Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

**1) Fourier's basis.** In the Fourier basis, a 2-periodic function $f(x)$ on $[-1, 1)$ is written as

$$f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\pi kx}, \qquad \text{where} \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi kx} f(x)\, dx.$$

**(a)** Compute the Fourier coordinates of $f(x) = \sin^3(\pi x)$, $g(x) = |x|$, and $h(x) = |\sin(\pi x)|^3$. Plot the magnitude of the Fourier coefficients $-250 \le k \le 250$ on a logarithmic scale. Based on the coefficient plots, roughly what accuracy do you expect if you approximate $g$ and $h$ by truncating their Fourier series, discarding terms with $|k| > 250$?

**(b)** Show that if $f$ is $n$-times continuously differentiable with $|f^{(n)}(x)| \le M$ on the periodic interval $[-1, 1)$, then $|\hat{f}_k| \le \sqrt{2}M/(\pi k)^n$. (**Hint:** integrate by parts.) If $f(x)$ is approximated by the truncated series $f_N(x) = \sum_{k=-N}^{N} \hat{f}_k e^{i\pi kx}$, how do you expect the approximation error $E_N = \max_{-1\le x\le 1} |f(x) - f_N(x)|$ to scale as $N$ is increased?

**(c)** If $a(x) = \sin^3(\pi x)$ and $f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\pi kx}$, then what are the Fourier coefficients of $a(x)f(x)$? Write down the (infinite) matrix representing "multiplication-by-$a(x)$" in the Fourier basis. How many nonzero entries are there in each row?

**2) Finite differences in 2D.** Consider Poisson's equation on the unit square:

$$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = f(x, y), \qquad \text{where} \qquad u(\pm 1, y) = 1 - y^2, \quad \text{and} \quad u(x, \pm 1) = 1.$$

The `poissonFD.ipynb` notebook accompanying Lecture 8 may be helpful in parts (a)-(d).

**(a)** Using centered second-order finite differences in $x$ and $y$ on an $N \times N$ grid, discretize the PDE (without boundary conditions) to obtain a matrix equation $D_2U + UD_2 = F$.

**(b)** Modify the right-hand side, $F$, of the matrix equation in part (a) to enforce the non-homogeneous boundary conditions $u(\pm 1, y) = 1 - y^2$ and $u(x, \pm 1) = 1$.

**(c)** Use the Kronecker product to rewrite the matrix equation from (a) and (b) in the standard form $Ax = b$, where $A$ is an $N^2 \times N^2$ matrix and $b$ is an $N^2 \times 1$ vector.

**(d)** Using the Gaussian right-hand side $f(x, y) = 5\exp(-10(x^2 + y^2))$, solve the discretized linear system in part (c) numerically and plot the solution on the $N \times N$ grid. Try increasing the value of $N$ until the numerical solution appears to converge. Should the solution satisfy a maximum or minimum principle? Explain your reasoning.

**3) Separation of variables.** Consider the exterior Laplace problem in polar coordinates,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] u(r, \theta) = 0, \qquad \text{where} \qquad r \ge 1 \quad \text{and} \quad u(1, \theta) = |\sin(\theta)|^3.$$

Use separation of variables in polar coordinates to find a *bounded* solution, $|u(r, \theta)| \le M$. Is your solution unique? Explain why or why not. If not, provide the general solution form.