Lecture 9

Numerical methods for American options

Lecture Notes
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American options

The holder of an **American option** has the right to exercise it at any moment up to maturity. If exercised at t

- an American call option has the payoff $(S_t - K)^+$,
- an American put option has the payoff $(K - S_t)^+$.

Theorem. If the underlying does not pay dividends, the price of an American call option with maturity T and exercise price K is equal to the price of a European call option with exercise price K expiring at T.

Important! It is only true if there are no dividends. Above theorem does not apply for options on the foreign exchange market and on stock indices.

American instruments

Definition. American type instrument with maturity T and payoff function f is a contingent claim that can be exercised at any moment up to T. Its payoff at t equals

$$f(S_t, t).$$

Theorem. The price of an American claim at time t can be written as

$$V(S_t, t)$$

for some function $V : (0, \infty) \times [0, T] \to \mathbb{R}$.

Properties

- $V(S_t, t)$ is the price of the instrument at t, so it values future payoffs from the instrument.
- $f(S_t, t)$ is the payoff of the instrument if it is exercised at t.

At the terminal time T

$$V(s, T) = f(s, T), \qquad s > 0.$$

To preclude arbitrage

$$V(s, t) \geq f(s, t), \qquad (s, t) \in (0, \infty) \times [0, T].$$

If the inequality is strict, i.e. $V(s,t) > f(s,t)$, the holder should not exercise the option. Because it is more profitable to sell the option for $V(s,t)$ than to exercise it for $f(s,t)$.

If $V(s,t) = f(s,t)$, it is optimal to exercise the option immediately. By holding it, the holder risks loosing money.

General rule. The holder should exercise the option as soon as

$$V(S_t, t) = f(S_t, t).$$

American option PDE

Up to the exercise of the option, its price forms a value process of the replicating strategy. It is therefore a martingale with respect to the risk-neutral measure. Applying Itô's formula gives

$$dV_t = d\big(V(S_t,t)\big) = \left(\sigma S_t \frac{\partial V(S_t,t)}{\partial s}\right) dW_t +$$

$$+ \left(r S_t \frac{\partial V(S_t,t)}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t,t)}{\partial s^2} - rV(S_t,t) + \frac{\partial V(S_t,t)}{\partial t}\right) dt.$$

This process is a martingale if and only if the term by dt is zero.

Until the optimal exercise we have

$$rs\frac{\partial V(s,t)}{\partial s}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V(s,t)}{\partial s^2}-rV(s,t)+\frac{\partial V(s,t)}{\partial t}=0.$$

And of course, we have $V(s,t)>f(s,t)$.

At the optimal exercise moment, we have

$$V(s,t)=f(s,t),$$

and

$$rs\frac{\partial V(s,t)}{\partial s}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V(s,t)}{\partial s^2}-rV(s,t)+\frac{\partial V(s,t)}{\partial t}\leq 0.$$

The inequality in the second formula comes from the fact that $V(s,t)$ may not represent the value process of some strategy after the optimal exercise moment.

Free-boundary problem

We summarize our findings in the following system of equations and inequalities called the **free-boundary problem**

$$\begin{cases} V(s,t) \geq f(s,t) \\ \\ \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} \leq 0 \\ \\ V(s,t) = f(s,t) \quad \text{or} \quad \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0 \\ \\ \text{boundary conditions...} \end{cases}$$

The "free-boundary" is the set of points where $V(s,t) = f(s,t)$. In these points the system is not governed by the equation with partial derivatives.

Boundary conditions

Boundary conditions for the **American put option**:

Terminal condition

$$V(s,T) = (K - s)^+.$$

Left-boundary condition

$$\lim_{s \to 0} V(s,t) = K.$$

Right-boundary condition

$$\lim_{s \to \infty} V(s,t) = 0.$$

Complete free-boundary problem

For $s > 0$, $t \in [0, T]$:

$$\begin{cases} V(s,t) \geq (K-s)^+ \\ \\ \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} \leq 0 \\ \\ V(s,t) = (K-s)^+ \quad \text{or} \quad \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) + \frac{\partial V(s,t)}{\partial t} = 0 \\ \\ V(s,T) = (K-s)^+ \\ \lim_{s\to 0} V(s,t) = K \\ \lim_{s\to\infty} V(s,t) = 0 \end{cases}$$

Simple example

We compute the price of an American put option by a simple numerical method.

