Massachusetts
Institute of
Technology

Green’s functions

Direct solution to PDEs

18.303 Linear Partial Differential Equations: Analysis and Numerics

Green's function methods

Consider a PDE

$$\mathcal{L}u(\mathsf{x}) = f(\mathsf{x}),$$

where $\mathsf{x} \in \Omega \subset \mathbb{R}^N$ and $\mathcal{L}$ is some linear differential operator. Furthermore, let's assume boundary conditions

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on the boundary $\partial\Omega$.

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- The solution is expressed as $u(\mathbf{x}) = \mathcal{G}f(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}$, where G is the Green's function
- G will also depend on the boundary condition

Properties of the Green's function

The operation $\mathcal{G}f$ is a linear operation. We also have

$$\mathcal{G}[\mathcal{L}u](\mathsf{x}) = \int_\Omega G(\mathsf{x},\mathsf{y})[\mathcal{L}u](\mathsf{y})\mathrm{d}\mathsf{y} = \int_\Omega G(\mathsf{x},\mathsf{y})f(\mathsf{y})\mathrm{d}\mathsf{y} = u(\mathsf{x})$$

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This can really be thought of as a generalization of solving a linear system: now the linear operations are generalizations of matrix-vector products, we just replace the discrete indices with a continuous x and y.

Example

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The Laplacian is translationally symmetric i.e. $\partial_x^2 u(x)\Big|_{x=x_0} = \partial_x^2 u(x+c)\Big|_{x=x_0-c}$ for all $c \in \mathbb{R}$. This means that the operator doesn't explicitly depend on the point it's evaluated at. This implies that also the Green's function has to be translationally invariant. Thus we can write $G(x,y) = G(x-y)$.

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We can calculate the Fourier transform of the equation for the Green's function giving

$$\int_{\mathbb{R}} e^{-ik(x-y)} \partial_x^2 G(x-y) \mathrm{d}x = \int_{\mathbb{R}} e^{-ik(x-y)} \delta(x-y) \mathrm{d}x = 1.$$

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We know that the Fourier transform of $f'(x)$ is $ik\hat{f}(k)$ giving the LHS. We have $-k^2 \hat{G}(k) = 1$.

Now we get

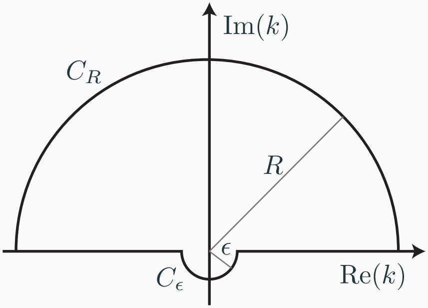
$$G(x-y) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ik(x-y)}}{k^2}\mathrm{d}k.$$

We can evaluate this integral as a limit of the contour shown on the right. We have

$$G(x-y) = \int_{\tilde{\mathbb{R}}} g(k)\ldots + \int_{C_R} \ldots + \int_{C_\epsilon} \ldots - \int_{C_\epsilon} \ldots$$

since the integral on the path C_R goes to zero as $R \to \infty$. The semicircle C_ϵ gives half of the Cauchy integral while the rest give the complete integral.
In the end we have

$$G(x,y) = \pi i \operatorname{Res}(g, 0).$$



The contour for integration.

We get

$$G(x-y) = -\frac{i}{2}\frac{\mathrm{d}e^{ik(x-y)}}{\mathrm{d}k}\bigg|_{k=0} = \frac{1}{2}\begin{cases} x-y, & x>y \\ y-x, & x<y \end{cases} = \frac{1}{2}|x-y|.$$

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We can check the solution by calculating

$$u(x) = \int_{\mathbb{R}} G(x-y)f(y)\mathrm{d}y = \frac{1}{2}\int_{\mathbb{R}} |x-y|f(y)\mathrm{d}y.$$

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Here we assume that $f(y)$ is integrable implying that the boundary term vanishes.

Calculating integration by parts again gives

$$u(x) = \frac{1}{2} \int_{\mathbb{R}} 2\delta(x-y) \int_{-\infty}^{y} \int_{-\infty}^{y'} f(y'')\mathrm{d}y''\mathrm{d}y'\mathrm{d}y.$$

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Here we notice that in distribution sense (weak sense), the derivative of the $\mathsf{sgn}(x-y)$ function is $-2\delta(x-y)$.

