The course of error due to distortion of input signal to the atan2 function

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Distortion of sine and cosine values, used for angle determination with the atan2 function, can result in numerical error. According to the performed review of literature, error is normally presented by taking only the basic harmonic into account. This paper however presents determination of error by taking into account also higher harmonics, which are non-negligible at larger distortion of sine or cosine. Error is going to be expressed with infinite series, which expand the domain of distortion parameter.

1 Introduction

These days the need for high quality motor regulation is present in numerous applications and has as a result become unavoidable. For a consistent and reliable measurement of rotation, position sensors are used [1], such as encoders and resolvers [2][3][4]. However, the output of which is a pair of quadrature sine and cosine signals. Angle information must first be calculated. The easiest way of doing so is by directly calculating with atan2 that returns a value between $[-\pi, \pi]$ [8].

Quadrature sine and cosine signals have not perfect form of mathematical functions sine and cosine. Quadrature sine and cosine signals can contain non-equal amplitudes, offsets or phase shifts. Despite non-perfect quadrature signals, angle is still calculated. Because of incorrect signal inputs of function atan2, calculated angel includes error.

Literature [5], [6], [7] analyzed the effect of incorrect input signals of atan2 function. However, angel error has been presented by most outstanding harmonic only. The course of error has been presented by linear function only. Results of literature are valid for relatively small incorrectness of input signals only. During research was noticed that error contains higher harmonics too. This paper presents angle error by infinite series. Amplitude and phase of harmonics depends on parameter, that describes incorrect quadrature sine and cosine signals which are inputs of atan2.

2 Methodology and results

Iinput signals to atan2 have form of sine and cosine. Nevertheless input signals could contain different amplitude, offsets or phase shifts(1)(2)

$$Sin = B_0 + B_1 \sin(\theta + \varphi_s) \qquad (1)$$
$$Cos = A_0 + A_1 \cos(\theta + \varphi_c) \qquad (2)$$

Ideal case is when offsets (B_0 and A_0) and phase shifts φ_s and φ_c are equal to zero. In ideal case amplitudes are same ($B_1 = A_1$). In this case error is zero. This research analyzed, course of error for each parmeter of (1) and (2) seperatly. Amplitude and phase can be depended on one parameter only. To input signals can be added signal with same frequency (Δ_c or Δ_s) (3)(4). Research was done for Δ_s and Δ_c separately too.

$$Sin = \sin(\theta) + \Delta_c \cos(\theta) + \Delta_s \sin(\theta) \qquad (3)$$
$$Cos = \cos(\theta) + \Delta_c \cos(\theta) + \Delta_s \sin(\theta) \qquad (4)$$

Definition of output from atan2d φ and error ε is:

$$\varphi = \text{atan2d}(Sin, Cos) \qquad (5)$$
$$\varepsilon = \varphi - \text{atan2d}(\sin(\theta), \cos(\theta)), \qquad (6)$$

where θ presents reference angle. In MATLAB is defined atan2d(), function for calculation of inverse tangent in four quadrant plane. Output is between $[-180°, 180°]$[9]. Error is presented by infinite series of harmonics (7).

$$\varepsilon(x) = C_0(x) + \sum_{n=1}^{\infty} C_n(x) \sin(n\Theta + \varphi_n(x)) \qquad (7)$$

Parameter x presents independent variable, that presents incorrectness of input signals (1) and (2). C_0 presents offset of error, C_n amplitude of individual harmonic and φ_n presents phase of individual harmonic of error. All functions depend on x.

Error is determinate with following approach. Limit of error is made of one parameter from (1), (2), (3), (4) to infinity or to the worst case. Output of atan2d φ become constant, error ε could be expressed as is described in (8).

$$\varepsilon(\theta) = \begin{cases} 90° - \theta, & \theta \in \{0°, 180°\} \\ 270° - \theta, & \theta \in \{180°, 360°\} \end{cases} \qquad (8)$$

Error is transformed to Fourier series. Transform shows to which harmonics parameter effect and shows convergence of amplitude and phase for each harmonic. Next

step is finding analytic function, that describes the course of the harmonics amplitude and the phase shift of an individual harmonic when the parameter changes. This research has shown that higher harmonies are potency depend on the basic harmonic of the error. Functions $C_0(x)$, $C_n(x)$, $\varphi_n(x)$ were analyzed with least square method. Sum squared error for functions that included parameters form (1) and (2)was less than $1,21 \cdot 10^{-7}$ degree. Sum squared error for functions that included parameters form (3) and (4)was less than $1,92 \cdot 10^{-5}$ degree.

