coefficients of a least squares regression line will be determined by the line of best fit through historical yield data points.

$$\Delta y_t^N = \alpha + \beta \Delta y_t^R + \varepsilon_t$$

where:
Δy_t^N = changes in the nominal yield
Δy_t^R = changes in the real yield

Recall that alpha represents the intercept term and beta represents the slope of the data plot. If least squares estimation determines the yield beta to be 1.0198, then this means that over the sample period, the nominal yield increases by 1.0198 basis points for every basis point increase in real yields.

LO 10.4: Calculate the face value of an offsetting position needed to carry out a regression hedge.

Defining F^R and F^N as the face amounts of the real and nominal bonds, respectively, and their corresponding DV01s as DV01^R and DV01^N, a DV01 hedge is adjusted by the hedge adjustment factor, or beta, as follows:

$$F^R = F^N \times \left(\frac{\text{DV01}^N}{\text{DV01}^R}\right) \times \beta$$

Now that we have determined the variability between the nominal and real yields, the hedge can be adjusted by the hedge adjustment factor of 1.0198:

$$F^R = 100\text{M} \times \left(\frac{0.068}{0.084}\right) \times 1.0198 = \$82.55 \text{ million}$$

This regression hedge approach suggests that for every $100 million sold in T-bonds, we should buy $82.55 million in TIPS. This will account for hedging not only the size of the underlying instrument, but also differences between nominal and real yields over time.

Note that in our example, the beta was close to one, so the resulting regression hedge did not change much from the DV01-neutral hedge. The regression hedge approach assumes that the hedge coefficient, β, is constant over time. This of course is not always the case, so it is best to estimate the coefficient over different time periods and make comparisons.

Two other factors should be also considered in our analysis: (1) the R-squared (i.e., the coefficient of determination), and (2) the standard error of the regression (SER). The

R-squared gives the percentage of variation in nominal yields that is explained by real yields. The standard error of the regression is the standard deviation of the realized error terms in the regression.

Two-Variable Regression Hedge

LO 10.5: Calculate the face value of multiple offsetting swap positions needed to carry out a two-variable regression hedge.

Regression hedging can also be conducted with two independent variables. For example, assume a trader in euro interest rate swaps buys/receives the fixed rate in a relatively illiquid 20-year swap and wishes to hedge this interest rate exposure. In this case, a regression hedge with swaps of different maturities would be appropriate. Since it may be impractical to hedge this position by immediately selling 20-year swaps, the trader may choose to sell a combination of 10- and 30-year swaps.

The trader is thus relying on a two-variable regression model to approximate the relationship between changes in 20-year swap rates and changes in 10- and 30-year swap rates. The following regression equation describes this relationship:

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \varepsilon_t$$

Similar to the single-variable regression hedge, this hedge of the 20-year euro swap can be expressed in terms of risk weights, which are the beta coefficients in the above equation:

$$\frac{(-F^{10} \times DV01^{10})}{(F^{20} \times DV01^{20})} = \text{change in 10-year swap rate, } \beta^{10}$$

$$\frac{(-F^{30} \times DV01^{30})}{(F^{20} \times DV01^{20})} = \text{change in 30-year swap rate, } \beta^{30}$$

The trader next does an initial regression analysis using data on changes in the 10-, 20-, and 30-year euro swap rates for a five-year time period. Assume the regression output is as follows:

Number of observations	1281
R-squared	99.8%
Standard error	0.14

Regression Coefficients	Value	Standard Error
Alpha	–0.0014	0.0040
Change in 10-year swap rate	0.2221	0.0034
Change in 30-year swap rate	0.7765	0.0037

Given these regression results and an illiquid 20-year swap, the trader would hedge 22.21% of the 20-year swap DV01 with a 10-year swap and 77.65% of the 20-year swap DV01 with a 30-year swap. Because these weights sum to approximately one, the regression hedge DV01 will be very close to the 20-year swap DV01.

The two-variable approach will provide a better hedge (in terms of R-squared) compared to a single-variable approach. However, regression hedging is not an exact science. There are several cases in which simply doing a one-security DV01 hedge, or a two-variable hedge with arbitrary risk weights, is not appropriate (e.g., hedging during a financial crisis).

Level and Change Regressions

LO 10.6: Compare and contrast level and change regressions.

When setting up and establishing regression-based hedges, there are two schools of thought. Some regress changes in yields on changes in yields, as demonstrated previously, but an alternative approach is to regress yields on yields.

Using a single-variable approach, the formula for a change-on-change regression with dependent variable *y* and independent variable *x* is as follows:

$$\Delta y_t = \alpha + \beta \Delta x_t + \Delta \varepsilon_t$$

where:
$\Delta y_t = y_t - y_{t-1}$
$\Delta x_t = x_t - x_{t-1}$

Alternatively, the formula for a level-on-level regression is as follows:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

With both approaches, the estimated regression coefficients are unbiased and consistent; however, the error terms are unlikely to be independent of each other. Thus, since the error terms are correlated over time (i.e., *serially correlated)*, the estimated regression coefficients are not efficient. As a result, there is a third way to model the relationship between two bond yields (for some constant correlation < 1):

$$\varepsilon_t = \rho \varepsilon_{t-1} + v_t$$

This formula assumes that today's error term consists of some part of yesterday's error term, plus a new random fluctuation.

Principal Components Analysis

LO 10.7: Describe principal component analysis and explain how it is applied to constructing a hedging portfolio.

Regression analysis focuses on yield changes among a small number of bonds. Empirical approaches, such as principal components analysis (PCA), take a different approach by providing a single empirical description of term structure behavior, which can be applied

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across all bonds. PCA attempts to explain all factor exposures using a small number of uncorrelated exposures which do an adequate job of capturing risk.

For example, if we consider the set of swap rates from 1 to 30 years, at annual maturities, the PCA approach creates 30 interest rate factors or components, and each factor describes a change in each of the 30 rates. This is in contrast to regression analysis, which looks at variances of rates and their pairwise correlations.

PCA sets up the 30 factors with the following properties:

1. The sum of the variances of the 30 principal components (PCs) equals the sum of the variances of the individual rates. The PCs thus capture the volatility of the set of rates.

2. The PCs are not correlated with each other.

3. Each PC is chosen to contain the highest possible variance, given the earlier PCs.

The advantage of this approach is that we only really need to describe the volatility and structure of the first three PCs since the sum of the variances of the first three PCs is a good approximation of the sum of the variances of all rates. Thus, the PCA approach creates three factors that capture similar data as a comprehensive matrix containing variances and covariances of all interest rate factors. Changes in 30 rates can now be expressed with changes in three factors, which is a much simpler approach.

Key Concepts

LO 10.1

A DV01-neutral hedge assumes the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points. However, empirically, a nominal yield will adjust by more than one basis point for every basis point adjustment in a real yield.

LO 10.2

A regression hedge adjusts for the extra movement in the projected nominal yield changes compared to the projected real yield changes.

LO 10.3

Least squares regression is conducted to analyze the changes in historical yields between nominal and real bonds.

$$\Delta y_t^N = \alpha + \beta \Delta y_t^R + \varepsilon_t$$

LO 10.4

A DV01 hedge is adjusted by the hedge adjustment factor, or beta, as follows:

$$F^R = F^N \times \left(\frac{DV01^N}{DV01^R} \right) \times \beta$$

This regression hedge approach gives an indication of the volatility of the hedged portfolio.

LO 10.5

Regression hedging can also be conducted with a two-variable regression model. The beta coefficients in the regression model represent risk weights, which are used to calculate the face value of multiple offsetting positions.

LO 10.6

The formula for a level-on-level regression with dependent variable *y* and independent variable *x* is as follows:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

The formula for a change-on-change regression is as follows:

$$y_t - y_{t-1} = \Delta y_t = \alpha + \beta \Delta x_t + \Delta \varepsilon_t$$

With both approaches, the error terms are unlikely to be independent of each other.

LO 10.7

Principal components analysis (PCA) provides a single empirical description of term structure behavior, which can be applied across all bonds. The advantage of this approach is that we only need to describe the volatility and structure of a small number of principal components, which approximate all movements in the term structure.

Concept Checkers

1. If a trader is creating a fixed income hedge, which hedging methodology would be least effective if the trader is concerned about the dispersion of the change in the nominal yield for a particular change in the real yield?
   A. One-variable regression hedge.
   B. DV01 hedge.
   C. Two-variable regression hedge.
   D. Principal components hedge.

2. Assume that a trader is making a relative value trade, selling a U.S. Treasury bond and correspondingly purchasing a U.S. TIPS. Based on the current spread between the two securities, the trader shorts $100 million of the nominal bond and purchases $89.8 million of TIPS. The trader then starts to question the amount of the hedge due to changes in yields on TIPS in relation to nominal bonds. He runs a regression and determines from the output that the nominal yield changes by 1.0274 basis points per basis point change in the real yield. Would the trader adjust the hedge, and if so, by how much?
   A. No.
   B. Yes, by $2.46 million (purchase additional TIPS).
   C. Yes, by $2.5 million (sell a portion of the TIPS).
   D. Yes, by $2.11 million (purchase additional TIPS).

3. What is a key advantage of using a regression hedge to fine tune a DV01 hedge?
   A. It assumes that term structure changes are driven by one factor.
   B. The proper hedge amount may be computed for any assumed change in the term structure.
   C. Bond price changes and returns can be estimated with proper measures of price sensitivity.
   D. It gives an estimate of the hedged portfolio's volatility over time.

4. What does the regression hedge assume about the hedge coefficient, beta?
   A. It moves in lockstep with real rates.
   B. It stays constant over time.
   C. It generally tracks nominal rates over time.
   D. It is volatile over time, similar to both real and nominal rates.

5. Traci York, FRM, is setting up a regression-based hedge and is trying to decide between a changes-in-yields-on-changes-in-yields approach versus a yields-on-yields approach. Which of the following is a correct statement concerning error terms in these two approaches?
   A. In both cases, the error terms are completely uncorrelated.
   B. With change-on-change, there is no correlation in error terms, while yield-on-yield error terms are completely correlated.
   C. Error terms are correlated over time with both approaches.
   D. With yield-on-yield, there is no correlation in error terms, while change-on-change error terms are completely correlated.

Concept Checker Answers

1. B The DV01 hedge assumes that the yield on the bond and the assumed hedging instruments rises and falls by the same number of basis points; so with a DV01 hedge, there is not much the trader can do to allow for dispersion between nominal and real yields.

2. B The trader would need to adjust the hedge as follows:

   $89.8 million × 1.0274 = $92.26 million

   Thus, the trader needs to purchase additional TIPS worth $2.46 million.

3. D A key advantage of using a regression approach in setting up a hedge is that it automatically gives an estimate of the hedged portfolio's volatility.

4. B It should be pointed out that while it is true that the regression hedge assumes a constant beta, this is not a realistic assumption; thus, it is best to estimate beta over several time periods and compare accordingly.

5. C With the level-on-level approach, error terms are somewhat correlated over time, while with the change-on-change approach, the error terms are completely correlated. Thus, error terms are correlated over time with both approaches.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Science of Term Structure Models

Topic 11

Exam Focus

The emphasis of this topic is the pricing of interest rate derivative contracts using a risk-neutral binomial model. The pricing process for interest rate derivatives requires intensive calculations and is very tedious. However, the relationship becomes straightforward when it is modeled to support risk neutrality. Understand the concepts of backward induction and how the addition of time steps will increase the accuracy of any bond pricing model. Bonds with embedded options are also discussed in this topic. Be familiar with the price-yield relationship of both callable and putable bonds. This topic incorporates elements of material from the FRM Part I curriculum where you valued options with binomial trees.

Interest Rate Tree (Binomial) Model

LO 11.1: Calculate the expected discounted value of a zero-coupon security using a binomial tree.

LO 11.2: Construct and apply an arbitrage argument to price a call option on a zero-coupon security using replicating portfolios.

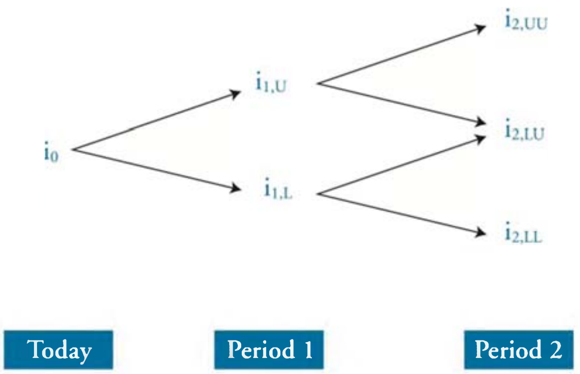
The binomial interest rate model is used throughout this topic to illustrate the issues that must be considered when valuing bonds with embedded options. A **binomial model** is a model that assumes that interest rates can take only one of two possible values in the next period.

This interest rate model makes assumptions about interest rate volatility, along with a set of paths that interest rates may follow over time. This set of possible interest rate paths is referred to as an **interest rate tree**.

Binomial Interest Rate Tree

The diagram in Figure 1 depicts a binomial interest rate tree.

Figure 1: 2-Period Binomial



To understand this 2-period binomial tree, consider the nodes indicated with the boxes in Figure 1. A node is a point in time when interest rates can take one of two possible paths—an upper path, *U*, or a lower path, *L*. Now consider the node on the right side of the diagram where the interest rate $i_{2,LU}$ appears. This is the rate that will occur if the initial rate, i_0, follows the lower path from node 0 to node 1 to become $i_{1,L}$, then follows the upper of the two possible paths to node 2, where it takes on the value $i_{2,LU}$. At the risk of stating the obvious, the upper path from a given node leads to a higher rate than the lower path. Notice also that an upward move followed by a downward move gets us to the same place on the tree as a down-then-up move, so $i_{2,LU} = i_{2,UL}$.

