Poisson noisy image deblurring

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Abstract

This project deals with the restoration of images corrupted by a non-invertible or ill-conditioned linear transform (blur) and Poisson noise. Poisson data typically occur in computed tomography where the images are obtained by counting particles, e.g., photons, that hit the image. This project is experimental and can be seen as a discussion of "as reported by Harizanov et al. (Epigraphical Projection for Solving Least Squares Anscombe Transformed Constrained Optimization Problems 2013)" [HPS13]. *The exposition there was in part incomplete and raises unanswered questions. We will go into implementation details and review some of the key formulas.*

Keywords: Poisson noise, Gaussian noise, deblurring, Anscombe transform, C++, image processing, tomography, primal-dual hybrid gradient algorithm, constraint optimization

1. Introduction

The presence of noise in images is unavoidable. It may be introduced by the image formation process, image recording, image transmission, etc. These random distortions make it difficult to perform any required picture processing. For example, industrial computed tomography (CT) scanning uses irradiation (usually with x-rays) to produce three-dimensional representations of the scanned object both externally and internally. The latter is derived from a large series of two-dimensional radiographic images taken around a single axis of rotation. To create each of the planar images, a heterogeneous beam of X-rays is produced and projected toward the object. A certain amount of X-ray is absorbed by the object, while the rest is captured behind by a detector (either photographic film or a digital detector). The local magnitudes of the detected X-ray amount determine the corresponding gray-scale pixel values of the radiographic image. In such processes, where images are obtained by counting particles, Poisson noise occurs. Even a small amount of noise is harmful when high accuracy is required, e.g. as in subcell (subpixel) image analysis.

Deblurring images corrupted by Poisson noise is a challenging process to which much research has been devoted, as in astronomical or biological imaging. In this project we review and experiment with state of the art techniques than discuss some of their harder to implement aspects.

2. Single-Constraint Optimization

We want to recover an original signal $\bar{u} \in [0, +\infty)^m$ from observations $f = P(H\bar{u})$, where P denotes an independent Poisson noise corruption process and $H \in [0, +\infty)^{n \times m}$ is a linear degradation operator, e. g. a blur.

The Poisson distribution exhibits a mean/variance relationship. This dependence can be reduced by using variance-stabilizing transformations, one of which is the Anscombe transform defined as

$$T(v) = 2\left(\sqrt{v_i + \frac{3}{8}}\right)_{1 \leq i \leq n}$$

It transforms Poisson noise to approximately Gaussian noise with zero-mean and unit variance (if the variance of the Poisson noise is large enough).

We adopt a constrained approach for minimizing the data fidelity in order to deblur $\bar{u}$. Where the data fidelity is equal to

$$\|T(H\bar{u}) - T(f)\|_2^2$$

And we impose that

$$\|T(H\bar{u}) - T(f)\|_2^2 < \tau$$

where $\tau \in [0, +\infty)$. Based on the statistical properties of the Anscombe transform and the law of large numbers, a consistent choice for the above bound is $\tau = n$, when the number of observations n is large.

3. Projections

Before diving into the algorithm we will be using we will list a few projections which it uses.

$$P_{V_\tau}(u) = \begin{cases} u_i, & \langle 1_n, u\rangle \le \tau \\ u_i + \dfrac{\tau - \langle 1_n, u\rangle}{n}, & else \end{cases}$$

$$for\ i\epsilon\{1,\dots,n\}$$

Where n is the number of pixels. P_{V_τ} is the formula for projecting an image onto the V_τ half-space.

$$V_\tau = \{\zeta \in R^n \colon \langle 1_n, \zeta\rangle \le \tau\}$$

$$P_C(u) = \max\{0, u_i\}\ for\ i\epsilon\{1,\dots,n\}$$

P_C is the formula for orthogonal projecting an image onto the nonnegative orthant of R^n. In fact we are doing a box constraint on the pixel values $P_C(u) = \max\{0, \min\{v, u_i\}\}\ for\ i\epsilon\{1,\dots,n\}$ so that we do not get overflow artefacts when we cast back to byte representation.

$$P_{epi_\varphi}(x,\zeta) = \begin{cases} (\max\{x,0\}, \zeta)\ if\ \ \varphi(\max\{x,0\}) \le \zeta, \\ \left(\left(\dfrac{t_+ + z}{2}\right)^2, t_+^2\right)\ if\ 4x \ge z^2, \\ \left(\left(\dfrac{t_- + z}{2}\right)^2, t_-^2\right)\ if\ 4x < z^2 \end{cases}$$

