Ling/CSC 439/539
Introduction to Probability Theory

Fall 2017

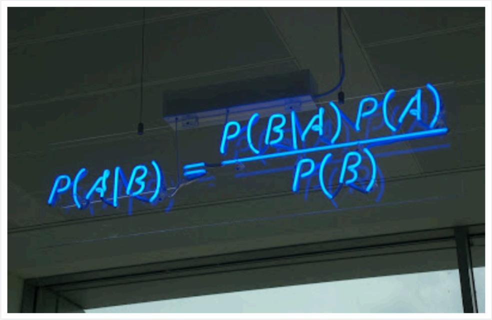
Take-away

Probability Theory Summary

- Representing conditional probabilities using contingency tables and probability trees
- The chain rule
- Bayes rule
- The law of total probability (aka the partition rule)
- Independent events

Motivation

Eleven Equations True Computer Science Geeks Should (at Least Pretend to) Know



http://www.elegantcoding.com/2011/11/eleven-equations-true-computer-science.html

How juries are fooled by statistics

http://www.ted.com/talks/lang/eng/peter_donnelly_shows_how_stats_fool_juries.html?

Outline

Definition

- How would you define "probability"?

Example: Blood Types

- Human blood is classified by the presence or absence of two antigens, called A and B. This gives rise to four types: O, A, B, and AB.

		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- Let A be the event "presence of antigen A" and B be the event "presence of antigen B"

Example: Blood Types

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	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- Exercise!

Outline

Examples of Conditional Probabilities

Conditional Probability

Bayes' Rule

Independence

Conditioning Changes the Sample Space

- Before we knew anything, anything in sample space S could occur.

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- After we know B happened, we are only choosing from within B.
- **The set B becomes our new sample space, with an updated probability of 1!**
- Instead of asking "In what proportion of S is D true?", we now ask "In what proportion of B is D true?"

Conditioning Changes the Sample Space

- We calculate conditional *proportions* from frequencies by restricting attention to ("conditioning on") a particular category, and dividing the joint frequency by the restricted total.
- Conditional probability is defined in exactly the same way, replacing proportions with probabilities.

Conditional Probability

To find the probability of *A given B*, consider the ways A can occur *in the context of B* (i.e., $A \cap B$), out of all the ways B can occur.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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- We can rearrange the formula for conditional probability.

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The "Chain Rule" of Probability

For any events, A and B, the joint probability $P(A \cap B)$ can be computed as

$$P(A \cap B) = P(A|B) \times P(B)$$

Or, since $P(A \cap B) = P(B \cap A)$

$$P(A \cap B) = P(B|A) \times P(A)$$

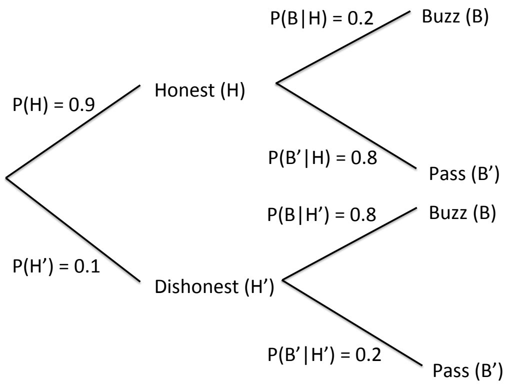
Example: Lie Detector

- A store owner discovers that some of her employees have taken cash. She decides to use a lie detector to discover who they are.
- Suppose that 10% of employees stole, but 100% say they didn't.
- The lie detector buzzes 80% of the time that someone lies, and 20% of the time that someone is telling the truth.
- If the detector buzzes, what's the probability that the person was lying?

Let's View These Rules Using Probability Trees

Probability tree == decision tree with probabilities attached to each decision

Let's View These Rules Using Probability Trees



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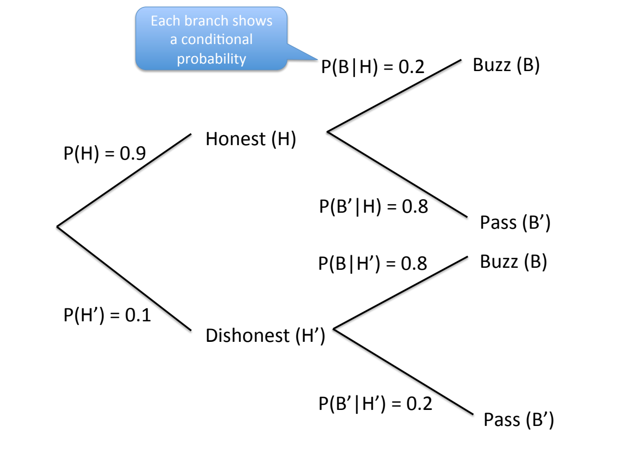

