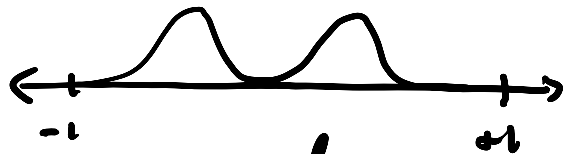
# The Wave Eqn (pt. 3)

$$\partial_t^2 u = c^2 \partial_x^2 u$$

$c$ "wave speed" $> 0$




periodic $[-1,+1)$

with initial conditions:

displacement $\Rightarrow$ $u(x,0) = g(x)$

velocity $\Rightarrow$ $\partial_t u(x,0) = h(x)$

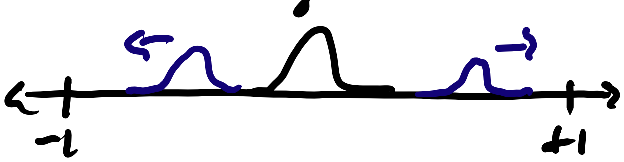
## Fourier Solution

$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} \Big[ \underbrace{\hat{g}_k}_{\langle e_k, g\rangle} \cos(\overbrace{c\pi k t}^{i\sqrt{\partial_{xx}}\, t}) + \underbrace{\frac{\hat{h}_k}{c\pi k}}_{\frac{\langle e_k, h\rangle}{i\sqrt{\partial_{xx}}}} \sin(\overbrace{c\pi k t}^{i\sqrt{\partial_{xx}}\, t}) \Big] \overbrace{e^{i\pi k x}}^{e_k(x)}$$

## D'Alembert's Solution

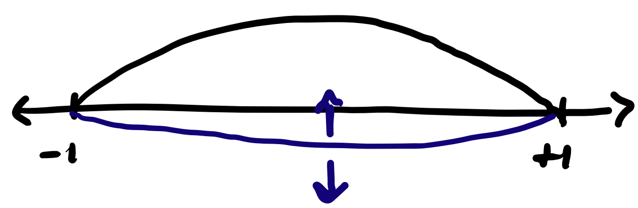
$$u(x,t) = \frac{1}{2}\left[ g(x+ct) + g(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} h(x')\,dx' \right]$$

### Traveling Waves




"Wave packets"

### Standing Waves



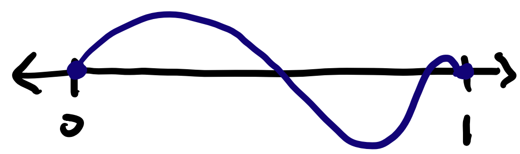

"vibrating string"

# Boundary Conditions

$$\partial_t^2 u = c^2 \partial_x^2 u$$
$$u(x,0) = g(x)$$
$$\partial_t u(x,0) = h(x)$$

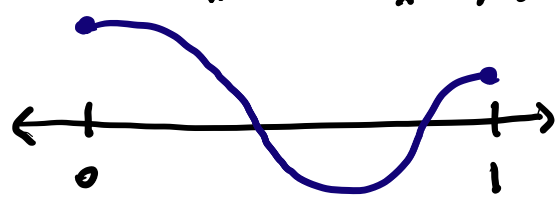
Dirichlet

$u(0,t) = u(1,t) = 0$




Neumann

$\partial_x u(0,t) = \partial_x u(1,t)$




## Dirichlet case

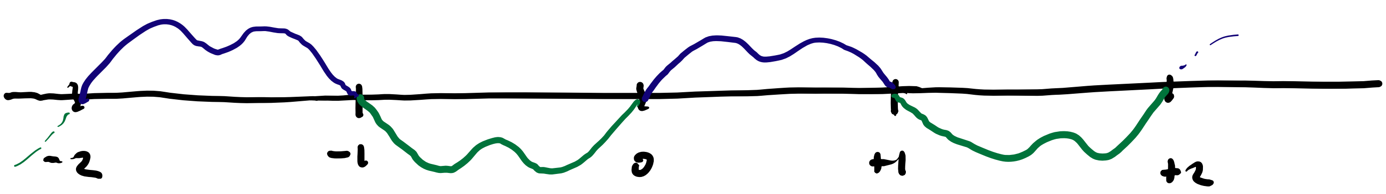
Eigenfunctions/Eigenvalues of $\partial_x^2$ are here

$$e_k(x) = \sin(k\pi x) \quad \text{and} \quad \lambda_k = -(ck\pi)^2, \quad k = 1, 2, 3, \ldots$$

wave operator

"exponential"

