CS5787: Exercises 3

https://github.com/mitkrieg/dl-assignment-3

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1 Theory: Question 1 [40 pts]

a) $\theta \neq 0$

- KL Divergence:
When $\theta = 0$ there is no overlap between P and Q. This means that whenever P is greater than zero, Q is zero. Applying this to the KL formula at $x = 0$ and assume that we can represent x as the Dirac delta function $\delta(x)$, and $u(y)$ is the uniform distribution over [0,1].

$$\begin{aligned} D_{KL}(P||Q) &= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} P(x,y)\log\frac{P(x,y)}{Q(x,y)}dydx \\ &= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \delta(x)\cdot u(y)\log\frac{\delta(x)\cdot u(y)}{\delta(x-\theta)\cdot u(y)}dydx \\ &= \int_{x=-\infty}^{\infty}\delta(x)\int_{y=0}^{1}\log(\frac{\delta(x)\cdot 1}{\delta(x-\theta)\cdot 1})dx \end{aligned} \tag{1}$$

Since, $\delta(x)$ is zero everywhere except $x = 0$

$$\begin{aligned} D_{KL}(P||Q) &= \int_{y=0}^{1}\log(\frac{\delta(0)}{\delta(-\theta)})dy \\ &= \frac{\delta(0)}{0} = \infty \end{aligned} \tag{2}$$

We can see from the above that because the logarithm in the equation is undefined, we get a value for KL Divergence that tends to ∞.

- JS Divergence:
The JS Divergence considers the KL Divergence between each distribution and the average of the two distributions. The average distribution $\frac{P+Q}{2}$ will split the distribution to be half at $x = 0$ and half at $x = \theta$. Therefore, The KL Divergence between P and the average will be:

$$\begin{aligned} D_{KL}(P||\frac{P+Q}{2}) &= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} P(x,y)\log\frac{P(x,y)}{\frac{1}{2}P(x,y)+\frac{1}{2}Q(x,y)}dydx \\ &= \int_{x=-\infty}^{\infty}\delta(x)\int_{y=0}^{1}\log(\frac{\delta(x)\cdot u(y)}{\frac{1}{2}\delta(x)\cdot u(y)+\frac{1}{2}\delta(x-\theta)\cdot u(y)}) \end{aligned} \tag{3}$$

Since, $\delta(x)$ is zero everywhere except $x = 0$, and $\delta(x-\theta) = 0$ at $x = 0$

$$\begin{aligned} D_{KL}(P||\frac{P+Q}{2}) &= \int_{y=0}^{1}\log(\frac{\delta(0))\cdot u(y)}{\frac{1}{2}\delta(0)\cdot u(y)+\frac{1}{2}\cdot 0\cdot u(y)}) \\ &= \log(\frac{1}{\frac{1}{2}}) \\ &= \log(2) \end{aligned} \tag{4}$$

$D_{KL}(Q||\frac{P+Q}{2})$ will give the same result. So applying the JS Divergence formula:

$$D_{JS}(P||Q) = \frac{1}{2}\cdot\log(2)+\frac{1}{2}\cdot\log(2) = \log(2) \tag{5}$$

- Wasserstein Distance:
The Wasserstein Distance is θ because both distributions have the same uniform distribution between [0,1] for y. Because they are identical except for the x-value of P is 0 and the x-value of Q is θ, the cost of moving P to Q would be θ.

b) $\theta = 0$

- KL Divergence:
Zero because the distributions are the same

$$\begin{aligned} D_{KL}(P||Q) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x)\log\frac{P(x,y)}{P(x,y)}dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y)\log 1 dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 0\cdot P(x,y)dydx \\ &= 0 \end{aligned} \tag{6}$$

- JS Divergence:
$\frac{P+Q}{2} = P = Q$ because both distributions are uniform between [0,1] for y and centered at $x = 0$. So the JS Divergence is zero because the distributions are the same

$$\begin{aligned} D_{JS}(P||Q) &= \frac{1}{2}D_{KL}(P||\frac{P+Q}{2}) + \frac{1}{2}D_{KL}(Q||\frac{P+Q}{2}) \\ &= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0 \end{aligned} \tag{7}$$

- Wasserstein Distance: Zero because both distributions are uniform between [0,1] for y and centered at $x = 0$. So there is no cost to transform P into Q.

