Please submit your solutions to the following problems. You may use one 3×5 study card.

Problem 1) For each question below, circle all answers that apply (if any).

(a) Which 2-periodic function has only finitely many nonzero Fourier coefficients?

(i) $f(x) = |\sin(\pi x)|$, (ii) $g(x) = |\sin(\pi x)|^3$, (iii) $\underline{h(x) = \sin^3(\pi x)}$

(b) Which 2-periodic solution $u(x,t)$ has a constant $L^2(-1,1)$ norm for all $t > 0$?

(i) $\partial_t u = \partial_x^2 u$, (ii) $\underline{i\partial_t u = -\partial_x^2 u}$, (iii) $\underline{\partial_t u = \partial_x u}$

(c) Which 2-periodic solution $u(x,t)$ has a vanishing $L^2(-1,1)$ norm as $t \to 0$?

(i) $\partial_t u = \partial_x^2 u$, (ii) $i\partial_t u = -\partial_x^2 u$, (iii) $\partial_t u = \partial_x u - u$

NOTE: This was supposed to be as $t \to \infty$, which only (iii) need satisfy (remember that (i) has a zero eigenvalue so the equilibrium solution may be a nonzero constant). In light of this typo, full credit is awarded for either answer: (iii) or "none."

(d) Consider Poisson's equation on the unit square, $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, with inhomogeneous Dirichlet boundary conditions. For which right-hand side must the solution achieve its maximum on the boundary?

(i) $\underline{\Delta u = \exp(-(x+y)^2)}$, (ii) $\Delta u = \sin(\pi xy)$, (iii) $\underline{\Delta u = \cos^2(\pi xy)}$

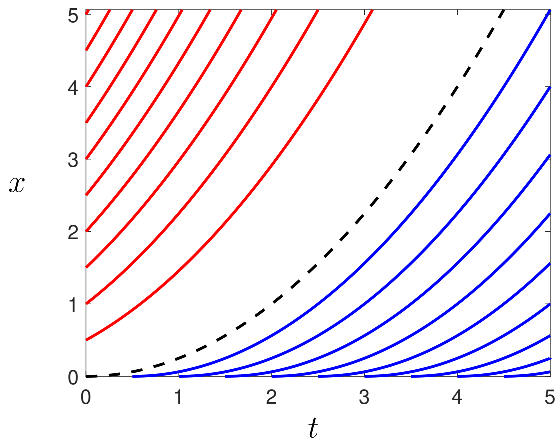

Figure 1: A selection of characteristics plotted from Problem 2(a). The characteristics that intersect the positive x-axis (red) lie above the parabola $t^2/4$ (dashed) while the characteristics that interesect the non-negative t-axis lie on and below the parabola (blue).

Problem 2) Consider the first-order linear transport PDE, given by

$$\partial_t u + \sqrt{x}\,\partial_x u = 0, \qquad \text{where} \qquad u(x,0) = 1/(1+x^2).$$

(a) Calculate the characteristic curves and plot them in the (x,t)-plane for $x > 0$, $t \geq 0$.

Solution: The characteristic curves, $(x(s), t(s))$, in the (x,t)-plane are defined by the differential equations

$$\frac{dt}{ds} = 1, \qquad \text{and} \qquad \frac{dx}{ds} = \sqrt{x},$$

with initial conditions $t(0) = 0$ and $x(0) = x_0$ (the value x_0 determines *which* characteristic curve by its starting point x_0 at time 0). Therefore, $t = s$ as usual. We can solve for $x(s)$ by separation of variables (for ODEs) and calculate that

$$\int_{x_0}^{x} \frac{dy}{\sqrt{y}} = \int_0^s d\sigma \implies 2\sqrt{x(s)} - 2\sqrt{x_0} = s.$$

Solving for $x(s)$ and substituting $s = t$, we find that the characteristic curves are

$$x(t) = (t/2 + \sqrt{x_0})^2, \qquad x_0 > 0.$$

Note that these characteristics lie strictly above the parabola $x(t) = t^2/4$ (see Figure 1).

