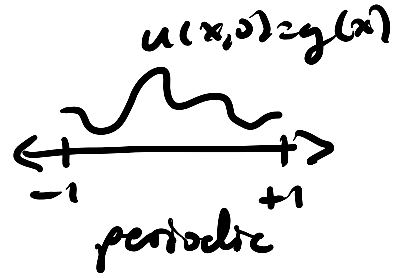
Numerical Stability of FDs

$$\partial_t u = \gamma \partial_x^2 u$$



Forward Euler (Explicit Scheme)

$$u_{j+1} = (I + \mu D_2) u_j \quad \text{where} \quad D = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

$n \times n$ matrix

$$\text{and} \quad \mu = \gamma \Delta t / (\Delta x)^2$$

Stable ($\|u_j\| \le M < \infty$ for all $j \ge 1$) if $0 \le \mu \le 1/2$

$$\Delta t \le \frac{(\Delta x)^2}{2\gamma}$$

Backward Euler

Idea, use backward difference in time:

now at step $j+1$

$$\frac{1}{\Delta t}(u_{j+1} - u_j) = \frac{\gamma}{(\Delta x)^2} D_2 u_{j+1}$$

$$u_{j+1} = (I - \mu D_2)^{-1} u_j$$

In practice, solve $(I-\mu D_2)u_{j+1} = u_j$ for $j=0,1,2,\ldots$

Stability Analysis

Just like last time, we have $u_{j+1} = A^{j+1}u_0$,

but now $A = (I-\mu D_2)^{-1}$.

$\Rightarrow$ For stability, need $|\underbrace{\lambda_k}_{\text{eigenvalues of } A}| \le 1$ (Lecture 17).

If $D_2 e_k = \alpha_k e_k$, then

$$(I-\mu D_2)e_k = (1-\mu\alpha_k)e_k \overset{\text{if } \mu\alpha_k \neq 1}{\Rightarrow} (I-\mu D_2)^{-1}e_k = (1-\mu\alpha_k)^{-1}e_k$$

$$\Rightarrow \lambda_k = \frac{1}{1-\mu\alpha_k}$$

From Lecture 17, we know $\alpha_k = -4\sin^2\left(\frac{\pi k}{n}\right)$

$$\Rightarrow \lambda_k = \frac{1}{1+\underbrace{4\mu\sin^2\left(\frac{\pi k}{n}\right)}_{\ge 0}} \le 1$$

Since $0 \le \lambda_k \le 1$ regardless of $\mu = \gamma \frac{\Delta t}{(\Delta x)^2}$,

the Back. Euler Scheme is unconditionally stable.

Backward Euler is an implicit time-stepping scheme.

$\Rightarrow$ Requires solving a linear system, which in general may be slower than matrix-vector multiplication.

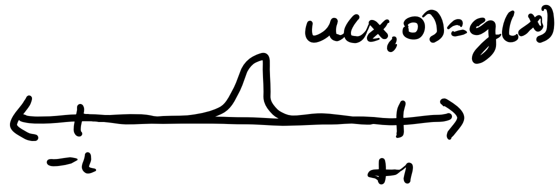
$\Rightarrow$ Can take (much) larger time-steps w/out numerical instability.

For heat eqn., $(I - \alpha D_2) u_{j+1} = u_j$ can be solved nearly as fast as a forward Euler step because $I - \alpha D_2$ has a special sparse structure.

Transport Equation

We can use the same techniques to analyze FD stability for Transport PDEs, when we are on a periodic domain.

$$\partial_t u = \overset{\text{wave speed}}{c}\, \partial_x u$$



FD Approximation

$$\partial_x \begin{bmatrix} u(x_1,t_j) \\ \vdots \\ u(x_n,t_j) \end{bmatrix} \approx \frac{1}{\Delta x} \underset{n\times n}{\overset{D_1}{\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & & -1 \end{bmatrix}}} \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix}$$

First-order forward diff in space

$$\partial_t u\Big|_{t_j} \approx \frac{u|_{t_{j+1}} - u|_{t_j}}{\Delta t}$$

first-order forward diff in time.

time step $\Rightarrow$

$$u_{j+1} = \Big(I + \overbrace{c\,\frac{\Delta t}{\Delta x}}^{\sigma} D_1\Big) u_j$$

Stability

Need eigenvalues of $A = I + \sigma D_1$ to have

$$|\lambda_k| \leq 1$$

If $D_1 e_k = \alpha_k e_k$, then $A e_k = (\overbrace{1+\sigma\alpha_k}^{\lambda_k}) e_k$, so we really want eigenvalues of D_1.