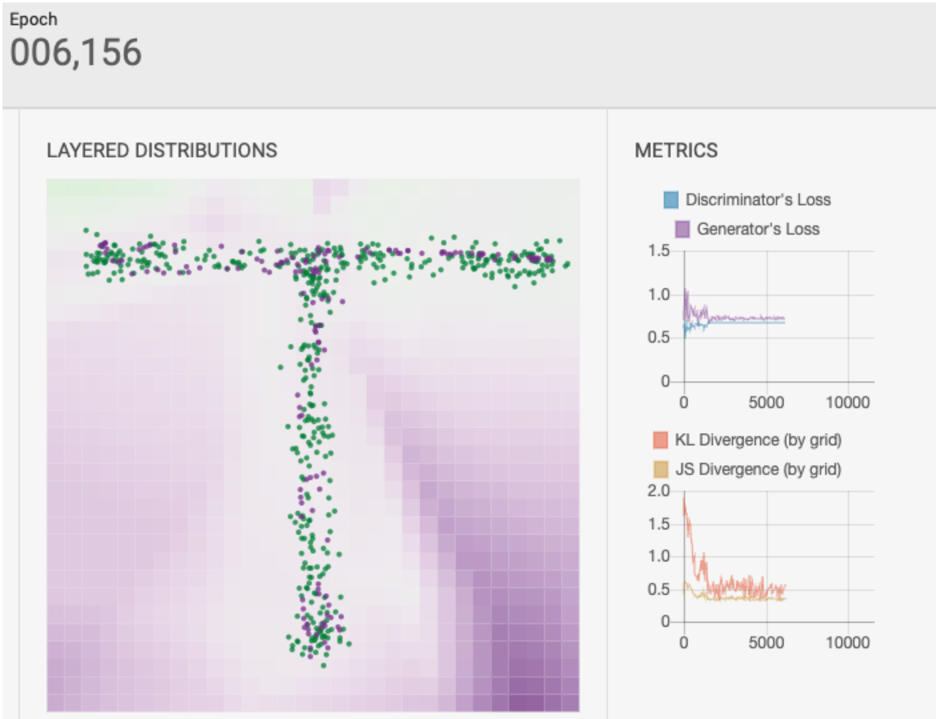
c) The advantage of the Wasserstein Distance is that it is both interpretable and stable when compared to KL and JS Divergence. When $\theta \neq 0$, KL is infinite and JS is $\log 2$, where as Wasserstein is θ. In this context, θ is a much clearer measure of the distances between the two distribution because it is simply the distance between the x-values of the distributions. In addition, both KL and JS have discontinuous jumps from when they overlap to when then don't, KL becomes infinite and JS although still finite jumps from 0 to $\log 2$. Wasserstein distance on the other hand, increases linearly as θ grows. These are big advantages because it would make the gradient in training a model more stable because it is continuous and not super sensitive to small changes (from not overlapping to overlapping).

2 Theory: Question 2 [10 pts]

LSTMs will process inputs sequentially. So at each time step, the LSTM will have to multiply the input by the hidden state, the previous hidden state by the current hidden state, and other smaller operations like the activation functions for the gates and addition for adding bias and cell state updates. The dominant operation here is the multiplication of the hidden states. This operation has a time complexity of $O(d^2)$ where d is the dimension of the hidden state. Because we do this at every time step t for all N, the total complexity is $O(d^2N)$. However, the size of the hidden state is a constant value, and not a function of the size of the input, so the total time complexity is linear $O(N)$.

