Massachusetts
Institute of
Technology

Distributions and Fourier transform

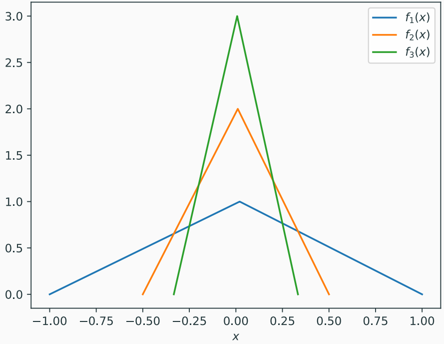
The infinite domain

18.303 Linear Partial Differential Equations: Analysis and Numerics

Distributions

Let's start with an example. Consider the series of functions

$$f_n(x) = \begin{cases} n(1 - |x|n), & -1/n < x < 1/n \\ 0, & \text{otherwise.} \end{cases}$$



Few of the functions f_n.

Distributions

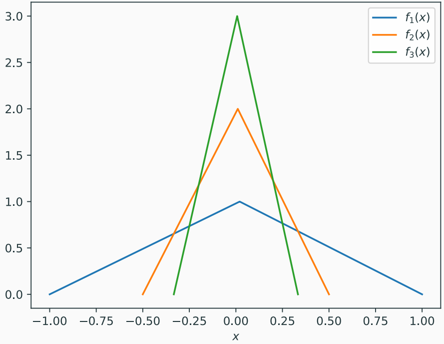
Let's start with an example. Consider the series of functions

$$f_n(x) = \begin{cases} n(1 - |x|n), & -1/n < x < 1/n \\ 0, & \text{otherwise.} \end{cases}$$

We see that f_n is not bounded by any number so it doesn't converge to an ordinary function. However,

$$\int_{\mathbb{R}} f_n(x)\mathrm{d}x = 1$$

for all n.



Few of the functions f_n.

Consider some integrable function $g : \mathbb{R} \to \mathbb{R}$. What would the inner product

$$\lim_{n\to\infty} \langle f_n, g\rangle = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) \mathrm{d}x$$

give?

Consider some integrable function $g : \mathbb{R} \to \mathbb{R}$. What would the inner product

$$\lim_{n\to\infty} \langle f_n, g \rangle = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) \mathrm{d}x$$

give?

The integral of f_n over the real numbers is 1 and $f_n \geq 0$. This implies that the inner product can be seen as some sort of weighted average of g.

Consider some integrable function $g : \mathbb{R} \to \mathbb{R}$. What would the inner product

$$\lim_{n\to\infty} \langle f_n, g\rangle = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\mathrm{d}x$$

give?

The integral of f_n over the real numbers is 1 and $f_n \geq 0$. This implies that the inner product can be seen as some sort of weighted average of g.

On the other hand, the support f_n (set where $f_n(x) \neq 0$) is the interval $(-1/n, 1/n)$, which is getting smaller and smaller with increasing n.

Consider some integrable function $g : \mathbb{R} \to \mathbb{R}$. What would the inner product

$$\lim_{n\to\infty} \langle f_n, g \rangle = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) \mathrm{d}x$$

give?

The integral of f_n over the real numbers is 1 and $f_n \geq 0$. This implies that the inner product can be seen as some sort of weighted average of g.

On the other hand, the support f_n (set where $f_n(x) \neq 0$) is the interval $(-1/n, 1/n)$, which is getting smaller and smaller with increasing n.

It can be shown that the limit of this sequence is $g(0)$.

Consider some integrable function $g : \mathbb{R} \to \mathbb{R}$. What would the inner product

$$\lim_{n\to\infty} \langle f_n, g \rangle = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) \mathrm{d}x$$

give?

The integral of f_n over the real numbers is 1 and $f_n \geq 0$. This implies that the inner product can be seen as some sort of weighted average of g.