P_{epi_φ} is the epigraphical projection of some point (x,ζ). Where the function $\varphi \in \Gamma_0(R)$ is defined as

$$\varphi(s) = \begin{cases} \left(2\sqrt{s} - z\right)^2, & s \ge 0 \\ +\infty, & else \end{cases}$$

$$epi\varphi = \{(v,\zeta) \in R^n \times R \colon \varphi(v) \le \zeta\}$$

where $z > 0$ check **Figure 1**. t_+ and t_- is the unique root in $[0, +\infty)$, respectfully in $(-z, 0)$ of the cubic polynomial

$$p(t) = 17t^3 + 3zt^2 +$$

$$(3z^2 - 16\zeta - 4x)t + z(z^2 - 4x)$$

It can be proofed that with a finite number of steps Newton's method for finding a zero of p converges.

$$t_0 = 2\sqrt{\max\{x,0\}}$$

$$p'(t_0) = 51t^2 + 6zt + 3z^2 - 4x - 16\zeta$$

$$t_{i+1} = t_i - \frac{p(t_i)}{p'(t_i)}$$

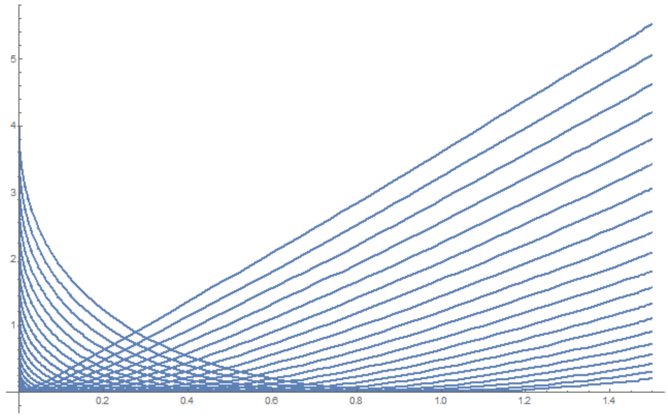

Figure 1: *Plots of $\varphi(s)$ for $z(0,1.5]$*

4. Gaussian Kernel

Applying a Gaussian blur to an image is the same as convolving the image with a Gaussian function. This is also known as a two-dimensional Weierstrass transform. By contrast, convolving by a circle (i.e., a circular box blur) would more accurately reproduce the bokeh effect. Since the Fourier transform of a Gaussian is another Gaussian, applying a Gaussian blur has the effect of reducing the image's high-frequency components a Gaussian blur is thus a low pass filter.

$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)^2}{\sigma_X\sigma_Y}\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

This is the bivariate normal distribution probability distribution function, where ρ is the correlation between X and Y, $\sigma_{X,Y} > 0$ and $\mu_{X,Y}$ are the variance and mean respectfully . In our use case $\rho = 0$, $\sigma_X = \sigma_Y = \sigma$ and $\mu_X = \mu_Y = \mu$, therefore we get

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\mu)^2 + (y-\mu)^2}{2\sigma^2}}$$

The kernel width is estimated from the 68–95–99.7 rule, so $width = 2\lceil 3\sigma\rceil + 1$. With it we can compute a convolution Gaussian kernel. Check **Table 1** for an example.

6.59e-06	0. 00042	0.001704	0. 00042	6.59e-06
0. 00042	0.027398	0.109878	0.027398	0. 00042
0.001704	0.109878	0.440655	0.109878	0.001704
0. 00042	0.027398	0.109878	0.027398	0. 00042
6.59e-06	0.023700	0.001704	0. 00042	6.59e-06

Table 1: *The values of Gaussian convolution filter width 5 elements with $\sigma_x = \sigma_Y = 0.6, \mu_X = \mu_y = 0$ and $\rho = 0$*

5. Problem

We will adopt a constrained approach by imposing that:

$$\|T(Hu) - T(f)\|_2^2 \leq \tau$$

Based on the statistical properties of the Anscombe transform and the law of large numbers, a consistent choice for the above bound is τ = n, when the number of observations n is large. The problem we need to solve is:

$$\min_{u \in C} \Phi(Lu) \, subject\ to\ \|T(Hu) - T(f)\|_2^2 \leq \tau$$

Where C is a nonempty closed convex subset of $[0,+\infty)^m$, $L \in R^{q \times m}$ and $\Phi: R^q \to (-\infty,+\infty]$ is a proper, lower-semicontinuous, convex function. This can be rewritten as:

$$\min_{(u,\zeta) \in R^m \times R^n} \iota_C(\mathrm{u}) + \Phi(Lu) + \sum_{i=1}^{n} \iota_{epi_{\varphi_i}}\left((Hu)_i + \frac{3}{8}, \zeta_i\right) + \iota_{V_\tau}(\zeta)$$

