The Binomial CEV Model and the Greeks

Aricson Cruz and José Carlos Dias*

This article compares alternative binomial approximation schemes for computing the option hedge ratios studied by Chung and Shackleton (2002), Chung, Hung, Lee, and Shih (2011), and Pelsser and Vorst (1994) under the lognormal assumption, but now considering the constant elasticity of variance (CEV) process proposed by Cox (1975) and using the continuous-time analytical Greeks recently offered by Larguinho, Dias, and Braumann (2013) as the benchmarks. Among all the binomial models considered in this study, we conclude that an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011). © 2016 Wiley Periodicals, Inc. Jrl Fut Mark 37:90–104, 2017

1. INTRODUCTION

Option traders need to repeatedly and accurately calculate options sensitivity measures (usually known as Greeks) to successfully implement hedging strategies in their risk management activities, especially in the case of naked short options positions. This is so mainly because the option's risk characteristics change dynamically as the underlying stock price and the remaining time to maturity change.

Given the absence of closed-form solutions for pricing and hedging many financial option contracts possessing early exercise features and/or exotic payoffs, binomial models—such as the one initially proposed by Cox, Ross, and Rubinstein (1979)—are commonly used by both academics and practitioners to value and hedge such derivative products. The computation of the required Greek measures is then often performed through a numerical differentiation procedure. However, it is well known that the use of such scheme for computing Greeks (and prices) may be flawed by the nature of the binomial discretization behavior observed in tree methods. See, for instance, Chung and Shackleton (2002, 2005), Chung et al. (2011), and Pelsser and Vorst (1994) for details under the geometric Brownian motion (henceforth, GBM) setup.

The main purpose of this article is to revisit the analysis performed by Chung and Shackleton (2002), Chung et al. (2011), and Pelsser and Vorst (1994) for choosing appropriate methods when calculating option hedge ratios under the GBM assumption, but now using the constant elasticity of variance (henceforth, CEV) diffusion process proposed by

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Cox (1975). We note that although there are several papers in the literature comparing the convergence behavior of alternative binomial schemes under the GBM assumption, such extension to the CEV model is still missing mainly due to the absence of analytical solutions for Greeks of European-style options under this modeling setup. Such extension to the CEV model is now possible given the closed-form solutions of Greeks recently offered by Larguinho, Dias, and Braumann (2013).

We recall that such state-dependent volatility modeling framework was introduced in the option pricing literature by Cox (1975) as a way to overcome the undesirable constant volatility assumption underlying the Black and Scholes (1973) and Merton (1973) setup. The importance of the CEV model for traders is justified by its ability to accommodate two empirical stylized facts commonly observed in options markets, namely: the existence of an inverse relation between stock returns and realized volatility (*leverage effect*), as highlighted, for instance, by Bekaert and Wu (2000) and Black (1976); and the negative correlation between the implied volatility and the strike price of an option contract (*implied volatility skew*), as documented, for example, in Dennis and Mayhew (2002). Therefore, it is with no surprise that the CEV model is still widely used nowadays in a variety of contexts, for example, by Ballestra and Cecere (2015), Chung and Shih (2009), Nunes (2009), and Ruas, Dias, and Nunes (2013) for pricing and hedging plain-vanilla American-style options, or by Chung, Shih, and Tsai (2013a,b), Dias, Nunes, and Ruas (2015), Nunes, Ruas, and Dias (2015), and Tsai (2014) in the case of barrier option contracts, just to mention a few.

In this study, we review the argument of Chung and Shackleton (2002), who demonstrated that the Binomial Black–Scholes (henceforth, BBS) model advocated by Broadie and Detemple (1996) outperforms either a straight extended tree or a BBS extended tree. Although such argument is true under the GBM setup considered in Chung and Shackleton (2002), we find that the use of a straight extended tree design is preferable for calculating Greeks under the state-dependent volatility CEV process, since it is better able to efficiently capture the leverage and volatility smile effects frequently found in the options markets. However, this is true only when we avoid the use of a Richardson extrapolation technique. Overall, we conclude that the use of an extended tree binomial CEV model possessing the smooth and monotonic convergence property substantially enhances the accuracy of Greeks because we can apply the extrapolation formula suggested by Chung et al. (2011).

Even though we are examining only approximation methods of Greeks for European-style options against their closed-form continuous-time benchmarks borrowed from Larguinho et al. (2013), the results should still be important for other option contracts. Options with early exercise features were not analyzed here given the absence of analytical solutions for prices and Greeks. However, our numerical experiments and discussions seem to suggest that the results highlighted in this article are a consequence of the method used for evaluation and not the option style itself. Hence, these results should be also of interest when pricing and hedging American-style option contracts.

The remainder of the article is organized as follows. Section 2 presents the theoretical CEV modeling setup and the binomial tree schemes that will be used for approximating the CEV continuous-time process, which are then numerically tested in Section 3. Finally, Section 4 summarizes the concluding remarks.

2. FIVE METHODS FOR COMPUTING GREEKS UNDER THE BINOMIAL CEV MODEL

For the analysis to remain self-contained, the next three subsections provide, respectively, a brief summary of the CEV model setup, the adopted binomial tree method for approximating the CEV diffusion process, and the five competing methods considered for calculating Greeks under the binomial CEV model.