First we make the following change of variables (the same as for the Black-Scholes PDE)

$$
\begin{aligned}
x &:= \log s + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \\
\tau &:= \frac{\sigma^2}{2}(T-t), \\
y(x,\tau) &:= e^{-r(T-\frac{2\tau}{\sigma^2})} V\left(e^{x-(\frac{2r}{\sigma^2}-1)\tau},\ T - \frac{2\tau}{\sigma^2}\right),
\end{aligned}
$$

where $x \in \mathbb{R}$ and $\tau \in \left[0, \frac{\sigma^2}{2}T\right]$.

Transformation of the problem

$$
\begin{cases}
y(x,\tau) \geq e^{-r(T-\frac{2\tau}{\sigma^2})}\left(K - e^{x-(\frac{2r}{\sigma^2}-1)\tau}\right)^+, \\
\frac{\partial y}{\partial \tau}(x,\tau) - \frac{\partial^2 y}{\partial x^2}(x,\tau) \geq 0, \\
y(x,\tau) = e^{-r(T-\frac{2\tau}{\sigma^2})}\left(K - e^{x-(\frac{2r}{\sigma^2}-1)\tau}\right)^+ \quad \text{or} \quad \frac{\partial y}{\partial t}(x,\tau) - \frac{\partial^2 y}{\partial x^2}(x,\tau) = 0, \\
y(x,0) = e^{-rT}(K - e^x)^+, \\
\lim_{x\to-\infty} y(x,\tau) = Ke^{-r(T-\frac{2\tau}{\sigma^2})}, \\
\lim_{x\to\infty} y(x,\tau) = 0.
\end{cases}
$$

Numerical method

Grid

- $x \in (-\infty, \infty)$, $\quad \tau \in [0, \frac{1}{2}\sigma^2 T]$
- discretization of time τ: $\delta\tau = \frac{\frac{1}{2}\sigma^2 T}{N}$

$$\tau_\nu := \nu \cdot \delta\tau \quad \text{for} \quad \nu = 0, 1, \ldots, N$$

- discretization of space x
  - x_{min}, x_{max}
  - $\delta x = \dfrac{x_{max} - x_{min}}{M}$

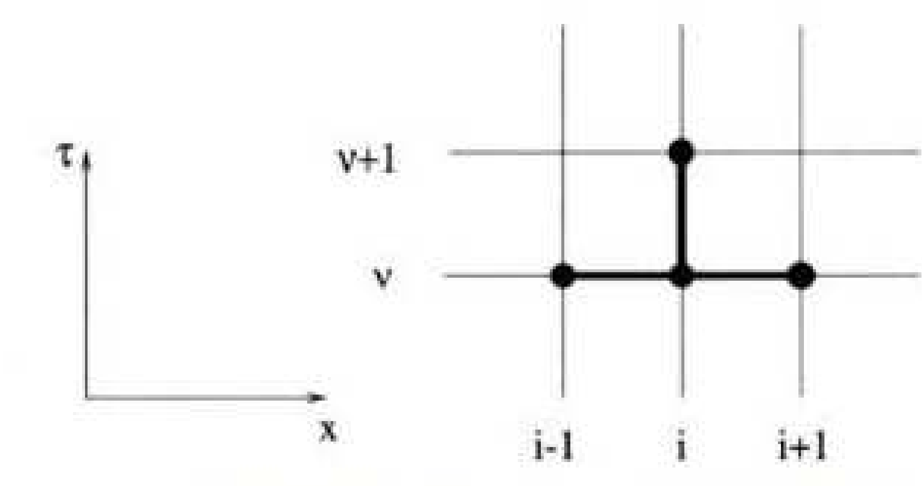
$$x_i := x_{min} + i \cdot \delta x \quad \text{for} \quad i = 0, 1, \ldots, M$$

- $w_{i,\nu}$ denotes the approximation of $y(x_i, \tau_\nu)$

Finite differences

We use **explicit method** (this is important !).

The consequence is a small time step for the algorithm to be stable.



