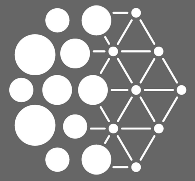


Some results on GAN dynamics

Ioannis Mitliagkas

Game dynamics are ~~weird~~
fascinating

Start with optimization dynamics

Optimization

$$\boldsymbol{\theta}^* \in \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\theta}} \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta})$$

Smooth, differentiable cost function, L

→ Looking for stationary (fixed) points (gradient is 0)

→ Gradient descent

Optimization

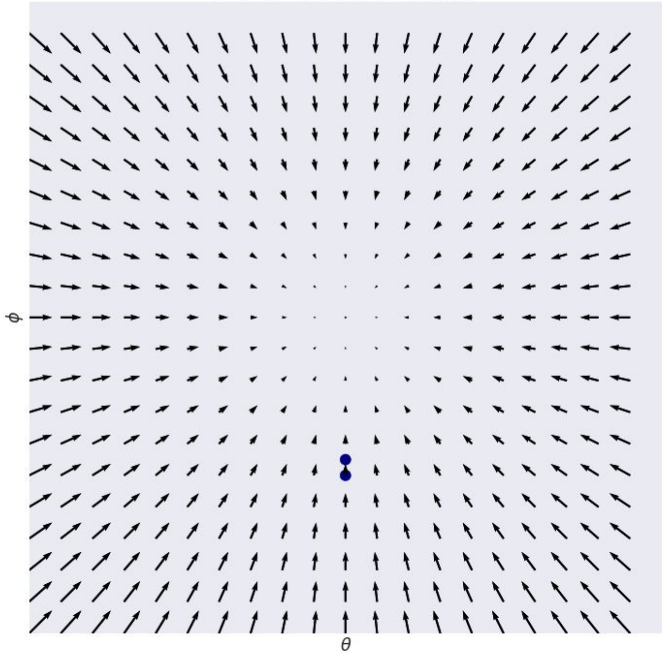
$$\boldsymbol{v}(\boldsymbol{\theta}) = \nabla \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta})$$

Conservative vector field

→

Straightforward dynamics

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \boldsymbol{v}(\boldsymbol{\theta}_t)$$



Gradient descent

$$\boldsymbol{v}(\boldsymbol{\theta}) = \nabla \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta})$$

Conservative vector field

→

Straightforward dynamics

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \boldsymbol{v}(\boldsymbol{\theta}_t)$$

Fixed-point analysis

$$F_\eta(\boldsymbol{\theta}) = \boldsymbol{\theta} - \eta \boldsymbol{v}(\theta)$$

Jacobian of operator

$$\nabla F_\eta(\boldsymbol{\theta}) = I - \eta \nabla \boldsymbol{v}(\theta)$$

Hessian of objective, L

Local convergence

Theorem 1 (Prop. 4.4.1 Bertsekas [1999]). *If the spectral radius* $\rho_{\max} \overset{def}{=} \rho(\nabla F_\eta(\boldsymbol{\omega}^*)) < 1$, *then, for* $\boldsymbol{\omega}_0$ *in a neighborhood of* $\boldsymbol{\omega}^*$, *the distance of* $\boldsymbol{\omega}_t$ *to the stationary point* $\boldsymbol{\omega}^*$ *converges at a linear rate of* $\mathcal{O}\left((\rho_{\max} + \epsilon)^t\right)$, $\forall \epsilon > 0$.

Eigenvalues of op. Jacobian

$$\lambda(\nabla F_\eta(\boldsymbol{\theta})) = 1 - \eta\lambda(\nabla \boldsymbol{v}(\theta))$$

If $\rho(\theta^*)=\max |\lambda(\theta^*)|<1$, then fast local convergence

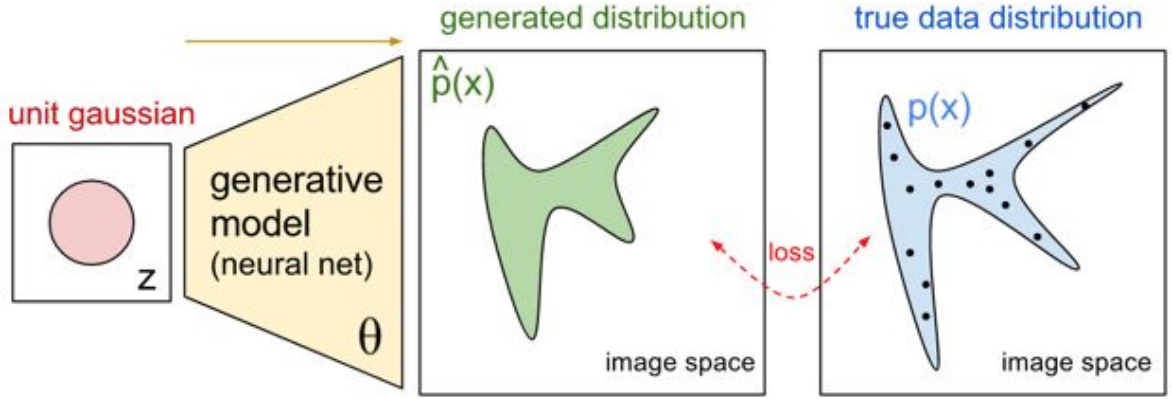
Jacobian of operator

$$\nabla F_\eta(\boldsymbol{\theta}) = I - \eta \nabla \boldsymbol{v}(\theta)$$

Hessian of objective, L
Symmetric, real-eigenvalues

Games

Implicit generative models



- Generative moment matching networks [Li et al. 2017]
- Other, domain-specific losses can be used
- Variational AutoEncoders [Kingma, Welling, 2014]
- Autoregressive models (PixelRNN [van den Oord, 2016])

