20.12.22

<u>Special case where $d_i = 1$ for all i and $c(e) = k$ for all edges e</u>

Since $c(e) = k \geq \sum_i d_i$ for every edge e, no congestion can occur. Hence, we only have to worry about maintaining transient flows.

We claim that the following algorithm solves the problem optimally:

1. Perform updates $r_1 = \overset{\bigcup_i V(F_i^u - F_i^o)\times\{i\}}{\bigcup_{i,b:\, b \text{ is an } i\text{-block},\, v\in F_i^u \cap b - y(b)}} (v,i)$ (if any) in a single round.
2. Perform updates $r_2 = \overset{\bigcup_i V(F_i^o \cap F_i^u)\times\{i\}}{\bigcup_{i,b:\, b \text{ is an } i\text{-block}}} (y(b), i)$ in a single round.
3. Perform updates $r_3 = \overset{\bigcup_i V(F_i^o - F_i^u)\times\{i\}}{\bigcup_{i,b:\, b \text{ is an } i\text{-block},\, v\in b - F_i^u}} (v,i)$ (if any) in a single round.

<u>Proof.</u> We need to show that our algorithm outputs a valid update sequence (r_1, r_2, r_3). Optimality then follows from Lemma 4.1. We need to show that our algorithm indeed outputs an update sequence (r_1, r_2, r_3), that is, that (1) $r_1 \cup r_2 \cup r_3 = \bigcup_i \underset{\cancel{(3)}}{V}(F_i^o \cup F_i^u) \times \{i\}$ and (2) $r_1, r_2,$ and r_3 are pairwise disjoint. Moreover, we need to show that $(\underset{(i)}{r_1}, \underset{(ii)}{r_2}, \underset{(iii)}{r_3})$ is valid, that is, that (3) ~~for every $j \in [3]$ and for every $S \subseteq r_j$, $U_j^S = S \cup \bigcup_{l=1}^{j} r_l$~~ $r_1, r_2,$ and r_3 obey the consistency rule.

<u>(1).</u> $r_1 \cup r_2 \cup r_3 = \bigcup_{i,b:\, b \text{ is an } i\text{-block},\, v\in F_i^u\cap b - y(b) \,\cup\, y(b) \,\cup\, b - F_i^u} (v,i) = \bigcup_{i,b:\, b \text{ is an } i\text{-block}} \bigcup_{v\in b} (v,i) = \bigcup_{i,b:\, b \text{ is an } i\text{-block}} b \times \{i\} = \bigcup_i V(F_i^o \cup F_i^u) \times \{i\}$. Note that this assumes a definition of blocks such that $\bigcup_{b:\, b \text{ is an } i\text{-block}} b = V(F_i^o \cup F_i^u)$, which would not hold ~~if we~~ for the definition in the papers.

<u>(2).</u> The proof depends on our definition of blocks but should work as long as we define i-blocks in a way such that they are pairwise disjoint.

<u>(3)(i).</u> Suppose r_1 does not obey the consistency rule. Hence $r_1 \neq \{\}$. Let $S \subseteq r_1$ be ~~minimal~~ minimal such that ~~$U_1^S = S$~~, for $U_1^S = S$, there is no transient flow T_{i,U_1^S} in $x_i(U_1^S, G)$ for some i. ~~Since S is minimal~~ Hence $S \neq \{\}$. Since S is minimal, $S \subseteq V(F_i^o \cup F_i^u) \times \{i\}$ and, for every $(v,i) \in S$, ~~$\exists$~~ there is a transient flow $T_{i,U_1^S-(v,i)}$ in $x_i(U_1^S - (v,i), G)$. Let $(v,i) \in S$. Since there is a transient flow $T_{i,U_1^S-(v,i)}$ in $x_i(U_1^S - (v,i), G)$ but no transient flow in $x_i(U_1^S, G)$, update (v,i) deactivates an edge $e \in E(F_i^o)$ with $\text{tail}(e) = v$. Hence $v \in V(F_i^o)$. We consider the cases (a) $v \notin V(F_i^u)$ and (b) $v \in V(F_i^u)$.

<u>Case (a).</u> Hence ~~...~~ $(v,i) \notin S \subseteq r_1$ by definition of r_1.

Case (b). Hence $v \in V(F_i^o \cap F_i^u)$. Hence $v = f(b)$ for some i-block b. Hence $(v,i) \in S \subseteq r_1$ by definition of r_1. We conclude that r_1 obeys the consistency rule.

(3)(ii). ... Since $(v,i) \in S \subseteq r_2$, $v = f(b)$ for some i-block b, by definition of r_2. Hence $v \in V(F_i^o \cup F_i^u)$.
By assumption, there is an (s,v)-path in $\alpha_i(U_2^s - (v,i), \delta)$ and $\alpha_i(U_2^s, \delta)$, a (v,t)-path in $\alpha_i(U_2^s - (v,i), \delta)$, but no (v,t)-path in $\alpha_i(U_2^s, \delta)$. Hence there is a blackhole u such that $v \xrightarrow{\alpha_i(U_2^s,\delta)} u$ there is a (v,u)-path in
Instead consider the cases $u \in V(F_i^u - F_i^o)$, $u \in V(F_i^o \cap F_i^u)$, $u \in V(F_i^u - F_i^o)$ and show a contradiction in each case.
$\alpha_i(U_2^s, \delta)$. We show $u \in V(F_i^u - F_i^o)$. Hence $(u,i) \in r_1$, ~~Hence~~ which contradicts that u is a blackhole.

~~$u \in V(F_i^u)$~~. If $u \in V(F_i^o \cap F_i^u)$, then u cannot be a blackhole in $\alpha_i(U, \delta)$ for any U. If $u \in V(F_i^o - F_i^u)$, then $(u,i) \in r_3$, that is, u has not been updated yet, and hence it cannot be a blackhole. Thus $u \in V(F_i^u - F_i^o)$.

(3)(iii). ... Since $(v,i) \in S \subseteq r_3$, $v \in V(F_i^o - F_i^u)$. ~~Now consider the (s,v)-path in $\alpha_i(U_3^s, \delta)$.~~ Since $s \in V(F_i^o \cap F_i^u)$, $v \neq s$. Now consider the vertex u preceding v on the (s,v)-path in $\alpha_i(U_3^s, \delta)$, that is, $(u,v) \in E(F_i^o)$.
Hence $u \in V(F_i^o)$. If $u \in V(F_i^o \cap F_i^u)$, then $(u,i) \in r_i$ and hence u would have been updated already, which contradicts the choice of u. Hence $u \in V(F_i^o - F_i^u)$. Inductively, $x \in V(F_i^o - F_i^u)$ holds for every vertex on the (s,v)-path in $\alpha_i(U_3^s, \delta)$, including s – a contradiction.