In the node $(x_i, \tau_{\nu+1})$ the expression

$$u_1 = \lambda w_{i-1,\nu} + (1-2\lambda) w_{i,\nu} + \lambda w_{i-1,\nu},$$

where $\lambda = \frac{\delta\tau}{(\delta x)^2}$, approximates the value of $y(x_i, \tau_{\nu+1})$ in the case of no exercise.

If this value is smaller than the payoff from the exercise

$$u_2 = e^{-r(T-\frac{2\tau_{\nu+1}}{\sigma^2})}\left(K - e^{x_i - (\frac{2r}{\sigma^2}-1)\tau_{\nu+1}}\right)^+$$

then it is optimal to exercise immediately and $w_{i,\nu+1} = u_2$.

This is written concisely as

$$\boxed{w_{i,\nu+1} = \max\Bigg(\lambda w_{i+1,\nu} + (1-2\lambda) w_{i,\nu} + \lambda w_{i-1,\nu},\ e^{-r(T-\frac{2\tau_{\nu+1}}{\sigma^2})}\left(K - e^{x_i - (\frac{2r}{\sigma^2}-1)\tau_{\nu+1}}\right)^+ \Bigg).}$$

Algorithm for an American put option

Input: x_{min}, x_{max}, M, N, K, T and the parameters of the model

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$\delta\tau = \frac{\sigma^2 T}{2N}$, $\qquad \delta x = \frac{x_{max} - x_{min}}{M}$

Calculate τ_ν, $\nu = 0, 1, \ldots, N$, and x_i, $i = 0, 1, \ldots, M$

For $i = 0, 1, \ldots, M$

$\quad w_{i,0} = e^{-rT}(K - e^{x_i})^+$

For $\nu = 0, 1, \ldots, N - 1$

$\quad w_{0,\nu+1} = Ke^{-r(T - \frac{2\tau_{\nu+1}}{\sigma^2})}$

$\quad w_{M,\nu+1} = 0$

$\quad$ For $i = 1, 2, \ldots, M - 1$

$\qquad u_1 = \lambda w_{i+1,\nu} + (1 - 2\lambda) w_{i,\nu} + \lambda w_{i-1,\nu}$

$\qquad u_2 = e^{-r(T - \frac{2\tau_{\nu+1}}{\sigma^2})} \left(K - e^{x_i - (\frac{2r}{\sigma^2} - 1)\tau_{\nu+1}} \right)^+$

$\qquad w_{i,\nu+1} = \max(u_1, u_2)$

Output: $w_{i,\nu}$ for $i = 0, 1, \ldots, M$, $\quad \nu = 0, 1, \ldots, N$

General American instrument

For a general American instrument we return to original variables only making the change of time $t \mapsto \tau = T - t$.

A general American instrument is characterized by a pay-off function $g(s,\tau)$.

The free-boundary problem for this instrument is given by

$$\begin{cases} \left(\frac{\partial V(s,\tau)}{\partial \tau} - rs\frac{\partial V(s,\tau)}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + rV(s,\tau)\right)\big(V(s,\tau) - g(s,\tau)\big) = 0, \\ V(s,\tau) - g(s,\tau) \geq 0, \\ \frac{\partial V(s,\tau)}{\partial \tau} - rs\frac{\partial V(s,\tau)}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + rV(s,\tau) \geq 0, \\ V(s,0) = g(s,0), \\ \lim_{s\to+\infty} V(s,\tau) = \lim_{s\to+\infty} g(s,\tau). \\ \lim_{s\to 0} V(s,\tau) = \lim_{s\to 0} g(s,\tau). \end{cases}$$

This problem is also called **the linear complementarity problem** for the American instrument defined by a pay-off function $g(s,\tau)$.

Penalty method

There are many numerical methods which solve the linear complementarity problem (LCP). We present here the method called **the penalty method**. This method is simple and efficient (this is particularly visible for more complicated instruments like barrier options). On the other hand the method is only first order (slow convergence).

The basic idea of the penalty method is simple. We replace the linear complementarity problem by the nonlinear PDE

$$\frac{\partial V(s,\tau)}{\partial \tau} = rs\frac{\partial V(s,\tau)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} - rV(s,\tau) + \rho \max\bigl(g(s,\tau) - V(s,\tau)\bigr)$$

where, in the limit as the positive penalty parameter $\rho \to \infty$, the solution satisfies $V - g \geq 0$.