The interest rates at each node in this interest rate tree are 1-period forward rates corresponding to the nodal period. Beyond the root of the tree, there is more than one 1-period forward rate for each nodal period (i.e., at year 1, we have two 1-year forward rates, $i_{1,U}$ and $i_{1,L}$). The relationship among the rates associated with each individual nodal period is a function of the interest rate volatility assumption of the model being employed to generate the tree.

Constructing the Binomial Interest Rate Tree

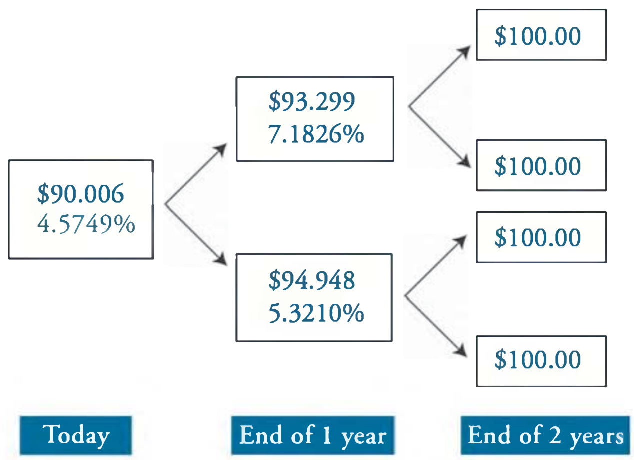
The construction of an interest rate tree, binomial or otherwise, is a tedious process. In practice, the interest rate tree is usually generated using specialized computer software. There is one underlying rule governing the construction of an interest rate tree: *The values for on-the-run issues generated using an interest rate tree should prohibit arbitrage opportunities.* This means that the value of an on-the-run issue produced by the interest rate tree must equal its market price. It should be noted that in accomplishing this, the interest rate tree must maintain the interest rate volatility assumption of the underlying model.

Valuing an Option-Free Bond With the Tree, Using Backward Induction

Backward induction refers to the process of valuing a bond using a binomial interest rate tree. The term "backward" is used because in order to determine the value of a bond at node 0, you need to know the values that the bond can take on at node 1. But to determine the values of the bond at node 1, you need to know the possible values of the bond at node 2, and so on. Thus, for a bond that has N compounding periods, the current value of the bond is determined by computing the bond's possible values at period N and working "backward" to node 0.

Consider the binomial tree shown in Figure 2 for a $100 face value, zero-coupon bond, with two years remaining until maturity, and a market price of $90.006. Starting on the top line, the blocks at each node include the value of the bond and the 1-year forward rate at that node. For example, at the upper path of node 1, the price is $93.299, and the 1-year forward rate is 7.1826%.

Figure 2: Valuing a 2-Year, Zero-Coupon, Option-Free Bond



Know that *the value of a bond at a given node in a binomial tree is the average of the present values of the two possible values from the next period.* The appropriate discount rate is the forward rate associated with the node under analysis.

Example: Valuing an option-free bond

Assuming the bond's market price is $90.006, **demonstrate** that the tree in Figure 2 is arbitrage free using backward induction.

Answer:

Consider the value of the bond at the *upper* node for period 1, $V_{1,U}$:

$$V_{1,U} = \frac{(\$100 \times 0.5) + (\$100 \times 0.5)}{1.071826} = \$93.299$$

Similarly, the value of the bond at the *lower* node for period 1, $V_{1,L}$ is:

$$V_{1,L} = \frac{(\$100 \times 0.5) + (\$100 \times 0.5)}{1.053210} = \$94.948$$

Now **calculate** V_0, the current value of the bond at node 0:

$$V_0 = \frac{(\$93.299 \times 0.5) + (\$94.948 \times 0.5)}{1.045749} = \$90.006$$

Since the computed value of the bond equals the market price, the binomial tree is arbitrage free.

Professor's Note: When valuing bonds with coupon payments, you need to add the coupons to the bond prices at each node. For example, with a $100 face value, 7% annual coupon bond, you would add the $7 coupon to each price before computing present values. Valuing coupon-paying bonds with a binomial tree will be illustrated in LO 11.5.

LO 11.3: Define risk-neutral pricing and apply it to option pricing.

LO 11.4: Distinguish between true and risk-neutral probabilities, and apply this difference to interest rate drift.

Using the 0.5 probabilities for up and down states as shown in the previous example may not produce an expected discounted value that exactly matches the market price of the bond. This is because the 0.5 probabilities are the assumed **true probabilities** of price movements. In order to equate the discounted value using a binomial tree and the market price, we need to use what is known as **risk-neutral probabilities**. Any difference between the risk-neutral and true probabilities is referred to as the **interest rate drift**.

Using the Risk-Neutral Interest Rate Tree

There are actually two ways to compute bond and bond derivative values using a binomial model. These techniques are referred to as **risk-neutral pricing**.

- The first method is to start with spot and forward rates derived from the current yield curve and then *adjust the interest rates* on the paths of the tree so that the value derived from the model is equal to the current market price of an on-the-run bond (i.e., the tree is created to be "arbitrage free"). This is the method we used in the previous example. Once the interest rate tree is derived for an on-the-run bond, we can use it to price derivative securities on the bond by calculating the expected discounted value at each node using the real-world probabilities.
- The second method is to take the rates on the tree as given and then *adjust the probabilities* so that the value of the bond derived from the model is equal to its current market price. Once we derive these *risk-neutral probabilities*, we can use them to price derivative securities on the bond by once again calculating the expected discounted value at each node using the risk-neutral probabilities and working backward through the tree.

The value of the derivative is the same under either method.

LO 11.5: Explain how the principles of arbitrage pricing of derivatives on fixed income securities can be extended over multiple periods.

There are three basic steps to valuing an option on a fixed-income instrument using a binomial tree:

Step 1: Price the bond value at each node using the projected interest rates.
Step 2: Calculate the intrinsic value of the derivative at each node at maturity.
Step 3: Calculate the expected discounted value of the derivative at each node using the risk-neutral probabilities and working backward through the tree.

Note that the option cannot be properly priced using expected discounted values because the call option value depends on the path of interest rates over the life of the option. Incorporating the various interest rate paths will prohibit arbitrage from occurring.

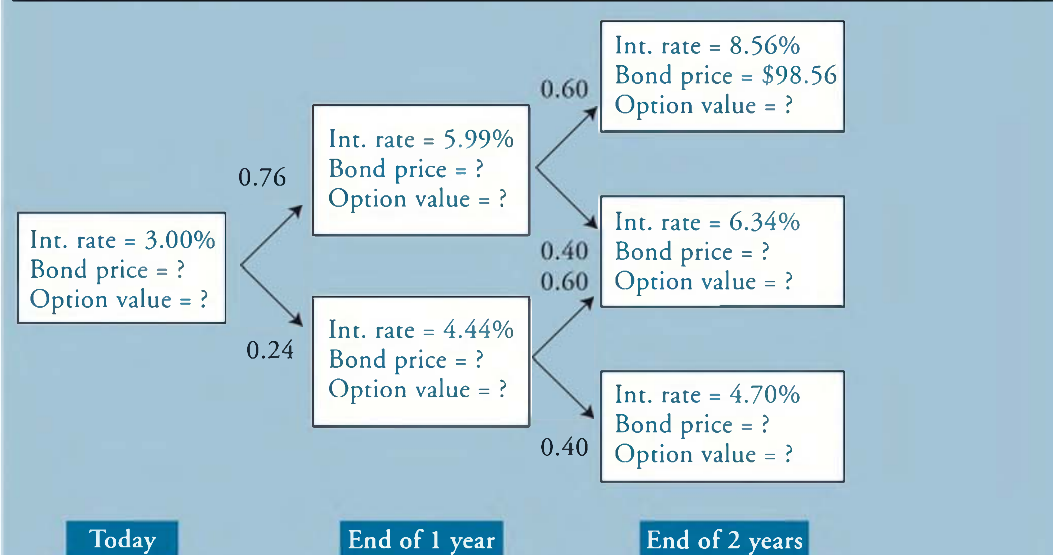
Example: Call option

Assume that you want to value a European call option with two years to expiration and a strike price of $100.00. The underlying is a 7%, annual-coupon bond with three years to maturity. Figure 3 represents the first two years of the binomial tree for valuing the underlying bond. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2.

Fill in the missing data in the binomial tree, and **calculate** the value of the European call option.

Professor's Note: Since the option is European, it can only be exercised at maturity.

Figure 3: Incomplete Binomial Tree for European Call Option on 3-Year, 7% Bond



Answer:

Step 1: Calculate the bond prices at each node using the backward induction methodology.

At the middle node in year 2, the price is \$100.62. You can calculate this by noting that at the end of year 2 the bond has one year left to maturity:

$$N = 1; I/Y = 6.34; PMT = 7; FV = 100; CPT \rightarrow PV = 100.62$$

At the bottom node in year 2, the price is \$102.20:

$$N = 1; I/Y = 4.70; PMT = 7; FV = 100; CPT \rightarrow PV = 102.20$$

At the top node in year 1, the price is \$100.37:

$$\frac{(\$105.56 \times 0.6) + (\$107.62 \times 0.4)}{1.0599} = \$100.37$$

At the bottom node in year 1, the price is \$103.65:

$$\frac{(\$107.62 \times 0.6) + (\$109.20 \times 0.4)}{1.0444} = \$103.65$$

Today, the price is \$105.01:

$$\frac{(\$107.37 \times 0.76) + (\$110.65 \times 0.24)}{1.03} = \$105.01$$

As shown here, the price at a given node is the expected discounted value of the cash flows associated with the two nodes that "feed" into that node. The discount rate that is applied is the prevailing interest rate at the given node. Note that since this is a European option, you really only need the bond prices at the maturity date of the option (end of year 2) if you are given the arbitrage-free interest rate tree. However, it's good practice to compute all the bond prices.

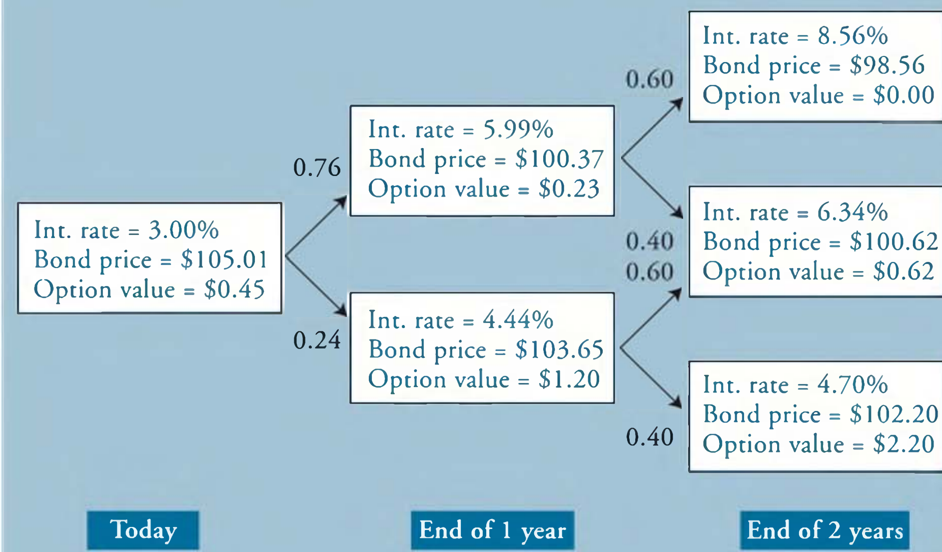
Step 2: Determine the intrinsic value of the option at maturity in each node. For example, the intrinsic value of the option at the bottom node at the end of year 2 is \$2.20 = \$102.20 – \$100.00. At the top node in year 2, the intrinsic value of the option is zero since the bond price is less than the call price.

Step 3: Using the backward induction methodology, calculate the option value at each node prior to expiration. For example, at the top node for year 1, the option price is \$0.23:

$$\frac{(\$0.00 \times 0.6) + (\$0.62 \times 0.4)}{1.0599} = \$0.23$$

Figure 4 shows the binomial tree with all values included.

Figure 4: Completed Binomial Tree for European Call Option on 3-Year, 7% Bond



The option value today is computed as:

$$\frac{(\$0.23 \times 0.76) + (\$1.20 \times 0.24)}{1.03} = \$0.45$$

Recombining and Nonrecombining

LO 11.7: Describe the rationale behind the use of recombining trees in option pricing.

In the previous example, the interest rate in the middle node of period two was the same (i.e., 6.34%) regardless of the path being up then down or down then up. This is known as a **recombining tree**. It may be the case, in a practical setting, that the up then down scenario produces a different rate than the down then up scenario. An example of this type of tree may result when any interest rate above a certain level (e.g., 3%) causes rates to move a fixed number of basis points, but any interest rate below that level causes rates to move at a pace that is below the up state's fixed amount. When rates move in this fashion, the movement process is known as **state-dependent volatility**, and it results in **nonrecombining trees**. From an economic standpoint, nonrecombining trees are appropriate; however, prices can be very difficult to calculate when the binomial tree is extended to multiple periods.

Constant Maturity Treasury Swap

LO 11.8: Calculate the value of a constant maturity Treasury swap, given an interest rate tree and the risk-neutral probabilities.

In addition to valuing options with binomial interest rate trees, we can also value other derivatives such as swaps. The following example calculates the price of a **constant maturity Treasury** (CMT) **swap**. A CMT swap is an agreement to swap a floating rate for a Treasury rate such as the 10-year rate.

