

Congestion-Free Network Updates: Algorithms and Complexity

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Table 1: Table of notations

\leq_G	The reachability relation for directed graph G
$b(v, P)$	The P -block containing vertex v
$P(b)$	The flow pair P such that block b is a P -block
$\mathcal{S}(b)$	The least vertex in block b w.r.t. reachability relation $\leq_{P(b) \circ \cup P(b)^u}$
$B^P(G)$	The set of P -blocks
$B(G)$	The set of blocks
$\mathcal{B}(b)$	The round in which block b is updated
$\mathcal{B}(v, P)$	The round in which block $b(v, P)$ is updated
B_i	The set of blocks updated before or in the i -th round

Part I

Preliminaries

Notation 1. For a directed graph G , $\leq_G = E(G)^*$ denotes the reachability relation of G . That is, for every two vertices $u, v \in V(G)$, $u \leq_G v$ iff there is a path in G from u to v .

Notice that for every directed graph G ,

1. reachability relation \leq_G is a partial order,
2. if G is a DAG, then \leq_G is antisymmetric, and
3. if G is a path graph, then \leq_G is a total order.

Definition 2. Let $G = (V, E, \mathcal{P}, s, t, c)$ be an update flow network and $P \in \mathcal{P}$ be a flow pair. Let v_1, \dots, v_ℓ be the set $V(P^o \cap P^u)$ ordered w.r.t. $\leq_{P^o \cup P^u}$. For every $i \in [\ell - 1]$, we define the i -th P -block as $b_i^P = \{v \mid v_i \leq_{P^o \cup P^u} v \leq_{P^o \cup P^u} v_{i+1}\}$.

Remark 3. There are multiple issues with this definition (see `../README.org`).

Notation 4. For a flow pair P and a vertex $v \in V(P)$, $b(v, P)$ denotes the P -block containing v .

Notation 5. For a block b , $P(b)$ denotes the flow pair P such that b is a P -block.

Notation 6. For a block b , $\mathcal{S}(b)$ denotes the *start* of b , that is, the least vertex in b w.r.t. $\leq_{P(b)^o \cup P(b)^u}$.

Notation 7. For an update flow network G and a flow pair P , $B^P(G)$ denotes the set of P -blocks.

Notation 8. For an update flow network G , $B(G) = \bigcup_{P \in \mathcal{P}} B^P(G)$ denotes the set of blocks.

Definition 9. A *block sequence* $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ is an ordered partition of the set of blocks.

Remark 10. We may ignore all blocks containing less than three vertices.

□ Flesh out and argue why.

Notation 11. For a block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ and a block b , $\mathcal{B}(b)$ denotes the index $i \in [\ell]$ such that b is contained in \mathcal{B}_i .

Notation 12. For a block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$, a flow pair P , and a vertex $v \in V(P)$, $\mathcal{B}(v, P) = \mathcal{B}(b(v, P))$ denotes the index $i \in [\ell]$ such that block $b(v, P)$ is contained in \mathcal{B}_i .

Notation 13. For a block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ and an index $i \in [\ell]$, $B_i = \bigcup_{j \leq i} \mathcal{B}_j$ denotes the set of blocks updated before or in the i -th round.

Definition 14. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence. For a flow pair P , an edge $(u, v) \in E(P^o \cup P^u)$, and an index $i \in [\ell]$, the *activation label* $\alpha_P((u, v), B_i)$ is defined as follows:

$$\alpha_P((u, v), B_i) = \begin{cases} \text{active} & \text{if } (u, v) \in E(P^o) \text{ and } b(u, P) \notin B_i \\ \text{active} & \text{if } (u, v) \in E(P^u) \text{ and } b(u, P) \in B_i \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Lemma 15. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence, P be a flow pair, $(u, v) \in E(P^o \cup P^u)$, and $i \in [\ell]$. Then:

1. If $(u, v) \in E(P^o \setminus P^u)$, then

$$\alpha_P((u, v), B_i) = \begin{cases} \text{active} & i < \mathcal{B}(u, P) \\ \text{inactive} & i \geq \mathcal{B}(u, P). \end{cases}$$

2. If $(u, v) \in E(P^o \cap P^u)$, then $\alpha_P((u, v), B_i) = \text{active}$.

3. If $(u, v) \in E(P^u \setminus P^o)$, then

$$\alpha_P((u, v), B_i) = \begin{cases} \text{active} & i \geq \mathcal{B}(u, P) \\ \text{inactive} & i < \mathcal{B}(u, P). \end{cases}$$

Notation 16. For a flow pair P and a P -block b , $U(b) = \{(v, P) \mid v \in b\}$ denotes the set of updates induced by b . Moreover, for a set B of blocks, $U(B) = \bigcup_{b \in B} U(b)$ denotes the set of updates induced by B .

The following lemma shows that for every block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$, every flow pair P , every edge $e \in E(P^o \cup P^u)$, and every $i \in [\ell]$, $\alpha_P(e, B_i) = \text{active}$ iff e is on the transient (s, t) -path for P after updating all blocks in B_i .

Lemma 17. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence, P be a flow pair, $e \in E(P^o \cup P^u)$, and $i \in [\ell]$. Then $\alpha_P(e, B_i) = \text{active}$ iff $e \in E(T_{P, U(B_i)})$.

Definition 18. A block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ is *feasible* if for every edge e and every index $i \in [\ell]$,

$$c(e) \geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P, \quad (1)$$

where we define \mathcal{B}_0 to be the empty set.

Remark 19. Let G be an update flow network with unit demand, that is, $d_P = 1$ for every flow pair P , and let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence. Then, for every edge e and every index $i \in [\ell]$, capacity constraint 1 simplifies to:

$$\begin{aligned} c(e) &\geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P \\ &= \sum_{P \in \mathcal{P}: \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} 1 \\ &= |\{P \in \mathcal{P} \mid \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}\}|. \end{aligned}$$

Lemma 20. *Let G be a not necessarily feasible update flow network and $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence. Then:*

1. *The old flow network is feasible if capacity constraint 1 is satisfied for every edge and $i = 1$.*
2. *The updated flow network is feasible if capacity constraint 1 is satisfied for every edge and $i = \ell$.*

Proof. Let G be a not necessarily feasible update flow network and $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence. Moreover, let e be an edge.

1. Suppose capacity constraint 1 is satisfied for e and $i = 1$. Then, since $\mathcal{B}_0 = \emptyset$, and by definitions of B_i and α_P :

$$\begin{aligned} c(e) &\geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_0) = \text{active or } \alpha_P(e, B_1) = \text{active}} d_P \\ &\geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_0) = \text{active}} d_P \\ &= \sum_{P \in \mathcal{P}: e \in E(P^o)} d_P. \end{aligned}$$

2. Suppose capacity constraint 1 is satisfied for e and $i = \ell$. Then, since \mathcal{B} partitions the set of blocks, and by definitions of B_i and α_P :

$$\begin{aligned} c(e) &\geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_{\ell-1}) = \text{active or } \alpha_P(e, B_\ell) = \text{active}} d_P \\ &\geq \sum_{P \in \mathcal{P}: \alpha_P(e, B_\ell) = \text{active}} d_P \\ &= \sum_{P \in \mathcal{P}: e \in E(P^u)} d_P. \end{aligned}$$

□

Corollary 21. *There is a feasible block sequence iff there is a feasible update sequence.*

Part II

NP-Hardness for $k = 3$

The goal of this section is to prove the following theorem.

Theorem 22. *The k -network flow update problem is **NP**-hard for $k = 3$.*

We will prove this theorem in two steps. First, we will prove the following theorem.

Theorem 23. *The k -network flow update problem, where every edge is used by at most three flow pairs, is **NP**-hard for $k = 10$.*

Then, we will (repeatedly) apply the following lemma to the flow update network we will have constructed in the proof of Theorem 23 to reduce the number of flow pairs from 10 to 3.

Lemma 24 (Merging Lemma). *Let G be an update flow network with $k \geq 2$ flow pairs, and let F, F' be two flow pairs such that*

1. $d_F = d_{F'}$,
2. F and F' have no common vertices other than s, t , that is, $V(F^o \cup F^u) \cap V(F'^o \cup F'^u) = \{s, t\}$, and
3. there are vertices $v_F, v_{F'}$ such that
 - (a) there is no edge from v_F to $v_{F'}$, that is, $(v_F, v_{F'}) \notin E$,
 - (b) (v_F, t) $((s, v_{F'}))$ is the last (first) edge on both F^o and F^u (F'^o and F'^u), that is, $(v_F, t) \in E(F^o \cap F^u)$ $((s, v_{F'}) \in E(F'^o \cap F'^u))$, and
 - (c) the capacity constraint for (v_F, t) $((s, v_{F'}))$ is trivially satisfied, that is,

$$c(e) \geq \sum_{P \in \mathcal{P}: e \in E(P^o \cup P^u)} d_P$$

for $e = (v_F, t)$ ($e = (s, v_{F'})$).

Then there is an update flow network \tilde{G} with $k - 1$ flow pairs such that $(|\tilde{G}| = O(|G|))$ and there is a feasible block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ for G iff there is a feasible block sequence $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ for \tilde{G} .

Remark 25. I'm confident we don't need property 3, but it significantly simplifies the proof.

Chapter 1

NP-Hardness for the Special Case

The proof of Theorem 23 is via reduction from 4-SAT and is based on the **NP**-hardness proof for $k = 6$ in (Amiri, Saeed A. and Dudycz, Szymon and Parham, Mahmoud and Schmid, Stefan and Wiederrecht, Sebastian, 2019).

Let C be a 4CNF formula with n variables x_1, \dots, x_n and m clauses C_1, \dots, C_m . W.l.o.g. every variable occurs both positively and negatively (otherwise, if a variable x_j occurs only positively (negatively), we can assign 1 (0) to x_j and remove all clauses containing literal x_j (\bar{x}_j)). We construct the corresponding update flow network G as follows.