Generative Adversarial Networks

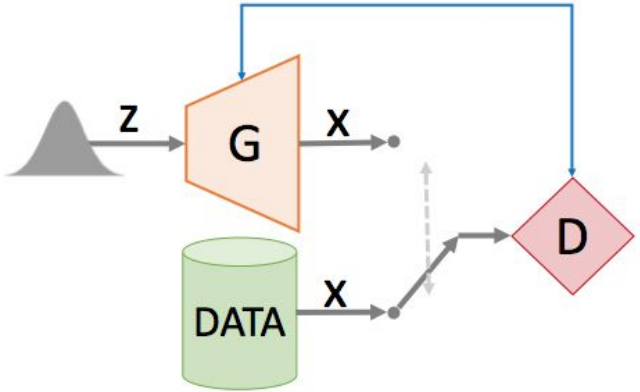
Both differentiable

Generator network, G

Given latent code, z, produces sample G(z)

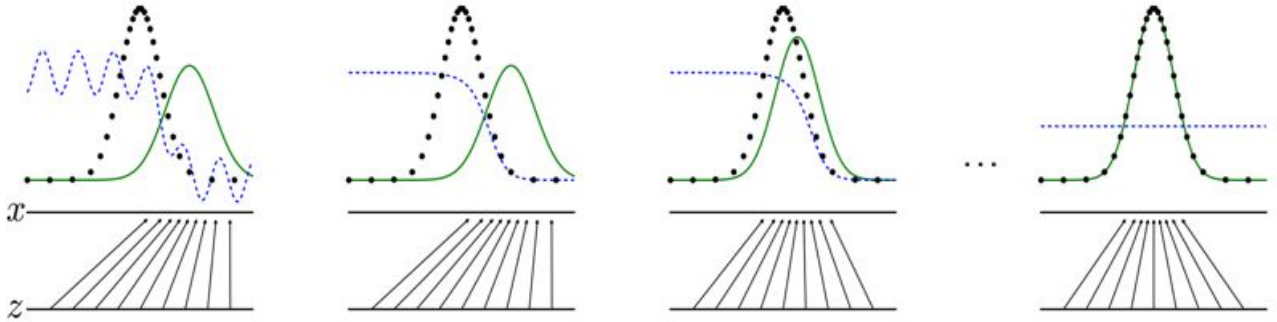
Discriminator network, D

Given sample x or G(z), estimates probability it is real



$$\min_G \max_D V(D,G) = \mathbb{E}_{x\sim\mathbb{P}_x}[\log D(x)] + \mathbb{E}_{z\sim\mathbb{P}_z}[\log(1-D(G(z)))]$$

Generative Adversarial Networks



$$\min_G \max_D V(D,G) = \mathbb{E}_{x\sim\mathbb{P}_x}[\log D(x)] + \mathbb{E}_{z\sim\mathbb{P}_z}[\log(1 - D(G(z)))]$$

Games

Nash Equilibrium

$$\boldsymbol{\theta}^* \in \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\theta}} \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\theta}, \boldsymbol{\varphi}^*)$$

$$\boldsymbol{\varphi}^* \in \arg\min_{\boldsymbol{\varphi} \in \boldsymbol{\varphi}} \mathcal{L}^{(\boldsymbol{\varphi})}(\boldsymbol{\theta}^*, \boldsymbol{\varphi})$$

Smooth, differentiable L

→ Looking for local Nash eq

→ Gradient descent

→ **Simultaneous**

→ **Alternating**

Game dynamics

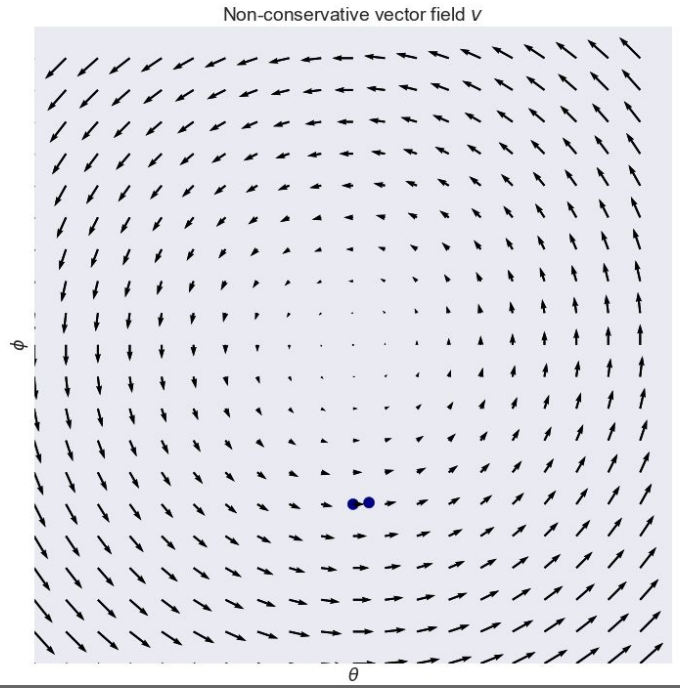
$$\boldsymbol{v}(\boldsymbol{\varphi}, \boldsymbol{\theta}) := \begin{bmatrix} \nabla_{\boldsymbol{\varphi}} \mathcal{L}^{(\boldsymbol{\varphi})}(\boldsymbol{\varphi}, \boldsymbol{\theta}) \\ \nabla_{\boldsymbol{\theta}} \mathcal{L}^{(\boldsymbol{\theta})}(\boldsymbol{\varphi}, \boldsymbol{\theta}) \end{bmatrix}$$

