14.01.23

Define merge.

Lemma: Let F_i, F_j, where $i \neq j$ be two flow pairs with $d_i = d_j$. Then F_i and F_j can be merged if F_i and F_j have no common vertices other than s, t, i.e. $V(F_i^o \cup F_i^u) \cap V(F_j^o \cup F_j^u) = \{s, t\}$.

Proof.

The idea is to append flow pair F_j to flow pair F_i ~~by redirecting edges~~.

What if $u_{F_i^o} = t$, $v_{F_i^o} = s$?

Let $u_{F_j^o}, u_{F_j^u}$ be such that $(s, u_{F_j^o}) \in E(F_j^o)$, $(s, u_{F_j^u}) \in E(F_j^u)$ and $v_{F_i^o}, v_{F_i^u}$ be such that $(v_{F_i^o}, t) \in E(F_i^o)$, $(v_{F_i^u}, t) \in E(F_i^u)$.

Update flow network $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\mathcal{P}}, s, t, \tilde{c})$

Give intuition.

We introduce five new vertices $\tilde{u}_{F_j^o}, \tilde{u}_{F_j^u}, \tilde{v}_{F_i^o}, \tilde{v}_{F_i^u}$, ~~w~~.

~~We introduce the following edges: $(s, \tilde{u}_{F_j^o}), (\tilde{u}_{F_j^o}, u_{F_j^o}), (s, \tilde{u}_{F_j^u}), ($~~

We replace edges $(s, u_{F_j^o}), (s, u_{F_j^u}), (v_{F_i^o}, t), (v_{F_i^u}, t)$ with the following edges:

- $(s, \tilde{u}_{F_j^o}), (s, \tilde{u}_{F_j^u}), (\tilde{v}_{F_i^o}, t), (\tilde{v}_{F_i^u}, t)$ with capacity ∞ (e.g. $\sum_i d_i$)
- $(\tilde{u}_{F_j^o}, u_{F_j^o})$, ~~$(\tilde{u}_{F_j^u}, u_{F_j^u})$~~ with capacity $c(s, u_{F_j^o})$
- $(\tilde{u}_{F_j^u}, u_{F_j^u})$ with capacity $c(s, u_{F_j^u})$
- $(v_{F_i^o}, \tilde{v}_{F_i^o})$ with capacity $c(v_{F_i^o}, t)$
- $(v_{F_i^u}, \tilde{v}_{F_i^u})$ with capacity $c(v_{F_i^u}, t)$

We update all flow pairs to use the new edges instead of the old eges, that is:

~~$E(\tilde{F}_\ell^o) = E(F_\ell^o) - \{(s, u_{F_j^o}), (s, u_{F_j^u}), (v_{F_i^o}, t), (v_{F_i^u}, t)\} \cup$~~

~~$\{(E(\tilde{F}_\ell^o) = E(F_\ell^o)$~~

$E(\tilde{F}_\ell^o) = E(F_\ell^o) - \{(s, u_{F_j^o}), (s, u_{F_j^u}), (v_{F_i^o}, t), (v_{F_i^u}, t)\} \cup$

$\{(s, \tilde{u}_{F_j^o}), (\tilde{u}_{F_j^o}, u_{F_j^o}) : (s, u_{F_j^o}) \in E(F_\ell^o)\} \cup$

$\{(s, \tilde{u}_{F_j^u}), (\tilde{u}_{F_j^u}, u_{F_j^u}) : (s, u_{F_j^u}) \in E(F_\ell^o)\} \cup$

$\{$~~$(v_{F_i^o}, t),$~~$(v_{F_i^o}, \tilde{v}_{F_i^o}), (\tilde{v}_{F_i^o}, t) :$

$\{(v_{F_i^u}, \tilde{v}_{F_i^u}), (\tilde{v}_{F_i^u}, t) :$

$E(\tilde{F}_\ell^u)$ is defined ~~similarly~~ analogously. We set $V(\tilde{F}_\ell^o) = V(E(\tilde{F}_\ell^o))$ and $V(\tilde{F}_\ell^u) = V(E(\tilde{F}_\ell^u))$.

Moreover, we introduce another new vertex w and edges $(\tilde{v}_{F_i^o}, w), (\tilde{v}_{F_i^u}, w), (w, \tilde{u}_{F_j^o}), (w, \tilde{u}_{F_j^u})$ with capacities $d_i = d_j$.

~~Overloaded~~ ~~notation $\tilde{F}_\ell^o$~~

~~We update flows the flow pairs $\tilde{F}_i, \tilde{F}_j$ as follows:~~ We replace flow pairs $\tilde{F}_i, \tilde{F}_j$ with a new flow pair $\tilde{F}$

$E(\tilde{F}^o) = E(\tilde{F}_i^o) - \{(\tilde{v}_{F_i^o}, t)\} \cup \{(\tilde{v}_{F_i^o}, w)\} \cup E(\tilde{F}_j^o) - (s, \tilde{u}_{F_j^o}) \cup \{(w, \tilde{u}_{F_j^o})\}$

$E(\tilde{F}^u)$

~~$E(F_j^o)$~~

We now show that there is a feasible update sequence σ for update flow network G iff there is a feasible update sequence $\tilde{\sigma}$ for update flow network $\tilde{G}$.