Finite differences in higher dimensions

Laplace and Poisson equations

18.303 Linear Partial Differential Equations: Analysis and Numerics

1d Poisson equation

Poisson's equation with Dirichlet boundaries

$$\begin{aligned}\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} &= f(x),\\ u(0) =& u_0,\ u(L) = u_{N+1}.\end{aligned}$$

Here $x \in (0, L)$. We discretize the space x as before: $x_k = k\Delta x$ and $u_k = u(x_k)$, where $k = 1, 2, ..., N$ and $\Delta x = L/(N+1)$.

We define the Dirichlet Laplacian

$$D^{(x)} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

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while the last equation gives

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We can move the known constants to the RHS of the equation. Now we get

$$D^{(x)}\mathbf{u} = \mathbf{f}_\mathrm{B},$$

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The first entry of $\mathbf{b} = u_0/\Delta x^2$ and the last one is $u_{N+1}/\Delta x^2$. Apart from that, it's zero.

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We can simply set $\mathbf{f} = 0$ and since $D^{(x)}$ is invertible, the equation can be solved.

Two dimensions

Now we have

$$\Delta u(x,y) = f(x,y),$$
$$u(x,0) = u^{(\mathrm{b})}(x), u(x,L_y) = u^{(\mathrm{t})}(x),$$
$$u(0,y) = u^{(\mathrm{l})}(y), u(L_x,y) = u^{(\mathrm{r})}(y).$$

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We discretize the space: $x_i = x(i\Delta x)$, $y_j = y(j\Delta y)$, $u_{i,j} = u(x_i, y_j)$ and so on. Here $i = 1, 2, ..., N$ and $j = 1, 2, 3, ..., M$. $\Delta x = L_x/(N+1)$ and $\Delta y = L_y/(M+1)$.

We can write the Laplace equation in indices as

$$\frac{1}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1}{\Delta y^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = f_{i,j}.$$

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We solve them in a similar way. The boundary vector is in indices

$$\frac{1}{\Delta x^2}(\delta_i^1 u_{0,j} + \delta_i^N u_{N+1,j}) + \frac{1}{\Delta y^2}(\delta_j^1 u_{i,0} + \delta_j^M u_{i,N+1}) = \frac{1}{\Delta x^2}(\delta_i^1 u_j^{(\mathrm{l})} + \delta_i^N u_j^{(\mathrm{r})}) + \frac{1}{\Delta y^2}(\delta_j^1 u_i^{(\mathrm{b})} + \delta_j^M u_i^{(\mathrm{t})})$$

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This will give a boundary matrix $b_{i,j}$ that has the discretized boundary function values on the boundary. Now the source for the equation $f^{(\mathrm{B})} = f - b$.

We can regroup the LHS of the equation

$$\frac{1}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) - \frac{2u_{i,j}}{\Delta y^2} + \frac{1}{\Delta y^2}(u_{i,j-1} + u_{i,j+1}) = f^{(B)}_{i,j}.$$

We notice that we have four terms for the column j on the left and two other terms for columns $j-1$ and $j+1$. Which linear operator gives the term in the first parentheses?

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Let us denote the jth column of u as $\mathbf{u}_j$ and the same for $f^{(\mathrm{B})}$. Now we have

$$D^{(x)}\mathbf{u}_j - \frac{2}{\Delta y^2}\mathbf{u}_j + \frac{1}{\Delta y^2}(\mathbf{u}_{j-1} + \mathbf{u}_{j+1}) = \mathbf{f}^{(\mathrm{B})}_j.$$

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Let us define $I_y = I/\Delta y^2$. Now we have

$$I_y\mathbf{u}_{j-1} + \underbrace{\left(D^{(x)} - 2I_y\right)}_{=B}\mathbf{u}_j + I_y\mathbf{u}_{j+1} = \mathbf{f}^{(\mathrm{B})}_j.$$

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We can make a matrix of matrices A called a block matrix. The equation will look like this

$$AU = F^{(\mathrm{B})},$$

where

$$A = \begin{pmatrix} B & I_y & & & \\ I_y & B & I_y & & \\ & \ddots & \ddots & \ddots & \\ & & I_y & B & I_y \\ & & & I_y & B \end{pmatrix}$$

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Note that this has the Dirichlet conditions in it. How many Bs are there in A? The dimension of y discretization i.e. M. A is called a block tridiagonal matrix.

$$AU = F^{(\mathrm{B})}.$$

How should we interpret U and $F^{(\mathrm{B})}$? If we write A by just filling in the matrices inside it, it will give A the dimensions $NM \times NM$. We can then express U as

$$U = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{N,1} \\ u_{1,2} \\ \vdots \\ u_{N,M} \end{pmatrix}.$$

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The vector $F^{(\mathrm{B})}$ is flattened in the same way.

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- Moreover, it can be hard to express u in any known eigenbasis making it hard to use spectral methods.
- Finite difference methods are easy to implement for simple geometries.
- The discrete linear equations can be solved efficiently using existing packages.
- Can be extended to non-uniform meshes but that requires quite a bit of effort.