The Heat Equation (pt. 2)

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$

-1 $+1$

$u(-1,t) = u(1,t)$

$\partial_x u(-1,t) = \partial_x u(1,t)$

$u(x,0) = g(x)$

"initial condition" + "boundary conditions"

$\Rightarrow$ Initial Boundary Value Problem

__Soln.__ $\gamma = 1$

$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{g}_k \, e^{-(\pi k)^2 t} e^{i\pi k x}$$

Fourier coeffs $\hat{g}_k = \frac{1}{\sqrt{2}} \int_{-1}^{+1} e^{-i\pi k x} g(x)\,dx$

$\Rightarrow$ For each $t \geq 0$, $u(x,t)$ has time-dependent Fourier coeffs that decay exponentially

$$\hat{u}(t) = \hat{g}_k e^{-(\pi k)^2 t}$$

This means that $u(x,t)$ is a very smooth function of x, infinitely differentiable (in fact it is analytic!)

$\Rightarrow$ As $t \to \infty$, the terms with larger k = "faster oscillation in x" decay faster than small k.

Only the $k=0$ term doesn't decay,

$$\lim_{t\to\infty} u(x,t) = \frac{1}{\sqrt{2}}\hat{g}_0 = \underbrace{\frac{1}{2}\int_{-1}^{1} g(x)\,dx}_{\text{mean value of } g(x)}$$

The equilibrium ($t\to\infty$) temperature distribution is uniform at the mean value of the initial temp. dist.

Operator Exponential

$$\frac{\partial u}{\partial t} = Au, \qquad u\big|_{t=0} = g$$

$\langle Af,g\rangle = \langle f,Ag\rangle$

A is a self-adjoint diff. op.

w/ eigenvalues $\lambda_1, \lambda_2, \ldots$

eigenfunctions $e_1, e_2, \ldots$

$$u(t) = e^{At}g = \sum_{k=1}^{\infty} e^{\lambda_k t}\langle e_k, g\rangle e_k$$

E.g. $\gamma \neq 1$ or heated ring ($k=\pm1, \pm2, \ldots$)

$$\lambda_k = -\gamma(\pi k)^2$$

$$e_k = \frac{1}{\sqrt{2}}e^{i\pi k x}$$

$$u(x,t) = \frac{1}{\sqrt{2}}\sum_{k=-\infty}^{\infty}\hat{g}_k e^{-\gamma(\pi k)^2 t} e^{i\pi k x}$$

as diffusivity increases, high frequency decay is faster.

Homogeneous B.C.'s

$$\frac{\partial u}{\partial t} = \overset{=1}{\gamma} \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$

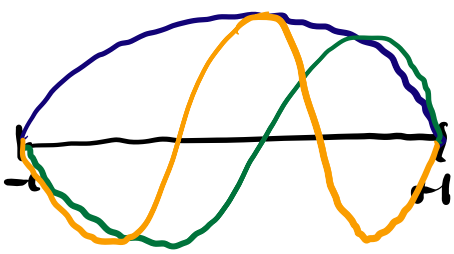
-1 $\qquad$ $+1$

$u(x,0) = g(x)$ $\qquad$ $u(-1,t) = u(1,0) = 0$

"Fixed temp at endpts"

Solution is $u(x,t) = \sum e^{\lambda_k t} \langle e_k, g \rangle e_k$, so we need to compute eigenvals/eigenfuns of $A = \frac{d^2 u}{dx^2}$ w/ $u(\pm 1) = 0$

From lecture 6, $\lambda_k = -\left(\frac{k\pi}{2}\right)^2$ $\quad k = 1, 2, 3, \ldots$



$$e_k(x) = \begin{cases} \cos\left(\frac{k\pi x}{2}\right) & k = 1, 3, 5, \ldots \\ \sin\left(\frac{k\pi x}{2}\right) & k = 2, 4, 6, \ldots \end{cases}$$