Non-conservative vector field

→

Rotational dynamics

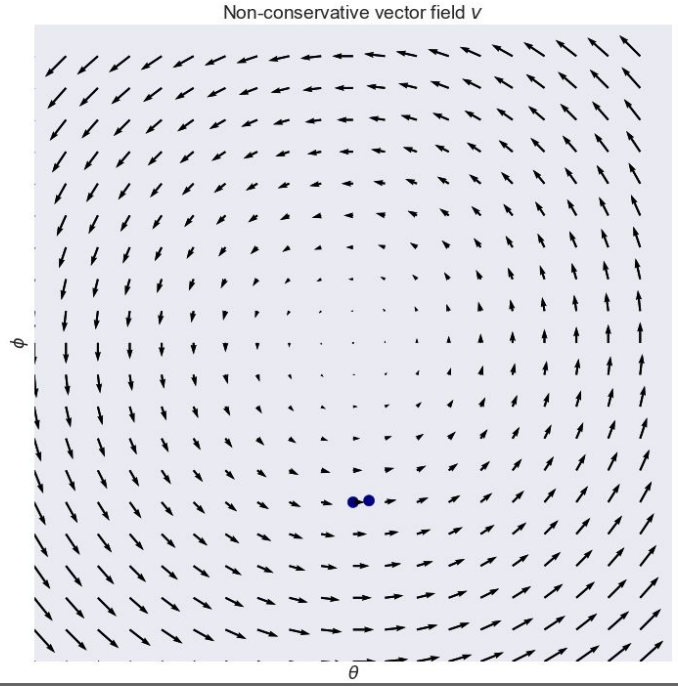
$$F_\eta(\boldsymbol{\varphi}, \boldsymbol{\theta}) \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{\varphi} & \boldsymbol{\theta} \end{bmatrix}^\top - \eta \, \boldsymbol{v}(\boldsymbol{\varphi}, \boldsymbol{\theta})$$



Game dynamics under gradient descent

$$F_\eta(\boldsymbol{\varphi}, \boldsymbol{\theta}) \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{\varphi} & \boldsymbol{\theta} \end{bmatrix}^\top - \eta \, \boldsymbol{v}(\boldsymbol{\varphi}, \boldsymbol{\theta})$$

Jacobian is non-symmetric, with complex eigenvalues → Rotations in decision space



Games demonstrate rotational dynamics.

The Numerics of GANs

by Mescheder, Nowozin, Geiger

A word on notation

$$\mathcal{L}^{(\phi)}(\phi, \theta) = -f(\phi, \theta)$$
$$\mathcal{L}^{(\theta)}(\phi, \theta) = -g(\phi, \theta)$$

