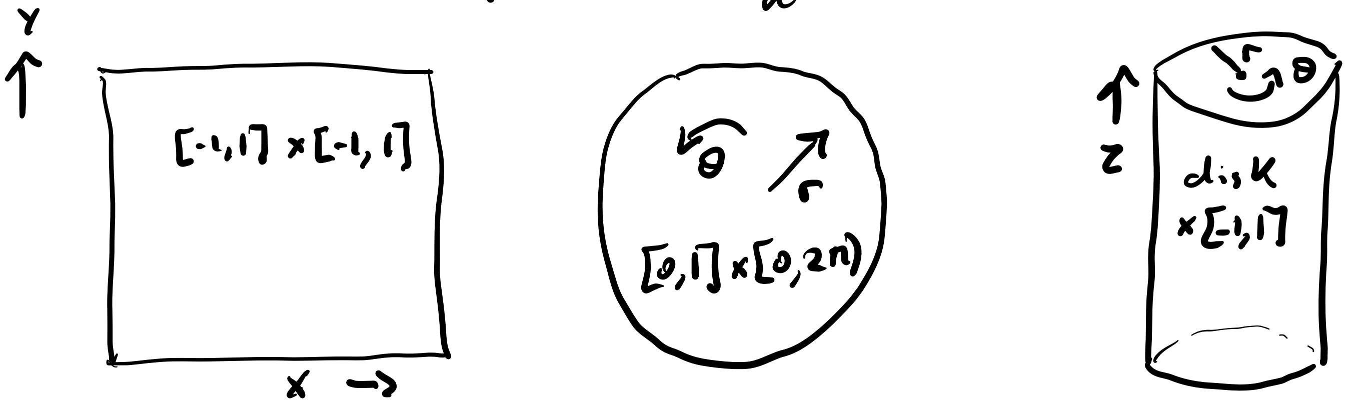
Laplace Equation



Poisson in seperable domain (Ω)

$$\Delta u = f \text{ in } \Omega, \qquad u|_{\partial\Omega} = g$$

prescribed values on $\partial\Omega$

boundary of Ω

General Solution: $\quad u = u_h + u_p$

u_h solves $\Delta u_h = 0$, $u_h|_{\partial\Omega} = g$

u_p solves $\Delta u_p = f$, $u_p|_{\partial\Omega} = 0$

$\Rightarrow$ Construct u_p systematically by diagonalizing

$$\Delta e_k = \lambda_k e_k \qquad u|_{\partial\Omega} = 0$$

and solving diagonal system:

$$u = \sum_k \lambda_k^{-1} \langle e_k, f \rangle e_k .$$

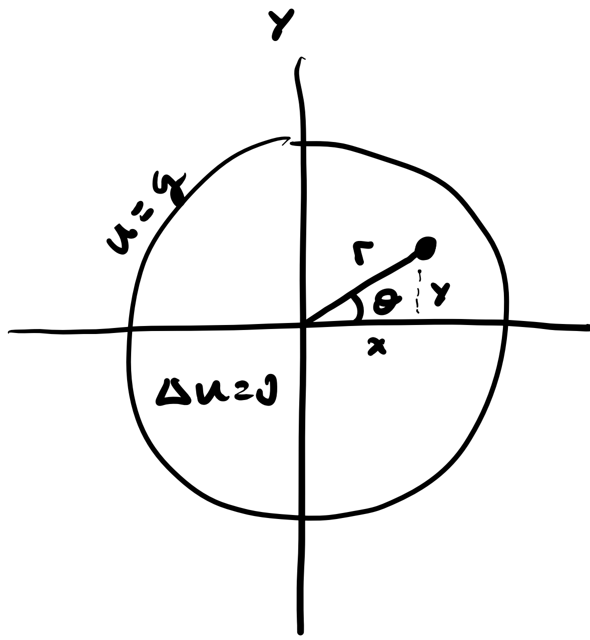
$\Rightarrow$ Construct u_h by solving Laplace Eqn., i.e., construct function in nullspace that satisfies inhomogeneous b.c.'s $u_n|_{\partial\Omega} = g$.

<u>Key tool:</u> In each case, separation of variables reduces the PDE to 2 ODEs, which are (usually) easier to solve.

