

The course of error due to distortion of input signal to the atan2 function

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Distortion of sine and cosine values, used for angle determination with the atan2 function, can result in numerical error. According to the performed review of literature, error is normally presented by taking only the basic harmonic into account. This paper however presents determination of error by taking into account also higher harmonics, which are non-negligible at larger distortion of sine or cosine. Error is going to be expressed with infinite series, which expand the domain of distortion parameter.

1 Introduction

These days the need for high quality motor regulation is present in numerous applications and has as a result become unavoidable. For a consistent and reliable measurement of rotation, position sensors are used [1], such as encoders and resolvers [2][3][4]. Because the output of such sensors is a pair of quadrature sine and cosine signals, angle must first be calculated. The easiest way of doing so is by directly calculating atan2, which returns a value between $[-\pi, \pi]$ [9].

Because position sensors aren't ideal, obtained sine and cosine signal can be deformed, phase shifted and DC offset. All of these imperfections cause the calculated angle to also include error.

Literature [5], [6], [7] and [8] analyses the impact of such imperfection for lower harmonics only and states that imperfections scale linearly. During our research it was found, that the frequency analysis also contains higher harmonics. The paper examines error waveform dependent on input signal mismatch with Fourier analysis.

2 Methodology and results

Output from a position sensor can be represented with

$$\text{Sin} = B_0 + B_1 \sin(\theta + \varphi_s) + \text{CMM} \quad (1)$$

$$\text{Cos} = A_0 + A_1 \cos(\theta + \varphi_c) + \text{CMM} \quad (2)$$

Where B_0 and A_0 represent DC offset, B_1 and A_1 signal amplitude, φ_s and φ_c phase shift and θ reference angle. Signals (1) and (2) can also have a common superimposed AC signal represented as CMM (3). CMM can be of cosine or sine form with Δ_c and Δ_s as amplitude.

$$\text{CMM} = \Delta_c \cos(\theta) + \Delta_s \sin(\theta) \quad (3)$$

By calculating atan2 for eq. (1) and (2)

$$\varphi = \text{atan2}(\text{Sin}, \text{Cos}) \quad (4)$$

and then subtracting it with an unaltered signal

$$\varepsilon = \varphi - \text{atan2}(\sin(\theta), \cos(\theta)) \quad (5)$$

we get error based on deformation. Because AC signal analysis is simpler in frequency domain we converted it with Fast Fourier Transform(FFT). By varying each parameter individually we examined the impact of the parameter in question on specter of error. Output of atan2 was examined by sending each parameter in eq. (1) and (2) to infinity or worst case with phase shift. In this case amplitude and phase of each harmonic approaches to a limit value.

The course of the amplitude and phase in dependence on the changing parameter was approximated by a function that best suited numerical waveform obtained from the error spectrum. By knowing the limit value to which each amplitude and phase approaches, the set of functions has decreased. We were searching for the best approximation with polynomials, rational functions, trigonometric and cyclometric functions. The best approximation was sought using the minimum squares method.

2.1 Defining of error at different amplitudes

In the first case we observed the dependence of error on the different amplitudes of the input signals. The output of the atan2 function is determined by the eq. (1) and (2) quotient. Since only the ratio of the amplitudes of the input signals needs to be preserved, we multiplied both signals by $\frac{1}{A_1}$. Taking into account only the amplitudes in (1) and (2), inputs are:

$$\text{Sin} = k \sin(\theta), \quad (6)$$

$$\text{Cos} = \cos(\theta), \quad (7)$$

where k represents $\frac{B_1}{A_1}$.

By varying the parameter k from 0 to infinity, we found that the error specter consists of even harmonics only. We also observed that phase shift didn't change. When k approaches infinity, the amplitude of harmonics

approaches $\frac{180}{\pi} \frac{2}{n}$, where n represents n -th harmonic. Eq. (8) contains even harmonics only.

$$\varepsilon(k \rightarrow \infty) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\theta \quad (8)$$

Because of highest amplitude of second harmonic, function that best approximate waveform was searched first (Figure 1). Best approximation with summed squared error (SSE) $1.18 \cdot 10^{-10}$ degrees was rational function (9).

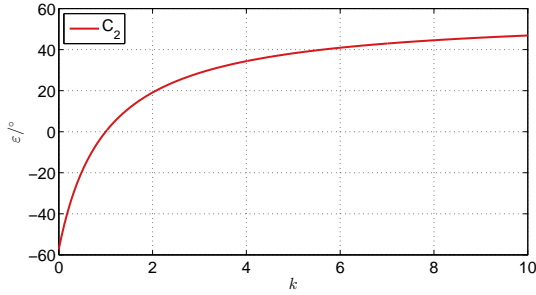


Figure 1: Waveform of the second harmonic depending on k

$$C_2(k) = \frac{180}{\pi} \cdot \frac{k-1}{k+1} \quad (9)$$

It was assumed, that higher order harmonics were correlated with based harmonic approximation. Function, that describe error depending on k is presented in Fourier series form (10).