Solution:

$$u(x,t) = \sum_{k=\text{odd}} e^{-\left(\frac{k\pi}{2}\right)^2 t} a_k \cos\frac{k\pi x}{2} + \sum_{k=\text{even}} e^{-\left(\frac{k\pi}{2}\right)^2 t} b_k \sin\frac{k\pi x}{2}$$

$$a_k = \int_{-1}^{1} \cos\frac{k\pi x}{2}\, g(x)\, dx$$

$$b_k = \int_{-1}^{1} \sin\frac{k\pi x}{2}\, g(x)\, dx$$

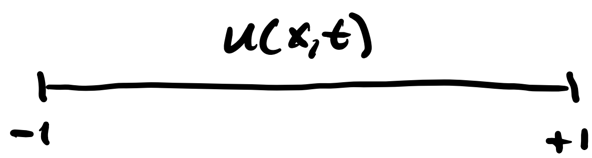
Note that $u(x,t)$ can be written as a standard Fourier series (complex exp. form) by plugging in $\cos\frac{k\pi x}{2} = \frac{1}{2}\left(e^{i\pi kx} + e^{-i\pi kx}\right)$

$$\sin\frac{k\pi x}{2} = \frac{1}{2i}\left(e^{i\pi kx} - e^{-i\pi kx}\right)$$

What happens as $t \to \infty$?

Inhomogeneous B.C.s

$$\frac{\partial u}{\partial t} = \gamma^{-1}\frac{\partial^2 u}{\partial x^2}$$



$u(x,0) = g(x)$ $\qquad$ $u(-1,t) = T_1,\quad u(1,0) = T_2$

"Fixed temp at endpts"

Idea: Split $u(x,t) = u_*(x) + \tilde{u}(x,t)$

"equilibrium"

$$\frac{\partial u_*}{\partial t} = 0$$

$u_*(-1) = T_1$

$u_*(+1) = T_2$

"time-dependent"

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2}$$

$\tilde{u}(\pm 1, t) = 0$

$\tilde{u}(x,0) = g - u_*$

Then, $u(x,t)$ solves heat equation w/inhom. B.C.'s.

Eqn $$\frac{\partial u}{\partial t} = \overset{=0}{\cancel{\frac{\partial u_*}{\partial t}}} + \frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \overset{=0}{\cancel{\frac{\partial^2 u_*}{\partial x^2}}} = \frac{\partial^2 u}{\partial x^2}$$ ✓

B.C.1 $\quad u(-1,0) = \underset{0}{\underset{\parallel}{\tilde{u}(-1,0)}} + \underset{T_1}{\underset{\parallel}{u_*(-1)}} = T_1$ ✓

B.C.2 $\quad u(+1,0) = \underset{0}{\underset{\parallel}{\tilde{u}(+1,0)}} + \underset{T_2}{\underset{\parallel}{u_*(+1)}} = T_2$ ✓

I.C. $\quad u(x,0) = \tilde{u}(x,0) + u_* = g(x)$ ✓

So we need to solve for u_* and $\tilde{u}$ to construct full solution to heat eqn w/ inhom. BCs.

Soln to Homogen. BVP w/ Inhom. BCs $\Rightarrow \quad u_*(x) = T_1 + T_2\left(\frac{1+x}{2}\right)$

$$\tilde{u}(x,t) = \sum_{k=\text{odd}} e^{-\left(\frac{\pi k}{2}\right)^2 t} a_k \cos\frac{\pi k x}{2} + \sum_{k=\text{even}} e^{-\left(\frac{\pi k}{2}\right)^2} b_k \sin\frac{\pi k x}{2}$$

$\Uparrow$

Soln to IBVP w/inhomogeneous BCs

Full Soln. $\Rightarrow \quad u(x,t) = u_*(x) + \tilde{u}(x,t)$

Notice similarity to solution decomposition for BVP (e.g. Poisson) w/inhomogeneous B.C.'s.