2.1. CEV Model Setup

The CEV process proposed by Cox (1975) assumes that the asset price $\{S_t, t \geq 0\}$ is governed (under the risk-neutral probability measure $\mathbb{Q}$) by the stochastic differential equation

$$dS_t = (r - q)\, S_t\, dt + \delta\, S_t^{\beta/2}\, dW_t^{\mathbb{Q}}, \tag{1}$$

for $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, $q \geq 0$ represents the dividend yield for the underlying asset price, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.[1]

The stochastic differential Equation (1) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$), as well as the absolute diffusion (when $\beta = 0$) and the square-root diffusion (for $\beta = 1$) models of Cox and Ross (1976), as special cases. We further notice that elasticity values of $\beta < 2$ (i.e., with a direct leverage effect) are observed for stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e., with an inverse leverage effect) are expected for some commodity spot prices and futures options with upward sloping implied volatility smiles, as documented, for instance, in Choi and Longstaff (1985), Davydov and Linetsky (2001), Dias and Nunes (2011), and Geman and Shih (2009).[2] In this article, we will focus on equity options and, hence, we assume a CEV process with $\beta < 2$.

2.2. Approximating the CEV Process Through a Lattice Scheme

To approximate the CEV diffusion process with a binomial tree method, we adopt the insights of Chung and Shih (2009, Page 2145) and Nelson and Ramaswamy (1990). First, we consider the x-transform $x(S) = S^{\alpha}/(\alpha\delta)$, with $\alpha = 1 - \beta/2$, and apply Itô's lemma to obtain

$$dx_t = \left[\frac{S_t^{\alpha-1}}{\delta}(r - q)\, S_t + \frac{\alpha - 1}{2}\delta S_t^{-\alpha}\right] dt + dW_t^{\mathbb{Q}}. \tag{2}$$

Replacing the inverse transform $S = (x\alpha\delta)^{1/\alpha}$ in Equation (2) results in a new process x with a constant volatility equal to 1:

$$dx_t = \left[x_t\alpha\,(r - q) + \frac{\alpha - 1}{2x_t\alpha}\right] dt + dW_t^{\mathbb{Q}}. \tag{3}$$

Figure 1 shows a simple two-period binomial x-tree with $x := x(S_{0,0})$, and where $S_{i,j}$ denotes the underlying asset value of an option contract in period i and state j.[3] As usual,

[1]We recall that the model parameter δ is a positive constant that can be interpreted as the scale parameter fixing the initial instantaneous volatility at the reference time $t_0 = 0$, that is, $\sigma_0 = \sigma(S_0) = \delta\, S_0^{\beta/2-1}$. This calibration procedure is standard in the literature and it ensures that the differences found between CEV models with different β values stem purely from the effect of the relationship between volatility and price levels.

[2]For additional background on the CEV process see, for instance, Cox (1975), Davydov and Linetsky (2001), Emanuel and MacBeth (1982), Larguinho et al. (2013), and Schroder (1989).

[3]We recall that the subscript j indicates the number of up moves that the underlying asset has made from its initial price $S_{0,0}$.

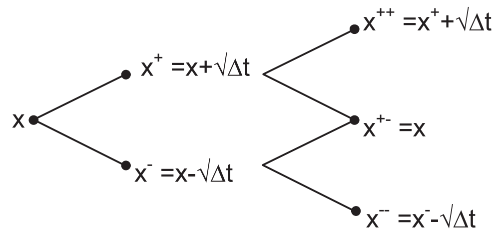

FIGURE 1
A Simple Two-Period Binomial Tree for the x-Transformed Process

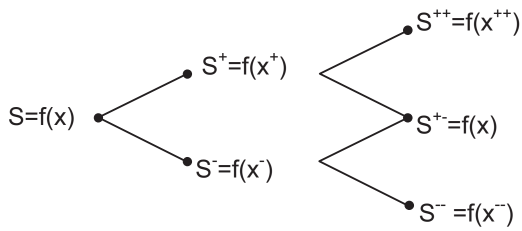

FIGURE 2
A Simple Two-Period Binomial Tree for the Inverse Transform S

the time to maturity of the option contract is divided into n evenly spaced time points such that the time between intervals is $\Delta t := (T - t_0)/n$. Given the current value of x, the approximation of x in the following time step is either $x^+ = x + \sqrt{\Delta t}$ for an up movement or $x^- = x - \sqrt{\Delta t}$ for a down movement. Repeating this procedure, we construct a recombined binomial tree for the x process.

Then, using the inverse transform $S^\pm = f(x^\pm) = (x^\pm \alpha\delta)^{1/\alpha}$ if $x^\pm > 0$, or $S^\pm = 0$ if $x^\pm \leq 0$, results in a recombined binomial grid for the underlying asset price as depicted in Figure 2.[4]

Following Chung and Shih (2009, Page 2145), the risk-neutral probability p^+ of an upward movement is then derived as

$$p^+ := \begin{cases} \frac{Se^{(r-q)\Delta t} - S^-}{S^+ - S^-} & \Leftarrow \quad x > 0 \text{ and } 0 \leq \frac{Se^{(r-q)\Delta t} - S^-}{S^+ - S^-} \leq 1 \\ 0 & \Leftarrow \quad x \leq 0 \text{ or } \frac{Se^{(r-q)\Delta t} - S^-}{S^+ - S^-} < 0 \\ 1 & \Leftarrow \quad x > 0 \text{ and } \frac{Se^{(r-q)\Delta t} - S^-}{S^+ - S^-} > 1 \end{cases} . \tag{4}$$

Finally, we compute the corresponding option values $V(S_{i,j})$ over all the nodes of the tree by applying the usual terminal condition $V(S_{n,j}) = \max(\phi K - \phi S_{n,j}, 0)$ of a call (if $\phi = -1$) or put (if $\phi = 1$), with K being the option's strike price, and then using the standard backward recursive procedure to obtain the time-t_0 option price $V(S_{0,0})$.

