17.01.23

Theorem: The k-network flow update problem where every edge is used by at most three flow pairs is NP-hard for $k=10$.

Proof by reduction from 4-SAT

Reduction

Given a 4CNF formula C with n variables $x_1,\dots,x_n$ and m clauses $C_1,\dots,C_m$, we construct the corresponding ~~10-network flow~~ update flow network G as follows.

First, we introduce a variable gadget for each variable and a clause gadget for each clause in a way that is independent of C. Then, we connect the variable and clause gadgets in a way that is particular to C. Finally, we introduce vertices s,t and ensure that G is indeed a feasible instance.

~~Variable gadget~~

~~The variable gadgets will be used by flow pairs $X,\bar{X}$, and B~~

Clause gadget

Start with ~~binary tree description~~ ~~and that~~ a particular update for u^i can only occur if at least one of u_1^i,u_2^i,u_3^i,u_4^i have been updated w.r.t. a particular flow before.

For each clause C_i, we introduce two vertices u^i,v^i and the edge (u^i,v^i). This edge is used by three flow pairs L,R, and B. The idea is to guarantee that clause C_i is satisfied iff for vertex u^i, flow pair L is updated before B or R is updated before B, that is, B cannot be updated unless at least one of L,R have been updated before. Moreover, the rest of the clause gadget will be such that if L ~~has been~~ (R) has been updated before B, then at least one of the first (last) two literals (the left ones) (Right) evaluate to ~~true~~. To accomplish this, we introduce four ~~pairs of~~ vertices u_1^i,u_2^i,u_3^i,u_4^i corresponding to the four literals in clause C_i. These vertices can be thought of the leaves of a binary tree rooted at vertex u^i, where ~~an~~ the update of ~~the~~ a parent will imply ~~at least one update~~ the update of at least one of its children.*

We proceed with the detailed description of the clause gadget for clause C_i.

Root level. We introduce two vertices u^i,v^i and ~~edo~~ add the edge (u^i,v^i) to flows L^o,R^o,B^u.

Intermediate level. We introduce flow pairs $\tilde{L},\tilde{R},\tilde{B}$ similarly to L,R,B in the root level.

Moreover, we introduce vertices $\tilde{u}_{12}^i,\tilde{v}_{12}^i,u_{12}^i,v_{12}^i,\tilde{u}_{34}^i,\tilde{v}_{34}^i,u_{12}^i,v_{12}^i,u_{34}^i,v_{34}^i$ and add edges

- $(\tilde{u}_{12}^i,\tilde{v}_{12}^i)$ to L^u and $\tilde{B}^o$
- $(\tilde{u}_{34}^i,\tilde{v}_{34}^i)$ to R^u and $\tilde{B}^o$
- (u_{12}^i,v_{12}^i) (u_{34}^i,v_{34}^i) to $\tilde{L}^o,\tilde{R}^o,\tilde{B}^u$

~~(u_{34}^i,v_{34}^i) to~~ Leaf level. We introduce vertices $u_1^i,v_1^i,u_2^i,v_2^i,u_3^i,v_3^i,u_4^i,v_4^i$ and add edges

*which then yields that if the clause vertex u^i has been updated, then at least one of the literal vertices u_1^i,u_2^i,u_3^i,u_4^i have been updated.

- $(u^i_1, v^i_1), (u^i_3, v^i_3)$ to $\tilde{L}^u$
- $(u^i_2, v^i_2), (u^i_4, v^i_4)$ to $\tilde{R}^u$

Finally, we add the following edges to connect the clause gadget:

- $(u^i, \tilde{u}^i_{12}), (\tilde{v}^i_{12}, v^i)$ to L^u
- $(u^i, \tilde{u}^i_{34}), (\tilde{v}^i_{34}, v^i)$ to R^u
- $(\tilde{v}^i_{12}, \tilde{u}^i_{34})$ to both $\tilde{B}^o$ and $\tilde{B}^u$
- $(\tilde{u}^i_{12}, u^i_{12}), (v^i_{12}, \tilde{v}^i_{12}), (\tilde{u}^i_{34}, u^i_{34}), (\tilde{v}^i_{34}, \tilde{v}^i_{34})$ to B^u
- (v^i_{12}, u^i_{34}) to $\tilde{L}^o, \tilde{L}^u, \tilde{R}^o, \tilde{R}^u$
- $(u^i_{12}, u^i_1), (u^i_{34}, u_3)$ ~~to $\tilde{L}^u$~~, $(v^i_1, v^i_{12}), (v^i_3, v^i_{34})$ to $\tilde{L}^u$
- $(u^i_{12}, u^i_2), (u^i_{34}, u_4)$ ~~to $\tilde{R}^u$~~, $(v^i_2, v^i_{12}), (v^i_4, v^i_{34})$ to $\tilde{R}^u$

