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Proof (ctd.)

TODO Show that instance $\tilde{G}$ is feasible iff G is feasible.

We now show that there is a feasible block sequence ~~$R = ($~~ $B = (b_1, \dots, b_k)$ for update flow network G iff there is a feasible ~~update~~ block sequence $\tilde{B} = (\tilde{b}_1, \dots, \tilde{b}_k)$ for update flow network $\tilde{G}$.

"$\Rightarrow$"

Let $B = (b_1, \dots, b_k)$ be a feasible block sequence for G. We define our block sequence $\tilde{B}$ for $\tilde{G}$ as follows

Map every vertex $\tilde{v}$ in $\tilde{G}$ to a vertex v in G such that $\tilde{B}(\tilde{b}) = B(b(v))$ if $\mathcal{I}(\tilde{b}) = \tilde{v}$.

$$\tilde{B}(b) = \begin{cases} B(b(s, \tilde{F}(b))) \text{ if } \mathcal{I}(b) = \tilde{u}_{F_j^o} \\ B(b(s, \tilde{F}(b))) \text{ if } \mathcal{I}(b) = \tilde{u}_{F_j^u} \\ B(b(v_{F_j^o}, \tilde{F}(b))) \text{ if } \mathcal{I}(b) = \tilde{v}_{F_j^o} \\ B(b(v_{F_j^u}, \tilde{F}(b))) \text{ if } \mathcal{I}(b) = \tilde{v}_{F_j^u} \\ B(b(s, F_j)) \text{ if } \mathcal{I}(b) = w \\ \cancel{B(b) \text{ otw.}}\ B(b(\mathcal{I}(b), \tilde{F}(b))) \text{ otw.} \end{cases}$$

$B(b(s, \tilde{F}(b)))$?

where $\tilde{F}(b) = \begin{cases} \tilde{F} \text{ if } F(b) = F_j \text{ or } F(b) = F_k \\ F(b) \text{ otw.} \end{cases}$

Suppose $\tilde{B}$ is not feasible. Then let $i \in [k]$ be minimum and $e \in E(\tilde{G})$ such that

$\tilde{c}(e) < \sum_{\tilde{P}: \alpha_{\tilde{P}}(e, \tilde{U}_{i-1}) = \text{active or } \alpha_{\tilde{P}}(e, \tilde{U}_i) = \text{active}} d_{\tilde{P}}$

$e \notin E$

Case $e \in E(\tilde{G}) - E(G)$. ~~Clearly~~ Then, by definition of $\tilde{c}$, e cannot be either of $(s, \tilde{u}_{F_j^o}), (s, \tilde{u}_{F_j^u})$, ~~$(v_{F_j^o}, \tilde{v}_{F_j^o}), (v_{F_j^u}, \tilde{v}_{F_j^u})$~~ $(\tilde{v}_{F_j^o}, t), (\tilde{v}_{F_j^u}, t), (\tilde{v}_{F_j^o}, w), (\tilde{v}_{F_j^u}, w), (w, \tilde{u}_{F_j^o}), (w, \tilde{u}_{F_j^u})$.

Case $e = (\tilde{u}_{F_j^o}, u_{F_j^o})$. ~~We show that~~

$\sum_{P: \alpha_P(e, U_{i-1}) = \text{active or } \alpha_P(e, U_i) = \text{active}} d_P = \sum_{P: \alpha_P(e, U_{i-1}) = \text{active}} d_P + \sum_{P: \alpha_P(e, U_{i-1}) = \text{inactive and } \alpha_P(e, U_i) = \text{active}} d_P \leq$

$\sum_{P: \alpha_P(e, U_{i-2}) = \text{active or } \alpha_P(e, U_{i-1}) = \text{active}} d_P + \sum_{P: \alpha_P(e, U_{i-1}) = \text{inactive and } \alpha_P(e, U_i) = \text{active}} d_P$

We show that $\alpha_{\tilde{P}}((\tilde{u}_{F_j^o}, u_{F_j^o}), \tilde{U}_i) = \alpha_{\hat{P}}((s, u_{F_j^o}), U_i)$, where $\hat{P} = P$ if $\tilde{P} \neq F$ and $\hat{P} = F_j$ otw.