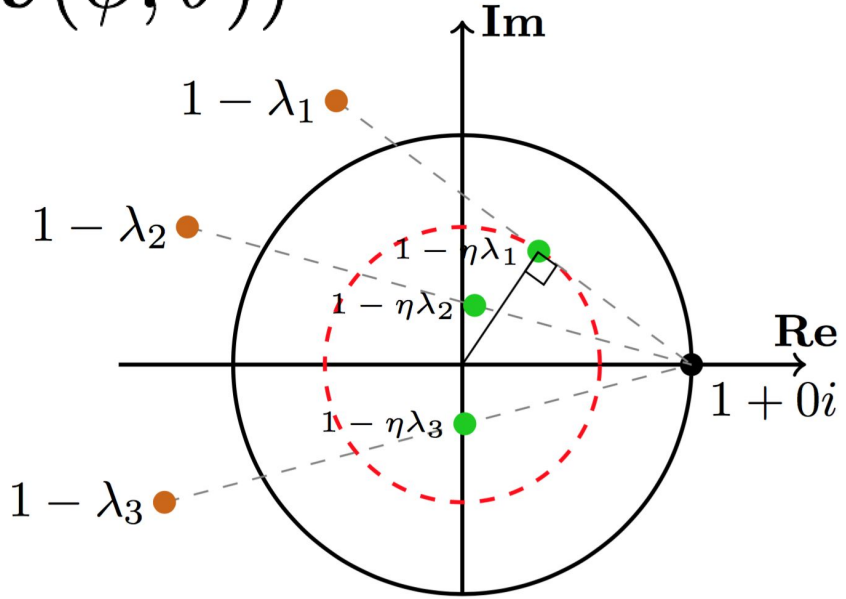
Warning: $\mathcal{L} \neq L$

Also: Maximization vs minimization

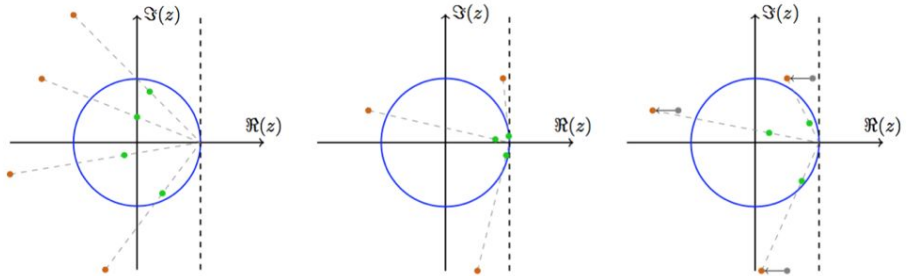
Eigen-analysis, 0 momentum

Theorem 1 (Prop. 4.4.1 Bertsekas [1999]). *If the spectral radius* $\rho_{\max} \overset{def}{=} \rho(\nabla F_\eta(\boldsymbol{\omega}^*)) < 1$, *then, for* $\boldsymbol{\omega}_0$ *in a neighborhood of* $\boldsymbol{\omega}^*$, *the distance of* $\boldsymbol{\omega}_t$ *to the stationary point* $\boldsymbol{\omega}^*$ *converges at a linear rate of* $\mathcal{O}\left((\rho_{\max} + \epsilon)^t\right)$, $\forall \epsilon > 0$.

$$\lambda(\nabla F_\eta(\boldsymbol{\phi}, \boldsymbol{\theta})) = 1 - \eta\lambda(\nabla \boldsymbol{v}(\phi, \theta))$$



The Numerics of GANs



(a) Illustration how the eigenvalues are projected into unit ball.

(b) Example where h has to be chosen extremely small.

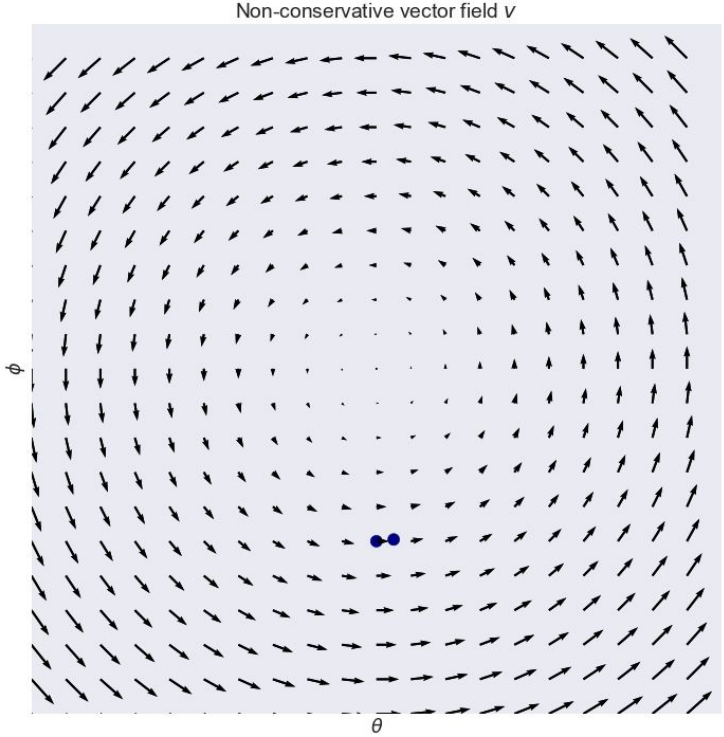
(c) Illustration how our method alleviates the problem.

Figure 1: Images showing how the eigenvalues of A are projected into the unit circle and what causes problems: when discretizing the gradient flow with step size h, the eigenvalues of the Jacobian at a fixed point are projected into the unit ball along rays from 1. However, this is only possible if the eigenvalues lie in the left half plane and requires extremely small step sizes h if the eigenvalues are close to the imaginary axis. The proposed method moves the eigenvalues to the left in order to make the problem better posed, thus allowing the algorithm to converge for reasonable step sizes.