<u>Variable gadgets</u>

For each variable x_j, we introduce a variable vertex x^j which ~~is~~ will be used by flow pairs $X, \overline{X}$, and B such that if X is updated before B, then variable x_j is assigned 1 and if $\overline{X}$ is updated before B, then variable x_j is assigned 0 and not both X and $\overline{X}$ can be updated before B. Notice that, so far, B is the only flow pair used in both the clause and variable gadgets. Indeed, the update flow network will be such that B blocks the update of any variable vertex x^j ~~before~~ until all clause vertices u^i have been updated.

We proceed with the detailed description of the variable gadget for variable x_j.

We introduce two vertices x^j, y^j and add the edge (x^j, y^j) to $X^u, \overline{X}^u$, and B^o. Moreover, we introduce flow pairs B_0 and B_1, where B_0 (B_1) will connect ~~each~~ the variable gadget to every clause gadget corresponding to clause C_i such that variable x_j occurs negatively (positively) in C_i. Furthermore, we introduce vertices $x^j_0, y^j_0, x^j_1, y^j_1$ and add edges

- (x^j_0, y^j_0) to $\overline{X}^o$ and B^u_0
- (x^j_1, y^j_1) to X^o and B^u_1

To connect the variable gadget, we add the following edges:

- $(x^j, x^j_0), (y^j_0, y^j)$ to $\overline{X}^o$
- $(x^j, x^j_1), (y^j_1, y^j)$ to X^o #2

<u>Connecting variable ~~gadgets to~~ and clause gadgets</u>

We now connect every variable gadget corresponding to variable x_j to all clause gadgets corresponding to clause C_i such that x_j occurs in C_i ~~an~~. Let $P_j = \{p^j_1, \dots, p^j_{c_j}\}$ be the set of indices of the clauses containing the literal x_j and

17.01.23

<u>Proof (ctd.)</u>

$\overline{P}_j = \{\overline{p}_1^j, \dots, \overline{p}_{\ell_j}^j\}$ be the set of indices of the clauses containing the literal $\overline{x}_j$. W.l.o.g., $\ell_j, \ell_j' > 0$ (otw. ~~the~~ variable x_j can be assigned true if $\ell_j' = 0$ and false if $\ell_j = 0$ and all clauses containing ~~of~~ variable x_j are satisfied and may be removed). Moreover, let $\pi(i,j)$ denote the position of literal ~~variable~~ x_j in clause C_i and $\overline{\pi}(i,j)$ denote the position of literal $\overline{x}_j$ in clause C_i.

As indicated above, we connect ~~flow pair B_0 (B_1) connects~~ vertex x_0^j (x_1^j) to all literals vertices corresponding to literal $\overline{x}_j$ (x_j) ~~as follows:~~ via flow pair B_0 (B_1) by adding the following edges:

- ~~$(x_0, u^{p_1^j}$~~ $(x_0^j, u^{p_1^j}_{\pi(p_1^j,j)})$, $(u^{p_i^j}_{\pi(p_i^j,j)}, v^{p_i^j}_{\pi(p_i^j,j)})$, $(v^{p_i^j}_{\pi(p_i^j,j)}, u^{p_{i+1}^j}_{\pi(p_{i+1}^j,j)})$, $(v^{p_\ell^j}_{\pi(p_\ell^j,j)}, y_0^j)$ to B_0^0
- ...

