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Lemma: Let G be an update flow network with $k \geq 2$ flow pairs, and let $F, F' \in \mathcal{P}$ be two flow pairs such that:

(i) $d_F = d_{F'}$,

(ii) F and F' have no common vertices other than s, t, that is $V(F) \cap V(F') = \{s, t\}$, and

(iii) there are edges $e_F, e_{F'}$ such that

(a) e_F ($e_{F'}$) is the last (first) edge on both F^o and F^u (F'^o and F'^u), and

(b) the capacity constraint for e_F ($e_{F'}$) is always satisfied, that is

$c(e_F) \geq \sum_{P \in \mathcal{P}: e_F \in E(P^o \cup P^u)} d_P$ ($c(e_{F'}) \geq \sum_{P \in \mathcal{P}: e_{F'} \in E(P^o \cup P^u)} d_P$.

Then there is an update flow network $\tilde{G}$ with $k-1$ flow pairs such that ($|\tilde{G}| = O(|G|)$ and) there is a feasible block sequence $B = (b_1, \ldots, b_\ell)$ for G iff there is a feasible block sequence $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_\ell)$ for $\tilde{G}$.

Proof.

~~Let $v_F, v_{F'}$ be two vertices such that~~ Let $G = (V, E, \mathcal{P}, s, t, c)$ be an update flow network with $|\mathcal{P}| \geq 2$, and let $F, F' \in \mathcal{P}$ be two flow pairs satisfying properties (i), (ii), (iii). Then let $v_F, v_{F'}$ be two vertices such that $(v_F, t) \in E(F^o \cap F^u)$, $(s, v_{F'}) \in E(F'^o \cap F'^u)$, and the capacity constraints for (v_F, t), $(s, v_{F'})$ are always satisfied. We construct an update flow network $\tilde{G} = (V, \tilde{E}, \tilde{\mathcal{P}}, s, t, \tilde{c})$ $|\tilde{\mathcal{P}}| = |\mathcal{P}| - 1$ such that there is a feasible block sequence $B = (b_1, \ldots, b_\ell)$ for G iff there is a feasible block sequence $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_\ell)$ for $\tilde{G}$ as follows. Intuitively, we merge flow pair F and F' into a single flow pair $\tilde{F}$ by concatenating F and F'. More precisely, $\tilde{F}$ will be the union of F and F' except that we will replace edges (v_F, t) and $(s, v_{F'})$ by edge $(v_F, v_{F'})$. More formally, we define $\tilde{F}$ as follows:

- $\tilde{E}(\tilde{F}^o) = E(F^o) - \{(v_F, t)\} \cup E(F'^o) - \{(s, v_{F'})\} \cup \{(v_F, v_{F'})\}$
- $\tilde{E}(\tilde{F}^u) = E(F^u) -$ "
- $d_{\tilde{F}} = d_F$

Update flow network $\tilde{G} = (V, \tilde{E}, \tilde{\mathcal{P}}, s, t, \tilde{c})$ is defined as follows:

Actually, we might have to remove (v_F, t), $(s, v_{F'})$. So just define $\tilde{E}$ as the union of all $E(\tilde{P}^o \cup \tilde{P}^u)$.

- $\tilde{E} = E \cup \{(v_F, v_{F'})\}$
- $\tilde{\mathcal{P}} = \mathcal{P} - \{F, F'\} \cup \{\tilde{F}\}$
- $\tilde{c}(e) = \begin{cases} \sum_{\tilde{P} \in \tilde{\mathcal{P}}: e \in E(\tilde{P}^o \cup \tilde{P}^u)} d_{\tilde{P}} & \text{if } e = (v_F, v_{F'}) \\ c(e) & \text{otw.} \end{cases}$

We actually need that $(v_F, v_{F'}) \notin E$.

Notice that $\tilde{G}$ is feasible iff G is feasible.

We now ~~that~~ show that there is a feasible block sequence $B=(b_1,\dots,b_\ell)$ for G iff there is a feasible block sequence $\tilde{B}=(\tilde{b}_1,\dots,\tilde{b}_\ell)$ for $\tilde{G}$.

