**An Atlas of Fourier Transforms**

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The relation between real and Fourier space is essential to the study of matter. This is especially true for electron microscopy which leverages real and reciprocal space for the measurement of materials, instrument operation, and data analysis. Thus, a deep intuition for Fourier transforms across dimensions is required for experts in the field. However, mastery of the frequency domain can take years, sometimes decades, to develop. Here, we introduce a library of Fourier transforms as an advanced and introductory reference or education manual.

*The Atlas of Fourier Transforms* is a curated compilation of 2D structures and their Fourier transforms. For beginners, the atlas is an advanced course in Fourier transforms—images graphically explain the underlying mathematics. As the atlas progresses, so does the complexity of each transform. Concepts in symmetry, rotation, translation, addition, multiplication, order, and disorder are introduced and combined. For the experienced, it provides an essential reference to a comprehensive dictionary of structures and their Fourier transform.

To illustrate, Figure 1 shows two representative pages taken from *The Atlas of Fourier Transforms*. Pages are presented as diptych pairs: on the left, the real-space structure and on the right, the corresponding Fourier-transform. While each real-space structure is made from real values, the structure in Fourier space contains complex values comprised of amplitude and phase. Here, the amplitude and phase are mapped to colors on the Munsell sphere: the amplitude is mapped to intensity value between black and white; phase is represented by the color hue. In the quantum mechanics of electron scattering only the amplitude is observable, however, the phase often plays an important role in the structure of the measured amplitude. Through computation, atlas provides referenceable intuition into the structure of phase in Fourier space.

Each row in Figure 1 teaches a concept about emergent structure in Fourier space. **The first row** of images highlights basic symmetry and linearity of Fourier transform. Fourier transform of a half circle is two-fold symmetric, like its real space structure. Reflecting half circle reflects the Fourier transform. Adding two half circles results in a full circle and becomes rotationally invariant. In reciprocal space, Airy disk—with same rotational symmetry—emerges as complex phases of half-circles cancel. **The second row** introduces periodicities. Placing two circles vertically, produce a vertical sinusoid, while placing circles horizontally forms a horizontal sinusoid in Fourier space. Note the wavelength is inversely proportional to the distance between circles. In column 3, we form a rectangular array that Fourier transforms into a lattice like structure hinting at the origin of crystalline diffraction patterns. **The third row** of images shows the relationship between convolution in real space and multiplication in the frequency domain. The image shows a periodic triangular mask when applied to a crystal lattice structure. This is a multiplication in real space, and thus results in a convolution of the two transforms in k-space (also known as frequency space). The image in the last column of the second row is the result when masking with the inverse of the periodic mask shown. **The last row** contains a subset of aperiodic structures contained in the atlas. **The last row** showcases a subset of aperiodic and disordered structures found in the Atlas. The first image shows regularly spaced circles along a spiral. The second image highlights polycrystallinity. The last image shows a topological defect with 5π phase winding. The atlas also includes other types of disorders and their transforms, such as stacking faults, interstitial disorder, and non-invertible crystal structures.

Diagram

Description automatically generated with medium confidence

**Figure 1:** Real space images alongside their 2D Fourier transforms. Real space images include simple shapes, periodic masking of crystalline structures, aperiodic structures and disordered structures.

References:

[1] G Harburn, C Taylor, and TR Welberry (1983). *Atlas of Optical Transforms*. London: Bell and Hyman.

[2] Dallapiccola, Ramona & Gopinath, Ashwin & Stellacci, Francesco & Negro, Luca. (2008). Quasi-periodic distribution of plasmon modes in two-dimensional Fibonacci arrays of metal nanoparticles. Optics express. 16. 5544-55. 10.1364/OE.16.005544.