Finite difference approximation

We use the same grid as for the Black-Scholes equation with V_i^n denoting an approximation to $V(s_i, \tau_n)$ and g_i^n an approximation to $g(s_i, \tau_n)$.

The nonlinear PDE for the penalty method becomes in the discrete version

$$
\begin{aligned}
V_i^{n+1} - V_i^n = \\
&(1-\theta)\left(\Delta\tau \sum_{j=i\pm 1} (\gamma_{ij} + \beta_{ij}) \left(V_j^{n+1} - V_i^{n+1}\right) - r\Delta\tau V_i^{n+1}\right) \\
&+ \theta\left(\Delta\tau \sum_{j=i\pm 1} (\gamma_{ij} + \beta_{ij}) \left(V_j^{n} - V_i^{n}\right) - r\Delta\tau V_i^{n}\right) \\
&+ P_i^{n+1}\left(g_i^{n+1} - V_i^{n+1}\right),
\end{aligned}
$$

where the choice of θ gives the implicit ($\theta = 0$) and the Crank-Nicolson ($\theta = \frac{1}{2}$) scheme.

Finite difference approximation – cont.

Coefficients from the previous slide are as follows:

$$P_i^{n+1} = \begin{cases} \rho, & \text{for } V_i^{n+1} < g_i^{n+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\gamma_{ij} = \frac{\sigma^2 s_i^2}{|s_j - s_i| \left(s_{i+1} - s_{i-1}\right)},$$

$$\beta_{ij} = \begin{cases} \frac{r s_i (j-i)}{s_{i+1} - s_{i-1}}, & \text{for } \sigma^2 s_i + r(j-i)|s_j - s_i| > 0, \\ \left(\frac{2 r s_i (j-i)}{s_{i+1} - s_{i-1}}\right)^+, & \text{otherwise,} \end{cases}$$

where $j = i \pm 1$ and ρ is a penalty factor (a large positive number).

Finite difference approximation – cont.

The numerical algorithm can be written in the concise form

$$\big(I + (1-\theta)\Delta\tau M + P(V^{n+1})\big)V^{n+1} = (I - \theta\Delta\tau M)V^n + P(V^{n+1})g^{n+1},$$

where V^n is a vector with entries V_i^n and g^n a vector with entries g_i^n,

$$[MV^n]_i = -\left(\sum_{j=i\pm1} (\gamma_{ij} + \beta_{ij})\left(V_j^n - V_i^n\right) - rV_i^n\right)$$

and $P(V^n)$ is a diagonal matrix with entries

$$[P(V^n)]_{i,i} = \begin{cases} \rho, & \text{for } V_i^n < g_i^n, \\ 0, & \text{otherwise.} \end{cases}$$

Finite difference approximation – cont.

Matrix M has the property of strict diagonal dominance. It has positive diagonal and non-positive off-diagonals with diagonal entries strictly dominating sum of absolute values of off-diagonal entries. This property of M is essential for the convergence of the method and is visible from the structure of the upper left corner of the matrix

$$M = \begin{pmatrix} r+\gamma_{12}+\beta_{12} & -\gamma_{12}-\beta_{12} & 0 & \dots \\ -\gamma_{21}-\beta_{21} & r+\gamma_{21}+\beta_{21}+\gamma_{23}+\beta_{23} & -\gamma_{23}-\beta_{23} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

Note that in the vector on the right hand side $(I-\theta\Delta\tau M)V^n$ the first and the last elements have to be modified to take into account the boundary conditions.

Convergence

Theorem.

Let us assume that

$$\gamma_{ij} + \beta_{ij} \geq 0,$$

$$2 - \theta\left(\Delta\tau \sum_{j=i\pm1} (\gamma_{ij} + \beta_{ij}) + r\Delta\tau\right) \geq 0,$$

$$\frac{\Delta\tau}{\Delta s} < const.$$

$$\Delta\tau, \Delta s \longrightarrow 0,$$

where $\Delta s = \min_i (s_{i+1} - s_i)$.

Convergence – cont.