<u>Claim:</u> $$B^{\tilde{P}}(\tilde{G})=\begin{cases} B^{F}(G)-\{\{v_F,t\}\}\cup B^{F'}(G)-\{\{s,v_{F'}\}\}\cup\{\{v_F,v_{F'}\}\} & \text{if } \tilde{P}=\tilde{F}\\ B^{P}(G) & \text{otw.}\end{cases}$$

<u>Proof.</u>

See scratch.

<u>„⇒"</u>

Let $B=(b_1,\dots,b_\ell)$ be a feasible block sequence for G. We define block sequence $\tilde{B}=(\tilde{b}_1,\dots,\tilde{b}_\ell)$ for $\tilde{G}$ as follows.
For every $\tilde{\mathcal{P}}$-block $\tilde{b}$,

$$\tilde{B}(\tilde{b})=\begin{cases} & \text{if } \tilde{P}(\tilde{b})=\tilde{F} \text{ and } \tilde{b}=\{\{v_F,v_{F'}\}\}\\ B(\tilde{b}) & \text{otw.}\end{cases}$$

I would like to avoid this case, as we don't need to schedule this block anyway.

~~We~~ Notice that if $\tilde{P}(\tilde{b})\neq\tilde{F}$ or $\tilde{b}\neq\{\{v_F,v_{F'}\}\}$, then, by Claim~, $\tilde{b}\in B(G)$.

We now show that $\tilde{B}$ is ~~indeed~~ feasible. Let $i\in[\ell]$ and $(u,v)\in\tilde{E}$. We show that inequality (1) is satisfied for $i,(u,v)$. If $(u,v)\in\{(v_F,t),(s,v_{F'}),(v_F,v_{F'})\}$, then $\tilde{c}(u,v)\geq\sum_{\tilde{P}\in\tilde{\mathcal{P}}:(u,v)\in E(\tilde{P}^{\circ}\cup\tilde{P}^{*})}d_{\tilde{P}}$. Hence it suffices to consider $(u,v)\notin\dots$ and $e\in\tilde{E}$.

Do we need ~~to~~ (v_F,t), $(s,v_{F'})$ here?

<u>Claim:</u> Let $i\in[\ell]$. Then:

(i) If $\tilde{\alpha}_{\tilde{F}}(e,\tilde{B}_i)=\text{active}$, then $\alpha_F(e,B_i)=\text{active}$ or $\alpha_{F'}(e,B_i)=\text{active}$.

(ii) Let $\tilde{P}\in\tilde{\mathcal{P}}-\{\tilde{F}\}$. Then if $\tilde{\alpha}_{\tilde{P}}(e,\tilde{B}_i)=\text{active}$, then $\alpha_{\tilde{P}}(e,B_i)=\text{active}$.

We now have

$\tilde{\mathcal{P}}=(\tilde{\mathcal{P}}-\{\tilde{F}\})\,\dot{\cup}\,\{\tilde{F}\}$

$\sum_{\tilde{P}\in\tilde{\mathcal{P}}:\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_{i-1})=\text{active or }\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_i)=\text{active}}d_{\tilde{P}}=$ ~~$\sum_{\tilde{P}\in\tilde{\mathcal{P}}-\{\tilde{F}\}:\alpha_{\tilde{P}}}$~~

$\sum_{\tilde{P}\in\tilde{\mathcal{P}}-\{\tilde{F}\}:\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_{i-1})=\text{active or }\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_i)=\text{active}}d_{\tilde{P}}+\sum_{\tilde{P}\in\{\tilde{F}\}:\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_{i-1})=\text{active or }\tilde{\alpha}_{\tilde{P}}((u,v),\tilde{B}_i)=\text{active}}d_{\tilde{P}}\leq$ Claim ~ (i, ii)

$\mathcal{P}=(\mathcal{P}-\{F,F'\})\,\dot{\cup}\,\{F,F'\}$

$\sum_{P\in\mathcal{P}-\{F,F'\}:\alpha_P((u,v),B_{i-1})=\text{active or }\alpha_P((u,v),B_i)=\text{active}}d_P+\sum_{P\in\{F,F'\}:\alpha_P((u,v),B_{i-1})=\text{active or }\alpha_P((u,v),B_i)=\text{active}}d_P$ $\quad d_{\tilde{P}}=d_{\tilde{F}}=d_F=d_{F'}$