[4]As highlighted by Davydov and Linetsky (2001, pg. 955), zero is an exit boundary whenever $1 \leq \beta < 2$, whereas for $\beta < 1$ zero is a regular boundary point that is specified as a killing boundary by adjoining a killing boundary condition. Hence, the inverse transform condition $S^\pm = 0$ if $x^\pm \leq 0$ is imposed to ensure that the CEV process is killed at the zero boundary.

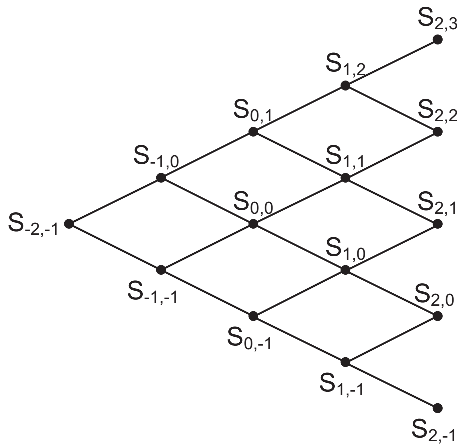

FIGURE 3
The Extended Binomial Tree Scheme

2.3. Five Methods for Computing Greeks

The *numerical differentiation method* for calculating the hedge ratio delta (Δ) relies on the introduction of a small perturbation parameter on the current asset value $S_{0,0}$. More specifically, one typically chooses a small positive number h and constructs new trees with novel initial underlying asset values $S_{0,0} = S_{0,0} + h$ and $S_{0,0} = S_{0,0} - h$. Assuming that $V(S_{0,0} + h, T)$ and $V(S_{0,0} - h, T)$ are the initial theoretical option values obtained from the corresponding trees for contracts expiring at time T, then Δ is approximated by

$$\Delta = \frac{V(S_{0,0} + h, T) - V(S_{0,0} - h, T)}{2h}, \tag{5}$$

whereas the second derivative of the option price with respect to the underlying asset, that is, gamma (Γ), is approximated by

$$\Gamma = 2\left[\frac{V(S_{0,0} + h, T) - V(S_{0,0}, T)}{h} - \frac{V(S_{0,0}, T) - V(S_{0,0} - k, T)}{k}\right] \Big/ (h + k), \tag{6}$$

with k being a second small (positive) perturbation parameter.[5]

Pelsser and Vorst (1994) consider also the *binomial extended tree method*, which extends the original lattice scheme to nodes where both i and j may now be negative for dates prior to time $t_0 = 0$, as shown in Figure 3.

After computing the option values over the whole set of nodes of the binomial extended tree, we can approximate Δ and Γ by

$$\Delta = \frac{V(S_{0,1}) - V(S_{0,-1})}{S_{0,1} - S_{0,-1}}, \tag{7}$$

[5]Typically k is set equal to h, but they might be different.

and

$$\Gamma = 2\left[\frac{V(S_{0,1}) - V(S_{0,0})}{S_{0,1} - S_{0,0}} - \frac{V(S_{0,0}) - V(S_{0,-1})}{S_{0,0} - S_{0,-1}}\right] \Big/ (S_{0,1} - S_{0,-1}). \tag{8}$$

Pelsser and Vorst (1994) compare both methods for calculating Δ and Γ under the GBM framework and conclude that the binomial extended tree method is not only faster than the numerical differentiation method, but also more accurate. As argued by Chung and Shackleton (2002), the implementation of the numerical differentiation method fails for Δ because the tree takes discrete payoffs, and, therefore, for a small perturbation parameter h, the option value is not convex in $S_{0,0}$. Hence, Δ is locally constant because the option price itself is a locally linear function of the underlying asset price. For the case of Γ, the numerical differentiation method might even result in approximations that make no sense at all.

Although Pelsser and Vorst (1994) focus their analysis on Δ and Γ, Chung and Shackleton (2002) consider also the partial derivative with respect to time, that is, theta (θ), which is approximated as

$$\theta = \frac{V(S_{0,0}, T - \tau) - V(S_{0,0}, T + \tau)}{2\tau}, \tag{9}$$

with τ being another small (positive) perturbation parameter, and

$$\theta = \frac{V(S_{-2,-1}) - V(S_{2,1})}{4\tau}, \tag{10}$$

under the numerical differentiation method and the binomial extended tree method, respectively.

The binomial extended tree method not only provides a faster computation, but also results in Greek sensitivity measures that do not suffer from the perturbation discreteness problem associated with the numerical differentiation method. However, the binomial extended tree method still possesses a remaining error that depends on the magnitude of the tree intervals chosen in time. Even though the use of fine trees instead of sparse trees reduces the discreteness errors attached to the binomial extended tree method, it has the shortcoming of increasing the corresponding computational burden.

As an alternative for computing Greeks, Chung and Shackleton (2002) suggest the use of the BBS method offered by Broadie and Detemple (1996), which essentially introduces Black–Scholes analytical option prices at the time step just before maturity originating a more accurate tree for option pricing purposes. Chung and Shackleton (2002) show that this method not only yields more accurate prices, but it also allows accurate calculation of Greek sensitivity measures under the GBM assumption. This is so mainly because the BBS method circumvents the pitfall of the piecewise linearity attached to the standard binomial option pricing model by introducing nonlinear and smoothly differentiable Black–Scholes functions into the continuation value of the final pricing nodes before maturity.

Another valuation approach explored in this study, usually known as the *binomial CEV method* (henceforth, BCEV method) is inspired by the BBS pricing scheme by including CEV option prices into the holding value of the penultimate nodes of the tree. To calculate such option values, we use the closed-form solutions offered by Schroder (1989) and the Benton and Krishnamoorthy (2003) algorithm for computing the required noncentral chi-square distribution functions, as suggested in Larguinho et al. (2013). Then, both the numerical differentiation and the binomial extended tree schemes are implemented via the BCEV pricing methodology.

