Congestion-Free Network Updates: Algorithms and Complexity

Mitja Daniel Krebs

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Contents

Table 1: Table of notations

$b(v, P)$	The P-block containing vertex v
$B^P(G)$	The set of P-blocks
$B(G)$	The set of blocks
$\mathcal{B}(b)$	The round in which block b is updated
$\mathcal{B}(v, P)$	The round in which block $b(v, P)$ is updated
B_i	The set of blocks updated before or in the i-th round

Part I

Preliminaries

Notation 1. For a flow pair P and a vertex $v \in V(P)$, $b(v, P)$ denotes the P-block containing v.

Notation 2. For an update flow network G and a flow pair P, $B^P(G)$ denotes the set of P-blocks.

Notation 3. For an update flow network G, $B(G) = \bigcup_{P\in\mathcal{P}} B^P(G)$

Definition 4. A *block sequence* $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ is an ordered partition of the set of blocks.

Notation 5. For a block sequence $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ and a block b, $\mathcal{B}(b)$ denotes the index $i \in [\ell]$ such that b is contained in $\mathscr{B}_i$.

Notation 6. For a block sequence $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$, a flow pair P, and a vertex $v \in V(P)$, $\mathcal{B}(v, P) = \mathcal{B}(b(v, P))$ denotes the index $i \in [\ell]$ such that block $b(v, P)$ is contained in $\mathscr{B}_i$.

Notation 7. For a block sequence $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ and an index $i \in [\ell]$, $B_i = \bigcup_{j\le i} \mathscr{B}_j$ denotes the set of blocks updated before or in the i-th round.

Definition 8. Let $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ be a block sequence. For a flow pair P, an edge $(u, v) \in E(P^o \cup P^u)$, and an index $i \in [\ell]$, the *activation label* $\alpha_P((u, v), B_i)$ is defined as follows:

$$\alpha_P((u,v),B_i) = \begin{cases} \text{active} & \text{if } (u,v) \in E(P^o) \text{ and } b(u,P) \notin B_i \\ \text{active} & \text{if } (u,v) \in E(P^u) \text{ and } b(u,P) \in B_i \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Lemma 9. *Let $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ be a block sequence, P be a flow pair, $(u, v) \in E(P^o \cup P^u)$, and $i \in [\ell]$. Then:*

1. *If $(u, v) \in E(P^o \setminus P^u)$, then*
$$\alpha_P((u,v),B_i) = \begin{cases} \text{active} & i < \mathcal{B}(u,P) \\ \text{inactive} & i \ge \mathcal{B}(u,P). \end{cases}$$
2. *If $(u, v) \in E(P^o \cap P^u)$, then $\alpha_P((u, v)B_i) = \text{active}$.*
3. *If $(u, v) \in E(P^u \setminus P^o)$, then*
$$\alpha_P((u,v),B_i) = \begin{cases} \text{active} & i \ge \mathcal{B}(u,P) \\ \text{inactive} & i < \mathcal{B}(u,P). \end{cases}$$

Definition 10. A block sequence $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_\ell)$ is *feasible* if for every edge e and every index $i \in [\ell]$,

$$c(e) \ge \sum\nolimits_{\mathrm{P\in P:\alpha_P(e,B_{i\text{-}1})=active\ or\ \alpha_P(e,B_i)=active}} d_{\mathrm{P}}, \tag{1}$$

where we define $\mathscr{B}_0 = \emptyset$.

Remark 11. Let G be an update flow network with unit demand equal to 1, that is, $d_P = 1$ for every flow pair P, and let $\mathcal{B} = (\mathscr{B}_1, \ldots, \mathscr{B}_\ell)$ be a block sequence. Then, for every edge e and every index $i \in [\ell]$, capacity constraint 1 simplifies to:

$$\begin{aligned} c(e) &\geq \sum_{P \in \mathcal{P} : \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} d_P \\ &= \sum_{P \in \mathcal{P} : \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}} 1 \\ &= |\{P \in \mathcal{P} \mid \alpha_P(e, B_{i-1}) = \text{active or } \alpha_P(e, B_i) = \text{active}\}|. \end{aligned}$$

Corollary 12. *There is a feasible block sequence iff there is a feasible update sequence.*

Part II

NP-Hardness for $k = 3$

The goal of this section is to prove the following theorem.

Theorem 13. *The k-network flow update problem is $\boldsymbol{NP}$-hard for $k = 3$.*

We will prove this theorem in two steps. First, we will prove the following theorem.

Theorem 14. *The k-network flow update problem, where every edge is used by at most three flow pairs, is $\boldsymbol{NP}$-hard for $k = 10$.*

Then, we will (repeatedly) apply the following lemma to the flow update network we will have constructed in the proof of Theorem 14 to reduce the number of flow pairs from 10 to 3.

Lemma 15. *Hehe.*

Chapter 1

NP-Hardness for the Special Case

The proof of Theorem 14 is via reduction from 4-SAT and is based on the **NP**-hardness proof for $k = 6$ in (Amiri, Saeed A. and Dudycz, Szymon and Parham, Mahmoud and Schmid, Stefan and Wiederrecht, Sebastian, 2019).

1.1 The Reduction

Let C be a 4CNF formula with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. W.l.o.g. every variable occurs both positively and negatively (otherwise, if a variable x_j occurs only positively (negatively), we can assign 1 (0) to x_j and remove all clauses containing literal x_j ($\bar{x}_j$)). We construct the corresponding update flow network G as follows. First, we introduce a *clause gadget* for each clause and a *variable gadget* for each variable. Then, we connect the variable and clause gadgets. Finally, we take the remaining steps necessary to ensure that G is indeed a feasible update flow network.

Clause gadgets. Let $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$ be a clause. We construct the corresponding clause gadget C^i as follows. The idea is to model the syntax tree for C_i depicted in Figure 1.1.

For the root operator node, we introduce a *clause vertex* u^i which is used by three flow pairs L, R, B. The idea is to guarantee that clause C_i is satisfied iff block $b(u^i, L)$ is updated before block $b(u^i, B)$ or block $b(u^i, R)$ is updated before $b(u^i, B)$. Equivalently, $b(u^i, B)$ cannot be updated unless at least one of $b(u^i, L), b(u^i, R)$ has been updated. Intuitively, if $b(u^i, L)$ ($b(u^i, R)$) is updated before $b(u^i, B)$, then the **L**eft half $(l_{i_1} \vee l_{i_2})$ (**R**ight half $(l_{i_3} \vee l_{i_4})$) of C_i is satisfied.

