Linear Algebra of Differentiation

Matrix

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \longrightarrow$$

Differential Op

$$v(x) = \left[x\partial_x^2 + \sin x\right] u(x)$$

Linear Transformations

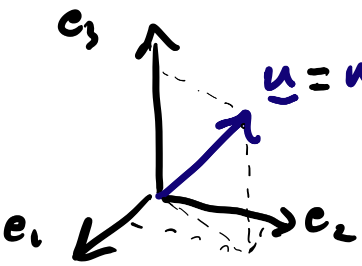
$$\underline{v} = A\,\underline{u}$$

$$A(\underline{u} + \alpha \underline{w}) = A\underline{u} + \alpha A \underline{w}$$

__Key Idea:__ use tools of linear algebra to analyze & solve ODEs/PDEs.

- => Fundamental Subspaces (general soln to $Ax = b$)
- => Orthogonal Bases (matrix representation for differential op.)
- => Factorizations (special bases in which diff. op. "simplifies")

Function Spaces & Bases



$$\underline{u} = u_1 e_1 + u_2 e_2 + u_3 e_3 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Orthonormal basis allows us to identify $\underline{u}$ with 3 coordinates.

Similarly, we can represent large classes of functions by their coordinates in a basis.

basis functions

E.g. $p(x) = c_0 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

coordinates

Every degree n polynomial can be written as a unique linear combo of $1, x, x^2, \ldots, x^n$

Basis

$n+1$ coordinates $c_0, c_1, \ldots, c_n$ uniquely identify $p(x)$.

$\mathbb{P}^n$ = space of degree n polynomials has dimension $n+1$.

Identifying a basis for our function space allows us to use linear algebra tools for DEs.

E.g.

linear operator $p(x) \to p'(x)$

$c_0 1 + c_1 x \quad {}_2 x^2 + \cdots + c_n x^n \to c_1 + 2c_2 x + \cdots + n c_n x^{n-1}$

$$\begin{bmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 2 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & n \\ 0 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$p'(x) = \frac{d}{dx} \quad p(x)$$

Differentiation is represented by an $(n+1) \times (n+1)$ matrix on $\mathbb{P}^n$ = deg n poly's.

<u>Nullspace:</u> soln's to $\frac{du}{dx} = 0 \Rightarrow u(x) = \text{const.}$

This corresponds to the first column of zeros:

$$\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 2 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & n \\ 0 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$0 = \frac{d}{dx} \quad \text{const}$$

<u>Range:</u> $\frac{d}{dx} p(x)$ = degree $n-1$ poly when $p \in \mathbb{P}_n$

This corresponds to the last row of zeros:

$$\begin{bmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 2 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & n \\ 0 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

coefficient of x^n always $= 0 \rightarrow$

General Solution

Most of the differential operators we encounter in this course are not invertible w/out auxillary conditions.

E.g. Trans't eqn $\partial_t u - \partial_x u = 0$ has infinitely many soln's $\Rightarrow$ They are the functions that are constant along characteristics and these make up the null-space of the diff. op. $A = \partial_t - \partial_x$. For a unique soln., we need to specify an initial condition, e.g.,

$$u(x,0) = \cos(x)$$

If basis for nullspace of A is $u_1, \dots, u_n$,

$$(*) \qquad u(x) = \underset{\substack{\uparrow \\ \text{particular} \\ \text{soln to } Au=f}}{u_p(x)} + \underbrace{c_1 u_1(x) + \cdots + c_n u_n(x)}_{\text{soln.'s to } Au=0}$$

describes all possible soln.'s to $Au=f$.

Note: finding u_p requires f in range of A.

E.g.

To solve $\frac{d}{dx}u(x) = f(x)$, we get

$$u(x) = \int f(x)\,dx + \text{const.}$$

particular solution to $\frac{d}{dx}u = f$

solution to $\frac{d}{dx}u = 0$

For unique solution, we need an auxillary eqn. like $u(-1) = 0$ to fix the constant of integration.

now have unique soln. $\Rightarrow u(x) = \int_{-1}^{x} f(x')\,dx'$

In general, initial/boundary/auxillary conditions for PDE allow us to specify a <u>unique</u> soln. among all possible soln's described by (*).