Chung et al. (2011) suggest that one can apply the standard Richardson extrapolation technique to enhance the accuracy of binomial Greeks if their convergence patterns are monotonic and smooth. The monotonic convergence is attractive because more time steps guarantee more accurate prices. Furthermore, smooth convergence is also desirable because an extrapolation formula can be used to enhance the accuracy. Therefore, our fifth and last method applies the two-point extrapolation formula provided by Chung et al. (2011) to the BCEV extended tree scheme, since this is a binomial model accommodating the monotonic and smooth convergence property.[6]

3. NUMERICAL RESULTS

Armed with the five aforementioned methods, it is now possible to compare the robustness of each one for computing Greeks of standard European-style option contracts under the CEV model against the analytical solutions of Greek measures borrowed from Larguinho et al. (2013). Hereafter, to provide a clear identification of each binomial method to be tested, we will name each one as follows:

(i) **NumDiff** computes Greeks through a standard binomial scheme using the numerical differentiation method.
(ii) **ExtTree** computes Greeks through an extended binomial tree scheme.
(iii) **NumDiffBCEV** computes Greeks through a standard binomial scheme using the numerical differentiation method, though calculating the continuation value of the penultimate nodes of the tree via the closed-form solutions of Schroder (1989).
(iv) **ExtTreeBCEV** computes Greeks through an extended binomial tree scheme, though calculating the continuation value of the penultimate nodes of the tree via the closed-form solutions of Schroder (1989).
(v) **ExtTreeBCEVR** computes Greeks by applying the two-point extrapolation formula suggested by Chung et al. (2011) to the ExtTreeBCEV method.
(vi) **Closed-form solution** stands for the analytical formulae of Greek measures recently offered by Larguinho et al. (2013).

We recall that the continuation values of the penultimate nodes of the NumDiffBCEV and ExtTreeBCEV trees are calculated using the Schroder (1989) option pricing solutions which require the computation of noncentral chi-square distribution functions. Similarly, the delta and theta analytical formulas provided by Larguinho et al. (2013) contain also such distribution laws, while their gamma solutions involve only probability density functions of a noncentral chi-square distribution. As expected, the numerical efficiency of the NumDiff-BCEV, ExtTreeBCEV, and ExtTreeBCEVR methods depends on the accuracy and speed of computation of the Schroder (1989) closed-form solutions. To the best of our knowledge—see, for instance, the numerical analysis performed by Larguinho et al. (2013,Section 4)—, the iterative procedure of Benton and Krishnamoorthy (2003,Algorithm 7.3) clearly offers the best speed-accuracy trade-off for evaluating option prices and Greeks under the (unrestricted) CEV model. Based on these insights and in order to make fair comparisons between

[6]As highlighted by Chung and Shih (2009, Footnote 11), the convergence pattern of the BCEV price to the accurate option price is the same as that of the BBS method, that is, the BCEV price converges monotonically and smoothly to the accurate value at a rate of $O(1/n)$. Therefore, it is possible to apply an extrapolation formula to enhance the accuracy of the extended tree BCEV prices and Greeks.

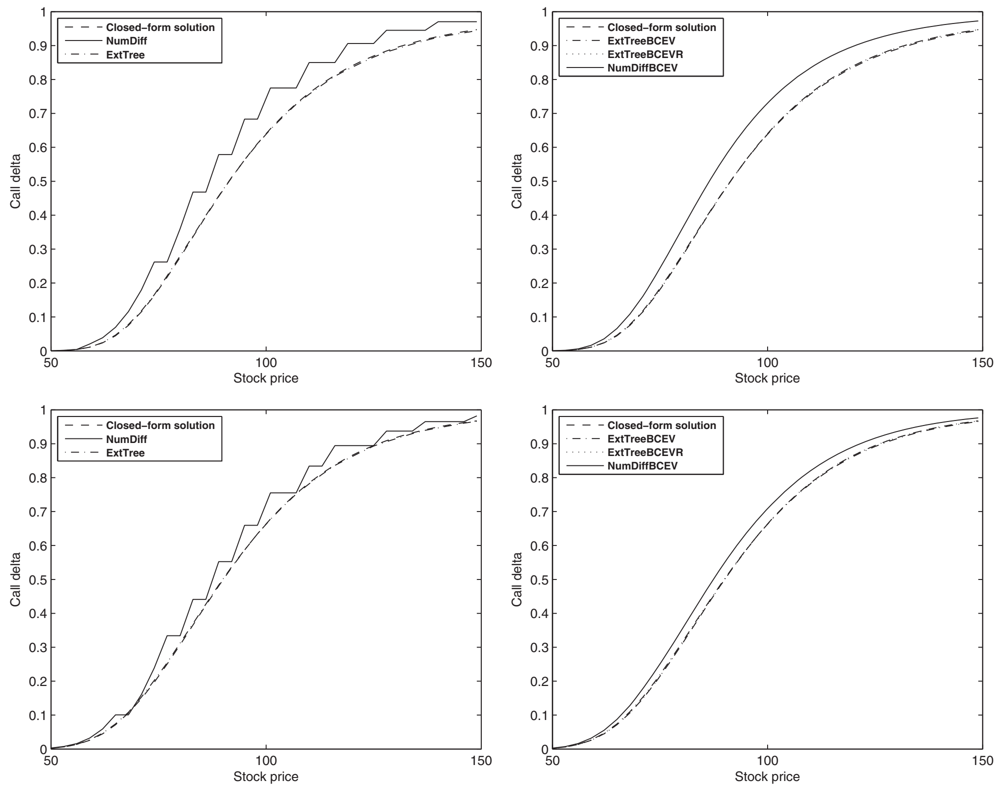

FIGURE 4
Call Delta of European-Style Options Computed via Alternative Binomial Methods Against the Benchmark Offered by Larguinho et al. (2013,Equation A7), with $\beta = 0$, for the Two Graphs at the Top, and $\beta = 1$, for the Two Graphs at the Bottom

competing methods, the necessary noncentral chi-square distribution functions appearing in the benchmark values of Larguinho et al. (2013) and in the NumDiffBCEV, ExtTreeBCEV, and ExtTreeBCEVR methods are all calculated via the Benton and Krishnamoorthy (2003) algorithm.[7]