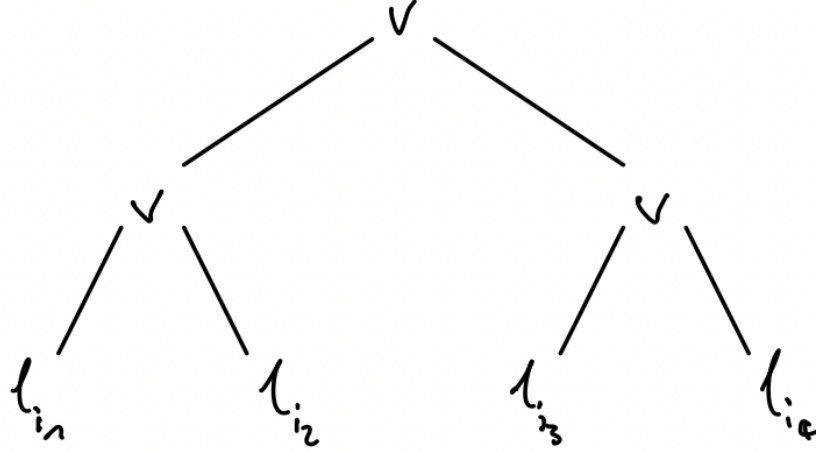
1.1 The Reduction

First, we introduce a *clause gadget* for each clause and a *variable gadget* for each variable. Then, we connect the variable and clause gadgets. Finally, we take the remaining steps necessary to ensure that G is indeed a feasible update flow network.

Clause gadgets. Let $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$ be a clause. We construct the corresponding clause gadget C^i as follows. The idea is to model the syntax tree for C_i depicted in Figure 1.1.

For the root operator node, we introduce a *clause vertex* u^i which is used by three flow pairs L, R, B . The idea is to guarantee that clause C_i is satisfied iff block $b(u^i, L)$ is updated before block $b(u^i, B)$ or block $b(u^i, R)$ is updated before $b(u^i, B)$. Equivalently, $b(u^i, B)$ cannot be updated unless at least one of $b(u^i, L), b(u^i, R)$ has been updated. Intuitively, if $b(u^i, L)$ ($b(u^i, R)$) is updated before $b(u^i, B)$, then the **Left** half ($l_{i_1} \vee l_{i_2}$) (**Right** half ($l_{i_3} \vee l_{i_4}$)) of C_i is satisfied.

Similarly, for the intermediate operator nodes of the syntax tree, we introduce clause vertices $u_{1,2}^i, u_{3,4}^i$, where $u_{1,2}^i$ corresponds to $(l_{i_1} \vee l_{i_2})$ and $u_{3,4}^i$

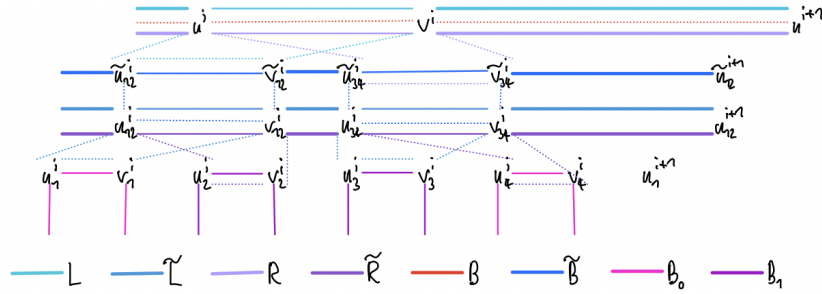
Figure 1.1: A syntax tree for clause $(l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$

corresponds to $(l_{i_3} \vee l_{i_4})$. Both clause vertices are used by flow pairs $\tilde{L}, \tilde{R}, \tilde{B}$ such that if $b(u_{1,2}^i, \tilde{L})$ ($b(u_{1,2}^i, \tilde{R})$) is updated before $b(u_{1,2}, \tilde{B})$, then the left half l_{i_1} (right half l_{i_2}) of $(l_{i_1} \vee l_{i_2})$ is satisfied, and analogously for $u_{3,4}^i$.

Moreover, for the operand nodes of the syntax tree, we introduce *literal vertices* $u_1^i, u_2^i, u_3^i, u_4^i$.

Finally, for every branch from a parent node to its left (right) child node, we add an edge to either L (R) (if the parent node is u^i) or \tilde{L} (\tilde{R}) (if the parent node is $u_{1,2}^i$ or $u_{3,4}^i$).

We now proceed with the detailed specification of clause gadget C^i (see Figure 1.2).

Figure 1.2: Clause gadget C^i

We introduce six flow pairs $L, R, B, \tilde{L}, \tilde{R}, \tilde{B}$, each with demand 1.

For the clause vertices, we introduce two vertices u^i, v^i and add edge (u^i, v^i) to flows L^o, R^o, B^u . Similarly, we introduce vertices $u_{1,2}^i, v_{1,2}^i, u_{3,4}^i, v_{3,4}^i$ and add edges $(u_{1,2}^i, v_{1,2}^i), (u_{3,4}^i, v_{3,4}^i)$ to flows $\tilde{L}^o, \tilde{R}^o, \tilde{B}^u$.

For the literal vertices, we introduce vertices $u_1^i, v_1^i, u_2^i, v_2^i, u_3^i, v_3^i, u_4^i, v_4^i$ and add edges $(u_1^i, v_1^i), (u_3^i, v_3^i)$ to flow \tilde{L}^u and $(u_2^i, v_2^i), (u_4^i, v_4^i)$ to \tilde{R}^u .

Moreover, we introduce auxiliary vertices $\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i, \tilde{u}_{3,4}^i, \tilde{v}_{3,4}^i$ and add edge $(\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i)$ to flows \tilde{L}^u, \tilde{B}^o and $(\tilde{u}_{3,4}^i, \tilde{v}_{3,4}^i)$ to \tilde{R}^u, \tilde{B}^o .

Finally, we add the following edges to connect clause gadget C^i :

- $(u^i, \tilde{u}_{1,2}^i), (\tilde{v}_{1,2}^i, v^i)$ to L^u
- $(u^i, \tilde{u}_{3,4}^i), (\tilde{v}_{3,4}^i, v^i)$ to R^u
- $(v_{1,2}^i, u_{3,4}^i)$ to $\tilde{L}^o, \tilde{L}^u, \tilde{R}^o, \tilde{R}^u$
- $(u_{1,2}^i, u_1^i), (v_1^i, v_{1,2}^i), (u_{3,4}^i, u_3^i), (v_3^i, v_{3,4}^i)$ to \tilde{L}^u
- $(u_{1,2}^i, u_2^i), (v_2^i, v_{1,2}^i), (u_{3,4}^i, u_4^i), (v_4^i, v_{3,4}^i)$ to \tilde{R}^u
- $(\tilde{v}_{1,2}^i, \tilde{u}_{3,4}^i)$ to \tilde{B}^o, \tilde{B}^u
- $(\tilde{u}_{1,2}^i, u_{1,2}^i), (v_{1,2}^i, \tilde{v}_{1,2}^i), (\tilde{u}_{3,4}^i, u_{3,4}^i), (v_{3,4}^i, \tilde{v}_{3,4}^i)$ to \tilde{B}^u

Variable gadgets. For every variable x_j , we construct the corresponding variable gadget X^j as follows. We introduce a *variable vertex* x^j which is used by three flow pairs X, \bar{X}, B . The idea is to guarantee the following:

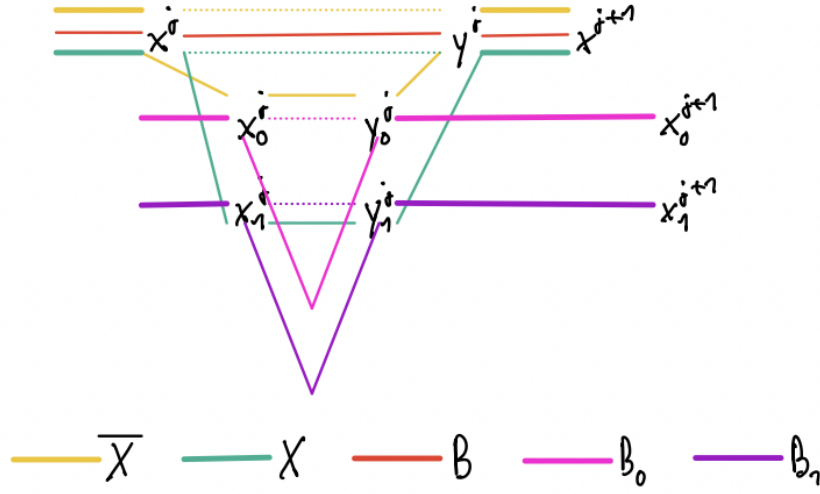
1. If block $b(x^j, X)$ is updated before block $b(x^j, B)$, then variable x_j is assigned 1.
2. If block $b(x^j, \bar{X})$ is updated before $b(x^j, B)$, then x_j is assigned 0.
3. Not both $b(x^j, X)$ and $b(x^j, \bar{X})$ can be updated before $b(x^j, B)$.

We now proceed with the detailed specification of variable gadget X^j (see Figure 1.3).

We introduce two flow pairs X, \bar{X} , each with demand 1. For the variable vertices, we introduce vertices x^j, y^j and add edge (x^j, y^j) to flows X^u, \bar{X}^u, B^o . Moreover, we introduce auxiliary vertices $x_0^j, y_0^j, x_1^j, y_1^j$ and add edge (x_0^j, y_0^j) to flow \bar{X}^o and (x_1^j, y_1^j) to X^o . Finally, to connect variable gadget X^j , we add edges $(x^j, x_0^j), (y_0^j, y^j)$ to flow \bar{X}^o and $(x^j, x_1^j), (y_1^j, y^j)$ to X^o .

Connecting variable with clause gadgets. For every $j \in [n]$ and every $i \in [m]$, we connect variable gadget X^j to clause gadget C^i if variable x_j occurs in clause C_i . More precisely, we introduce two flow pairs B_0, B_1 , each with demand 1, such that B_0 (B_1) connects vertex x_0^j (x_1^j) to all literal vertices corresponding to literal \bar{x}_j (x_j).