Example: CMT swap

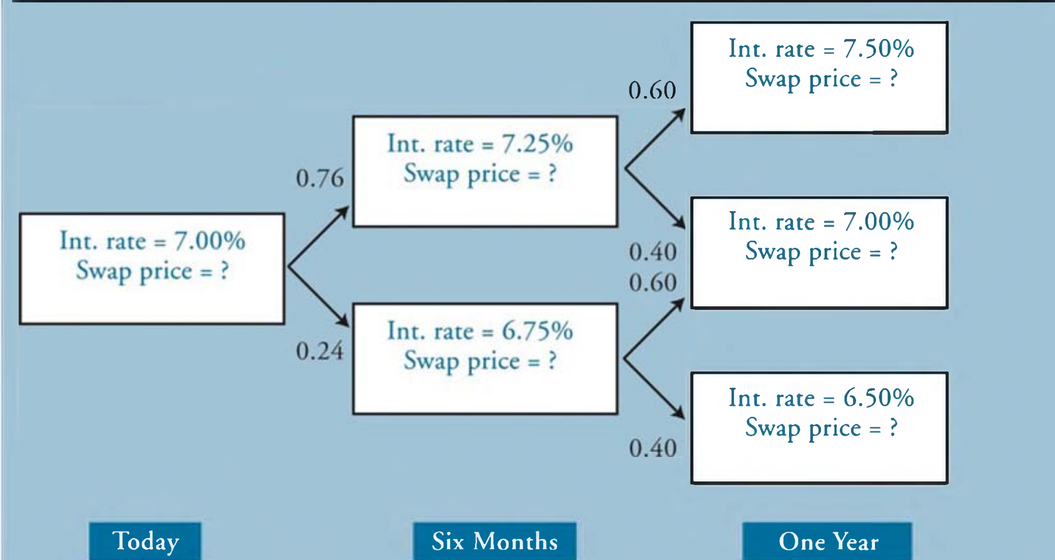
Assume that you want to value a constant maturity Treasury (CMT) swap. The swap pays the following every six months until maturity:

$$\left(\frac{\$1{,}000{,}000}{2}\right)\times\left(y_{\text{CMT}}-7\%\right)$$

y_{CMT} is a semiannually compounded yield, of a predetermined maturity, at the time of payment (y_{CMT} is equivalent to 6-month spot rates). Assume there is a 76% risk-neutral probability of an increase in the 6-month spot rate and a 60% risk-neutral probability of an increase in the 1-year spot rate.

Fill in the missing data in the binomial tree, and **calculate** the value of the swap.

Figure 5: Incomplete Binomial Tree for CMT Swap



Answer:

In six months, the top node and bottom node payoffs are, respectively:

$$\text{payoff}_{1,U} = \frac{\$1{,}000{,}000}{2} \times (7.25\% - 7.00\%) = \$1{,}250$$

$$\text{payoff}_{1,L} = \frac{\$1{,}000{,}000}{2} \times (6.75\% - 7.00\%) = -\$1{,}250$$

Similarly in one year, the top, middle, and bottom payoffs are, respectively:

$$\text{payoff}_{2,U} = \frac{\$1{,}000{,}000}{2} \times (7.50\% - 7.00\%) = \$2{,}500$$

$$\text{payoff}_{2,M} = \frac{\$1{,}000{,}000}{2} \times (7.00\% - 7.00\%) = \$0$$

$$\text{payoff}_{2,L} = \frac{\$1{,}000{,}000}{2} \times (6.50\% - 7.00\%) = -\$2{,}500$$

The possible prices in six months are given by the expected discounted value of the 1-year payoffs under the risk-neutral probabilities, *plus* the 6-month payoffs ($1,250 and –$1,250). Hence, the 6-month values for the top and bottom node are, respectively:

$$V_{1,U} = \frac{(\$2{,}500 \times 0.6) + (\$0 \times 0.4)}{1 + \frac{0.0725}{2}} + \$1{,}250 = \$2{,}697.53$$

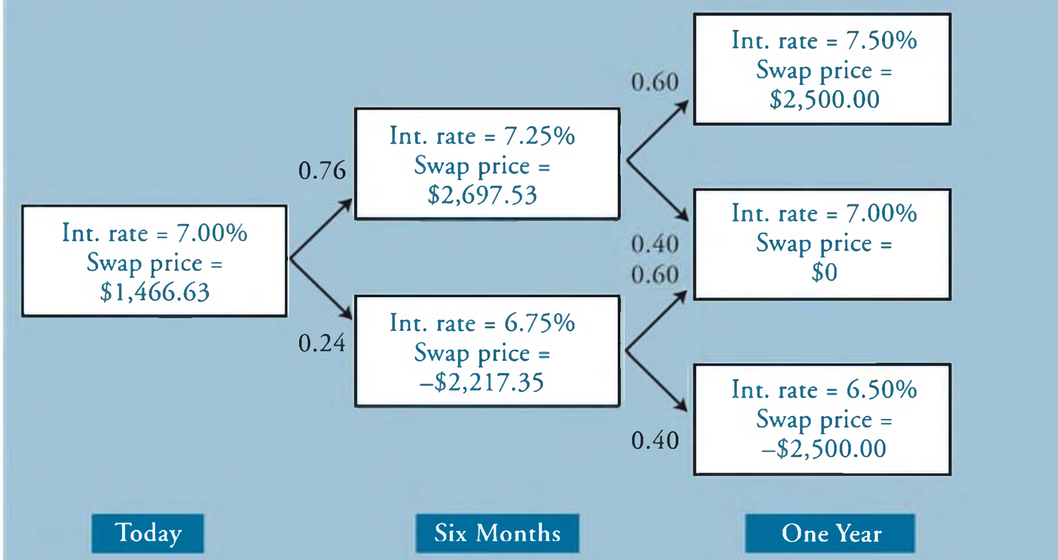
$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2{,}500 \times 0.4)}{1 + \frac{0.0675}{2}} - \$1{,}250 = -\$2{,}217.35$$

Today the price is $1,466.63, calculated as follows:

$$V_0 = \frac{(\$2{,}697.53 \times 0.76) + (-\$2{,}217.35 \times 0.24)}{1 + \frac{0.07}{2}} = \$1{,}466.63$$

Figure 6 shows the binomial tree with all values included.

Figure 6: Completed Binomial Tree for CMT Swap



Option-Adjusted Spread

LO 11.6: Define option-adjusted spread (OAS) and apply it to security pricing.

The option-adjusted spread (OAS) is the spread that makes the model value (calculated by the present value of projected cash flows) equal to the current market price. In the previous CMT example, the model price was equal to $1,466.63. Now assume that the market price of the CMT swap was instead $1,464.40, which is $2.23 less than the model price. In this case, the OAS to be added to each discounted risk-neutral rate in the CMT swap binomial tree turns out to be 20 basis points. In six months, the rates to be adjusted are 7.25% in the up node and 6.75% in the down node. Incorporating the OAS into the six-month rates generates the following new swap values:

$$V_{1,U} = \frac{(\$2{,}500 \times 0.6) + (\$0 \times 0.4)}{1 + \frac{0.0745}{2}} + \$1{,}250 = \$2{,}696.13$$

$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2{,}500 \times 0.4)}{1 + \frac{0.0695}{2}} - \$1{,}250 = -\$2{,}216.42$$

Notice that the only rates adjusted by the OAS spread are the rates used for discounting values. The OAS does not impact the rates used for estimating cash flows. The final step in this CMT swap valuation is to adjust the interest rate used to discount the price back to today. In this example, the discounted rate of 7% is adjusted by 20 basis points to 7.2%. The updated initial CMT swap value is:

$$V_0 = \frac{(\$2{,}696.13 \times 0.76) + (-\$2{,}216.42 \times 0.24)}{1 + \frac{0.072}{2}} = \$1{,}464.40$$

Now we can see that adding the OAS to the discounted risk-neutral rates in the binomial tree generates a model price ($1,464.40) that is equal to the market price ($1,464.40). In this example, the market price was initially less than the model price. This means that the security was trading *cheap*. If the market price were instead higher than the model price we would say that the security was trading *rich*.

Time Steps

LO 11.9: Evaluate the advantages and disadvantages of reducing the size of the time steps on the pricing of derivatives on fixed income securities.

For the sake of simplicity, the previous example assumed periods of six months. However, in reality, the time between steps should be much smaller. As you can imagine, the smaller the time between steps, the more complicated the tree and calculations become. Using daily time steps will greatly enhance the accuracy of any model but at the expense of additional computational complexity.

Fixed-Income Securities and Black-Scholes-Merton

LO 11.10: Evaluate the appropriateness of the Black-Scholes-Merton model when valuing derivatives on fixed income securities.

The Black-Scholes-Merton model is the most well-known equity option-pricing model. Unfortunately, the model is based on three assumptions that do not apply to fixed-income securities:

1. The model's main shortcoming is that it assumes there is no upper limit to the price of the underlying asset. However, bond prices do have a maximum value. This upper limit occurs when interest rates equal zero so that zero-coupon bonds are priced at par and coupon bonds are priced at the sum of the coupon payments plus par.

2. It assumes the risk-free rate is constant. However, changes in short-term rates do occur, and these changes cause rates along the yield curve and bond prices to change.

3. It assumes bond price volatility is constant. With bonds, however, price volatility decreases as the bond approaches maturity.

Bonds With Embedded Options

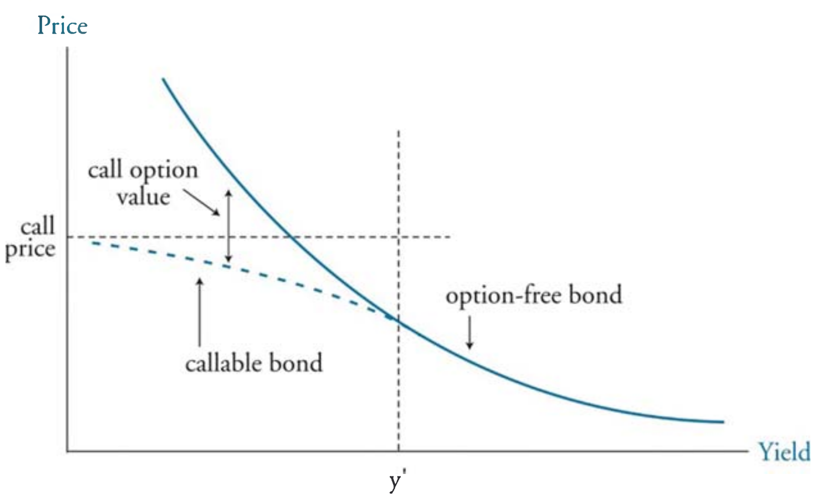
LO 11.11: Describe the impact of embedded options on the value of fixed income securities.

Fixed-income securities are often issued with **embedded options**, such as a call feature. In this case, the price-yield relationship will change, and so will the price volatility characteristics of the issue.

Callable Bonds

A call option gives the issuer the right to buy back the bond at fixed prices at one or more points in the future, prior to the date of maturity. Since the investor takes a short position in the call, the right to purchase rests with the issuer. Such bonds are deemed to be *callable* (note that a call provision on a bond is analytically similar to a prepayment option).

Figure 7: Price-Yield Function of Callable Bond



For an option-free noncallable bond, prices will fall as yields rise, and prices will rise unabated as yields fall—in other words, they'll move in line with yields. That's not the case, however, with **callable bonds**. As you can see in Figure 7, the decline in callable bond yield will reach the point where the rate of increase in the price of the callable bond will start slowing down and eventually level off.

This is known as **negative convexity**. Such behavior is due to the fact that the issuer has the right to retire the bond prior to maturity at some specified call price. The call price, in effect, acts to hold down the price of the bond (as rates fall) and causes the price-yield curve to flatten. The point where the curve starts to flatten is at (or near) a yield level of y'. Note that as long as yields remain above y', a callable bond will behave like any option-free (noncallable) issue and exhibit positive convexity. That's because at high yield levels, there is little chance of the bond being called.

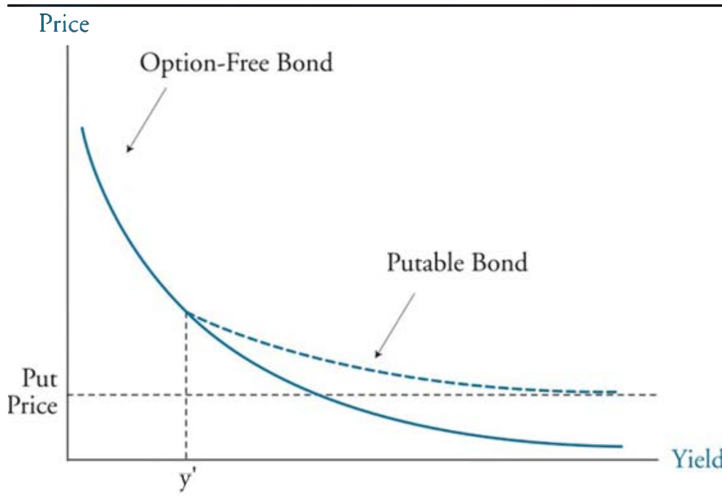
Below y', investors begin to anticipate that the firm may call the bond, in which case investors will receive the call price. Therefore, as yield levels drop, the bond's market value is bounded from above by the call price. Thus, callability effectively caps the investor's capital gains as yields fall. Moreover, it exacerbates reinvestment risk since it increases the cash flow that must be reinvested at lower rates (i.e., without the call or prepayment option, the cash flow will only be the coupon; with the option, the cash flow is the coupon plus the call price).