Induction base $i = 0$. $\alpha_{\tilde{P}}((\tilde{u}_{F_j^o}, u_{F_j^o}), \tilde{U}_0) = \text{active}$ iff $(\tilde{u}_{F_j^o}, u_{F_j^o}) \in E(\tilde{P}^o)$ iff $(s, u_{F_j^o}) \in E(\hat{P}^o)$ iff $\alpha_{\hat{P}}((s, u_{F_j^o}), U_0) = \text{active}$ ✓

TODO Prove

Induction step $i-1 \to i$. $\alpha_{\tilde{P}}((\tilde{u}_{F_j^o}, u_{F_j^o}), \tilde{U}_i) = \text{active}$ iff ~~$(\tilde{u}_{F_j^o}, \tilde{P}) \notin \tilde{U}_i$ and $(\tilde{u}_{F_j^o}, u_{F_j^o}) \in E(\tilde{P}^o)$ or $(\tilde{u}_{F_j^o}, \tilde{P}) \in \tilde{U}_i$ and $(\tilde{u}_{F_j^o}, u_{F_j^o}) \in E(\tilde{P}^u)$~~ $\alpha_{\tilde{P}}((\tilde{u}_{F_j^o}, u_{F_j^o}), \tilde{U}_{i-1}) = \text{active}$ and $(\tilde{u}_{F_j^o}, \tilde{P}) \notin U(\tilde{b}_i)$ or $\alpha_{\tilde{P}}((\tilde{u}_{F_j^o}, u_{F_j^o}), \tilde{U}_{i-1}) = \text{inactive}$ and $(\tilde{u}_{F_j^o}, \tilde{P}) \in U(\tilde{b}_i)$ iff $\alpha_{\hat{P}}((s, u_{F_j^o}), U_{i-1}) = \text{active}$ and $\tilde{b}(\tilde{u}_{F_j^o}, \tilde{P}) \notin \tilde{b}_i$ or $\alpha_{\hat{P}}((s, u_{F_j^o}), U_{i-1}) = \text{inactive}$ and $\tilde{b}(\tilde{u}_{F_j^o}, \tilde{P}) \in \tilde{b}_i$ iff

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$\alpha_{\hat{P}}((s, u_{F_t^0}), U_{i-1}) = \text{active}$ and $b(s, \hat{P}) \notin C_i$ or $\alpha_{\hat{P}}((s, u_{F_t^0}), U_{i-1}) = \text{inactive}$ and $b(s, \hat{P}) \in C_i$ iff

$\alpha_{\hat{P}}((s, u_{F_t^0}), U_i) = \text{active}$ ✓

<u>Claim:</u> $\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P}) \in \tilde{C}_i$ iff $b(s, \hat{P}) \in C_i$.

~~$\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P}) = \{$~~ $\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P}) \in \tilde{C}_i$ iff $\tilde{B}(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = i$. By construction, $f(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) \in \{s, \tilde{u}_{F_t^0}, w\}$. Create appropriate Lemma

Case $f(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = s$. Then $\tilde{B}(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = B(b(s, \hat{P}))$. ✓

Case $f(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = \tilde{u}_{F_t^0}$. Then $\tilde{B}(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = B(b(s, \hat{P}))$. ✓

Case $f(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = w$. Then $\tilde{B}(\tilde{b}(\tilde{u}_{F_t^0}, \tilde{P})) = B(b(s, F_t))$. Since $\tilde{F}$ is the only flow pair containing w, $\tilde{P} = \tilde{F}$ and hence $\hat{P} = F_t$.

<u>Case $e \in E$.</u> We show that $\alpha_{\tilde{P}}(e, \tilde{U}_i) = \alpha_{\hat{P}}(e, U_i)$.

Induction base $i = 0$. $\alpha_{\tilde{P}}(e, \tilde{U}_0) = \text{active}$ iff $e \in E(\tilde{P}^0)$ iff $e \in E(\hat{P})$ iff $\alpha_{\hat{P}}(e, U_0) = \text{active}$. ✓

Induction step $i-1 \to i$. Let $(u, v) = e$.

Claim: $\tilde{b}(u, \tilde{P}) \in \tilde{C}_i$ iff $b(u, \hat{P}) \in C_i$.