Similarly, for the intermediate operator nodes of the syntax tree, we introduce clause vertices $u^i_{1,2}, u^i_{3,4}$, where $u^i_{1,2}$ corresponds to $(l_{i_1} \vee l_{i_2})$ and $u^i_{3,4}$

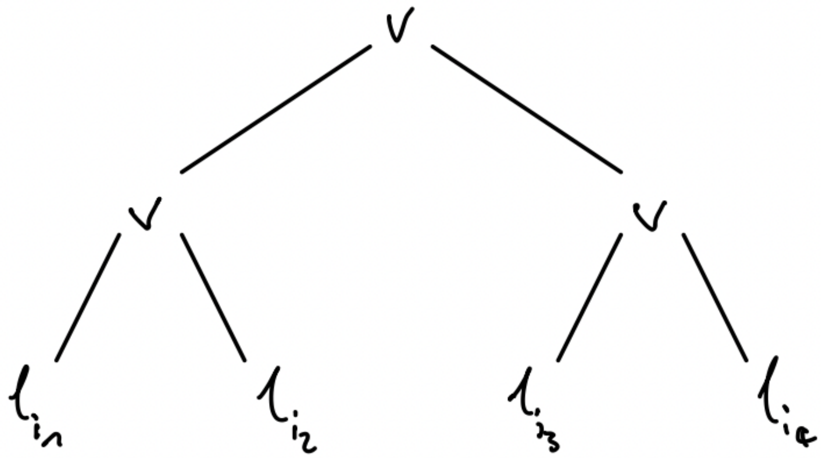

Figure 1.1: A syntax tree for clause $(l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$

corresponds to $(l_{i_3} \vee l_{i_4})$. Both clause vertices are used by flow pairs $\tilde{L}, \tilde{R}, \tilde{B}$ such that if $b(u^i_{1,2}, \tilde{L})$ ($b(u^i_{1,2}, \tilde{R})$) is updated before $b(u_{1,2}, \tilde{B})$, then the left half l_{i_1} (right half l_{i_2}) of $(l_{i_1} \vee l_{i_2})$ is satisfied, and analogously for $u^i_{3,4}$.

Moreover, for the operand nodes of the syntax tree, we introduce *literal vertices* $u^i_1, u^i_2, u^i_3, u^i_4$.

Finally, for every branch from a parent node to its left (right) child node, we add an edge to either L (R) (if the parent node is u^i) or $\tilde{L}$ ($\tilde{R}$) (if the parent node is $u^i_{1,2}$ or $u^i_{3,4}$).

We now proceed with the detailed specification of clause gadget C^i (see Figure 1.2).

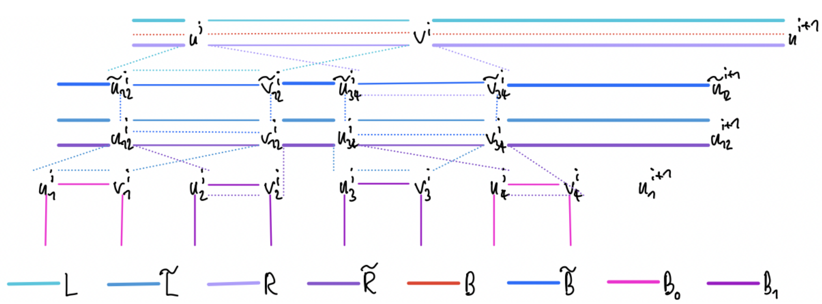

Figure 1.2: Clause gadget C^i

We introduce six flow pairs $L, R, B, \tilde{L}, \tilde{R}, \tilde{B}$, each with demand 1.

For the clause vertices, we introduce two vertices u^i, v^i and add edge (u^i, v^i) to flows L^o, R^o, B^u. Similarly, we introduce vertices $u^i_{1,2}, v^i_{1,2}, u^i_{3,4}, v^i_{3,4}$ and add edges $(u^i_{1,2}, v^i_{1,2}), (u^i_{3,4}, v^i_{3,4})$ to flows $\tilde{L}^o, \tilde{R}^o, \tilde{B}^u$.

For the literal vertices, we introduce vertices $u^i_1, v^i_1, u^i_2, v^i_2, u^i_3, v^i_3, u^i_4, v^i_4$ and add edges $(u^i_1, v^i_1), (u^i_3, v^i_3)$ to flow $\tilde{L}^u$ and $(u^i_2, v^i_2), (u^i_4, v^i_4)$ to $\tilde{R}^u$.

Moreover, we introduce auxiliary vertices $\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}, \tilde{u}^i_{3,4}, \tilde{v}^i_{3,4}$ and add edge $(\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2})$ to flows $\tilde{L}^u, \tilde{B}^o$ and $(\tilde{u}^i_{3,4}, \tilde{v}^i_{3,4})$ to $\tilde{R}^u, \tilde{B}^o$.

Finally, we add the following edges to connect clause gadget C^i:

- $(u^i, \tilde{u}^i_{1,2}), (\tilde{v}^i_{1,2}, v^i)$ to L^u
- $(u^i, \tilde{u}^i_{3,4}), (\tilde{v}^i_{3,4}, v^i)$ to R^u
- $(v^i_{1,2}, u^i_{3,4})$ to $\tilde{L}^o, \tilde{L}^u, \tilde{R}^o, \tilde{R}^u$
- $(u^i_{1,2}, u^i_1), (v^i_1, v^i_{1,2}), (u^i_{3,4}, u^i_3), (v^i_3, v^i_{3,4})$ to $\tilde{L}^u$
- $(u^i_{1,2}, u^i_2), (v^i_2, v^i_{1,2}), (u^i_{3,4}, u^i_4), (v^i_4, v^i_{3,4})$ to $\tilde{R}^u$
- $(\tilde{v}^i_{1,2}, \tilde{u}^i_{3,4})$ to $\tilde{B}^o, \tilde{B}^u$
- $(\tilde{u}^i_{1,2}, u^i_{1,2}), (v^i_{1,2}, \tilde{v}^i_{1,2}), (\tilde{u}^i_{3,4}, u^i_{3,4}), (v^i_{3,4}, \tilde{v}^i_{3,4})$ to $\tilde{B}^u$

Variable gadgets. For every variable x_j, we construct the corresponding variable gadget X^j as follows. We introduce a *variable vertex* x^j which is used by three flow pairs $X, \bar{X}, B$. The idea is to guarantee the following:

1. If block $b(x^j, X)$ is updated before block $b(x^j, B)$, then variable x_j is assigned 1.
2. If block $b(x^j, \bar{X})$ is updated before $b(x^j, B)$, then x_j is assigned 0.
3. Not both $b(x^j, X)$ and $b(x^j, \bar{X})$ can be updated before $b(x^j, B)$.

We now proceed with the detailed specification of variable gadget X^j (see Figure 1.3).

