Advanced MCMC, Computer Classes

Mateusz K. Łącki

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We now plunge into the last topic: the Sequential Monte Carlo. This programme is sponsored by letter S.

Task 1 Consider the Stochastic Volatility model of the form

$$\begin{aligned} X_n &= \alpha X_{n-1} + \sigma V_n, \\ Y_n &= \beta \exp{(X_n/2)} W_n \end{aligned}$$

where $n = 1, \dots, T$, only Y's are observed and V_n and W_n are standard gaussians.

We assume additionally that $X_1 \sim \mathcal{N}(\mu, \sigma^2)$.

Set up values of parameters to $\alpha = .2$, $\sigma^2 = .1$, $\beta = .1$, $\mu = 1$. Generate three sets of data from this distribution for $T = 5, 10, 30$.

Note that $g_n(x_n|x_{1:n-1}) = h(x_n|\alpha x_{n-1}, \sigma^2)$ and $f_n(y_n|x_n) = h(y_n|0, \beta^2 \exp(x_n))$, where $h(x|m, \delta^2)$ is gaussian density with mean m and variance δ^2.

Task 2 Implement the Sequential Importance Sampling algorithm for the above problem. The algorithm looks like this:

- For $n = 1$:
  $X_1^i \sim q_1(x_1)$
  compute $w_1(X_1^i)$ for all particles i, where $w_1(x_1) = \frac{\gamma_1(x_1)}{q_1(x_1)}$
  normalize them: $W_1^i \propto w(X_1^i)$
- For $n > 1$:
  $X_n^i \sim q_n(x_n|x_{1:n-1})$
  compute $w_n(X_1^i) = w_{n-1}(X_{1:n-1}^i)\alpha(X_{1:n}^i)$ for all particles i
  normalize them: $W_1^i \propto w(X_1^i)$

where the incremental weights equal $\alpha(x_{1:n}) = \frac{\gamma_n(x_{1:n})}{\gamma_{n-1}(x_{1:n-1})q_n(x_n|x_{1:n-1})}$.

The incremental weight is chosen so that $w_n(x_{1:n}) = \frac{\gamma_n(x_{1:n})}{q_n(x_{1:n})}$, as $w_n(x_{1:n}) = w_{n-1}(x_{1:n-1})\alpha(x_{1:n})$.

- Establish the form of $\alpha_n(x_{1:n})$ when we choose $\gamma_n(x_{1:n}) = p(x_{1:n}, y_{1:n})$.
- Plug in proposition $q_n(x_n|x_{1:n-1}) = g_n(x_n|x_{n-1})$ from Stochastic Volatility, under the same parameters.
- Plot the results of the simulation against the three sampled trajectories.
- For problem with 10 periods plot the histogram of the values of weights $(W_n^i)_i$ obtained for $n = 2, 5$, and 10.

We should note that the weights are tending towards zero, leaving only a handful of them bigger than others. This supposingly introduces larger variance of the estimates (see the arguement in the last paper I sent you). We want to minimize this problem. This can be done by spreading these particles with higher weights.

Task 3 Implement the Sequential Importance Resampling for the same problem to avoid the problem of the disappearing weights.

- For $n = 1$:

  $X_1^i \sim q_1(x_1)$

  compute $w_1(X_1^i)$ for all particles i, where $w_1(x_1) = \frac{\gamma_1(x_1)}{q_1(x_1)}$

  normalize them: $W_1^i \propto w(X_1^i)$

  resample the points:

  - Draw $N_1, \dots, N_I \sim \text{Multi}(N; w(X_1^1), w(X_1^2), \dots, w(X_1^I))$
  - Consider a new sample s.t. there are N_i particles $\bar{X}_1^j$ equal to X_1^i (draw a sample with replacement from set of points $(X_1^i)_i$ with probabilities equal to $(w(X_1^i))_i$)

- For $n > 1$:

  $X_n^i \sim q_n(x_n|x_{1:n-1})$

  compute $w_n(X_1^i) = \cancel{w_{n-1}(X_{1:n-1}^i)}\alpha(X_{1:n}^i)$ for all particles i

  normalize them: $W_1^i \propto w(X_1^i)$

  resample $\{(W_n^i, X_{1:n}^i)\}$ to obtain new equally-weighted particles $\{(1/N, \bar{X}_{1:n}^i)\}$

where the incremental weights equal $\alpha(x_{1:n}) = \frac{\gamma_n(x_{1:n})}{\gamma_{n-1}(x_{1:n-1})q_n(x_n|x_{1:n-1})}$.

(Observe that we do not multiply the weights by previous values)

Perform the calculations on the Stochastic Volatility model.

Please send the answers by the end of semester.
Best wishes, Matteo