More formally, for every $j \in [n]$, let $P_j = \{p_1^j, \dots, p_{\ell_j}^j\}$ denote the set of indices of the clauses containing literal x_j and $\bar{P}_j = \{\bar{p}_1^j, \dots, \bar{p}_{\ell_j}^j\}$ denote the set of indices of the clauses containing literal \bar{x}_j . Moreover, for every $j \in [n]$ and

Figure 1.3: Variable gadget X^j

every $i \in [m]$, let $\pi(i, j)$ denote the position of literal x_j in clause C_i and $\bar{\pi}(i, j)$ denote the position of literal \bar{x}_j in C_i . For every $j \in [n]$, we add the following edges:

- $(x_0^j, u_{\bar{\pi}(\bar{p}_1^j, j)}^{\bar{p}_1^j}), (u_{\bar{\pi}(\bar{p}_\ell^j, j)}^{\bar{p}_\ell^j}, v_{\bar{\pi}(\bar{p}_\ell^j, j)}^{\bar{p}_\ell^j})$ for every $\ell \in [\ell'_j]$, $(v_{\bar{\pi}(\bar{p}_\ell^j, j)}^{\bar{p}_\ell^j}, u_{\bar{\pi}(\bar{p}_{\ell+1}^j, j)}^{\bar{p}_{\ell+1}^j})$ for every $\ell \in [\ell'_j - 1]$, and $(v_{\bar{\pi}(\bar{p}_{\ell'_j}^j, j)}^{\bar{p}_{\ell'_j}^j}, y_0^j)$ to B_0^o
- $(x_1^j, u_{\pi(p_1^j, j)}^{p_1^j}), (u_{\pi(p_\ell^j, j)}^{p_\ell^j}, v_{\pi(p_\ell^j, j)}^{p_\ell^j})$ for every $\ell \in [\ell_j]$, $(v_{\pi(p_\ell^j, j)}^{p_\ell^j}, u_{\pi(p_{\ell+1}^j, j)}^{p_{\ell+1}^j})$ for every $\ell \in [\ell_j - 1]$, and $(v_{\pi(p_{\ell_j}^j, j)}^{p_{\ell_j}^j}, y_1^j)$ to B_1^o

Completing the update flow network. We introduce vertices s, t and create (s, t) -paths for all flows by adding the following edges:

- $(s, u^1), (v^m, t)$ to L^o, L^u, R^o, R^u
- (v^i, u^{i+1}) for every $i \in [m - 1]$ to L^o, L^u, R^o, R^u, B^u
- $(s, u_{1,2}^1), (v_{3,4}^i, u_{1,2}^{i+1})$ for every $i \in [m - 1]$, and $(v_{3,4}^m, t)$ to $\tilde{L}^o, \tilde{L}^u, \tilde{R}^o, \tilde{R}^u$
- $(s, \tilde{u}_{1,2}^1), (\tilde{v}_{3,4}^i, \tilde{u}_{1,2}^{i+1})$ for every $i \in [m - 1]$, and $(\tilde{v}_{3,4}^m, t)$ to \tilde{B}^o, \tilde{B}^u
- $(s, x^1), (y^n, t)$ to $X^o, X^u, \bar{X}^o, \bar{X}^u, B^o, B^u$
- (y^j, x^{j+1}) for every $j \in [n - 1]$ to $X^o, X^u, \bar{X}^o, \bar{X}^u, B^o$

- $(x^1, u^1), (v^m, y^n)$ to B^u
- $(s, x_0^1), (y_0^j, x_0^{j+1})$ for every $j \in [n-1]$, and (y_0^n, t) to B_0^o, B_0^u
- $(s, x_1^1), (y_1^j, x_1^{j+1})$ for every $j \in [n-1]$, and (y_1^n, t) to B_1^o, B_1^u

See Figure 1.4 for the complete update flow network and Table 1.1 for all (s, t) -flows.

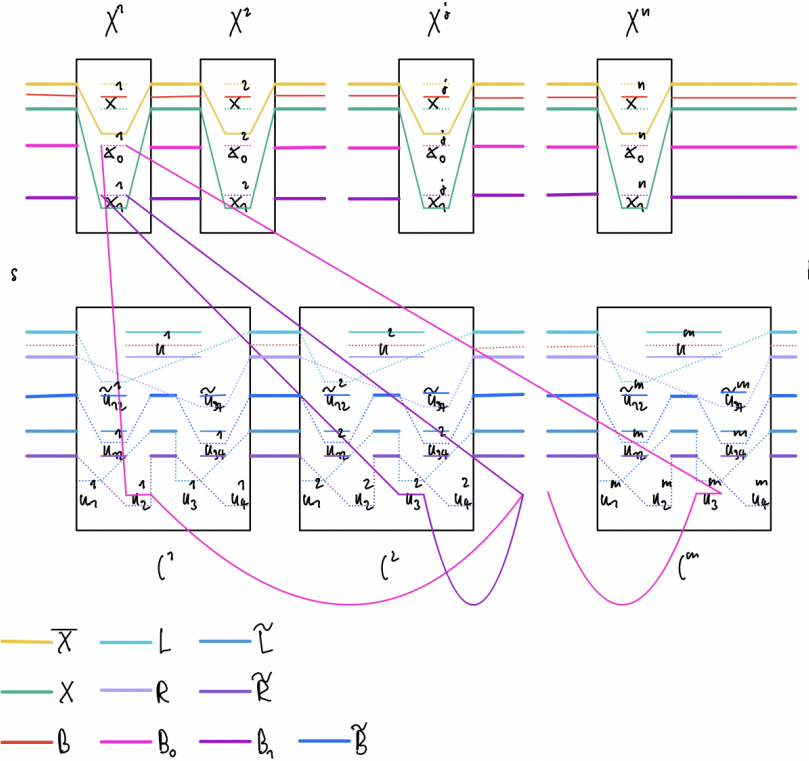


Figure 1.4: The update flow network

Edge capacities are defined as follows.

- We set the capacity to 2 for edges $(u^i, v^i), (u_{1,2}^i, v_{1,2}^i), (u_{3,4}^i, v_{3,4}^i), (x^j, y^j)$ for every $i \in [m]$ and every $j \in [n]$.
- We set the capacity to 1 for edges $(u_1^i, v_1^i), (u_2^i, v_2^i), (u_3^i, v_3^i), (u_4^i, v_4^i), (\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i), (\tilde{u}_{3,4}^i, \tilde{v}_{3,4}^i), (x_0^j, y_0^j), (x_1^j, y_1^j)$ for every $i \in [m]$ and every $j \in [n]$.
- All remaining edge capacities are set to 10, that is, the number of flow pairs, which equals the sum of all demands.

Table 1.1: All (s, t) -flows

Flow	(s, t) -path
X^o	$s, x^1, x_0^1, y_0^1, y^1, x^2, \dots, y^n, t$
\bar{X}^u	$s, x^1, y^1, x^2, \dots, y^n, t$
L^o	$s, u^1, v^1, u^2, \dots, v^m, t$
L^u	$s, u^1, \tilde{u}_{1,2}^1, \tilde{v}_{1,2}^1, v^1, u^2, \dots, v^m, t$
\tilde{L}^o	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$
\tilde{L}^u	$s, u_{1,2}^1, u_1^1, v_1^1, v_{1,2}^1, u_{3,4}^1, u_3^1, v_3^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$
X^o	$s, x^1, x_1^1, y_1^1, y^1, x^2, \dots, y^n, t$
X^u	$s, x^1, y^1, x^2, \dots, y^n, t$
R^o	$s, u^1, v^1, u^2, \dots, v^m, t$
R^u	$s, u^1, \tilde{u}_{3,4}^1, \tilde{v}_{3,4}^1, v^1, u^2, \dots, v^m, t$
\tilde{R}^o	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$
\tilde{R}^u	$s, u_{1,2}^1, u_2^1, v_2^1, v_{1,2}^1, u_{3,4}^1, u_4^1, v_4^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$
B^o	$s, x^1, y^1, x^2, \dots, y^n, t$
B^u	$s, x^1, u^1, v^1, u^2, \dots, v^m, y^n, t$
\tilde{B}^o	$s, \tilde{u}_{1,2}^1, \tilde{v}_{1,2}^1, \tilde{u}_{3,4}^1, \tilde{v}_{3,4}^1, \tilde{u}_{1,2}^2, \dots, \tilde{v}_{3,4}^m, t$
\tilde{B}^u	$s, \tilde{u}_{1,2}^1, u_{1,2}^1, v_{1,2}^1, \tilde{v}_{1,2}^1, \tilde{u}_{3,4}^1, u_{3,4}^1, v_{3,4}^1, \tilde{v}_{3,4}^1, \tilde{u}_{1,2}^2, \dots, \tilde{v}_{3,4}^m, t$
B_0^o	$s, x_0^1, u_{\bar{\pi}(\bar{p}_1^1, 1)}^{\bar{p}_1^1}, v_{\bar{\pi}(\bar{p}_1^1, 1)}^{\bar{p}_1^1}, u_{\bar{\pi}(\bar{p}_2^1, 1)}^{\bar{p}_2^1}, \dots, v_{\bar{\pi}(\bar{p}_{l_1}^1, 1)}^{\bar{p}_{l_1}^1}, y_0^1, x_0^2, \dots, y_0^n, t$
B_0^u	$s, x_0^1, y_0^1, x_0^2, \dots, y_0^n, t$
B_1^o	$s, x_1^1, u_{\pi(p_1^1, 1)}^{p_1^1}, v_{\pi(p_1^1, 1)}^{p_1^1}, u_{\pi(p_2^1, 1)}^{p_2^1}, \dots, v_{\pi(p_{l_1}^1, 1)}^{p_{l_1}^1}, y_1^1, x_1^2, \dots, y_1^n, t$
B_1^u	$s, x_1^1, y_1^1, x_1^2, \dots, y_1^n, t$

We remark that vertices $\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i, \tilde{u}_{3,4}^i, \tilde{v}_{3,4}^i$ are not necessary for this proof. Instead, we could directly connect clause vertices $u^i, u_{1,2}^i$ via flow pair L and $u^i, u_{3,4}^i$ via R . Similarly, vertices $x_0^j, y_0^j, x_1^j, y_1^j$ as well as flow pairs B_0, B_1 are not necessary. We could instead directly connect variable vertex x^j to literal vertex, say u_1^i , via X (\bar{X}) if $l_{i_1} = x_j$ ($l_{i_1} = \bar{x}_j$). The vertices and flow pairs are necessary, however, for the proof of Theorem 22.

Let us quickly verify that G is a feasible update flow network.

To verify that every flow is indeed an (s, t) -path, see Table 1.1. Recall we assumed every variable x_j occurs both negatively and positively in formula C . Hence both P_j and \bar{P}_j are non-empty. Thus both B_0^o and B_1^o form (s, t) -paths.

To verify that every flow pair forms a DAG, again consider Table 1.1.

Using Lemma 20, we will show that all capacity constraints are satisfied for both the old flow network and the updated flow network in the if part of the proof of Theorem 23.

1.2 The Proof

Before we prove Theorem 23, let us show that every feasible block sequence for the update flow network specified in the previous section satisfies the following properties.

