

# The course of error due to distortion of input signal to the atan2 function

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*Distortion of sine and cosine values, used for angle determination with the atan2 function, can result in numerical error. According to the performed review of literature, error is normally presented by taking only the basic harmonic into account. This paper however presents determination of error by taking into account also higher harmonics, which are non-negligible at larger distortion of sine or cosine. Error is going to be expressed with infinite series, which expand the domain of distortion parameter.*

## 1 Introduction

These days the need for high quality motor regulation is present in numerous applications and has as a result become unavoidable. For a consistent and reliable measurement of rotation, position sensors are used [1], such as encoders and resolvers [2][3][4]. Because the output of such sensors is a pair of quadrature sine and cosine signals, angle must first be calculated. The easiest way of doing so is by directly calculating atan2, which returns a value between  $[-\pi, \pi]$  [9].

Because position sensors aren't ideal, obtained sine and cosine signal can be deformed, phase shifted and DC offset. All of these imperfections cause the calculated angle to also include error.

Literature [5], [6], [7] and [8] analyses the impact of such imperfection for lower harmonics only and states that imperfections scale linearly. During our research it was found, that the frequency analysis also contains higher harmonics. The paper examines error waveform dependent on input signal mismatch with Fourier analysis.

## 2 Methodology and results

Output from a position sensor can be represented with

$$\text{Sin} = B_0 + B_1 \sin(\theta + \varphi_s) + CMM \quad (1)$$

$$\text{Cos} = A_0 + A_1 \cos(\theta + \varphi_c) + CMM \quad (2)$$

Where  $B_0$  and  $A_0$  represent DC offset,  $B_1$  and  $A_1$  signal amplitude,  $\varphi_s$  and  $\varphi_c$  phase shift and  $\theta$  reference angle. Signals (1) and (2) can also have a common superimposed AC signal represented as CMM (3). CMM can be of cosine or sine form with  $\Delta_c$  and  $\Delta_s$  as amplitude.

$$CMM = \Delta_c \cos(\theta) + \Delta_s \sin(\theta) \quad (3)$$

By calculating atan2 for eq. (1) and (2)

$$\varphi = \text{atan2}(\text{Sin}, \text{Cos}) \quad (4)$$

and then subtracting it with an unaltered signal

$$\varepsilon = \varphi - \text{atan2}(\sin(\theta), \cos(\theta)) \quad (5)$$

we get error based on deformation. Because AC signal analysis is simpler in frequency domain, error was converted with Fast Fourier Transform(FFT). By varying each parameter individually we examined the impact of the parameter in question on specter of error. Output of atan2 was examined by sending each parameter in eq. (1) and (2) to infinity or worst case with phase shift. In this case amplitude and phase of each harmonic approaches to a limit value.

The course of the amplitude and phase in dependence on the changing parameter was approximated by a function, that best suited numerical waveform obtained from the error spectrum. By knowing the limit value to which each amplitude and phase approaches, the set of functions has decreased. We were searching for the best approximation with polynomials, rational functions, exponential, trigonometric and cyclometric functions. The best approximation was sought using the minimum squares method.

### 2.1 Defining of error at different amplitudes

In the first case was observed the dependence of error on the different amplitudes of the input signals. The output of the atan2 function is determined by the eq. (1) and (2) quotient. Since only the ratio of the amplitudes of the input signals needs to be preserved, we multiplied both signals by  $\frac{1}{A_1}$ . Taking into account only the amplitudes in (1) and (2), inputs are:

$$\text{Sin} = k \sin(\theta), \quad (6)$$

$$\text{Cos} = \cos(\theta), \quad (7)$$

where  $k$  represents  $\frac{B_1}{A_1}$ .

By varying the parameter  $k$  from 0 to infinity, was found that, the error specter consists of even harmonics only. It was also observed that phase shift didn't change.