There is a second set of characteristics that do not intersect the positive x-axis. These characteristics arise from the infinitely many solutions to the initial value problem $dx/ds = \sqrt{x}$ when $x_0 = 0$. Although $x = 0$ is not in the domain of our problem, the characteristics curves from these solutions pass through the domain $x > 0$, $t > 0$:

$$x(t) = \begin{cases} (t - t_0)^2/4, & t \geq t_0, \\ 0, & t < t_0. \end{cases}$$

The parameter t_0 describes where these characteristics intersect the non-negative t-axis.

(b) Write down a formula for the solution $u(x,t)$ that is valid for any $x > 0$, $t \geq 0$.

Solution: Since $du/ds = 0$, we have that $u(s) = u(x(s), t(s)) = u(x_0, 0)$ is constant along characteristics. If $x > t^2/4$, we can trace the characteristic back to the initial $x_0 > 0$ to find the solution value at (x,t). To do this, we solve for x_0 in terms of x and t:

$$x_0 = (\sqrt{x} - t/2)^2, \qquad \text{when} \qquad x > t^2/4.$$

Therefore, for $x > t^2/4$, the solution is given explicitly by

$$u(x,t) = u((\sqrt{x} - t/2)^2, 0) = \frac{1}{1 + (\sqrt{x} - t/2)^4}$$

When $x \leq t^2/4$, we cannot follow characteristics back to the positive x-axis at $t = 0$ to find the value of the solution. In other words, the initial data prescribed on the positive x-axis does not determine the solution along the characteristics on and below the parabola $x = t^2/4$. So, how should we extend the solution into this region?

There are two particularly natural ways to extend the solution to all $x > 0$, $t > 0$. The first is to simply use the formula above on both sides of the parabola. The formula is well-defined and solves the differential equation for all $x > 0$, $t > 0$ because it is constant along the characteristics on and below the parabola (with value $1/(1+t_0^4/16)$).

The second is to set the solution to a constant and choose the constant function so that, based on the initial data $u(x,0)$ and the solution above the parabola $x = t^2/4$, the solution will be continuously differentiable across the boundary $x = t^2/4$. In this case, the full solution valid for all $x > 0$, $t \geq 0$ is

$$u(x,t) = \begin{cases} \frac{1}{1+(\sqrt{x}-t/2)^4}, & x > t^2/4, \\ 1, & 0 < x \leq t^2/4. \end{cases}$$

(c) How do the characteristics change if the domain is $x \geq 0$? Is the solution unique?

Solution: If the domain includes the origin, $x = 0$, then the characteristics on and below the parabola $x = t^2/4$ share some points in common along the non-negative t-axis. In particular, the value of the solution along all these characteristics is determined by the value of the solution at $x_0 = 0$, which (for us) is $u(0, 0) = 1$. This corresponds to the second "extend-by-constant" solution mentioned in part (b). However, in this case, it is now the unique solution to our transport equation, since we no longer have the freedom to specify other values along the characteristics on and below the parabola.

Problem 3) Consider the exterior Laplace problem in polar coordinates,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] u(r,\theta) = 0, \qquad \text{where} \qquad r \geq 1.$$

(a) Find the general bounded solution to the exterior Laplace equation on $r \geq 1$.

Solution: In Lecture 7, we derived a general solution for the Laplace equation in polar coordinates, which had the form (recall that the sum is over all nonzero integers $k \neq 0$)

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + B_0 \log(r) + \sum_{k\neq 0}(A_k r^k + B_k r^{-k})e^{ik\theta}\right].$$

Any solution that is bounded in the exterior of the unit disk, $r \geq 1$, must have $A_k = 0$ for $k \geq 1$ and $B_k = 0$ for $k \leq 0$ to eliminate terms that grow as $r \to \infty$. We are left with

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + \sum_{k\geq 1} r^{-k}\left(A_{-k}e^{-ik\theta} + B_k e^{ik\theta}\right)\right].$$

This is the general form of bounded solutions to the exterior Laplace problem.

(b) Derive a particular solution that satisfies a Neumann condition, $\partial_r u(1,\theta) = \sin(\theta)$, and a decay condition, $\lim_{r\to\infty} |u(r,\theta)| = 0$. Is this particular solution unique?