Idea 1: Minimize the norm of the gradient

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}) = \frac{1}{2} \|v(\boldsymbol{\phi}, \boldsymbol{\theta})\|^2$$

Make vector field “more conservative”

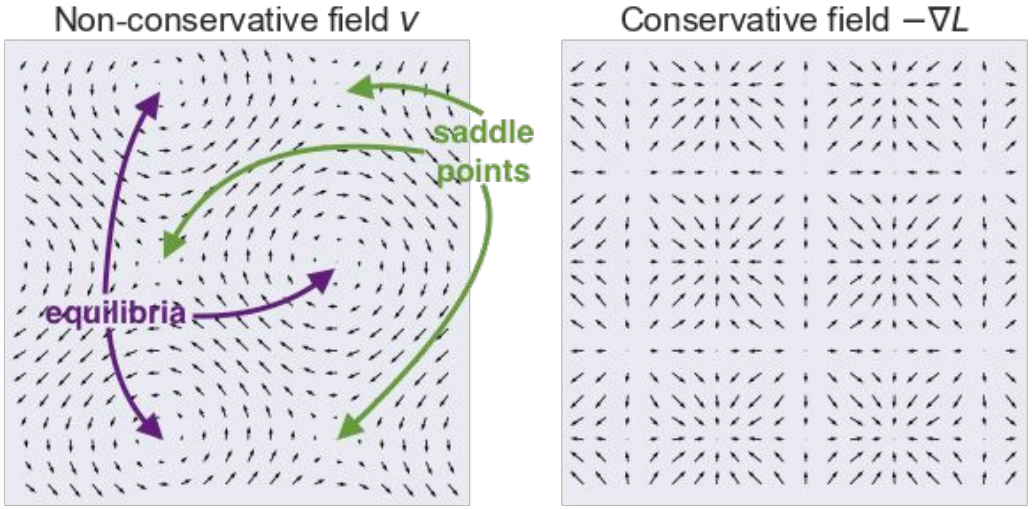


Idea 1: Minimize the norm of the gradient

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}) = \frac{1}{2}\|v(\boldsymbol{\phi}, \boldsymbol{\theta})\|^2$$

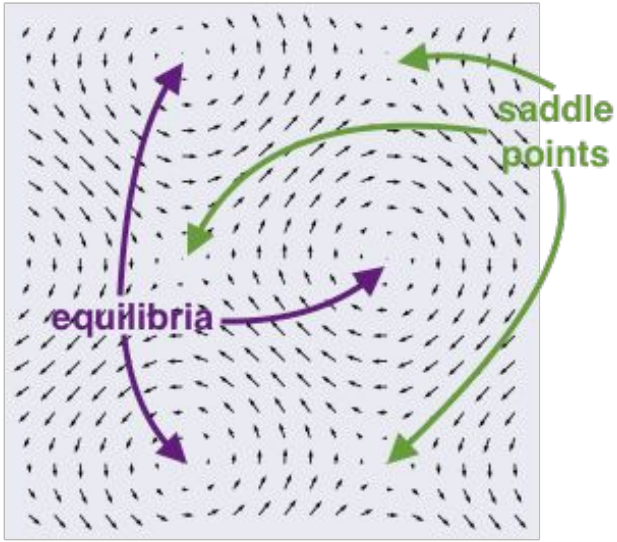
Idea 1: Minimize vector field norm

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}) = \frac{1}{2}\|v(\boldsymbol{\phi}, \boldsymbol{\theta})\|^2$$

