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<u>Proof (ctd.)</u>

We now show that there is a satisfying assignment σ for 4CNF formula C iff there is a feasible block sequence B for the corresponding update flow network G, which, by Lemma ~, is the case iff there is a feasible update sequence for G. We will choose σ, B, respectively, such that σ assigns 1 to variable x_j iff $B(b(x^j, \overline{X})) > B(b(s,B))$.

Notation

Let us first determine the set ~~$\mathcal{B}$~~ of blocks of G: See Table ~ for the set of vertices for each flow and Table ~ for the blocks for each flow pair.

We need to remark why we may ignore all other blocks.

<u>„⇐"</u>

~~Let B be a feasible block sequence for G. We define assignment σ as follows: For every variable x_j, we assign 1 to x_j iff $B(b(x^j, \overline{X})) > B(b(s,B))$. We now show that σ is indeed a satisfying assignment for C.~~

Let B be a feasible block sequence for G. Let us first prove some properties of B.

<u>Claim:</u> For every feasible block sequence B for update flow network G, the following holds:

(1) For every $i \in [m]$, $B(b(u^i, L)) < B(b(s,B))$ or $B(b(u^i,R)) < B(b(s,B))$.

(2) (i) $B(\tilde{u}^i_{12}, \tilde{B}) < B(u^i, L)$ and (ii) $B(\tilde{u}^i_{34}, \tilde{B}) < B(u^i, R)$.

(3) (i) $B(u^i_{12}, \tilde{L}) < B(\tilde{u}^i_{12}, \tilde{B})$ or $B(u^i_{12}, \tilde{R}) < B(\tilde{u}^i_{12}, \tilde{B})$ and ... (ii)

(4) (i) For every $j \in [n]$, and ~~for and~~ $i \in [m]$, ~~and $\ell \in [4]$~~, if $C_{i4} = \overline{x_j}$, then $B(x^j_0, B_0) < B(u^i_{12}, \tilde{L})$, and if $C_{i4} = x_j$, then $B(x^j_1, B_1) < B(u^i_{12}, \tilde{L})$. (ii) (iii) (iv)

(5) For every $j \in [n]$, $B(x^j, \overline{X}) < B(x^j_0, B_0)$ and $B(x^j, X) < B(x^j_1, B_1)$.

(6) $B(s,B) < B(x^j, \overline{X})$ or $B(s,B) < B(x^j, X)$

<u>Proof.</u>

(1), (3). We only show (1); the proof for (3) is analogous. Suppose not, that is, there is $i \in [m]$ such that both $B(u^i, L) \geq B(s,B)$ and $B(u^i, R) \geq B(s,B)$. ~~Then Hence~~ We show that the capacity constraint for edge (u^i, v^i) is violated for $B(s,B)$. We have that (a) $\alpha_L((u^i,v^i), B_{B(s,B)-1}) =$ active, since $b(u^i, L) \notin B_{B(s,B)-1}$ and $(u^i, v^i) \in E(L^o)$, (b) $\alpha_R((u^i,v^i), B_{B(s,B)-1}) =$ active, since $b(u^i, R) \notin B_{B(s,B)-1}$ and $(u^i,v^i) \in E(R^o)$, and (c) $\alpha_B((u^i,v^i), B_{B(s,B)}) =$ active, since $b(u^i, B)$ ~~$\in E(B^u)$~~ $= b(s,B) \in B_{B(s,B)}$ and $(u^i,v^i) \in E(B^u)$. Hence ~~$\sum_{P\in\mathcal{P}}$~~ $|\{P \in \mathcal{P} : \alpha_P((u^i,v^i), B_{B(s,B)-1}) = \text{active or } \alpha_P((u^i,v^i), B_{B(s,B)}) = \text{active}\}| \geq |\{L, R, B\}| = 3 > 2 = c(u^i, v^i)$.

Need to remark that this corresponds to the load $[\sum_P d_P]$.

(2), (4), (5) ...

(6) ...

We define assignment σ as follows: For every variable x_i, we assign 1 to x_i iff $B(x^i, \overline{X}) > B(s, B)$.

