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Proof.

By Lemma ~, for every $P \in \mathcal{P}$, every $i \in [\ell+2]$, and every $S \subseteq r_i$, $P$ is transient for $U_i^S$. Hence it is sufficient to show the following: ~~For every $i \in [\ell]$ let $P \in \mathcal{P}$.~~

(1) For every $S \subseteq r_1$ and every edge $e \in E$, $c(e) \geq \sum_{P \in \mathcal{P}: e \in E(T_{P,U_1^S})} d_P$.

(2) For every $i \in [\ell]$, ~~every $S \subseteq r_{i+1}$~~, and every $e \in E$, $c(e) \geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P$ iff for every $S \subseteq r_{i+1}$, $c(e) \geq \sum_{P \in \mathcal{P}: e \in E(T_{P,U_{i+1}^S})} d_P$.

(3) For every $S \subseteq r_{\ell+2}$ and every edge $e \in E$, $c(e) \geq \sum_{P \in \mathcal{P}: e \in E(T_{P,U_{\ell+2}^S})} d_P$ iff $c(e) \geq \sum_{P \in \mathcal{P}: e \in E(T_{P,U_{\ell+2}^{\emptyset}})} d_P$.

Claim: For every $P \in \mathcal{P}$, $i \in [\ell+2]$, $S \subseteq r_i$, and type-1 (type-3) update $u \in S$, $P$ is transient for $U_i^S$ iff $P$ is transient for $U_i^{S-u}$ and indeed $T_{P,U_i^S} = T_{P,U_i^{S-u}}$ (if they exist).

(1). By Claim ~ and definition of $r_1$, $T_{P,U_1^S} = T_{P,U_1^{\emptyset}} = T_{P,\emptyset} = P^o$ for every $P \in \mathcal{P}$.

(3). By Claim ~ and definition of $r_{\ell+2}$, $T_{P,U_{\ell+2}^S} = T_{P,U_{\ell+2}^{\emptyset}} = T_{P,U_{\ell+1}^{r_{\ell+1}}}$ for every $P \in \mathcal{P}$.

(2). „$\Rightarrow$": Let $S \subseteq r_{i+1}$. ~~By Claim ~, $T_{P,U_{i+1}^S} = T_{P,U_{i+1}^{S'}}$~~

Claim: Let $P \in \mathcal{P}$, $v \in V(P^o \cap P^u)$. Then $v \in V(T_{P,U})$ for every set $U$ of updates (if $T_{P,U}$ exists).

Claim: Let $P \in \mathcal{P}$, $(u,v) \in E(P^u)$, $u \in V(P^o \cap P^u)$, and $(u,P) \in U$. Then $(u,v) \in E(T_{P,U'})$ for every set $U' \supseteq U$ of updates (if $T_{P,U'}$ exists).

Claim: $\alpha_P(e, B_i) = \text{active}$ iff $e \in E(T_{P,U_{i+1}^{r_{i+1}}})$.

Proof.

By definition, $U(B_i) = U_i r_i \cup r_{i+1}|_{2,3} \cup r_{i+2}|_3$. Hence $\alpha_P(e, B_i) = \text{active}$ $\overset{\text{Lemma ~}}{\text{iff}}$ $e \in E(T_{P,U(B_i)})$ iff $e \in E(T_{P,U_i r_i \cup r_{i+1}|_{2,3} \cup r_{i+2}|_3})$ $\overset{\text{Claim ~}}{\text{iff}}$ $e \in E(T_{P,U_i r_i \cup r_{i+1}})$ iff $e \in E(T_{P,U_{i+1}^{r_{i+1}}})$.

Requires generalization of Claim ~

Claim: Let $P \in \mathcal{P}$, $(u,v) \in E(P^o)$, $u \in V(P^o \cap P^u)$, and $(u,P) \notin U$. Then $(u,v) \in E(T_{P,U'})$ for every set $U' \subseteq U$ of updates (if $T_{P,U'}$ exists).

This still doesn't follow from above lemmas; we need something stronger, for all edges induced by the blocks and in $E(P^u)$, $E(P^o)$, respectively.

Claim: If $e \in E(T_{P,U_i^S})$, then $e \in E(T_{P,U_i^{\emptyset}})$ or $e \in E(T_{P,U_i^{r_i}})$.