<u>Ensuring feasibility</u>

Finally, we add vertices s, t, and create s,t-paths for all flow pairs, by adding the following edges:

- (s, x^1), (y^i, x^{i+1}), (y^n, t) to ~~$X, \overline{X}$, and B~~ $X^0, X^u, \overline{X}^0, \overline{X}^u$, and B^0
- (s, x_0^1), (y_0^i, x_0^{i+1}), (y_0^n, t) to B_0^0 and B_0^u
- (s, x_1^1), (y_1^i, x_1^{i+1}), (y_1^n, t) to B_1^0 and B_1^u
- (s, u^1), (v^i, u^{i+1}), (v^m, t) to L^0, L^u, R^0, R^u, B^u
- $(s, \tilde{u}_{12}^1)$, $(\tilde{v}_{34}^i, \tilde{u}_{12}^{i+1})$, $(\tilde{v}_{34}^m, t)$ to $\tilde{B}^0, \tilde{B}^u$
- (s, u_{12}^1), (v_{34}^i, u_{12}^{i+1}), (v_{34}^m, t) to $\tilde{L}^0, \tilde{L}^u, \tilde{R}^0, \tilde{R}^u$

We now have ten flow pairs $X, \overline{X}, L, R, \tilde{L}, \tilde{R}, B, B_0, B_1, \tilde{B}$, all with demand set to 1.
It remains to specify the edge capacities:

- We set the capacity of edges (x^i, y^i), (u^i, v^i), (u_{12}^i, v_{12}^i), (u_{34}^i, v_{34}^i) to 2.
- (x_0^i, y_0^i), (x_1^i, y_1^i), $(\tilde{u}_{12}^i, \tilde{v}_{12}^i)$, $(\tilde{u}_{34}^i, \tilde{v}_{34}^i)$, (u_1^i, v_1^i), (u_2^i, v_2^i), (u_3^i, v_3^i), (u_4^i, v_4^i) to 1.
- All remaining edge capacities are set to 10, that is, the number of flows. Hence they cannot violate any capacity constraints.

TODO Verify feasibility.

*2 We remark that vertices $x_0^i, y_0^i, x_1^i, y_1^i$ as well as flow pairs B_0, B_1 are unnecessary for this reduction (we could instead just use $X, \overline{X}$ to connect the variable and clause gadgets). They are necessary, however, for the reduction used to prove Theorem ~.

We now show that there is a satisfying assignment for ~~C~~ σ for C iff there is a feasible update sequence respectively $\mathcal{R}$ for G. We will choose $\sigma, \mathcal{R}$ such that σ assigns 1 to variable x_i iff $\mathcal{R}(x^i, \overline{X}) > \mathcal{R}(s, B)$. $\geq$?

„⇒"

Let σ be a satisfying assignment for C. We construct a feasible block sequence, which induces a feasible update sequence.

For every flow pair P, we determine $V(P^o \cap P^u) - \{s,t\}$, which induces the set of blocks:

TODO Specify all blocks in a table.

- $V(X^o \cap X^u) - \{s,t\} = \{x_i, y_i\}_i$
- $V(\overline{X}^o \cap \overline{X}^u) - \{s,t\} = \{x_i, y_i\}_i$
- $V(B^o \cap B^u) - \{s,t\} = \{\}$
- $V(B_0^o \cap B_0^u) - \{s,t\} = \{x_0^i, y_0^i\}_i$
- $V(B_1^o \cap B_1^u) - \{s,t\} = \{x_1^i, y_1^i\}_i$
- $V(L^o \cap L^u) - \{s,t\} = \{u^i, v^i\}_i$
- $V(R^o \cap R^u) - \{s,t\} = \{u^i, v^i\}_i$
- $\tilde{L}: \{u_{12}^i, v_{12}^i, u_{34}^i, v_{34}^i\}_i$
- $\tilde{R}: \{u_{12}^i, v_{12}^i, u_{34}^i, v_{34}^i\}_i$
- $\tilde{B}: \{\tilde{u}_{12}^i, \tilde{v}_{12}^i, \tilde{u}_{34}^i, \tilde{v}_{34}^i\}_i$

We define our block sequence $\mathcal{B} = \{b_1, \dots, b_n\}$ as follows:

