Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Mixed boundary conditions. Solve the heat equation in the unit square, $\Omega = [-1,1] \times [-1,1]$, when no heat flux is permitted through the vertical boundaries $x = \pm 1$ and the temperature is held constant along the horizontal boundaries $y = \pm 1$. That is, solve

$$\partial_t u = \Delta u, \quad \text{where} \quad \partial_x u|_{x=\pm 1} = u|_{y=1} = 0, \quad u|_{y=-1} = 1, \quad \text{and} \quad u|_{t=0} = g.$$

(a) Find eigenfunctions of Δ that satisfy homogeneous Neumann boundary conditions on the vertical boundaries and Dirichlet boundary conditions on the horizontal boundaries.

(b) Find an equilibrium solution to the heat equation, which satisfies $\Delta u_* = 0$, that satisfies the mixed Neumann and Dirichlet boundary conditions in the problem statement.

(c) Using your work from part (a) and (b), derive a series solution to the heat equation that satisfies the initial condition and boundary conditions in the problem statement.

2) Advection and diffusion. Consider the advection-diffusion equation, given by

$$\partial_t u = \alpha \partial_x^2 u + \beta \partial_x u, \qquad \text{such that} \qquad u = \text{periodic on } [-1,1),$$

with initial condition $u(x,0) = g(x)$ and non-negative constants α and β. The notebook `hw1_soln.ipynb` on the 18.303 course repository may be helpful in part (d).

(a) Find a Fourier series solution using the operator exponential for the right-hand side.

(b) If $\alpha > 0$, derive an equilibrium solution from the Fourier series solution. How does the case $\beta > 0$ (advection-diffusion) compare to the case $\beta = 0$ (pure heat equation)?

(c) Discretize the advection-diffusion equation in the problem statement using second-order finite differences in space and *backward* Euler's method in time. Solve the PDE numerically with $\beta = 1$, initial condition $g(x) = 5\exp(-10\cos^2(\pi x))$, and $\alpha = 0.1$, 0.01, and 0.001. Try 100 gridpoints in space and a time-step of 0.01 for $0 \le t \le 5$. How do your observations compare with your results from part (b)?