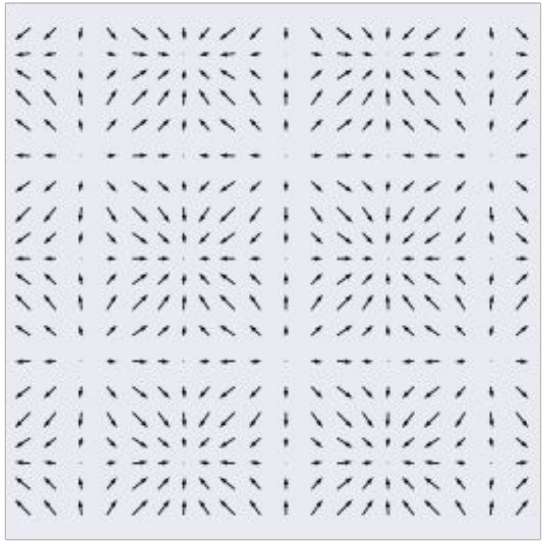


Idea 2: use L as regularizer

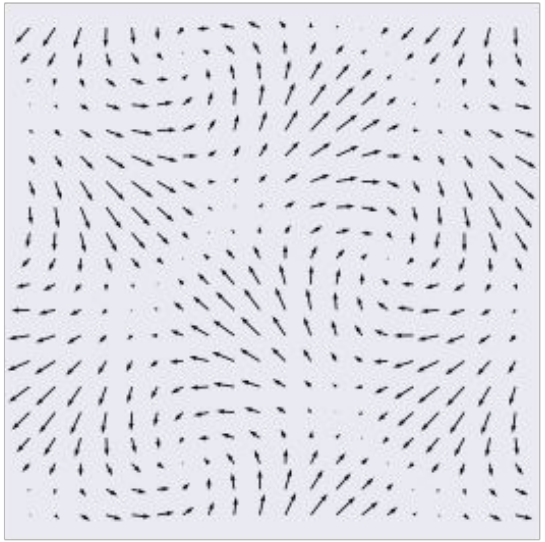
Non-conservative field v



Conservative field $-\nabla L$



Combined field $v - 0.6\nabla L$

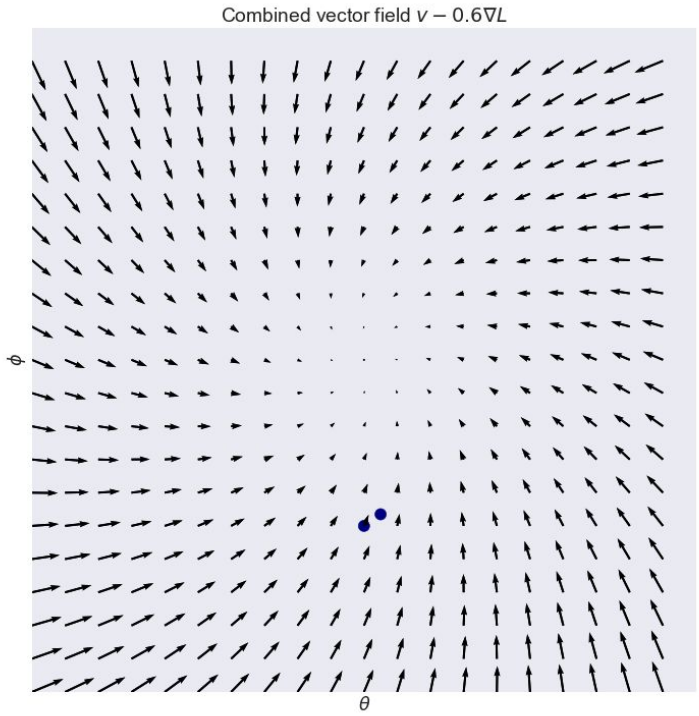


Idea 2: use L as regularizer

Algorithm 2 Consensus optimization

while not converged **do**
  $v_\phi \leftarrow \nabla_\phi(f(\theta,\phi) - \gamma L(\theta,\phi))$
  $v_\theta \leftarrow \nabla_\theta(g(\theta,\phi) - \gamma L(\theta,\phi))$
  $\phi \leftarrow \phi + h v_\phi$
  $\theta \leftarrow \theta + h v_\theta$
end while

Idea 2: use L as regularizer



Other ways to control these rotations?

Momentum (heavy ball, Polyak 1964)

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \boldsymbol{v}(\boldsymbol{\theta}_t) + \beta(\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})$$

Jacobian of momentum operator

$$\nabla F_{\eta,\beta}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}) = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0}_n \\ \boldsymbol{I}_n & \boldsymbol{0}_n \end{bmatrix} - \eta \begin{bmatrix} \nabla \boldsymbol{v}(\boldsymbol{\theta}_t) & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{0}_n \end{bmatrix} + \beta \begin{bmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{0}_n & \boldsymbol{0}_n \end{bmatrix}$$

Non-symmetric, with complex eigenvalues
→ Rotations in augmented state-space

Summary

Positive momentum can be bad for adversarial games

Practice that was very common when GANs were first invented.

- → Recent work reduced the momentum parameter.
- → Not an accident

Université de Montréal

Negative Momentum for Improved Game Dynamics

Gidel, Askari Hemmat, Pezeshki, Huang,
Lepriol, Lacoste-Julien, Mitliagkas

AISTATS 2019

Our results

Negative momentum is optimal on simple bilinear game

Negative momentum values are locally preferrable near 0 on a more general class of games

Negative momentum is empirically best for certain zero sum games like “saturating GANs”

Momentum on games

Recall Polyak's momentum (on top of simultaneous grad. desc.):

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \eta \boldsymbol{v}(\boldsymbol{x}_t) + \beta(\boldsymbol{x}_t - \boldsymbol{x}_{t-1}), \quad \boldsymbol{x}_t = (\boldsymbol{\theta}_t, \boldsymbol{\phi}_t)$$

Fixed point operator requires a **state augmentation**:
(because we need previous iterate)

$$F_{\eta,\beta}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) := \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0}_n \\ \boldsymbol{I}_n & \boldsymbol{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{x}_{t-1} \end{bmatrix} - \eta \begin{bmatrix} \boldsymbol{v}(\boldsymbol{x}_t) \\ \boldsymbol{0}_n \end{bmatrix} + \beta \begin{bmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{0}_n & \boldsymbol{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{x}_{t-1} \end{bmatrix}$$

Bilinear game

$$\min_{\boldsymbol{\theta}} \max_{\boldsymbol{\varphi}} \boldsymbol{\theta}^\top \boldsymbol{A} \boldsymbol{\varphi}$$

Method	β	Bounded	Converges
Simultaneous	$\beta \in \mathbb{R}$	✗	✗
Alternated	>0	✗	✗
	0	✓	✗
	<0	✓	✓