For illustrative purposes, we consider the constellation of parameters used by Chung and Shackleton (2002) and Pelsser and Vorst (1994), but augmented by the β parameter. More specifically, we assume that $\{K, T, \sigma_0, r, q\} = \{100, 1, 0.25, 0.09, 0\}$ over a small range of current asset values $S_{0,0}$ using $n = 50$. Moreover, we set the perturbation parameters $h = k = 0.01$ and $\tau = 0.001$. Even though we will concentrate our numerical analysis on calls, the results for puts are similar.

Figure 4 plots the call delta of European-style options computed via the aforementioned binomial methods against the benchmark offered by Larguinho et al. (2013, Equation A7), with $\beta = 0$, for the two graphs at the top, and $\beta = 1$, for the two graphs at the bottom. The two left-hand side graphs show that, similarly to what was pointed out by Pelsser and Vorst (1994) under the GBM assumption, the ExtTree method provides a much better approximation to the true call delta under the CEV model than the NumDiff method, since the latter

[7]For completeness, we further note that the probability density functions of a noncentral chi-square distribution law appearing in the analytical solutions of Greeks borrowed from Larguinho et al. (2013) are computed through the *ncx2pdf* built-in function available in Matlab.

results in option deltas that are highly discrete in $S_{0,0}$. The two right-hand side graphs perform essentially the same calculations, but now comparing the three versions of the BCEV approach against the benchmark. Even though we observe that the NumDiffBCEV method does not yield discrete values of delta, it is still much less accurate than the ExtTreeBCEV and ExtTreeBCEVR methods. Thus, while the inclusion of nonlinear and smoothly differentiable Schroder (1989) CEV functions into the final pricing nodes of the tree before maturity removes the discreteness problem associated with numerical differentiation, the extension of the original lattice scheme to nodes with dates prior to time $t_0 = 0$ seems to be a relevant feature under the CEV model as it reduces significantly the Δ estimation error of the ExtTree, ExtTreeBCEV, and ExtTreeBCEVR methods.

This implies that the conclusion pointed out by Chung and Shackleton (2002)—for the same set of parameters but under the GBM framework—that the addition of a smooth function in the BBS tree and numerical differentiation with $h = 0.01$ outperforms either a standard extended tree or a BBS extended tree is not valid under the CEV model. As expected, as we augment the number of n evenly spaced time points all the methods tend to converge to the true value. However, a straight extended tree or a BCEV extended tree both offer advantages in terms of computational burden as the NumDiff and NumDiffBCEV methods require the tree to be recalculated, whereas the ExtTree and ExtTreeBCEV methods do not.

Figure 4 highlights also that as we move further away from the limiting GBM process (i.e., $\beta = 2$) both the NumDiff and NumDiffBCEV methods amplify the magnitude of the overestimation errors when approximating option deltas under CEV models with alternative β values. This suggests that the ExtTree, ExtTreeBCEV, and ExtTreeBCEVR methods accommodate better the leverage and volatility smile effects—both of which are commonly observed across options markets and are captured by the CEV model specification—when approximating Δ values.

Other numerical results not reported here, but available upon request, show that the same line of reasoning occurs also for the CEV model with an inverse leverage effect (i.e., with $\beta > 2$). More specifically, as we move further away to the right of $\beta = 2$ the NumDiff and NumDiffBCEV methods produce again significant biases, but now with increasing underestimation errors, whereas the ExtTree, ExtTreeBCEV, and ExtTreeBCEVR methods are still very accurate. Therefore, while the introduction of nonlinear and smoothly differentiable Schroder (1989) CEV functions into the final pricing nodes of the tree before maturity removes the wavy erratic behavior of the NumDiff method, the NumDiffBCEV method still contains substantial approximation errors due to the distribution and nonlinearity errors discussed in Figlewski and Gao (1999).

Figure 5 illustrates the call (and put) gamma of European-style options, for $\beta = 1$, calculated through the mentioned binomial methods against the true value provided by Larguinho et al. (2013, Equation A12). The left-hand side graph shows that the ExtTree method leads to very good approximations of the CEV gamma.[8] Even though the NumDiffBCEV method is exhibited in the right-hand side graph, it reveals also very poor approximations for Γ. By contrast, the ExtTreeBCEV and ExtTreeBCEVR models originate very good approximations for Γ calculations. Once again, the extension of the tree to nodes with dates prior to time $t_0 = 0$ seems to be a key feature for computing accurate gammas under binomial CEV schemes.

[8]We note, however, that the NumDiff method gives approximations that do not make any sense: it provides gamma values equal to zero everywhere except near the point where the underlying spot price is 100. At this point, the gamma value is very large when compared with the benchmark gamma value. For this reason, the NumDiff method is not plotted in Figure 5. This is in line with the observation made by Pelsser and Vorst (1994) when calculating gamma values under the GBM assumption.