On the other hand, the support f_n (set where $f_n(x) \neq 0$) is the interval $(-1/n, 1/n)$, which is getting smaller and smaller with increasing n.

It can be shown that the limit of this sequence is $g(0)$.

The object that we get as the limit of f_n is called a distribution. It can be understood through inner products with ordinary functions.

The delta distribution

In fact, the limit $\lim_{n\to\infty} f_n = \delta$ is a very important distribution and is called the delta distribution. For the delta distribution we have (as we reasoned before)

$$\int_{\mathbb{R}} g(x)\delta(x)\mathrm{d}x = g(0).$$

The delta distribution

In fact, the limit $\lim_{n\to\infty} f_n = \delta$ is a very important distribution and is called the delta distribution. For the delta distribution we have (as we reasoned before)

$$\int_{\mathbb{R}} g(x)\delta(x)\mathrm{d}x = g(0).$$

More generally, we can write

$$\int_{\mathbb{R}} g(x)\delta(x-y)\mathrm{d}x = \int_{\mathbb{R}} g(x)\delta(y-x)\mathrm{d}x = g(y).$$

The delta distribution

In fact, the limit $\lim_{n\to\infty} f_n = \delta$ is a very important distribution and is called the delta distribution. For the delta distribution we have (as we reasoned before)

$$\int_{\mathbb{R}} g(x)\delta(x)\mathrm{d}x = g(0).$$

More generally, we can write

$$\int_{\mathbb{R}} g(x)\delta(x-y)\mathrm{d}x = \int_{\mathbb{R}} g(x)\delta(y-x)\mathrm{d}x = g(y).$$

This distribution lives in a vector space and we have

$$\int_{\mathbb{R}} (\alpha\delta(x-y) + \beta\delta(x-z))\, g(x)\mathrm{d}x = \alpha \int_{\mathbb{R}} \delta(x-y)g(x)\mathrm{d}x + \beta \int_{\mathbb{R}} \delta(x-z)g(x)\mathrm{d}x = \alpha g(y) + \beta g(z).$$

The delta function has a scaling property

$$\delta(ax) = \delta(x)/|a|.$$

The delta function has a scaling property

$$\delta(ax) = \delta(x)/|a|.$$

This can be in fact generalized to

$$\delta(f(x)) = \sum_{n:f(x_n)=0} \frac{\delta(x - x_n)}{|f'(x_n)|},$$

where the sum goes through all the zeroes of f.

The delta function has a scaling property

$$\delta(ax) = \delta(x)/|a|.$$

This can be in fact generalized to

$$\delta(f(x)) = \sum_{n:f(x_n)=0} \frac{\delta(x - x_n)}{|f'(x_n)|},$$

where the sum goes through all the zeroes of f.

See if you can prove this by calculating

$$\int_{\mathbb{R}} \delta(f(x))g(x)\mathrm{d}x.$$

The delta distribution can also be obtained as a sequence of functions that do not have a finite support. For example, for the Gaussian distribution we have

$$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-x')^2}{2\sigma^2}} \xrightarrow[\sigma\to 0]{} \delta(x-x').$$

The delta distribution can also be obtained as a sequence of functions that do not have a finite support. For example, for the Gaussian distribution we have

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x')^2}{2\sigma^2}} \xrightarrow[\sigma \to 0]{} \delta(x - x').$$

The sufficient condition is that the integral of f_n approaches 1 and that for any open interval Ω not containing 0 and any $\epsilon > 0$, there's a N s.t. for all $n > N$, the integral

$$\int_\Omega f_n(x)\mathrm{d}x < \epsilon.$$

The delta distribution can also be obtained as a sequence of functions that do not have a finite support. For example, for the Gaussian distribution we have

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x')^2}{2\sigma^2}} \xrightarrow[\sigma\to 0]{} \delta(x-x').$$

The sufficient condition is that the integral of f_n approaches 1 and that for any open interval Ω not containing 0 and any $\epsilon > 0$, there's a N s.t. for all $n > N$, the integral

$$\int_\Omega f_n(x)\mathrm{d}x < \epsilon.$$

The limiting functions don't even have to be even. We could have chosen e.g.

$$f_n(x) = \begin{cases} 2(1-nx)n, & 0 \geq x < 1/n \\ 0, & \text{otherwise} \end{cases}$$

and this would still give us the delta distribution $\delta(0)$.