Thus, in Figure 7, as long as yields remain below *y'*, callable bonds will exhibit price compression, or *negative convexity*; however, at yields above *y'*, those same callable bonds will exhibit all the properties of *positive convexity*.

Putable Bonds

The put feature in **putable bonds** is another type of embedded option. The put feature gives the bondholder the right to sell the bond back to the issuer at a set price (i.e., the bondholder can "put" the bond to the issuer). The impact of the put feature on the price-yield relationship is shown in Figure 8.

Figure 8: Price-Yield Function of a Putable Bond



At low yield levels relative to the coupon rate, the price-yield relationship of putable and nonputable bonds is similar. However, as shown in Figure 8, if yields rise above *y'*, the price of the putable bond does not fall as rapidly as the price of the option-free bond. This is because the put price serves as a floor value for the price of the bond.

Key Concepts

LO 11.1

Backward-induction methodology with a binomial model requires discounting of the cash flows that occur at each node in an interest rate tree (bond value plus coupon payment) backward to the root of the tree.

LO 11.2

The values for on-the-run issues generated using an interest rate tree should prohibit arbitrage opportunities.

LO 11.3

Risk-neutral, or no-arbitrage, binomial tree models are used to allow for proper valuation of bonds with embedded options.

LO 11.4

Using the average of present values of the two possible values from the next period may not produce an expected discounted bond value that exactly matches the market price of the bond.

Using 0.5 probabilities for up and down states are the assumed true probabilities of price movements. In order to equate the discounted value using a binomial tree and the market price, we need to use risk-neutral probabilities.

LO 11.5

To value an option on a fixed-income instrument using a binomial tree:

1. Price the bond at each node using projected interest rates.
2. Calculate the intrinsic value of the derivative at each node at maturity.
3. Calculate the expected discounted value of the derivative at each node using the risk-neutral probabilities and working backward through the tree.

Callable bonds can be valued by modifying the cash flows at each node in the interest rate tree to reflect the cash flow prescribed by the embedded call option.

LO 11.6

The option-adjusted spread (OAS) allows a security's model price to equal its market price. It is added to any rate in the interest rate tree that is used for discounting purposes.

LO 11.7

Nonrecombining trees result when the up-down scenario produces a different rate than the down-up scenario.

LO 11.8

A constant maturity Treasury (CMT) swap is an agreement to swap a floating rate for a Treasury rate.

LO 11.9

The precision of a model can be improved by reducing the length of the time steps, but the trade-off is increased complexity.

LO 11.10

The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following unreasonable assumptions:

- There is no upper price bound.
- The risk-free rate is constant.
- Bond volatility is constant.

LO 11.11

Fixed-income securities are often issued with embedded options. When embedded options are present, the price-yield relationship will change, and so will the price volatility characteristics of the issue.

Concept Checkers

1. A European put option has two years to expiration and a strike price of $101.00. The underlying is a 7% annual coupon bond with three years to maturity. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2. The current interest rate is 3.00%. At the end of year 1, the rate will either be 5.99% or 4.44%. If the rate in year 1 is 5.99%, it will either rise to 8.56% or rise to 6.34% in year 2. If the rate in one year is 4.44%, it will either rise to 6.34% or rise to 4.70%. The value of the put option today is closest to:
   A. $1.17.
   B. $1.30.
   C. $1.49.
   D. $1.98.

2. The Black-Scholes-Merton option pricing model is not appropriate for valuing options on corporate bonds because corporate bonds:
   A. have credit risk.
   B. have an upper price bound.
   C. have constant price volatility.
   D. are not priced by arbitrage.

3. Which of the following regarding the use of small time steps in the binomial model is true?
   A. Less realistic model.
   B. More accurate model.
   C. Less complicated computations.
   D. Less computational expense.

4. Which of the following statements about callable bonds compared to noncallable bonds is false?
   A. They have less price volatility.
   B. They have negative convexity.
   C. Capital gains are capped as yields rise.
   D. At low yields, reinvestment rate risk rises.

5. Which of the following statements concerning the calculation of value at a node in a fixed income binomial interest rate tree is most accurate? The value at each node is the:
   A. present value of the two possible values from the next period.
   B. average of the present values of the two possible values from the next period.
   C. sum of the present values of the two possible values from the next period.
   D. average of the future values of the two possible values from the next period.

Concept Checker Answers

1. **A** This is the same underlying bond and interest rate tree as in the call option example from this topic. However, here we are valuing a put option.

   The option value in the upper node at the end of year 1 is computed as:

$$\frac{(\$2.44\times0.6)+(\$0.38\times0.4)}{1.0599}=\$1.52$$

   The option value in the lower node at the end of year 1 is computed as:

$$\frac{(\$0.38\times0.6)+(\$0.00\times0.4)}{1.0444}=\$0.22$$

   The option value today is computed as:

$$\frac{(\$1.52\times0.76)+(\$0.22\times0.24)}{1.0300}=\$1.17$$

2. **B** The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following assumptions, which are not reasonable for valuing fixed-income securities:
   - There is no upper price bound.
   - The risk-free rate is constant.
   - Bond volatility is constant.

3. **B** The use of small time steps in the binomial model yields a more realistic model, a more accurate model, more complicated computations, and more computational expense.

4. **C** Callable bonds have the following characteristics:
   - *Less* price volatility.
   - Negative convexity.
   - Capital gains are capped as yields *fall*.
   - Exhibit increased reinvestment rate risk when yields fall.

5. **B** The value at any given node in a binomial tree is the average present value of the cash flows at the two possible states immediately to the right of the given node, discounted at the 1-period rate at the node under examination.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Evolution of Short Rates and the Shape of the Term Structure

Topic 12

Exam Focus

This topic discusses how the decision tree framework is used to estimate the price and returns of zero-coupon bonds. This decision tree framework illustrates how interest rate expectations determine the shape of the yield curve. For the exam, candidates should understand how current spot rates and forward rates are determined by the expectations, volatility, and risk premiums of short-term rates. Furthermore, the use of Jensen's inequality should be understood, and candidates should be prepared to use this formula to demonstrate how maturity and volatility increase the convexity of zero-coupon bonds.

Interest Rate Expectations

LO 12.1: Explain the role of interest rate expectations in determining the shape of the term structure.

Expectations of future interest rates are based on uncertainty. For example, an investor may expect that interest rates over the next period will be 8%. However, this investor may realize there is also a high probability that interest rates could be 7% or 9% over the next period.

Expectations play an important role in determining the shape of the yield curve and can be illustrated by examining yield curves that are flat, upward-sloping, and downward-sloping. In the following yield curve examples assume future interest rates are known and that there is no uncertainty in rates.

Flat Yield Curve

Suppose the 1-year interest rate is 8% and future 1-year forward rates are 8% for the next two years. Given these interest rate expectations, the present values of 1-, 2-, and 3-year zero-coupon bonds with $1 face values assuming annual compounding are calculated as follows:

$$\text{price of 1-year zero-coupon bond} = \frac{\$1}{1.08} = \$0.92593$$

$$\text{price of 2-year zero-coupon bond} = \frac{\$1}{(1.08)(1.08)} = \$0.85734$$

$$\text{price of 3-year zero-coupon bond} = \frac{\$1}{(1.08)(1.08)(1.08)} = \$0.79383$$

In this example, investors expect the 1-year spot rates for the next three years to be 8%. Thus, the yield curve is flat and investors are willing to lock in interest rates for two or three years at 8%.

Upward-Sloping Yield Curve

Now suppose the 1-year interest rate will remain at 8%, but investors expect the 1-year rate in one year to be 10% and the 1-year rate in two years to be 12%. The 2-year spot rate, $\hat{r}(2)$, must satisfy the following equation:

$$\text{price of 2-year zero-coupon bond} = \frac{\$1}{(1.08)(1.10)} = \frac{\$1}{(1+\hat{r}(2))^2}$$

Cross multiplying, taking the square root of each side, and subtracting one from each side results in the following equation:

$$\hat{r}(2) = \sqrt[2]{(1.08)(1.10)} - 1 = \sqrt{1.188} - 1$$

Solving this equation reveals that the 2-year spot rate is 8.995%.

The 3-year spot rate can be solved in a similar fashion:

$$\text{price of 3-year zero-coupon bond} = \frac{\$1}{(1.08)(1.10)(1.12)} = \frac{\$1}{(1+\hat{r}(3))^3}$$

Thus, the 3-year spot rate is computed as follows:

$$\hat{r}(3) = \sqrt[3]{(1.08)(1.10)(1.12)} - 1 = (1.33056)^{1/3} - 1 = 9.988\%$$

If expected 1-year spot rates for the next three years are 8%, 10%, and 12%, then this results in an upward-sloping yield curve of 1-, 2-, and 3-year spot rates of 8%, 8.995%, and 9.988%, respectively. Thus, the expected future spot rates will determine the upward-sloping shape of the yield curve.

Downward-Sloping Yield Curve

Now suppose the 1-year interest rate will remain at 8%, but investors expect the 1-year rate in one year to be 6% and the 1-year rate in two years to be 4%. The 2-year and 3-year spot rates are computed as follows:

$$\hat{r}(2) = \sqrt[2]{(1.08)(1.06)} - 1 = 6.995\%$$

$$\hat{r}(3) = \sqrt[3]{(1.08)(1.06)(1.04)} - 1 = 5.987\%$$

These calculations imply a downward-sloping yield curve for 1-, 2-, and 3-year spot rates of 8%, 6.995%, and 5.987%, respectively.

These three examples illustrate that expectations of future interest rates can describe the shape and level of the term structure for short-term horizons. If expected 1-year spot rates for the next three years are r_1, r_2, and r_3, then the 2-year and 3-year spot rates are computed as:

$$\hat{r}(2) = \sqrt[2]{(1+r_1)(1+r_2)} - 1$$

$$\hat{r}(3) = \sqrt[3]{(1+r_1)(1+r_2)(1+r_3)} - 1$$

In the short run, expected future spot rates determine the shape of the yield curve. In the long run, however, it is less likely that investors have confidence in 1-year spot rates several years from now (e.g., 30 years). Thus, expectations are unable to describe the shape of the term structure for long-term horizons. However, it is reasonable to assume that real rates and inflation rates are relatively constant over the long run. For example, a short-term rate of 5% may imply a long-run real rate of interest of 3% and a long-run inflation rate of 2%. Thus, interest rate expectations can describe the level of interest rates for long-term horizons.

Interest Rate Volatility

LO 12.2: Apply a risk-neutral interest rate tree to assess the effect of volatility on the shape of the term structure.

Suppose an investor is risk-neutral and has uncertainty regarding expectations of future interest rates. The decision tree in Figure 1 represents expectations that are used to determine the 1-year rate. If there is a 50% probability that the 1-year rate in one year will be 10% and a 50% probability that the 1-year rate in one year will 6%, the expected 1-year rate in one year can be calculated as:

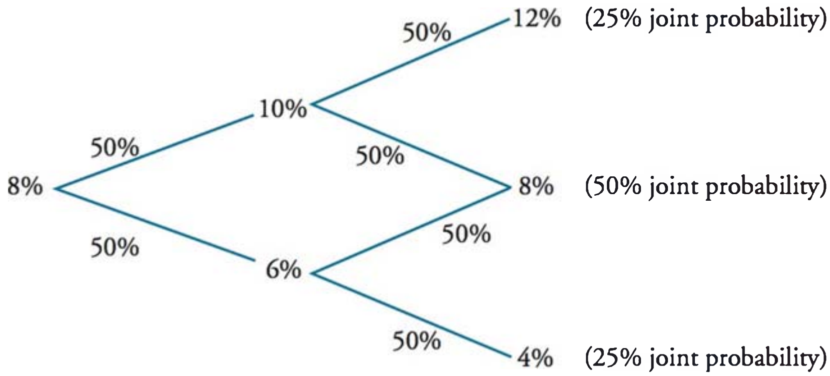
$$(0.5 \times 10\%) + (0.5 \times 6\%) = 8\%$$

Using the expected rates in Figure 1, the price of a 1-year zero-coupon bond is 0.92593% of par [calculated as $1 / 1.08 with a $1 par value].

The last column of the decision tree in Figure 1 illustrates the joint probabilities of 12%, 8%, or 4% 1-year rates in two years. The 12% rate in the upper node occurs 25% of the time (50% × 50%), the 8% rate in the middle node occurs 50% of the time (50% × 50% + 50% × 50%), and the 4% rate in the bottom node occurs 25% of the time (50% × 50%). Thus, the expected 1-year rate in two years is calculated as:

$$(0.25 \times 12\%) + (0.50 \times 8\%) + (0.25 \times 4\%) = 8\%$$

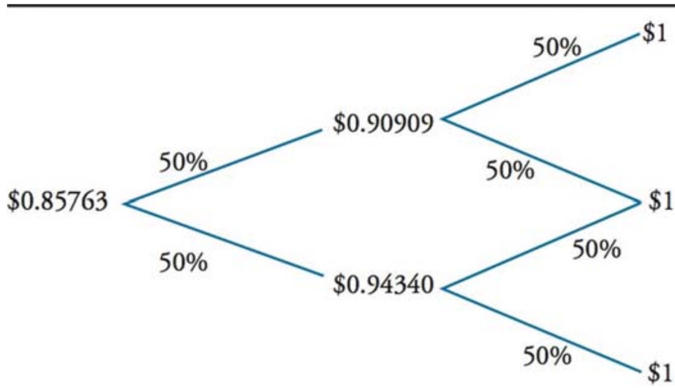
Figure 1: Decision Tree Illustrating Expected 1-Year Rates for Two Years



Assuming risk-neutrality, the decision tree in Figure 2 illustrates the calculation of the price of a 2-year zero-coupon bond with a face value of \$1 using the expected rates given in the Figure 1 decision tree. The expected price in one year for the upper node is \$0.90909, calculated as \$1 / 1.10. The expected price in one year for the lower node is \$0.94340, calculated as \$1 / 1.06. Thus, the current price of the 2-year zero-coupon bond is calculated as:

$$[0.5 \times (\$0.90909 / 1.08)] + [0.5 \times (\$0.94340 / 1.08)] = \$0.85763$$

Figure 2: Risk-Neutral Decision Tree for a 2-Year Zero-Coupon Bond



With the present value of the 2-year zero-coupon bond, we can computed the implied 2-year spot rate by solving for $\hat{r}(2)$ as follows:

$$\$0.85763 = \frac{\$1}{(1+\hat{r}(2))^2}$$

$$\hat{r}(2) = \sqrt{\frac{1}{0.85763}} - 1 = 0.079816 \text{ or } 7.9816\%$$

Alternatively, this can also be computed using a financial calculator as follows:

PV = –0.85763; FV = 1; PMT = 0; N = 2; CPT → I/Y = 7.9816%

This example illustrates that when there is uncertainty regarding expected rates, the volatility of expected rates causes the future spot rates to be lower. With the implied rate, we can compute the value of convexity for the 2-year zero-coupon bond as: 8% – 7.9816% = 0.0184% or 1.84 basis points.