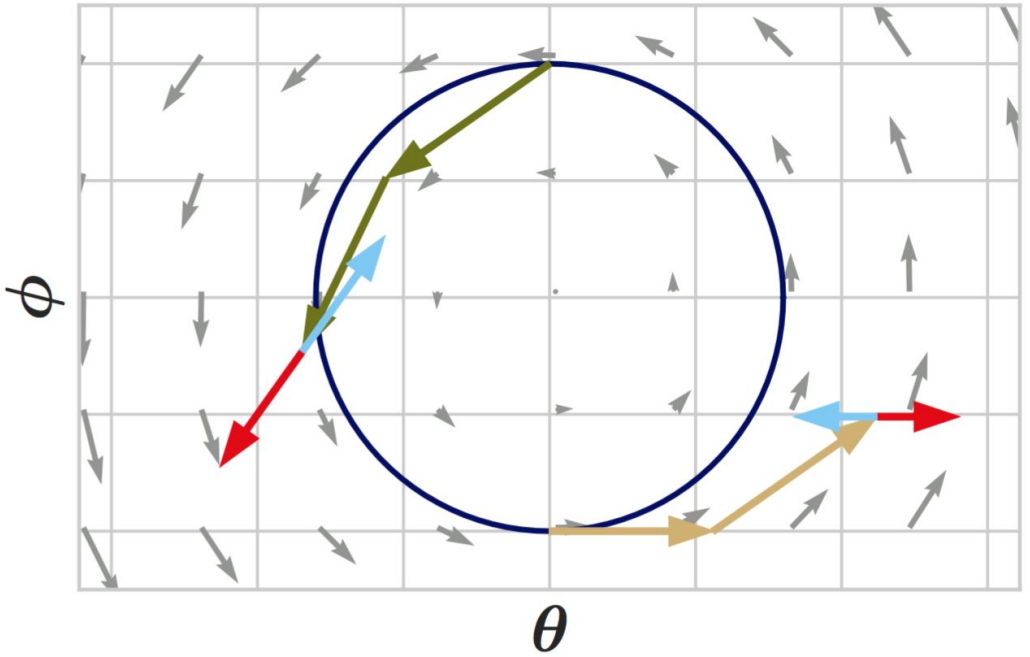
“Proof by picture”

Gradient descent

- → **Simultaneous**
- → **Alternating**

Momentum

- → **Positive**
- → **Negative**

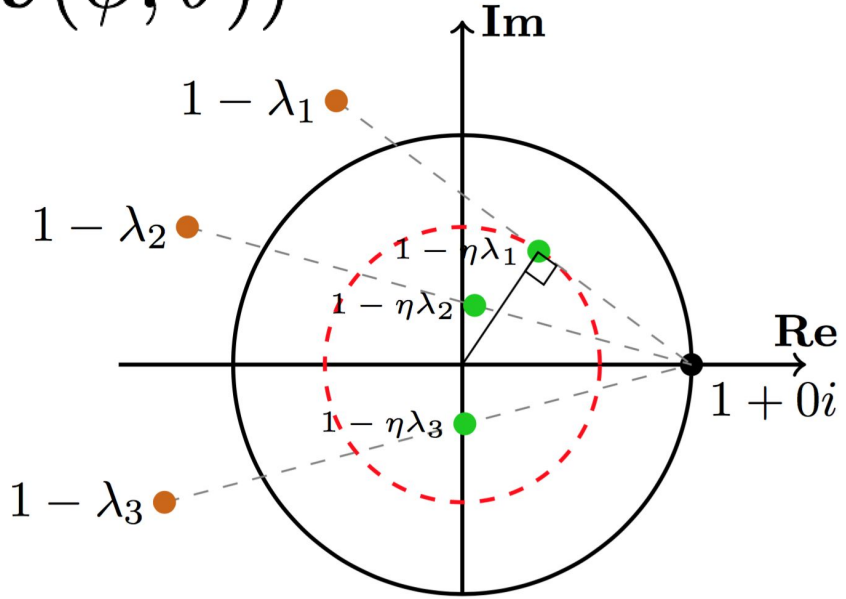


General games

Eigen-analysis, 0 momentum

Theorem 1 (Prop. 4.4.1 Bertsekas [1999]). *If the spectral radius* $\rho_{\max} \overset{def}{=} \rho(\nabla F_\eta(\boldsymbol{\omega}^*)) < 1$, *then, for* $\boldsymbol{\omega}_0$ *in a neighborhood of* $\boldsymbol{\omega}^*$, *the distance of* $\boldsymbol{\omega}_t$ *to the stationary point* $\boldsymbol{\omega}^*$ *converges at a linear rate of* $\mathcal{O}\left((\rho_{\max} + \epsilon)^t\right)$, $\forall \epsilon > 0$.

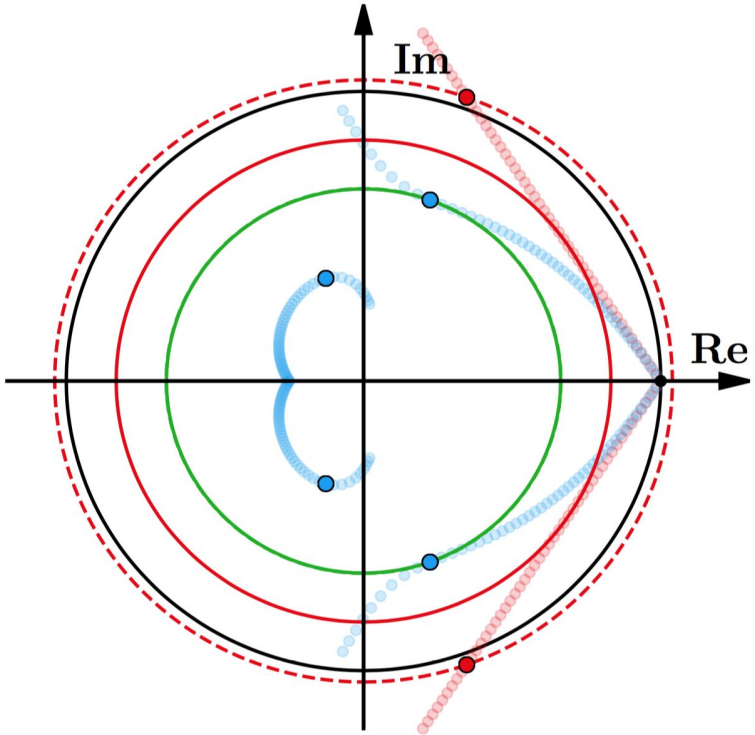
$$\lambda(\nabla F_\eta(\boldsymbol{\phi}, \boldsymbol{\theta})) = 1 - \eta\lambda(\nabla \boldsymbol{v}(\phi, \theta))$$