When  $k$  approaches infinity, the amplitude of harmonics approaches  $\frac{180}{\pi} \frac{2}{n}$ , where  $n$  represents  $n$ -th harmonic. Eq. (8) contains even harmonics only.

$$\varepsilon(k \rightarrow \infty) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\theta \quad (8)$$

Because of highest amplitude of second harmonic, function, that best approximate waveform of second harmonic was searched first (Figure 1). Best approximation with summed squared error (SSE)  $1.18 \cdot 10^{-10}$  degrees was rational function (9).

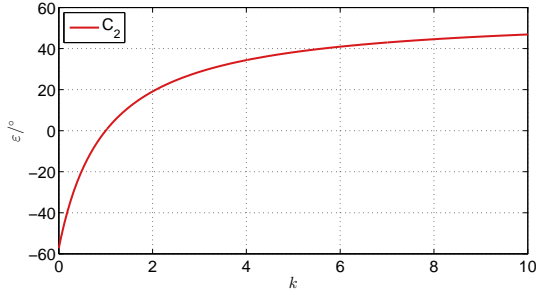


Figure 1: Waveform of the second harmonic depending on  $k$

$$C_2(k) = \frac{180}{\pi} \cdot \frac{k-1}{k+1} \quad (9)$$

It was assumed, that higher order harmonics were correlated with based harmonic approximation. Function, that describe error depending on  $k$  is presented in Fourier series form (10).

$$\varepsilon(k) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{k-1}{k+1} \right)^n \sin 2n\theta \quad (10)$$

Replacement of  $k$  for ratio of amplitudes  $\frac{B_1}{A_1}$  presents final equation for prediction of error due to known amplitudes of input signals.

$$\varepsilon(A_1, B_1) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{B_1 - A_1}{B_1 + A_1} \right)^n \sin 2n\theta \quad (11)$$

(11) is valid only for positive ratio of amplitudes.

$$\frac{B_1}{A_1} \geq 0.$$

## 2.2 Defining of error at non-orthogonality

Amplitudes were fixed to one, eq. (1) and (2) were depended to phases only. Input signals are represented as:

$$\text{Sin} = \sin(\theta + \varphi_s) \quad (12)$$

$$\text{Cos} = \cos(\theta + \varphi_c) \quad (13)$$

Error was analyzed for each parameter individually and in the end results have been merged.

First analysis was made for parameter  $\varphi_s$ . Worst case of error is, when phase parameter approaches  $90^\circ$ . Error

can be presented as Fourier series (14). Error contains DC component and even harmonics only.

$$\varepsilon(\varphi_s \rightarrow 90^\circ) = 45^\circ - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta) \quad (14)$$

By varying  $\varphi_s$  between  $0^\circ$  and  $90^\circ$  was found correlation of amplitudes of harmonics with tangent function (Figure 2). DC component and phase of error were changed linearly. Tangent function approximate second harmonic with SSE of  $1.18 \cdot 10^{-10}$  degrees.

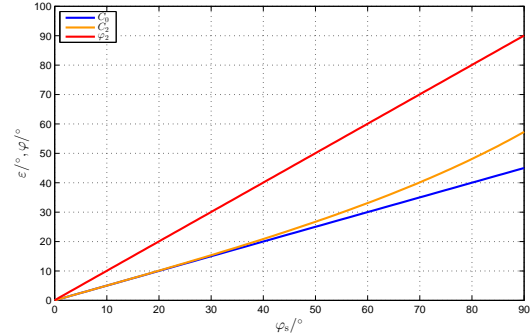


Figure 2: The waveforms of DC component  $C_0$ , amplitude of second harmonic  $C_2$  and phase of second harmonic of error  $\varphi_2$  due to ideal cosine signal, depend to phase shift  $\varphi_s$

Same procedure was made for parameter  $\varphi_c$ . Function, that describes correlation of phase shifts of (12) and (13) to error is presented by Fourier function form (15).

$$\begin{aligned} \varepsilon(\varphi_s, \varphi_c) &= \frac{\varphi_s + \varphi_c}{2} + \\ &\frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \tan \frac{\varphi_s - \varphi_c}{2} \right)^n \sin(2n\theta + n(90^\circ + \varphi_s + \varphi_c)) \end{aligned} \quad (15)$$