We now show that σ is indeed a satisfying assignment for C. Let $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$ be a clause.

We show that C_i ~~is satisfied by σ~~ σ satisfies by obtaining a literal in C_i that evaluates to 1. Consider $B(s, B)$.

By Claim ~(1), $B(s,B) > B(u^i, L)$ or $B(s,B) > B(u^i, R)$. We only consider the case $B(s,B) > B(u^i, L)$; the other case is analogous. By Claim ~(2)(i), $B(u^i, L) > B(\alpha^i_{12}, \overline{B})$. By Claim ~(3)(i), $B(\alpha^i_{12}, \overline{B}) > B(u^i_{12}, \tilde{L})$ or $B(\alpha^i_{12}, \overline{B}) > B(u^i_{12}, \tilde{R})$. We only consider the case $B(\alpha^i_{12}, \overline{B}) > B(u^i_{12}, \tilde{R})$; the other case is analogous.

Let ~~variable x_j be such that~~ x_j be the variable corresponding to literal l_{i_2}, that is, $l_{i_2} = x_j$ or $l_{i_2} = \overline{x_j}$.

<u>Case $l_{i_2} = \overline{x_j}$.</u> By Claim ~(4)(ii), $B(u^i_{12}, \tilde{R}) > B(x^j_0, B_0)$. By Claim ~(5)(i), $B(x^j_0, B_0) > B(x^j, \overline{X})$.

Putting everything together, we now have the following chain of inequalities

$B(x^j, \overline{X}) < B(x^j_0, B_0) < B(u^i_{12}, \tilde{R}) < B(\alpha^i_{12}, \overline{B}) < B(u^i, L) < B(s, B)$

Hence, by definition of our assignment σ, ~~we~~ variable x_j is assigned 0. Hence $l_{i_2} = \overline{x_j}$ evaluates to 1.

<u>Case $l_{i_2} = x_j$.</u> By Claim ~(4)(ii), $B(u^i_{12}, \tilde{R}) > B(x^j_1, B_1)$. By Claim ~(5)(ii), $B(x^j_1, B_1) > B(x^j, X)$.

Putting ... $B(x^j, X) < \ldots < B(s, B)$. Hence, by Claim ~(6), $B(x^j, \overline{X}) > B(s, B)$. Hence ...

<u>„$\Rightarrow$"</u>

Let σ be a satisfying assignment for C. We construct a feasible block sequence $B = (b_1, \ldots, b_{11})$ as follows.

The basic idea is to update blocks induced by variable ~~vertices~~ (literal, clause) vertices corresponding to variables ~~that assigned 1 (~~ that are assigned 1 (literals that evaluate to 1, satisfied clauses) before we update block $b(s, B)$ and all other blocks afterwards. We now ~~describe~~ specify $b_1, \ldots, b_{11}$ in detail: How to do this properly?

- b_1: For every variable x_j, if x_j is assigned 1, we add block $b(x^j, X)$ to b_1, otherwise we add $b(x^j, \overline{X})$.

  That is, $b_1 = \ldots$

- b_2: For every variable x_j, if x_j is assigned 1, we add block $b(x^j_1, B_1)$ to b_2, otherwise we add $b(x^j_0, B_0)$.

  That is, $b_2 = \ldots$

- b_3: For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$,
  - if l_{i_1} evaluates to 1, we add $b(u^i_{12}, \tilde{L})$,

    l_{i_2} $b(u^i_{12}, \tilde{R})$,

    l_{i_3} $b(u^i_{34}, \tilde{L})$, and

    l_{i_4} $b(u^i_{34}, \tilde{R})$.

  That is, $b_3 = \ldots$

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Proof (ctd.)

- t_4: For every clause $C_i = (\ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3} \vee \ell_{i_4})$, if ℓ_{i_1} or ℓ_{i_2} evaluate to 1, we add block $b(\bar{x}^i_{12}, \tilde{B})$ to t_4, and if ℓ_{i_3} or ℓ_{i_4} evaluate to 1, we add $b(\bar{x}^i_{34}, \tilde{B})$. That is, ...
- t_5: For every clause $C_i = (\ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3} \vee \ell_{i_4})$, if ℓ_{i_1} or ℓ_{i_2} evaluate to 1, we add block $b(u^i, L)$ to t_5, and if ℓ_{i_3} or ℓ_{i_4} evaluate to 1, we add $b(u^i, R)$. That is, ...
- $t_6 = \{b(s, B)\}$.
- t_7: ...
- t_8: ...
- t_9: ...
- t_{10}: ...
- t_{11}: ...

