The Wave Eqn

$$\partial_t^2 u = c^2 \partial_x^2 u$$

$c>0$, "wave speed"

$u(+1,t) = u(-1,t)$

$\partial_x u(+1,t) = \partial_x u(-1,t)$

"Periodic B.C.'s"

Initial displacement $\Rightarrow$ $u(x,0) = g(x)$

& velocity $\Rightarrow$ $\partial_t u(x,0) = h(x)$

Since ∂_t^2 appears on the left, we cannot apply the operator exponential directly. But, we can still diagonalize $\partial_x^2 u$ to decouple & simplify solution.

$\partial_x^2 e_k = \lambda_k e_k$ w/periodic B.C.'s $\Rightarrow$ $e_k(x) = \frac{1}{\sqrt{2}} e^{i\pi k x}$, $\lambda_k = -(\pi k)^2$

<u>Expand in basis $\{e_k\}$:</u> $u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e_k(x)$

<u>Diagonalize wave eqn:</u>

$$\partial_t^2 u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{u}_k''(t) e_k(x), \qquad \partial_x^2 u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \lambda_k \hat{u}_k(t) e_k(x)$$

$$\partial_t^2 u = c^2 \partial_x^2 u \quad \underset{\text{coordinates}}{\overset{\text{eigenvector}}{\Rightarrow}} \quad \langle e_k, \partial_t^2 u \rangle = c^2 \langle e_k, \partial_x^2 u \rangle$$

$k = 0, \pm 1, \pm 2, \ldots$ decoupled ODEs $\Rightarrow \hat{u}_k''(t) = c^2 \lambda_k \hat{u}_k(t)$

$$\hat{u}_k''(t) = -(c\pi k)^2 \hat{u}_k(t)$$

two linear indep. solutions $\Rightarrow$ $\hat{u}_k(t) = \underbrace{c_1^{(k)}}_{\text{constants}} e^{i(c\pi k)t} + \underbrace{c_2^{(k)}}_{\text{constants}} e^{-i(c\pi k)t}$

$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \left[c_1^{(k)} e^{i(c\pi k)t} + c_2^{(k)} e^{-i(c\pi k)t} \right] e_k(x)$$

How to choose constants $c_1^{(k)}$ & $c_2^{(k)}$?

Initial conditions: $u(x,0) = g(x)$, $u'(x,0) = h(x)$

$$\underbrace{\hat{g}_k}_{\text{Fourier coeff of } g} = \langle e_k, g \rangle = \langle e_k, u(x,0) \rangle = \hat{u}_k(0) = c_1^{(k)} + c_2^{(k)}$$

$$\underbrace{\hat{h}_k}_{\text{Fourier coeff of } h} = \langle e_k, h \rangle = \langle e_k, \partial_t u(x,0) \rangle = \hat{u}_k'(0) = i c_1^{(k)}(c\pi k) - i c_2^{(k)}(c\pi k)$$

$$\begin{bmatrix} 1 & 1 \\ ic\pi k & -ic\pi k \end{bmatrix} \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \end{bmatrix} = \begin{bmatrix} \hat{g}_k \\ \hat{h}_k \end{bmatrix}, \quad \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \end{bmatrix} = \frac{1}{-2ic\pi k} \begin{bmatrix} -ic\pi k & -1 \\ -ic\pi k & 1 \end{bmatrix} \begin{bmatrix} \hat{g}_k \\ \hat{h}_k \end{bmatrix}$$

$$c_1^{(k)} = \frac{1}{2}\hat{g}_k + \frac{1}{2icnk}\hat{h}_k$$

$$c_2^{(k)} = \frac{1}{2}\hat{g}_k - \frac{1}{2icnk}\hat{h}_k$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \Big[\Big(\frac{1}{2}\hat{g}_k + \frac{1}{2icnk}\hat{h}_k\Big) e^{icnkt}$$

$$+ \Big(\frac{1}{2}\hat{g}_k - \frac{1}{2icnk}\hat{h}_k\Big) e^{-icnkt} \Big] e_k(x)$$

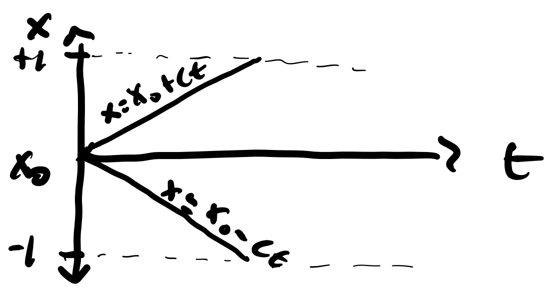
$$= \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \Big[\hat{g}_k \cos(cnkt) + \frac{\hat{h}_k}{cnk} \sin(cnkt) \Big] e^{inkx}$$

Full soln Wave Equation on periodic $[-1,1]$.

__Characteristics__

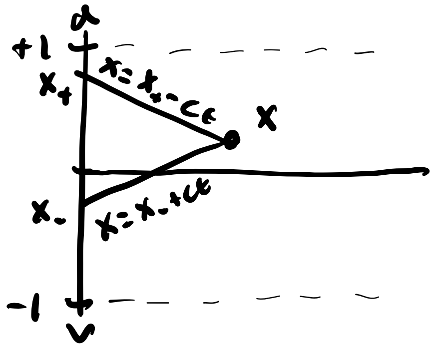
$$u(x,t) = \underbrace{\frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_1^{(k)} e^{ink(ct+x)}}_{\substack{\text{const. along} \\ ct+x=\text{const.} \\ F_1(x,t)=F_1(x+ct)}} + \underbrace{\frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} c_2^{(k)} e^{ink(-ct+x)}}_{\substack{\text{const along} \\ -ct+x=\text{const.} \\ F_2(x,t)=F_2(x-ct)}}$$

Wave Equation has two sets of characteristics.



Information const. carried along soln. F_1 and F_2 at speed $c > 0$

Solution $u(x,t) = F_1(x+ct) + F_2(x-ct)$



Solution decouples into contributions from two sets of characteristics.