12.02.23

Notation: ~~For~~ Given a graph G, we denote by $\le_G$ the reachability relation of G. That is, for every two vertices $u,v\in V(G)$, $u\le_G v$ iff there is a (directed) path in G from u to v.

Notice that if G is a DAG, then $\le_G$ is antisymmetric reachability relation, and if G is a path graph, then $\le_G$ is a total order.

Notation: Let $u\le_{P^o} v$ ($u\le_{P^u} v$). Then we denote by $p^o(u,v,P)$ ($p^u(u,v,P)$) the (unique) directed path in P^o (P^u) from u to v.

Ch. **Remark.** Let $u,v\in V(P^o\cap P^u)$. Then $u\le_{P^o} v$ iff $u\le_{P^u} v$ and $u\le_{P^o\cup P^u} v$ iff $u\le_{P^o} v$ and $u\le_{P^u} v$.

Remark: $\le_{P^o\cup P^u}|_{V(P^o\cap P^u)}$ is a total order on $V(P^o\cap P^u)$.

Definition: Let $\{v_1,\dots,v_\ell\}$ be the intersection of $V(P^o)$ and $V(P^u)$ w.r.t. $\le_{P^o\cup P^u}$. For every $i<\ell$, we define the i-th P-block $b_i^P=V(p^o(v_i,v_{i+1},P)\cup p^u(v_i,v_{i+1},P))$.

Notation: We denote by $B^P(G)=\bigcup_{i<|V(P^o\cap P^u)|}\{b_i\}$ the set of P-blocks of G, and by $B(G)=\bigcup_P B^P(G)$ the set of blocks of G.

I guess we need that $b_i\cap b_{i+1}=\{v_{i+1}\}$ and for $i+1<j$, $b_i\cap b_j=\{\}$.

Notation: We denote by $b(v,P)$ the P-block containing vertex v. More precisely, ~~let $b_1,\dots,b_\ell$ be $B^P=\{\}$~~ let $\{v_1,\dots,v_\ell\}$ be the intersection of $V(P^o)$ and $V(P^u)$ w.r.t. $\le_{P^o\cup P^u}$ and $B^P(G)=\{b_1,\dots,b_{\ell-1}\}$. If $v\in V(P^o\cap P^u)$, then $b(v,P)$ denotes the P-block b_i such that $v=v_i$, that is, the P-block b_i such that $v=s(b_i)$. Otherwise, $b(v,P)$ denotes the P-block b_i such that $v\in b_i$.

Lemma: Let G be an update flow network and $F,F'\in\mathcal{P}$ be two flow pairs such that $d_F=d_{F'}$ and F and F' have no common vertices other than s,t. Then there is an update flow network $\tilde G$ such that $|\tilde G|=O(|G|)$, $|\tilde{\mathcal P}|=|\mathcal P|-1$, and there is a feasible block sequence $B=(b_1,\dots,b_\ell)$ for G iff there is a feasible block sequence $\tilde B=(\tilde b_1,\dots,\tilde b_\ell)$ for $\tilde G$.

Proof.

Construction of $\tilde G$ as in 22.01.23.

„$\Rightarrow$"

Let $B=(b_1,\dots,b_\ell)$ be a feasible block sequence for G. We construct a feasible block sequence $\tilde B=(\tilde b_1,\dots,\tilde b_\ell)$ for $\tilde G$ as follows. The idea is to map every block $\tilde b\in\tilde B(\tilde G)$ to a block $f(\tilde b)\in B(G)$ such that $\tilde b\in\tilde b_i$ iff $f(\tilde b)\in b_i$. We define mappings $f_V:\tilde V\to V$ and $f:\tilde B(\tilde G)\to B(G)$ as follows.

$$f_V(\tilde v)=\begin{cases} u_{F^o} & \text{if } \tilde v=\tilde u_{F^o}\\ u_{F^u} & \text{if } \tilde v=\tilde u_{F^u}\\ s & \text{if } \tilde v=\tilde v_{F^o} \text{ or } \tilde v=\tilde v_{F^u} \text{ or } \tilde v=w\\ v & \text{otw. } [\tilde v=v]\end{cases}$$

Ch.

$$\cancel{f(\tilde{b})=}\ f_s(\tilde{v}, \tilde{P}) = \begin{cases} F \text{ if } \cancel{f_v(\tilde{v}) \in}\ \tilde{P} = \tilde{F} \text{ and } f_v(\tilde{v}) \in V(F^{0} \cup F^{u}) - \{t\} \\ F' \text{ if } \tilde{P} = \tilde{F} \text{ and } f_v(\tilde{v}) \in V(F'^{0} \cup F'^{u}) - \{s\} \\ P \text{ else} \end{cases}$$

Remove?

$$f(\tilde{b}) = b(f_v(\tilde{J}(\tilde{b})), f_s(f_v(\tilde{J}(\tilde{b})), \tilde{P}(\tilde{b})))$$

We define our block sequence $\tilde{B}$ as $\tilde{B}(\tilde{b}) = B(f(\tilde{b}))$.

Let $i \in [\ell]$ and $(\tilde{u}, \tilde{v}) \in \tilde{E}$. We show that inequality (1) is satisfied for $i, (\tilde{u}, \tilde{v})$. If $(\tilde{u}, \tilde{v}) \in \{(\tilde{u}_{F0}, t), (\tilde{u}_{F1}, t), (s, \tilde{v}_{F0}), (s, \tilde{v}_{F1}), (\tilde{u}_{F0}, w), (\tilde{u}_{F1}, w), (w, \tilde{v}_{F0}), (w, \tilde{v}_{F1})\}$, then $c(\tilde{u}, \tilde{v}) = \infty$. Hence it suffices to consider the cases $(\tilde{u}, \tilde{v}) = (u_{F0}, \tilde{u}_{F0})$, $(\tilde{u}, \tilde{v}) = (u_{F1}, \tilde{u}_{F1})$, $(\tilde{u}, \tilde{v}) = (\tilde{v}_{F0}, v_{F0})$, $(\tilde{u}, \tilde{v}) = (\tilde{v}_{F1}, v_{F1})$, and $(\tilde{u}, \tilde{v}) \in E$. Ch.

Case $(\tilde{u}, \tilde{v}) = (u_{F0}, \tilde{u}_{F0})$.