We introduce two flow pairs $X, \bar{X}$, each with demand 1. For the variable vertices, we introduce vertices x^j, y^j and add edge (x^j, y^j) to flows $X^u, \bar{X}^u, B^o$. Moreover, we introduce auxiliary vertices $x^j_0, y^j_0, x^j_1, y^j_1$ and add edge (x^j_0, y^j_0) to flow $\bar{X}^o$ and (x^j_1, y^j_1) to X^o. Finally, to connect variable gadget X^j, we add edges $(x^j, x^j_0), (y^j_0, y^j)$ to flow $\bar{X}^o$ and $(x^j, x^j_1), (y^j_1, y^j)$ to X^o.

Connecting variable with clause gadgets. For every $j \in [n]$ and every $i \in [m]$, we connect variable gadget X^j to clause gadget C^i if variable x_j occurs in clause C_i. More precisely, we introduce two flow pairs B_0, B_1, each with demand 1, such that B_0 (B_1) connects vertex x^j_0 (x^j_1) to all literal vertices corresponding to literal $\bar{x}_j$ (x_j).

More formally, for every $j \in [n]$, let $P_j = \{p^j_1, \ldots, p^j_{\ell_j}\}$ denote the set of indices of the clauses containing literal x_j and $\bar{P}_j = \{\bar{p}^j_1, \ldots, p^j_{\ell'_j}\}$ denote the set of indices of the clauses containing literal $\bar{x}_j$. Moreover, for every $j \in [n]$ and

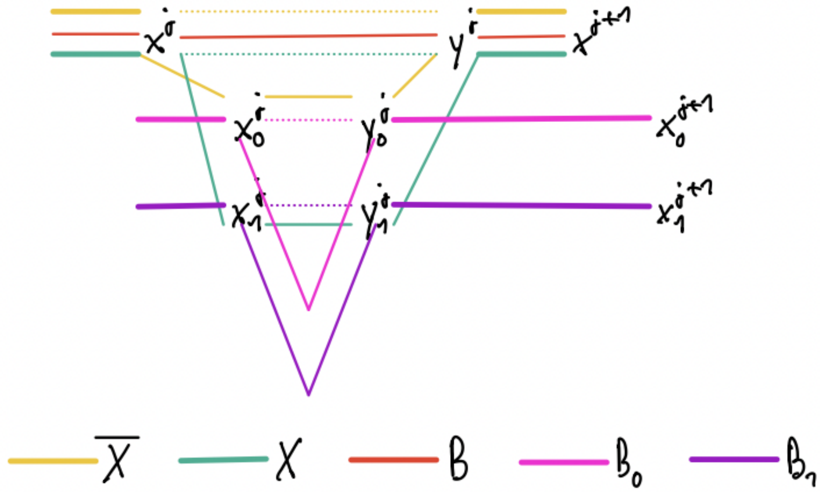

Figure 1.3: Variable gadget X^j

every $i \in [m]$, let $\pi(i,j)$ denote the position of literal x_j in clause C_i and $\bar{\pi}(i,j)$ denote the position of literal $\bar{x}_j$ in C_i. For every $j \in [n]$, we add the following edges:

- $(x_0^j, u^{\bar{p}_1^j}_{\bar{\pi}(\bar{p}_1^j,j)})$, $(u^{\bar{p}_\ell^j}_{\bar{\pi}(\bar{p}_\ell^j,j)}, v^{\bar{p}_\ell^j}_{\bar{\pi}(\bar{p}_\ell^j,j)})$ for every $\ell \in [\ell'_j]$, $(v^{\bar{p}_\ell^j}_{\bar{\pi}(\bar{p}_\ell^j,j)}, u^{\bar{p}_{\ell+1}^j}_{\bar{\pi}(\bar{p}_{\ell+1}^j,j)})$ for every $\ell \in [\ell'_j - 1]$, and $(v^{\bar{p}_{\ell'_j}^j}_{\bar{\pi}(\bar{p}_{\ell'_j}^j,j)}, y_0^j)$ to B_0^o
- $(x_1^j, u^{p_1^j}_{\pi(p_1^j,j)})$, $(u^{p_\ell^j}_{\pi(p_\ell^j,j)}, v^{p_\ell^j}_{\pi(p_\ell^j,j)})$ for every $\ell \in [\ell_j]$, $(v^{p_\ell^j}_{\pi(p_\ell^j,j)}, u^{p_{\ell+1}^j}_{\pi(p_{\ell+1}^j,j)})$ for every $\ell \in [\ell_j - 1]$, and $(v^{p_{\ell_j}^j}_{\pi(p_{\ell_j}^j,j)}, y_1^j)$ to B_1^o

Completing the update flow network. We introduce vertices s, t and create (s,t)-paths for all flows by adding the following edges:

- $(s, u^1), (v^m, t)$ to L^o, L^u, R^o, R^u
- (v^i, u^{i+1}) for every $i \in [m-1]$ to L^o, L^u, R^o, R^u, B^u
- $(s, u^1_{1,2})$, $(v^i_{3,4}, u^{i+1}_{1,2})$ for every $i \in [m-1]$, and $(v^m_{3,4}, t)$ to $\tilde{L}^o, \tilde{L}^u, \tilde{R}^o, \tilde{R}^u$
- $(s, \tilde{u}^1_{1,2})$, $(\tilde{v}^i_{3,4}, \tilde{u}^{i+1}_{1,2})$ for every $i \in [m-1]$, and $(\tilde{v}^m_{3,4}, t)$ to $\tilde{B}^o, \tilde{B}^u$
- $(s, x^1), (y^n, t)$ to $X^o, X^u, \bar{X}^o, \bar{X}^u, B^o, B^u$
- (y^j, x^{j+1}) for every $j \in [n-1]$ to $X^o, X^u, \bar{X}^o, \bar{X}^u, B^o$

- $(x^1, u^1), (v^m, y^n)$ to B^u
- (s, x_0^1), (y_0^j, x_0^{j+1}) for every $j \in [n-1]$, and (y_0^n, t) to B_0^o, B_0^u
- (s, x_1^1), (y_1^j, x_1^{j+1}) for every $j \in [n-1]$, and (y_1^n, t) to B_1^o, B_1^u

See Figure 1.4 for the complete update flow network, Table 1.1 for all (s,t)-flows, and Table 1.2 for the set of blocks grouped by flow pair.

