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Lemma: Let $P \in \mathcal{P}$. ~~Then~~ If ~~there~~ is an (s,t)-flow $F \subseteq P^\circ \cup P^u$ ~~$F \subseteq P^\circ \cup P^u$~~ in G, then for every P-block b, $F|_b$ is a $(\mathfrak{s}(b), \mathcal{E}(b))$-flow in $G|_b$.

Lemma: Let $P \in \mathcal{P}$. If for every P-block b, $F_b \subseteq P^\circ \cup P^u$ is a $(\mathfrak{s}(b), \mathcal{E}(b))$-flow in $G|_b$, then $\bigcup_b F_b$ is an (s,t)-flow in G.

$F = P^\circ|_b$ or $F = P^u|_b$?

Lemma: Let $P \in \mathcal{P}'$ ~~and~~ b be a P-block, and F be a $(\mathfrak{s}(b), \mathcal{E}(b))$-flow in $G|_b$. Then $F \subseteq P^\circ$ or $F \subseteq P^u$.

Lemma: Let b be a P-block. Then:

(i) If for every edge $e \in E(P^\circ|_b)$, $\alpha_P(e, U) = \text{active}$, then $P^\circ|_b$ is a $(\mathfrak{s}(b), \mathcal{E}(b))$-flow in $\alpha_P(U, G)|_b$.

(ii) " "

Proof. (ctd.)

... Since S is minimal, $P' = P$ and P is transient for $U_{i+1}^{S-\{(v,P)\}}$. For every P-block b other than $b(v,P)$, ~~we have~~ by Lemma ~, there is a $(\mathfrak{s}(b), \mathcal{E}(b))$-flow F_b in $\alpha_P(U_{i+1}^S, G)|_b$ (since, for every edge $e \in E(P|_b)$, $\alpha_P(e, U_{i+1}^S) = \alpha_P(e, U_{i+1}^{S-\{(v,P)\}})$). We show that $P^u|_{b(v,P)}$ is a $(\mathfrak{s}(b(v,P)), \mathcal{E}(b(v,P)))$-flow in $\alpha_P(U_{i+1}^S, G)|_{b(v,P)}$. Then, by Lemma ~, $\bigcup_b F_b \cup P^u|_{b(v,P)}$ is an (s,t)-flow in $\alpha_P(U_{i+1}^S, G)$ which contradicts that P is not transient for U_{i+1}^S.

We show that for every edge $e \in E(P^u|_{b(v,P)})$, $\alpha_P(e, U_{i+1}^S) = \text{active}$, which then, by Lemma ~, implies the claim. We show that for every vertex $u \in V(P^u) \cap b(v,P)$, $(u,P) \in U_{i+1}^S$.

We need something like $e \in E(P^u)$ iff $\text{tail}(e) \in V(P^u)$ and $e = e^u(\text{tail}(e))$.

We first show that $v = \mathfrak{s}(b(v,P))$, by showing that (v,P) is a type-2 update.

Suppose (v,P) is a type-1 update. By definition of R_3, $(\mathfrak{s}(b_\circ(v,P)), P) \notin U_{i+1}^{S-\{(v,P)\}}$. Hence, by Lemma ~(ii), P is transient for $U_{i+1}^{S-\{(v,P)\}} \cup \{(v,P)\} = U_{i+1}^S$, a contradiction.

Suppose (v,P) is a type-3 update. By definition of R_3, $(\mathfrak{s}(b(v,P)), P) \in r_i \subseteq U_{i+1}^{S-\{(v,P)\}}$. Hence, by Lemma ~(i), P is transient for $U_{i+1}^{S-\{(v,P)\}} \cup \{(v,P)\} = U_{i+1}^S$, a contradiction.

We conclude that (v,P) is a type-2 update. Hence, by definition of R_3, every type-1 update $(u,P) \in r_i \subseteq U_{i+1}$ $u \in b(v,P)$. Hence, for every vertex $u \in V(P^u) \cap b(v,P) = V(P^u - P^\circ) \cap b(v,P) \cup V(P^\circ \cap P^u) \cap b(v,P) = V(P^u - P^\circ) \cap b(v,P) \cup \mathfrak{s}(b(v,P)) = V(P^u - P^\circ) \cap b(v,P) \cup \{v\}$, $(u,P) \in U_{i+1}^S$.

<u>Proof</u> of lemma "... then the induced update sequence $\mathcal{R}_\mathcal{S} = (r_1, \dots, r_{\ell+2})$ is feasible."