Convexity Effect

LO 12.3: Estimate the convexity effect using Jensen's inequality.

LO 12.4: Evaluate the impact of changes in maturity, yield, and volatility on the convexity of a security.

The convexity effect can be measured by applying a special case of Jensen's inequality as follows:

$$E\left[\frac{1}{(1+r)}\right] > \frac{1}{E[1+r]}$$

Example: Applying Jensen's inequality

Assume that next year there is a 50% probability that 1-year spot rates will be 10% and a 50% probability that 1-year spot rates will be 6%. **Demonstrate** Jensen's inequality for a 2-year zero-coupon bond with a face value of $1 assuming the previous interest rate expectations shown in Figure 1.

Answer:

The left-hand side of Jensen's inequality is the expected price in one year using the 1-year spot rates of 10% and 6%.

$$E\left[\frac{\$1}{(1+r)}\right] = 0.5 \times \frac{\$1}{(1.10)} + 0.5 \times \frac{\$1}{(1.06)} = \$0.92624$$

The expected price in one year using an expected rate of 8% computes the right-hand side of the inequality as:

$$\frac{\$1}{0.5 \times 1.10 + 0.5 \times 1.06} = \frac{\$1}{1.08} = 0.92593$$

Thus, the left-hand side is greater than the right-hand side, $0.92624 > $0.92593.

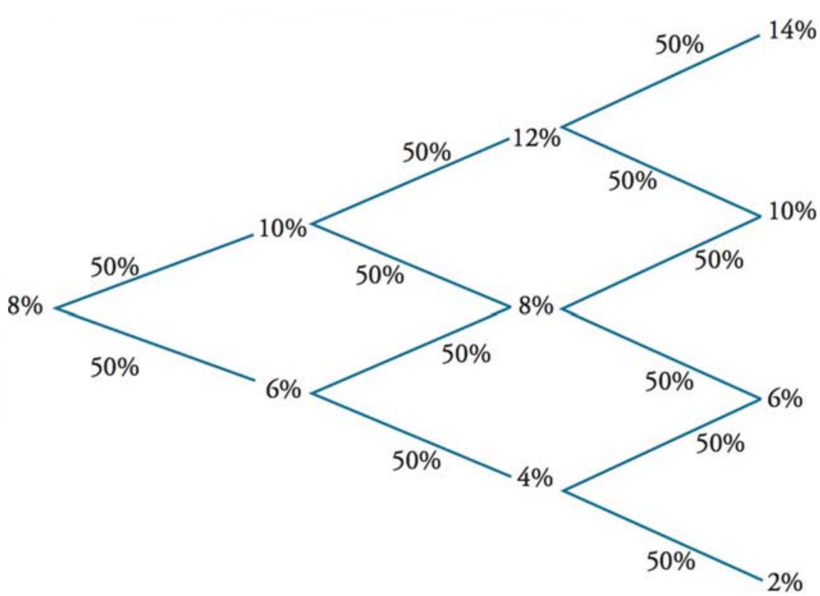
If the current 1-year rate is 8%, then the price of a 2-year zero-coupon bond is found by simply dividing each side of the equation by 1.08. In other words, discount the expected 1-year zero-coupon bond price for one more year at 8% to find the 2-year price. The price of the 2-year zero-coupon bond on the left-hand side of Jensen's inequality equals \$0.85763 (calculated as \$0.92624 / 1.08). The right-hand side is calculated as the price of a 2-year zero-coupon bond discounted for two years at the expected rate of 8%, which equals \$0.85734 (calculated as \$1 / 1.08^2).

The left-hand side is again greater than the right-hand side, \$0.85763 > \$0.85734.

This demonstrates that the price of the 2-year zero-coupon bond is greater than the price obtained by discounting the \$1 face amount by 8% over the first period and by 8% over the second period. Therefore, we know that since the 2-year zero-coupon price is higher than the price achieved through discounting, its implied rate must be lower than 8%.

Extending the above example out for one more year illustrates that convexity increases with maturity. Suppose an investor expects the spot rates to be 14%, 10%, 6%, or 2% in three years. Assuming each expected return has an equal probability of occurring results in the decision tree shown in Figure 3.

Figure 3: Risk-Neutral Decision Tree Illustrating Expected 1-Year Rates for Three Years



The decision tree in Figure 4 uses the expected spot rates from the decision tree in Figure 3 to calculate the price of a 3-year zero-coupon bond.

The price of a 1-year zero-coupon bond in two years with a face value of \$1 for the upper node is \$0.89286 (calculated as \$1 / 1.12). The price of a 1-year zero coupon bond in two years for the middle node is \$0.92593 (calculated as \$1 / 1.08). The price of a 1-year zero coupon bond in two years for the bottom node is \$0.96154 (calculated as \$1 / 1.04).

The price of a 2-year zero-coupon bond in one year using the upper node expected spot rates is calculated as:

$$[0.5 \times (\$0.89286 / 1.10)] + [0.5 \times (\$0.92593 / 1.10)] = \$0.82672$$

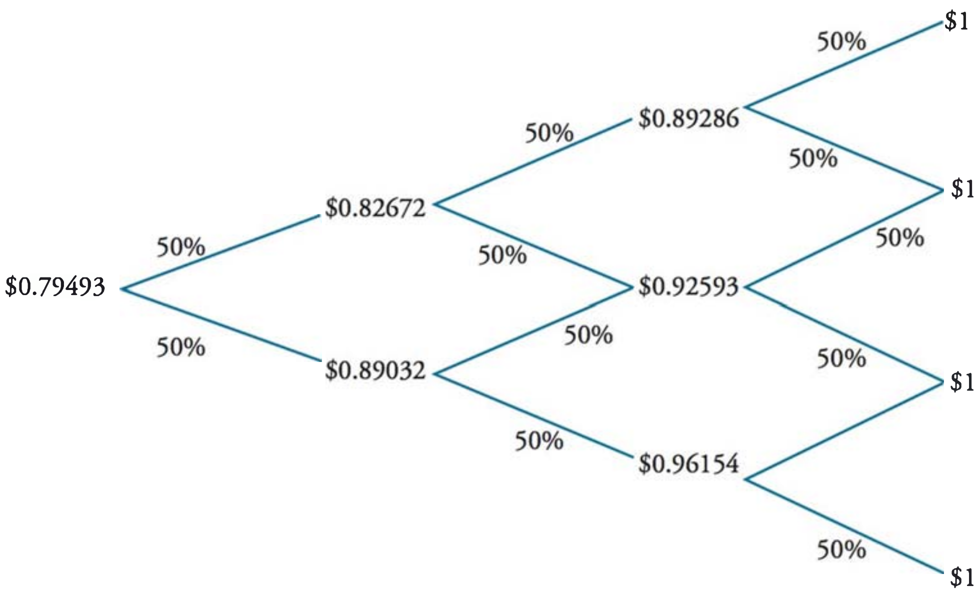
The price of a 2-year zero-coupon bond in one year using the bottom node expected spot rates is calculated as:

$$[0.5 \times (\$0.92593 / 1.06)] + [0.5 \times (\$0.96154 / 1.06)] = \$0.89032$$

Lastly, the price of a 3-year zero-coupon bond today is calculated as:

$$[0.5 \times (\$0.82672 / 1.08)] + [0.5 \times (\$0.89032 / 1.08)] = \$0.79493$$

Figure 4: Risk-Neutral Decision Tree for a 3-Year Zero-Coupon Bond



To measure the convexity effect, the implied 3-year spot rate is calculated by solving for $\hat{r}(3)$ in the following equation:

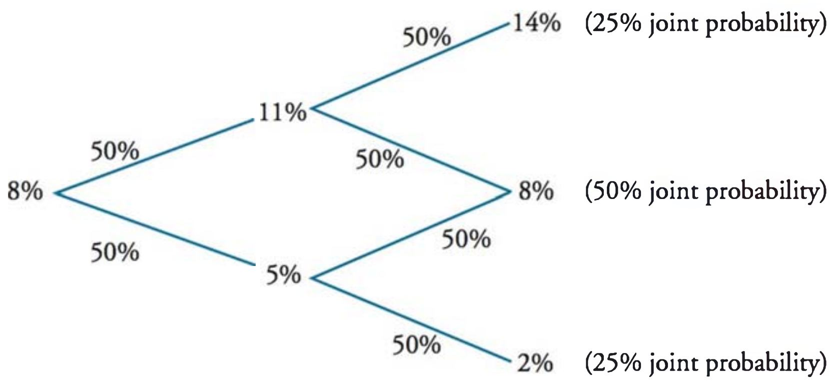
$$0.79493 = \frac{1}{(1 + r(3))^3}$$

$$\hat{r}(3) = \sqrt[3]{\frac{1}{0.79493}} - 1 = 0.0795 \text{ or } 7.95\%$$

Notice that convexity lowers bond yields and that this reduction in yields is equal to the value of convexity. For the 3-year zero-coupon bond, the value of convexity is 8% – 7.95% = 0.05% or 5 basis points. Recall that the value of convexity for the 2-year zero-coupon bond was only 1.84 basis points. Therefore, *all else held equal, the value of convexity* *increases with maturity*. In other words, as the maturity of a bond increases, the price-yield relationship becomes more convex.

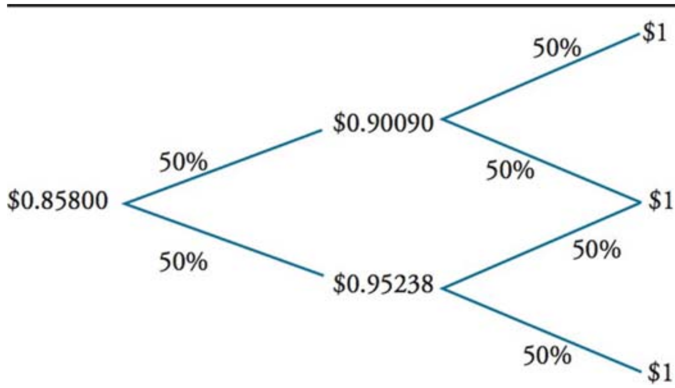
This convexity occurs due to volatility. Thus, we can also say that the value of convexity increases with volatility. The following decision trees in Figures 5 and 6 illustrate the impact of increasing the volatility of interest rates. In this example, the 1-year spot rate in one year in Figure 5 ranges from 2% to 14% instead of 4% to 12% as was shown in Figure 1.

Figure 5: Risk-Neutral Decision Tree Illustrating Volatility Effect on Convexity



Using the same methodology as before, the price of a 2-year zero-coupon bond with the listed expected interest rates in Figure 5 is \$0.858.

Figure 6: Price of a 2-Year Zero-Coupon Bond with Increased Volatility



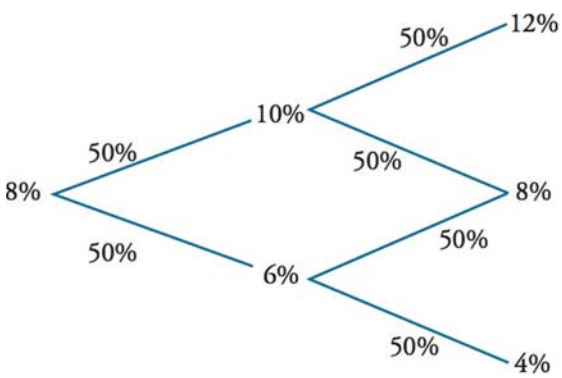
This price results in a 2-year implied spot rate of 7.958%. Thus, the value of convexity is 8% – 7.958% = 0.042% or 4.2 basis points. This is higher than the previous 2-year example where the value of convexity was 1.84 basis points when expected spot rates ranged from 4% to 12%, instead of 2% to 14%. Therefore, *the value of convexity increases with both volatility and time to maturity*.

Risk Premium

LO 12.5: Calculate the price and return of a zero coupon bond incorporating a risk premium.

Suppose an investor expects 1-year rates to resemble those in Figure 7. In this example, there is volatility of 400 basis point of rates per year where 1-year rates in one year range from 4% to 12% in the second year.

