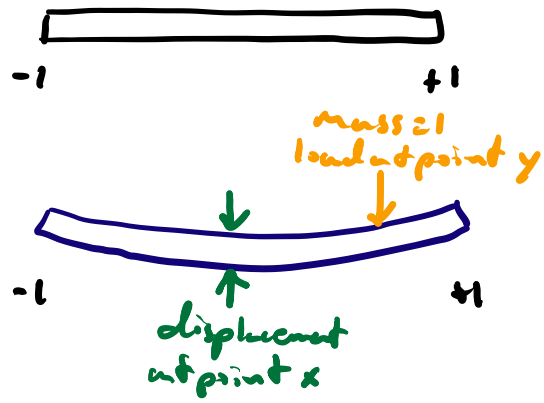
Green's Functions & Distributions

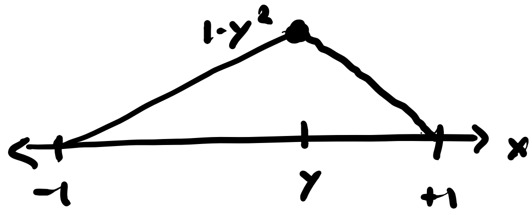
"Thin Elastic Beam"

$$-u_y''(x) = \delta(x-y)$$

$$\text{s.t.} \quad u_y(\pm 1) = 0$$



Green's Function



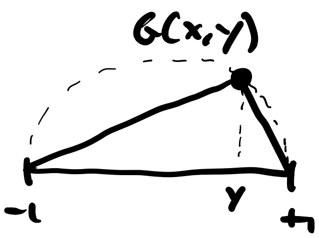
$$G(x,y) = u_y(x) = \begin{cases} x - y + \frac{1}{2}(1+y)(1-x) & x < y \\ \frac{1}{2}(1+y)(1-x) & y < x \end{cases}$$

See Lecture 21

Q: Where should we place load to achieve max disp at $x = \frac{1}{4}$?

$$\Rightarrow \underset{y \in [-1,1]}{\arg\max}\ G\left(\tfrac{1}{4}, y\right) = \tfrac{1}{4} \quad \to \quad G\left(\tfrac{1}{4}, \tfrac{1}{4}\right) = \tfrac{15}{16}$$

Q: How does the maximum displacement depend on the location of the load?



$$\max_{-1 < x < 1} G(x,y) = 1 - y^2 \begin{cases} \to 0 & \text{as } y \to 1 \\ \to 1 & \text{as } y \to 0 \\ \to 0 & \text{as } y \to -1 \end{cases}$$

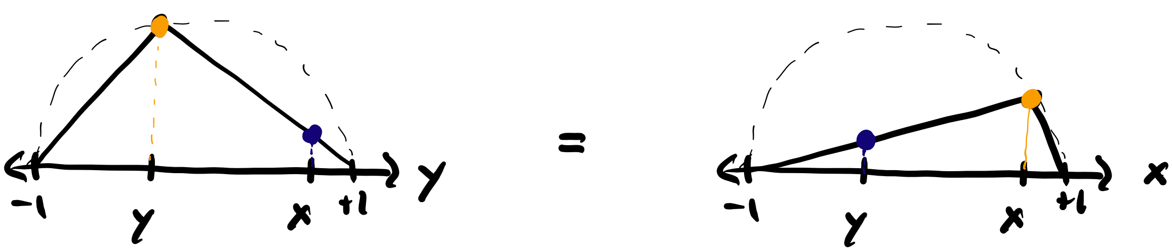
Q: What happens if $x \leftrightarrows y$? Physical interpretation?

$$G(x,y) = u_y(x) = \begin{cases} x - y + \frac{1}{2}(1+y)(1-x) & x<y \\ \frac{1}{2}(1+y)(1-x) & y<x \end{cases}$$

$$= \left.\begin{cases} \frac{1}{2}(1 + x - y - xy) & x<y \\ \frac{1}{2}(1 + y - x - xy) & y<x \end{cases}\right\} = 1 - xy - |x-y|$$

<u>$x<y$</u> $\qquad$ <u>$y<x$</u>

$G(x,y) = \frac{1}{2}(1 + x - y - xy) = G(y,x)$, $\quad G(x,y) = \frac{1}{2}(1 + y - x - xy) = G(y,x)$



$G(x,y)$ $\qquad\qquad$ $G(y,x)$

$\Rightarrow$ Displacement at x due to point-mass load at y is equal to displacement at y due to load at x.

$\Rightarrow$ Symmetry of G is due to selfadjoint diff. op. $u \mapsto -u''$ with $u(\pm 1) = 0$! Analogous to "Symmetric matrix has symmetric inverse."

Inverse Operator

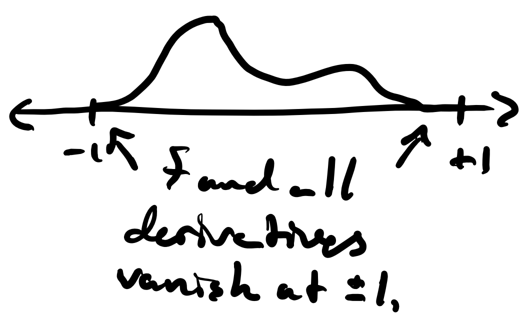
$$u(x) = \int_{-1}^{1} G(x,y)\, f(y)\, dy \quad \text{solves}$$

$$-u''(x) = f(x) \quad \text{s.t.} \quad u(\pm 1) = 0$$

Delta Function as a Distribution

$$(*) \quad \int_{-1}^{1} \delta(y)\, f(y) = f(0) \qquad \text{for all} \quad f \in C_0^{\infty}(-1,1)$$

"Smooth & Compact supp in $(-1,1)$"



δ is not a "function" in $L^2(-1,1)$, there is no such function with property $(*)$! Instead δ is a continuous linear functional, also called a distribution or generalized function.

We'll study the "calculus" of distributions so we can correctly calculate w/ δ and Green's functions.