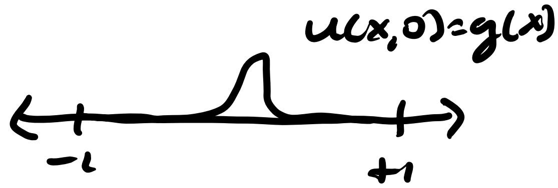
Numerical Stability of FDs (pt. 2)

$$\partial_t u = \overset{\text{wave speed}}{c}\,\partial_x u$$



1st Order Explicit

$$U_{j+1} = \Big(I + \overbrace{c\frac{\Delta t}{\Delta x}}^{G} D_1\Big) U_j$$

$$\hookrightarrow D_1 = \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & 1 \\ 1 & & & -1 \end{bmatrix}$$

Stability

$$0 < G \le 1 \quad \Rightarrow \quad 0 < \Delta t \le \frac{\Delta x}{c}$$

Numerical Diffusion

Note that when $G = c\frac{\Delta t}{\Delta x} = 1$, the eigvals of $A = I + GD_1$ live <u>on</u> the unit circle. Since A also has a full set of eigenvectors, it is a unitary matrix: $\|Au\| = \|u\|$ for any vector u. Just like the true PDE solution

$$\|u_{j+1}\| = \|A^{j+1}u_0\| = \|u_0\|$$

The norm is conserved at every time step.

However, when $G = c\frac{\Delta t}{\Delta x} < 1$, the eigenvals of A have $|\lambda_k| < 1$ (except $\lambda_n = 1$) and so $\|U_{j+1}\|$ typically decreases as $j \to \infty$.

This phenomenon is called numerical or or artificial diffusion, because it mimics the behavior of a diffusion term in the PDE (although there is <u>is no</u> such term in our model).

<u>Boundary Conditions</u>

wave speed

$$\partial_t u = c\,\partial_x u$$

Hom. Dirichlet B.C. $\quad u(+1,t) = 0$

<u>1st order Explicit</u>

$$U_{j+1} = \Big(I + \overbrace{c\,\frac{\Delta t}{\Delta x}}^{G} D_1\Big) U_j$$

To implement B.C., D_1 changes

$$D_1 = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & -1 \end{bmatrix}$$

no periodicity => <u>not</u> "circulant"

D_1 no longer has orthogonal eigenvectors. In fact it only has one linearly independent eigenvector! Its eigenvalues are all $\alpha_k = -1$ $k = 1, \dots, n$

$$\Rightarrow \lambda_k = 1 + \sigma \alpha_k = 1 - \sigma$$

According to eigenvalues, stability requires

$$0 < \sigma \leq 2 \; ?$$

But numerical experiments show blow up if

$$\sigma = 1.5, \; 2.0,$$

Eigenvalue analysis fails b/c D_1 doesn't have a good (complete & well-conditioned) eigenbasis.

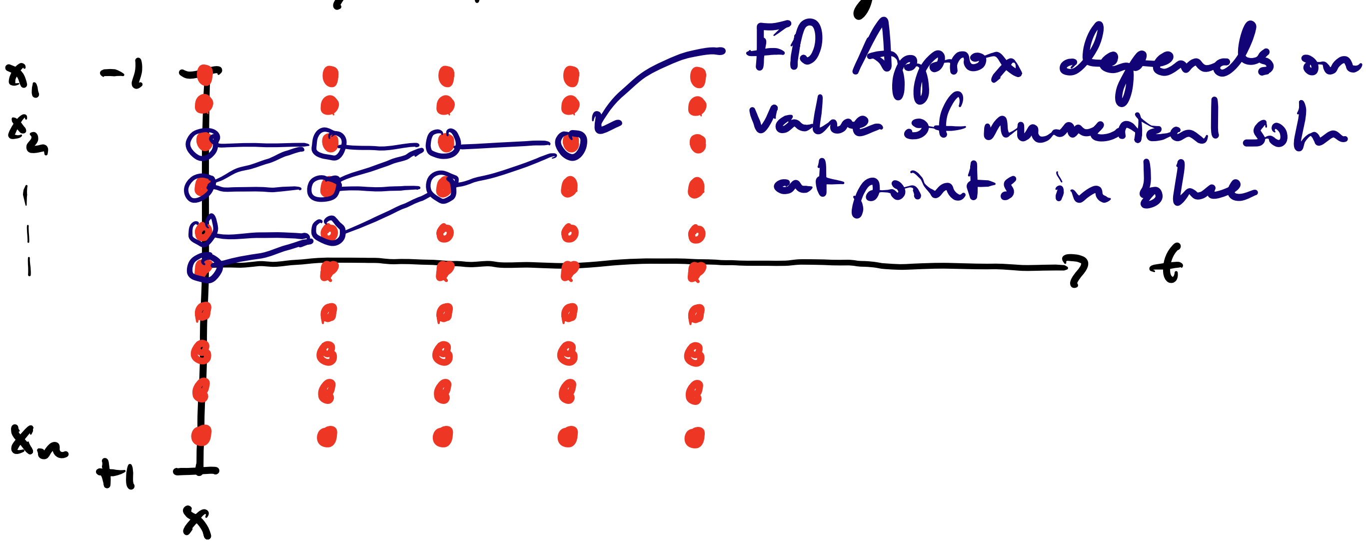
How can we understand stability in this case?