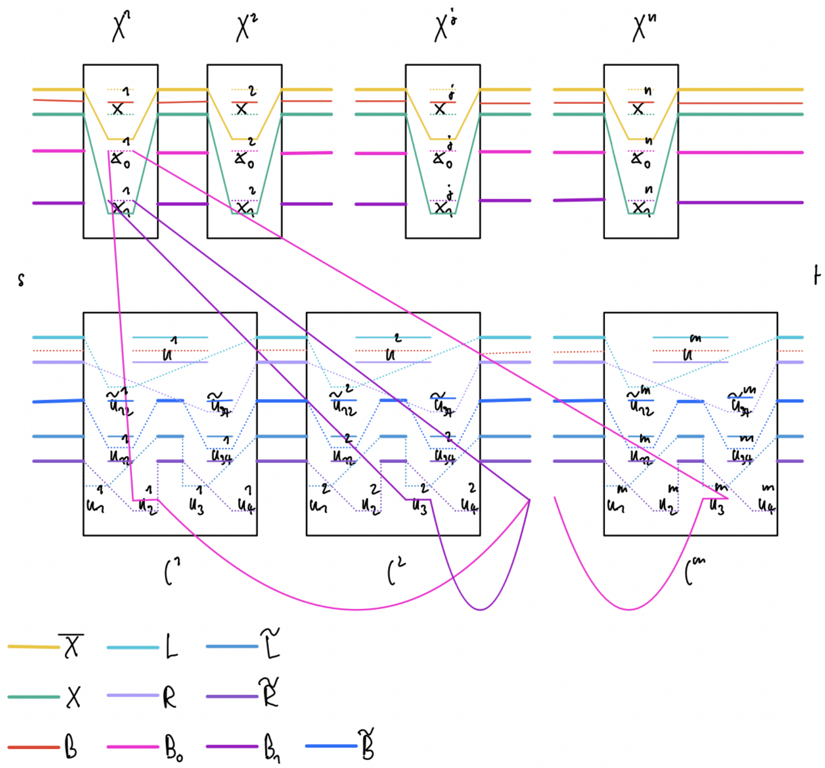

Figure 1.4: The update flow network

Edge capacities are defined as follows.

- We set the capacity to 2 for edges $(u^i, v^i), (u^i_{1,2}, v^i_{1,2}), (u^i_{3,4}, v^i_{3,4}), (x^j, y^j)$ for every $i \in [m]$ and every $j \in [n]$.
- We set the capacity to 1 for edges $(u^i_1, v^i_1), (u^i_2, v^i_2), (u^i_3, v^i_3), (u^i_4, v^i_4), (\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}), (\tilde{u}^i_{3,4}, \tilde{v}^i_{3,4}), (x^j_0, y^j_0), (x^j_1, y^j_1)$ for every $i \in [m]$ and every $j \in [n]$.
- All remaining edge capacities are set to 10, that is, the number of flow pairs, which equals the sum of all demands.

Table 1.1: All (s,t)-flows

Flow	(s,t)-path
$\bar{X}^o$	$s, x^1, x_0^1, y_0^1, y^1, x^2, \ldots, y^n, t$
$\bar{X}^u$	$s, x^1, y^1, x^2, \ldots, y^n, t$
L^o	$s, u^1, v^1, u^2, \ldots, v^m, t$
L^u	$s, u^1, \tilde{u}_{1,2}^1, \tilde{v}_{1,2}^1, v^1, u^2, \ldots, v^m, t$
$\tilde{L}^o$	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \ldots, v_{3,4}^m, t$
$\tilde{L}^u$	$s, u_{1,2}^1, u_1^1, v_1^1, v_{1,2}^1, u_{3,4}^1, u_3^1, v_3^1, v_{3,4}^1, u_{1,2}^2, \ldots, v_{3,4}^m, t$
X^o	$s, x^1, x_1^1, y_1^1, y^1, x^2, \ldots, y^n, t$
X^u	$s, x^1, y^1, x^2, \ldots, y^n, t$
R^o	$s, u^1, v^1, u^2, \ldots, v^m, t$
R^u	$s, u^1, \tilde{u}_{3,4}^1, \tilde{v}_{3,4}^1, v^1, u^2, \ldots, v^m, t$
$\tilde{R}^o$	$s, u_{1,2}^1, v_{1,2}^1, u_{3,4}^1, v_{3,4}^1, u_{1,2}^2, \ldots, v_{3,4}^m, t$
$\tilde{R}^u$	$s, u_{1,2}^1, u_2^1, v_2^1, v_{1,2}^1, u_{3,4}^1, u_4^1, v_4^1, v_{3,4}^1, u_{1,2}^2, \ldots, v_{3,4}^m, t$
B^o	$s, x^1, y^1, x^2, \ldots, y^n, t$
B^u	$s, x^1, u^1, v^1, u^2, \ldots, v^m, y^n, t$
$\tilde{B}^o$	$s, \tilde{u}_{1,2}^1, \tilde{v}_{1,2}^1, \tilde{u}_{3,4}^1, \tilde{v}_{3,4}^1, \tilde{u}_{1,2}^2, \ldots, \tilde{v}_{3,4}^m, t$
$\tilde{B}^u$	$s, \tilde{u}_{1,2}^1, u_{1,2}^1, v_{1,2}^1, \tilde{v}_{1,2}^1, \tilde{u}_{3,4}^1, u_{3,4}^1, v_{3,4}^1, \tilde{v}_{3,4}^1, \tilde{u}_{1,2}^2, \ldots, \tilde{v}_{3,4}^m, t$
B_0^o	$s, x_0^1, u_{\bar{\pi}(\bar{p}_1^1,1)}^{\bar{p}_1^1}, v_{\bar{\pi}(\bar{p}_1^1,1)}^{\bar{p}_1^1}, u_{\bar{\pi}(\bar{p}_2^1,1)}^{\bar{p}_2^1}, \ldots, v_{\bar{\pi}(\bar{p}_{l_1'}^1,1)}^{\bar{p}_{l_1'}^1}, y_0^1, x_0^2, \ldots, y_0^n, t$
B_0^u	$s, x_0^1, y_0^1, x_0^2, \ldots, y_0^n, t$
B_1^o	$s, x_1^1, u_{\pi(p_1^1,1)}^{p_1^1}, v_{\pi(p_1^1,1)}^{p_1^1}, u_{\pi(p_2^1,1)}^{p_2^1}, \ldots, v_{\pi(p_{l_1}^1,1)}^{p_{l_1}^1}, y_0^1, x_0^2, \ldots, y_0^n, t$
B_1^u	$s, x_1^1, y_1^1, x_1^2, \ldots, y_1^n, t$

We remark that vertices $\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}, \tilde{u}^i_{3,4}, \tilde{v}^i_{3,4}$ are not necessary for this proof. Instead, we could directly connect clause vertices $u^i, u^i_{1,2}$ via flow pair L and $u^i, u^i_{3,4}$ via R. Similarly, vertices $x^j_0, y^j_0, x^j_1, y^j_1$ as well as flow pairs B_0, B_1 are not necessary. We could instead directly connect variable vertex x^j to literal vertex, say u^i_1, via X ($\bar{X}$) if $l_{i_1} = x_j$ ($l_{i_1} = \bar{x}_j$). The vertices and flow pairs are necessary, however, for the proof of Theorem 13.

Let us quickly verify that G is indeed a feasible update flow network.

☐ Feasibility.