Zero vs negative momentum

Momentum

- → **Zero**
- → **Negative**



Negative Momentum

Theorem 3. *The eigenvalues of* $\nabla F_{\eta,\beta}(\boldsymbol{\phi}^*, \boldsymbol{\theta}^*)$ *are*

$$\mu_{\pm}(\beta, \eta, \lambda) := (1 - \eta\lambda + \beta)\frac{1 \pm \Delta^{\frac{1}{2}}}{2}, \tag{9}$$

where $\Delta := 1 - \frac{4\beta}{(1-\eta\lambda+\beta)^2}$, $\lambda \in Sp(\nabla \boldsymbol{v}(\boldsymbol{\phi}^*, \boldsymbol{\theta}^*))$ *and* $\Delta^{\frac{1}{2}}$ *is the* complex *square root of* Δ *with positive real part*[3]*. Moreover we have the following Taylor approximation,*

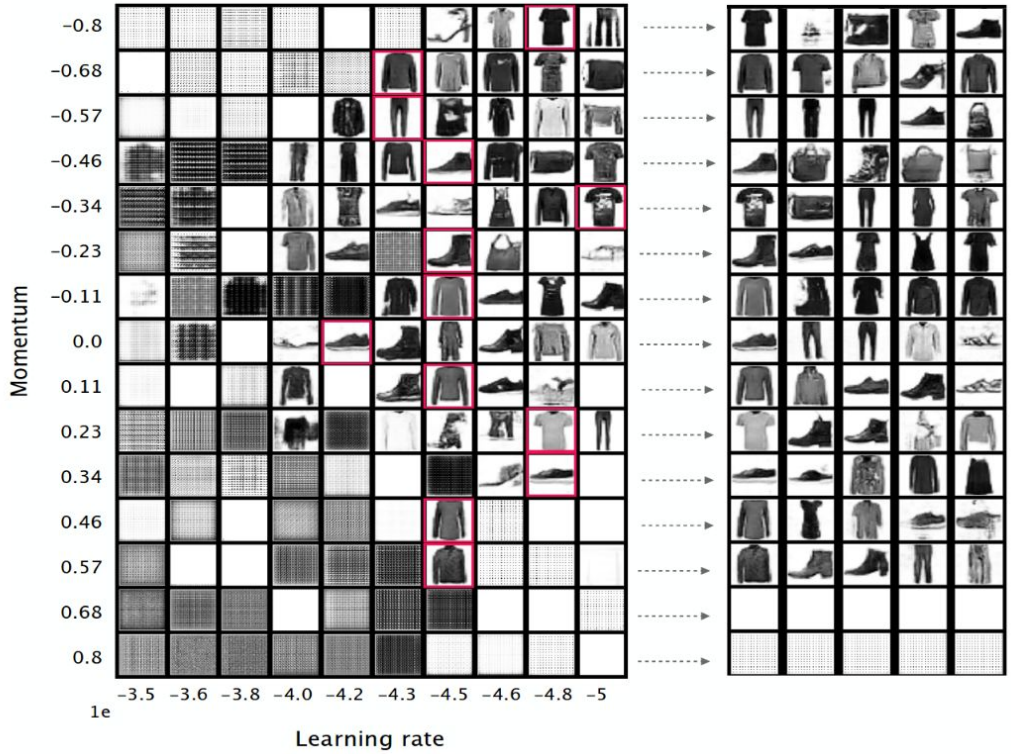
$$\mu_+(\beta, \eta, \lambda) = 1 - \eta\lambda - \beta\frac{\eta\lambda}{1-\eta\lambda} + O(\beta^2) \quad \textit{and} \quad \mu_-(\beta, \eta, \lambda) = \frac{\beta}{1-\eta\lambda} + O(\beta^2) \tag{10}$$

[3] If Δ is a negative real number we set $\Delta^{\frac{1}{2}} := i\sqrt{-\Delta}$

Empirical results

What happens in practice ?

Fashion MNIST:



What happens in practice ?

CIFAR-10:

Negative Momentum

To sum up:

- Negative momentum seems to improve the behaviour due to “bad” eigenvalues.
- Optimal for a class of games
- Empirically optimal on “saturating” GANs