

A bit of vector calculus

Some integration rules

18.303 Linear Partial Differential Equations: Analysis and Numerics

Divergence and gradient theorems

For $g: \mathbb{R}^N \to \mathbb{R}$ we have

$$\int_{\gamma} \nabla g(\mathbf{x}) \cdot d\mathbf{x} = g(\mathbf{x}_{e}) - g(\mathbf{x}_{s}).$$

Here γ is a differentiable path embedded in \mathbb{R}^N and $g(\mathbf{x}_s)$ and $g(\mathbf{x}_e)$ are the start and endpoints of the path. This is called the gradient theorem or the fundamental theorem of calculus for line integrals.

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For a compact $\Omega \subset \mathbb{R}^N$ and a function $\mathbf{f}: \mathbb{R}^N \to \mathbb{R}^N$ we have

$$\int_{\Omega} \nabla \cdot f(x) \mathrm{d}V = \int_{\partial \Omega} f(x) \cdot \mathrm{d}S = \int_{\partial \Omega} f(x) \cdot \hat{n}(x) \mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is the unit normal of a given point on the boundary of the region Ω . We assume here that the boundary is piecewise smooth. This is called the divergence theorem or Gauss's theorem.

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Stokes' theorem

Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a sufficiently smooth vector field. Let Σ be a simply connected smooth two-dimensional subset of \mathbb{R}^3 with a piecewise smooth boundary.

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We have

$$\int_{\Sigma} \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_{\partial \Sigma} \mathbf{A} \cdot d\mathbf{x}.$$

Here the integral is contracted against the the tangent $\boldsymbol{.}$

Green's first identity

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$$\int_{\Omega} \psi \Delta \varphi + \nabla \psi \cdot \nabla \varphi dV = \int_{\partial \Omega} \psi \nabla \varphi \cdot dS.$$

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This can be also written as

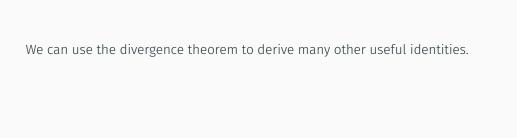
$$\int_{\Omega} \nabla \psi \cdot \nabla \varphi dV = \int_{\partial \Omega} \psi \nabla \varphi \cdot d\mathbf{S} - \int_{\Omega} \psi \Delta \varphi dV,$$

so it becomes a rule for integration by parts for vector calculus.

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Exercise

Derive Green's first identity using the divergence theorem. Hint, choose $\mathbf{f}=\psi\nabla\varphi$.



We can use the divergence theorem to derive many other useful identities.

E.g. choosing $\mathbf{f} = \psi \mathbf{I}$, where \mathbf{I} is the identity matrix gives

$$\int_{\Omega} \nabla \cdot (\psi \mathbf{I}) dV = \int_{\Omega} \nabla \psi dV = \int_{\partial \Omega} \psi \mathbf{I} \cdot d\mathbf{S} = \int_{\partial \Omega} \psi d\mathbf{S}.$$