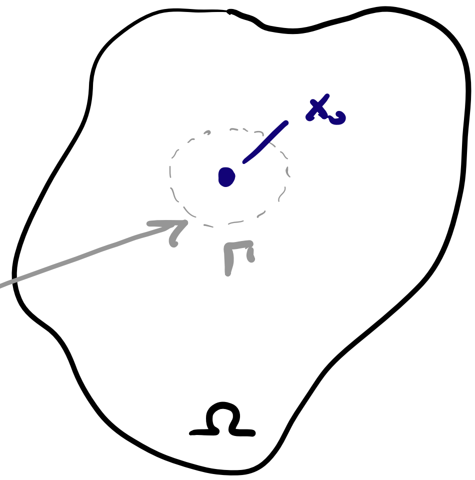
More on Poisson's Eq.

Laplace Eq. $\Rightarrow$ $\Delta u = 0$

$$u|_{\partial\Omega} = g$$

$$u|_{\partial\Gamma} = \tilde{g}$$



Poisson Formula (Lecture 7) $\Rightarrow$

$$u(x_0) = \frac{1}{2\pi}\int_0^{2\pi} G(0,\theta,\theta')\,\tilde{g}(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \tilde{g}(\theta)\,d\theta$$

The solution at x_0 is an average of the solution's values on the circle Γ.

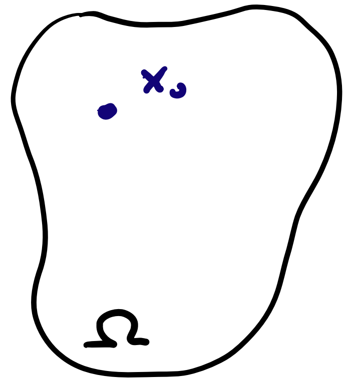
$\Rightarrow$ The maximum and minimum values of the solution are obtained on the boundary of the domain Ω.

Maximum (& Minimum) Principles

We can make similar statements about

Poisson's Eq. $\Rightarrow$ $\Delta u = f$

$u|_{\partial\Omega} = g$



Max: If $f \geq 0$ on Ω, then the maximum of u must be on $\partial\Omega$.

Min: If $f \leq 0$ on Ω, then the minimum of u must be on $\partial\Omega$.

Idea: At "regular" maximum, we have

at least one is <

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_0} \leq 0 \quad \text{and} \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{x_0} \leq 0.$$

which contradicts $\Delta u = f \geq 0$. More care needed to deal with irregular maxima where both $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_0} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y_0} = 0$.

Generalization: maximum/minimum principles can be derived for elliptic definite PDEs in open region $\Omega \subset \mathbb{R}^n$:

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} = f(x,y)$$

The functions a_{ij}, b_i should be continuous and the matrix $(A)_{ij} = a_{ij}(x)$ should be symmetric positive-definite for each $x \in \Omega$.

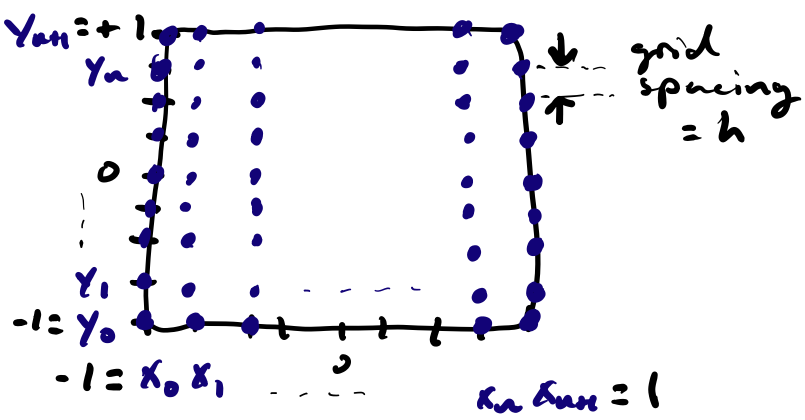
They also come in "strong" & "weak" form

"Strong = Only constant function achieves max/min inside Ω.

"Weak" = Max achieved on $\partial\Omega$.