1.2 The Proof

Before we prove Theorem 14, let us show that every feasible block sequence for the update flow network specified in the previous section satisfies the following properties.

Lemma 16. *Let $\mathcal{B}$ be a feasible block sequence for update flow network G. Then:*

1. *For every $i \in [m]$, $\mathcal{B}(u^i, L) < \mathcal{B}(x^1, B)$ or $\mathcal{B}(u^i, R) < \mathcal{B}(x^1, B)$.*
2. *For every $i \in [m]$,*
   (a) *$\mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) < \mathcal{B}(u^i, L)$, and*
   (b) *$\mathcal{B}(\tilde{u}^i_{3,4}, \tilde{B}) < \mathcal{B}(u^i, R)$.*
3. *For every $i \in [m]$,*
   (a) *$\mathcal{B}(u^i_{1,2}, \tilde{L}) < \mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B})$ or $\mathcal{B}(u^i_{1,2}, \tilde{R}) < \mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B})$, and*
   (b) *$\mathcal{B}(u^i_{3,4}, \tilde{L}) < \mathcal{B}(\tilde{u}^i_{3,4}, \tilde{B})$ or $\mathcal{B}(u^i_{3,4}, \tilde{R}) < \mathcal{B}(\tilde{u}^i_{3,4}, \tilde{B})$.*
4. *For every $j \in [n]$, $\mathcal{B}(x^1, B) < \mathcal{B}(x^j, \bar{X})$ or $\mathcal{B}(x^1, B) < \mathcal{B}(x^j, X)$.*
5. *For every $i \in [m]$ and every $j \in [n]$,*
   (a) *if $l_{i_1} = \bar{x}_j$, then $\mathcal{B}(x^j_0, B_0) < \mathcal{B}(u^i_{1,2}, \tilde{L})$, and if $l_{i_1} = x_j$, then $\mathcal{B}(x^j_1, B_1) < \mathcal{B}(u^i_{1,2}, \tilde{L})$,*
   (b) *if $l_{i_2} = \bar{x}_j$, then $\mathcal{B}(x^j_0, B_0) < \mathcal{B}(u^i_{1,2}, \tilde{R})$, and if $l_{i_2} = x_j$, then $\mathcal{B}(x^j_1, B_1) < \mathcal{B}(u^i_{1,2}, \tilde{R})$,*
   (c) *if $l_{i_3} = \bar{x}_j$, then $\mathcal{B}(x^j_0, B_0) < \mathcal{B}(u^i_{3,4}, \tilde{L})$, and if $l_{i_3} = x_j$, then $\mathcal{B}(x^j_1, B_1) < \mathcal{B}(u^i_{3,4}, \tilde{L})$,*
   (d) *if $l_{i_4} = \bar{x}_j$, then $\mathcal{B}(x^j_0, B_0) < \mathcal{B}(u^i_{3,4}, \tilde{R})$, and if $l_{i_4} = x_j$, then $\mathcal{B}(x^j_1, B_1) < \mathcal{B}(u^i_{3,4}, \tilde{R})$.*
6. *For every $j \in [n]$,*

Table 1.2: All blocks grouped by flow pair

P	$V(P^o \cap P^u)$ ordered w.r.t. $\leq_{P^o \cup P^u}$	$B^P(G)$
$\bar{X}$	$s, x^1, y^1, x^2, \dots, y^n, t$	$\{s, x^1\}$, $\{x^j, x^j_0, y^j_0, y^j\}, j \in [n]$, $\{y^j, x^{j+1}\}, j \in [n-1]$, $\{y^n, t\}$
L	$s, u^1, v^1, u^2, \dots, v^m, t$	$\{s, u^1\}$, $\{u^i, \tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}, v^i\}, i \in [m]$, $\{v^i, u^{i+1}\}, i \in [m-1]$, $\{v^m, t\}$
$\tilde{L}$	$s, u^1_{1,2}, v^1_{1,2}, u^1_{3,4}, v^1_{3,4}, u^2_{1,2}, \dots, v^m_{3,4}, t$	$\{s, u^1_{1,2}\}$, $\{u^i_{1,2}, u^i_1, v^i_1, v^i_{1,2}\}, i \in [m]$, $\{v^i_{1,2}, u^i_{3,4}\}, i \in [m]$, $\{u^i_{3,4}, u^i_3, v^i_3, v^i_{3,4}\}, i \in [m]$, $\{v^i_{3,4}, u^{i+1}_{1,2}\}, i \in [m-1]$, $\{v^m_{3,4}, t\}$
X	$s, x^1, y^1, x^2, \dots, y^n, t$	$\{s, x^1\}$, $\{x^j, x^j_1, y^j_1, y^j\}, j \in [n]$, $\{y^j, x^{j+1}\}, j \in [n-1]$, $\{y^n, t\}$
R	$s, u^1, v^1, u^2, \dots, v^m, t$	$\{s, u^1\}$, $\{u^i, \tilde{u}^i_{3,4}, \tilde{v}^i_{3,4}, v^i\}, i \in [m]$, $\{v^i, u^{i+1}\}, i \in [m-1]$, $\{v^m, t\}$
$\tilde{R}$	$s, u^1_{1,2}, v^1_{1,2}, u^1_{3,4}, v^1_{3,4}, u^2_{1,2}, \dots, v^m_{3,4}, t$	$\{s, u^1_{1,2}\}$, $\{u^i_{1,2}, u^i_2, v^i_2, v^i_{1,2}\}, i \in [m]$, $\{v^i_{1,2}, u^i_{3,4}\}, i \in [m]$, $\{u^i_{3,4}, u^i_4, v^i_4, v^i_{3,4}\}, i \in [m]$, $\{v^i_{3,4}, u^{i+1}_{1,2}\}, i \in [m-1]$, $\{v^m_{3,4}, t\}$
B	s, x^1, y^n, t	$\{s, x^1\}$, $\{x^j, y^j, u^i, v^i \mid j \in [n], i \in [m]\}$, $\{y^n, t\}$
$\tilde{B}$	$s, \tilde{u}^1_{1,2}, \tilde{v}^1_{1,2}, \tilde{u}^1_{3,4}, \tilde{v}^1_{3,4}, \tilde{u}^2_{1,2}, \dots, \tilde{v}^m_{3,4}, t$	$\{s, \tilde{u}^1_{1,2}\}$, $\{\tilde{u}^i_{1,2}, u^i_{1,2}, v^i_{1,2}, \tilde{v}^i_{1,2}\}, i \in [m]$, $\{\tilde{v}^i_{1,2}, \tilde{u}^i_{3,4}\}, i \in [m]$, $\{\tilde{u}^i_{3,4}, u^i_{3,4}, v^i_{3,4}, \tilde{v}^i_{3,4}\}, i \in [m]$, $\{\tilde{v}^i_{3,4}, \tilde{u}^{i+1}_{1,2}\}, i \in [m-1]$, $\{\tilde{v}^m_{3,4}, t\}$
B_0	$s, x^1_0, y^1_0, x^2_0, \dots, y^n_0, t$	$\{s, x^1_0\}$, $\{x^j_0, u^i_{\bar{\pi}(i,j)}, v^i_{\bar{\pi}(i,j)}, y^j_0 \mid i \in \bar{P}_j\}, j \in [n]$, $\{y^n_0, t\}$
B_1	$s, x^1_1, y^1_1, x^2_1, \dots, y^n_1, t$	$\{s, x^1_1\}$, $\{x^j_1, u^i_{\pi(i,j)}, v^i_{\pi(i,j)}, y^j_1 \mid i \in P_j\}, j \in [n]$, $\{y^n_1, t\}$