Lemma 26. *Let \mathcal{B} be a feasible block sequence for update flow network G . Then:*

1. For every $i \in [m]$, $\mathcal{B}(u^i, L) < \mathcal{B}(x^1, B)$ or $\mathcal{B}(u^i, R) < \mathcal{B}(x^1, B)$.
2. For every $i \in [m]$,
 - (a) $\mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) < \mathcal{B}(u^i, L)$, and
 - (b) $\mathcal{B}(\tilde{u}_{3,4}^i, \tilde{B}) < \mathcal{B}(u^i, R)$.
3. For every $i \in [m]$,
 - (a) $\mathcal{B}(u_{1,2}^i, \tilde{L}) < \mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B})$ or $\mathcal{B}(u_{1,2}^i, \tilde{R}) < \mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B})$, and
 - (b) $\mathcal{B}(u_{3,4}^i, \tilde{L}) < \mathcal{B}(\tilde{u}_{3,4}^i, \tilde{B})$ or $\mathcal{B}(u_{3,4}^i, \tilde{R}) < \mathcal{B}(\tilde{u}_{3,4}^i, \tilde{B})$.
4. For every $j \in [n]$, $\mathcal{B}(x^1, B) < \mathcal{B}(x^j, \bar{X})$ or $\mathcal{B}(x^1, B) < \mathcal{B}(x^j, X)$.
5. For every $i \in [m]$ and every $j \in [n]$,
 - (a) if $l_{i_1} = \bar{x}_j$, then $\mathcal{B}(x_0^j, B_0) < \mathcal{B}(u_{1,2}^i, \tilde{L})$, and if $l_{i_1} = x_j$, then $\mathcal{B}(x_1^j, B_1) < \mathcal{B}(u_{1,2}^i, \tilde{L})$,
 - (b) if $l_{i_2} = \bar{x}_j$, then $\mathcal{B}(x_0^j, B_0) < \mathcal{B}(u_{1,2}^i, \tilde{R})$, and if $l_{i_2} = x_j$, then $\mathcal{B}(x_1^j, B_1) < \mathcal{B}(u_{1,2}^i, \tilde{R})$,
 - (c) if $l_{i_3} = \bar{x}_j$, then $\mathcal{B}(x_0^j, B_0) < \mathcal{B}(u_{3,4}^i, \tilde{L})$, and if $l_{i_3} = x_j$, then $\mathcal{B}(x_1^j, B_1) < \mathcal{B}(u_{3,4}^i, \tilde{L})$,

Table 1.2: All blocks grouped by flow pair

P	$V(P^o \cap P^u)$ ordered w.r.t. $\leq_{P^o \cup P^u}$	$B^P(G)$
\bar{X}	$s, x^1, y^1, x^2, \dots, y^n, t$	$\{s, x^1\},$ $\{x^j, x_0^j, y_0^j, y^j\}, j \in [n],$ $\{y^j, x^{j+1}\}, j \in [n-1],$ $\{y^n, t\}$
L	$s, u^1, v^1, u^2, \dots, v^m, t$	$\{s, u^1\},$ $\{u^i, \tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i, v^i\}, i \in [m],$ $\{v^i, u^{i+1}\}, i \in [m-1],$ $\{v^m, t\}$
\tilde{L}	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$	$\{s, u_{1,2}^1\},$ $\{u_{1,2}^i, u_1^i, v_1^i, v_{1,2}^i\}, i \in [m],$ $\{v_{1,2}^i, u_{3,4}^i\}, i \in [m],$ $\{u_{3,4}^i, u_3^i, v_3^i, v_{3,4}^i\}, i \in [m],$ $\{v_{3,4}^i, u_{1,2}^{i+1}\}, i \in [m-1],$ $\{v_{3,4}^m, t\}$
X	$s, x^1, y^1, x^2, \dots, y^n, t$	$\{s, x^1\},$ $\{x^j, x_1^j, y_1^j, y^j\}, j \in [n],$ $\{y^j, x^{j+1}\}, j \in [n-1],$ $\{y^n, t\}$
R	$s, u^1, v^1, u^2, \dots, v^m, t$	$\{s, u^1\},$ $\{u^i, \tilde{u}_{3,4}^i, \tilde{v}_{3,4}^i, v^i\}, i \in [m],$ $\{v^i, u^{i+1}\}, i \in [m-1],$ $\{v^m, t\}$
\tilde{R}	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \dots, v_{3,4}^m, t$	$\{s, u_{1,2}^1\},$ $\{u_{1,2}^i, u_2^i, v_2^i, v_{1,2}^i\}, i \in [m],$ $\{v_{1,2}^i, u_{3,4}^i\}, i \in [m],$ $\{u_{3,4}^i, u_4^i, v_4^i, v_{3,4}^i\}, i \in [m],$ $\{v_{3,4}^i, u_{1,2}^{i+1}\}, i \in [m-1],$ $\{v_{3,4}^m, t\}$
B	s, x^1, y^n, t	$\{s, x^1\}, \{x^j, y^j, u^i, v^i \mid j \in [n], i \in [m]\}, \{y^n, t\}$
\tilde{B}	$s, \tilde{u}_{1,2}^1, \tilde{v}_{1,2}^1, \tilde{u}_{3,4}^1, \tilde{v}_{3,4}^1, \tilde{u}_{1,2}^2, \dots, \tilde{v}_{3,4}^m, t$	$\{s, \tilde{u}_{1,2}^1\},$ $\{\tilde{u}_{1,2}^i, u_{1,2}^i, v_{1,2}^i, \tilde{v}_{1,2}^i\}, i \in [m],$ $\{\tilde{v}_{1,2}^i, \tilde{u}_{3,4}^i\}, i \in [m],$ $\{\tilde{u}_{3,4}^i, u_{3,4}^i, v_{3,4}^i, \tilde{v}_{3,4}^i\}, i \in [m],$ $\{\tilde{v}_{3,4}^i, \tilde{u}_{1,2}^{i+1}\}, i \in [m-1],$ $\{\tilde{v}_{3,4}^m, t\}$
B_0	$s, x_0^1, y_0^1, x_0^2, \dots, y_0^n, t$	$\{s, x_0^1\},$ $\{x_0^j, u_{\pi(i,j)}^i, v_{\pi(i,j)}^i, y_0^j \mid i \in \bar{P}_j\}, j \in [n],$ $\{y_0^n, t\}$
B_1	$s, x_1^1, y_1^1, x_1^2, \dots, y_1^n, t$	$\{s, x_1^1\},$ $\{x_1^j, u_{\pi(i,j)}^i, v_{\pi(i,j)}^i, y_1^j \mid i \in P_j\}, j \in [n],$ $\{y_1^n, t\}$

(d) if $l_{i_4} = \bar{x}_j$, then $\mathcal{B}(x_0^j, B_0) < \mathcal{B}(u_{3,4}^i, \tilde{R})$, and if $l_{i_4} = x_j$, then $\mathcal{B}(x_1^j, B_1) < \mathcal{B}(u_{3,4}^i, \tilde{R})$.

6. For every $j \in [n]$,

- (a) $\mathcal{B}(x^j, \bar{X}) < \mathcal{B}(x_0^j, B_0)$, and
- (b) $\mathcal{B}(x^j, X) < \mathcal{B}(x_1^j, B_1)$.

Proof. We show every property by contradiction. More precisely, for every property, we assume it doesn't hold and then obtain an edge and a round such that the corresponding capacity constraint is violated, which contradicts the feasibility of block sequence \mathcal{B} .

Since every flow pair has demand 1, we may use 19 to argue about capacity constraints.

1, 3. We only show 1; the proofs for 3a and 3b are analogous. Suppose not. Then obtain $i \in [m]$ such that both $\mathcal{B}(u^i, L) \geq \mathcal{B}(x^1, B)$ and $\mathcal{B}(u^i, R) \geq \mathcal{B}(x^1, B)$. We show that the capacity constraint for edge (u^i, v^i) is violated for round $\mathcal{B}(x^1, B)$.

We have that

1. $\alpha_L((u^i, v^i), B_{\mathcal{B}(x^1, B)-1}) = \text{active}$, since $b(u^i, L) \notin B_{\mathcal{B}(x^1, B)-1}$ and $(u^i, v^i) \in E(L^o)$,
2. $\alpha_R((u^i, v^i), B_{\mathcal{B}(x^1, B)-1}) = \text{active}$, since $b(u^i, R) \notin B_{\mathcal{B}(x^1, B)-1}$ and $(u^i, v^i) \in E(R^o)$, and
3. $\alpha_B((u^i, v^i), B_{\mathcal{B}(x^1, B)}) = \text{active}$, since $b(u^i, B) = b(x^1, B) \in B_{\mathcal{B}(x^1, B)}$ and $(u^i, v^i) \in E(B^u)$.

Hence

$$|\{P \in \mathcal{P} \mid \alpha_P((u^i, v^i), B_{\mathcal{B}(x^1, B)-1}) = \text{active or} \\ \alpha_P((u^i, v^i), B_{\mathcal{B}(x^1, B)}) = \text{active}\}| \geq |\{L, R, B\}| = 3 > 2 = c(u^i, v^i)$$

2, 5, 6. We only show 2a; the proofs for 2b, 5a, 5b, 5c, 5d, 6a, and 6b are similar. Suppose not. Then obtain $i \in [m]$ such that $\mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) \geq \mathcal{B}(u^i, L)$. We show that the capacity constraint for edge $(\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i)$ is violated for round $\mathcal{B}(u^i, L)$.

We have that

1. $\alpha_{\tilde{B}}((\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i), B_{\mathcal{B}(u^i, L)-1}) = \text{active}$, since $b(\tilde{u}_{1,2}^i, \tilde{B}) \notin B_{\mathcal{B}(u^i, L)-1}$ and $(\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i) \in E(\tilde{B}^o)$, and
2. $\alpha_L((\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i), B_{\mathcal{B}(u^i, L)}) = \text{active}$, since $b(\tilde{u}_{1,2}^i, L) = b(u^i, L) \in B_{\mathcal{B}(u^i, L)}$ and $(\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i) \in E(L^u)$.

Hence

$$|\{P \in \mathcal{P} \mid \alpha_P((\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i), B_{\mathcal{B}(u^i, L)-1}) = \text{active or} \\ \alpha_P((\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i), B_{\mathcal{B}(u^i, L)}) = \text{active}\}| \geq |\{\tilde{B}, L\}| = 2 > 1 = c(\tilde{u}_{1,2}^i, \tilde{v}_{1,2}^i)$$

4. Suppose not. Then obtain $j \in [n]$ such that both $\mathcal{B}(x^1, B) \geq \mathcal{B}(x^j, \bar{X})$ and $\mathcal{B}(x^1, B) \geq \mathcal{B}(x^j, X)$. We show that the capacity constraint for edge (x^j, y^j) is violated for round $\mathcal{B}(x^1, B)$.

We have that

1. $\alpha_B((x^j, y^j), B_{\mathcal{B}(x^1, B)-1}) = \text{active}$, since $b(x^j, B) = b(x^1, B) \notin B_{\mathcal{B}(x^1, B)-1}$ and $(x^j, y^j) \in E(B^o)$,
2. $\alpha_{\bar{X}}((x^j, y^j), B_{\mathcal{B}(x^1, B)}) = \text{active}$, since $b(x^j, \bar{X}) \notin B_{\mathcal{B}(x^1, B)}$ and $(x^j, y^j) \in E(\bar{X}^u)$, and
3. $\alpha_X((x^j, y^j), B_{\mathcal{B}(x^1, B)}) = \text{active}$, since $b(x^j, X) \notin B_{\mathcal{B}(x^1, B)}$ and $(x^j, y^j) \in E(X^u)$.