$$\varepsilon(k) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{k-1}{k+1} \right)^n \sin 2n\theta \quad (10)$$

Replacement of k for ratio of amplitudes $\frac{B_1}{A_1}$ presents final equation for prediction of error due to known amplitudes of input signals.

$$\varepsilon(A_1, B_1) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{B_1 - A_1}{B_1 + A_1} \right)^n \sin 2n\theta \quad (11)$$

(11) is valid only for positive ratio of amplitudes.

$$\frac{B_1}{A_1} \geq 0.$$

2.2 Defining of error at non-orthogonality

Amplitudes was fixed to one, eq. (1) and (2) were depended to phases only. Input signals are presented as:

$$\text{Sin} = \sin(\theta + \varphi_s) \quad (12)$$

$$\text{Cos} = \cos(\theta + \varphi_c) \quad (13)$$

Error was analyzed for each parameter individually and in the end results have been merged.

First analysis was made for parameter φ_s . Worst case of error is, when phase parameter approaches 90° . Error

can be presented as Fourier series (14). Error contains DC component and even harmonics only.

$$\varepsilon(\varphi_s \rightarrow 90^\circ) = 45^\circ - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta) \quad (14)$$

By varying φ_s between 0° and 90° was found correlation of amplitudes of harmonics with tangent function (Figure 2). DC component and phase of error were changed linearly. Tangent function approximate second harmonic with SSE of $1.18 \cdot 10^{-10}$ degrees.

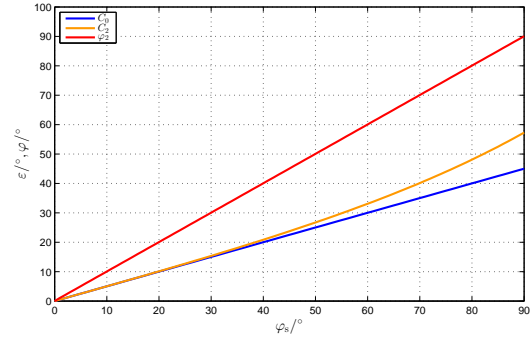


Figure 2: The waveforms of DC component C_0 , amplitude of second harmonic C_2 and phase of error φ_2 depend to ideal cosine signal, due to phase shift φ_s

Same procedure was made for parameter φ_c . Function that describes correlation of phase shifts of (12) and (13) to error is represented by Fourier function form (15).

$$\begin{aligned} \varepsilon(\varphi_s, \varphi_c) &= \frac{\varphi_s + \varphi_c}{2} + \\ &+ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\tan \frac{\varphi_s - \varphi_c}{2} \right)^n \sin(2n\theta + n(90^\circ + \varphi_s + \varphi_c)) \end{aligned} \quad (15)$$

Expression (15) is valid only for:

$$\varphi_s - \varphi_c \in [-90^\circ, 90^\circ]$$

2.3 Course of error for DC component in one input signal only

Input signals can contain DC component. Considering, changing parameters of DC components in (1) and (2) only, present input signals as:

$$\text{Sin} = \sin(\theta) + B_0 \quad (16)$$

$$\text{Cos} = \cos(\theta) + A_0. \quad (17)$$

First was analyzed offset in Cos signal.

$$\varepsilon(A_0 \rightarrow \infty) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{2}{n} \sin(n\theta + n180^\circ). \quad (18)$$

Error does not include offset component, highest amplitude has first harmonic. The waveform from figure 3, was

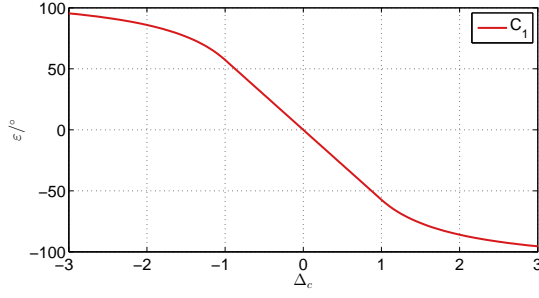


Figure 3: The course of amplitude of first harmonic due to offset A_0 , where input signals have amplitude of 1

split to 3 parts. Expression which best approximate waveform of first harmonic of error with SSE of $1.21 \cdot 10^{-7}$ degrees is in exponential and linear form:

$$\varepsilon(A_0) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (2 - |\frac{A_0}{A_1}|^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (\frac{A_0}{A_1})^n \sin(n\theta), & |\frac{A_0}{A_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (2 - (\frac{A_0}{A_1})^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \geq 1 \end{cases} \quad (19)$$

Same procedure was done for DC component in *Sin* signal. Result is eq. (20).

$$\varepsilon(B_0) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\frac{B_0}{B_1}|^{-n}) \sin(n\theta - 90^\circ n), & \frac{B_0}{B_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{B_0}{B_1})^n \sin(n\theta + 90^\circ n), & |\frac{B_0}{B_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\frac{B_0}{B_1})^{-n}) \sin(n\theta + 90^\circ n), & \frac{B_0}{B_1} \geq 1 \end{cases} \quad (20)$$