Expression (15) is valid only for:

$$\varphi_s - \varphi_c \in [-90^\circ, 90^\circ]$$

## 2.3 Coure of error for DC component in one input signal only

Input signals can contain DC component. Considering, changing parameters of DC components in (1) and (2) only, represent input signals as:

$$\text{Sin} = \sin(\theta) + B_0 \quad (16)$$

$$\text{Cos} = \cos(\theta) + A_0. \quad (17)$$

First was analyzed offset in  $\text{Cos}$  signal. Parameter  $A_0$  approaches infinity and error was decribed by eq. (18).

$$\varepsilon(A_0 \rightarrow \infty) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{2}{n} \sin(n\theta + n180^\circ). \quad (18)$$

Error does not contain DC component, highest amplitude has first harmonic. The waveform from figure 3, was split

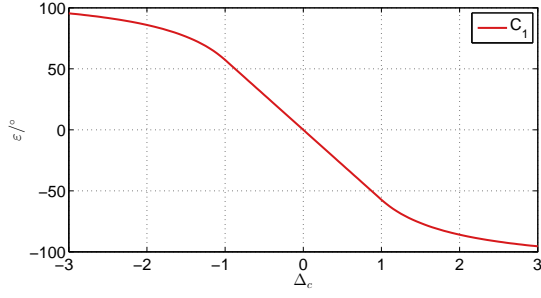


Figure 3: The waveform of amplitude of first harmonic depending to offset  $A_0$ , where input signals have amplitude of 1

to 3 parts. Expression which best approximate waveform of first harmonic of error with SSE of  $1.21 \cdot 10^{-7}$  degrees is in exponential and linear form:

$$\varepsilon(A_0, A_1) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\frac{A_0}{A_1}|^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (\frac{A_0}{A_1})^n \sin(n\theta), & |\frac{A_0}{A_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (2 - (\frac{A_0}{A_1})^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \geq 1 \end{cases} \quad (19)$$

Same procedure was done for DC component in *Sin* signal. Result is eq. (20).

$$\varepsilon(B_0, B_1) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\frac{B_0}{B_1}|^{-n}) \sin(n\theta - 90^\circ n), & \frac{B_0}{B_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{B_0}{B_1})^n \sin(n\theta + 90^\circ n), & |\frac{B_0}{B_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\frac{B_0}{B_1})^{-n}) \sin(n\theta + 90^\circ n), & \frac{B_0}{B_1} \geq 1 \end{cases}$$

## 2.4 Curse of error for same DC component in both input signals

Input signals can also contain same DC component. Same analysis was made and result is presented in eq. (21).

$$\varepsilon(A_0, B_0 = A_0, A_1) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\sqrt{2} \frac{A_0}{A_1}|^{-n}) \sin(n\theta - 45^\circ n), & \frac{A_0}{A_1} \leq -\frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\sqrt{2} \frac{A_0}{A_1})^n \sin(n\theta + 135^\circ n), & |\frac{A_0}{A_1}| \leq \frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\sqrt{2} \frac{A_0}{A_1})^{-n}) \sin(n\theta + 135^\circ n), & \frac{A_0}{A_1} \geq \frac{\sqrt{2}}{2} \end{cases} \quad (20)$$

## 2.5 Impact of CCM signal to error

Added common mode signal to input signals was analyzed for each parameter individually. This paper presents procedure for CCM of cosinu signal only. Result of CCM sinusoidal form is added only. Procedure to achieve result was the same.

Procedure started by limitation of error of  $\Delta_c$  to infinity. CCM effects to amplitudes and phases of input signals, so was expected that error will contain DC component and even harmonics only. Error when  $\Delta_c$  approaches infinity can be represented in Fourier series

form (22).

$$\varepsilon = 45^\circ - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta). \quad (22)$$

By varying  $\Delta_c$  and analyzing error specter, none result was found. Than error specter was split to sine and cosine part. In that case waveform of amplitude of second harmonic of error was presented as:

$$C_{2s}(\Delta_c) \cdot \sin(2\theta) + C_{2c}(\Delta_c) \cdot \cos(2\theta) \quad (23)$$

Figure 4 represents waveforms of DC component and split waveform of second harmonic of error.

