Advection problems vol. 2

Flow equations etc.

18.303 Linear Partial Differential Equations: Analysis and Numerics

Massachusetts Institute of Technology

Method of characteristics

Consider a differential equation

$$a(x, y, z)z_x + b(x, y, z)z_y = c(x, y, z),$$

where a, b, and c are known functions. We want to solve for $z(x, y)$.

Method of characteristics

Consider a differential equation

$$a(x, y, z)z_x + b(x, y, z)z_y = c(x, y, z),$$

where a, b, and c are known functions. We want to solve for $z(x, y)$.

The solution can be comprised of characteristic curves that are parametrized curves in $\mathbb{R}^3$ satisfying

$$\begin{aligned} \dot{x}(t) &= a(x, y, z), \\ \dot{y}(t) &= b(x, y, z), \\ \dot{z}(t) &= c(x, y, z). \end{aligned}$$

Method of characteristics

Consider a differential equation

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z),$$

where a, b, and c are known functions. We want to solve for $z(x,y)$.

The solution can be comprised of characteristic curves that are parametrized curves in $\mathbb{R}^3$ satisfying

$$\begin{aligned}\dot{x}(t) &= a(x,y,z),\\ \dot{y}(t) &= b(x,y,z),\\ \dot{z}(t) &= c(x,y,z).\end{aligned}$$

Substituting the equations in the original PDE gives

$$\dot{x}(t)z_x + \dot{y}(t)z_y = \dot{z}(t),$$

which is just the chain rule for taking the derivative $\dot{z}$. We see that this is consistent as long as we have some known initial point $z(x(t_0), y(t_0))$.

Another way of looking at this is to calculate the normal to the surface $z(x, y)$. We get a normal by calculating the cross product of two tangent vectors for the points (x, y, z) given by

$$\partial_x(x, y, z) = (1, 0, z_x),\ \partial_y(x, y, z) = (0, 1, z_y).$$

Another way of looking at this is to calculate the normal to the surface $z(x, y)$. We get a normal by calculating the cross product of two tangent vectors for the points (x, y, z) given by

$$\partial_x(x, y, z) = (1, 0, z_x),\ \partial_y(x, y, z) = (0, 1, z_y).$$

The normal is given by

$$(0, 1, z_y) \times (1, 0, z_x) = \mathbf{e}_y \times \mathbf{e}_x + z_x \mathbf{e}_y \times \mathbf{e}_z + z_y \mathbf{e}_z \times \mathbf{e}_x = (z_x, z_y, -1).$$

Another way of looking at this is to calculate the normal to the surface $z(x, y)$. We get a normal by calculating the cross product of two tangent vectors for the points (x, y, z) given by

$$\partial_x(x, y, z) = (1, 0, z_x),\ \partial_y(x, y, z) = (0, 1, z_y).$$

The normal is given by

$$(0, 1, z_y) \times (1, 0, z_x) = \mathbf{e}_y \times \mathbf{e}_x + z_x \mathbf{e}_y \times \mathbf{e}_z + z_y \mathbf{e}_z \times \mathbf{e}_x = (z_x, z_y, -1).$$

Calculating the dot product

$$(a, b, c) \cdot (z_x, z_y, -1) = a z_x + b z_y - c = 0$$

because of the original PDE. This shows that the vector field (a, b, c) is tangent to the surface $z(x, y)$. Therefore we can parametrize a curve on z such that its tangent is $\partial_t(x(t), y(t), z(t)) = (a, b, c)$.

Burgers' equation

We have

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0$$

with some initial condition $u(0,x) = u_0(x)$. Here $t \geq 0$ and $x \in \mathbb{R}$.

Burgers' equation

We have

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0$$

with some initial condition $u(0,x) = u_0(x)$. Here $t \geq 0$ and $x \in \mathbb{R}$.

Using the method of characteristics we find the characteristic curves

$$\begin{aligned}
\partial_\xi t(\xi) &= 1, \\
\partial_\xi x(\xi) &= u(t(\xi), x(\xi)), \\
\partial_\xi u(t(\xi), x(\xi)) &= 0.
\end{aligned}$$

We have

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0$$

with some initial condition $u(0,x) = u_0(x)$. Here $t \geq 0$ and $x \in \mathbb{R}$.