$\sum_{P\in\mathcal{P}:\alpha_P((u,v),B_{i-1})=\text{active or }\alpha_P((u,v),B_i)=\text{active}}d_P\leq$ ~~$c(u,v)$~~ ↯ B feasible

$c(u,v)=(u,v)\neq(v_F,v_{F'})$

$\tilde{c}(u,v)$

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<u>Proof (ctd.)</u>

<u>"$\Leftarrow$"</u>

Let $\tilde{B} = (\tilde{b}_1, \dots, \tilde{b}_\ell)$ be a feasible block sequence for $\tilde{G}$. We define block sequence $B = (b_1, \dots, b_\ell)$ for G as follows. We don't need to update F-block $\{v_{F_1}, t\}$ and F'-block $\{s, v_{F'}\}$. For every other block b, we define $B(b) = \tilde{B}(b)$. Notice that $b \in \tilde{B}(\tilde{G})$ by Claim ~.

We now show that B is feasible. Let $i \in [\ell]$ and $(u,v) \in E$. We show that inequality (1) is satisfied for $i, (u,v)$. If $(u,v) \in \{(v_F, t), (s, v_{F'})\}$, then $c(u,v) \geq \sum_{P \in \mathcal{P} : (u,v) \in E(P^{\circ} \cup P^{\circ\circ})} d_P$ by assumption (iii)(b).

Now let $(u,v) \in E - \{(v_{F_1}, t), (s, v_{F'})\}$. Hence $(u,v) \in \tilde{E}$.

<u>Claim:</u> Let $i \in [\ell]$ and $e \in E - \{(v_{F_1}, t), (s, v_{F'})\}$. Then:

(i) If $\alpha_F(e, B_{i-1}) = \text{active}$, then $\tilde{\alpha}_{\tilde{F}}(e, \tilde{B}_i) = \text{active}$, and $\alpha_{F'}(e, \overset{B}{B_i}) = \text{inactive}$ for every B, as $e \notin E(F'^{\circ} \cup F'^{\circ\circ})$.

(ii) If $\alpha_{F'}$ $\qquad \tilde{F} \qquad F$

(iii) Let $P \in \mathcal{P} - \{F, F'\}$. Then if $\alpha_P(e, B_i) = \text{active}$, then $\tilde{\alpha}_P(e, \tilde{B}_i) = \text{active}$.

We now have

$$\sum_{P \in \mathcal{P} : \alpha_P((u,v), B_{i-1}) = \text{active or } \alpha_P((u,v), B_i) = \text{active}} d_P \overset{\mathcal{P} = \dots}{=}$$

$$\sum_{P \in \mathcal{P} - \{F, F'\} : \alpha_P((u,v), B_{i-1}) = \text{active or } \alpha_P((u,v), B_i) = \text{active}} d_P + \sum_{P \in \{F, F'\} : \alpha_P((u,v), B_{i-1}) = \text{active or } \alpha_P((u,v), B_i) = \text{active}} d_P \underset{\mathcal{P} = \dots}{\overset{\text{Claim} \sim}{\leq}} d_{\tilde{F}} = d_F =$$

$$\sum_{P \in \tilde{\mathcal{P}} - \{\tilde{F}\} : \tilde{\alpha}_P((u,v), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_P((u,v), \tilde{B}_i) = \text{active}} d_P + \sum_{P \in \{\tilde{F}\} : \tilde{\alpha}_P((u,v), \tilde{B}_{i-1}) = \text{active or } \tilde{\alpha}_P((u,v), \tilde{B}_i) = \text{active}} d_P =$$

$$\sum_{P \in \tilde{\mathcal{P}} : \tilde{\alpha}_P((u,v), B_{i-1}) = \text{active or } \tilde{\alpha}_P((u,v), B_i) = \text{active}} d_P \leq \tilde{B} \text{ feasible}$$

$\tilde{c}(u,v) = (u,v) \in E$

$c(u,v)$

Ch.