(a) $\mathcal{B}(x^j, \bar{X}) < \mathcal{B}(x_0^j, B_0)$*, and*

(b) $\mathcal{B}(x^j, X) < \mathcal{B}(x_1^j, B_1)$*.*

Proof. We show every property by contradiction. More precisely, for every property, we assume it doesn't hold and then obtain an edge and a round such that the corresponding capacity constraint is violated, which contradicts the feasibility of block sequence $\mathcal{B}$.

Since every flow pair has demand 1, we may use 11 to argue about capacity constraints.

1, 3. We only show 1; the proofs for 3a and 3b are analogous. Suppose not. Then obtain $i \in [m]$ such that both $\mathcal{B}(u^i, L) \geq \mathcal{B}(x^1, B)$ and $\mathcal{B}(u^i, R) \geq \mathcal{B}(x^1, B)$. We show that the capacity constraint for edge (u^i, v^i) is violated for round $\mathcal{B}(x^1, B)$.

We have that

1. $\alpha_L((u^i, v^i), B_{\mathcal{B}(x^1,B)-1}) = \text{active}$, since $b(u^i, L) \notin B_{\mathcal{B}(x^1,B)-1}$ and $(u^i, v^i) \in E(L^o)$,
2. $\alpha_R((u^i, v^i), B_{\mathcal{B}(x^1,B)-1}) = \text{active}$, since $b(u^i, R) \notin B_{\mathcal{B}(x^1,B)-1}$ and $(u^i, v^i) \in E(R^o)$, and
3. $\alpha_B((u^i, v^i), B_{\mathcal{B}(x^1,B)}) = \text{active}$, since $b(u^i, B) = b(x^1, B) \in B_{\mathcal{B}(x^1,B)}$ and $(u^i, v^i) \in E(B^u)$.

Hence

$$\begin{aligned}|\{P \in \mathcal{P} \,|\alpha_P((u^i, v^i), B_{\mathcal{B}(x^1,B)-1}) &= \text{active or} \\ \alpha_P((u^i, v^i), B_{\mathcal{B}(x^1,B)}) &= \text{active}\}| \geq |\{L, R, B\}| = 3 > 2 = c(u^i, v^i)\end{aligned}$$

2, 5, 6. We only show 2a; the proofs for 2b, 5a, 5b, 5c, 5d, 6a, and 6b are similar. Suppose not. Then obtain $i \in [m]$ such that $\mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) \geq \mathcal{B}(u^i, L)$. We show that the capacity constraint for edge $(\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2})$ is violated for round $\mathcal{B}(u^i, L)$.

We have that

1. $\alpha_{\tilde{B}}((\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}), B_{\mathcal{B}(u^i,L)-1}) = \text{active}$, since $b(\tilde{u}^i_{1,2}, \tilde{B}) \notin B_{\mathcal{B}(u^i,L)-1}$ and $(\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}) \in E(\tilde{B}^o)$, and
2. $\alpha_L((\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}), B_{\mathcal{B}(u^i,L)}) = \text{active}$, since $b(u^i, L) \in B_{\mathcal{B}(u^i,L)}$ and $(\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}) \in E(L^u)$.

Hence

$$\begin{aligned}|\{P \in \mathcal{P} \,|\alpha_P((\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}), B_{\mathcal{B}(u^i,L)-1}) &= \text{active or} \\ \alpha_P((\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2}), B_{\mathcal{B}(u^i,L)}) &= \text{active}\}| \geq |\{\tilde{B}, L\}| = 2 > 1 = c(\tilde{u}^i_{1,2}, \tilde{v}^i_{1,2})\end{aligned}$$

4. Suppose not. Then obtain $j \in [n]$ such that both $\mathcal{B}(x^1, B) \geq \mathcal{B}(x^j, \bar{X})$ and $\mathcal{B}(x^1, B) \geq \mathcal{B}(x^j, X)$. We show that the capacity constraint for edge (x^j, y^j) is violated for round $\mathcal{B}(x^1, B)$.

We have that

1. $\alpha_B((x^j, y^j), B_{\mathcal{B}(x^1,B)-1}) =$ active, since $b(x^j, B) = b(x^1, B) \notin B_{\mathcal{B}(x^1,B)-1}$ and $(x^j, y^j) \in E(B^o)$,
2. $\alpha_{\bar{X}}((x^j, y^j), B_{\mathcal{B}(x^1,B)}) =$ active, since $b(x^j, \bar{X}) \notin B_{\mathcal{B}(x^1,B)}$ and $(x^j, y^j) \in E(\bar{X}^u)$, and
3. $\alpha_X((x^j, y^j), B_{\mathcal{B}(x^1,B)}) =$ active, since $b(x^j, X) \notin B_{\mathcal{B}(x^1,B)}$ and $(x^j, y^j) \in E(X^u)$.

Hence

$$
\begin{aligned}
|\{P \in \mathcal{P} \, | \alpha_P((x^j, y^j), B_{\mathcal{B}(x^1,B)-1}) &= \text{active or} \\
\alpha_P((x^j, y^j), B_{\mathcal{B}(x^1,B)}) &= \text{active}\}| \geq |\{B, \bar{X}, X\}| = 3 > 2 = c(x^j, y^j)
\end{aligned}
$$

□

We are now ready to prove Theorem 14.

Proof of Theorem [[thm:np-hardness-special-case.]] We show that there is a satisfying assignment σ for 4CNF formula C iff there is a feasible block sequence $\mathcal{B}$ for the corresponding update flow network G, which, by Corollary 12, is the case iff there is a feasible update sequence for G. We will choose σ, $\mathcal{B}$, respectively, such that σ assigns 1 to variable x_j iff $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$.

Only-if part. Let $\mathcal{B}$ be a feasible block sequence for G. We define assignment σ as follows: For every variable x_j, we assign 1 to x_j iff $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$. We now show that σ is indeed a satisfying assignment for C.

Let $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$ be a clause. We show that σ satisfies C_i by obtaining a literal that evaluates to 1.