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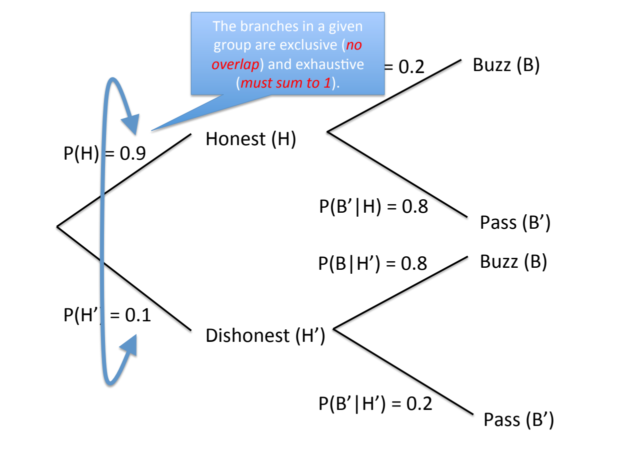

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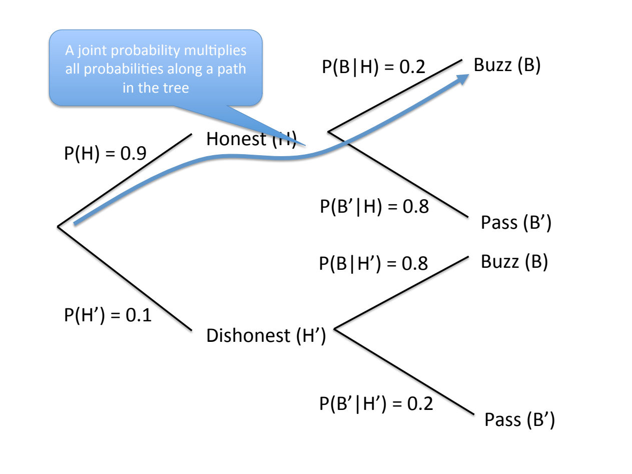

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- What is $P(\text{Pass} \cap \text{Dishonest})$?

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- Computing conditional probabilities going against the flow of the tree is a bit tricky. We will see a better way next!

Outline

Reversing Conditional Probabilities

- 16% of the students are Nutrition Science majors and 55% are female. Of the Nutrition Science majors, 54% are female. What proportion of female students in the class are Nutrition Science majors?

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- So altogether, I have

$$P(N|F) = \frac{P(N)P(F|N)}{P(F)} = \frac{0.54 \times 0.16}{0.55}$$

Bayes' Rule

We can reverse conditional probabilities using **Bayes' Rule**

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For any two events, A and B, we have

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- Remember: use Bayes Rule to reverse conditional probabilities.
- Useful to infer causes from effects (inferential statistics)!
- Very easy to derive from the chain rule, so remember that first.

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For any two events, A and B, we have

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

- The **prior** (baseline) probability of having Lycanthropy is 1 in 1000.
- A test has been developed which gives a positive result for 9 in 10 werewolves and 1 in 20 non-werewolves.
- What's the conditional probability of having Lycanthropy, given a positive test result?

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- and 1 in 20 non-werewolves: $P() = 1/20$

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Bayes' Rule

Bayes' Rule for Werewolves

We have

$$P(L|T) = \frac{P(L)P(T|L)}{P(T)}$$

- Do we have everything we need?

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- We also have $P(T|L')$, which we haven't used yet...
- Can we find $P(T)$ from what we have?

How to Compute $P(T)$?

- Two equivalent ways of doing it:
  - Contingency tables
  - Probability trees
- Use whichever is easiest for you.
  - In this course we will use probability trees

To the Probability Tree!

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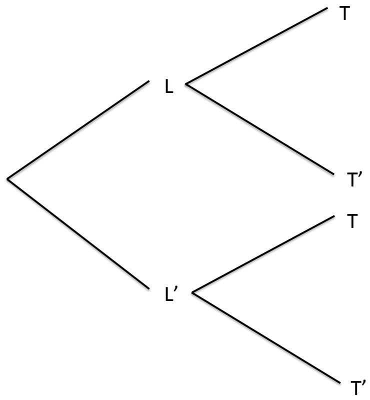

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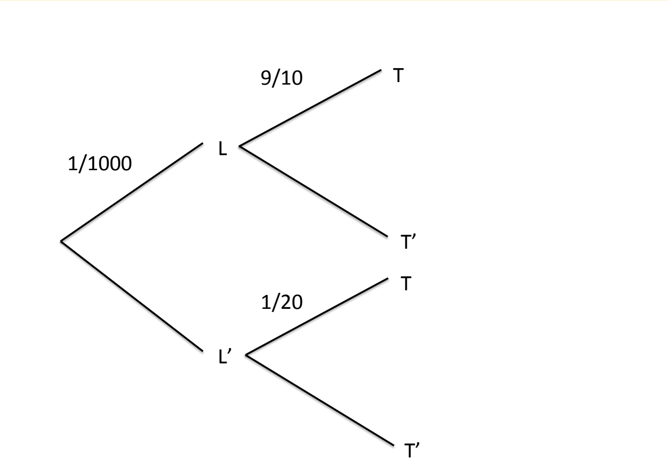