Then the numerical scheme for the LCP from the previous slides solves

$$
\begin{aligned}
&\frac{\partial V(s,\tau)}{\partial \tau} - rs\frac{\partial V(s,\tau)}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + rV(s,\tau) \geq 0, \\
&V_i^{n+1} - g_i^{n+1} \geq -\frac{C}{\rho},\ C > 0, \\
&\left(\frac{\partial V(s,\tau)}{\partial \tau} - rs\frac{\partial V(s,\tau)}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + rV(s,\tau) = 0\right) \vee \\
&\vee \left(|V_i^{n+1} - g_i^{n+1}| \leq \frac{C}{\rho}\right),
\end{aligned}
$$

where C is independent of ρ, $\Delta\tau$ and ΔS.

Iterative solution

Since we get a nonlinear equation for V^{n+1} it has to be solved by iterations. We shall use here the simple iteration method.

Let $(V^{n+1})^{(k)}$ be the k-th estimate for V^{n+1}. For notational convenience, we write

$$V^{(k)} \equiv (V^{n+1})^{(k)} \text{ and } P^{(k)} \equiv P\big((V^{n+1})^{(k)}\big).$$

If $V^{(0)} = V^n$, then we have the following algorithm of Penalty American Constraint Iteration.

Algorithm

Input: V^n, tolerance tol.

$$V^{(0)} = V^n$$

For $k = 0, \ldots$ until convergence
solve

$$\big(I + (1-\theta)\Delta\tau M + P^{(k)}\big)V^{(k+1)} = (I - \theta\Delta\tau M)V^n + P^{(k)} g^{n+1}.$$

If

$$\max_i \frac{|V_i^{(k+1)} - V_i^{(k)}|}{\max(1, |V_i^{(k+1)}|)} < tol$$

or

$$P^{(k+1)} = P^{(k)}$$

quit.

$$V^{n+1} = V^{(k+1)}$$

Output: V^{n+1}

Convergence of iterations

Theorem.
Let

$$\gamma_{ij} + \beta_{ij} \geq 0,$$

then

- the nonlinear iteration converges to the unique solution to the numerical algorithm of the penalized problem for any initial iterate $V^{(0)}$;
- the iterates converge monotonically, i.e., $V^{(k+1)} \geq V^{(k)}$ for $k \geq 1$;
- the iteration has finite termination; i.e. for an iterate sufficiently close to the solution of the penalized problem, convergence is obtained in one step

Size of ρ

In theory, if we are taking the limit as $\Delta s, \Delta\tau \to 0$, then we should have

$$\rho = O\left(\frac{1}{\min\big((\Delta s)^2, (\Delta\tau)^2\big)}\right).$$

This means that any error in the penalized formulation tends to zero at the same rate as the discretization error. However, in practice it seems easier to specify the value of ρ in terms of the required accuracy. Then we should take

$$\rho \approx \frac{1}{tol}.$$

Speed up iterates convergence

Although the simple iterates converge to the solution of the nonlinear problem its speed of convergence is rather slow.

To make the convergence more rapid we can use the **Newton iterates**. This requires to write the nonlinear equation in the form $F(x) = 0$ and solve the iterative procedure

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k),$$

where $F'(x)$ is the Jacobian of F.

In the penalty method algorithm the only nonlinear term which requires differentiation in order to obtain F' is P_i^n. Unfortunately, this term is discontinuous. A good convergence can be obtained when we define the derivative of the penalty term as

$$\frac{\partial P_i^{n+1}(g_i^{n+1} - V_i^{n+1})}{\partial V_i^{n+1}} = \begin{cases} -\rho, & \text{for } V_i^{n+1} < g_i^{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

American call option

Solving linear complementarity problem for an American call option we have to impose a boundary condition on artificial boundary S_{max}. Let us recall that for an Europen call option the suggested boundary condition is

$$\left(S_{max} - Ke^{-r(T-t)}\right)^{+}$$

or for a stock paying dividend of constant rate d

$$\left(S_{max}e^{-d(T-t)} - Ke^{-r(T-t)}\right)^{+}.$$

Due to the early exercise possibility for an American call option the boundary condition on artificial boundary S_{max} should be

$$\left(S_{max} - K\right)^{+}.$$

American call option – cont.

For large S_{max} and $r > 0$ we have

$$\left(S_{max} - K\right)^+ < \left(S_{max} - Ke^{-r(T-t)}\right)^+.$$

Hence for large S, the price of an American call option is smaller than the price of an European call option, which contradicts the theorem that these prices are equal.