Reference mentioning above

Observe that all other blocks, that is, blocks not contained in ~~$t_1, \dots, t_{11}$~~ t_ℓ for $\ell = 1, \dots, 11$, neither activate nor deactivate any edges and may thus be added to any ~~of $t_1, \dots, t_{11}$~~ t_ℓ.

We now show that block sequence $B = (t_1, \dots, t_{11})$ is indeed feasible by verifying that Inequality ~ is satisfied for every edge with capacity less than 10 and every $\ell \in [11]$.

- (x^i, y^i): Edge (x^i, y^i) is used by flow pairs $\bar{X}, X, B$.
  - $\sigma(x_i) = 1$.
    - $\bar{X}$: $\alpha_{\bar{X}}((x^i, y^i), B_\ell) =$ inactive ~~for~~ if $\ell < 7$ since $b(x^i, \bar{X}) \in t_7$ and $(x^i, y^i) \in E(\bar{X}^u)$. active if $\ell \geq 7$
    - X: $\alpha_X((x^i, y^i), B_\ell) =$ inactive if $\ell < 1$ since $b(x^i, X) \in t_1$ and $(x^i, y^i) \in E(X^u)$. active if $\ell \geq 1$
    - B: $\alpha_B((x^i, y^i), B_\ell) =$ active if $\ell < 6$ since $b(x^i, B) = b(s, B) \in t_6$ and $(x^i, y^i) \in E(B^o)$. inactive if $\ell \geq 6$

Notation

$$\text{Hence } F((x^i, y^i), B_\ell) = \begin{cases} \{B\} & \text{if } \ell < 1 \\ \{X, B\} & \text{if } 1 \leq \ell < 6 \\ \{X\} & \text{if } \ell = 6 \\ \{X, \bar{X}\} & \text{if } \ell \geq 7 \end{cases}$$

Do we need both? Also overloaded notation.

$$\text{Hence } F((x^i, y^i), B_\ell) = \begin{cases} \{X, B\} & \text{if } \ell < \cancel{6}\,7 \\ \{\bar{X}, X\} & \text{if } \ell \geq 7 \end{cases}$$

Hence $|F((x^i, y^i), B_\ell)| \leq 2 = c(x^i, y^i)$ for every $\ell \in [11]$.

- $\sigma(x_i) = 0$ ...

See Table ~ for a compact overview. This concludes the proof.

~~Thm~~ Corollary: The k-network flow update problem is NP-hard for $k=3$.

Proof. by reduction from 4-SAT.

Given a 4CNF formula C, we apply the reduction used in the proof of Theorem ~ to obtain an update flow network G with ten flow pairs such that there is a satisfying assignment for C iff there is a feasible block sequence B for G. Then, we apply lemma ~ seven times to obtain an update flow network $\tilde{G}$ with three flow pairs such that there is a feasible block sequence B for G iff there is a feasible block sequence $\tilde{B}$ for $\tilde{G}$.

Update flow network G comprises ten flow pairs $\cancel{X, X,}$ $\bar{X}, L, \tilde{L}, X, R, \tilde{R}, B, B_0, B_1, \tilde{B}$, all with demand 1. We apply lemma ~ to $\bar{X}, L, \tilde{L}$, to $X, R, \tilde{R}$, and to $B, B_0, B_1, \tilde{B}$ to reduce the number of flows pairs to three.

~~- $\bar{X}, L, \tilde{L}$~~: Observe that $\cancel{\forall (X \in}$ $\bar{X}, L$, and $\tilde{L}$ do not have any common vertices other than s, t (confirm ~~Table ~ 1. Hence~~

WAITING We need some remark or lemma that generalizes lemma ~ to set of flow pairs that have no common vertices other than s, t (pairwise).