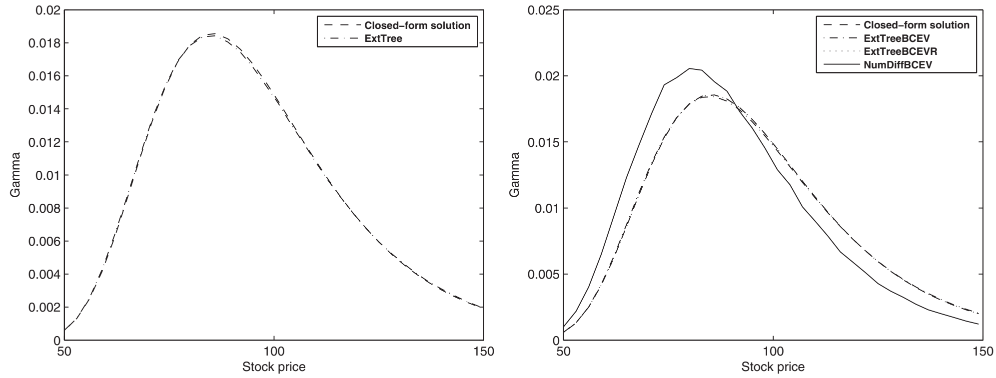

FIGURE 5
Call (and Put) Gamma of European-Style Options Computed via Alternative Binomial Methods Against the Benchmark Offered by Larguinho et al. (2013, Equation A12), with $\beta = 1$

Figure 6 plots the call theta of a European-style in-the-money option, that is, with $S_{0,0} = 105$ as in Chung and Shackleton (2002, Figures 3 and 4), and for $\beta = 1$, using the two extended tree methods against the benchmark offered by Larguinho et al. (2013,Equation A16). Following the recommendation of Chung and Shackleton (2002), the graphs plot the right- and left-hand derivatives and their average that can be calculated from the extended tree as follows:

$$\theta_- = \frac{V(S_{-2,-1}) - V(S_{0,0})}{2\tau}, \tag{11}$$

$$\theta_+ = \frac{V(S_{0,0}) - V(S_{2,1})}{2\tau}, \tag{12}$$

while the central (or symmetric) difference $\theta = \frac{\theta_+ + \theta_-}{2}$ is given by Equation (10).

The left-hand side graph of Figure 6 shows the ExtTree method with backward (θ_-), forward (θ_+), and central (θ) approximating schemes compared to the theoretical θ values. We conclude that the average of forward and backward differences performs best when

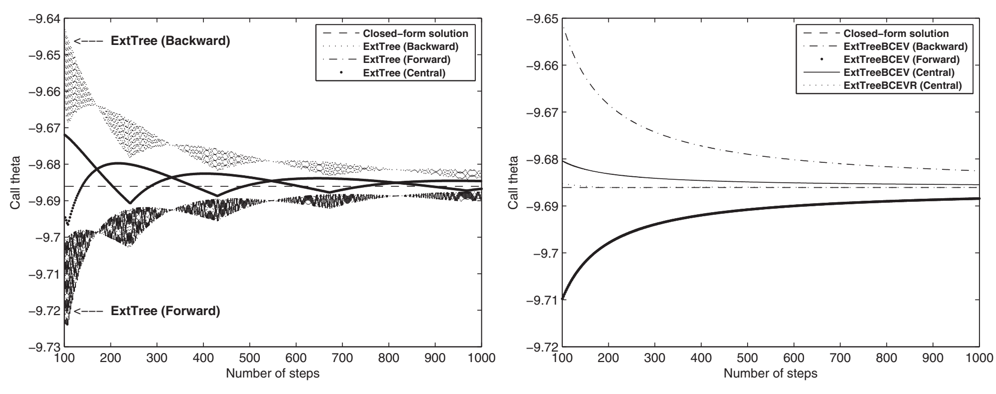

FIGURE 6
Call Theta of European-Style Options Computed via Extended Tree Schemes Against the Benchmark Offered by Larguinho et al. (2013, Equation A16), with $\beta = 1$

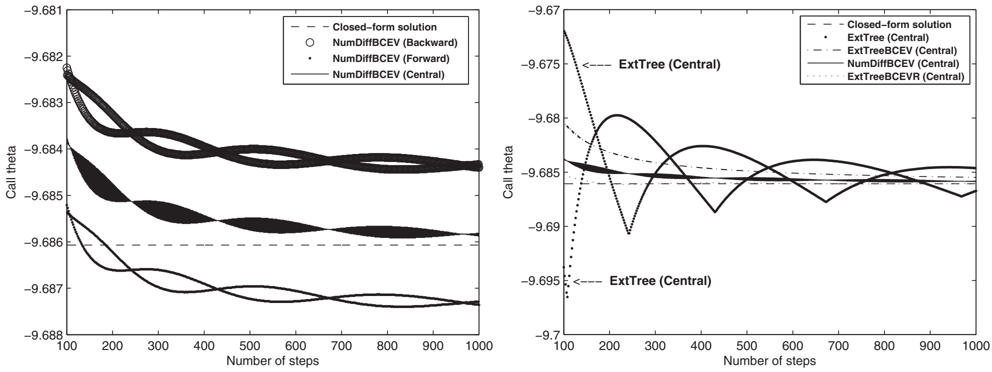

FIGURE 7

Call Theta of European-Style Options Computed via Alternative Binomial Methods Against the Benchmark Offered by Larguinho et al. (2013, Equation A16), with $\beta = 1$

compared with the benchmark value, though some oscillatory behavior is observed as the number of time steps rise. By contrast, the right-hand side graph of Figure 6 reveals that no oscillation is found as we increase the number of tree steps under the ExtTreeBCEV method due to the insertion of a smooth function before maturity. Moreover, the average of forward and backward methods is still the preferable scheme for computing thetas as it provides closer values to the continuous-time benchmark. The graph reveals also that the theta value obtained through the ExtTreeBCEV method seems to converge to the benchmark value monotonically and smoothly. Therefore, the use of the extrapolation formula derived by Chung et al. (2011) is able to enhance significantly the accuracy of the theta value, as highlighted by the ExtTreeBCEVR method.