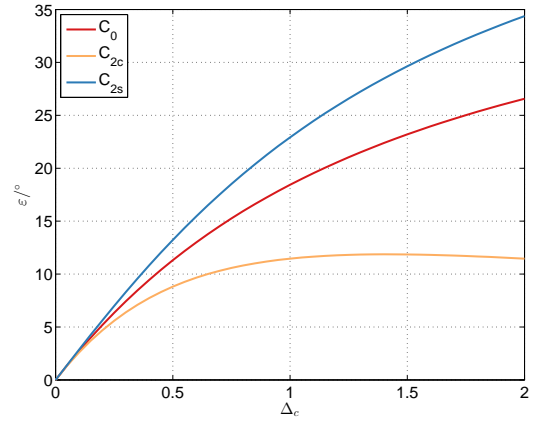


Figure 4: The course of offset and amplitude of second harmonic of error due to  $\Delta_c$

DC component was approximated with SSE of  $1.94 \cdot 10^{-22}$  degrees with atan function. Waveform of  $C_{2s}$  and  $C_{2c}$  were approximated with SSE less then  $1.15 \cdot 10^{-9}$  degrees. Functions were merged to function that represent course of amplitude as  $\sqrt{C_{2s}^2 + C_{2c}^2}$  and course of phase as  $\text{atan} \frac{C_{2c}}{C_{2s}}$ . It was assumed that same correlation applies to higher harmonics. Final function that described error depending on  $\Delta_c$  is presented in Fourier series form (24).

$$\varepsilon(\Delta_c) = \text{atan} \frac{\Delta_c}{\Delta_c + 2A_1} + \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\Delta_c}{\sqrt{\Delta_c^2 + 2A_1\Delta_c + 2A_1^2}} \right)^n \sin(2n\theta + n(90^\circ + \text{atan}(\frac{\Delta_c + A_1}{A_1}))) \quad (24)$$

$$\varepsilon(\Delta_s) = \text{atan} \frac{-\Delta_s}{\Delta_s + 2A_1} + \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\Delta_s}{\sqrt{\Delta_s^2 + 2A_1\Delta_s + 2A_1^2}} \right)^n \sin(2n\theta + n(90^\circ + \text{atan}(\frac{\Delta_s + A_1}{A_1}))) \quad (25)$$

$$\Delta_s, \Delta_c > -A_1$$

### 3 Comment on results

In test were used first 15 components of potency series. Difference between error predicted by results and actual error is only numeric (Figure 5). I made FFT of predicted error and actual error. Difference between amplitude of harmonics is numeric only. By increasing parameter error, actual error limit to discretion (nezveznosti). Error can not be fitted using first 15 components only. It is necessary to mention that despite the derivation, the presented types of errors of individual deformations still depends on each other.

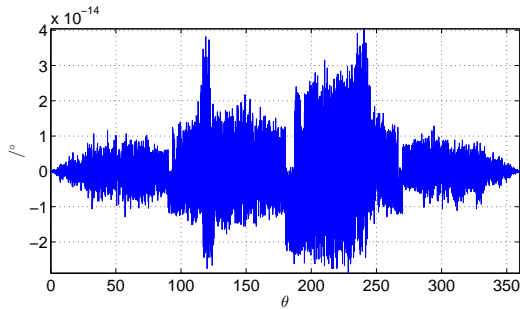


Figure 5: Difference between predicted (10) and actual error at  $k = 1.1$

### 4 Conclusion

This paper presents courses of error due to different amplitudes, different offsets, phase shifts and combination of parameters in input signals. Error includes higher harmonics, which become non-negligible at bigger distortion. For low distortion approximation, linear function can be adequate. Literature confirmed results that was calculated at low distortion [6]. With those expression can be found reason of inappropriate installation of position sensor or actuator. Expressions can be used in applications where user do not have access to measured signals as are sine and cosine. Input signals can include higher harmonics too. Higher harmonics in input signals have impact to output signal and error. The influence of the distortion of the input signals in the atan2 function to the output error offers many challenges for further work.

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