AMP for SLOPE

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Abstract

The aim of this project is to investigate the the approximate message passing algorithm for SLOPE regularization problem based on (Bu et al. 2019) and compare it with classical convex optimization methods. Some numerical experiments regarding the cases that do not fit into the theoretical framework of (Bu et al. 2019) are also performed and analyzed.

Theoretical bacground

Introduction

We are interested in solving the standard linear inverse problem

$$y = Ax + w, \tag{1}$$

where $y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times p}$ are known parameters of the model, $w \in \mathbb{R}^n$ is a random noise vector and $x \in \mathbb{R}^p$ is an unknown vector of paramteres we wish to estimate. We assume $p \gg n$, i.e. the number of features is much greater than the size of the sample data and whence there might be many potential solutions to the problem (1).

To resolve this issue and prevent overfitting, we introduce a penalty function ϕ which faforizes sparse solutions of (1), i.e. now we are looking among the minimizers of the following form

$$\widehat{x} = \underset{x}{\operatorname{argmin}}\{\, \|Ax - y\|_2^2 + \phi(x)\,\}. \tag{2}$$

The usual choices of ϕ are scaled l^2 penalty (Tikhonov regularization) and l^1 penalty (LASSO). This note concerns SLOPE regularization, introduced for the first time in (Bogdan et al. 2015), which assumes ϕ to be the sorted l^1 penalty, i.e.

$$\phi(x) = \phi_\theta(x) = \sum_{i=1}^{n} \theta_i x_i^{\downarrow}, \tag{3}$$

where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$ is the ordered permutation of the vector $(|x_1|, |x_2|, \ldots, |x_n|)$ and $\theta \in \mathbb{R}_+^n$ is a hyperparameter of the model. Such a choice is a generalization of the l^1 regularization, as can be seen by taking $\theta_i = \text{const}$.

The fact that ϕ_θ is non-separable makes the analysis of its teoretical properties much more onerous than in case of classical (separable) models, cf. (Bogdan et al. 2015, Bu et al. (2019)). Nonetheless, it turns out that SLOPE has two advantages over other regularization methods such as LASSO and knocoffs, namely:

1. it achieves certain minimax estimation properties under particular random designs *without* requiring any knowledge of the sparsity degree of $\widehat{x}$, cf. (Bellec, Lecué, and Tsybakov 2018);
2. it controls the false discovery rate in the case of independent predictors, cf. (Bogdan et al. 2015).

Since the objective function in (2) is convex but not smooth, one can not apply directly gradient descent

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Donoho, Maleki, and Montanari (2009), Bu et al. (2019), Zdeborová and Krzakala (2016), Bayati and Montanari (2011).

Convex optimization methods

Numerical experiments

Comparison with convex optimization methods

References

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