Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Mixed boundary conditions. Solve the heat equation in the unit square, $\Omega = [-1,1]\times[-1,1]$, when no heat flux is permitted through the vertical boundaries $x = \pm 1$ and the temperature is held constant along the horizontal boundaries $y = \pm 1$. That is, solve

$$\partial_t u = \Delta u, \quad \text{where} \quad \partial_x u|_{x=\pm 1} = u|_{y=1} = 0, \quad u|_{y=-1} = 1, \quad \text{and} \quad u|_{t=0} = g.$$

(a) Find eigenfunctions of Δ that satisfy homogeneous Neumann boundary conditions on the vertical boundaries and Dirichlet boundary conditions on the horizontal boundaries.

Solution: We can use separation of variables to solve the eigenvalue problem, $\Delta v = \lambda v$, with mixed homogeneous boundary conditions. With the separable ansatz $v(x,y) = X(x)Y(y)$ and the usual separation of variables argument, we find that

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda, \qquad \implies X'' = \alpha X, \qquad \text{and} \qquad Y'' = \beta Y,$$

where $\alpha + \beta = \lambda$. Applying the homogeneous Dirichlet boundary conditions at $y = \pm 1$ and solving the one-dimensional eigenvalue problem for Y, we find that the solutions for Y are the Dirichlet eigenfunctions of the second derivative operator (see Lecture 6):

$$Y_n(y) = \begin{cases} \cos(n\pi y/2), & n = 1,3,5,\dots, \\ \sin(n\pi y/2), & n = 2,4,6,\dots, \end{cases}$$

with eigenvalues $\beta_n = -(\pi n/2)^2$. The equation for X is also a one-dimensional eigenvalue problem for the second derivative, but with Neumann boundary conditions. The solutions are

$$X_m(x) = \begin{cases} 1/2, & m = 0, \\ \sin(m\pi x/2), & m = 1,3,5,\dots, \\ \cos(m\pi x/2), & m = 2,4,6,\dots, \end{cases}$$

with eigenvalues $\alpha_m = -(\pi m/2)^2$. Therefore, the eigenvalues of Δ on the unit square with the mixed boundary conditions are $\lambda_{n,m} = (\pi n/2)^2 + (\pi m/2)^2$ and the eigenfunctions are the separable products $e_{n,m}(x,y) = X_m(x)Y_n(y)$.

(b) Find an equilibrium solution to the heat equation, which satisfies $\Delta u_* = 0$, that satisfies the mixed Neumann and Dirichlet boundary conditions in the problem statement.

Solution: In general, we can compute the general series solution to the Laplace equation in the unit square by separation of variables and then select the coefficients of the series to find a particular solution that satisfies the inhomogeneous boundary conditions. However, in this case, the boundary conditions are simple constants: vanishing derivatives and solution on three sides and a constant solution on the third side. Therefore, we can pick the linear function $u_*(x,y) = (1-y)/2$ as the equilibrium solution.

(c) Using your work from part (a) and (b), derive a series solution to the heat equation that satisfies the initial condition and boundary conditions in the problem statement.

Solution: Given the eigenvalues/eigenfunctions of Δ satisfying the homogeneous mixed boundary conditions (from part (a)) and the equilibrium solution solution satisfying the inhomogeneous mixed boundary conditions (from part (b)), the series solution is given by the operator exponential (see Lecture 12) applied to a modified initial condition,

$$u(x,y,t) = \sum_{n,m} e^{\lambda_{n,m}t}\langle e_{n,m}, \tilde{g}\rangle e_{n,m}(x,y),$$

where the modified initial condition is $\tilde{g}(x,y) = g(x,y) - u_*(x,y)$.

2) Advection and diffusion. Consider the advection-diffusion equation, given by

$$\partial_t u = \alpha\partial_x^2 u + \beta\partial_x u, \qquad \text{such that} \qquad u = \text{periodic on } [-1,1),$$

with initial condition $u(x,0) = g(x)$ and non-negative constants α and β. The notebook `hw1_soln.ipynb` on the 18.303 course repository may be helpful in part (d).

(a) Find a Fourier series solution using the operator exponential for the right-hand side.

Solution: The eigenfunctions of both ∂_x and ∂_x^2 on the periodic interval $[-1,1)$ are the complex exponentials (see Lecture 5)

$$e_k(x) = \frac{1}{\sqrt{2}}e^{i\pi kx}, \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

with eigenvalues $i\pi k$ and $-(\pi k)^2$, respectively. Therefore, the eigenfunctions of the right-hand side are these same complex exponentials with eigenvalues

$$\lambda_k = -\alpha\pi^2 k^2 + i\beta\pi k.$$

Therefore, the operator exponential provides the Fourier series solution

$$u(x,t) = \frac{1}{\sqrt{2}}\sum_{k=-\infty}^{\infty}\hat{g}_k\, e^{(-\alpha\pi^2k^2+i\beta\pi k)t}e^{i\pi kx}, \qquad \text{where} \qquad \hat{g}_k = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^{-ik\pi x}g(x)\,dx.$$

(b) If $\alpha > 0$, derive an equilibrium solution from the Fourier series solution. How does the case $\beta > 0$ (advection-diffusion) compare to the case $\beta = 0$ (pure heat equation)?

Solution: When $\alpha > 0$, the real part of each eigenvalue with $k \neq 0$ is negative. Therefore, the (time-dependent) Fourier coefficients of $u(x,t)$ become exponentially small as t becomes large:

$$|\hat{u}_k| = |\langle e_k, u\rangle| = |\hat{g}_k|e^{-\alpha\pi^2k^2t} \to 0, \qquad \text{as} \qquad t\to\infty.$$

By Parseval's identity (see Lecture 13), this means that $|u(x,t) - \hat{g}_0/\sqrt{2}| \to 0$ as $t \to \infty$. Therefore, the equilibrium solution is

$$\frac{\hat{g}_0}{\sqrt{2}} = \frac{1}{2}\int_{-1}^{1} g(x)\,dx = \text{mean value of } g(x).$$

This is the same equilibrium solution that we found in class for the case $\beta = 0$ (see Lecture 11). In other words, the advection term does not effect the equilibrium solution.

(c) Discretize the advection-diffusion equation in the problem statement using second-order finite differences in space and *backward* Euler's method in time. Solve the PDE numerically with $\beta = 1$, initial condition $g(x) = 5\exp(-10\cos^2(\pi x))$, and $\alpha = 0.1$, 0.01, and 0.001. Try 100 gridpoints in space and a time-step of 0.01 for $0 \le t \le 5$. How do your observations compare with your results from part (b)?

Solution: For each value of α, the oscillatory initial condition travels to the left at the same speed due to the advection term, while the oscillations are "damped" out slower when α (diffusion constant) is smaller. In each case, the solution approaches the constant equilibrium solution, as predicted in part (b). See `hw3_soln.ipynb`.