Hence

$$|\{P \in \mathcal{P} \mid \alpha_P((x^j, y^j), B_{\mathcal{B}(x^1, B)-1}) = \text{active or} \\ \alpha_P((x^j, y^j), B_{\mathcal{B}(x^1, B)}) = \text{active}\}| \geq |\{B, \bar{X}, X\}| = 3 > 2 = c(x^j, y^j)$$

□

We are now ready to prove Theorem 23.

Proof of Theorem [thm:np-hardness-special-case]. We show that there is a satisfying assignment σ for 4CNF formula C iff there is a feasible block sequence \mathcal{B} for the corresponding update flow network G , which, by Corollary 21, is the case iff there is a feasible update sequence for G . We will choose σ , \mathcal{B} , respectively, such that σ assigns 1 to variable x_j iff $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$.

Only-if part. Let \mathcal{B} be a feasible block sequence for G . We define assignment σ as follows: For every variable x_j , we assign 1 to x_j iff $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$. We now show that σ is a satisfying assignment for C .

Let $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$ be a clause. We show that σ satisfies C_i by obtaining a literal that evaluates to 1.

Consider round $\mathcal{B}(x^1, B)$. By Lemma 26 1, $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L)$ or $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, R)$. We only consider the former case $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L)$; the latter one is analogous.

By Lemma 26 2a, $\mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B})$. By Lemma 26 3a, $\mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) > \mathcal{B}(u_{1,2}^i, \tilde{L})$ or $\mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) > \mathcal{B}(u_{1,2}^i, \tilde{R})$. We only consider the latter case $\mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) > \mathcal{B}(u_{1,2}^i, \tilde{R})$; the former one is analogous.

Let x_j be the variable corresponding to literal l_{i_2} . We consider the cases $l_{i_2} = \bar{x}_j$ and $l_{i_2} = x_j$ separately.

Case $l_{i_2} = \bar{x}_j$. By Lemma 26 5b, $\mathcal{B}(u_{1,2}^i, \tilde{R}) > \mathcal{B}(x_0^j, B_0)$. By Lemma 26 6a, $\mathcal{B}(x_0^j, B_0) > \mathcal{B}(x^j, \bar{X})$. Putting everything together yields the following chain of inequalities:

$$\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) > \mathcal{B}(u_{1,2}^i, \tilde{R}) > \mathcal{B}(x_0^j, B_0) > \mathcal{B}(x^j, \bar{X})$$

Hence, by definition of our assignment, variable x_j is assigned 0. Hence literal $l_{i_2} = \bar{x}_j$ evaluates to 1.

Case $l_{i_2} = x_j$. By Lemma 26 5b, $\mathcal{B}(u_{1,2}^i, \tilde{R}) > \mathcal{B}(x_1^j, B_1)$. By Lemma 26 6b, $\mathcal{B}(x_1^j, B_1) > \mathcal{B}(x^j, X)$. Putting everything together yields the following chain of inequalities:

$$\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}_{1,2}^i, \tilde{B}) > \mathcal{B}(u_{1,2}^i, \tilde{R}) > \mathcal{B}(x_1^j, B_1) > \mathcal{B}(x^j, X)$$

Hence, by Lemma 26 4, $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$. Hence, by definition of our assignment, variable x_j is assigned 1. Hence literal $l_{i_2} = x_j$ evaluates to 1.

If part. Let σ be a satisfying assignment for C . We construct a feasible block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_{11})$ for G as follows. The basic idea is to update blocks induced by

- variable vertices corresponding to variables that are assigned 1 and
- clause vertices corresponding to satisfied clauses

before we update block $b(x^1, B)$, and all other blocks afterwards. We now specify $\mathcal{B}_1, \dots, \mathcal{B}_{11}$ in detail.

1. For every variable x_j , if x_j is assigned 1, we add block $b(x^j, X)$ to \mathcal{B}_1 , otherwise we add $b(x^j, \bar{X})$. That is,

$$\mathcal{B}_1 = \{b(x^j, X) \mid \sigma(x_j) = 1\} \cup \{b(x^j, \bar{X}) \mid \sigma(x_j) = 0\}.$$

2. For every variable x_j , if x_j is assigned 1, we add block $b(x_1^j, B_1)$ to \mathcal{B}_2 , otherwise we add $b(x_0^j, B_0)$. That is,

$$\mathcal{B}_2 = \{b(x_1^j, B_1) \mid \sigma(x_j) = 1\} \cup \{b(x_0^j, B_0) \mid \sigma(x_j) = 0\}.$$

3. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$,

- (a) if l_{i_1} evaluates to 1, we add block $b(u_{1,2}^i, \tilde{L})$ to \mathcal{B}_3 ,
- (b) if l_{i_2} evaluates to 1, we add $b(u_{1,2}^i, \tilde{R})$,
- (c) if l_{i_3} evaluates to 1, we add $b(u_{3,4}^i, \tilde{L})$, and
- (d) if l_{i_4} evaluates to 1, we add $b(u_{3,4}^i, \tilde{R})$.

That is,

$$\mathcal{B}_3 = \{b(u_{1,2}^i, \tilde{L}) \mid \sigma(l_{i_1}) = 1\} \cup \{b(u_{1,2}^i, \tilde{R}) \mid \sigma(l_{i_2}) = 1\} \cup \\ \{b(u_{3,4}^i, \tilde{L}) \mid \sigma(l_{i_3}) = 1\} \cup \{b(u_{3,4}^i, \tilde{R}) \mid \sigma(l_{i_4}) = 1\}.$$

4. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is satisfied, we add block $b(\tilde{u}_{1,2}^i, \tilde{B})$ to \mathcal{B}_4 , and if the right half $(l_{i_3} \vee l_{i_4})$ is satisfied, we add $b(\tilde{u}_{3,4}^i, \tilde{B})$. That is,

$$\mathcal{B}_4 = \{b(\tilde{u}_{1,2}^i, \tilde{B}) \mid \sigma(l_{i_1}) = 1 \text{ or } \sigma(l_{i_2}) = 1\} \cup \\ \{b(\tilde{u}_{3,4}^i, \tilde{B}) \mid \sigma(l_{i_3}) = 1 \text{ or } \sigma(l_{i_4}) = 1\}.$$

5. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is satisfied, we add block $b(u^i, L)$ to \mathcal{B}_5 , and if the right half $(l_{i_3} \vee l_{i_4})$ is satisfied, we add $b(u^i, R)$. That is,

$$\mathcal{B}_5 = \{b(u^i, L) \mid \sigma(l_{i_1}) = 1 \text{ or } \sigma(l_{i_2}) = 1\} \cup \\ \{b(u^i, R) \mid \sigma(l_{i_3}) = 1 \text{ or } \sigma(l_{i_4}) = 1\}.$$

6. $\mathcal{B}_6 = \{b(x^1, B)\}$.

7. For every variable x_j , if x_j is assigned 0, we add block $b(x^j, X)$ to \mathcal{B}_7 , otherwise we add $b(x^j, \bar{X})$. That is,

$$\mathcal{B}_7 = \{b(x^j, X) \mid \sigma(x_j) = 0\} \cup \{b(x^j, \bar{X}) \mid \sigma(x_j) = 1\}.$$

8. For every variable x_j , if x_j is assigned 0, we add block $b(x_1^j, B_1)$ to \mathcal{B}_8 , otherwise we add $b(x_0^j, B_0)$. That is,

$$\mathcal{B}_8 = \{b(x_1^j, B_1) \mid \sigma(x_j) = 0\} \cup \{b(x_0^j, B_0) \mid \sigma(x_j) = 1\}.$$

9. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$,

- (a) if l_{i_1} evaluates to 0, we add block $b(u_{1,2}^i, \tilde{L})$ to \mathcal{B}_9 ,
- (b) if l_{i_2} evaluates to 0, we add $b(u_{1,2}^i, \tilde{R})$,
- (c) if l_{i_3} evaluates to 0, we add $b(u_{3,4}^i, \tilde{L})$, and
- (d) if l_{i_4} evaluates to 0, we add $b(u_{3,4}^i, \tilde{R})$.

That is,

$$\mathcal{B}_9 = \{b(u_{1,2}^i, \tilde{L}) \mid \sigma(l_{i_1}) = 0\} \cup \{b(u_{1,2}^i, \tilde{R}) \mid \sigma(l_{i_2}) = 0\} \cup \\ \{b(u_{3,4}^i, \tilde{L}) \mid \sigma(l_{i_3}) = 0\} \cup \{b(u_{3,4}^i, \tilde{R}) \mid \sigma(l_{i_4}) = 0\}.$$

10. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is unsatisfied, we add block $b(\tilde{u}_{1,2}^i, \tilde{B})$ to \mathcal{B}_{10} , and if the right half $(l_{i_3} \vee l_{i_4})$ is unsatisfied, we add $b(\tilde{u}_{3,4}^i, \tilde{B})$. That is,

$$\begin{aligned} \mathcal{B}_{10} = & \{b(\tilde{u}_{1,2}^i, \tilde{B}) \mid \sigma(l_{i_1}) = 0 \text{ and } \sigma(l_{i_2}) = 0\} \cup \\ & \{b(\tilde{u}_{3,4}^i, \tilde{B}) \mid \sigma(l_{i_3}) = 0 \text{ and } \sigma(l_{i_4}) = 0\}. \end{aligned}$$

11. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is unsatisfied, we add block $b(u^i, L)$ to \mathcal{B}_{11} , and if the right half $(l_{i_3} \vee l_{i_4})$ is unsatisfied, we add $b(u^i, R)$. That is,

$$\begin{aligned} \mathcal{B}_{11} = & \{b(u^i, L) \mid \sigma(l_{i_1}) = 0 \text{ and } \sigma(l_{i_2}) = 0\} \cup \\ & \{b(u^i, R) \mid \sigma(l_{i_3}) = 0 \text{ and } \sigma(l_{i_4}) = 0\}. \end{aligned}$$

By Remark 10, we may ignore all other blocks.

We now show that block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_{11})$ is feasible by verifying that the capacity constraint is satisfied for every edge and every $\ell \in [11]$. Since every flow pair has demand 1, we

- may use remark 19 again to argue about capacity constraints, and
- only have to consider edges with capacity less than 10, that is, the number of flow pairs.