- b_1: For every variable x_i, if $\sigma(x_i)=1$, then add block $b(x^i, X)$, otherwise add $b(x^i, \overline{X})$.
- b_2: For every variable x_i, if $\sigma(x_i)=1$, then add block $b(x_1^i, B_1)$, otherwise add $b(x_0^i, B_0)$.
- b_3: For every clause $C_i = (l_1 \vee l_2 \vee l_3 \vee l_4)$,
  - if $\sigma(l_1)=1$, then add $b(u_{12}^i, \tilde{L})$,
  - if $\sigma(l_2)=1$, then add $b(u_{12}^i, \tilde{R})$,
  - $\sigma(l_3)$ $b(u_{34}^i, \tilde{L})$,
  - $\sigma(l_4)$ $b(u_{34}^i, \tilde{R})$
- b_4: For every clause $C_i = (l_1 \vee l_2 \vee l_3 \vee l_4)$, if $\sigma(l_1)=1$ or $\sigma(l_2)=1$, then add $b(\tilde{u}_{12}^i, \tilde{B})$, ~~otherwise~~ if $\sigma(l_3)=1$ or $\sigma(l_4)=1$, then add $b(\tilde{u}_{34}^i, \tilde{B})$.
- b_5: For every clause $C_i = (l_1 \vee l_2 \vee l_3 \vee l_4)$, if $\sigma(l_1)=1$ or $\sigma(l_2)=1$, then add $b(u^i, L)$, if $\sigma(l_3)=1$ or $\sigma(l_4)=1$, then add $b(u^i, R)$
- $b_6 = \{b(s, B)\}$

17.01.23

<u>Proof (ctd.)</u>

The sets of blocks $b_7, b_8, b_9, b_{10}, b_{11}$ are mirrors of b_1, b_2, b_3, b_4, b_5, respectively, that is, for example, b_7 contains all blocks $b(x^i, X)$, $b(x^i, \overline{X})$, that is, all X- and $\overline{X}$-blocks, that were not added to b_1.

Observe that all other blocks, that is, blocks not contained in b_ℓ for $\ell = 1, \dots, 11$, do not induce any updates that activate or deactivate any edges and may ~~be~~ thus be added to any b_ℓ.

We now show that block sequence B is indeed feasible by verifying that inequality ~ is satisfied for every $\ell \in [11]$, and every edge with capacity less than 10.

~~- b_1 activates edges (x^i, y^i)~~ For every $\ell \in [11]$, it suffices to consider the edges that were activated by b_ℓ; ~~the capacity constraints for all other edges cannot be violated~~ if the capacity constraint for another edge is violated for ℓ, then so is it for $\ell - 1$, where the case for $b_0 := \{\}$ follows because instance G is feasible.

~~- b_1 activates edges (x^i, y^i) for X if $\sigma(x_i) = 1$ and X for every i.~~

~~$c(x^i, y^i) = 3 \Rightarrow$ Before b_1, edge (x^i, y^i) is used only by flow pair B. After b_1, (x^i, y^i) is also used by X if $\sigma(x_i) = 1$ and $\overline{X}$ otw.. Hence it is used by exactly 2 = $c(x^i, y^i)$ flow pairs.~~

- (x_0^i, y^i): ~~In G, edge (x^i, y^i) is active only for flow pair B. After b_1~~
  - In G, edge (x^i, y^i) is active only for flow pair B.
  - ~~After b_1,~~ b_1 activates (x^i, y^i) for X if $\sigma(x_i) = 1$ and $\overline{X}$ otw.
  - The flow pairs using (x^i, y^i) remain unchanged until b_6, which deactivates (x^i, y^i) for B.
  - b_7 activates (x^i, y^i) for $\overline{X}$ if $\sigma(x_i) = 1$ and X otw.
- (x_0^i, y_0^i):
  - In G,