$\sum_{P \in \mathcal{P}: e \in E(T_{P,U_{i+1}^S})} d_P \overset{\text{Claim ~}}{\leq} \sum_{P: e \in E(T_{P,U_{i+1}^{\emptyset}}) \text{ or } e \in E(T_{P,U_{i+1}^{r_{i+1}}})} d_P \overset{\text{Claim ~}}{=} \sum_{P: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P \leq c(e)$

„$\Leftarrow$": ~~Suppose not. Then there is $i \in [\ell]$, $e \in E$ such that $c(e) < \sum_{P \in \mathcal{P}: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P$.~~

~~We set $S := \{(\text{tail}(e), P) \in r_{i+1} : \alpha_P(e, B_{i-1}) = \text{inactive and } \alpha_P(e, B_i) = \text{active}\}$ and show~~

~~$c(e) < \sum_{P \in \mathcal{P}: e \in E(T_{P,U_{i+1}^S})} d_P$.~~

__Lemma:__ Let $v \in V(F^o \cap F^u)$ and $U$ be a set of updates. Then

(1) If $(v, F) \in U$, then for every $e \in E(b(v, F)) \cap E(F^u)$ and $U' \supseteq U$, $e \in E(T_{F, U'})$ (if $T_{F, U'}$ exists).

(2) If $(v, F) \notin U$, $F^o$ $\leq$

__Lemma:__ Let $R_B = (r_1, \dots, r_{\ell+2})$ be the update sequence induced by block sequence $B$. Then for every $i \in [\ell]$, $P \in \mathcal{P}$, $S \subseteq r_{i+1}$, $(u,v) = e \in E(T_{P, U^S_{i+1}})$ iff $(f(b(~~tail(e)~~ u, P)), P) \notin U^S_{i+1}$ and $(u, v) \in E(P^o)$ or $(f(b(u, P)), P) \in U^S_{i+1}$ and $(u, v) \in E(P^u)$.

We denote by $S_e := \{(f(b(tail(e))), P) : b(tail(e), P) \in \mathcal{B}_i \text{ and } e \in P^u\}$ the worst case set of updates $\subseteq r_{i+1}$ for edge $e$. Notice that $S_e \subseteq r_{i+1}$.

__Claim:__ If $e \in E(T_{P, U^{S_B}_{i+1}})$, then $e \in E(T_{P, U^{S_e}_{i+1}})$ and if $e \in E(T_{P, U^{r_{i+1}}_{i+1}})$, then $e \in E(T_{P, U^{S_e}_{i+1}})$.

__Proof.__

Suppose $e \in E(T_{P, U^{S_B}_{i+1}})$.

Case $(f(b(tail(e), P)), P) \in U^{S_B}_{i+1}$. Then, by lemma ~, $e \in E(P^u)$. Hence, by lemma ~, $e \in E(T_{P, U^{S_e}_{i+1}})$.

Case $\notin$ . $e \in E(P^o)$.

Subcase ~~Case~~ $(f(b(tail(e), P)), P) \in U^{S_e}_{i+1}$. Then $(f(b(tail(e), P)), P) \in S_e$. Hence, by definition of $S_e$, $e \in E(P^u)$. Hence, by lemma ~, $e \in E(T_{P, U^{S_e}_{i+1}})$.

Subcase $(f(b(tail(e), P)), P) \notin U^{S_e}_{i+1}$. Then, since $e \in E(P^o)$, $e \in E(T_{P, U^{S_e}_{i+1}})$.

Suppose $e \in E(T_{P, U^{r_{i+1}}_{i+1}})$.

Case $(f(b(tail(e), P)), P) \in U^{r_{i+1}}_{i+1}$. Then, by lemma ~, $e \in E(P^u)$.

Subcase $(f(b(tail(e), P)), P) \in U^{S_e}_{i+1}$. Then, since $e \in E(P^u)$, $e \in E(T_{P, U^{S_e}_{i+1}})$.

Subcase $\notin$ . Then $e \in E(P^o)$. Hence, by lemma ~, $e \in E(T_{P, U^{S_e}_{i+1}})$.

Case $(f(b(tail(e), P)), P) \notin U^{r_{i+1}}_{i+1}$. Then, by lemma ~, $e \in E(P^o)$. Hence, by lemma ~, $e \in E(T_{P, U^{S_e}_{i+1}})$.

We now have $\sum_{P \in \mathcal{P} : x_P(e, B_{i-1}) = \text{active or } x_P(e, B_i) = \text{active}} d_P = \sum_{P : e \in E(T_{P, U^{r_i}_i}) \text{ or } e \in E(T_{P, U^{r_{i+1}}_{i+1}})} d_P \overset{\text{Claim~}}{\leq} \sum_{P : e \in E(T_{P, U^{S_e}_{i+1}})} d_P \overset{\text{assumption}}{\leq} c(e)$.

This concludes the proof of (2).