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<u>Lemma:</u> Let $e \in E(T_{P,U})$. Then:

(i) If $e \in E(P^o)$, then $e \in E(T_{P,U'})$ for every $U' \subseteq U$ (if $T_{P,U'}$ exists).

(ii) P^u $U' \supseteq U$

<u>Proof.</u>

Let $e \in E(T_{P,U})$. If $e \in E(P^o \cap P^u)$, then $e \in E(T_{P,U'})$ for every U' (if $T_{P,U'}$ exists). Hence it is sufficient...

(i) ~~Suppose~~ Let $U' \subseteq U$. Suppose $T_{P,U'}$ exists and $e \in E(P^o - P^u)$. We denote by $b := b(tail(e), P)$ the P-block inducing edge e. We show that $T_{P,U'}|_b = T_{P,U}|_b$, which, since $e \in E(T_{P,U}|_b)$, implies $e \in E(T_{P,U'})$.

<u>Claim:</u> $T_{P,U}|_b = P^o - P^u|_b$.

<u>Proof.</u>

By Lemma ~, $T_{P,U}|_b$ is a $(f(b), \mathcal{E}(b))$-flow in $\alpha_P(U,G)|_b$. By Lemma ~, $T_{P,U}|_b = P^o - P^u|_b$ or $T_{P,U} = P^u - P^o|_b$.

Since $e \in E(T_{P,U}|_b)$ and $e \in E(P^o - P^u)$, $T_{P,U}|_b = P^o - P^u|_b$.

<u>Claim:</u> $P^o - P^u|_b = T_{P,U'}|_b$.

<u>Proof.</u>

We show that $P^o - P^u|_b$ is a $(f(b), \mathcal{E}(b))$-flow in $\alpha_P(U',G)|_b$. It then follows that $T_{P,U'}|_b = P^o - P^u|_b$; otherwise, we could substitute $T_{P,U'}|_b$ by $P^o - P^u|_b$ in $T_{P,U'}$ to obtain an (s,t)-flow in $\alpha_P(U',G)$ distinct from $T_{P,U'}$, which contradicts the uniqueness of $T_{P,U'}$.

Suppose that $P^o - P^u|_b$ is not a $(f(b), \mathcal{E}(b))$-flow in $\alpha_P(U',G)|_b$. By Lemma ~(i), there is an edge $(u,v) \in E(P^o - P^u$ such that $\alpha_P((u,v), U') =$ inactive. Hence, $(u,P) \in U'$. Since $U' \subseteq U$, $(u,P) \in U$. Hence, $\alpha_P((u,v), U) =$ inactive.

Hence $(u,v) \notin E(T_{P,U}|_b) = E(P^o - P^u|_b)$, a contradiction.

The second inference does not ~~fo~~ immediately follow from the lemma. We probably should handle the case $P^o \cap P^u$ separately and then assume $P^o - P^u$ or $P^u - P^o$ for the rest of the thesis.

(ii) ...

<u>Lemma:</u> Let P be transient for U and $(u,v) \in E(P^o \cup P^u)$. Then:

(i) If $(u,v) \in E(P^u - P^o)$, then $(u,v) \in E(T_{P,U})$ iff $(f(b(u,P)), P) \in U$.

(ii) $P^o \cap P^u$

(iii) $P^o - P^u$ iff $\notin U$.

<u>Proof</u>

By assumption, $T_{P,U}$ is an (s,t)-flow in $\alpha_P(U,G)$. We set $b := b(u,P)$. By Lemma ~, $T_{P,U}|_b$ is a $(f(b), \mathcal{E}(b))$-flow in $\alpha_P(U,G)|_b$. By Lemma ~, $T_{P,U}|_b = P^o - P^u|_b$ or $T_{P,U}|_b = P^u - P^o|_b$.

(i). ~~Let f~~ Suppose $(u,v) \in E(P^u - P^o)$. We first show the if-part.

Suppose $(u,v) \in E(T_{P,U})$. Hence $T_{P,U}|_b = P^u - P^o|_b$. Hence, for every edge $e \in E(P^u - P^o|_b)$, $\alpha_P(e,U) =$ active.