Consider round $\mathcal{B}(x^1, B)$. By Lemma 16 1, $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L)$ or $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, R)$. We only consider the former case $\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L)$; the latter one is analogous.

By Lemma 16 2a, $\mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B})$. By Lemma 16 3a, $\mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) > \mathcal{B}(u^i_{1,2}, \tilde{L})$ or $\mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) > \mathcal{B}(u^i_{1,2}, \tilde{R})$. We only consider the latter case $\mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) > \mathcal{B}(u^i_{1,2}, \tilde{R})$; the former one is analogous.

Let x_j be the variable corresponding to literal l_{i_2}. We consider the cases $l_{i_2} = \bar{x}_j$ and $l_{i_2} = x_j$ separately.

Case $l_{i_2} = \bar{x}_j$. By Lemma 16 5b, $\mathcal{B}(u^i_{1,2}, \tilde{R}) > \mathcal{B}(x^j_0, B_0)$. By Lemma 16 6a, $\mathcal{B}(x^j_0, B_0) > \mathcal{B}(x^j, \bar{X})$. Putting everything together yields the following chain of inequalities:

$$
\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) > \mathcal{B}(u^i_{1,2}, \tilde{R}) > \mathcal{B}(x^j_0, B_0) > \mathcal{B}(x^j, \bar{X})
$$

Hence, by definition of our assignment, variable x_j is assigned 0. Hence literal $l_{i_2} = \bar{x}_j$ evaluates to 1.

Case $l_{i_2} = x_j$. By Lemma 16 5b, $\mathcal{B}(u^i_{1,2}, \tilde{R}) > \mathcal{B}(x^j_1, B_1)$. By Lemma 16 6b, $\mathcal{B}(x^j_1, B_1) > \mathcal{B}(x^j, X)$. Putting everything together yields the following chain of inequalities:

$$\mathcal{B}(x^1, B) > \mathcal{B}(u^i, L) > \mathcal{B}(\tilde{u}^i_{1,2}, \tilde{B}) > \mathcal{B}(u^i_{1,2}, \tilde{R}) > \mathcal{B}(x^j_1, B_1) > \mathcal{B}(x^j, X)$$

Hence, by Lemma 16 4, $\mathcal{B}(x^j, \bar{X}) > \mathcal{B}(x^1, B)$. Hence, by definition of our assignment, variable x_j is assigned 1. Hence literal $l_{i_2} = x_j$ evaluates to 1.

If part. Let σ be a satisfying assignment for C. We construct a feasible block sequence $\mathcal{B} = (\mathscr{B}_1, \ldots, \mathscr{B}_{11})$ for G as follows. The basic idea is to update blocks induced by

- variable vertices corresponding to variables that are assigned 1 and
- clause vertices corresponding to satisfied clauses

before we update block $b(x^1, B)$, and all other blocks afterwards. We now specify $\mathscr{B}_1, \ldots, \mathscr{B}_{11}$ in detail.

1. For every variable x_j, if x_j is assigned 1, we add block $b(x^j, X)$ to $\mathscr{B}_1$, otherwise we add $b(x^j, \bar{X})$. That is,

$$\mathscr{B}_1 = \{b(x^j, X) \mid \sigma(x_j) = 1\} \cup \{b(x^j, \bar{X} \mid \sigma(x_j) = 0\}.$$

2. For every variable x_j, if x_j is assigned 1, we add block $b(x^j_1, B_1)$ to $\mathscr{B}_2$, otherwise we add $b(x^j_0, B_0)$. That is,

$$\mathscr{B}_2 = \{b(x^j_1, B_1) \mid \sigma(x_j) = 1\} \cup \{b(x^j_0, B_0 \mid \sigma(x_j) = 0\}.$$

3. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$,

   (a) if l_{i_1} evaluates to 1, we add block $b(u^i_{1,2}, \tilde{L})$ to $\mathscr{B}_3$,

   (b) if l_{i_2} evaluates to 1, we add $b(u^i_{1,2}, \tilde{R})$,

   (c) if l_{i_3} evaluates to 1, we add $b(u^i_{3,4}, \tilde{L})$, and

   (d) if l_{i_4} evaluates to 1, we add $b(u^i_{3,4}, \tilde{R})$.

   That is,

$$\begin{aligned}\mathscr{B}_3 =& \{b(u^i_{1,2}, \tilde{L}) \mid \sigma(l_{i_1}) = 1\} \cup \{b(u^i_{1,2}, \tilde{R}) \mid \sigma(l_{i_2}) = 1\} \cup \\ & \{b(u^i_{3,4}, \tilde{L}) \mid \sigma(l_{i_3}) = 1\} \cup \{b(u^i_{3,4}, \tilde{R}) \mid \sigma(l_{i_4}) = 1\}.\end{aligned}$$

4. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is satisfied, we add block $b(\tilde{u}^i_{1,2}, \tilde{B})$ to $\mathscr{B}_4$, and if the right half $(l_{i_3} \vee l_{i_4})$ is satisfied, we add $b(\tilde{u}^i_{3,4}, \tilde{B})$. That is,

$$\begin{aligned}\mathscr{B}_4 = &\{b(\tilde{u}^i_{1,2}, \tilde{B}) \mid \sigma(l_{i_1}) = 1 \text{ or } \sigma(l_{i_2}) = 1\} \cup \\ &\{b(\tilde{u}^i_{3,4}, \tilde{B}) \mid \sigma(l_{i_3}) = 1 \text{ or } \sigma(l_{i_4}) = 1\}.\end{aligned}$$

5. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is satisfied, we add block $b(u^i, L)$ to $\mathscr{B}_5$, and if the right half $(l_{i_3} \vee l_{i_4})$ is satisfied, we add $b(u^i, R)$. That is,

$$\begin{aligned}\mathscr{B}_5 = &\{b(u^i, L) \mid \sigma(l_{i_1}) = 1 \text{ or } \sigma(l_{i_2}) = 1\} \cup \\ &\{b(u^i, R) \mid \sigma(l_{i_3}) = 1 \text{ or } \sigma(l_{i_4}) = 1\}.\end{aligned}$$

6. $\mathscr{B}_6 = \{b(x^1, B)\}$.