Using the method of characteristics we find the characteristic curves

$$\begin{aligned} \partial_\xi t(\xi) &= 1, \\ \partial_\xi x(\xi) &= u(t(\xi), x(\xi)), \\ \partial_\xi u(t(\xi), x(\xi)) &= 0. \end{aligned}$$

We can solve the first equation by integrating ξ from 0 to ξ giving $t(\xi) = \xi + t(0)$. We see that we can choose the parameter $\xi = t$ since ξ and t are linearly related.

The third equation gives us $\partial_t u = 0$. This means that the function $u(t, x(t))$ is a constant, i.e. the velocity doesn't vary along the trajectory $(t, x(t))$. Furthermore, from the initial condition we know that $u(t, x(t)) = u(0, x(0)) = u_0(x(0)) = u_0(x_0)$.

The third equation gives us $\partial_t u = 0$. This means that the function $u(t, x(t))$ is a constant, i.e. the velocity doesn't vary along the trajectory $(t, x(t))$. Furthermore, from the initial condition we know that $u(t, x(t)) = u(0, x(0)) = u_0(x(0)) = u_0(x_0)$.

The second equation gives

$$\partial_t x(t) = u(t, x(t)).$$

The third equation gives us $\partial_t u = 0$. This means that the function $u(t, x(t))$ is a constant, i.e. the velocity doesn't vary along the trajectory $(t, x(t))$. Furthermore, from the initial condition we know that $u(t, x(t)) = u(0, x(0)) = u_0(x(0)) = u_0(x_0)$.

The second equation gives

$$\partial_t x(t) = u(t, x(t)).$$

This just means that u is indeed the velocity at $(t, x(t))$. Substituting the solution for $u(t, x(t))$ gives

$$\partial_t x(t) = u_0(x_0),$$

which can be integrated from 0 to t giving

$$x(t) = tu_0(x_0) + x_0.$$

We learn that the characteristic curves are actually two-dimensional lines $(t, tu_0(x_0) + x_0)$ along which the velocity field is a constant.

We learn that the characteristic curves are actually two-dimensional lines $(t, tu_0(x_0) + x_0)$ along which the velocity field is a constant.

These curves are parametrized by the initial point x_0. So, what is $u(t, x)$ given any (t, x). We have the map

$$x(t; x_0) = tu_0(x_0) + x_0,$$

which can be formally inverted to give $x_0(t, x(t))$. Now the solution is given by $u(t, x) = u_0(x_0(t, x(t)))$.

We learn that the characteristic curves are actually two-dimensional lines $(t, tu_0(x_0) + x_0)$ along which the velocity field is a constant.

These curves are parametrized by the initial point x_0. So, what is $u(t, x)$ given any (t, x). We have the map

$$x(t; x_0) = tu_0(x_0) + x_0,$$

which can be formally inverted to give $x_0(t, x(t))$. Now the solution is given by $u(t, x) = u_0(x_0(t, x(t)))$.

Can $x(t; x_0)$ really be inverted? It's possible if $x(t; x_0)$ is a *unique* function of x_0 i.e. for any pair x, y we have $x(t; x_0) \neq x(t; x_1)$, when $x_0 \neq x_1$.

We learn that the characteristic curves are actually two-dimensional lines $(t, tu_0(x_0) + x_0)$ along which the velocity field is a constant.

These curves are parametrized by the initial point x_0. So, what is $u(t, x)$ given any (t, x). We have the map

$$x(t; x_0) = tu_0(x_0) + x_0,$$

which can be formally inverted to give $x_0(t, x(t))$. Now the solution is given by $u(t, x) = u_0(x_0(t, x(t)))$.

Can $x(t; x_0)$ really be inverted? It's possible if $x(t; x_0)$ is a *unique* function of x_0 i.e. for any pair x, y we have $x(t; x_0) \neq x(t; x_1)$, when $x_0 \neq x_1$.