Figure 7: Decision Tree Illustrating Expected 1-Year Rates for Two Years



Next year, the 1-year return will be either 10% or 6%. A risk-neutral investor calculates the price of a 2-year zero-coupon bond with a face value of $1 as follows:

$$\frac{\left[\frac{\$1}{1.10}+\frac{\$1}{1.06}\right]\times 0.5}{1.08}=\frac{[\$0.90909+\$0.94340]\times 0.5}{1.08}=\$0.85763$$

In this example, the price of $0.85763 implies a 1-year expected return of 8%. However, this is only the average return. The actual return will be either 6% or 10%. Risk-averse investors would require a higher rate of return for this investment than an investment that has a certain 8% return with no variability. Thus, risk-averse investors require a risk premium for bearing this interest rate risk, and demand a return greater than 8% for buying a 2-year zero-coupon bond and holding it for the next year.

Example: Incorporating a risk premium

Calculate the price and return for the zero-coupon bond using the expected returns in Figure 7 and assuming a risk premium of 30 basis points for each year of interest rate risk.

Answer:

The price of a 2-year zero-coupon bond with a 30 basis point risk premium included is calculated as:

$$\frac{\left[\frac{\$1}{1.103}+\frac{\$1}{1.063}\right]\times 0.5}{1.08}=\frac{[\$0.90662+\$0.94073]\times 0.5}{1.08}=\$0.85525$$

Notice that this price is less than the $0.85763 price calculated previously for the risk-neutral investor. Next year, the price of the 2-year zero-coupon bond will either be $0.90909 or $0.94340, depending on whether the 1-year rate is either 10% or 6%, respectively. Thus, the expected return for the next year of the 2-year zero-coupon bond is 8.3%, calculated as follows:

$$\frac{(\$0.90909+\$0.94340)\times 0.5-\$0.85525}{\$0.85525}=0.083$$

Therefore, risk-averse investors require a 30 basis point premium or 8.3% return to compensate for one year of interest rate risk. For a 3-year zero-coupon bond, risk-averse investors will require a 60 basis point premium or 8.6% return given two years of interest rate risk.

Professor's Note: In the previous example, it is assumed that rates can change only once a year, so in the first year there is no uncertainty of interest rates. There is only uncertainty in what the 1-year rate will be one and two years from today.

Key Concepts

LO 12.1

If expected 1-year spot rates for the next three years are r_1, r_2, and r_3, then the 2-year spot rate, $\hat{r}(2)$, is computed as $\hat{r}(2)=\sqrt[2]{(1+r_1)(1+r_2)}-1$, and the 3-year spot rate, $\hat{r}(3)$, is computed as $\hat{r}(3)=\sqrt[3]{(1+r_1)(1+r_2)(1+r_3)}-1$.

LO 12.2

The volatility of expected rates creates convexity, which lowers future spot rates.

LO 12.3

The convexity effect can be measured by using Jensen's inequality: $E\left[\frac{1}{(1+r)}\right]>\frac{1}{E[1+r]}$.

LO 12.4

Convexity lowers bond yields due to volatility. This reduction in yields is equal to the value of convexity. Thus, we can say that the value of convexity increases with volatility. The value of convexity will also increase with maturity, because the price-yield relationship will become more convex over time.

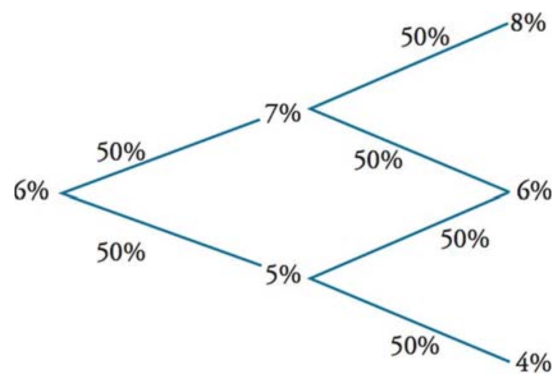
LO 12.5

Risk-averse investors will price bonds with a risk premium to compensate them for taking on interest rate risk.

Concept Checkers

1. An investor expects the current 1-year rate for a zero-coupon bond to remain at 6%, the 1-year rate next year to be 8%, and the 1-year rate in two years to be 10%. What is the 3-year spot rate for a zero-coupon bond with a face value of $1, assuming all investors have the same expectations of future 1-year rates for zero-coupon bonds?
   A. 7.888%.
   B. 7.988%.
   C. 8.000%.
   D. 8.088%.

2. Suppose investors have interest rate expectations as illustrated in the decision tree below where the 1-year rate is expected to be 8%, 6%, or 4% in the second year and either 7% or 5% in the first year for a zero-coupon bond.



   If investors are risk-neutral, what is the price of a $1 face value 2-year zero-coupon bond today?
   A. $0.88113.
   B. $0.88634.
   C. $0.89007.
   D. $0.89032.

3. If investors are risk-neutral and the price of a 2-year zero-coupon bond is $0.88035 today, what is the implied 2-year spot rate?
   A. 4.339%.
   B. 5.230%.
   C. 5.827%.
   D. 6.579%.

4. What is the impact on the bond price-yield curve if, all other factors held constant, the maturity of a zero-coupon bond increases? The pricing curve becomes:
   A. less concave.
   B. more concave.
   C. less convex.
   D. more convex.

5. Suppose an investor expects that the 1-year rate will remain at 6% for the first year for a 2-year zero-coupon bond. The investor also projects a 50% probability that the 1-year spot rate will be 8% in one year and a 50% probability that the 1-year spot rate will be 4% in one year. Which of the following inequalities most accurately reflects the convexity effect for this 2-year bond using Jensen's inequality formula?
A. \$0.89031 > \$0.89000.
B. \$0.89000 > \$0.80000.
C. \$0.94340 > \$0.89031.
D. \$0.94373 > \$0.94340.

Concept Checker Answers

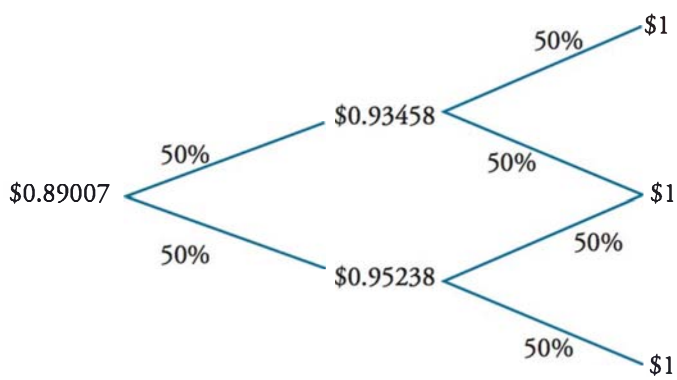
1. B The 3-year spot rate can be solved for using the following equation:

$$\frac{\$1}{(1.06)(1.08)(1.10)} = \frac{\$1}{(1+\hat{r}(3))^3}$$

Solving for $\hat{r}(3) = \sqrt[3]{(1.06)(1.08)(1.10)} - 1 = 7.988\%$

2. C Assuming investors are risk-neutral, the following decision tree illustrates the calculation of the price of a 2-year zero-coupon bond using the expected rates given. The expected price in one year for the upper node is \$0.93458, calculated as \$1 / 1.07. The expected price in one year for the lower node is \$0.95238, calculated as \$1 / 1.05. Thus, the current price is \$0.89007, calculated as:

[0.5 × (\$0.93458 / 1.06)] + [0.5 × (\$0.95238 / 1.06)] = \$0.89007



3. D The implied 2-year spot rate is calculated by solving for $\hat{r}(2)$ in the following equation:

$$\$0.88035 = \frac{\$1}{(1+\hat{r}(2))^2}$$

$$\hat{r}(2) = \sqrt{\frac{1}{0.88035}} - 1 = 0.06579 \text{ or } 6.579\%$$

Alternatively, this can also be computed using a financial calculator as follows:

PV = –0.88035; FV = 1; PMT = 0; N = 2; CPT → I/Y = 6.579%.

4. D As the maturity of a bond increases, the price-yield relationship becomes more convex.

5. A The left-hand side of Jensen's inequality is the expected price in one year using the 1-year spot rates of 8% and 4%.

$$E\left[\frac{\$1}{(1+r)}\right] = 0.5 \times \frac{\$1}{(1.08)} + 0.5 \times \frac{\$1}{(1.04)} = 0.5 \times 0.92593 + 0.5 \times \$0.96154 = \$0.94373$$

The expected price in one year using an expected rate of 6% computes the right-hand side of the inequality as:

$$\frac{\$1}{0.5 \times 1.08 + 0.5 \times 1.04} = \frac{\$1}{1.06} = 0.94340$$

Next, divide each side of the equation by 1.06 to discount the expected 1-year zero-coupon bond price for one more year at 6%. The price of the 2-year zero-coupon bond equals \$0.89031 (calculated as \$0.94373 / 1.06), which is greater than \$0.89000 (the price of a 2-year zero-coupon bond discounted for two years at the expected rate of 6%). Thus, Jensen's inequality reveals that \$0.89031 > \$0.89000.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Art of Term Structure Models: Drift

Topic 13

Exam Focus

This topic introduces different term structure models for estimating short-term interest rates. Specifically, we will discuss models that have no drift (Model 1), constant drift (Model 2), time-deterministic drift (Ho-Lee), and mean-reverting drift (Vasicek). For the exam, understand the differences between these short rate models, and know how to construct a two-period interest rate tree using model predictions. Also, know how the limitations of each model impact model effectiveness. For the Vasicek model, understand how to convert a nonrecombining tree into a combining tree.

LO 13.1: Construct and describe the effectiveness of a short term interest rate tree assuming normally distributed rates, both with and without drift.

Term Structure Model with No Drift (Model 1)

This topic begins with the simplest model for predicting the evolution of short rates (Model 1), which is used in cases where there is no drift and interest rates are normally distributed. The continuously compounded instantaneous rate, denoted r_t, will change (over time) according to the following relationship:

$$dr = \sigma dw$$

where:
dr = change in interest rates over small time interval, dt
dt = small time interval (measured in years) (e.g., one month = 1/12)
σ = annual basis-point volatility of rate changes
dw = normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$

Given this definition, we can build an interest rate tree using a binomial model. The probability of up and down movements will be the same from period to period (50% up and 50% down) and the tree will be recombining. Since the tree is recombining, the up-down path ends up at the same place as the down-up path in the second time period.

For example, consider the evolution of interest rates on a monthly basis. Assume the current short-term interest rate is 6% and annual volatility is 120bps. Using the above notation, $r_0 = 6\%$, $\sigma = 1.20\%$, and $dt = 1/12$. Therefore, *dw* has a mean of 0 and standard deviation of $\sqrt{1/12} = 0.2887$.

After one month passes, assume the random variable *dw* takes on a value of 0.2 (drawn from a normal distribution with mean = 0 and standard deviation = 0.2887). Therefore, the

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change in interest rates over one month is calculated as: dr = 1.20% × 0.2 = 0.24% = 24 basis points. Since the initial rate was 6% and interest rates "changed" by 0.24%, the new spot rate in one month will be: 6% + 0.24% = 6.24%.

LO 13.2: Calculate the short-term rate change and standard deviation of the rate change using a model with normally distributed rates and no drift.

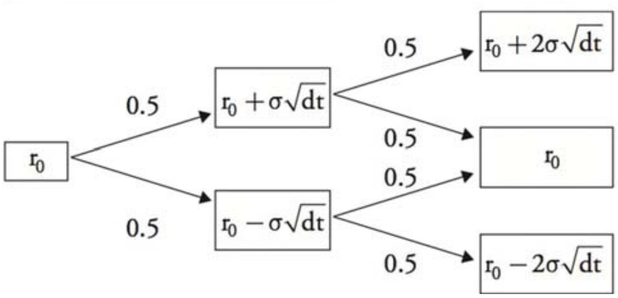
In Model 1, since the expected value of *dw* is zero [i.e., E(dw) = 0], the drift will be zero. Also, since the standard deviation of dw = $\sqrt{dt}$, the volatility of the rate change = $\sigma\sqrt{dt}$. This expression is also referred to as the standard deviation of the rate.

In the preceding example, the standard deviation of the rate is calculated as:

$$1.2\% \times \sqrt{1/12} = 0.346\% = 34.6 \text{ basis points}$$

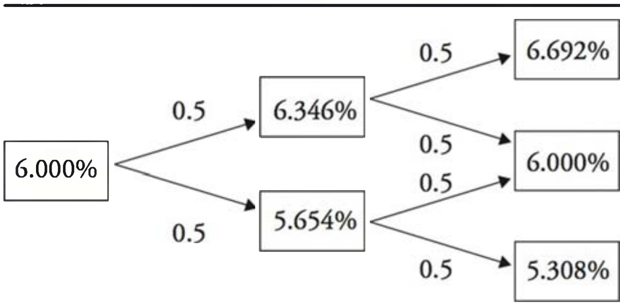
Returning to our previous discussion, we are now ready to construct an interest rate tree using Model 1. A generic interest rate tree over two periods is presented in Figure 1. Note that this tree is recombining and the ending rate at time 2 for the middle node is the same as the initial rate, r_0. Hence, the model has no drift.

Figure 1: Interest Rate Tree with No Drift



The interest rate tree using the previous numerical example is shown in Figure 2. One period from now, the observed interest rate will either increase with 50% probability to: 6% + 0.346% = 6.346% or decrease with 50% probability to: 6% – 0.346% = 5.654%. Extending to two periods completes the tree with upper node: 6% + 2(0.346%) = 6.692%, middle node: 6% (unchanged), and lower node: 6% – 2(0.346%) = 5.308%.

Figure 2: Numerical Example of Interest Rate Tree with No Drift



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LO 13.3: Describe methods for addressing the possibility of negative short-term rates in term structure models.

