13.12.22

<u>Special case where $d_i = 1$ for all i and $c(e) = k$ for all edges e</u>

Proof (ctd.).

(1).

$U_1 \cup U_2 \cup U_3 = \bigcup_{i, b : b \text{ is an } i\text{-block}} \bigcup_{v \in F_i^u \cap b - \gamma(b) \cup \gamma(b) \cup b - F_i^u} (v, i) = \bigcup_{i, b : b \text{ is an } i\text{-block}} \bigcup_{v \in b} (v, i) =$

$\bigcup_{i, b : b \text{ is an } i\text{-block}} b \times \{i\} = \bigcup_i V(F_i^o \cup F_i^u) \times \{i\}$.

~~(2)~~. Note that this assumes $\bigcup_{b : b \text{ is an } i\text{-block}} b = V(F_i^o \cup F_i^u)$, which would not be the case if we defined blocks as in the paper (then $\bigcup_{b : b \text{ is an } i\text{-block}} b = V(F_i^o \cup F_i^u) - V(F_i^o \cap F_i^u)$ would hold and we would have to update $V(F_i^o \cap F_i^u)$ separately).

(2). The proof depends on our definition of blocks but should work as long as we define blocks in a way such that they are pairwise disjoint.

(3). Let o be a schedule output by the algorithm, that is, a schedule that agrees with U_1, U_2 and U_3, and let $j \in [6]$ be such that $U_j = \bigcup_{\ell=1}^{j} o_\ell$ does not obey the consistency rule.

Let j be minimal among all ~~not~~ such indices.

<u>Claim:</u> After every update $U \subseteq V \times \{i\}$, there is at most one ~~valid~~ (s, t)-flow F_i for every i.

This follows from our definition of updates, that is, after performing the update (v, i), v has at most one outgoing edge w.r.t. the i-th flow pair.

Let $o_j = (v, i)$. Since we chose j minimal, there is a valid (s, t)-flow $T_{i, U_{j-1}}$, where we define $U_0 = \{\}$ and $T_{i, \{\}} = F_i^o$. By the claim, there is ~~an~~ a path from s to v in both $G(i, U_{j-1})$ and $G(i, U_j)$, a path from v to t in $G(i, U_{j-1})$, but no path from v to t in $G(i, U_j)$. Hence No! We have to consider the case $v \in V(F_i^o - F_i^u)$.

Has to follow from our definition of blocks.

$v \in V(F_i^o \cap F_i^u)$. <u>Hence there is an i-block b with $\gamma(b) = v$.</u> Let u be maximal (w.r.t. $\geq_i$) such that $v \to^{+} u$ but $u \not\to^{*} t$.

<u>Lemma:</u> Let u, v be two vertices such that $u \in V(F_i^o \cap F_i^u)$, $v \in V(F_i^o \cup F_i^u)$, $u < v$. Then $u \to^{+}_{F_i^o} v$ or $u \to^{+}_{F_i^u} v$.

Proof. Case $v \in V(F_i^o)$. Case $v \in V(F_i^u)$ is analogous. Since both $u, v \in V(F_i^o)$, either $u \to^{+}_{F_i^o} v$, $u = v$, or $v \to^{+}_{F_i^o} u$. The latter two cases contradict $u < v$.

<u>Corollary:</u> Let $v \in b$. Then $\gamma(b) \to^{*}_{F_i^o} v$ or $\gamma(b) \to^{*}_{F_i^u} v$.

Proof. By definition, $\gamma(b) \in V(F_i^o \cap F_i^u)$. ~~and $v \in V(F_i^o \cup F_i^u)$. By as~~. By assumption, $v \in V(F_i^o \cup F_i^u)$ and $\gamma(b) \leq v$. The claim follows with above lemma for $\gamma(b), v$.

<u>Invariant:</u> $V(F_i^o \cap F_i^u) \subseteq T_{i, U_j}$ for all j.

Step 1 versus step 2 in the algorithm.

We show that $u \in F_i^u \cap b$. Hence the update (u, i) occurred before the update $(\gamma(b), i) = o_i$. This contradicts the maximality of u. To show $u \in F_i^u \cap b$, we show both $u \in F_i^u$ and $u \in b$. For both, we obtain a vertex $x \in V(F_i^o \cap F_i^u)$ that is reachable from v in $G(i, U_j)$. ~~Hence~~ Since $x \in T_{i, U_{j-1}}$, this implies a path from s to v to x to t in $G(i, U_j)$, a contradiction.

Prove instead that every vertex on the path from v to u in $G(i, U_j)$ is in $V(F_i^u - F_i^o)$.

<u>$u \in F_i^u$.</u> Suppose not. Hence $u \in V(F_i^o - F_i^u)$. The head of the new edge (introduced by ~~up~~ the update (v, i)) is in $V(F_i^u - F_i^o)$. Hence, there is a vertex $x \in V(F_i^o \cap F_i^u)$ such that $v \to^+_{G(i, U_j)} x$ and $x \to^+_{G(i, U_j)} u$.

<u>$u \in b$.</u> Suppose not. Hence there is a vertex $x \in V(F_i^o \cap F_i^u)$ such that $v < x \leq u$ and $u \in b(x)$, where $b(x)$ ~~denotes~~ denotes the i-block containing x. Since the path from v to u in $G(i, U_j)$ is completely contained in F_i^u, x is on this path (if $x \to^+_{F_i^u} v$, then $x < v$; if $u \to^+ x$, then $u < x$).

We now consider the case $v \in V(F_i^o - F_i^u)$, that is, the update (v, i) ~~deletes~~ removes the unique outgoing edge from v in $F_i^o \cup F_i^u$. Hence $(v, i) \in U_3$. Hence the update $(\gamma(b(v)), i)$ occurred before the update (v, i). Inductively, the entire path from $\gamma(b(v))$ to v (including v!) in $G(i, U_j)$ has to be contained in F_i^u, a contradiction. (Otherwise, there would be a vertex in $V(F_i^o \cap F_i^u)$ between $\gamma(b(v))$ and v, which contradicts $v \in b(v)$.)