2.1 Defining of error at different amplitudes

If both input signals are multiplied by same coefficient, φ will not be changed. Multiply both signals with $\frac{1}{A_1}$ and ratio $\frac{B_1}{A_1}$ define as k. Setting offsets and phases to zero, the input signals are defined as:

$$Sin = k\sin(\theta) \tag{9}$$

$$Cos = \cos(\theta). \tag{10}$$

Limiting error where k goes to infinity, error is shown in figure 1.

$$\lim_{k \to \infty} (\text{atan2}(k\sin\theta, \cos\theta) - \text{atan2}(\sin\theta, \cos\theta)) \tag{11}$$
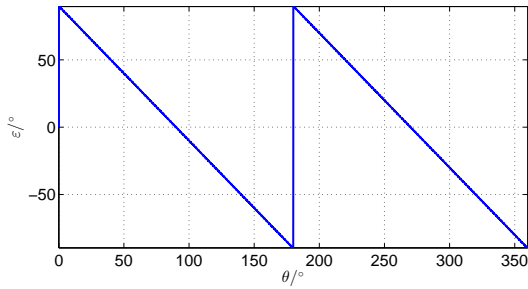
Error can be transformed to Fourier series:

Figure 1: Limit of ε at k goes to infinity

$$\varepsilon = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\theta. \tag{12}$$

By calculating Fourier series of error from figure 1, is presented by even harmonics, of which the second harmonics is the largest. Because of number 2 in argument of sine in (12), C_1 presents function of amplitude for second harmonic.

Using Curve Fitting Toolbox, best fit is rational function. Error can be expressed as:

$$\varepsilon(k) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{k-1}{k+1})^n \sin 2n\theta \tag{13}$$

Expression is valid for k bigger than 0.

$$k \geq 0$$

Instead of k, in (14) is inserted the ratio of amplitudes

$$\varepsilon(k) = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{B_1 - A_1}{B_1 + A_1})^n \sin 2n\theta \tag{14}$$
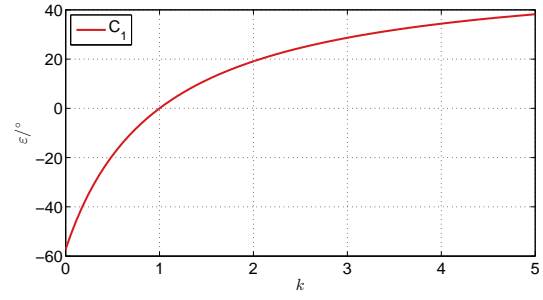

Figure 2: The course of the second harmonic depending on k

where (14) is valid for:

$$\frac{B_1}{A_1} \geq 0. $$

2.2 Defining of error at non-orthogonality

Input signals are defined as:

$$Sin = \sin(\theta + \varphi_s) \tag{15}$$

$$Cos = \cos(\theta + \varphi_c) \tag{16}$$

Error is determinate for each parameter separately. Other paramether is set to zero. At the end of equations are merged. For limitation of equation is not obligatory limit to infinity, just to the worst case. Limit is at $\pm 90°$:

$$\varepsilon = \lim_{\varphi_s \to 90°} \text{atan2}(Sin, Cos) - \text{atan2d}(\sin(\theta), \cos(\theta)) \tag{17}$$

Error can be transform and presented in Fourier series.

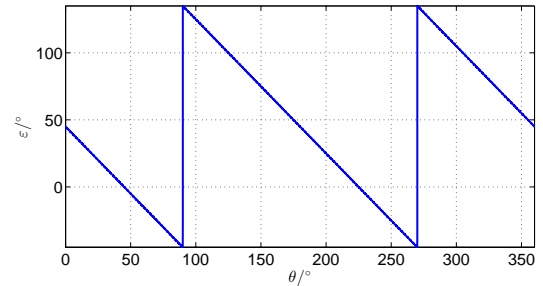Figure 3: Error ε of limiting $\varphi_s \to 90°$

$$\varepsilon = 45° - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta) \tag{18}$$