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
(x^r, y^r)	B	B X $\sigma(x_i)=1$ $\overline{X}$ otw.					~~B~~ X $\sigma(x_i)=1$ $\overline{X}$ otw.	X $\overline{X}$				
(x_0^r, y_0^r)	$\overline{X}$	$\overline{X}$ $\sigma(x_i)=1$	$\overline{X}$ $\sigma(x_i)=1$ B_0 $\sigma(x_i)=0$					B_0 $\sigma(x_i)=0$	B_0			
~~(x_0^r, y_0^r)~~ (x_1^r, y_1^r)	X	X $\sigma(x_i)=0$	X $\sigma(x_i)=0$ B_1 $\sigma(x_i)=1$					B_1 $\sigma(x_i)=1$	B_1			
~~(x_1^r, y_1^r)~~ (u^i, v^i)	L R					L $\sigma(\ell_1)=0 \wedge$ R $\sigma(\ell_3)=0 \wedge$	L $\sigma(\ell_1)=0 \wedge$ R $\sigma(\ell_3)=0 \wedge$ B					B
(u_{12}^i, v_{12}^i)	$\tilde{L}$ $\tilde{R}$	~~$\tilde{L}$ $\sigma(\ell_1)=0$~~ ~~$\tilde{R}$ $\sigma(\ell_2)=0$~~		$\tilde{L}$ $\sigma(\ell_1)=0$ $\tilde{R}$ $\sigma(\ell_2)=0$	$\tilde{L}$ $\sigma(\ell_1)=0$ $\tilde{R}$ $\sigma(\ell_2)=0$ $\tilde{B}$ $\sigma(\ell_1)=1 \vee$					$\tilde{B}$ $\sigma(\ell_1)=1 \vee \tilde{B}$		
(u_{34}^i, v_{34}^i)	$\tilde{L}$ $\tilde{R}$	~~$\tilde{L}$ $\sigma(\ell_3)=0$~~ ~~$\tilde{R}$ $\sigma(\ell_4)=0$~~		$\tilde{L}$ $\sigma(\ell_3)=0$ $\tilde{R}$ $\sigma(\ell_4)=0$	$\tilde{L}$ $\sigma(\ell_3)=0$ $\tilde{R}$ $\sigma(\ell_4)=0$ $\tilde{B}$ $\sigma(\ell_3)=1 \vee$					$\tilde{B}$ $\sigma(\ell_3)=1 \vee \tilde{B}$		
$(\tilde{u}_{12}^i, \tilde{v}_{12}^i)$	$\tilde{B}$				$\tilde{B}$ $\sigma(\ell_1)=0 \wedge$	$\tilde{B}$ $\sigma(\ell_1)=0 \wedge$ L $\sigma(\ell_1)=1 \vee$					L $\sigma(\ell_1)=1 \vee L$	
$(\tilde{u}_{34}^i, \tilde{v}_{34}^i)$	$\tilde{B}$				$\tilde{B}$ $\sigma(\ell_3)=0 \wedge$	$\tilde{B}$ $\sigma(\ell_3)=0 \wedge$ R $\sigma(\ell_3)=1 \vee$					R $\sigma(\ell_3)=1 \vee R$	
(u_1^i, v_1^i)	B_0 ℓ_1 neg B_1 ℓ_1 pos		B_0 $\sigma(\ell_1)=0$ B_1 $\sigma(\ell_1)=0$	B_0 $\sigma(\ell_1)=0$ B_1 $\sigma(\ell_1)=0$ $\tilde{L}$ $\sigma(\ell_1)=1$					$\tilde{L}$ $\sigma(\ell_1)=1$	$\tilde{L}$		
~~(u_2^i, v_2^i)~~ (u_2^i, v_2^i) ~~(u_3^i, v_3^i)~~	B_0 ℓ_2 neg B_1 ℓ_2 pos		B_0 $\sigma(\ell_2)=0$ B_1 $\sigma(\ell_2)=0$	B_0 $\sigma(\ell_2)=0$ B_1 $\sigma(\ell_2)=0$ $\tilde{R}$ $\sigma(\ell_2)=1$					$\tilde{R}$ $\sigma(\ell_2)=1$	$\tilde{R}$		
(u_3^i, v_3^i)	B_0 ℓ_3 neg B_1 ℓ_3 pos		B_0 $\sigma(\ell_3)=0$ B_1 $\sigma(\ell_3)=0$	B_0 $\sigma(\ell_3)=0$ B_1 $\sigma(\ell_3)=0$ $\tilde{L}$ $\sigma(\ell_3)=1$					$\tilde{L}$ $\sigma(\ell_3)=1$	$\tilde{L}$		
(u_4^i, v_4^i)	B_0 ℓ_4 neg		B_0 $\sigma(\ell_4)=0$	B_0 $\sigma(\ell_4)=0$ $\tilde{R}$ $\sigma(\ell_4)=1$					$\tilde{R}$ $\sigma(\ell_4)=1$	$\tilde{R}$		