For every such edge e , we proceed as follows.

1. First, for every $\ell \in \{0, \dots, 11\}$ and every flow pair P , we determine if e is on the transient (s, t) -path for P after updating all blocks in B_ℓ , that is, we determine if $\alpha_P(e, B_\ell) = \text{active}$.
2. Next, for every $\ell \in \{0, \dots, 11\}$, we determine the set of flow pairs P such that $\alpha_P(e, B_\ell) = \text{active}$, that is, we determine the set $\mathcal{P}(e, \ell) := \{P \in \mathcal{P} \mid \alpha_P(e, B_\ell) = \text{active}\}$.
3. Then, for every $\ell \in [11]$, we determine the set $\mathcal{P}'(e, \ell) := \mathcal{P}(e, \ell - 1) \cup \mathcal{P}(e, \ell) = \{P \in \mathcal{P} \mid \alpha_P(e, B_{\ell-1}) = \text{active} \text{ or } \alpha_P(e, B_\ell) = \text{active}\}$.
4. Finally, for every $\ell \in [11]$, we verify that the cardinality of the set $\mathcal{P}'(e, \ell)$ obtained in the previous step is at most $c(e)$.

(x^j, y^j) Let $j \in [n]$. Then edge (x^j, y^j) is used by flow pairs \bar{X}, X, B .

Since $(x^j, y^j) \in E(\bar{X}^u \setminus \bar{X}^o)$, by Lemma 15,

$$\alpha_{\bar{X}}((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \text{if } \sigma(x_j) = 1 \text{ and } \ell \geq 7 \\ \text{active} & \text{if } \sigma(x_j) = 0 \text{ and } \ell \geq 1 \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Since $(x^j, y^j) \in E(X^u \setminus X^o)$, by Lemma 15,

$$\alpha_X((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \text{if } \sigma(x_j) = 1 \text{ and } \ell \geq 1 \\ \text{active} & \text{if } \sigma(x_j) = 0 \text{ and } \ell \geq 7 \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Since $(x^j, y^j) \in E(B^o \setminus B^u)$ and $b(x^j, B) = b(x^1, B) \in \mathcal{B}_6$, by Lemma 15,

$$\alpha_B((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \ell < 6 \\ \text{inactive} & \ell \geq 6. \end{cases}$$

Hence,

$$\mathcal{P}((x^j, y^j), \ell) = \begin{cases} \{B\} & \ell < 1 \\ \{X, B\} & \sigma(x_j) = 1 \text{ and } 1 \leq \ell < 6 \\ \{X\} & \sigma(x_j) = 1 \text{ and } \ell = 6 \\ \{\bar{X}, B\} & \sigma(x_j) = 0 \text{ and } 1 \leq \ell < 6 \\ \{\bar{X}\} & \sigma(x_j) = 0 \text{ and } \ell = 6 \\ \{\bar{X}, X\} & \ell \geq 7. \end{cases}$$

Hence,

$$\mathcal{P}'((x^j, y^j), \ell) = \begin{cases} \{X, B\} & \sigma(x_j) = 1 \text{ and } \ell < 7 \\ \{\bar{X}, B\} & \sigma(x_j) = 0 \text{ and } \ell < 7 \\ \{\bar{X}, X\} & \ell \geq 7. \end{cases}$$

Hence $|\mathcal{P}'((x^j, y^j), \ell)| = 2 = c(x^j, y^j)$ for every $\ell \in [11]$.

□ Repeat for other edges.

□

Chapter 2

Merging Flow Pairs

We now prove the Merging Lemma.

Let $G = (V, E, \mathcal{P}, s, t, c)$ be an update flow network with $|\mathcal{P}| \geq 2$, and let $F, F' \in \mathcal{P}$ and $v_F, v_{F'} \in V$ such that they satisfy properties 1, 2, and 3 (see Figure 2.1). We construct an update flow network $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\mathcal{P}}, s, t, \tilde{c})$ with $|\tilde{\mathcal{P}}| = |\mathcal{P}| - 1$ such that there is a feasible block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ for G iff there is a feasible block sequence $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ for \tilde{G} as follows.

2.1 The Construction

Intuitively, we merge flow pairs F and F' into a single flow pair \tilde{F} by concatenating F and F' . More precisely, \tilde{F} will be the union of F and F' except that we replace edges (v_F, t) and $(s, v_{F'})$ by edge $(v_F, v_{F'})$ (see Figure 2.2 for an illustration). More formally, we define flow pair \tilde{F} as follows:

$$\begin{aligned}\tilde{E}(\tilde{F}^o) &= (E(F^o) \setminus \{(v_F, t)\}) \cup (E(F'^o) \setminus \{(s, v_{F'})\}) \cup \{(v_F, v_{F'})\} \\ \tilde{E}(\tilde{F}^u) &= (E(F^u) \setminus \{(v_F, t)\}) \cup (E(F'^u) \setminus \{(s, v_{F'})\}) \cup \{(v_F, v_{F'})\} \\ \tilde{V}(\tilde{F}^o) &= \tilde{V}(\tilde{E}(\tilde{F}^o)) \\ \tilde{V}(\tilde{F}^u) &= \tilde{V}(\tilde{E}(\tilde{F}^u)) \\ \tilde{d}_{\tilde{F}} &= d_F\end{aligned}$$

Update flow network $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\mathcal{P}}, s, t, \tilde{c})$ is defined as follows:

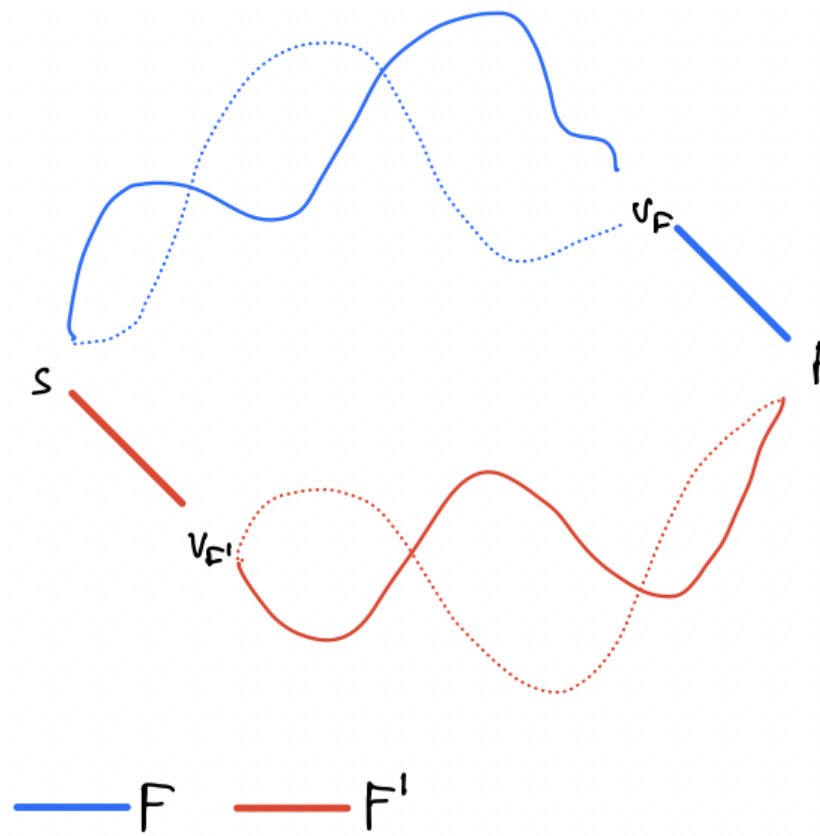


Figure 2.1: Flow pairs F and F' in update flow network G

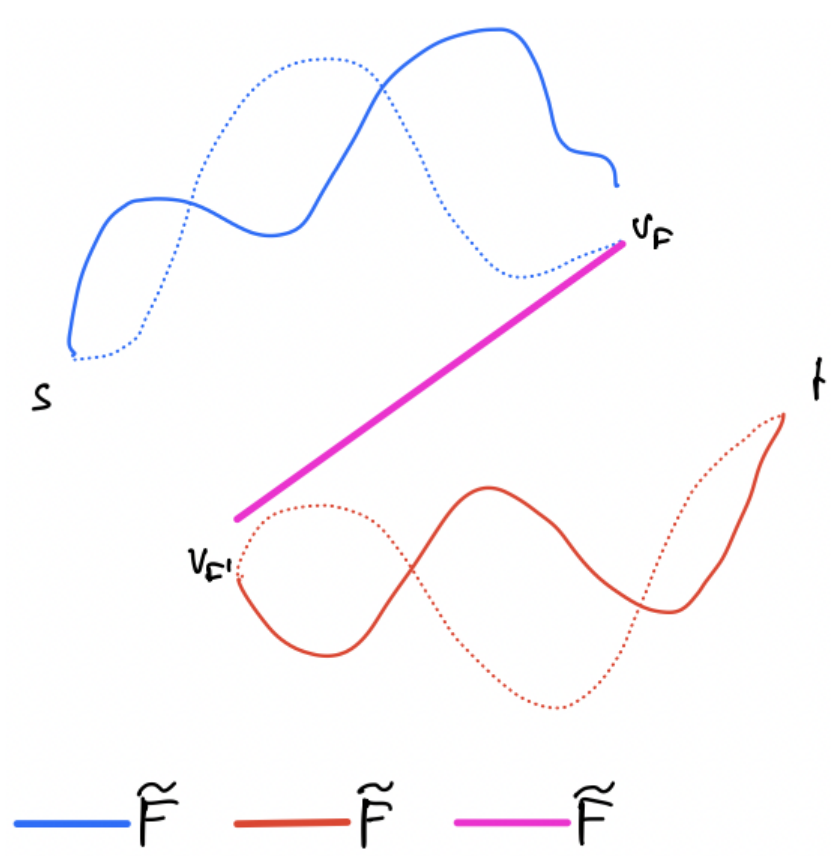


Figure 2.2: Flow pair \tilde{F} in update flow network \tilde{G}

$$\begin{aligned}
\tilde{\mathcal{P}} &= \mathcal{P} \setminus \{F, F'\} \cup \{\tilde{F}\} \\
\tilde{V} &= \bigcup_{\tilde{P} \in \tilde{\mathcal{P}}} \tilde{V}(\tilde{P}^o \cup \tilde{P}^u) \\
\tilde{E} &= \bigcup_{\tilde{P} \in \tilde{\mathcal{P}}} \tilde{E}(\tilde{P}^o \cup \tilde{P}^u) \\
\tilde{c}(\tilde{e}) &= \begin{cases} \sum_{\tilde{P} \in \tilde{\mathcal{P}}: \tilde{e} \in \tilde{E}(\tilde{P}^o \cup \tilde{P}^u)} \tilde{d}_{\tilde{P}} & \text{if } \tilde{e} = (v_F, v_{F'}) \\ c(\tilde{e}) & \text{otherwise} \end{cases}
\end{aligned}$$

Let us quickly verify that \tilde{G} is a feasible update flow network.