Solution: To satisfy the boundary condition $\partial_r u(1,\theta) = \sin(\theta)$, we must choose the remaining coefficients so that

$$\partial_r u(1,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\geq 1} (-k) \left(A_{-k} e^{-ik\theta} + B_k e^{ik\theta}\right) = \sin(\theta).$$

To do this, we can write the complex Fourier series for $\sin(\theta)$ using Euler's identity:

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

Since periodic square-integrable functions are determined uniquely by their coefficients, for $u(1,\theta) = \sin(\theta)$, we must have $A_{-1} = \sqrt{2\pi}/(2i)$, $B_1 = -\sqrt{2\pi}/(2i)$, and $A_{-k} = B_k = 0$ for all $k \geq 2$. This leaves us with a particular solution given by

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + r^{-1}\left(\frac{\sqrt{2\pi}}{2i} e^{-i\theta} - \frac{\sqrt{2\pi}}{2i} e^{i\theta}\right)\right].$$

To satisfy the decay condition $\lim_{r\to\infty}$, we must have $A_0 = 0$. Therefore,

$$u(r,\theta) = \frac{1}{\sqrt{2\pi} r}\left(\frac{\sqrt{2\pi}}{2i} e^{-i\theta} - \frac{\sqrt{2\pi}}{2i} e^{i\theta}\right) = \frac{-1}{r}\sin(\theta).$$

This particular solution is unique. After enforcing the boundary and decay conditions, there are no free parameters left over from the general solution.

(c) Identify the maximum and minimum values of the solution and where they are located.

Solution: From the solution in part (b), the max and min both occur on the unit circle at the extrema of $\sin(\theta)$. The maximum value of 1 is achieved at $\theta = 3\pi/2$ and the minimum value of -1 is achieved at $\theta = \pi/2$.

Problem 4) Solve the one-dimensional heat equation on the interval $\Omega = [0, 1]$, given by

$$\partial_t u = \partial_x^2 u, \qquad \text{where} \qquad \partial_x u|_{\partial\Omega} = 0, \qquad \text{and} \qquad u(x, 0) = g(x).$$

(a) Write down the homogeneous Neumann eigenvalues/eigenfunctions of ∂_x^2.

Solution: The general solutions to $\partial_x^2 u = \lambda u$ are $u(x) = \exp(\pm\sqrt{\lambda} x)$. Since the derivative of u must be zero at both endpoints, we must have $\text{real}(\pm\sqrt{\lambda}) = 0$. Therefore, $\sqrt{\lambda}$ is pure imaginary and, for convenience, we denote $\omega = \text{imag}(\sqrt{\lambda})$. To make the derivative vanish at $x = 0$, we can take a combination of the form $u(x) = e^{i\omega x} + e^{-i\omega x} = 2\cos(\omega x)$, so that $u'(0) = -2\sin(\omega(0)) = 0$. To make the derivative vanish at $x = 1$, we take $\omega = \pi k$ with $k =$ integer, since then $u'(1) = -2\sin(\pi k) = 0$. We conclude that the homogeneous Neumann eigenvalues and eigenfunctions of ∂_x^2 are

$$\lambda_k = -(\pi k)^2, \qquad \text{with} \qquad e_k(x) = \begin{cases} 1 & k = 0, \\ \sqrt{2}\cos(\pi k x), & k = 1, 2, 3, \ldots. \end{cases}$$

The normalization factor of $\sqrt{2}$ is chosen so that $\langle e_k, e_k \rangle = 1$ for $k = 1, 2, 3, \ldots$.

(b) Write down a series solution and give a formula for the series coefficients.

Solution: With the eigenvalues and (normalized) eigenfunctions from part (a), we can write our solution using the operator exponential:

$$u(x,t) = \sum_{k=0}^{\infty} \langle e_k, g \rangle e^{\lambda_k t} e_k(x).$$

With $\hat{g}_0 = \int_0^1 g(x)\,dx$ and $\hat{g}_k = \sqrt{2}\int_0^1 \cos(\pi k x) g(x)\,dx$ for $k \geq 1$, we calculate that

$$u(x,t) = \hat{g}_0 + \sqrt{2}\sum_{k=1}^{\infty} \hat{g}_k e^{-\pi^2 k^2 t} \cos(\pi k x).$$