The function $x(t; x_0)$ is a continuous map (as long as u_0 is continuous) so in case there are no crossings it preserves the order i.e. if $x_0 < x_1 < x_2$ it implies that $x(t; x_0) < x(t; x_1) < x(t; x_2)$ for any t.

We learn that the characteristic curves are actually two-dimensional lines $(t, tu_0(x_0) + x_0)$ along which the velocity field is a constant.

These curves are parametrized by the initial point x_0. So, what is $u(t, x)$ given any (t, x). We have the map

$$x(t; x_0) = tu_0(x_0) + x_0,$$

which can be formally inverted to give $x_0(t, x(t))$. Now the solution is given by $u(t, x) = u_0(x_0(t, x(t)))$.

Can $x(t; x_0)$ really be inverted? It's possible if $x(t; x_0)$ is a *unique* function of x_0 i.e. for any pair x, y we have $x(t; x_0) \neq x(t; x_1)$, when $x_0 \neq x_1$.

The function $x(t; x_0)$ is a continuous map (as long as u_0 is continuous) so in case there are no crossings it preserves the order i.e. if $x_0 < x_1 < x_2$ it implies that $x(t; x_0) < x(t; x_1) < x(t; x_2)$ for any t.

Therefore it suffices to see if $x(t; x_0) < x(t; x_0 + \epsilon)$ for some $\epsilon > 0$.

Assume there are no crossings i.e. $x(t; x_0) < x(t; x_0 + \epsilon)$. Substituting the solution for x gives

$$tu_0(x_0) + x_0 < tu_0(x_0 + \epsilon) + x_0 + \epsilon.$$

Assume there are no crossings i.e. $x(t; x_0) < x(t; x_0 + \epsilon)$. Substituting the solution for x gives

$$tu_0(x_0) + x_0 < tu_0(x_0 + \epsilon) + x_0 + \epsilon.$$

Simplifying a bit gives

$$-t\frac{u_0(x_0 + \epsilon) - u_0(x_0)}{\epsilon} < 1.$$

Taking $\epsilon \to 0$ gives a derivative and we have

$$u_0'(x_0) > -\frac{1}{t}.$$

Assume there are no crossings i.e. $x(t; x_0) < x(t; x_0 + \epsilon)$. Substituting the solution for x gives

$$tu_0(x_0) + x_0 < tu_0(x_0 + \epsilon) + x_0 + \epsilon.$$

Simplifying a bit gives

$$-t\frac{u_0(x_0 + \epsilon) - u_0(x_0)}{\epsilon} < 1.$$

Taking $\epsilon \to 0$ gives a derivative and we have

$$u_0'(x_0) > -\frac{1}{t}.$$

We see that there *will* be a crossing with the crossing time $t = -1/u_0'(x_0)$ if $u_0'(x_0) < 0$.

What does this mean in practice?

What does this mean in practice?

To answer this question let's calculate

$$\partial_x u(t,x) = \frac{\partial x_0}{\partial x} u_0'(x_0).$$

What does this mean in practice?

To answer this question let's calculate

$$\partial_x u(t,x) = \frac{\partial x_0}{\partial x} u_0'(x_0).$$

From the equation for x we get

$$\frac{\partial x}{\partial x_0} = t u_0'(x_0) + 1 = \left(\frac{\partial x_0}{\partial x}\right)^{-1}.$$

What does this mean in practice?

To answer this question let's calculate

$$\partial_x u(t,x) = \frac{\partial x_0}{\partial x} u_0'(x_0).$$

From the equation for x we get

$$\frac{\partial x}{\partial x_0} = t u_0'(x_0) + 1 = \left(\frac{\partial x_0}{\partial x}\right)^{-1}.$$

Substituting gives

$$\partial_x u(t,x) = \frac{u_0'(x_0)}{t u_0'(x_0) + 1}.$$

What does this mean in practice?