Let $\tilde{P} \in \tilde{\mathcal{P}}$. If $\tilde{P} \neq \tilde{F}$, then $\tilde{P} \in \mathcal{P}$ and hence, by feasibility of update flow network G , both \tilde{P}^o and \tilde{P}^u are (s, t) -paths in \tilde{G} and \tilde{P} forms a DAG. Now suppose $\tilde{P} = \tilde{F}$. By feasibility of G and construction of \tilde{F} , \tilde{F}^o (\tilde{F}^u) comprises the (s, v_F) -path in F^o (F^u), edge $(v_F, v_{F'})$, and the $(v_{F'}, t)$ -path in F'^o (F'^u), and hence forms an (s, t) -path. Moreover, since, again by feasibility of G , both F and F' form DAGs, and edge $(v_F, v_{F'})$ does not introduce a cycle, as F and F' have no common vertices other than s, t by assumption, \tilde{F} forms a DAG.

Using Lemma 20, we will show that all capacity constraints are satisfied for both the old flow network and the updated flow network in the if part of the proof of the Merging Lemma.

We denote notations such as $b(v, P)$, B_i , and $\alpha_P(e, B)$ referring to update flow network \tilde{G} by $\tilde{b}(v, P)$, \tilde{B}_i , and $\tilde{\alpha}_P(e, B)$.

2.2 The Proof

Our goal is to show that there is a feasible block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ for G iff there is a feasible block sequence $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ for \tilde{G} . We will choose $\mathcal{B}, \tilde{\mathcal{B}}$, respectively, such that, for every block b contained in both G and \tilde{G} , b is updated in round $i \in [\ell]$ in \mathcal{B} iff it is updated in round i in $\tilde{\mathcal{B}}$, that is, $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. The key insight is that it is indeed sufficient to consider such blocks.

Lemma 27.

1. Let $\tilde{u} \in \tilde{V}(\tilde{F}^o \cup \tilde{F}^u) \setminus \{v_F\}$. Then:

- (a) If $\tilde{u} \in V(F^o \cup F^u)$, then $\tilde{b}(\tilde{u}, \tilde{F}) = b(\tilde{u}, F)$.
- (b) If $\tilde{u} \in V(F'^o \cup F'^u) \setminus \{s\}$, then $\tilde{b}(\tilde{u}, \tilde{F}) = b(\tilde{u}, F')$.

2. For every $\tilde{P} \in \tilde{\mathcal{P}} \setminus \{\tilde{F}\}$ and every $\tilde{u} \in \tilde{V}(\tilde{P}^o \cup \tilde{P}^u)$, $\tilde{b}(\tilde{u}, \tilde{P}) = b(\tilde{u}, \tilde{P})$.

Remark 28. The proof is very technical and tedious—and hence omitted for now—and I hope we can come up with a better characterization of blocks (see [../README.org](#)) which significantly simplifies the proof.

Corollary 29.

1. For every block $\tilde{b} \in \tilde{B}(\tilde{G}) \setminus \{\{v_F, v_{F'}\}\}$, $\tilde{b} \in B(G)$.
2. For every block $b \in B(G) \setminus \{\{v_F, t\}, \{s, v_{F'}\}\}$, $b \in \tilde{B}(\tilde{G})$.

Proof.

1. Let $\tilde{b} \in \tilde{B}(\tilde{G}) \setminus \{\{v_F, v_{F'}\}\}$, $\tilde{P} = \tilde{P}(\tilde{b})$, and $\tilde{u} = \tilde{\mathcal{S}}(\tilde{b})$. If $\tilde{P} = \tilde{F}$, then, by assumption, $\tilde{u} \neq v_F$ and hence, by construction of \tilde{F} and Lemma 27 1, $\tilde{b} = b(\tilde{u}, \tilde{F}) \in B(G)$ or $\tilde{b} = b(\tilde{u}, \tilde{F}') \in B(G)$. If $\tilde{P} \neq \tilde{F}$, then, by Lemma 27 2, $\tilde{b} = b(\tilde{u}, \tilde{P}) \in B(G)$.
2. Let $b \in B(G) \setminus \{\{v_F, t\}, \{s, v_{F'}\}\}$, $P = P(b)$, and $u = \mathcal{S}(b)$. If $P = F$, then, by assumption, $u \neq v_F$ and hence, by construction of \tilde{F} and Lemma 27 1a, $b = \tilde{b}(u, \tilde{F}) \in \tilde{B}(\tilde{G})$. If $P = F'$, then, by assumption, $u \notin \{v_F, s\}$ and hence, by construction of \tilde{F} and Lemma 27 1b, $b = \tilde{b}(u, \tilde{F}) \in \tilde{B}(\tilde{G})$. If $P \in \mathcal{P} \setminus \{F, F'\}$, then $P \in \tilde{\mathcal{P}} \setminus \{\tilde{F}\}$ and hence, by Lemma 27 2, $b = \tilde{b}(u, P) \in \tilde{B}(\tilde{G})$. \square

To show that block sequences $\mathcal{B}, \tilde{\mathcal{B}}$ as chosen above are feasible, we will verify that capacity constraint 1 is satisfied for every edge $e \in E$, $\tilde{e} \in \tilde{E}$, respectively, and every $i \in [\ell]$. We now show that for every edge \tilde{e} other than $(v_F, t), (s, v_{F'}), (v_F, v_{F'})$ and every $i \in [\ell]$, \tilde{e} is on some transient (s, t) -path in \tilde{G} after updating all blocks in \tilde{B}_i iff it is on some transient (s, t) -path in G after updating all blocks in B_i .

Lemma 30. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence for G and $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ be a block sequence for \tilde{G} such that for every block b contained in both G and \tilde{G} , $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. Moreover, let $(\tilde{u}, \tilde{v}) \in \tilde{E} \setminus \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$ and $i \in [\ell]$. Finally, let $\tilde{P} \in \tilde{\mathcal{P}}$ such that $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{P}^o \cup \tilde{P}^u)$. Then:

1. If $\tilde{P} = \tilde{F}$, then $\tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}$ iff either $\alpha_F((\tilde{u}, \tilde{v}), B_i) = \text{active}$ or $\alpha_{F'}((\tilde{u}, \tilde{v}), B_i) = \text{active}$.
2. If $\tilde{P} \neq \tilde{F}$, then $\tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i)$.

Proof. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence for G and $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ be a block sequence for \tilde{G} such that for every block b satisfying both $b \in B(G)$ and $b \in \tilde{B}(\tilde{G})$, $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. Let $(\tilde{u}, \tilde{v}) \in \tilde{E} \setminus \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$ and $i \in [\ell]$. Let $\tilde{P} \in \tilde{\mathcal{P}}$ such that $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{P}^o \cup \tilde{P}^u)$.

1. Suppose $\tilde{P} = \tilde{F}$. By definition of \tilde{F} and since $(\tilde{u}, \tilde{v}) \in \tilde{E} \setminus \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$, $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^o)$ iff $(\tilde{u}, \tilde{v}) \in E(F^o)$ or $(\tilde{u}, \tilde{v}) \in E(F'^o)$. Similarly, $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^u)$ iff $(\tilde{u}, \tilde{v}) \in E(F^u)$ or $(\tilde{u}, \tilde{v}) \in E(F'^u)$. We show $\tilde{\alpha}_{\tilde{F}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}$ iff $\alpha_F((\tilde{u}, \tilde{v}), B_i) = \text{active}$ or $\alpha_{F'}((\tilde{u}, \tilde{v}), B_i) = \text{active}$. Notice that this implies 1, since, by assumption, F, F' are edge-disjoint: Otherwise, either

1. F and F' have a common vertex other than s, t , or
2. $F^o \cup F^u$ and $F'^o \cup F'^u$ both consist of the single edge (s, t) , in which case $v_F = s$ and $v_{F'} = t$, which contradicts that $(v_F, v_{F'}) \notin E$.

We first show the if part. Suppose $\tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}$. By assumption, $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^o)$ or $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^u)$. Hence $(\tilde{u}, \tilde{v}) \in E(F^o)$ or $(\tilde{u}, \tilde{v}) \in E(F^u)$ or $(\tilde{u}, \tilde{v}) \in E(F'^o)$ or $(\tilde{u}, \tilde{v}) \in E(F'^u)$. We only consider case $(\tilde{u}, \tilde{v}) \in E(F^o)$; case $(\tilde{u}, \tilde{v}) \in E(F^u)$ is similar, and cases $(\tilde{u}, \tilde{v}) \in E(F'^o)$, $(\tilde{u}, \tilde{v}) \in E(F'^u)$ are analogous to cases $(\tilde{u}, \tilde{v}) \in E(F^o)$, $(\tilde{u}, \tilde{v}) \in E(F^u)$, respectively.

Suppose $(\tilde{u}, \tilde{v}) \in E(F^o)$. Hence $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^o)$. Hence $\tilde{b}(\tilde{u}, \tilde{F}) \in \tilde{B}_i$. Moreover, by Lemma 27 1a, $\tilde{b}(\tilde{u}, \tilde{F}) = b(\tilde{u}, F)$. Hence, by assumption, $b(\tilde{u}, F) \in B_i$. Thus $\alpha_F((\tilde{u}, \tilde{v}), B_i) = \text{active}$.

We now show the only-if part. Suppose $\alpha_F((\tilde{u}, \tilde{v}), B_i) = \text{active}$ or $\alpha_{F'}((\tilde{u}, \tilde{v}), B_i) = \text{active}$. We only consider the former case; the latter one is analogous.

Suppose $\alpha_F((\tilde{u}, \tilde{v}), B_i) = \text{active}$. Hence $(\tilde{u}, \tilde{v}) \in E(F^o)$ or $(\tilde{u}, \tilde{v}) \in E(F^u)$. We again only consider the former case; the latter one is similar.

Suppose $(\tilde{u}, \tilde{v}) \in E(F^o)$. Hence $b(\tilde{u}, F) \in B_i$. Moreover, by Lemma 27 1a, $b(\tilde{u}, F) = \tilde{b}(\tilde{u}, \tilde{F})$. Hence, by assumption, $\tilde{b}(\tilde{u}, \tilde{F}) \in \tilde{B}_i$. Moreover, $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{F}^o)$. Thus $\tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}$.