6. Primal-Dual Algorithm

We have implemented the **p**rimal-**d**ual **h**ybrid **g**radient algorithm with an extrapolation (**m**odification) on the dual variable [STE12] [JCP12]. Based on the following reformulation of the above stated problem:

$$\min_{(u,\zeta)(v_1,v_2,\eta)} \iota_C(\mathrm{u}) + \iota_{V_\tau}(\zeta) + \Phi(v_2) + \sum_{i=1}^{n} \iota_{epi_{\varphi_i}}(v_{1,i}, \eta_i) \ subject\ to$$

$$\begin{pmatrix} H & 0 \\ L & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ \zeta \end{pmatrix} + \begin{pmatrix} 3/8 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \eta \end{pmatrix}$$

The algorithm reads:

For $k = 1, \dots$ repeat until stop criterion is reached

$$u^{(k+1)} = P_C\left(u^{(k)} - \sigma\rho\left(H * \bar{p}_1^{(k)} + L * \bar{p}_2^{(k)}\right)\right)$$

$$\zeta^{(k+1)} = P_{V_\tau}(\zeta^{(k)} - \sigma\rho\bar{p}_3^{(k)})$$

$$\left(v_{1,i}^{(k+1)}, \eta_i^{(k+1)}\right) = P_{epi\varphi_i}\left(p_{1,i}^{(k)} + \left(Hu^{(k+1)}\right)_i + \frac{3}{8}, p_{3,i}^{(k)} + \zeta_i^{(k+1)}\right) \quad for\ i = 1, \dots, n$$

$$v_2^{(k+1)} = prox_{\sigma^{-1}\phi}\left(p_2^{(k)} + Lu^{(k+1)}\right)$$

$$p_1^{(k+1)} = p_1^{(k)} + Hu^{(k+1)} + \frac{3}{8} - v_1^{(k+1)}$$

$$p_2^{(k+1)} = p_2^{(k)} + Lu^{(k+1)} - v_2^{(k+1)}$$

$$p_3^{(k+1)} = p_3^{(k)} + \zeta^{(k+1)} - \eta^{(k+1)}$$

$$\bar{p}_j^{(k+1)} = p_j^{(k+1)} + \theta\left(p_j^{(k+1)} - p_j^{(k)}\right) \ for\ j = 1,2,3$$

Where for a stop criterion we are using $k < 1000$ and $u^{(0)}$ is the initial corrupted image. Also $(\sigma,\rho) \in (0,+\infty)^2$ and $\sigma\rho < \frac{1}{\max\{1, \|H*H+L*L\|_2\}}$, however the exact values are a work of art and magic. In the last section we will see that $\tau = n$, where n is the number of pixels in the image is a good choice. For $\theta \in (0,1]$ we are using $\theta = 1$. The algorithm is initialized with $\zeta^{(0)}, \left(p_j^{(0)}\right)_{1 \leq j \leq 3} = \left(\bar{p}_j^{(0)}\right)_{1 \leq j \leq 3}$ with all elements set to zero.

$$\Phi = l_{2,1}$$

$$prox_{\sigma^{-1}\phi}(x) = x\left(1 - \frac{\sigma^{-1}}{\max\{x^2, \sigma^{-1}\}}\right)$$

The proximation can be performed by coupled soft shrinkage with threshold σ^{-1} if we use the $l_{2,1} - norm$ [YIN16].

7. Numerical Experiments

In this section we demonstrate the performance of the studied algorithms by numerical example implemented in C++ (Intel Core i7-4510U Processor with 4M Cache, 2.00 GHz, 6 GB physical memory). We have tested the original images $\bar{u}$, namely 'cameraman' (256x256), 'lena' (512x512), 'peppers' (256x256) and 'boats' (720x576). Refer to **Figure 2**.

The images were blurred by matrix H corresponding to a convolution with a Gaussian kernel with standard deviation 1.3 and mirrored boundary (we have then $m = n$). Their gray values are interpreted as photon counts in the range $[0, \nu]$, where ν is the intensity of the images. We tested with intensity of 255.

We add Poisson noise to our test image by using C++'s build in std::poisson_distribution random number generator by using the current pixel's intensity for distribution's mean and sampling it. We do not do any form of value scaling as it is done in MATLAB's imnoise('poisson'). Noise is added while the image is stored as an array of 8-bit integer values.