Fourier transform

We define the Fourier transform for functions f as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} f(t) \mathrm{d}t.$$

Fourier transform

We define the Fourier transform for functions f as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} f(t) \mathrm{d}t.$$

It has the inverse transform defined as

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \int_{\mathbb{R}} e^{2\pi i \xi t} \hat{f}(\xi) \mathrm{d}\xi.$$

Notice that the sign of the exponential changes here.

Fourier transform

We define the Fourier transform for functions f as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} f(t) \mathrm{d}t.$$

It has the inverse transform defined as

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \int_{\mathbb{R}} e^{2\pi i \xi t} \hat{f}(\xi) \mathrm{d}\xi.$$

Notice that the sign of the exponential changes here.

Fourier transform exists for all functions with $\int_{\mathbb{R}} |f(x)| \mathrm{d}x < \infty$. This condition is sufficient but not necessary. Writing the sufficient condition requires a bit more math that we'll cover during this class.

Fourier transforms can be generalized to $\mathbb{R}^n$ by making ξ and t vectors and their multiplication a dot product i.e.

$$f(\mathsf{x}) = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi} \cdot \mathsf{x}} \mathrm{d}\mathsf{x}.$$

Fourier transforms can be generalized to $\mathbb{R}^n$ by making ξ and t vectors and their multiplication a dot product i.e.

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \mathrm{d}\mathbf{x}.$$

You might also see Fourier transforms like

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) \mathrm{d}x,$$

that has the inverse

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} f(k) \mathrm{d}k.$$

Fourier transforms can be generalized to $\mathbb{R}^n$ by making ξ and t vectors and their multiplication a dot product i.e.

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}.$$

You might also see Fourier transforms like

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) \mathrm{d}x,$$

that has the inverse

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} f(k) \mathrm{d}k.$$

Other variants have the normalization constant in the $\frac{1}{2\pi}$ in the transform instead of the inverse transform. There's also a unitary version with $\frac{1}{\sqrt{2\pi}}$ normalization for both directions.

Fourier transforms can be generalized to $\mathbb{R}^n$ by making ξ and t vectors and their multiplication a dot product i.e.

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \mathrm{d}\mathbf{x}.$$

You might also see Fourier transforms like

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) \mathrm{d}x,$$

that has the inverse

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} f(k) \mathrm{d}k.$$

Other variants have the normalization constant in the $\frac{1}{2\pi}$ in the transform instead of the inverse transform. There's also a unitary version with $\frac{1}{\sqrt{2\pi}}$ normalization for both directions.

This might seem confusing but in the end the difference between these conventions is always about multiplying the result with some π dependent constant.

There's an important property of the basis functions $e^{2\pi i\xi x}$ that we will derive here. Consider

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i\xi x} \mathrm{d}\xi.$$

There's an important property of the basis functions $e^{2\pi i\xi x}$ that we will derive here. Consider

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi.$$

Replacing $\hat{f}$ with its definition gives

$$f(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x') e^{-2\pi i \xi x'} \mathrm{d}x' \right) e^{2\pi i \xi x} \mathrm{d}\xi.$$

There's an important property of the basis functions $e^{2\pi i\xi x}$ that we will derive here. Consider

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i\xi x} \mathrm{d}\xi.$$

Replacing $\hat{f}$ with its definition gives

$$f(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x') e^{-2\pi i\xi x'} \mathrm{d}x' \right) e^{2\pi i\xi x} \mathrm{d}\xi.$$