Transform includes offset and even harmonics only. Figure 4 presents course of offset C_0, amplitude of second harmonic C_1 and phase of second harmonic φ_1 due to φ_s. y axis is in degrees. For C_0 and C_1 degrees presents amplitude of error harmonics, for φ_1 degrees presents phase. Offset is best fitted by linear function. Second harmonic is tangent function. Phase of second harmonic increases linear but for presentation with sine
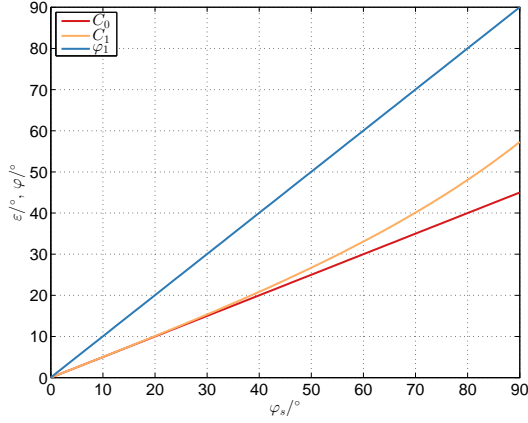
Figure 4: The course of offset component C_0, amplitude of second harmonic C_1 and phase φ_1 depend of ideal cosine signal, due to phase shift φ_s

form must be added $90°$. Same derivation can be done for phase of cosine signal. Equations can be marged and result is presented (19).

$$\varepsilon(\varphi_s, \varphi_c) = \frac{\varphi_s + \varphi_c}{2} +$$
$$\frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\tan\frac{\varphi_s - \varphi_c}{2})^n \sin(2n\theta + n(90° + \varphi_s + \varphi_c)) \tag{19}$$

Expression is valid only for:

$$\varphi_s - \varphi_c \in [-90°, 90°]$$

2.3 Defining of error at offsets

Define amplitudes of input signals to 1, phase shift is set to zero. Let limit parameter A_0 to infinity, error is transformed to Fourier series and expressed as:

$$\varepsilon = \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{2}{n} \sin(n\theta + 90°n). \tag{20}$$

Error does not include offset component, highest amplitude has first harmonic. The course from figure 5 it is
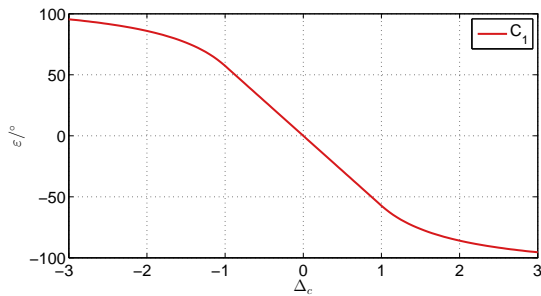

Figure 5: The course of amplitude of first harmonic due to offset A_0, where input signals have amplitude of 1

split to 3 parts and expression that best fit the curve is:

$$\varepsilon(A_0) =$$
$$\begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (2 - |\frac{A_0}{A_1}|^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (\frac{A_0}{A_1})^n \sin(n\theta), & |\frac{A_0}{A_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (2 - (\frac{A_0}{A_1})^{-n}) \sin(n\theta), & \frac{A_0}{A_1} \geq 1 \end{cases} \tag{21}$$

Same derivation can be done for B_0 (22) and for fitting error, when sine and cosine include same offset ($A_0 = B_0$) (23).

$$\varepsilon(B_0) =$$
$$\begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\frac{B_0}{B_1}|^{-n}) \sin(n\theta - 90°n), & \frac{B_0}{B_1} \leq -1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{B_0}{B_1})^n \sin(n\theta + 90°n), & |\frac{B_0}{B_1}| \leq 1 \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\frac{B_0}{B_1})^{-n}) \sin(n\theta° + 90n), & \frac{B_0}{B_1} \geq 1 \end{cases} \tag{22}$$

$$\varepsilon(A_0, B_0 = A_0) =$$
$$\begin{cases} \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - |\sqrt{2}\frac{A_0}{A_1}|^{-n}) \sin(n\theta + 90°n), & \frac{A_0}{A_1} \leq -\frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\sqrt{2}\frac{A_0}{A_1})^n \sin(n\theta - 90°n), & |\frac{A_0}{A_1}| \leq \frac{\sqrt{2}}{2} \\ \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (2 - (\sqrt{2}\frac{A_0}{A_1})^{-n}) \sin(n\theta - 90°n), & \frac{A_0}{A_1} \geq \frac{\sqrt{2}}{2} \end{cases} \tag{23}$$

