Derivation for multi-dimensional case

1. $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} A & B \\ 0 & D-CA^{-1}B \end{bmatrix}$ (1) assuming A is invertable

$= \det A \det[D-CA^{-1}B]$

2. $$\begin{bmatrix} y_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} I-\alpha H+uI & -uI \\ I & 0 \end{bmatrix}\begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}$$

let $\det\begin{bmatrix} I-\alpha H+uI-\lambda I & -uI \\ I & -\lambda I \end{bmatrix} = 0$.

From (1) we have

$$\det(I-\alpha H+uI-\lambda I)\det[-\lambda I+[I-\alpha H+uI-\lambda I]^{-1}u] = 0$$

For the second det, we have

$$\lambda^2 I - \lambda[I-\alpha H+uI]+uI = 0$$

i.e.

$$[\lambda I - \tfrac{1}{2}(I-\alpha H+uI)]^2 = \tfrac{1}{4}(I-\alpha H+uI)^T(I-\alpha H+uI) - uI$$

If the largest eigenvalue $\le 2\sqrt{u}$, we have

$\frac{1}{4}(I-\alpha H+uI)^T(I-\alpha H+uI)-uI$ is negative semidefinite.

Then we have $\|\lambda\| = \sqrt{u}$ (2) again for multi-dimensional case

as (2) $\iff -2\sqrt{u}I \le I-\alpha H+uI \le 2\sqrt{u}I$

$\iff (1-\sqrt{u})^2 \le \alpha\,\sigma(H) \le (1+\sqrt{u})^2$