Green's Functions & Distributions (Pt. 2)

$\delta(x)$ is not a function, but a distribution.

Distributions act on smooth functions in $C_0^\infty(\underbrace{-1,1}_{\text{or other domains in } \mathbb{R}^n})$

$$\int_{-1}^{+1} \delta(x) \underbrace{f(x)}_{\substack{\text{smooth}\\\text{function}}} dx = \underbrace{f(0)}_{\substack{\text{value}\\\text{at zero}}}$$

Distributions are linear:

$$\int_{-1}^{+1} \delta(x) \big(\underbrace{\alpha f(x) + \beta g(x)}_{\substack{\text{linear combo}\\\text{of inputs}}}\big) dx = \underbrace{\alpha f(0) + \beta f(0)}_{\substack{\text{linear combo}\\\text{of outputs}}}$$

They can be "differentiated":

$$\int_{-1}^{+1} \underbrace{\delta'(x)}_{\substack{\text{"distributional"}\\\text{derivative of } \delta}} f(x)\, dx = -\underbrace{f'(0)}_{\substack{\text{derivative}\\\text{of } f \text{ at } x=0}}$$

The derivative is a new distribution, i.e. maps $f(x)$ to $f'(0)$, which is another linear map.

Distributions can also be integrated:

$$\int_{-1}^{+1}\int_{-1}^{x}\delta(y)\,dy\, f(x)\,dx = \int_{0}^{+1} f(x)\,dx$$

Algebra

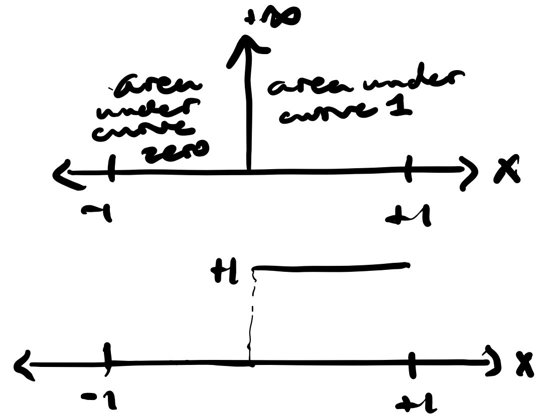
$$\int_{-1}^{+1}\int_{-1}^{+1}\delta(y)\,\chi_{(-1,x)}(y)\,dy\, f(x)\,dx$$

$$= \int_{-1}^{+1}\delta(y)\int_{-1}^{+1} f(x)\,\chi_{(-1,x)}(y)\,dx\,dy$$

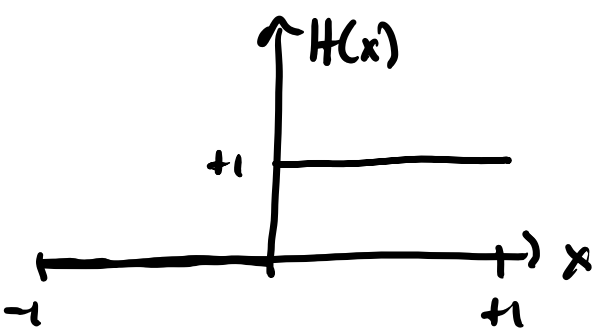
$$= \int_{-1}^{+1}\delta(y)\int_{y}^{+1} f(x)\,dx\,dy$$

$$= \int_{0}^{+1} f(x)\,dx$$

Picture



The integral of $\delta(x)$, $\int_{-1}^{x}\delta(y)\,dy$, acts like the Heaviside function $H(x)$, a piecewise constant function



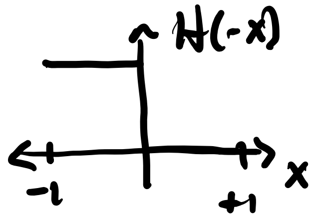
$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\int_{-1}^{+1}\int_{-1}^{x}\delta(y)\,dy\, f(x)\,dx = \int_{-1}^{+1} H(x) f(x)\,dx = \int_{0}^{1} f(x)\,dx$$

Similarly, we can derive other integrals of the δ function like

they are the same distribution

$$\int_x^1 \delta(y)\,dy \overset{\downarrow}{=} H(-x) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$$