The problem comes from the replacement of an original boundary condition for $S \to \infty$ by a condition at artificial boundary S_{max}. To avoid the problem we should impose the following boundary condition at S_{max}

$$\max\left(\left(S_{max} - K\right)^+, \left(S_{max}e^{-d(T-t)} - Ke^{-r(T-t)}\right)^+\right).$$

American barrier options

Contrary to European barrier options for American barrier options we do not have *in-out* parity. In general, the sum of prices of knock-in and knock-out options is not equal to the price of the corresponding American vanilla option.

Let

$$\tau_X^{(D)} = \inf\{t \leq T \colon S_t \leq X\} \quad \tau_X^{(U)} = \inf\{t \leq T \colon S_t \geq X\}$$

be the first time the price of the underlying asset falls below (rise above) the barrier X. Then for options with payoff $g(S_t, t)$ we have

$$\sup_{\tau_1 \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-r\tau_1} g(S_{\tau_1}, \tau_1) 1_{\{\tau_1 < \tau_X^{(\cdot)}\}}\right] + \sup_{\tau_2 \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-r\tau_2} g(S_{\tau_2}, \tau_2) 1_{\{\tau_2 > \tau_X^{(\cdot)}\}}\right] \geq \sup_{\tau_3 \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-r\tau_3} g(S_{\tau_3}, \tau_3)\right]$$

and the equality holds only when all stopping times τ_1, τ_2 and τ_3 are equal.

American knock-in options

Down-and-in option.

The option condition is: when the asset price S_t falls below X the holder of the option receives an American vanilla option, with maturity date T and strike price K.

To price that option we consider only the interval $[X, +\infty)$. In this interval there is no early exercise possibility and we gave an European option which fulfils the Black-Scholes equation

$$\frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - rV(S,t) = 0.$$

with terminal condition

$$V(S,T) = 0 \text{ for } S > X.$$

and boundary conditions

$$V(S,t) \to 0 \text{ as } S \to \infty, \quad V(X,t) = C(X,t),$$

where $C(S,t)$ is the price of the corresponding American vanila option.

American knock-in options – cont.

Up-and-in option.

The option condition is: when the asset price S_t rises above X the holder of the option receives an American vanilla option, with maturity date T and strike price K.

We price the option in the interval $[0, X]$. In this interval there is no early exercise possibility and we gave an European option which fulfils the Black-Scholes equation with terminal condition

$$V(S,T) = 0 \text{ for } S < X.$$

and boundary conditions

$$V(S,t) \to 0 \text{ as } S \to 0,$$
$$V(X,t) = C(X,t),$$

where $C(S,t)$ is the price of the corresponding American vanila option.

American knock-out options

These options are options with early exercise provision. Terminal conditions are as for American vanila options modified by limitations coming from boundary conditions.

Boundary conditions:

Down-and-out option

$$V(S,t) = h(X,t), \text{ for } S = X,$$
$$V(S,t) = \text{boundary value for vanila option}, \text{ for } S > X.$$

Up-and-out option

$$V(S,t) = \text{boundary value for vanila option}, \text{ for } S < X,$$
$$V(S,t) = h(X,t), \text{ for } S = X.$$

Here $h(X,t)$ is a payoff function on barrier X (see the next slide).

American knock-out options – cont.

From both financial and mathematical point of view, payoff function $h(X,t)$ is equal to zero.

But before the stock price crosses the trigger X, the American option is active and its price must be no less than the option's payoff function $g(S_t,t)$ ($(S_t - K)^+$ for call option and $(K - S_t)^+$ for put option). Hence, up to the barrier the price of the option must be bounded from below by the payoff function $g(S_t,t)$.

Assuming $h(X,t) = 0$ will create a jump on the barrier since before the stock price crosses the trigger X the price $V(S,t) \geq g(S,t)$. If the option price is continuous up to the barrier then we shall get $V(X,t) \geq g(X,t)$. That suggests to take as the payoff function on the barrier $h(X,t) = g(X,t)$.

This choice does not change the resulting price $V(S,t)$ but has obvious numerical adventage: we do not solve a problem with jumps on a boundary. As a result we obtain a numerical algorithm which is more stable (jumps on a boundary can create oscillations in numerical solutions).