Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Second-order transport. Consider the transport equation on a periodic domain, i.e.,

$$\partial_t u = c\,\partial_x u, \qquad \text{such that} \qquad u = \text{periodic on } [-1, 1),$$

with initial condition $u(x, 0) = g(x)$ and wave-speed $c \in \mathbb{R}$.

(a) Write down an explicit time-stepping scheme that uses a first-order forward difference in time and a second-order central difference in space. What is the CFL condition? Use eigenvalue analysis to show the scheme is **unstable** for all real wave speeds.

(b) Derive a second-order accurate time-stepping scheme. **Hint:** Use a second-order Taylor expansion of $\partial_t u$ and use the transport equation to replace the time-derivatives with spatial derivatives. Approximate the spatial derivatives to second order on the grid.

(c) Given a uniform grid spacing, what time-steps make the scheme in part (b) stable?

(d) Implement the scheme in part (b) in Julia and experiment with the Gaussian initial condition $g(x) = 2\exp(-10\cos^2(\pi x))$. Do you observe any numerical diffusion? Compare your observations with the first-order schemes in `FD_stability.ipynb`.

2) Wave-in-a-box. Consider the 2-dimensional wave equation in a box, given by

$$\partial_t^2 u = c^2\,\Delta u, \qquad \text{such that} \qquad u(\pm 1, y, t) = u(x, \pm 1, t) = 0,$$

with initial displacement $u(x, t, 0) = g(x, y)$ and velocity $\partial_t u(x, y, 0) = h(x, y)$.

(a) Discretize the wave-equation on an $n \times n$ spatial grid with second-order centered differences. Use the "vec" and Kronecker operations to derive an $n^2 \times n^2$ matrix D such that $\partial_t^2 \mathbf{u} \approx c^2 D\mathbf{u}$, where $\mathbf{u}$ is an $n^2 \times 1$ vector of values on the computational grid.

(b) Use a second-order centered difference approximation in time to derive an analogue of the time-stepping scheme for the wave equation derived at the end of the Lecture 19 notes. That is, derive a time-stepping scheme of the form $\mathbf{u}_{j+1} = (2I + \sigma^2 D)\mathbf{u}_j - \mathbf{u}_{j-1}$.

(c) Explain how to initialize your scheme with second-order accuracy using the the initial displacement $g(x, y)$ and velocity $h(x, y)$ provided in the problem statement.

(d) Implement the scheme in part (b) in Julia and experiment with the initial displacement $g(x, y) = 4\exp(-15(x^2 + y^2))$ and velocity $h(x, y) = 0$. Compare your numerical solution with the solution to the 1D wave equation in `FD_stability.ipynb`.

3) Green's functions. The displacement of a uniform beam of unit length subject to a load $f(x)$, for $x \in [0, 1]$, is given by a function $u(x)$ which satisfies

$$-u''(x) = f(x), \qquad \text{where} \qquad 0 \le x \le 1.$$

Find the Green's function for the differential operator on the left-hand side when:

(a) The left end is fixed, $u(0) = 0$, and the right end is free, $u'(1) = 0$.

(b) The left end is fixed, $u(0) = 0$, and the right end satisfies $u(1) = 2u'(1)$.

(c) Explain why there is no Green's function when the left end is fixed, $u(0) = 0$, and the right end satisfies $u(1) = u'(1)$.