By reordering the integral we get

$$f(x) = \int_{\mathbb{R}} f(x') \left(\int e^{2\pi i\xi(x-x')} \mathrm{d}\xi \right) \mathrm{d}x'.$$

There's an important property of the basis functions $e^{2\pi i\xi x}$ that we will derive here. Consider

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i\xi x} \mathrm{d}\xi.$$

Replacing $\hat{f}$ with its definition gives

$$f(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x') e^{-2\pi i\xi x'} \mathrm{d}x' \right) e^{2\pi i\xi x} \mathrm{d}\xi.$$

By reordering the integral we get

$$f(x) = \int_{\mathbb{R}} f(x') \left(\int e^{2\pi i\xi(x-x')} \mathrm{d}\xi \right) \mathrm{d}x'.$$

Question: what does the integral in the parentheses give?

There's an important property of the basis functions $e^{2\pi i\xi x}$ that we will derive here. Consider

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi.$$

Replacing $\hat{f}$ with its definition gives

$$f(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x') e^{-2\pi i \xi x'} \mathrm{d}x' \right) e^{2\pi i \xi x} \mathrm{d}\xi.$$

By reordering the integral we get

$$f(x) = \int_{\mathbb{R}} f(x') \left(\int e^{2\pi i \xi (x - x')} \mathrm{d}\xi \right) \mathrm{d}x'.$$

Question: what does the integral in the parentheses give?

It gives us the delta function

$$\int e^{2\pi i \xi (x - x')} \mathrm{d}\xi = \delta(x - x').$$

This tells us that the basis $\{e^{2\pi i \xi t}\}_{\xi}$ is orthonormal.

Fourier transforms have other important properties. It can for example be used to calculate the derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi = \int_{\mathbb{R}} 2\pi i \xi \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}x = \mathcal{F}^{-1} \left[2\pi i \xi \hat{f} \right] (x).$$

Fourier transforms have other important properties. It can for example be used to calculate the derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi = \int_{\mathbb{R}} 2\pi i \xi \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}x = \mathcal{F}^{-1}\left[2\pi i \xi \hat{f}\right](x).$$

Notice here that if we would use the basis functions e^{ikx}, we wouldn't have the deal here with the 2π coefficients. This is the main reason this other definition is used.

Fourier transforms have other important properties. It can for example be used to calculate the derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi = \int_{\mathbb{R}} 2\pi i \xi \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}x = \mathcal{F}^{-1}\left[2\pi i \xi \hat{f}\right](x).$$

Notice here that if we would use the basis functions e^{ikx}, we wouldn't have the deal here with the 2π coefficients. This is the main reason this other definition is used.

Another property worth mentioning is *Plancherel theorem*. It states that

$$\int_{\mathbb{R}} g^*(x) f(x) \mathrm{d}x = \int_{\mathbb{R}} \hat{g}^*(\xi) \hat{f}(\xi) \mathrm{d}\xi,$$

which we can also write as

$$\langle g, f \rangle = \langle \hat{g}, \hat{f} \rangle.$$

Here we have to assume that $\int_{\mathbb{R}} |g(x)|^2 \mathrm{d}x$ and $\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x$ are finite.

Exercise 1

Derive Plancherel theorem using the orthonormality of the basis $\{e^{2\pi i\xi x}\}_\xi$.

Fourier coefficients and the Fourier transform

Fourier coefficients are closely related to the Fourier transform.

Fourier coefficients and the Fourier transform

Fourier coefficients are closely related to the Fourier transform.

Consider a function that is zero outside the interval $(-L/2, L/2)$. We have the Fourier coefficients

$$\hat{F}_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{2\pi ixn/L}\mathrm{d}x.$$

Fourier coefficients and the Fourier transform

Fourier coefficients are closely related to the Fourier transform.