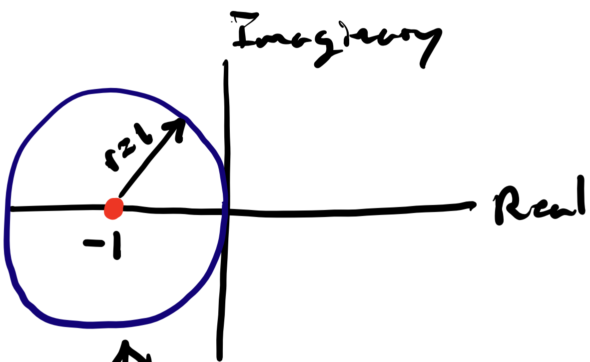
$$D_1 = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & -1 \end{bmatrix} = \text{"Circulant matrix" again!}$$

$\Rightarrow$ Complete orthogonal set of eigenvectors $V^{-1} = V^*$ and $\|V\|\|V^{-1}\| = 1$.

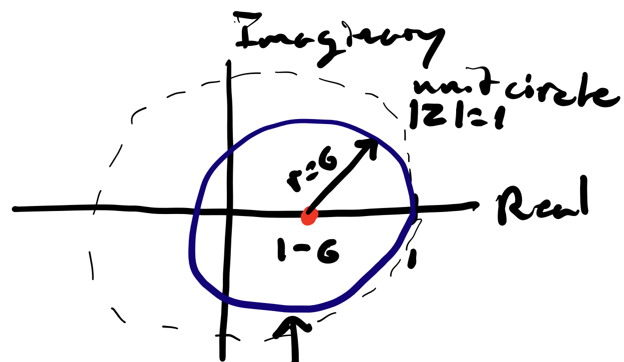
eigenvalues known

$$\alpha_k = -1 + e^{2\pi i k/n} \quad \Rightarrow \quad \lambda_k = 1 - \sigma(1 - e^{2\pi i k/n})$$

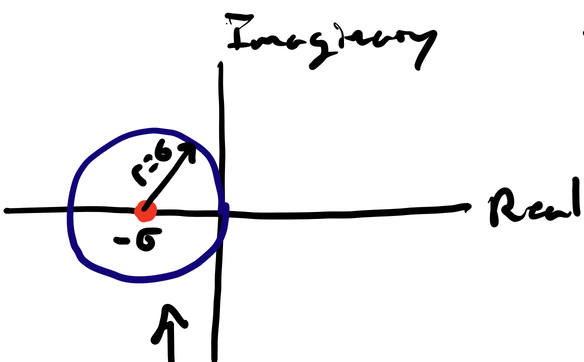
Since these eigenvalues are complex, let's plot:



eigs of D_1 live on this circle in complex plane



eigs of $I + \sigma D_1$ live here



eigs of σD_1 live here

The blue circle on the right always intersects the real point $z=1$ and as long as $0<\sigma\leq 1$, the blue circle never leaves the unit circle. So for stability, we need to have