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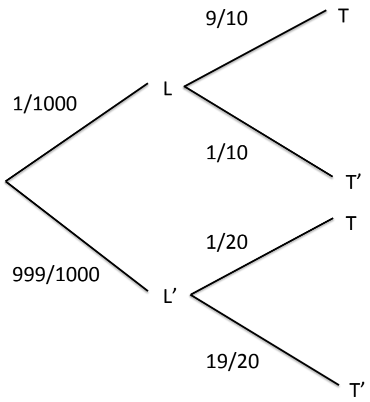

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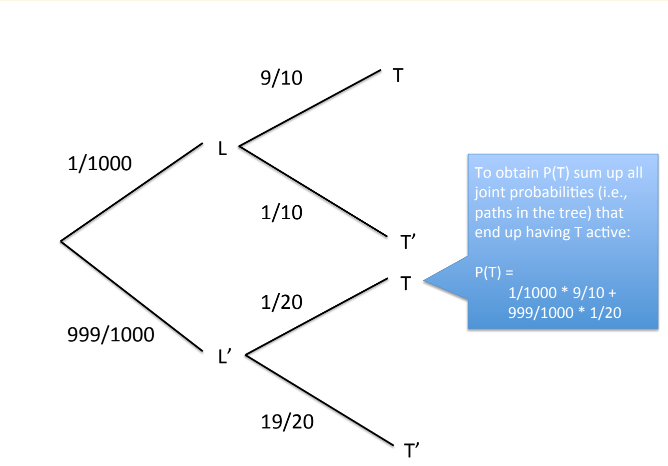

The Law of Total Probability

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  2. To find the joint probabilities, we multiplied a marginal by a conditional, according to the "chain rule".

The Law of Total Probability

For any events A and B, we have $P(B) = P(A \cap B) + P(A' \cap B)$. Together with two applications of the chain rule, this gives us

$$P(B) = P(A)P(B|A) + P(A')P(B|A')$$

Bayes' Rule Revisited

Incorporating the Law of Total Probability into Bayes' Rule, we get

Bayes Rule (version 2)

For any events A and B, we have

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(A \cap B) + P(A' \cap B)} \\ &= \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A')P(B|A')} \end{aligned}$$

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Probabilistic Independence

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Probabilistic Independence

- 10% of employees are dishonest.
- There's a 5% chance of rain tomorrow.
- What's the probability an employee is dishonest if it rains tomorrow?
- Probably your intuition is that one conveys no information about the other. What does this mean about the relationship between the conditional probability of D given R, and the marginal probability of D?

Probabilistic Independence

Independent Events

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We say that event A is **independent** of event B if conditioning on B does not change the probability of A, that is if

$$P(A|B) = P(A)$$

- If the employee is dishonest, what's the probability that it will rain tomorrow?
- It seems like independence should be symmetric. Is it?

Probabilistic Independence

- If A is independent of B, then $P(A|B) = P(A)$. Is $P(B|A)$ also equal to $P(B)$?

Probabilistic Independence

- If A is independent of B, then $P(A|B) = P(A)$. Is $P(B|A)$ also equal to $P(B)$?
- Using Bayes' rule, we have

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

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- So independence is in fact symmetric.

Probabilistic Independence

- It turns out that if A and B are independent, then their joint probability has a particularly simple form

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If A and B are independent events, then

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Independence (version 2)

If A and B are independent events, then

$$P(A \cap B) = P(A)P(B)$$

- What is $P(A \cup B)$ for independent events?

Are these Independent Events?

1. E: First coin comes up heads, F: Second coin comes up heads

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4. E: Draw a yellow ball on first pick, F: Draw a yellow ball on second pick (with replacement)

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5. E: Draw a yellow ball on first pick, F: Draw a yellow ball on second pick (without replacement)

Are these Independent Events?

1. We remove socks from a drawer with black and white, one after another, until we get a match. Is the color of the first sock independent of the color of the second?

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1. We remove socks from a drawer with black and white, one after another, until we get a match. Is the color of the first sock independent of the color of the second?
2. E_1: it rains today. E_2: today is Thursday.

Exercise: the Case of the Two Classes

- The Health Club is wondering how best to market their new yoga class, and the Head of Marketing wonders if someone who goes swimming is more likely to go to a yoga class. "Maybe we could offer some sort of discount to the swimmers to get them to try our yoga."
- The CEO disagrees: "I think you're wrong. I think people who go swimming and people who go to yoga are independent. I don't think people who go swimming are more likely to do yoga than anybody else."
- They ask a group of 96 people whether they go to swimming or yoga classes. Out of this group, 32 go to yoga and 72 go swimming. 24 people go to both.
- Are the two classes dependent or independent?

Take-away

Probability Theory Summary

- Representing conditional probabilities using contingency tables and probability trees
- The chain rule
- Bayes rule
- The law of total probability (aka the partition rule)
- Independent events

For more see this course + textbook:

Math 363:
`http://math.arizona.edu/~jwatkins/math363f15.htm`