Transformers on the other hand use self-attention, which requires every value of the input to be compared to all other values in the input. This is an $O(N^2)$ operation. The other components of transformers, such as the feed forward network, encoder/decoder and positional encoding are all generally $O(N)$, so they get dominated by the self-attention mechanism's computational complexity. So the total time complexity is quadratic at $O(N^2)$

3 Practical: Question 3 [5 pts]



4 Practical: Question 4 [20 pts]

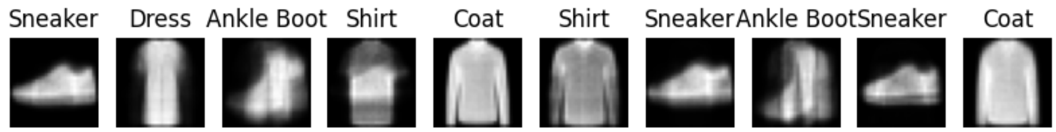
The goal for this problem was to implement the M1 architecture described in "Semi-supervised Learning with Deep Generative Models" (Kingsma et al.) for the Fashion MNIST dataset. M1 attempts to build a generative model that can create images in the likeness of the the data using a Variational autoencoder (VAE) with an SVM head. This model was trained using a semi-supervised method. The encoder in the VAE aims to predict the mean and variance of the distribution over the latent variable z. However, because of the random nature of z, we are unable to take the derivative of it which means it can't be used in back propagation. To address this, the reparameterization trick is used, where instead of directly sampling from this distribution we sample from the standard gaussian distribution and then shift it by the mean and variance predicted by the encoder. Then the decoder uses this sample to reconstruct an image.

In addition, an SVM with a radial basis function kernel is trained as a head for this model to classify the generated images using the latent representations created by the encoder as input. Two SVMs were trained for each model using different regularization parameters, and the one with the better accuracy was chosen.

Four VAE models were trained each with an increasing number of labeled samples 100, 600, 1000 and 3000, and the results are described in the table below:

N	SVM Head Test Set Accuracy
100	0.5957
600	0.7015
1000	0.7093
6000	0.7291

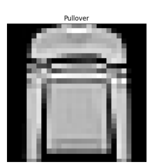
An example of the generated output and the SVM prediction is below:

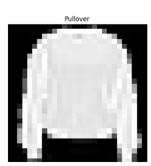


5 Practical: Question 5 [25 pts]

The goal of was to also build a generative model that can create images in the likeness of the Fashion MNIST dataset but this time using 3 GAN methods (DCGAN, WGAN with clipping, WGAN-GP) over 2 different architectures. Both architectures used have a discriminator/critic that consists of 4 convolutional layers with a kernel size of 4, stride of 2 and padding of 1 (except for the final layer has zero padding), followed by a fully connected layer. The generator has these same layers in reverse order. The main difference between the two architectures is the use of normalization and choice of activation function. In architecture A, Leaky ReLU was used with a negative slope of 0.2 as an activation function. In addition, BatchNorm (or LayerNorm in WGAN-GP) was used for normalization. Architecture B on the other hand, uses an adjusted softplus, $\frac{\text{softplus}(2x+2)}{2}$ and has no normalization.

Figure 1: Real Pullovers compared with Generated Pullovers





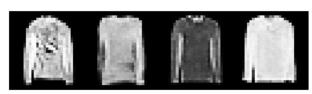


Figure 2: DCGAN

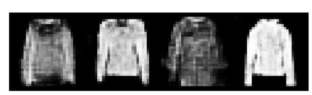


Figure 3: WGAN with Weight Clipping

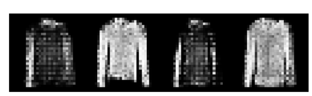
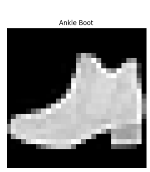


Figure 4: WGAN-GP

Figure 5: Real Ankle Boot compared with Generated Ankle Boot



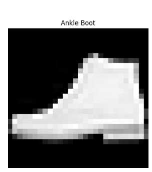


Figure 6: DCGAN

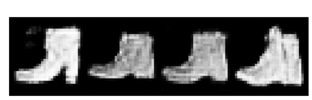


Figure 7: WGAN with Weight Clipping

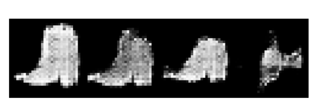


Figure 8: WGAN-GP

Architecture A had consistently better visual results after training for 10 epochs. The figures above display real samples for two classes (ankle boot and pullover) on the left compared with generated samples on the right for architecture A with each GAN method. All three models, generate images that are in the likeness of Fashion MNIST.