7. For every variable x_j, if x_j is assigned 0, we add block $b(x^j, X)$ to $\mathscr{B}_7$, otherwise we add $b(x^j, \bar{X})$. That is,

$$\mathscr{B}_7 = \{b(x^j, X) \mid \sigma(x_j) = 0\} \cup \{b(x^j, \bar{X} \mid \sigma(x_j) = 1\}.$$

8. For every variable x_j, if x_j is assigned 0, we add block $b(x^j_1, B_1)$ to $\mathscr{B}_8$, otherwise we add $b(x^j_0, B_0)$. That is,

$$\mathscr{B}_8 = \{b(x^j_1, B_1) \mid \sigma(x_j) = 0\} \cup \{b(x^j_0, B_0 \mid \sigma(x_j) = 1\}.$$

9. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$,

   (a) if l_{i_1} evaluates to 0, we add block $b(u^i_{1,2}, \tilde{L})$ to $\mathscr{B}_9$,

   (b) if l_{i_2} evaluates to 0, we add $b(u^i_{1,2}, \tilde{R})$,

   (c) if l_{i_3} evaluates to 0, we add $b(u^i_{3,4}, \tilde{L})$, and

   (d) if l_{i_4} evaluates to 0, we add $b(u^i_{3,4}, \tilde{R})$.

   That is,

$$\begin{aligned}\mathscr{B}_9 = &\{b(u^i_{1,2}, \tilde{L}) \mid \sigma(l_{i_1}) = 0\} \cup \{b(u^i_{1,2}, \tilde{R}) \mid \sigma(l_{i_2}) = 0\} \cup \\ &\{b(u^i_{3,4}, \tilde{L}) \mid \sigma(l_{i_3}) = 0\} \cup \{b(u^i_{3,4}, \tilde{R}) \mid \sigma(l_{i_4}) = 0\}.\end{aligned}$$

10. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is unsatisfied, we add block $b(\tilde{u}^i_{1,2}, \tilde{B})$ to $\mathscr{B}_{10}$, and if the right half $(l_{i_3} \vee l_{i_4})$ is unsatisfied, we add $b(\tilde{u}^i_{3,4}, \tilde{B})$. That is,

$$\begin{aligned}\mathscr{B}_{10} = &\{b(\tilde{u}^i_{1,2}, \tilde{B}) \mid \sigma(l_{i_1}) = 0 \text{ and } \sigma(l_{i_2}) = 0\} \cup \\ &\{b(\tilde{u}^i_{3,4}, \tilde{B}) \mid \sigma(l_{i_3}) = 0 \text{ and } \sigma(l_{i_4}) = 0\}.\end{aligned}$$

11. For every clause $C_i = (l_{i_1} \vee l_{i_2} \vee l_{i_3} \vee l_{i_4})$, if the left half $(l_{i_1} \vee l_{i_2})$ of C_i is unsatisfied, we add block $b(u^i, L)$ to $\mathscr{B}_{11}$, and if the right half $(l_{i_3} \vee l_{i_4})$ is unsatisfied, we add $b(u^i, R)$. That is,

$$\begin{aligned}\mathscr{B}_{11} =&\{b(u^i, L) \mid \sigma(l_{i_1}) = 0 \text{ and } \sigma(l_{i_2}) = 0\} \cup \\ &\{b(u^i, R) \mid \sigma(l_{i_3}) = 0 \text{ and } \sigma(l_{i_4}) = 0\}.\end{aligned}$$

□ Remark that we may ignore all other blocks.

We now show that block sequence $\mathcal{B} = (\mathscr{B}_1, \dots, \mathscr{B}_{11})$ is indeed feasible by verifying that the capacity constraint is satisfied for every edge and every $\ell \in [11]$. Since every flow pair has demand 1, we

- may use remark 11 again to argue about capacity constraints, and
- only have to consider edges with capacity less than 10, that is, the number of flow pairs.

For every such edge e, we proceed as follows.

1. First, for every $\ell \in \{0, \dots, 11\}$ and every flow pair P, we determine if e is on the (s,t)-path for P after updating all blocks in B_ℓ, that is, we determine if $\alpha_P(e, B_\ell) = \text{active}$.
2. Next, for every $\ell \in \{0, \dots, 11\}$, we determine the set of flow pairs P such that $\alpha_P(e, B_\ell) = \text{active}$, that is, we determine the set $\mathcal{P}(e, \ell) := \{P \in \mathcal{P} \mid \alpha_P(e, B_\ell) = \text{active}\}$.
3. Then, for every $\ell \in [11]$, we determine the set $\mathcal{P}'(e, \ell) := \mathcal{P}(e, \ell - 1) \cup \mathcal{P}(e, \ell) = \{P \in \mathcal{P} \mid \alpha_P(e, B_{\ell-1}) = \text{active or } \alpha_P(e, B_\ell) = \text{active}\}$.
4. Finally, for every $\ell \in [11]$, we verify that the cardinality of the set $\mathcal{P}'(e, \ell)$ obtained in the previous step is at most $c(e)$.

(x^j, y^j) Let $j \in [n]$. Then edge (x^j, y^j) is used by flow pairs $\bar{X}, X, B$.
Since $(x^j, y^j) \in E(\bar{X}^u \setminus \bar{X}^o)$, by Lemma 9,

$$\alpha_{\bar{X}}((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \text{if } \sigma(x_j) = 1 \text{ and } \ell \geq 7 \\ \text{active} & \text{if } \sigma(x_j) = 0 \text{ and } \ell \geq 1 \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Since $(x^j, y^j) \in E(X^u \setminus X^o)$, by Lemma 9,

$$\alpha_X((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \text{if } \sigma(x_j) = 1 \text{ and } \ell \geq 1 \\ \text{active} & \text{if } \sigma(x_j) = 0 \text{ and } \ell \geq 7 \\ \text{inactive} & \text{otherwise.} \end{cases}$$

Since $(x^j, y^j) \in E(B^o \setminus B^u)$ and $b(x^j, B) = b(x^1, B) \in \mathscr{B}_6$, by Lemma 9,

$$\alpha_B((x^j, y^j), B_\ell) = \begin{cases} \text{active} & \ell < 6 \\ \text{inactive} & \ell \geq 6. \end{cases}$$

Hence,

$$\mathcal{P}((x^j, y^j), \ell) = \begin{cases} \{B\} & \ell < 1 \\ \{X, B\} & \sigma(x_j) = 1 \text{ and } 1 \leq \ell < 6 \\ \{X\} & \sigma(x_j) = 1 \text{ and } \ell = 6 \\ \{\bar{X}, B\} & \sigma(x_j) = 0 \text{ and } 1 \leq \ell < 6 \\ \{\bar{X}\} & \sigma(x_j) = 0 \text{ and } \ell = 6 \\ \{\bar{X}, X\} & \ell \geq 7. \end{cases}$$

Hence,

$$\mathcal{P}'((x^j, y^j), \ell) = \begin{cases} \{X, B\} & \sigma(x_j) = 1 \text{ and } \ell < 7 \\ \{\bar{X}, B\} & \sigma(x_j) = 0 \text{ and } \ell < 7 \\ \{\bar{X}, X\} & \ell \geq 7. \end{cases}$$

Hence $|\mathcal{P}'((x^j, y^j), \ell)| = 2 = c(x^j, y^j)$ for every $\ell \in [11]$.

☐ Other edges.

☐