Although a straight numerical differentiation scheme (i.e., the NumDiff method) is known to perform poorly for approximating θ values (even under the GBM setup), the left-hand side graph of Figure 7 shows that the use of the NumDiffBCEV method with central differences improves significantly the θ estimates.[9] A direct comparison between methods is provided in the right-hand side graph of Figure 7. Even though the NumDiffBCEV method requires the calculation of a second tree, the obtained θ estimates are significantly more accurate than the ExtTreeBCEV method for a limited number of time steps, at least for the specific contract under analysis. This finding is consistent with Chung and Shackleton (2002, Figure 5) who show that the central numerical differentiation of the BBS tree also improves significantly the accuracy of θ under the GBM assumption. However, the ExtTree-BCEVR method is preferable as it provides theta values that are closer to the benchmark value.

Figures 4–7 plot Greek measures that do not represent a sufficiently large enough sample to take more robust conclusions, thus giving only a preliminary flavor of the results. Hence, to better assess the speed-accuracy trade-off between the competing methods, we follow the guidelines of Broadie and Detemple (1996) by conducting a careful large sample evaluation of 2500 randomly generated contracts. To accomplish this purpose, we fix the initial asset price at $S_{t_0} = 100$ and take the strike price K to be uniform between 70 and 130. The β parameter is distributed uniformly between -4 and 2. The volatility σ_0 is distributed

[9] For completeness, we recall that the forward and backward differences for the numerical differentiation scheme are calculated as $\theta_+ = \frac{V(S_{0,0},T-\tau)-V(S_{0,0},T)}{\tau}$ and $\theta_- = \frac{V(S_{0,0},T)-V(S_{0,0},T+\tau)}{\tau}$, respectively, whereas the central difference $\theta = \frac{\theta_+ + \theta_-}{2}$ results in Equation (9).

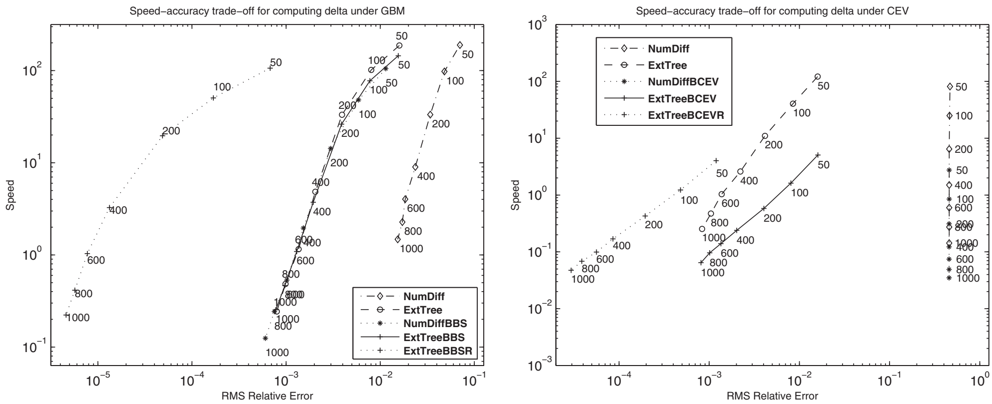

FIGURE 8

Speed-Accuracy Trade-Off of Alternative Binomial Methods for Computing Δ Under the GBM and CEV Models Using a Random Sample of 2500 and 2474 Contracts, Respectively

Note. The RMS relative error is defined by $\sqrt{\frac{1}{m}\sum_{i=1}^{m}((\widehat{\Delta}_i - \Delta_i)/\Delta_i)^2}$, where Δ_i is the true delta value computed via the closed-form solutions of Black and Scholes (1973) and Larguinho et al. (2013, Equation A7), respectively, $\widehat{\Delta}_i$ is the approximate delta value estimated by the corresponding binomial method, and m is the number of contracts. Speed is measured as the number of delta values calculated per second (on a 2.50 GHz i3-3120M Toshiba Satellite). Preferred methods are in the upper-left corner.

uniformly between 0.10 and 0.60, and the scale parameter δ is then computed. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. The dividend yield q is uniform between 0.0 and 0.1. The riskless rate r is uniform between 0.0 and 0.1.

The graph on the left-hand side of Figure 8 compares the speed-accuracy trade-off between the alternative valuation methods—all binomial models in this graph are based on the standard binomial scheme of Cox et al. (1979)—for computing deltas under the GBM assumption using the whole set of 2500 contracts. The plot highlights that the speed-accuracy trade-off between the ExtTree, NumDiffBBS, and ExtTreeBBS methods are almost indistinguishable. We also observe that the NumDiffBBS method requires less evenly spaced time points to achieve the same root mean square (henceforth, RMS) relative error obtained by the ExtTree and ExtTreeBBS methods. These results confirm the argument stated by Chung and Shackleton (2002) that the inclusion of nonlinear and smoothly differentiable Black–Scholes functions into the holding value of the final pricing nodes before maturity improves significantly the straight NumDiff method under the GBM framework. Nevertheless, the ExtTreeBBS method is a binomial model with the smooth and monotonic convergence property and, therefore, we may apply the two-point extrapolation formula derived by Chung et al. (2011). Consequently, the ExtTreeBBSR method clearly offers the best speed-accuracy trade-off among the methods tested here under the GBM setup.[10]

The graph on the right-hand side of Figure 8 also compares the speed-accuracy trade-off between the alternative valuation methods for computing deltas, but now under the CEV model. Following the insight outlined by Larguinho et al. (2013, Page 911), we excluded option parameter configurations where the abscissa value and the noncentrality

[10]See Chung et al. (2011) for thorough discussions on other recent binomial models under the GBM assumption whose binomial option prices converge to the true value monotonically and smoothly.