2.4 Curse of error for same DC component in both input signals

Signals from encoder or resolver can contain equal DC component.

$$\varepsilon(A_0, B_0 = A_0) = \begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\sqrt{2} \frac{A_0}{A_1}|^{-n}) \sin(n\theta + 90^\circ n), & \frac{A_0}{A_1} \leq -\frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\sqrt{2} \frac{A_0}{A_1})^n \sin(n\theta - 90^\circ n), & |\frac{A_0}{A_1}| \leq \frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\sqrt{2} \frac{A_0}{A_1})^{-n}) \sin(n\theta - 90^\circ n), & \frac{A_0}{A_1} \geq \frac{\sqrt{2}}{2} \end{cases} \quad (21)$$

2.5 Impact of different amplitude and phase due to one parameter

Transform to Fourier series of limit (5) to infinity of Δ_c , where inputs are (??) and (??) is

$$\varepsilon = 45^\circ - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta). \quad (22)$$

The course of second harmonic due to Δ_c , can be express as sum of sine and cosine signal. Each signal is presented in figure 4.

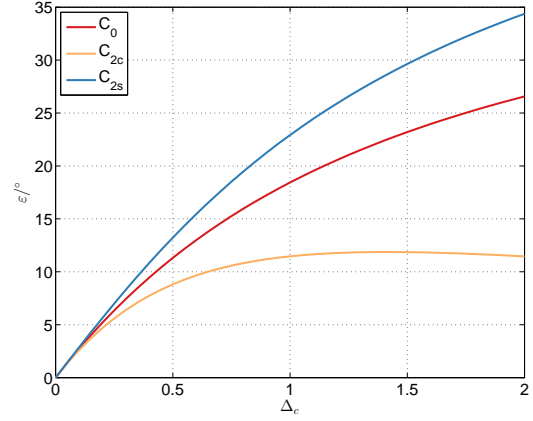


Figure 4: The course of offset and amplitude of second harmonic of error due to Δ_c

$$C_{1s}(\Delta_c) \cdot \sin(2\theta) + C_{1c}(\Delta_c) \cdot \cos(2\theta) \quad (23)$$

Offset of error is fitted to invert tangent function. C_{1s} and C_{1c} are best fitted to rational function. Total amplitude is geometrical summation $C_n = \sqrt{C_{ns}^2 + C_{nc}^2}$, phase fitted to sine signal is calculated by $\varphi_n = \text{atan}(\frac{C_{nc}}{C_{ns}})$. Final equation due to Δ_c and Δ_s is expressed in (24) and (25)

$$\varepsilon(\Delta_c) = \text{atan} \frac{\Delta_c}{\Delta_c + 2A_1} + \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\Delta_c}{\sqrt{\Delta_c^2 + 2A_1\Delta_c + 2A_1}} \right)^n \sin(2n\theta + n(90^\circ + \text{atan}(\frac{\Delta_c + A_1}{A_1}))) \quad (24)$$

$$\varepsilon(\Delta_s) = \text{atan} \frac{-\Delta_s}{\Delta_s + 2A_1} + \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\Delta_s}{\sqrt{\Delta_s^2 + 2A_1\Delta_s + 2A_1}} \right)^n \sin(2n\theta + n(90^\circ + \text{atan}(\frac{\Delta_s + A_1}{A_1}))) \quad (25)$$

$\Delta_s, \Delta_c > -A_1$

3 Comment on results

In test were used first 15 components of potency series. Difference between error predicted by results and actual error is only numeric (Figure 5). I made FFT of predicted error and actual error. Difference between amplitude of harmonics is numeric only. By increasing parameter error, actual error limit to discretion (nezveznosti). Error can not be fitted using first 15 components only. It is necessary to mention that despite the derivation, the presented types of errors of individual deformations still depends on each other.

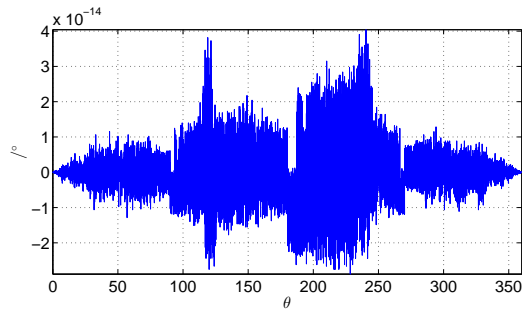


Figure 5: Difference between predicted (10) and actual error at $k = 1.1$

4 Conclusion

This paper presents courses of error due to different amplitudes, different offsets, phase shifts and combination of parameters in input signals. Error includes higher harmonics, which become non-negligible at bigger distortion. For low distortion approximation, linear function can be adequate. Literature confirmed results that was calculated at low distortion [6]. With those expression can be found reason of inappropriate installation of position sensor or actuator. Expressions can be used in applications where user do not have access to measured signals as are sine and cosine. Input signals can include higher harmonics too. Higher harmonics in input signals have impact to output signal and error. The influence of the distortion of the input signals in the atan2 function to the output error offers many challenges for further work.

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