Loss for each model's generator and discriminator/critic has been plotted below as well. For all three models, we can see that the loss curves for architecture A converges much better than B (there was an order of magnitude difference between A and B loss for WGAN and WGAN-GP so zoomed in plots are provided on the following page). WGAN-GP's loss oscillates more frequently and has larger swings in later epochs suggesting that we may have been able to achieve the same results in less epochs. To train these models, it was easier to find convergence on WGAN and WGAN-GP than on DCGAN. For DCGAN, training was run 5 or 6 times before finding convergence. Where as with WGAN and WGAN-GP there was only 1 failed convergence and the rest converged (even if it was to a poor result).

Figure 9: Loss Curves

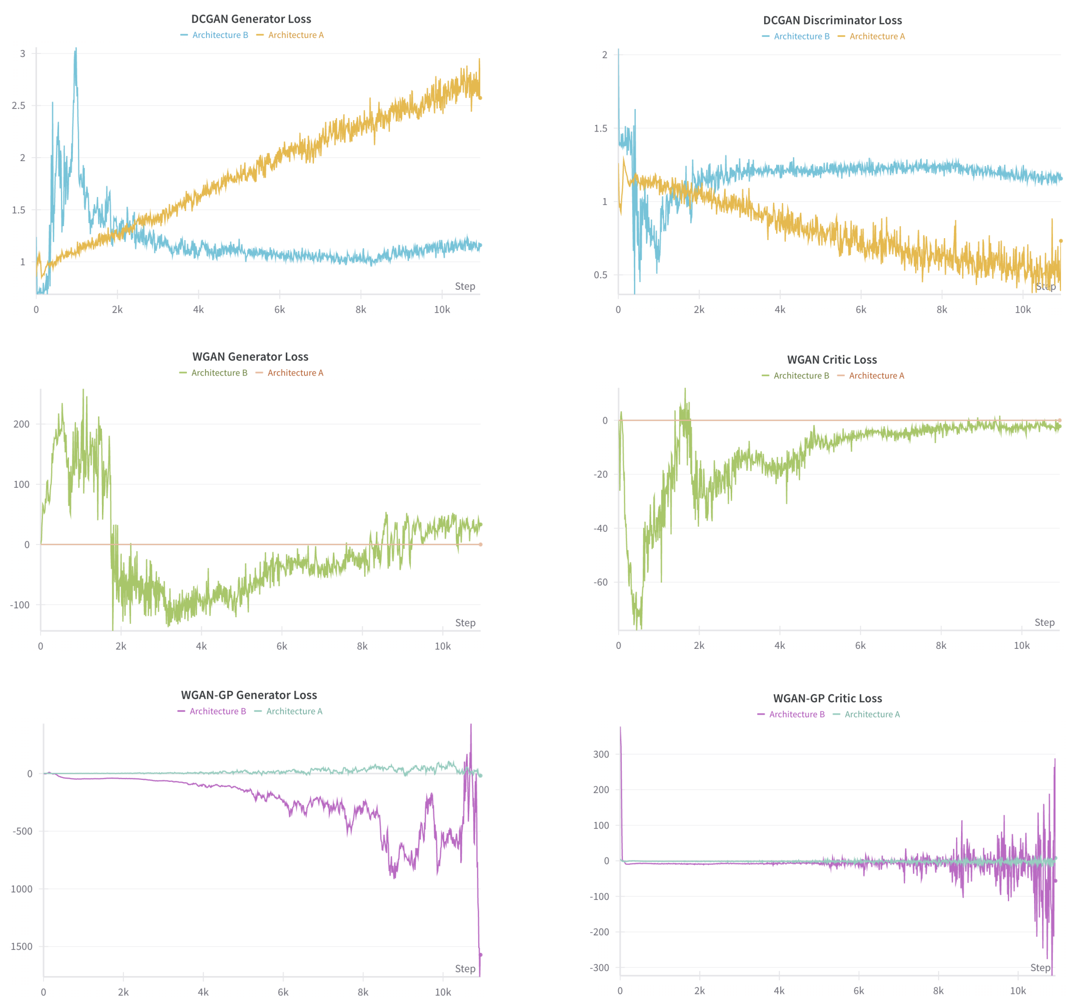


Figure 10: Zoom-In: Loss Curve for WGAN and WGAN-GP Architecture A