By lemma ~, for every $P \in \mathcal{P}$, $i \in [\ell+2]$, $S \subseteq r_i$, P is transient for U_i^S. Hence it is sufficient to show the following. Let $e \in E$ be an edge. Then:

(1) For every $S \subseteq r_1$, $c(e) \geq \sum_{P \in \mathcal{P} : e \in E(T_{P, U_1^S})} d_P$.

(2) For every $i \in [\ell]$ and $S \subseteq r_{i+1}$, $c(e) \geq \sum_{P : e \in E(T_{P, U_{i+1}^S})} d_P$.

(3) For every $S \subseteq r_{\ell+2}$, $c(e) \geq \sum_{P : e \in E(T_{P, U_{\ell+2}^S})} d_P$.

~~Proof~~ (1). By definition of $\mathcal{R}_\mathcal{S}$, r_1 contains only type-1 updates. Hence, by (repeated application of) lemma ~(i),

$T_{P, U_1^S} = T_{P,S} = T_{P,\emptyset} = P^o$ for every P.

(3). By definition of $\mathcal{R}_\mathcal{S}$, $r_{\ell+2}$ contains only type-3 updates (and $r_{\ell+1}$ contains the corresponding type-2 updates). Hence, by (repeated application of) lemma ~(ii), $T_{P, U_{\ell+2}^S} = T_{P, U_{\ell+2}^{r_{\ell+2}}} = P^n$ for every P.

(2). Let $i \in [\ell]$ and $S \subseteq r_{i+1}$.

<u>Claim:</u> If $e \in E(T_{P, U_{i+1}^S})$, then $\alpha_P(e, B_{i-1}) = \text{active}$ or $\alpha_P(e, B_i) = \text{active}$.

<u>Fact:</u> $U(B_i) = \bigcup_{j=1}^{i} r_j \cup r_{i+1}|_{2,3} \cup r_{i+2}|_3$

We probably need to prove this (via induction?). Also, notation $r_i|_{1,2,3}$

<u>Proof</u> of Claim ~.

Case $e \in E(P^o)$. ~~Hence,~~ By ~~Claim~~ lemma ~(i), $e \in E(T_{P, U_{i+1}^{S \cup r_{i+1}|_3}})$. By Fact ~, $U(B_{i-1}) = \bigcup_{j=1}^{i-1} r_j \cup r_i|_{2,3} \cup r_{i+1}|_3 = U_i^{r_i|_{2,3}} \cup r_{i+1}|_3 \subseteq U_i^{r_i} \cup r_{i+1}|_3 = U_{i+1}^{r_{i+1}|_3} \subseteq U_{i+1}^{S \cup r_{i+1}|_3}$. Hence, by lemma ~(i), $e \in E(T_{P, U(B_{i-1})})$. Hence, by lemma ~, $\alpha_P(e, B_{i-1}) = \text{active}$.

Case $e \in E(P^n)$. By ~~Claim~~ lemma ~(ii), $e \in E(T_{P, U_{i+1}^{S - r_{i+1}|_1}})$. By Fact ~, $U_{i+1}^{S - r_{i+1}|_1} \subseteq U_{i+1}^{r_{i+1}|_{2,3}} = \bigcup_{j=1}^{i} r_j \cup r_{i+1}|_{2,3} \subseteq U(B_i)$.

Hence, by lemma ~(ii), $e \in E(T_{P, U(B_i)})$. Hence, by lemma ~, $\alpha_P(e, B_i) = \text{active}$.

Now, by Claim ~ and assumption, $\sum_{P : e \in E(T_{P, U_{i+1}^S})} d_P \leq \sum_{P : \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P \leq c(e)$.

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<u>Lemma:</u> If there is a feasible update sequence, then there is a feasible block sequence.

<u>Proof.</u>

Let $\mathcal{R}$ be a feasible update sequence. We apply Lemma 2 to obtain a feasible update sequence $\mathcal{R}'=(r_1,\dots,r_\ell)$ which updates every block in at most 3 consecutive rounds. We define the block sequence $B_{\mathcal{R}'}=(b_1,\dots,b_\ell)$ induced by $\mathcal{R}'$ as follows: For every $i\in[\ell]$, ~~we set $b_i=$~~ $b_i:=\{b:(f(b),P(b))\in r_i\}$. We show that $B_{\mathcal{R}'}$ is feasible ~~by contradiction~~.