Note that the terminal nodes in the two-period model generate three possible ending rates: $r_0 + 2\sigma\sqrt{dt}$, r_0, and $r_0 - 2\sigma\sqrt{dt}$. This discrete, finite set of outcomes does not technically represent a normal distribution. However, our knowledge of probability distributions tells us that as the number of steps increases, the terminal distribution at the nodes will approach a continuous normal distribution.

One obvious drawback to Model 1 is that there is always a positive probability that interest rates could become negative. On the surface, negative interest rates do not make much economic sense (i.e., lending $100 and receiving less than $100 back in the future). However, you could plausibly rationalize a small negative interest rate if the safety and/or inconvenience of holding cash were sufficiently high.

The negative interest rate problem will be exacerbated as the investment horizon gets longer, since it is more likely that forecasted interest rates will drop below zero. As an illustration, assume a ten-year horizon and a standard deviation of terminal interest rates of $1.2\% \times \sqrt{10} = 3.79\%$. It is clear that negative interest rates will be well within a two standard deviation confidence interval when centered around a current rate of 6%. Also note that the problem of negative interest rates is greater when the current level of interest rates is low (e.g., 4% instead of the original 6%).

There are two reasonable solutions for negative interest rates. First, the model could use distributions that are always non-negative, such as lognormal or chi-squared distributions. In this way, the interest rate can never be negative, but this action may introduce other non-desirable characteristics such as skewness or inappropriate volatilities. Second, the interest rate tree can "force" negative interest rates to take a value of zero. In this way, the original interest rate tree is adjusted to constrain the distribution from being below zero. This method may be preferred over the first method because it forces a change in the original distribution only in a very low interest rate environment whereas changing the entire distribution will impact a much wider range of rates.

As a final note, it is ultimately up to the user to decide on the appropriateness of the model. For example, if the purpose of the term structure model is to price coupon-paying bonds, then the valuation is closely tied to the average interest rate over the life of the bond and the possible effect of negative interest rates (small probability of occurring or staying negative for long) is less important. On the other hand, option valuation models that have asymmetric payoffs will be more affected by the negative interest rate problem.

Model 1 Effectiveness

Given the no-drift assumption of Model 1, we can draw several conclusions regarding the effectiveness of this model for predicting the shape of the term structure:

- The no-drift assumption does not give enough flexibility to accurately model basic term structure shapes. The result is a downward-sloping predicted term structure due to a larger convexity effect. Recall that the convexity effect is the difference between the model par yield using its assumed volatility and the par yield in the structural model with assumed zero volatility.

- Model 1 predicts a flat term structure of volatility, whereas the observed volatility term structure is hump-shaped, rising and then falling.
- Model 1 only has one factor, the short-term rate. Other models that incorporate additional factors (e.g., drift, time-dependent volatility) form a richer set of predictions.
- Model 1 implies that any change in the short-term rate would lead to a parallel shift in the yield curve, again, a finding incongruous with observed (non-parallel) yield curve shifts.

Term Structure Model with Drift (Model 2)

Casual term structure observation typically reveals an upward-sloping yield curve, which is at odds with Model 1, which does not incorporate drift. A natural extension to Model 1 is to add a positive drift term that can be economically interpreted as a positive risk premium associated with longer time horizons. We can augment Model 1 with a constant drift term, which yields Model 2:

$$dr = \lambda dt + \sigma dw$$

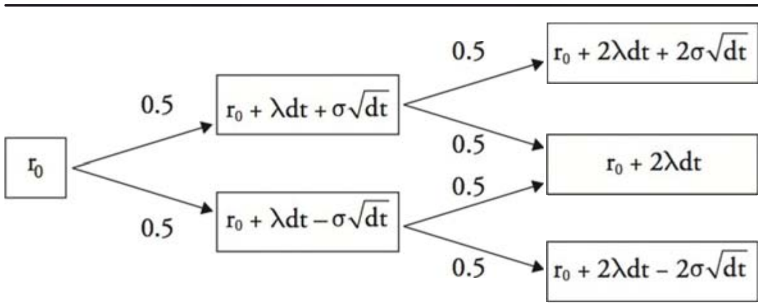
Let's continue with a new example assuming a current short-term interest rate, r_0, of 5%, drift, λ, of 0.24%, and standard deviation, σ, of 1.50%. As before, the *dw* realization drawn from a normal distribution (with mean = 0 and standard deviation = 0.2887) is 0.2. Thus, the change in the short-term rate in one month is calculated as:

$$dr = 0.24\% \times (1/12) + 1.5\% \times 0.2 = 0.32\%$$

Hence, the new rate, r_1, is computed as: 5% + 0.32% = 5.32%. The monthly drift is 0.24% × 1/12 = 0.02% and the standard deviation of the rate is $1.5\% \times \sqrt{1/12} = 0.43\%$ (i.e., 43 basis points per month). The 2bps drift per month (0.02%) represents any combination of expected changes in the short-term rate (i.e., true drift) and a risk premium. For example, the 2bps observed drift could result from a 1.5bp change in rates coupled with a 0.5bp risk premium.

The interest rate tree for Model 2 will look very similar to Model 1, but the drift term, λdt, will increase by λdt in the next period, $2\lambda dt$ in the second period, and so on. This is visually represented in Figure 3. Note that the tree recombines at time 2, but the value at time 2, $r_0 + 2\lambda dt$, is greater than the original rate, r_0, due to the positive drift.

Figure 3: Interest Rate Tree with Constant Drift



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Model 2 Effectiveness

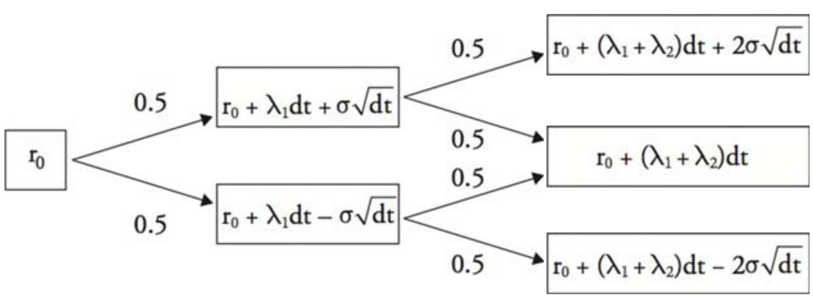
As you would expect, Model 2 is more effective than Model 1. Intuitively, the drift term can accommodate the typically observed upward-sloping nature of the term structure. In practice, a researcher is likely to choose r_0 and λ based on the calibration of observed rates. Hence, the term structure will fit better. The downside of this approach is that the estimated value of drift could be relatively high, especially if considered as a risk premium only. On the other hand, if the drift is viewed as a combination of the risk premium and the expected rate change, the model suggests that the expected rates in year 10 will be higher than year 9, for example. This view is more appropriate in the short run, since it is more difficult to justify increases in expected rates in the long run.

Ho-Lee Model

LO 13.4: Construct a short-term rate tree under the Ho-Lee Model with time-dependent drift.

The Ho-Lee model further generalizes the drift to incorporate time-dependency. That is, the drift in time 1 may be different than the drift in time 2; additionally, each drift does not have to increase and can even be negative. Thus, the model is more flexible than the constant drift model. Once again, the drift is a combination of the risk premium over the period and the expected rate change. The tree in Figure 4 illustrates the interest rate structure and effect of time-dependent drift.

Figure 4: Interest Rate Tree with Time-Dependent Drift



It is clear that if $\lambda_1 = \lambda_2$ then the Ho-Lee model reduces to Model 2. Also, it should not be surprising that λ_1 and λ_2 are estimated from observed market prices. In other words, r_0 is the observed one-period spot rate. λ_1 could then be estimated so that the model rate equals the observed two-period market rate. λ_2 could be calibrated from using r_0 and λ_1 and the observed market rate for a three-period security, and so on.

Arbitrage-Free Models

LO 13.5: Describe uses and benefits of the arbitrage-free models and assess the issue of fitting models to market prices.

Broadly speaking, there are two types of models: arbitrage-free models and equilibrium models. The key factor in choosing between these two models is based on the need to match market prices. Arbitrage models are often used to quote the prices of securities that are illiquid or customized. For example, an arbitrage-free tree is constructed to properly price on-the-run Treasury securities (i.e., the model price must match the market price). Then, the arbitrage-free tree is used to predict off-the-run Treasury securities and is compared to market prices to determine if the bonds are properly valued. These arbitrage models are also commonly used for pricing derivatives based on observable prices of the underlying security (e.g., options on bonds).

There are two potential detractors of arbitrage-free models. First, calibrating to market prices is still subject to the suitability of the original pricing model. For example, if the parallel shift assumption is not appropriate, then a better fitting model (by adding drift) will still be faulty. Second, arbitrage models assume the underlying prices are accurate. This will not be the case if there is an external, temporary, exogenous shock (e.g., oversupply of securities from forced liquidation, which temporarily depresses market prices).

If the purpose of the model is relative analysis (i.e., comparing the value of one security to another), then using arbitrage-free models, which assume both securities are properly priced, is meaningless. Hence, for relative analysis, equilibrium models would be used rather than arbitrage-free models.

Vasicek Model

LO 13.6: Describe the process of constructing a simple and recombining tree for a short-term rate under the Vasicek Model with mean reversion.

The Vasicek model assumes a mean-reverting process for short-term interest rates. The underlying assumption is that the economy has an equilibrium level based on economic fundamentals such as long-run monetary supply, technological innovations, and similar factors. Therefore, if the short-term rate is above the long-run equilibrium value, the drift adjustment will be negative to bring the current rate closer to its mean-reverting level. Similarly, short-term rates below the long-run equilibrium will have a positive drift adjustment. Mean reversion is a reasonable assumption but clearly breaks down in periods of extremely high inflation (i.e., hyperinflation) or similar structural breaks.

The formal Vasicek model is as follows:

$$dr = k(\theta - r)dt + \sigma dw$$

where:
k = a parameter that measures the speed of reversion adjustment
θ = long-run value of the short-term rate assuming risk neutrality
r = current interest rate level

In this model, k measures the speed of the mean reversion adjustment; a high k will produce quicker (larger) adjustments than smaller values of k. A larger differential between the long-run and current rates will produce a larger adjustment in the current period.

Similar to the previous discussion, the drift term, λ, is a combination of the expected rate change and a risk premium. The risk neutrality assumption of the long-run value of the short-term rate allows θ to be approximated as:

$$\theta \approx r_l + \frac{\lambda}{k}$$

where:
r_l = the long-run true rate of interest

Let's consider a numerical example with a reversion adjustment parameter of 0.03, annual standard deviation of 150 basis points, a true long-term interest rate of 6%, a current interest rate of 6.2%, and annual drift of 0.36%. The long-run value of the short-term rate assuming risk neutrality is approximately:

$$\theta \approx 6\% + \frac{0.36\%}{0.03} = 18\%$$

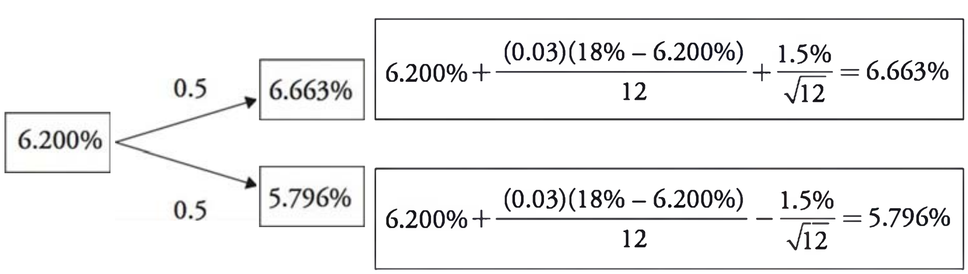
It follows that the forecasted change in the short-term rate for the next period is:

$$0.03(18\% - 6.2\%)(1/12) = 0.0295\%$$

The volatility for the monthly interval is computed as $1.5\% \times \sqrt{1/12} = 0.43\%$ (43 basis points per month).

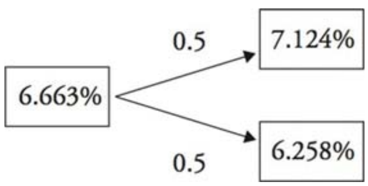
The next step is to populate the interest rate tree. Note that this tree will not recombine in the second period because the adjustment in time 2 after a downward movement in interest rates will be larger than the adjustment in time 2 following an upward movement in interest rates (since the lower node rate is further from the long-run value). This can be illustrated directly in the following calculations. Starting with r_0 = 6.2%, the interest rate tree over the first period is:

Figure 5: First Period Upper and Lower Node Calculations



If the interest rate evolves upward in the first period, we would turn to the upper node in the second period. The interest rate process can move up to 7.124% or down to 6.258%.

Figure 6: Second Period Upper Node Calculations

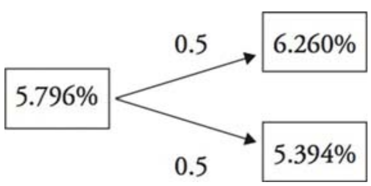


$$6.663\% + \frac{(0.03)(18\% - 6.663\%)}{12} + \frac{1.5\%}{\sqrt{12}} = 7.124\%$$

$$6.663\% + \frac{(0.03)(18\% - 6.663\%)}{12} - \frac{1.5\%}{\sqrt{12}} = 6.258\%$$

If the interest evolves downward in the first period, we would turn to the lower node in the second period. The interest rate process can move up to 6.260% or down to 5.394%.