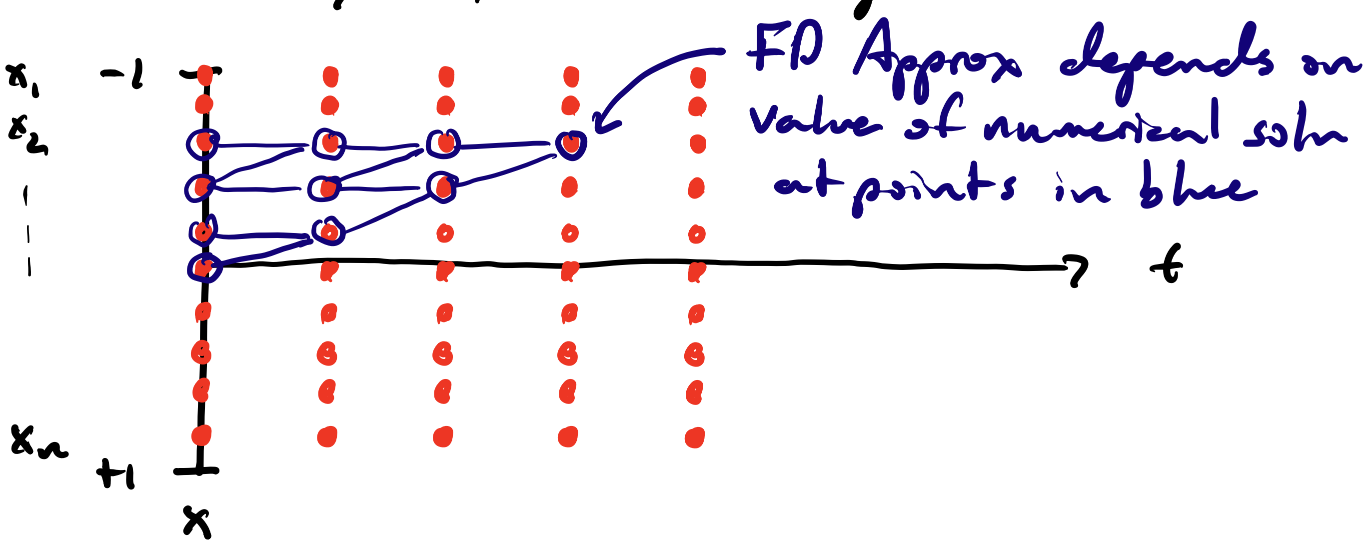
$$0 < c\frac{\Delta t}{\Delta x} \leq 1 \qquad\qquad \Delta t \leq \frac{\Delta x}{c}$$

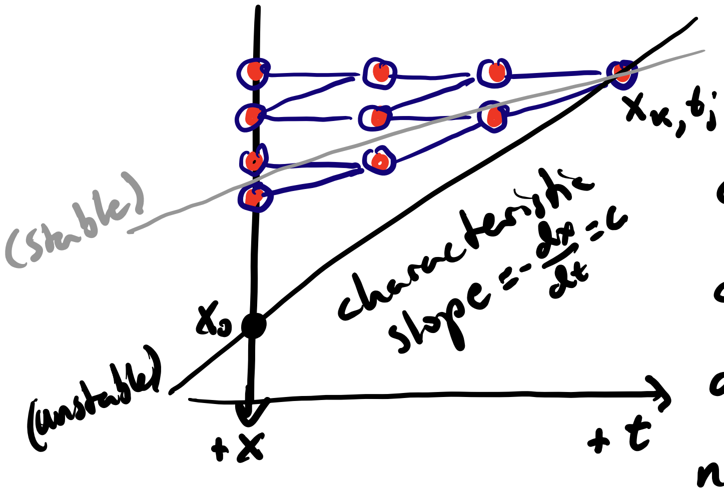
In particular, notice that we need $c>0$!

$\Rightarrow$ For right-moving transport ($c<0$), the scheme is unstable. Would need to use a backward FD approx in space instead.

CFL Condition

We can get another vantage point on instability by considering characteristics.





If we change initial condition ONLY in a small area around x_0, true soln. at (x_k, t_j) changes but numerical solution doesn't change at all! So numerical solution can be arbitrarily bad $\Rightarrow$ unstable

In particular, for right-moving characteristics $(c<0)$, same argument applies $\Rightarrow$ unstable

CFL Condition $\Rightarrow$ Characteristic through (x_k, t_j) must pass between x_k and x_{k-1} at time t_j (for forward diff. approx in space).

$$\Delta t, \Delta x > 0 \text{ and } \left|\frac{\Delta x}{\Delta t}\right| \geq c \Rightarrow 0 < \frac{\Delta t}{\Delta x} \leq \frac{1}{c}$$

which is precisely our earlier restriction.

Numerical Diffusion

Note that when $\sigma = c\frac{\Delta t}{\Delta x} = 1$, the eigenvals of $A = I + \sigma D_1$ live <u>on</u> the unit circle. Since A also has a full set of eigenvectors, it is a unitary matrix: $\|Au\| = \|u\|$ for any vector u. Just like the true PDE solution,

$$\|u_{j+1}\| = \|A^{j+1}u_0\| = \|u_0\|$$

The norm is conserved at every time step.

However, when $\sigma = c\frac{\Delta t}{\Delta x} < 1$, the eigenvals of A have $|\lambda_k| < 1$ (except $\lambda_n = 1$) and so $\|u_{j+1}\|$ typically decreases as $j \to \infty$.

This phenomenon is called numerical or or artificial diffusion, because it mimics the behavior of a diffusion term in the PDE (although there <u>is no</u> such term in our model).