A bit of vector calculus

Some integration rules

18.303 Linear Partial Differential Equations: Analysis and Numerics

Divergence and gradient theorems

For $g : \mathbb{R}^N \to \mathbb{R}$ we have

$$\int_\gamma \nabla g(\mathsf{x}) \cdot \mathrm{d}\mathsf{x} = g(\mathsf{x}_\mathrm{e}) - g(\mathsf{x}_\mathrm{s}).$$

Here γ is a differentiable path embedded in $\mathbb{R}^N$ and $g(\mathsf{x}_\mathrm{s})$ and $g(\mathsf{x}_\mathrm{e})$ are the start and endpoints of the path. This is called the gradient theorem or the fundamental theorem of calculus for line integrals.

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For a compact $\Omega \subset \mathbb{R}^N$ and a function $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^N$ we have

$$\int_\Omega \nabla \cdot \mathbf{f}(\mathbf{x}) \mathrm{d}V = \int_{\partial\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{S} = \int_{\partial\Omega} \mathbf{f}(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) \mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is the unit normal of a given point on the boundary of the region Ω. We assume here that the boundary is piecewise smooth. This is called the divergence theorem or Gauss's theorem.

Stokes' theorem

Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be a sufficiently smooth vector field. Let Σ be a *simply connected* smooth two-dimensional subset of $\mathbb{R}^3$ with a piecewise smooth boundary.

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We have

$$\int_\Sigma \nabla \times \mathsf{A} \cdot \mathrm{d}\mathsf{S} = \oint_{\partial\Sigma} \mathsf{A} \cdot \mathrm{d}\mathsf{x}.$$

Here the integral is contracted against the the tangent .

Green's first identity

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This can be also written as

$$\int_\Omega \nabla \psi \cdot \nabla \varphi \mathrm{d}V = \int_{\partial\Omega} \psi \nabla \varphi \cdot \mathrm{d}\mathbf{S} - \int_\Omega \psi \Delta \varphi \mathrm{d}V,$$

so it becomes a rule for integration by parts for vector calculus.

Exercise

Derive Green's first identity using the divergence theorem. Hint, choose $\mathsf{f} = \psi \nabla \varphi$.

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E.g. choosing $\mathbf{f} = \psi\mathbf{I}$, where $\mathbf{I}$ is the identity matrix gives

$$\int_{\Omega} \nabla \cdot (\psi \mathbf{I}) \mathrm{d}V = \int_{\Omega} \nabla \psi \mathrm{d}V = \int_{\partial\Omega} \psi \mathbf{I} \cdot \mathrm{d}\mathbf{S} = \int_{\partial\Omega} \psi \mathrm{d}\mathbf{S}.$$