To answer this question let's calculate

$$\partial_x u(t,x) = \frac{\partial x_0}{\partial x} u_0'(x_0).$$

From the equation for x we get

$$\frac{\partial x}{\partial x_0} = t u_0'(x_0) + 1 = \left(\frac{\partial x_0}{\partial x}\right)^{-1}.$$

Substituting gives

$$\partial_x u(t,x) = \frac{u_0'(x_0)}{t u_0'(x_0) + 1}.$$

This approaches infinity as $t u_0'(x_0) \to -1$ i.e. crossing of the characteristic curves just means that the spatial derivatives of the velocity field blow up. This is known as shock formation.

Another way to see this is to define a density. Assume we have a uniform distribution of something along the x coordinate. How much stuff do we have between 0 and x at time t?

Another way to see this is to define a density. Assume we have a uniform distribution of something along the x coordinate. How much stuff do we have between 0 and x at time t?

The answer is the length of the line segment $[x_1, x_2]$ for which

$$0 = tu_0(x_1) + x_1, \; x = tu_0(x_2) + x_2$$

i.e. the stuff that is transported between points 0 and x at time t by the flow. Mathematically we have

$$\text{stuff} = \int_{x_1}^{x_2} \mathrm{d}x_0 = \int_0^x \frac{\partial x_0}{\partial x} \mathrm{d}x.$$

Another way to see this is to define a density. Assume we have a uniform distribution of something along the x coordinate. How much stuff do we have between 0 and x at time t?

The answer is the length of the line segment $[x_1, x_2]$ for which

$$0 = tu_0(x_1) + x_1, \; x = tu_0(x_2) + x_2$$

i.e. the stuff that is transported between points 0 and x at time t by the flow. Mathematically we have

$$\text{stuff} = \int_{x_1}^{x_2} \mathrm{d}x_0 = \int_0^x \frac{\partial x_0}{\partial x} \mathrm{d}x.$$

Therefore

$$\frac{\partial x_0}{\partial x} = \frac{1}{tu_0'(x_0) + 1}$$

can be seen as a *density* that approaches infinity when a blow-up occurs. This means that stuff gets packed to a single point.

18.303 summary

The most important takeaway from this class is that functions can be seen as *vectors* and differentials as *linear operators*. Therefore we can always express our functions in a suitable basis.

18.303 summary

The most important takeaway from this class is that functions can be seen as *vectors* and differentials as *linear operators*. Therefore we can always express our functions in a suitable basis.

These transformations between bases can be used analytically and numerically. For example if

$$\partial_t f(t, \mathsf{x}) = \mathcal{L} f(t, \mathsf{x}).$$

with some boundary and initial conditions, we can express f in the eigenbasis of $\mathcal{L}$ given by the differential equation

$$\mathcal{L}\psi(\mathsf{x}) = \lambda\psi(\mathsf{x})$$

with the same boundary conditions for x.

18.303 summary

The most important takeaway from this class is that functions can be seen as *vectors* and differentials as *linear operators*. Therefore we can always express our functions in a suitable basis.

These transformations between bases can be used analytically and numerically. For example if

$$\partial_t f(t, \mathsf{x}) = \mathcal{L} f(t, \mathsf{x}).$$

with some boundary and initial conditions, we can express f in the eigenbasis of $\mathcal{L}$ given by the differential equation

$$\mathcal{L}\psi(\mathsf{x}) = \lambda\psi(\mathsf{x})$$

with the same boundary conditions for x.

After such a change of basis it is almost trivial to solve for the time evolution. The eigenbasis can be usually found analytically for symmetric domains and simple enough operators.

Transforms

We introduced a bunch of different transformations that can be seen as a change of bases.

Transforms

We introduced a bunch of different transformations that can be seen as a change of bases.

Among these we have

- Fourier transform for functions on the real line (or $\mathbb{R}^N$)

Transforms

We introduced a bunch of different transformations that can be seen as a change of bases.

Among these we have

- Fourier transform for functions on the real line (or $\mathbb{R}^N$)
- Laplace transform for functions on $\mathbb{R}_+$

Transforms

We introduced a bunch of different transformations that can be seen as a change of bases.