Figure 7: Second Period Lower Node Calculations

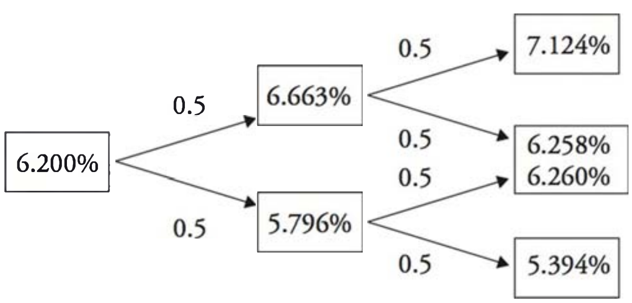


$$5.796\% + \frac{(0.03)(18\% - 5.796\%)}{12} + \frac{1.5\%}{\sqrt{12}} = 6.260\%$$

$$5.796\% + \frac{(0.03)(18\% - 5.796\%)}{12} - \frac{1.5\%}{\sqrt{12}} = 5.394\%$$

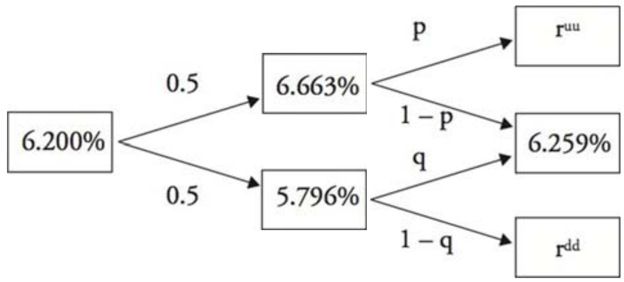
Finally, we complete the 2-period interest rate tree with mean reversion. The most interesting observation is that the model is not recombining. The up-down path leads to a 6.258% rate while the down-up path leads to a 6.260% rate. In addition, the down-up path rate is larger than the up-down path rate because the mean reversion adjustment has to be larger for the down path, as the initial interest rate was lower (5.796% versus 6.663%).

Figure 8: 2-Period Interest Rate Tree with Mean Reversion



At this point, the Vasicek model has generated a 2-period non-recombining tree of short-term interest rates. It is possible to modify the methodology so that a recombining tree is the end result. There are several ways to do this, but we will outline one straightforward method. The first step is to take an average of the two middle nodes (6.258% + 6.260%) / 2 = 6.259%. Next, we remove the assumption of 50% up and 50% down movements by generically replacing them with (p, $1 - p$) and (q, $1 - q$) as shown in Figure 9.

Figure 9: Recombining the Interest Rate Tree



The final step for recombining the tree is to solve for p and q and r^{uu} and r^{dd}. p and q are the respective probabilities of up movements in the trees in the second period after the up and down movements in the first period. r^{uu} and r^{dd} are the respective interest rates from successive (up, up and down, down) movements in the tree.

We can solve for the unknown values using a system of equations. First, we know that the average of $p \times r^{uu}$ and $(1 - p) \times 6.259\%$ must equal:

$$6.663\% + 0.03(18\% - 6.663\%)(1/12) = 6.691\%$$

Second, we can use the definition of standard deviation to equate:

$$\sqrt{p(r^{uu} - 6.691\%)^2 + (1 - p)(6.259\% - 6.691\%)^2} = 1.50\% \times \sqrt{\frac{1}{12}}$$

We would then repeat the process for the bottom portion of the tree, solving for q and r^{dd}. If the tree extends into a third period, the entire process repeats iteratively.

LO 13.7: Calculate the Vasicek Model rate change, standard deviation of the rate change, expected rate in T years, and half life.

The previous discussion encompassed the rate change in the Vasicek model and the computation of the standard deviation when solving for the parameters in the recombining tree. In this section, we turn our attention to the forecasted rate in *T* years.

To continue with the previous example, the current short-term rate is 6.2% with the mean-reversion parameter, k, of 0.03. The long-term mean-reverting level will eventually reach 18%, but it will take a long time since the value of k is quite small. Specifically, the current rate of 6.2% is 11.8% from its ultimate natural level and this difference will decay exponentially at the rate of mean reversion (11.8% is calculated as 18% – 6.2%). To forecast the rate in 10 years, we note that $11.8\% \times e^{(-0.03 \times 10)} = 8.74\%$. Therefore, the expected rate in 10 years is 18% – 8.74% = 9.26%.

In the Vasicek model, the expected rate in T years can be represented as the weighted average between the current short-term rate and its long-run horizon value. The weighting factor for the short-term rate decays exponentially by the speed of the mean-reverting parameter, θ:

$$r_0e^{-kT} + \theta(1 - e^{-kT})$$

A more intuitive measure for computing the forecasted rate in T years uses a factor's half-life, which measures the number of years to close half the distance between the starting rate and mean-reverting level. Numerically:

$$(18\% - 6.2\%)e^{-0.03\tau} = ½(18\% - 6.2\%)$$

$$e^{-0.03\tau} = ½ \rightarrow \tau = \ln(2) / 0.03 = 23.1 \text{ years}$$

Professor's Note: A larger mean reversion adjustment parameter, k, *will result in a shorter half-life.*

Vasicek Model Effectiveness

LO 13.8: Describe the effectiveness of the Vasicek Model.

There are some general comments that we can make to compare mean-reverting (Vasicek) models to models without mean reversion. In development of the mean-reverting model, the parameters r_0 and θ were calibrated to match observed market prices. Hence, the mean reversion parameter not only improves the specification of the term structure, but also produces a specific term structure of volatility. Specifically, the Vasicek model will produce a term structure of volatility that is declining. Therefore, short-term volatility is overstated and long-term volatility is understated. In contrast, Model 1 with no drift generates a flat volatility of interest rates across all maturities.

Furthermore, consider an upward shift in the short-term rate. In the mean-reverting model, the short-term rate will be impacted more than long-term rates. Therefore, the Vasicek model does not imply parallel shifts from exogenous liquidity shocks. Another interpretation concerns the nature of the shock. If the shock is based on short-term economic news, then the mean reversion model implies the shock dissipates as it approaches the long-run mean. The larger the mean reversion parameter, the quicker the economic news is incorporated. Similarly, the smaller the mean reversion parameter, the longer it takes for the economic news to be assimilated into security prices. In this case, the economic news is long-lived. In contrast, shocks to short-term rates in models without drift affect all rates equally regardless of maturity (i.e., produce a parallel shift).

Key Concepts

LO 13.1

Model 1 assumes no drift and that interest rates are normally distributed. The continuously compounded instantaneous rate, r_t, will change according to:

$$dr = \sigma dw$$

Model 1 limitations:

- The no-drift assumption is not flexible enough to accommodate basic term structure shapes.
- The term structure of volatility is predicted to be flat.
- There is only one factor, the short-term rate.
- Any change in the short-term rate would lead to a parallel shift in the yield curve.

Model 2 adds a constant drift: $dr = \lambda dt + \sigma dw$. The new interest rate tree increases each node in the next time period by λdt. The drift combines the expected rate change with a risk premium. The interest rate tree is still recombining, but the middle node rate at time 2 will not equal the initial rate, as was the case with Model 1.

Model 2 limitations:

- The calibrated values of drift are often too high.
- The model requires forecasting different risk premiums for long horizons where reliable forecasts are unrealistic.

LO 13.2

The interest rate tree for Model 1 is recombining and will increase/decrease each period by the same 50% probability.

LO 13.3

The normality assumption of the terminal interest rates for Model 1 will always have a positive probability of negative interest rates. One solution to eliminate this negative rate problem is to use non-negative distributions, such as the lognormal distribution; however, this may introduce other undesirable features into the model. An alternative solution is to create an adjusted interest rate tree where negative interest rates are replaced with 0%, constraining the data from being negative.

LO 13.4

The Ho-Lee model introduces even more flexibility than Model 2 by allowing the drift term to vary from period to period (i.e., time-dependent drift). The recombined middle node at time 2 = $r_0 + (\lambda_1 + \lambda_2)dt$.

LO 13.5

Arbitrage models are often used to price securities that are illiquid or off-market (e.g., uncommon maturity for a swap). The more liquid security prices are used to develop a consistent pricing model, which in turn is used for illiquid or non-standard securities. Because arbitrage models assume the market price is "correct," the models will not be effective if there are short-term imbalances altering bond prices. Similarly, arbitrage-free models cannot be used in relative valuation analysis because the securities being compared are already assumed to be properly priced.

LO 13.6

The Vasicek model assumes mean reversion to a long-run equilibrium rate. The specific functional form of the Vasicek model is as follows:

$$dr = k(\theta - r)dt + \sigma dw$$

The parameter k measures the speed of the mean reversion adjustment; a high k will produce quicker (larger) adjustments than smaller values of k. Assuming there is a long-run interest rate of r_l, the long-run mean-reverting level is:

$$\theta \approx r_l + \frac{\lambda}{k}$$

The Vasicek model is not recombining. The tree can be approximated as recombining by averaging the unequal two nodes and recalibrating the associated probabilities (i.e., no longer using 50% probabilities for the up and down moves).

LO 13.7

The expected rate in T years can be forecasted assuming exponential decay of the difference between the current level and the mean-reverting level. The half-life, τ, can be computed as the time to move halfway between the current level and the mean-reverting level:

$$(\theta - r_0)e^{-k\tau} = ½(\theta - r_0)$$

LO 13.8

The Vasicek model not only improves the specification of the term structure, but also produces a downward-sloping term structure of volatility. Model 1, on the other hand, predicts flat volatility of interest rates across all maturities. Model 1 implies parallel shifts from exogenous shocks while the Vasicek model does not. Long- (short-) lived economic shocks have low (high) mean reversion parameters. In contrast, in Model 1, shocks to short-term rates affect all rates equally regardless of maturity.

Concept Checkers

1. Using Model 1, assume the current short-term interest rate is 5%, annual volatility is 80bps, and *dw*, a normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$, has an expected value of zero. After one month, the realization of *dw* is –0.5. What is the change in the spot rate and the new spot rate?

	Change in Spot	New Spot Rate
A.	0.40%	5.40%
B.	–0.40%	4.60%
C.	0.80%	5.80%
D.	–0.80%	4.20%

2. Using Model 2, assume a current short-term rate of 8%, an annual drift of 50bps, and a short-term rate standard deviation of 2%. In addition, assume the ex-post realization of the *dw* random variable is 0.3. After constructing a 2-period interest rate tree with annual periods, what is the interest rate in the middle node at the end of year 2?
   A. 8.0%.
   B. 8.8%.
   C. 9.0%.
   D. 9.6%.

3. The Bureau of Labor Statistics has just reported an unexpected short-term increase in high-priced luxury automobiles. What is the most likely anticipated impact on a mean-reverting model of interest rates?
   A. The economic information is long-lived with a low mean-reversion parameter.
   B. The economic information is short-lived with a low mean-reversion parameter.
   C. The economic information is long-lived with a high mean-reversion parameter.
   D. The economic information is short-lived with a high mean-reversion parameter.

4. Using the Vasicek model, assume a current short-term rate of 6.2% and an annual volatility of the interest rate process of 2.5%. Also assume that the long-run mean-reverting level is 13.2% with a speed of adjustment of 0.4. Within a binomial interest rate tree, what are the upper and lower node rates after the first month?

	Upper node	Lower node
A.	6.67%	5.71%
B.	6.67%	6.24%
C.	7.16%	6.24%
D.	7.16%	5.71%

5. John Jones, FRM, is discussing the appropriate usage of mean-reverting models relative to no-drift models, models that incorporate drift, and Ho-Lee models. Jones makes the following statements:

Statement 1: Both Model 1 (no drift) and the Vasicek model assume parallel shifts from changes in the short-term rate.

Statement 2: The Vasicek model assumes decreasing volatility of future short-term rates while Model 1 assumes constant volatility of future short-term rates.

Statement 3: The constant drift model (Model 2) is a more flexible model than the Ho-Lee model.

How many of his statements are correct?
A. 0.
B. 1.
C. 2.
D. 3.

Concept Checker Answers

1. **B** Model 1 has a no-drift assumption. Using this model, the change in the interest rate is predicted as:

   $dr = \sigma dw$
   $dr = 0.8\% \times (-0.5) = -0.4\% = -40$ basis points

   Since the initial rate was 5% and $dr = -0.40\%$, the new spot rate in one month is:

   $5\% - 0.40\% = 4.60\%$

2. **C** Using Model 2 notation:

   current short-term rate, $r_0 = 8\%$
   drift, $\lambda = 0.5\%$
   standard deviation, $\sigma = 2\%$
   random variable, $dw = 0.3$
   change in time, $dt = 1$

   Since we are asked to find the interest rate at the second period middle node using Model 2, we know that the tree will recombine to the following rate: $r_0 + 2\lambda dt$.

   $8\% + 2 \times 0.5\% \times 1 = 9\%$

3. **D** The economic news is most likely short-term in nature. Therefore, the mean reversion parameter is high so the mean reversion adjustment per period will be relatively large.

4. **D** Using a Vasicek model, the upper and lower nodes for time 1 are computed as follows:

   $$\text{upper node} = 6.2\% + \frac{(0.4)(13.2\% - 6.2\%)}{12} + \frac{2.5\%}{\sqrt{12}} = 7.16\%$$

   $$\text{lower node} = 6.2\% + \frac{(0.4)(13.2\% - 6.2\%)}{12} - \frac{2.5\%}{\sqrt{12}} = 5.71\%$$

5. **B** Only Statement 2 is correct. The Vasicek model implies decreasing volatility and non-parallel shifts from changes in short-term rates. The Ho-Lee model is actually more general than Model 2 (the no drift and constant drift models are special cases of the Ho-Lee model).