2.4 Impact of different amplitude and phase due to one parameter

Transform to Fourier series of limit (6) to infinity of Δ_c, where inputs are (3) and (4) is

$$\varepsilon = 45° - \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\theta). \tag{24}$$

The course of second harmonic due to Δ_c, can be express as sum of sine and cosine signal. Each signal is presented in figure 6.

$$C_{1s}(\Delta_c) \cdot \sin(2\theta) + C_{1c}(\Delta_c) \cdot \cos(2\theta) \tag{25}$$

Offset of error is fitted to invert tangent function. C_{1s} and C_{1c} are best fitted to rational function. Total amplitude is geometrical summation $C_n = \sqrt{C_{ns}^2 + C_{nc}^2}$, phase fitted to sine signal is calculated by $\varphi_n = atan(\frac{C_{nc}}{C_{ns}})$. Final equation due to Δ_c and Δ_s is expressed in (26) and (27)

$$\varepsilon(\Delta_c) =$$
$$\text{atand}\frac{\Delta_c}{\Delta_c + 2A_1} + \frac{180}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{\Delta_c}{\sqrt{\Delta_c^2 + 2A_1\Delta_c + 2A_1}})^n$$
$$\sin(2n\theta + n(90° + \text{atan}(\frac{\Delta_c + A_1}{A_1}))) \tag{26}$$
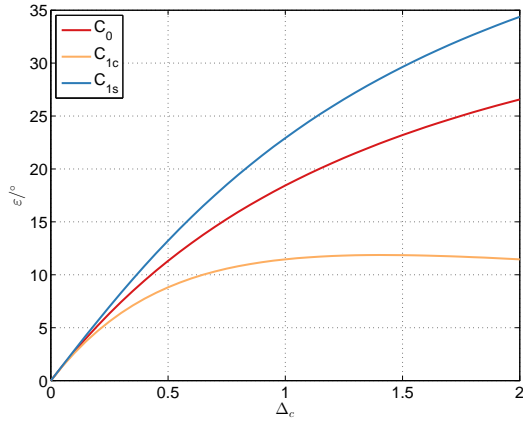
Figure 6: The course of offset and amplitude of second harmonic of error due to Δ_c

$$\varepsilon(\Delta_s) =$$
$$\text{atand}\frac{-\Delta_s}{\Delta_s + 2A_1} + \frac{180}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\frac{\Delta_s}{\sqrt{\Delta_s^2 + 2A_1\Delta_s + 2A_1}})^n$$
$$\sin(2n\theta + n(90° + \text{atan}(\frac{\Delta_s + A_1}{A_1}))) \quad (27)$$

$$\Delta_s, \Delta_c > -A_1$$

3 Comment on results

In test were used first 15 components of potency series. Difference between error predicted by results and actual error is only numeric (Figure 7). I made FFT of predicted error and actual error. Difference between amplitude of harmonics is numeric only. By increasing parameter error, actual error limit to discretion (nezveznosti). Error can not be fitted using first 15 components only. It is necessary to mention that despite the derivation, the presented types of errors of individual deformations still depends on each other.
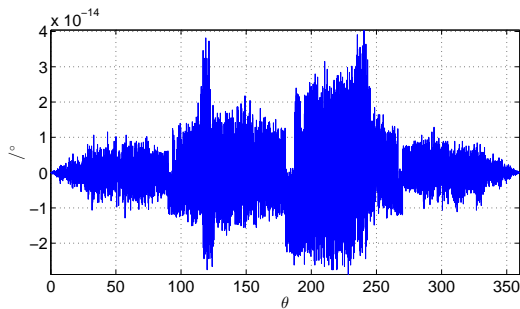

Figure 7: Difference between predicted (13) and actual error ati $k = 1.1$

4 Conclusion

This paper presents courses of error due to different amplitudes, different offsets, phase shifts and combination of parameters in input signals. Error includes higher harmonics, which become non-negligible at bigger distortion. For low distortion approximation, linear function can be adequatable. Literature confirmed results that was calculated at low distortion [5]. With those expression can be found reason of inappropriate installation of position sensor or actuator. Expressions can be used in applications where user do not have access to measured signals as are sine and cosine. Input signals can include higher harmonics too. Higher harmonics in input signals have impact to output signal and error. The influence of the distortion of the input signals in the atan2 function to the output error offers many challenges for further work.

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