Among these we have

- Fourier transform for functions on the real line (or $\mathbb{R}^N$)
- Laplace transform for functions on $\mathbb{R}_+$
- Fourier series for periodic data on a bounded interval

Transforms

We introduced a bunch of different transformations that can be seen as a change of bases.

Among these we have

- Fourier transform for functions on the real line (or $\mathbb{R}^N$)
- Laplace transform for functions on $\mathbb{R}_+$
- Fourier series for periodic data on a bounded interval
- Sine series for Dirichlet problems on bounded intervals

We introduced a bunch of different transformations that can be seen as a change of bases.

Among these we have

- Fourier transform for functions on the real line (or $\mathbb{R}^N$)
- Laplace transform for functions on $\mathbb{R}_+$
- Fourier series for periodic data on a bounded interval
- Sine series for Dirichlet problems on bounded intervals
- Discrete Fourier transform for discrete data

Different numerical schemes

- **Finite difference methods:** easy to implement and very versatile.

Different numerical schemes

- **Finite difference methods:** easy to implement and very versatile.
- **Spectral methods (especially in Fourier basis):** will evaluate derivatives very accurately but cannot be used for arbitrary domains and boundary conditions (at least easily).

Different numerical schemes

- **Finite difference methods:** easy to implement and very versatile.
- **Spectral methods (especially in Fourier basis):** will evaluate derivatives very accurately but cannot be used for arbitrary domains and boundary conditions (at least easily).
- **Finite element methods:** a bit harder to implement but efficient for complicated domains and boundary conditions.

Different numerical schemes

- **Finite difference methods:** easy to implement and very versatile.
- **Spectral methods (especially in Fourier basis):** will evaluate derivatives very accurately but cannot be used for arbitrary domains and boundary conditions (at least easily).
- **Finite element methods:** a bit harder to implement but efficient for complicated domains and boundary conditions.
- Many great commercial and open source software packages exist for all the different methods.

Different numerical schemes

- **Finite difference methods:** easy to implement and very versatile.
- **Spectral methods (especially in Fourier basis):** will evaluate derivatives very accurately but cannot be used for arbitrary domains and boundary conditions (at least easily).
- **Finite element methods:** a bit harder to implement but efficient for complicated domains and boundary conditions.
- Many great commercial and open source software packages exist for all the different methods.
- In the end we just have a bunch of linear algebra problems that computers can solve efficiently using existing packages.

Some analytical techniques

- Separation of variables for partial differential equations.

Some analytical techniques

- Separation of variables for partial differential equations.
- Green’s functions (there’s a lot more to these than what we covered in this class)

Some analytical techniques

- Separation of variables for partial differential equations.
- Green's functions (there's a lot more to these than what we covered in this class)
- Method of characteristics for non-linear first order PDEs

Some analytical techniques

- Separation of variables for partial differential equations.
- Green's functions (there's a lot more to these than what we covered in this class)
- Method of characteristics for non-linear first order PDEs
- We also covered basic properties of distributions and how they can be used to make things simple

Map between linear algebra and differential equations

Homogeneous problems:

$$\mathcal{L}u(\mathsf{x}) = 0 \rightarrow A\mathbf{u} = 0.$$

Map between linear algebra and differential equations

Homogeneous problems:

$$\mathcal{L}u(\mathsf{x}) = 0 \rightarrow A\mathsf{u} = 0.$$

Inhomogeneous problems:

$$\mathcal{L}u(\mathsf{x}) = f(\mathsf{x}) \rightarrow A\mathsf{u} = \mathsf{f}.$$

Map between linear algebra and differential equations

Homogeneous problems:

$$\mathcal{L}u(\mathsf{x}) = 0 \to A\mathbf{u} = 0.$$

Inhomogeneous problems:

$$\mathcal{L}u(\mathsf{x}) = f(\mathsf{x}) \to A\mathbf{u} = \mathbf{f}.$$

Eigenproblems

$$\mathcal{L}\phi_n = \lambda_n\phi_n \to A\boldsymbol{\phi}_n = \lambda_n\boldsymbol{\phi}_n.$$