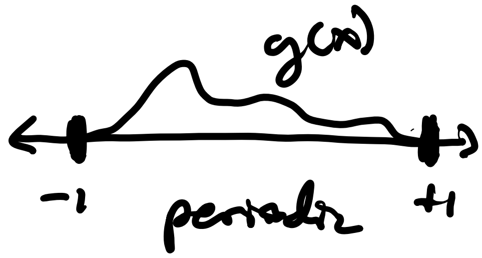
Numerical Stability of FDs

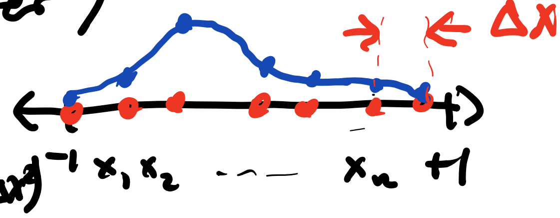
$$\partial_t u = \gamma \partial_x^2 u$$

$\uparrow$ Diffusivity



$u(x,0) = g(x)$

Numerical Approximation (Forward Euler)



$$\partial_x u(x_k, t_j) = \frac{u_{k-1,j} - 2u_{k,j} + u_{k+1,j}}{(\Delta x)^2} + O(\Delta x^2)$$

$u_{k,j} = u(x_k, t_j)$

$\downarrow$

$0 = t_0 < t_1 < t_2 < t_3$

$t_{k+1} - t_k = \Delta t$

$$\underset{\substack{\parallel \\ u_j'}}{\begin{bmatrix} u'_{1,j} \\ u'_{2,j} \\ \vdots \\ u'_{n,j} \end{bmatrix}} = \frac{1}{(\Delta x)^2} \underbrace{\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}}_{D_2} \underset{\substack{\parallel \\ u_j}}{\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix}}$$

$$\partial_t u(x_k, t_j) = \frac{1}{\Delta t}(u_{j+1,k} - u_{j,k}) + O(\Delta t) \xrightarrow[\text{at once}]{\text{all gridpts}} \frac{1}{\Delta t}(u_{j+1} - u_j)$$

$$\frac{1}{\Delta t}(u_{j+1} - u_j) = \gamma \frac{1}{(\Delta x)^2} D_2 u_j$$

$$u_{j+1} = I u_j + \gamma \frac{\Delta t}{(\Delta x)^2} D_2 u_j = \overbrace{\left(I + \gamma \frac{\Delta t}{(\Delta x)^2} D_2\right)}^{A} u_j$$

Stability Analysis (Diagonalizable A)

Accuracy $\Rightarrow$ $\mathcal{O}(\Delta t)$ accuracy in time
$\mathcal{O}((\Delta x)^2)$ accuracy in space

error on order of
Heuristic $\Rightarrow$ $\Delta t \approx (\Delta x)^2$ or $\min(\Delta t, \Delta x^2)$

But actually, sometimes $\|u_j\| \to \infty$ as $j \to \infty$?

$$u_{j+1} = A u_j = A^{j+1} u_0$$

u_0 ← initial condition

$$A = \left(I + \gamma \frac{\Delta t}{(\Delta x)^2} D\right)$$

$$\hat{A} = \underset{\text{eigenvectors}}{V} \overset{\text{eigenvalues}}{\Lambda} V^{-1} \qquad V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\|u_{j+1}\| = \|A^{j+1} u_0\| \le \underbrace{\|A^{j+1}\|}_{\sup_{x \in \mathbb{R}^n} \frac{\|A^{j+1} x\|}{\|x\|}} \underbrace{\|u_0\|}_{\text{doesn't depend on } j}$$

$$\|A^{j+1}\| = \|(V \Lambda V^{-1})^{j+1}\| = \|V \Lambda^{j+1} V^{-1}\|$$

$$(V\Lambda V^{-1})(V\Lambda V^{-1}) \cdots (V\Lambda V^{-1}) = V \Lambda^{j+1} V^{-1}$$

$$\leq \underbrace{\|V\|\,\|V^{-1}\|}_{\text{don't depend on } j} \; \underbrace{\|\Lambda^{j+1}\|}_{\text{all growth or decay}}$$

$$\Lambda^{j+1} = \begin{bmatrix} \lambda_1^{j+1} & & & \\ & \lambda_2^{j+1} & & \\ & & \ddots & \\ & & & \lambda_n^{j+1} \end{bmatrix} \Rightarrow \text{If } |\lambda_k| \leq 1$$

then,

$$\|\Lambda^{j+1}\| \leq 1$$

Eigenvalues of A

$$A = I + \gamma \frac{\Delta t}{(\Delta x)^2} D$$

If $De_k = \alpha_k e_k \Rightarrow Ae_k = \overbrace{\left(1 + \gamma \frac{\Delta t}{(\Delta x)^2} \alpha_k\right)}^{\lambda_k} e_k$

$$D = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & 1 & -2 \end{bmatrix} = \text{"Circulant Matrix"}$$

$$\alpha_k = -2 + 1e^{2\pi i k/n} + 1e^{(2\pi i k/n)(n-1)}$$

$$= -2 + e^{2\pi i k/n} + e^{-2\pi i k/n} \underbrace{e^{2\pi i k}}_{1}$$

$$= -2 + 2\cos(2\pi k/n)$$

$$A \Rightarrow \lambda_k = 1 + \gamma \frac{\Delta t}{(\Delta x)^2} \alpha_k$$

$$= 1 - 2\gamma \frac{\Delta t}{(\Delta x)^2} \underbrace{\left(1 - \cos(2\pi k/n)\right)}_{\text{power reduction}}$$

$$= 1 - 4\gamma \frac{\Delta t}{(\Delta x)^2} \left(\sin^2(\pi k/n)\right)$$

$$-1 \le 1 - 4\gamma \frac{\Delta t}{(\Delta x)^2} \sin^2(\pi k/n) \le 1$$

$$0 \le \gamma \frac{\Delta t}{(\Delta x)^2} \underbrace{\sin^2\left(\frac{\pi k}{n}\right)}_{\le 1} \le \frac{1}{2}$$

$$\boxed{\Delta t \le \frac{(\Delta x)^2}{2\gamma}}$$

Conditionally stable

see notebook on 18.303 course repo

$\Delta x = .02 \qquad \gamma = 0.1$

$$\Delta t \le \frac{(.02)^2}{2(0.1)} = \frac{.0004}{0.2}$$

$$= \underline{.002}$$

$\Delta t = .0004$ ✓

$.004$ ✗ ⇐

$.002$ ✓ ⇐

fd_stability.ipynb for numerical experiments

Discretization $\Longleftrightarrow$? **Operator**

$$\alpha_k = -2\left(1-\cos\left(\frac{2\pi k}{n}\right)\right) = -4\sin^2\left(\frac{\pi k}{n}\right) \qquad -(\pi k)^2$$

$$\frac{\alpha_k}{(\Delta x)^2} = \frac{-4}{\Delta x^2}\sin^2\left(\frac{\pi k}{n}\right)$$

small angle approx

$\sin\theta \approx \theta$, $|\theta| \ll 1 \implies$ when $k \ll n$

$$\approx \frac{-4}{(\Delta x)^2}\left(\frac{\pi k}{n}\right)^2 = \frac{-4}{(\Delta x)^2}\left(\frac{\pi k}{2/\Delta x}\right)^2 = \frac{-4}{(\Delta x)^2}\frac{(\Delta x)^2}{4}(\pi k)^2$$

$n = 2/\Delta x$ $\qquad = -(\pi k)^2$ ✓