2D Finite Differences

$$\Delta u = f$$
$$u|_{\partial\Omega} = 0$$



Idea: Approximate derivatives on grid using finite differences on "interior grid" $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.

Apply boundary conditions on outer layer of "boundary" gridpoints.

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j,y_k} \approx \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{h^2}, \qquad \left.\frac{\partial^2 u}{\partial y^2}\right|_{x_j,y_k} = \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{h^2}$$

In matrix notation:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j,y_k} \approx \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}}_{D = \text{"2nd deriv matrix"}} \begin{array}{cc} & \text{"row coord"} \\ \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \\ \begin{matrix} y_1 & y_2 & \cdots & y_n \end{matrix} & \\ \text{"column coord"} & \end{array}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{x_j,y_k} \approx \begin{array}{cc} \text{"row coord"} & \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \\ \text{"column coord"} & \begin{matrix} y_1 & y_2 & \cdots & y_n \end{matrix} \end{array} \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}}_{D = \text{"2nd deriv matrix"}}$$

Discretize Poisson's Equation!

$$\overbrace{D\,U + U\,D}^{\text{2nd deriv. matrix}} = F$$

$$\tau(U)_{jk} = u_{jk} \qquad \tau(F)_{jk} = F(x_j, y_k)$$

Boundary Conditions: At nodes with x_1, x_n, y_1, y_n

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_1, y_k} \approx \frac{\overbrace{\cancel{u_{0,k}}}^{=0} - 2u_{1,k} + u_{2,k}}{h^2} = \frac{-2u_{1,k} + u_{2,k}}{h^2}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_n, y_k} \approx \frac{u_{n-1,k} - 2u_{n,k} + \overbrace{\cancel{u_{n+1,k}}}^{=0}}{h^2} = \frac{u_{n-1,k} - 2u_{n,k}}{h^2}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{x_j, y_1} \approx \frac{\overbrace{\cancel{u_{j,0}}}^{=0} - 2u_{j,1} + u_{j,2}}{h^2} = \frac{-2u_{j,1} + u_{j,2}}{h^2}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{x_j, y_n} \approx \frac{u_{j,n-1} - 2u_{j,n} + \overbrace{\cancel{u_{j,n+1}}}^{=0}}{h^2} = \frac{u_{j,n-1} - 2u_{j,n}}{h^2}$$

<u>HW2 Q:</u> How to enforce $u|_{\partial\Omega} = g$?

Now, how do we solve the matrix eq.

$$DU + UD = F \quad ?$$

Solve: we can rewrite this matrix equation as a linear system $Ax=b$, by using the "vec" and "kron" operations.

$$\mathrm{vec}(F) = \begin{bmatrix} F_{11} \\ \vdots \\ F_{n1} \\ F_{12} \\ \vdots \\ F_{n2} \\ \vdots \\ F_{1n} \\ \vdots \\ F_{nn} \end{bmatrix} \begin{array}{l} 1^{st} \\ \text{column} \\ \\ 2^{nd} \\ \text{column} \\ \\ \vdots \\ n^{th} \\ \text{column} \\ \\ \end{array}$$

vector $n^2 \times 1$

"Stack columns of F vertically"

"kron": Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}B & - & - & a_{nn}B \end{bmatrix}$$

matrix $n^2 \times n^2$

Key Identity: $\mathrm{vec}(BXA) = (A^T \otimes B)\,\mathrm{vec}(X)$

$n \times n$ identity

$$\Rightarrow \quad DUI + IUD = F$$

"vec" both sides $\Rightarrow$ $(I^T \otimes D)\,\mathrm{vec}(U) + (D^T \otimes I)\,\mathrm{vec}(U) = \mathrm{vec}(F)$

$$\underbrace{\left(I \otimes D + D \otimes I\right)}_{A} \underbrace{\mathrm{vec}(U)}_{x} = \underbrace{\mathrm{vec}(F)}_{b}$$

$\Rightarrow$ Solve $n^2 \times n^2$ system for vec(U).

$\Rightarrow$ Reshape vec(U) to $n \times n$ matrix for plotting on 2D grid (typically).