2. Suppose $\tilde{P} \neq \tilde{F}$. By definition of \tilde{G} , $\tilde{P} \in \mathcal{P} \setminus \{F, F'\}$ and hence both $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{P}^o)$ iff $(\tilde{u}, \tilde{v}) \in E(\tilde{P}^o)$ and $(\tilde{u}, \tilde{v}) \in \tilde{E}(\tilde{P}^u)$ iff $(\tilde{u}, \tilde{v}) \in E(\tilde{P}^u)$. Hence $\tilde{b}(\tilde{u}, \tilde{P}) \in \tilde{B}(\tilde{G})$ and $\tilde{b}(\tilde{u}, \tilde{P}) \in B(G)$. Hence, by assumption, $\tilde{b}(\tilde{u}, \tilde{P}) \in \tilde{B}_i$ iff $\tilde{b}(\tilde{u}, \tilde{P}) \in B_i$, and, by Lemma 27 2, $\tilde{b}(\tilde{u}, \tilde{P}) = b(\tilde{u}, \tilde{P})$. Hence $\tilde{b}(\tilde{u}, \tilde{P}) \in \tilde{B}_i$ iff $b(\tilde{u}, \tilde{P}) \in B_i$. The claim now follows by definitions of $\tilde{\alpha}_{\tilde{P}}, \alpha_{\tilde{P}}$. \square

Lemma 31. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence for G and $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ be a block sequence for \tilde{G} such that for every block b contained in both G and \tilde{G} , $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. Moreover, let $(\tilde{u}, \tilde{v}) \in \tilde{E} \setminus \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$ and $i \in [\ell]$. Then

$$\sum_{\tilde{P} \in \tilde{\mathcal{P}}: \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_{i-1}) = \text{active} \text{ or } \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}} \tilde{d}_{\tilde{P}} = \sum_{\tilde{P} \in \mathcal{P}: \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_{i-1}) = \text{active} \text{ or } \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i) = \text{active}} d_{\tilde{P}}.$$

Proof. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a block sequence for G and $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ be a block sequence for \tilde{G} such that for every block b satisfying both $b \in B(G)$ and $b \in \tilde{B}(\tilde{G})$, $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. Let $(\tilde{u}, \tilde{v}) \in \tilde{E} \setminus \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$ and $i \in [\ell]$. By definition of \tilde{G} , $\tilde{\mathcal{P}} = \mathcal{P} \setminus \{F, F'\} \cup \{\tilde{F}\}$, $d_{\tilde{F}} = d_F$, and $\tilde{d}_{\tilde{P}} = d_{\tilde{P}}$ for every $\tilde{P} \in \tilde{\mathcal{P}} \setminus \{\tilde{F}\}$. Moreover, by assumption, $d_F = d_{F'}$. Hence, by Lemma 30, we have

$$\begin{aligned}
& \sum_{\tilde{P} \in \tilde{\mathcal{P}}: \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}} \tilde{d}_{\tilde{P}} = \\
& \sum_{\tilde{P} \in \tilde{\mathcal{P}} \setminus \{\tilde{F}\}: \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}} \tilde{d}_{\tilde{P}} \\
& + \sum_{\tilde{P} \in \{\tilde{F}\}: \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}} \tilde{d}_{\tilde{P}} = \\
& \sum_{\tilde{P} \in \mathcal{P} \setminus \{F, F'\}: \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_{i-1}) = \text{active or } \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i) = \text{active}} d_{\tilde{P}} \\
& + \sum_{\tilde{P} \in \{F, F'\}: \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_{i-1}) = \text{active or } \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i) = \text{active}} d_{\tilde{P}} = \\
& \sum_{\tilde{P} \in \mathcal{P}: \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_{i-1}) = \text{active or } \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i) = \text{active}} d_{\tilde{P}}.
\end{aligned}$$

□

We are now ready to prove the Merging Lemma.

Proof of Lemma [lem.merging-flow-pairs.] We show that there is a feasible block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ for G iff there is a feasible block sequence $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ for \tilde{G} .

If part. Let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ be a feasible block sequence for G . We define block sequence $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ for \tilde{G} as follows. By Remark 10, we may ignore block $\{v_F, v_{F'}\}$. For every other block $\tilde{b} \in \tilde{B}(\tilde{G}) \setminus \{\{v_F, v_{F'}\}\}$, we define $\tilde{\mathcal{B}}(\tilde{b}) = \mathcal{B}(\tilde{b})$. Notice that if $\tilde{b} \in \tilde{B}(\tilde{G}) \setminus \{\{v_F, v_{F'}\}\}$, then, by Lemma 29 1, $\tilde{b} \in B(G)$ and hence $\mathcal{B}(\tilde{b})$ is defined.

We now show that $\tilde{\mathcal{B}}$ is feasible. Let $(\tilde{u}, \tilde{v}) \in \tilde{E}$ and $i \in [\ell]$. We show that the capacity constraint for (\tilde{u}, \tilde{v}) and i is satisfied.

If $(\tilde{u}, \tilde{v}) = (v_F, v_{F'})$, then, by definition of \tilde{G} ,

$$\tilde{c}(\tilde{u}, \tilde{v}) \geq \sum_{\tilde{P} \in \tilde{\mathcal{P}}: (\tilde{u}, \tilde{v}) \in E(\tilde{P}^o \cup \tilde{P}^u)} d_{\tilde{P}}$$

and hence the capacity constraint is satisfied.

Now suppose $(\tilde{u}, \tilde{v}) \neq (v_F, v_{F'})$. Hence, by definition of \tilde{G} , $(\tilde{u}, \tilde{v}) \in E$ and $\tilde{c}(\tilde{u}, \tilde{v}) = c(\tilde{u}, \tilde{v})$. If $(\tilde{u}, \tilde{v}) \in \{(v_F, t), (s, v_{F'})\}$, then by assumption 3c, the capacity constraint is satisfied.

Now suppose $(\tilde{u}, \tilde{v}) \notin \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$. Hence, by Lemma 31 and since block sequence \mathcal{B} is feasible, we have

$$\begin{aligned}
& \sum_{\tilde{P} \in \tilde{\mathcal{P}}: \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_{\tilde{P}}((\tilde{u}, \tilde{v}), \tilde{B}_i) = \text{active}} \tilde{d}_{\tilde{P}} = \\
& \sum_{\tilde{P} \in \mathcal{P}: \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_{i-1}) = \text{active or } \alpha_{\tilde{P}}((\tilde{u}, \tilde{v}), B_i) = \text{active}} d_{\tilde{P}} \leq \\
& c(\tilde{u}, \tilde{v}) = \tilde{c}(\tilde{u}, \tilde{v}).
\end{aligned}$$

Only-if part. Let $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_\ell)$ be a feasible block sequence for \tilde{G} . We define block sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_\ell)$ for G as follows. By Remark 10, we may ignore blocks $\{v_F, t\}, \{s, v_{F'}\}$. For every other block $b \in B(G) \setminus \{\{v_F, t\}, \{s, v_{F'}\}\}$, we define $\mathcal{B}(b) = \tilde{\mathcal{B}}(b)$. Notice that if $b \in B(G) \setminus \{\{v_F, t\}, \{s, v_{F'}\}\}$, then, by Lemma 29 2, $b \in \tilde{B}(\tilde{G})$ and hence $\tilde{\mathcal{B}}(b)$ is defined.

We now show that \mathcal{B} is feasible. Let $(u, v) \in E$ and $i \in [\ell]$. We show that the capacity constraint for (u, v) and i is satisfied.

If $(u, v) \in \{(v_F, t), (s, v_{F'})\}$, then, by assumption 3c, the capacity constraint is satisfied.

Now suppose $(u, v) \notin \{(v_F, t), (s, v_{F'})\}$. Hence, by assumption 3a, $(u, v) \notin \{(v_F, t), (s, v_{F'}), (v_F, v_{F'})\}$. Hence, by definition of \tilde{G} , $(u, v) \in \tilde{E}$ and $c(u, v) = \tilde{c}(u, v)$. Hence, by Lemma 31 and since block sequence $\tilde{\mathcal{B}}$ is feasible, we have

$$\begin{aligned}
& \sum_{P \in \mathcal{P}: \alpha_P((u, v), B_{i-1}) = \text{active or } \alpha_P((u, v), B_i) = \text{active}} d_P = \\
& \sum_{P \in \tilde{\mathcal{P}}: \tilde{\alpha}_P((u, v), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_P((u, v), \tilde{B}_i) = \text{active}} \tilde{d}_P \leq \\
& \tilde{c}(u, v) = c(u, v).
\end{aligned}$$

□

Remark 32. Let G, F, F' and \tilde{G}, \tilde{F} be as specified in the proof of the Merging Lemma. Then notice that:

1. For every flow pair $P \in \mathcal{P} \setminus \{F, F'\}$, if properties 1, 2, and 3 are satisfied for both F, P and F', P in G , then they are also satisfied for \tilde{F}, P in \tilde{G} .
2. For every two flow pairs $P, P' \in \mathcal{P} \setminus \{F, F'\}$, if properties 1, 2, and 3 are satisfied for P, P' in G , then they are also satisfied for P, P' in \tilde{G} .

Chapter 3

NP-Hardness for the General Case

We are finally ready to prove Theorem 22.

Proof of Theorem [thm:np-hardness-k-eq-3]. || Let C be a 4CNF formula and G be the corresponding update flow network as specified in the proof of Theorem 23. Then G comprises 10 flow pairs $\bar{X}, L, \tilde{L}, X, R, \tilde{R}, B, \tilde{B}, B_0, B_1$, each with demand 1.

By definition, flow pairs \bar{X}, L, \tilde{L} pairwise satisfy assumptions 1, 2, and 3 of the Merging Lemma (confirm Table 1.1). Hence, by Remark 32 1, we may apply the Merging Lemma twice to obtain from G an update flow network G_8 with 8 flow pairs.

Similarly, flow pairs X, R, \tilde{R} pairwise satisfy assumptions 1, 2, and 3 of the Merging Lemma in G . By Remark 32 2, they also satisfy these assumptions in G_8 . Hence, again by Remark 32 1, we may apply the Merging Lemma twice to obtain from G_8 an update flow network G_6 with 6 flow pairs.

Finally, as flow pairs B, \tilde{B}, B_0, B_1 again pairwise satisfy all three assumptions, we apply the Merging Lemma three times to obtain from G_6 an update flow network G_3 with 3 flow pairs.

Putting everything together, we have that there is a satisfying assignment for 4CNF formula C iff there is a feasible block sequence for update flow network G iff there is a feasible block sequence for G_8 iff there is a feasible block sequence for G_6 iff there is a feasible block sequence for G_3 , which, by Corollary 21, is the case iff there is a feasible update sequence for G_3 . \square