Figure 2: Original, corrupted and restored versions of each test image

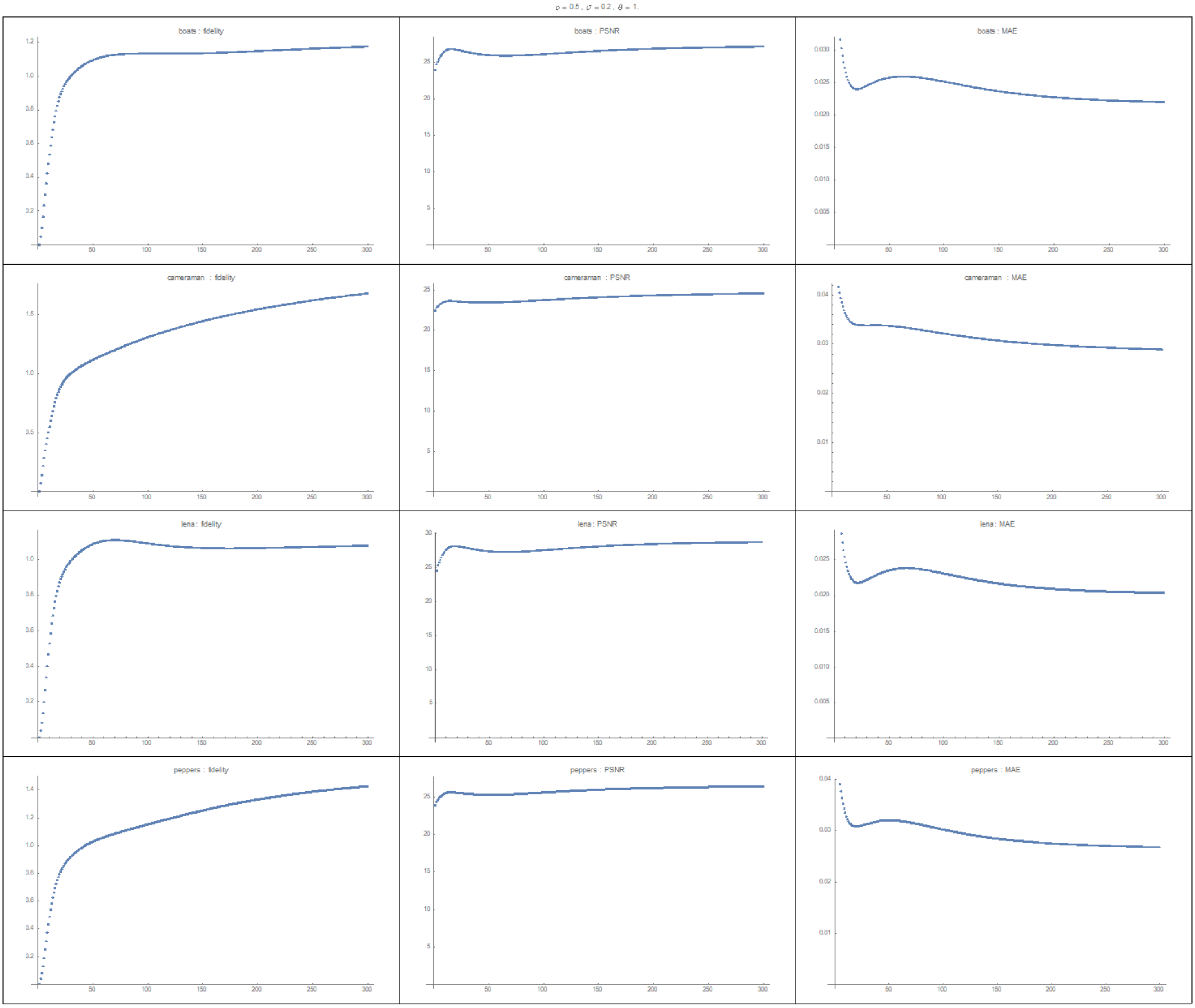

Figure 3: The values of $fidelity$, $PSNR$and MAE for each image

For a quantitative comparison of the images, we computed the peak signal to noise ratio ($PSNR$) and the MAE defined by $PSNR = 10\log_{10}\frac{|\max\bar{u}-\min\bar{u}|^2}{\frac{1}{n}\|u-\bar{u}\|_2^2}$ and $MAE = \frac{1}{nv}\|\bar{u}-u\|_1$. We also compute the value of data fidelity $\frac{\|T(H\bar{u})-T(f)\|_2^2}{n}$. Check **Figure 3** for values.

8. Summary

We have studied, implemented and explained a constrained restoration model for images corrupted by a linear transform and Poisson noise by making use of the Anscombe transform. We have detailed all needed formulas used in the algorithm for finding minimizer of the model, which.

As future work one can try various restriction techniques for the constrained set in order to improve the resulting image [Har16]. One of particular interest consists of a simple block subdivision of the spatial domain.

9. References

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