Spectral methods and the weak formulation

Working with bases of functions

18.303 Linear Partial Differential Equations: Analysis and Numerics

Massachusetts
Institute of
Technology

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- This amounts to high efficiency since the derivatives are evaluated accurately even with a small number of modes (the basis needs to be truncated for computations)

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- Hermite polynomials for $x \in \mathbb{R}$

Weak formulation

This far we have considered differential equations

$$\mathcal{L}u(\mathsf{x}) = f(\mathsf{x}),$$

on some domain $x \in \Omega$ with boundary conditions

$$\mathcal{G}u = b(\mathsf{x})$$

on the boundary $x \in \partial\Omega$. We say that the functions fulfilling the boundary condition are in some vector space V.

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The functions φ are test functions and it suffices that $V' = C_0^\infty$ i.e. the space of smooth functions (infinitely differentiable) whose all derivatives and values are 0 at the boundary.

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Let's assume for the time being that we have some basis functions $\{\phi_i\}$ that satisfy the boundary conditions. We can approximate both the test functions and the u in this basis writing

$$u(\mathrm{x}) = \sum_{j=0}^{N-1} \hat{u}_j \phi_j(\mathrm{x}),\ \varphi(\mathrm{x}) = \sum_{i=0}^{N-1} \hat{\varphi}_i \phi_i(\mathrm{x}).$$

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The weak form becomes

$$\sum_{i,j=0}^{N-1} \hat{\varphi}_i \hat{u}_j \langle \phi_i, \mathcal{L}\phi_j \rangle = \sum_{i=0}^{N-1} \hat{\varphi}_i \langle \phi_i, f \rangle$$

for all $\hat{\varphi}_i$. The only way the equation holds is if

$$\sum_{j=0}^{N-1} \hat{u}_j \langle \phi_i, \mathcal{L}\phi_j \rangle = \langle \phi_i, f \rangle$$

for all i. We assume here that $\mathcal{L} : V_N \to V_N$ when restricted to just N modes.

We write again $L = \langle \phi_i, \mathcal{L}\phi_j \rangle$. This becomes a matrix equation

$$L\hat{\mathbf{u}} = \hat{\mathbf{f}},$$

where L is some square matrix with dimensions $N \times N$.

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This is called the Galerkin method for solving a linear PDE. The error of the solution $\epsilon = u - u_N$ satisfies

$$\langle \phi_k, \mathcal{L}\epsilon \rangle = \langle \phi_k, \mathcal{L}u \rangle - \langle \phi_k, \mathcal{L}u_N \rangle = \langle \phi_k, f \rangle - \langle \phi_k, f_N \rangle = 0.$$

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This means that the error lives in the orthogonal space so solving in the truncated space V_N is the best projection of the solution onto $V_N \subset V$.

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This is better illustrated with an example.

Example

Let the differential operator be $\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the interval $x \in (-1, 1)$ and assume we have $u(-1) = u_-$ and $u(1) = u_+$. Let our basis functions be some polynomials for which the degree of ϕ_n is n.

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It turns out the Null space of L has dimension 2. From the boundary conditions we have

$$u_- = \sum_{j=0}^{N-1} \hat{u}_j \phi(-1), \; u_+ = \sum_{j=0}^{N-1} \hat{u}_j \phi(1).$$

Adding these equations solves the coefficients uniquely.