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We can get rid of one integral using the delta distribution giving

$$u(x) = \int_{-\infty}^{x}\int_{-\infty}^{y'} f(y'')\mathrm{d}y''\mathrm{d}y'.$$

This is just the double integral of f, which makes sense since we wanted to solve

$$\partial_x^2 u(x) = f(x).$$

For 2d Laplacian we have

$$G(r) = \frac{1}{2\pi}\log(r)$$

and in 3d

$$G(r) = -\frac{1}{4\pi r}.$$

Here $r = |\mathbf{x} - \mathbf{y}|$.

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Important: these equations are for the whole space: boundary conditions will change the Green's function.

Example with boundaries

Assume we have a Dirichlet problem for the Laplacian i.e.

$$\Delta u(\mathsf{x}) = f(\mathsf{x}),$$

when $\mathsf{x} \in \Omega \subset \mathbb{R}^n$ with boundary condition

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Assume we know a function for which

$$\Delta_{\mathsf{x}} G(\mathsf{x} - \mathsf{y}) = \delta(\mathsf{x} - \mathsf{y}).$$

We have

$$u(\mathsf{x}) = \int_\Omega \delta(\mathsf{x}-\mathsf{y})u(\mathsf{y})\mathrm{d}\mathsf{y} = \int_\Omega \Delta_{\mathsf{x}} G(\mathsf{x}-\mathsf{y})u(\mathsf{y})\mathrm{d}\mathsf{y}.$$

We have

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We can change the derivative to act on y since G is a function of the distance $\mathsf{x}-\mathsf{y}$. Integrating by parts gives

$$u(\mathsf{x}) = \int_{\partial\Omega} \underbrace{u(\mathsf{y})}_{=g(\mathsf{y})} \nabla_\mathsf{y} G \cdot \mathrm{d}\mathsf{S} - \int_\Omega \nabla_\mathsf{y} G \cdot \nabla_\mathsf{y} u(\mathsf{y})\mathrm{d}\mathsf{y}.$$

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After integrating by parts again we get

$$u(\mathsf{x}) = \int_{\partial\Omega} g(\mathsf{y})\nabla_\mathsf{y} G \cdot \mathrm{d}\mathsf{S} - \int_{\partial\Omega} G\nabla_\mathsf{y} u(\mathsf{y}) \cdot \mathrm{d}\mathsf{S} + \int_\Omega G\, \underbrace{\Delta_\mathsf{y} u(\mathsf{y})}_{=f(\mathsf{y})}\, \mathrm{d}\mathsf{y}.$$

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The second term is zero since $G(\mathsf{x}-\mathsf{y}) = 0$ if either x or y is on the boundary (we don't solve for the boundary points).

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$$u(\mathsf{x}) = \int_{\partial\Omega} g(\mathsf{y})\nabla_{\mathsf{y}} G \cdot \mathrm{d}\mathsf{S} + \int_{\Omega} G(\mathsf{x}-\mathsf{y}) f(\mathsf{y}) \mathrm{d}\mathsf{y}.$$

Notice how the boundary plays a role here.

Eigenbases and the Green's function

We have a Poisson type equation

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We have $\hat{u}_{\mathsf{n}} = \hat{f}_{\mathsf{n}}/\lambda_{\mathsf{n}}$. Calculating the solution u gives

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On the other hand we have

$$\hat{f}_{\mathsf{n}} = \langle \phi_{\mathsf{n}}, f \rangle = \int_{\Omega} \phi_{\mathsf{n}}(\mathsf{y}) f(\mathsf{y}) \mathrm{d}\mathsf{y}.$$

Plugging this in gives

$$u(\mathsf{x}) = \sum_{\mathsf{n}} \frac{1}{\lambda_{\mathsf{n}}} \int_{\Omega} \phi_{\mathsf{n}}(\mathsf{y}) f(\mathsf{y}) \mathrm{d}\mathsf{y} \phi_{\mathsf{n}}(\mathsf{x}).$$

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Changing the order between summation and integration results in

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where we identify the Green's function in the brackets.

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This shows that in principle we can calculate the Green's function if we have solved the eigenproblem.