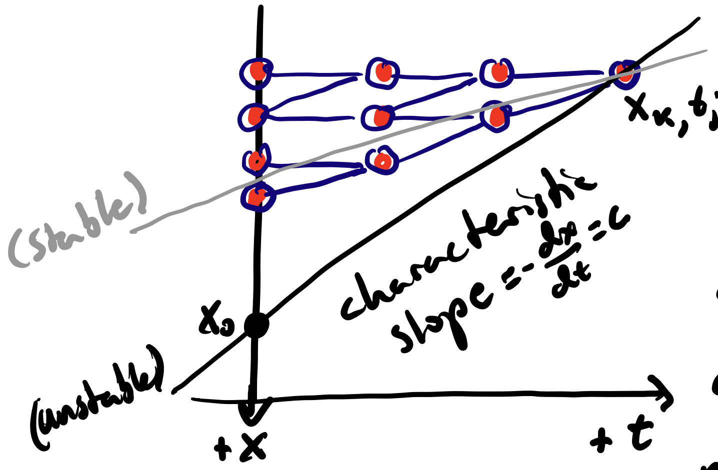
$\Rightarrow$ In general, more advanced techniques for estimating norms of matrix functions like $e^{D_1 t}$ are useful (e.g. "pseudospectra")

$\Rightarrow$ For certain wave equations, the famous Courant-Friedrichs-Lewy (CFL) condition provides useful guidance

CFL Condition

We can get another vantage point on instability by considering characteristics.





If we change initial condition ONLY in a small area around x_0, true soln. at (x_k, t_j) changes but numerical solution doesn't change at all! So numerical solution can be arbitrarily bad $\Rightarrow$ unstable

In particular, for right-moving characteristics ($c<0$), same argument applies $\Rightarrow$ unstable

CFL Condition $\Rightarrow$ Characteristic through (x_k, t_j) must pass between x_k and x_{k-1} at time t_j (for forward diff. approx in space).

$\Delta t, \Delta x > 0$ and $\left|\frac{\Delta x}{\Delta t}\right| \geq c \quad \Rightarrow \quad 0 < \frac{\Delta t}{\Delta x} \leq \frac{1}{c}$

This agrees with our earlier prediction for the periodic case and explains our observations for the homogeneous Dirichlet B.C. case above.

Note that, in general, CFL only provides a "necessary" condition for stability (see "centered difference" example on pg. 199 of Olver). A scheme can satisfy CFL and still be unstable.

Wave Equation

$$\partial_t^2 u = c^2 \partial_x^2 u$$

$$\left.\begin{aligned} u(x,0) &= g(x) \\ \partial_t u(x,0) &= h(x) \end{aligned}\right\} \begin{aligned}&\text{initial displ.} \\ &\text{vel.}\end{aligned}$$

$u(\pm 1, t) = 0$

"Clamped B.C."

$$\left.\partial_x^2 u\right|_{t=t_j} \approx \frac{1}{(\Delta x)^2} \underbrace{\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{D_2}$$

$$\left.\partial_t^2 u\right|_{t=t_j} \approx \frac{1}{(\Delta t)^2}\left(u\big|_{t=t_{j+1}} - 2u\big|_{t=t_j} + u\big|_{t=t_{j-1}} \right)$$

$$u_{j+1} - 2u_j + u_{j-1} = \overbrace{c^2\frac{(\Delta t)^2}{(\Delta x)^2}}^{=\sigma^2} D_2 u_j$$

$$u_{j+1} = (2I + \sigma^2 D_2) u_j - u_{j-1}$$

To start, need $u|_{t=t_0} = g$ and $u|_{t=t_1} = ?$

$$u\big|_{t=t_1} = u\big|_{t=t_0} + \left(\partial_t u\big|_{t=t_0}\right)(t_1 - t_0) + \frac{1}{2}\left(\partial_t^2 u\big|_{t=t_0}\right)(t_1 - t_0)^2$$

$$= g + h\,(t_1 - t_0) + \frac{c^2}{2}\left(\partial_x^2 u\big|_{t=t_0}\right)(t - t_0)^2$$