In particular, $e^u(f(b),P) \in E(P^u - P^o|_b)$ and $\alpha_P(e^u(f(b),P),U) =$ active. Hence $(f(b),P) \in U$.

Notation. Also, do we need to prove $e^u(f(b),P) \in E(P^u - P^o|$

To show the only-if-part, suppose $(u,v) \notin E(T_{P,U})$. Hence $T_{P,U}|_b = P^o - P^u|_b$. Hence, for every edge ~~the~~ $e \in E(P^o - P^u|_b)$, $\alpha_P(e,U) =$ active. In particular, ...

(ii). Clearly, $E(T_{P,U}|_b) \subseteq E(b)$. Since $f(b) \neq \varepsilon(b)$, the claim follows with the following lemma.

<u>Lemma:</u> If $(u,v) \in E(P^o \cap P^u)$, then $b(u,P) = \{u\}$ (and $E(b(u,P)) = \{(u,v)\}$).

<u>Lemma:</u> Let P be transient for U and (v,P) be a type-1 update such that $(f(b(v,P)),P) \notin U$. Then:

(i) P is transient for $U \cup \{(v,P)\}$ and $T_{P,U \cup \{(v,P)\}} = T_{P,U}$.

(ii) — $T_{P,U - \{(v,P)\}}$

This lemma should be a corollary of the previous one, but to apply the previous one, we need that P is transient for $U \cup \{(v,P)\}$

<u>Proof.</u>

We show that $T_{P,U}$ is still an (s,t)-path in $\alpha_P(U \cup \{(v,P)\}, G)$, $\alpha_P(U - \{(v,P)\}, G)$, respectively.

Let b be a P-block. By lemma ~, $T_{P,U}|_b$ is a $(f(b), \varepsilon(b))$-flow in $\alpha_P(U,G)|_b$. We show that $T_{P,U}|_b$ is still a $(f(b), \varepsilon(b))$-flow in $\alpha_P(U \cup \{(v,P)\}, G)$, $\alpha_P(U - \{(v,P)\}, G)$, respectively. Then, by lemma ~, $T_{P,U}$ is an (s,t)-flow in $\alpha_P(U \cup \{(v,P)\}, G)$, $\alpha_P(U - \{(v,P)\}, G)$, respectively.

(i). Case $b = b(v,P)$. Since $(f(b),P) \notin U$, by lemma ~(iii), $T_{P,U}|_b = P^o - P^u|_b$. Let $e \in E(P^o - P^u|_b)$. Then ~~Hence, for every edge $e \in E(P^o - P^u|_b)$,~~ $\alpha_P(e,U) =$ active. Hence $(tail(e),P) \notin U$. ~~Hence~~ Hence $(tail(e),P) \notin U \cup \{(v,P)\}$, as $v \in V(P^u - P^o)$. Hence $\alpha_P(e, U \cup \{(v,P)\}) =$ active. Hence, by lemma ~, $\overset{= T_{P,U}|_b}{P^o - P^u|_b}$ is a $(f(b), \varepsilon(b))$-flow in $\alpha_P(U \cup \{(v,P)\}, G)|_b$.

Move "(i)." up. We may need lemma ~ here.

Let P be trans(u,v)

<u>Lemma:</u> $\alpha_P(\overset{(u,v)}{\cancel{e}}, B) =$ active iff $\cancel{e} \in E(T_{P,U(B)})$.

<u>Proof.</u>

~~"⇐": Suppose $e \in E(T_{P,U(B)})$. Let u,v be vertices~~

Case $(u,v) \in E(P^u - P^o)$. Then, by lemma ~(i), $(u,v) \in E(T_{P,U(B)})$ iff $(f(b(u,P)),P) \in U(B)$, which holds iff $b(u,P) \in B$, which, by assumption, holds iff $\alpha_P((u,v),B) =$ active.

Case $(u,v) \in E(P^o \cap P^u)$. By lemma ~(ii), $(u,v) \in E(T_{P,U(B)})$. Moreover, by definition, $\alpha_P((u,v),B) =$ active.