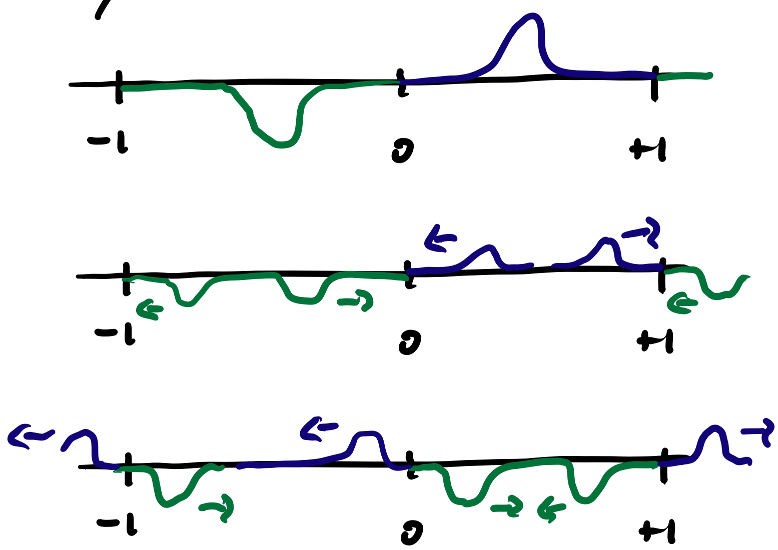
$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} \left[ \tilde{g}_k \cos(c\pi k t) + \frac{\tilde{h}_k}{c\pi k} \sin(c\pi k t) \right] \sin(k\pi x)$$




"Odd periodic extension"

By taking $x$ outside of $[0,1)$ in $u(x,t)$, the solution extends to an odd 2-periodic function.

So, what happens to a traveling wave packet at the boundary?




The wave packet appears to be reflected and flipped upside down at the boundaries $x=0,1$.

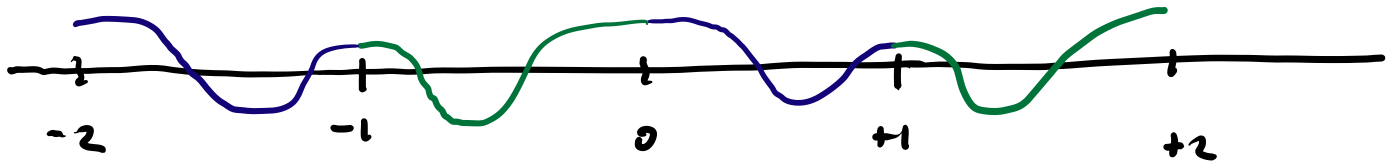
## Neumann case

Eigenvalues/Eigenfunctions of $\partial_x^2$ are here

$$e_k(x) = \begin{cases} 1/2 & k=0 \\ \cos(k\pi x) & \end{cases} \quad \text{and} \quad \lambda_k = -(c\pi k)^2, \quad k=0,1,2,\dots$$

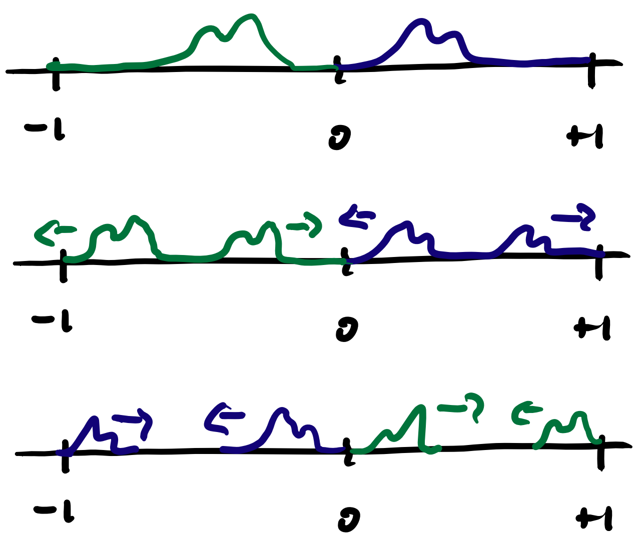
$$\Rightarrow u(x,t) = \langle e_0, g\rangle + \langle e_0, h\rangle t \qquad \text{(unstable modes)}$$

$$+ \sum_{k=1}^{\infty} \left[ \langle e_k, g\rangle \cos(c\pi k t) + \frac{\langle e_k, h\rangle}{c\pi k} \sin(c\pi k t) \right] \cos(k\pi x)$$




"even periodic extension"

By taking x outside of $[0,1)$ in $u(x,t)$, the solution extends to an even 2-periodic function.




The wave packet <u>appears</u> to be reflected and flipped left-to-right at the boundaries $x=0,1$.