Consider a function that is zero outside the interval $(-L/2, L/2)$. We have the Fourier coefficients

$$\hat{F}_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{2\pi ixn/L}\mathrm{d}x.$$

Since $f(x)$ is zero outside this interval, we can extend the integral to the whole line. We notice then that this is just the Fourier transform of f at n/L i.e.

$$\hat{F}_n = \frac{1}{L}\hat{f}\left(\frac{n}{L}\right).$$

Fourier coefficients and the Fourier transform

Fourier coefficients are closely related to the Fourier transform.

Consider a function that is zero outside the interval $(-L/2, L/2)$. We have the Fourier coefficients

$$\hat{F}_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{2\pi ixn/L}\mathrm{d}x.$$

Since $f(x)$ is zero outside this interval, we can extend the integral to the whole line. We notice then that this is just the Fourier transform of f at n/L i.e.

$$\hat{F}_n = \frac{1}{L}\hat{f}\left(\frac{n}{L}\right).$$

We write the function f using the Fourier coefficients as

$$f(x) = \sum_n \hat{F}_n e^{2\pi ixn/L}.$$

Fourier coefficients and the Fourier transform

Fourier coefficients are closely related to the Fourier transform.

Consider a function that is zero outside the interval $(-L/2, L/2)$. We have the Fourier coefficients

$$\hat{F}_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{2\pi ixn/L}\mathrm{d}x.$$

Since $f(x)$ is zero outside this interval, we can extend the integral to the whole line. We notice then that this is just the Fourier transform of f at n/L i.e.

$$\hat{F}_n = \frac{1}{L}\hat{f}\left(\frac{n}{L}\right).$$

We write the function f using the Fourier coefficients as

$$f(x) = \sum_n \hat{F}_n e^{2\pi ixn/L}.$$

Substituting $\hat{F}_n$ gives

$$f(x) = \sum_n \frac{1}{L}\hat{f}\left(\frac{n}{L}\right)e^{2\pi ixn/L}.$$

Writing $\xi_n = n/L$ gives

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{1}{L} \hat{f}(\xi_n) e^{2\pi i \xi_n x}.$$

Writing $\xi_n = n/L$ gives

$$f(x) = \sum_{n\in\mathbb{Z}} \frac{1}{L}\hat{f}(\xi_n)e^{2\pi i \xi_n x}.$$

We notice that $1/L = \xi_{n+1} - \xi_n = \Delta\xi$. As $L \to \infty$, this sum tends to the Riemann integral

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i \xi x}\mathrm{d}\xi.$$

Writing $\xi_n = n/L$ gives

$$f(x) = \sum_{n\in\mathbb{Z}} \frac{1}{L}\hat{f}(\xi_n)e^{2\pi i \xi_n x}.$$

We notice that $1/L = \xi_{n+1} - \xi_n = \Delta\xi$. As $L \to \infty$, this sum tends to the Riemann integral

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}\mathrm{d}\xi.$$

The connection we derived between the Fourier coefficients and the Fourier transform tells us also that we can easily approximate the Fourier transform in a discretized setting using some algorithm for calculating the Fourier coefficients.

Writing $\xi_n = n/L$ gives

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{1}{L} \hat{f}(\xi_n) e^{2\pi i \xi_n x}.$$

We notice that $1/L = \xi_{n+1} - \xi_n = \Delta\xi$. As $L \to \infty$, this sum tends to the Riemann integral

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \mathrm{d}\xi.$$

The connection we derived between the Fourier coefficients and the Fourier transform tells us also that we can easily approximate the Fourier transform in a discretized setting using some algorithm for calculating the Fourier coefficients.

Last time we noticed that the time domain in the wave equation was not bounded to any box. This calculation tells us that we can interpret the function $f(t)$ in this continuous basis as a superposition of oscillating solutions where the oscillation frequency is not bounded from below.

In other words, a single oscillation may take infinite amount of time.

No lecture next Tuesday due to a student holiday!