Suppose not. Then let $e\in E$ and $i\in[\ell]$ such that $c(e)<\sum_{P:\alpha_P(e,B_{i-1})=\text{active or }\alpha_P(e,B_i)=\text{active}} d_P$.

We construct a set $S_e\subseteq r_i$ such that $c(e)<\sum_{P:e\in E(T_{P,U_i^{S_e}})} d_P$ which contradicts the feasibility of $\mathcal{R}'$.

~~We set $S_e:=\{$~~ Intuitively, $S_e\subseteq r_i$ can be thought of as the worst-case set of updates for edge e.

More precisely, S_e contains all type-2 updates corresponding to blocks $b\in b_i$ such that $\alpha_{P(b)}(e,B_{i-1})=\text{inactive}$ and $\alpha_{P(b)}(e,B_i)=\text{active}$ and none of the type-2 updates corresponding to blocks $b\in b_i$ such that $\alpha_{P(b)}(e,B_{i-1})=\text{active}$ and $\alpha_{P(b)}(e,B_i)=\text{inactive}$. More formally, we set

$S_e:=\{(f(b),P(b)):b\in b_i \text{ and } e\in E(P(b)^u|_b)\}$.

<u>Claim:</u> (i) If $\alpha_P(e,B_{i-1})=\text{active}$, then $e\in E(T_{P,U_i^{S_e}})$, and (ii) if $\alpha_P(e,B_i)=\text{active}$, then $e\in E(T_{P,U_i^{S_e}})$.

<u>Proof.</u>

(i). Suppose $\alpha_P(e,B_{i-1})=\text{active}$. ~~Hence, by lemma ~, $e\in E(T_{P,U(B_{i-1})})$. We consider the cases~~

We set $b:=b(\text{tail}(e),P)$. We consider the cases $b\in B_{i-1}$, $b\in b_i$, and $b\notin B_i$.

Case $b\in B_{i-1}$. By assumption and definition of α_P, $e\in E(P^u)$. Moreover, by definition of B, $(f(b),P)\in U_i^{\le 3}$. Hence, by lemma ~(i, ii), $e\in E(T_{P,U_i^{\le 3}})$. Hence, by lemma ~(ii), $e\in E(T_{P,U_i^{S_e}})$.

Case $b\in b_i$. Hence $b\notin B_{i-1}$. By assumption and definition of α_P, $e\in E(P^o)$. Hence, by definition of S_e, $(f(b),P)\notin S_e$. Hence $(f(b),P)\notin U_i^{S_e}$. Hence, by lemma ~(ii, iii), $e\in E(T_{P,U_i^{S_e}})$.

Case $b\notin B_i$. Hence $b\notin B_{i-1}$ and, by definition of B, $(f(b),P)\notin U_i^{\le}$. Hence $e\in E(P^o)$. Hence, by lemma ~(i), $e\in E(T_{P,U_i^{\le}})$. Hence, by lemma ~(i), $e\in E(T_{P,U_i^{S_e}})$.

(ii). Suppose $\alpha_P(e,B_i)=\text{active}$. We set $b:=b(\text{tail}(e),P)$ and consider the cases $b\in B_{i-1}$, $b\in b_i$, and $b\notin B_i$.

Case $b\in B_{i-1}$. Hence $e\in E(P^u)$ and $(f(b),P)\in U_i^{\le 3}$. Hence, by lemma ~(i, ii), $e\in E(T_{P,U_i^{\le 3}})$. Hence, by lemma ~(ii), $e\in E(T_{P,U_i^{S_e}})$.

Case $b\in b_i$. Hence $b\in B_i$. Hence $e\in E(P^u)$. Hence $(f(b),P)\in S_e$. Hence $(f(b),P)\in U_i^{S_e}$. Hence, by lemma ~(i, ii), $e\in E(T_{P,U_i^{S_e}})$.

case $b \notin B_i$. Hence $e \in E(P^0)$ and, by definition of $B_{\mathcal{S}'}$, $(s(b), P) \notin U_i^{\mathcal{S}}$. Hence, by Lemma ~ (ii, iii), $e \in E(T_{P, U_i^{\mathcal{S}}})$.

Hence, by Lemma ~ (i), $e \in E(T_{P, U_i^{\mathcal{S}'}})$.

Now, by Claim ~, $\sum_{P : \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P \leq \sum_{P : e \in E(T_{P, U_i^{\mathcal{S}'}})} d_P \leq c(e)$.

<u>Lemma:</u> There is a feasible update sequence iff there is a feasible block sequence.

<u>Proof</u>

Follows immediately from Lemmas ~ and ~.