Laplace's Eqn. in a disk

$$\Delta u = 0$$

$$\text{s.t. } u\big|_{r=1} = g(\theta)$$



Polar Coords = separable domain

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$\Rightarrow$ Solve w/separation of variables

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Theta \frac{\partial^2 R}{\partial r^2} + \frac{\Theta}{r}\frac{\partial R}{\partial r} + \frac{R}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

$$\underbrace{\frac{1}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right]}_{=-\lambda} + \underbrace{\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}}_{=\lambda} = 0$$

<u>ODE for Θ:</u> $\quad \frac{\partial^2 \Theta}{\partial \theta^2} = \lambda \Theta \qquad \Theta(0) = \Theta(2\pi)$

Θ is 2π-periodic in polar coords

$$\Rightarrow \Theta_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta} \qquad \lambda_k = -k^2 \qquad k = 0, \pm 1, \pm 2, \ldots$$

<u>ODE for R:</u> $\quad \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \underbrace{\lambda}_{=-k^2} R = 0$

Euler equation: $\quad R(r) = \begin{cases} A_0 + B_0 \log r & k = 0 \\ A_k r^k + B_k r^{-k} & k = \pm 1, \pm 2, \ldots \end{cases}$

General Soln: $\quad u(r,\theta) = \frac{1}{\sqrt{2\pi}}\Big[A_0 + B_0 \log r + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (A_k r^k + B_k r^{-k}) e^{ik\theta} \Big]$

<u>But</u>, we need $u(r,\theta)$ smooth in disk, so we can discard terms w/singularity at the origin $\Rightarrow \log r$, r^{-k}.

General soln in disk

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + \sum_{k=1}^{\infty} (A_k e^{ik\theta} + B_{-k} e^{-ik\theta}) r^k \right]$$

<u>Idea:</u> Choose coefficients A_k, B_{-k} so that $u(1,\theta) = g(\theta)$, satisfying b.c.s.

$$g(\theta) = u(1,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + \sum_{k=1}^{\infty} A_k e^{ik\theta} + B_{-k} e^{-ik\theta}\right]$$

IFF $$\tilde{g}_k = \begin{cases} A_k & k=0,1,2,\ldots \\ B_k & k=-1,-2,-3,\ldots \end{cases}$$

Fourier coeffs of $g(\theta)$

Therefore, $$A_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\theta} g(\theta)\,d\theta, \quad k=0,1,\ldots$$

$$B_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\theta} g(\theta)\,d\theta, \quad k=-1,-2,\ldots$$

<u>E.g.</u> Suppose $g(\theta) = \cos^2\theta$.

$$\tilde{g}_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\theta}\cos^2\theta\,d\theta$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\theta}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\theta}\left(\frac{1}{2} + \frac{1}{4}e^{i2\theta} + \frac{1}{4}e^{-i2\theta}\right)d\theta$$

$$= \begin{cases} \sqrt{\frac{\pi}{2}} & k=0 \\ \frac{1}{2}\sqrt{\frac{\pi}{2}} & k=\pm 2 \end{cases}$$

$$\Rightarrow \quad u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[\sqrt{\frac{\pi}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(e^{i2\theta} + e^{-i2\theta}\right)r^2\right]$$

$$= \frac{1}{2} + \frac{r^2}{2}\cos 2\theta$$

Check

note that $u(1,\theta) = \frac{1}{2} + \frac{1}{2}\cos 2\theta = \cos^2\theta$ ✓

and $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$$= \cos 2\theta + \cos 2\theta - 2\cos 2\theta = 0 \checkmark$$

Our $u(r,\theta)$ solves $\Delta u = 0$ and $u|_{\partial\Omega} = \cos^2\theta$.

Poisson Formula

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{g}_k \, r^{|k|} e^{ik\theta}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ik\theta'} g(\theta')\, d\theta' \; r^{|k|} e^{ik\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik(\theta-\theta')}}_{G(r,\theta,\theta')} g(\theta')\, d\theta'$$

Poisson Kernel (disk)

$$G(r,\theta,\theta') = \frac{1}{2\pi}\sum_{k=0}^{\infty} \left(re^{i(\theta-\theta')}\right)^k + \frac{1}{2\pi}\sum_{k=1}^{\infty} \left(re^{-i(\theta-\theta')}\right)^k$$

$$= \frac{1}{2\pi}\left[1 + 2\,\mathrm{Real}\left[\sum_{k=1}^{\infty} \left(re^{i(\theta-\theta')}\right)^k\right]\right]$$

(sum geometric series)

$$= \frac{1}{2\pi}\left[1 + 2\,\mathrm{Real}\left(\frac{re^{i(\theta-\theta')}}{1-re^{i(\theta-\theta')}}\right)\right]$$

(Take real part)

$$= \frac{1}{2\pi}\left[1 + 2\,\frac{r\cos(\theta-\theta')}{1-2r\cos(\theta-\theta')+r^2}\right]$$

(simplify)

$$= \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}$$

"Inverse" of Δ on disk w/ $u|_{\partial\Omega} = g$

$$Kg = \int_0^{2\pi} G(r,\theta,\theta')\, g(\theta)\, d\theta$$