First-Order Linear PDEs

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)$$

partial differential eqn.

$\Rightarrow$ Only first derivatives of $u(x,t)$ appear

$\Rightarrow$ Only linear operations are applied to $u(x,t)$

E.g. $$\cos t \frac{\partial u}{\partial t} + e^x \frac{\partial u}{\partial x} = \sin t$$

Is this a first-order, linear PDE? (yes!)

Solution via Method-of-Characteristics

IDEA: Instead of a PDE, solve a collection of simpler ordinary differential eqn's (ODEs).

$$\frac{dx}{ds} = a(x,y) \qquad \frac{dy}{ds} = b(x,y) \qquad \frac{du}{ds} = c(x,y)$$

"Characteristic" Curves

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$c(x,y) = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \qquad \checkmark$$

Along characteristic curves $(x(s), y(s))$, the soln. $u(x(s), y(s))$ satisfies ODE $\frac{du}{ds} = c(x(s), y(s))$.

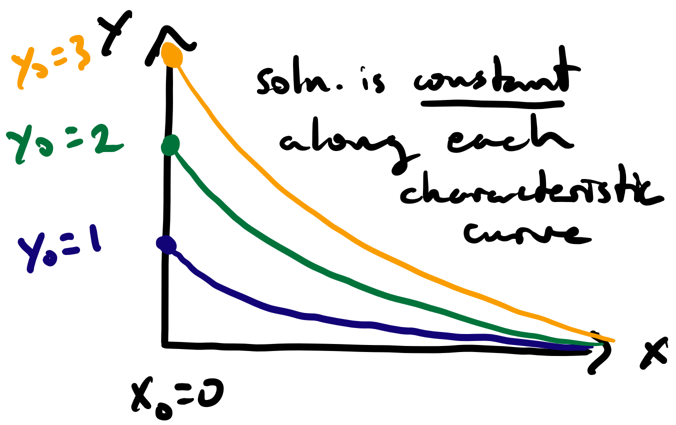
<u>Example</u>

$$\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

char. curves $\Rightarrow$ $\frac{dx}{ds} = 1$, $\frac{dy}{ds} = -y$, $\frac{du}{ds} = 0$

$$x(s) = s + x_0, \quad y(s) = y_0 e^{-s}, \quad u(s) = u_0$$

$$u(s+x_0, y_0 e^{-s}) = u(x_0, y_0)$$

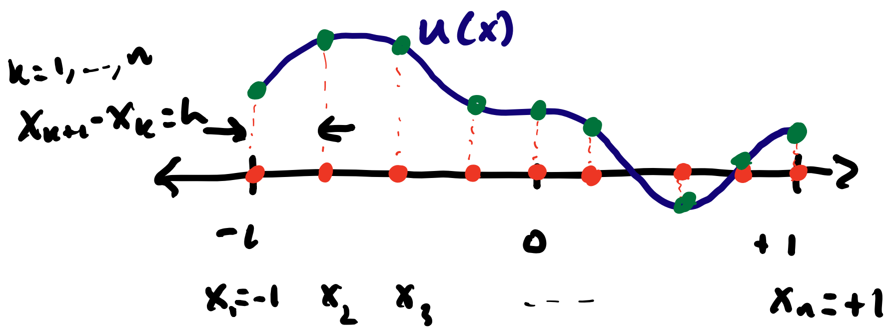


For <u>unique</u> solution, specify data $u(x_0, y_0)$ for <u>one pt</u> on each characteristic curve.

In general, we can't solve the ODEs analytically every time. Instead, we can solve them numerically.

Finite Differences

To solve ODEs/PDEs on the computer, we need to represent functions and their derivatives with a finite set of numbers.



Idea 1 "Sample" function values on a discrete grid with spacing h.

$$x_1, \ldots, x_n \implies u_1 = u(x_1), \ldots u_n = u(x_n)$$

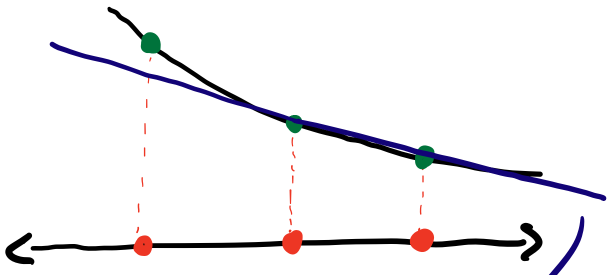
sample points — function samples

"Forward Difference Quotient"

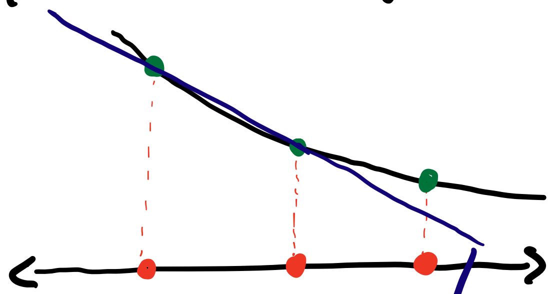
$$u'(x_k) \approx \underbrace{\frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k}}_{\text{any grid}} = \underbrace{\frac{u_{k+1} - u_k}{h}}_{\text{equispaced grid}}$$

"Backward" Difference Quotient

$$u'(x_k) \approx \frac{u(x_k) - u(x_{k-1})}{x_k - x_{k-1}} = \frac{u_k - u_{k-1}}{h}$$



Forward diff. approximates $u'(x_k)$ by slope of secant between x_k and x_{k+1}



Backward diff. approximates $u'(x_k)$ by slope of secant between x_{k-1} and x_k

Both converge to $u'(x_k)$ = slope of <u>tangent</u> line at x_k, provided that $u(x)$ has a well-defined tangent at x_k, i.e., is differentiable.