In matrix picture of $\mathbb{P}_n$, suppose

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ & & & & & 0 \\ 0 & 0 & & & 0 & n \\ 0 & 0 & \cdots & & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \\ c_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix}$$

replace → (last row)

← need $f \in \text{range}(A)$

replace last eqn. $0 = 0$ with boundary condition $u(-1) = 0$ to get now enforces boundary condition →

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ & & & & & 0 \\ 0 & 0 & & & 0 & n \\ 1 & -1 & \cdots & & & (-1)^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \\ c_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix}$$

with $\quad p(-1) = c_0 1 + c_1(-1) + c_2(1) + \dots + c_n(-1)^n = 0$

The matrix is now invertible and solution is given by (check it yourself!)

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/3 & \cdots & (-1)^{n+1} \\ 1 & 0 & & & \\ & \frac{1}{2} & 0 & & \\ & & \ddots & \ddots & \\ & & & 1/n & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix}$$

This is precisely the matrix representation of

$$[Kf](x) = \int_{-1}^{x} f(y)\,dy$$

mapping $\quad f_1 1 + \dots + f_{n-1} x^{n-1} \to c_0 1 + c_1 x + \dots + c_n x^n$.

K is the inverse of diff op $A = \frac{d}{dx}$ acting on differentiable functions with $u(-1) = 0$.

Inner Products and Norms

In $\mathbb{R}^n$, we usually represent vectors in an orthonormal basis. Orthonormal bases play a key role in analysis and numerical solution of PDEs too.

"Dot" Product of real vectors	$\longrightarrow$	Inner Product of real functions
$\langle u, v\rangle = u_1 v_1 + \cdots + u_n v_n = v^T u$		$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,dx$

In general, inner product on a function space satisfies:

1) $\overset{\text{conjugate}}{\text{Symmetry}}$ $\langle u, v\rangle = \overline{\langle v, u\rangle}$ ← complex conjugate if functions are complex
2) Linearity $\langle u+\alpha w, v\rangle = \langle u, v\rangle + \alpha\langle w, v\rangle$
3) Positive-definite $\langle u, u\rangle > 0$ IFF $u \neq 0$.

These make the inner product behave like the familiar $v^T u = u_1 v_1 + \cdots + u_n v_n$ and allows us to measure the "size" of a function

"norm" $\quad \|u\| = \sqrt{\langle u,u\rangle} = \left[\int_{-1}^{1} |f(x)|^2 dx\right]^{1/2}$

e.g.

Useful, familiar properties from $\mathbb{R}^n$ are:

Homogeneity $\quad \|\alpha u\| = |\alpha|\,\|u\|$

Triangle Inequality $\quad \|u+v\| \leq \|u\| + \|v\|$

Cauchy-Schwartz $\quad |\langle u,v\rangle| \leq \|u\|\|v\|$

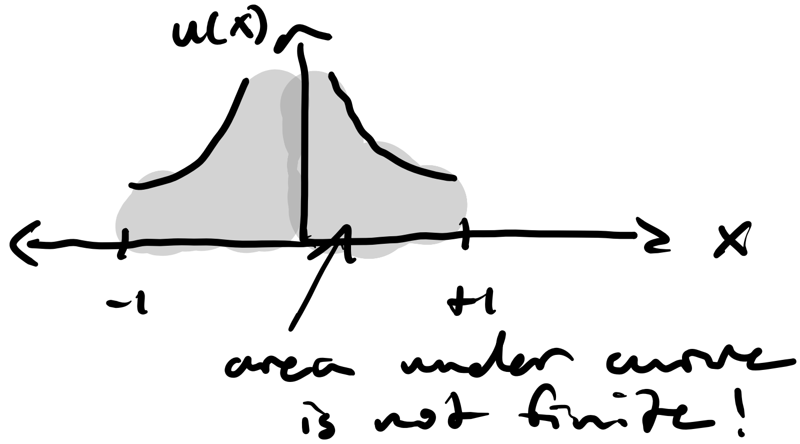
We say that u and v are orthogonal if $\langle u,v\rangle = 0$ and $u_1, \ldots, u_n$ are an orthogonal basis for a space of function if

$$\langle u_j, u_k\rangle = 0 \quad \text{for } j \neq k.$$

If $\langle u_j, u_j\rangle = 1$, then the basis is orthonormal.

Caution! Unlike vectors in $\mathbb{R}^n$, many functions on $[-1,1]$ may not be integrable!

E.g. $u(x) = x^{-1}$



$$\langle 1, u \rangle = \int_{-1}^{1} \frac{dx}{x} = +\infty$$

$$\|u\| = \left[\int_{-1}^{1} \frac{dx}{x^2}\right]^{1/2} = +\infty$$

To do linear Algebra with functions, we restrict ourselves (usually) to functions where

$$\|u\| = \sqrt{\langle u, u \rangle} = \text{finite}$$

This is called a Hilbert Space of functions.