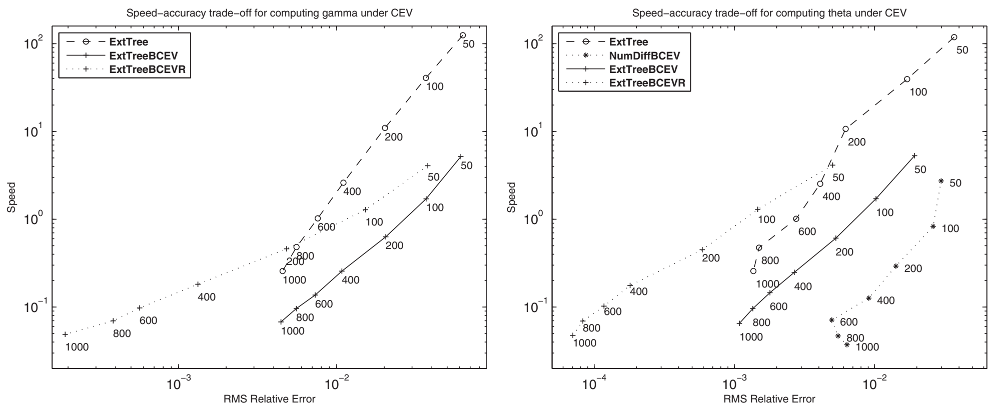

FIGURE 9
Speed-Accuracy Trade-Off of Alternative Binomial Methods for Computing Γ and θ Under the CEV Model Using a Random Sample of 2474 Contracts

Note. The RMS relative error is defined by $\sqrt{\frac{1}{m}\sum_{i=1}^{m}((\widehat{f_i}-f_i)/f_i)^2}$, with $f \in \{\Gamma, \theta\}$ and where f_i is the true Γ or θ value computed via Larguinho et al. (2013, Equation A12 or A16), respectively, $\widehat{f_i}$ is the approximate Γ or θ value estimated by the corresponding binomial method, and m is the number of contracts. Speed is measured as the number of Γ or θ values calculated per second (on a 2.50 GHz i3-3120M Toshiba Satellite). Preferred methods are in the upper-left corner.

parameter of the noncentral chi-square distribution functions appearing in the analytical solutions provided by Schroder (1989) are both ≥ 5000. Out of the 2500 randomly generated contracts, 2474 did not satisfy this criteria.[11] The plot reveals that both the NumDiff and NumDiffBCEV methods perform poorly. Hence, contrary to what it is observed under the GBM case, the inclusion of nonlinear and smoothly differentiable CEV option pricing analytical solutions into the penultimate pricing nodes of the tree does not reduce significantly the errors of the standard NumDiff method, though it removes the discreteness problem associated with the straight numerical differentiation.

The conclusions to be taken from this large sample confirm that the extension of the original lattice scheme to nodes prior to time $t_0 = 0$ is an important feature to obtain accurate delta values via binomial CEV models. We observe that the ExtTree and ExtTreeBCEV methods offer a similar accuracy for the same number of time steps though the primer scheme is faster than the latter. This means that without extrapolation, the ExtTree provides a better speed-accuracy trade-off. However, we can apply the two-point extrapolation formula to the ExtTreeBCEV method to significantly enhance the accuracy of the delta estimation. For instance, the ExtTreeBCEVR method requires less than 100 time steps to achieve the same RMS relative error as that of the ExtTree and ExtTreeBCEV methods with 1000 time steps. Hence, the ExtTreeBCEVR method is clearly the most efficient one.

Figure 9 compares the speed-accuracy trade-off between the alternative valuation methods for computing gammas—left-hand side graph—and thetas—right-hand side graph—under the CEV model. As shown before, both the NumDiff and NumDiffBCEV methods are inadequate for computing gamma sensitivity measures. Therefore, these methods are not plotted here. Similarly to the delta case, the ExtTree method offers a better speed-accuracy

[11]The remaining 26 randomly generated contracts are excluded from the sample to be used for performing speed-accuracy tests under CEV because they are all associated to β parameters that are almost indistinguishable from the limiting GBM value of $\beta = 2$ coupled with low-volatility values.

trade-off if we do not make use of the extrapolation scheme, though both the ExtTree and ExtTreeBCEV methods provide an identical accuracy for the same number of evenly spaced time points. Nevertheless, the ExtTreeBCEVR method provides a superior accuracy for computing gammas when $n > 200$.

Regarding the theta case, the graph reveals that without extrapolation the ExtTree method is still the preferable one. However, the ExtTreeBCEV procedure requires significantly fewer time-steps to achieve the same RMS relative error. Moreover, while Figure 7 highlighted that the use of numerical differentiation can be reinstated for computing theta values by simply including nonlinear and smoothly differentiable CEV option pricing analytical solutions into the penultimate pricing nodes of the tree, the results in Figure 9 show that the NumDiffBCEV method is less efficient than the other methods. Overall, we may conclude that the ExtTreeBCEVR is the best choice for computing thetas under the CEV model.

4. CONCLUSIONS

This article examines the choice of method for computing the option hedge ratios studied by Chung and Shackleton (2002), Chung et al. (2011), and Pelsser and Vorst (1994) but assumes the underlying stock price is governed by a CEV diffusion process. Contrary to what was found by Chung and Shackleton (2002) under the GBM assumption, we show that, under the CEV model, an extended tree design is the key feature for generating accurate and fast calculations of Greeks if one ignores the use of a Richardson extrapolation technique. However, an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011).

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