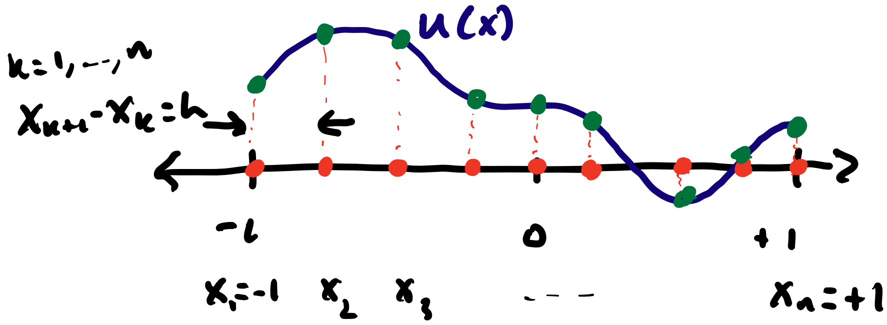
Finite Differences

To solve ODEs/PDEs on the computer, we need to represent functions and their derivatives with a finite set of numbers.



"Forward" Difference Quotient
$$u'(x_k) \approx \frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k} = \frac{u_{k+1} - u_k}{h}$$

any grid — equispaced grid

"Backward" Difference Quotient
$$u'(x_k) \approx \frac{u(x_k) - u(x_{k-1})}{x_k - x_{k-1}} = \frac{u_k - u_{k-1}}{h}$$

"Central Difference" Quotient
$$u'(x_k) \approx \frac{u(x_{k+1}) - u(x_{k-1})}{x_{k+1} - x_{k-1}} = \frac{1}{2}\,\frac{u_{k+1} - u_{k-1}}{h}$$

Error estimates (what does "$\approx$" mean?)

If $u(x)$ is differentiable at x_k, then

Taylor's Thm

$$u(x_{k+1}) = u(x_k) + u'(x_k)\overset{h = x_{k+1} - x_k}{h} + g(h)h$$

for some function with $\lim_{h \to 0} g(h) = 0$

$$\Rightarrow \quad \frac{u(x_{k+1}) - u(x_k)}{h} = u'(x_k) + \underbrace{g(h)}_{\to 0 \text{ as } h \to 0}$$

To estimate error explicitly, we need to understand how big $g(x)$ is for fixed $h > 0$.

Taylor's Remainder

If $u''(x)$ is continuous in $[x_k, x_{k+1}]$,

$$g(h) = \frac{1}{2} u''(y_h)h \quad \text{for some} \quad x_k \leq y_h \leq x_{k+1}.$$

Since $u''(y)$ is continuous on $[x_k, x_{k+1}]$, it is bounded there, and $g(x)$ must decrease at least proportionally to h as $h \to 0$.

Therefore, forward difference error is

$$\left| \frac{u(x_{k+1}) - u(x_k)}{h} - u'(x_k) \right| \le \frac{1}{2} \max_{x_k \le y \le x_{k+1}} |u''(y)|\, h$$

at worst, proportional to h.

We call this a "first-order" finite difference.

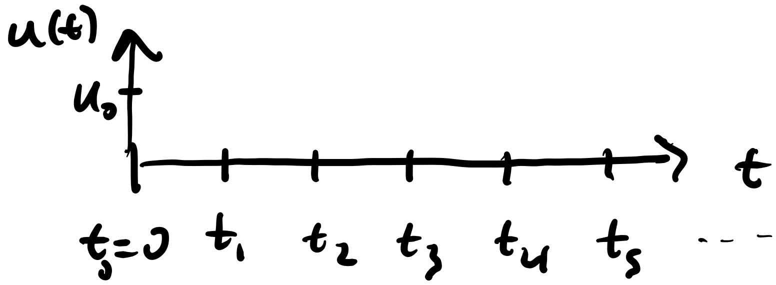
HW Design a 2^{nd} order finite difference approx for the first derivative using samples $u(x_{k-1})$, $u(x_k)$, $u(x_{k+1})$.

Computing w/ finite differences

IDEA 1 (Time-Stepping)

initial condition

Solve $\quad \dfrac{du}{dt} = f(t,u) \quad$ s.t. $\quad u(0) = u_0$



step-size
$t_{k+1} - t_k = h$

$$\frac{u_1 - u_0}{h} = f(t, u_0)$$

forward diff

Step 1 $\qquad u_1 = u_0 + h f(t, u_0)$

Step 2 $\qquad u_2 = u_1 + h f(t, u_1)$

$$\vdots$$

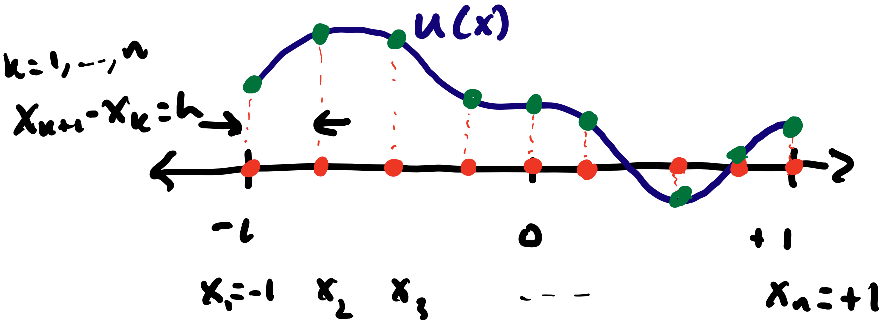
$\Rightarrow$ At each step, an error of order h is introduced. Be careful how these accumulate!

Local Truncation Error

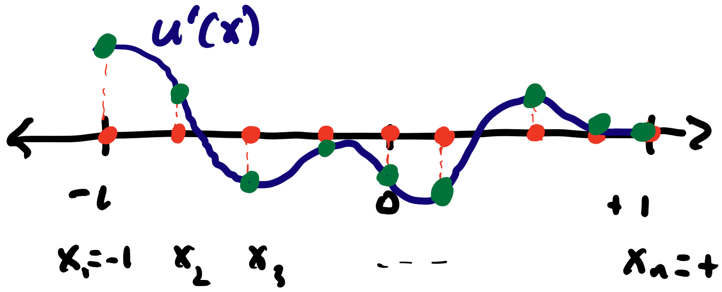
$\Rightarrow$ This simple approach is called **Forward Euler**.

$\Rightarrow$ Higher order methods use careful estimates of intermediate values of $u(t)$ from differences with smaller step-size and combines them to acheive smaller local truncation error.

Idea 2 (Spatial Discretization)



$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$



$$\underline{u}' = \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix}$$

$$\left.\frac{du}{dx}\right|_{x_k} \approx \frac{u_{k+1} - u_{k-1}}{2h}$$

$2 \leq k \leq n-1$

Central Difference

Forward

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

Backward

$\Rightarrow$ On grid w/FD Approx, differentiation (a linear operation) is represented by a matrix

for PDEs with mix of time and space variables, common to use time-stepping for time variable and spatial discretization for space variables.

$\Rightarrow$ Higher order FD Approximations lead to matrices with larger "bandwidth."