$\tilde{b}(u, \tilde{P}) \in \tilde{C}_i$ iff $\tilde{B}(\tilde{b}(u, \tilde{P})) = i$.

Case $f(\tilde{b}(u, \tilde{P})) = \tilde{u}_{F_t^0}$. Hence $u = \tilde{u}_{F_t^0} \notin V$ ↯ TODO Prove

Case $f(\tilde{b}(u, \tilde{P})) = \tilde{v}_{F_i^0}$. Hence $u = \tilde{v}_{F_i^0} \notin V$ ↯ TODO Prove

Case $f(\tilde{b}(u, \tilde{P})) = w$. Then $\tilde{B}(\tilde{b}(u, \tilde{P})) = B(b(s, F_t))$. ??? [$f(\tilde{b}(u, \tilde{P})) = w \Rightarrow f(b(u, \hat{P})) = s$ TODO

[Case otw. Then $\tilde{B}(\tilde{b}(u, \tilde{P})) = \cancel{B(b(f(b)), \hat{P})}$ $B(b(f(\tilde{b}(u, \tilde{P})), \hat{P}))$. ??? [otw. $\Rightarrow f(\tilde{b}(u, \tilde{P})) = f(b(u, \hat{P}))$ TODO

Hence ... [~~chain itt~~ iff chain].

<u>„$\Leftarrow$"</u>

Let $\tilde{B} = (\tilde{C}_1, \dots, \tilde{C}_\ell)$ be a feasible block sequence for $\tilde{G}$. We define block sequence B for G as follows:

$B(b) = \tilde{B}(\tilde{b}(f(b), \tilde{F}(b)))$, where $\tilde{F}(b) = \tilde{F}$ if $F(b) = F_i$ or $F(b) = F_t$ and $\tilde{F}(b) = F(b)$ otw.

We define the following mapping $f : E(G) \to E(\tilde{G})$:

$$f(e) = \begin{cases} (\tilde{u}_{F_t^0}, u_{F_t^0}) & \text{if } e = (s, u_{F_t^0}) \\ (\tilde{u}_{F_t^u}, u_{F_t^u}) & \text{if } e = (s, u_{F_t^u}) \\ (v_{F_i^0}, \tilde{v}_{F_i^0}) & \text{if } e = (v_{F_i^0}, t) \\ (v_{F_i^u}, \tilde{v}_{F_i^u}) & \text{if } e = (v_{F_i^u}, t) \\ e & \text{otw.} \end{cases}$$

Claim: $\alpha_P(e, U_i) = \alpha_{\tilde{P}}(f(e), \tilde{U}_i)$.

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<u>Proof (ctd.)</u>

Induction base $i=0$. $\alpha_P(e, U_0) = \text{active}$ iff $e \in E(P^\circ)$ $\overset{\text{case distinction}}{\text{iff}}$ $f(e) \in E(\tilde{P}^\circ)$ iff $\alpha_{\tilde{P}}(f(e), \tilde{U}_0) = \text{active}$. ✓

Let $(u,v)=e$.

Induction step ~~for~~ $i-1 \to i$. $\alpha_P(e, U_i) = \text{active}$ iff $\alpha_P(e, U_{i-1}) = \text{active}$ and $(u, P) \notin U(t_i)$ or

$\alpha_P(e, U_{i-1}) = \text{inactive}$ and $(u, P) \in U(t_i)$ iff $\alpha_{\tilde{P}}(f(e), \tilde{U}_{i-1}) = \text{active}$ and

Claim: $b(u, P) \in t_i$ iff $\tilde{b}(u, \tilde{P}) \in \tilde{t}_i$. [$b(\text{tail}(e), P) \in t_i$ iff $\tilde{b}(\widetilde{\text{tail}}(f(e)), \tilde{P}) \in \tilde{t}_i$?

TODO $b(u, P) \in t_i$ iff $B(b(u,P)) = i$ iff $\tilde{B}(\tilde{b}(f(b(u,P)), \tilde{P})) = i$ iff $\tilde{b}(f(b(u,P)), \tilde{P}) \in \tilde{t}_i$ $\overset{???}{\text{iff}}$ $\tilde{b}(u, \tilde{P}) \in \tilde{t}_i$.