Since distributions are defined by their action on test functions in $C_0^\infty(-1,1)$, what do we mean by

$$-u_y''(x) = \delta(x-y) \quad ?$$
$$u_y(\pm 1) = 0$$

The equality means $-u_y''$ and δ should be the same distribution, that is, we are looking for a <u>distribution</u> u_y s.t. $-u_y'' = \delta$:

$$\int_{-1}^{+1} -u_y''(x)\,\phi(x)\,dx = \int_{-1}^{+1} \delta(x-y)\,\phi(x)\,dx$$

$$= \phi(y)$$

This equivalent to $u(x) = \int_{-1}^{+1} G(x,y)\,f(y)\,dy$ solving $-u''(x) = f(x)$ for all $f \in C_0^\infty(-1,1)$, i.e., $G(x,y)$ being the inverse of $-d^2/dx^2$.

Properties of Green's Functions

Green's Functions can often be constructed by patching together functions in the null-space of L with continuity & jump conditions.

For $\quad [Lu](x) = p(x)\dfrac{d^2u}{dx^2} + q(x)\dfrac{du}{dx} + r(x)u(x)$

with $u(a)=u(b)=0$, $p,q,r \in \overset{\text{continuous}}{C(a,b)}$ and $p(x)\neq 0$.

the Green's function $G(x,y) = u_y(x)$:

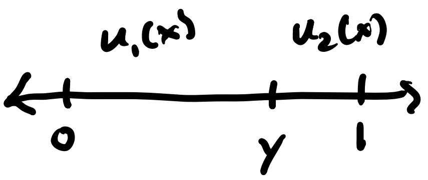
1) Solves $Lu_y(x)=0$ for all $x\neq y$,
2) Satisfies $u_y(a)=u_y(b)=0$,
3) is continuous in both x and y,
4) has, for each fixed $a<y<b$, continuous derivative $\dfrac{\partial G}{\partial x}$ when $x\neq y$ and $\dfrac{\partial G}{\partial x}$ has a jump of magnitude 1 at $x=y$.

Example $\quad -u_y''(x) + \omega^2 u_y(x) = \delta(x-y)$

$$u_y(0) = u_y(1) = 0$$

for $x \neq y$, $-u_y''(x) + \omega^2 u_y(x) = 0$ and $u_y(0) = u_y(1) = 0$
by (1) ; (2)

general solutions $\Rightarrow$ $e^{\omega x}$ and $e^{-\omega x}$



$0<x<y$

free constant

$$u_1(x) = \frac{a}{2}(e^{\omega x} - e^{-\omega x})$$

So that $u_1(0) = 0$

$y<x<1$

so that $u_2(1) = 0$

$$u_2(x) = \frac{b}{2}(e^{-\omega} e^{\omega x} - e^{\omega} e^{-\omega x})$$

$$= \frac{b}{2}(e^{\omega(x-1)} - e^{-\omega(x-1)})$$

Now choose a and b so that

$$u(x) = \begin{cases} u_1(x) & x<y \\ u_2(x) & x>y \end{cases}$$

satisfies continuity (3) and jump (4) conditions

$\Rightarrow u_1(x) = a \sinh \omega x$

$\Rightarrow u_2(x) = b \sinh \omega(x-1)$

(3) Continuity at $x=y$

$$u_1(y) = a \sinh \omega y = b \sinh \omega(y-1) = u_2(y)$$

$$\Rightarrow \quad a \sinh \omega y - b \sinh \omega(y-1) = 0$$

(4) Jump at $x=y$

$$1 + u_1'(y) = 1 + a\omega \cosh \omega y = b\omega \cosh \omega(y-1)$$

$$\Rightarrow \; -a\omega \cosh \omega y + b\omega \cosh \omega(y-1) = 1$$

$$\begin{bmatrix} -\omega\cosh\omega y & \omega\cosh\omega(y-1) \\ \sinh\omega y & -\sinh\omega(y-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solve & simplify $\Rightarrow$

$$a = -\frac{\sinh\omega(y-1)}{\omega\sinh\omega}$$

$$b = -\frac{\sinh\omega y}{\omega\sinh\omega}$$

$$G(x,y) = u_y(x) = \begin{cases} \dfrac{\sinh\omega x \sinh\omega(1-y)}{\omega\sinh\omega} & x \le y \\ \dfrac{\sinh\omega(1-x)\sinh\omega y}{\omega\sinh\omega} & x \ge y \end{cases}$$