Massachusetts Institute of Technology

Lotka-Volterra equations

An example of a non-linear system

18.303 Linear Partial Differential Equations: Analysis and Numerics

Lotka-Volterra equations

The Lotka-Volterra system is given by

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= ax(t) - bx(t)y(t)\\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= cx(t)y(t) - dy(t),\end{aligned}$$

where a, b, c, and d are non-negative parameters.

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- a controls exponential growth of the prey i.e. the growth is proportional to the number of prey.
- The term with b is the rate at which the predators eat prey: it's proportional to both of the population sizes.
- c controls the proliferation of predators, it's again proportional to both population sizes.
- d is the ”death rate” of the predators proportional to the population size.

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$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = dz(t)y(t) - dy(t).$$

We can define $\tau = dt$ giving the change of the time derivatives $\partial_t = d\partial_\tau$.

Now we have

$$\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = \alpha z(\tau) - \beta z(\tau)y(\tau)$$
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Now we are facing the problem of solving the system, which in general can't be done analytically. However, we can still say something about it.

Conserved quantities

We can ask ourselves if there are conserved quantities. We have

$$\frac{\dot{z}}{\dot{y}} = \frac{z(\alpha - \beta y)}{y(z-1)}.$$

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Integrating with respect to τ from 0 to τ gives

$$z(\tau) - z(0) - \log(z(\tau)) + \log(z(0)) = \alpha\log(y(\tau)) - \alpha\log(y(0)) - \beta y(\tau) + \beta y(0).$$

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$$z(\tau) + \beta y(\tau) - \log(z(\tau)y(\tau)^{\alpha}) = -\log(z(0)y(0)^{\alpha}) + z(0) + \beta y(0) =: V.$$

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We have discovered a conserved quantity V. You can also show that the solutions are closed curves on the (z, y) plane.

Fixed points

Another thing we can do is to examine the system close to where the time evolution is zero. By requiring $\dot{z}, \dot{y} = 0$ we get

$$\begin{aligned} z(\alpha - \beta y) &= 0, \\ y(z - 1) &= 0. \end{aligned}$$

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We find two fixed points, namely $(z, y) = (0, 0)$ and $(z, y) = (1, \alpha/\beta)$. We can linearize the system near the fixed point by applying a small perturbation $z = z_p + \epsilon\phi$, $y = y_p + \epsilon\psi$, where z_p and y_p are the fixed point coordinates.

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For small ϵ we can write the differential equations up to linear order. For the point $(0, 0)$ we get

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We see immediately that the small time behavior of this fixed point is the prey population exploding and the predator population decaying exponentially.

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The solution is of the form

$$\phi(\tau) = A\cos(\sqrt{\alpha}\tau - \varphi),\ \psi(\tau) = A\frac{\sqrt{\alpha}}{\beta}\sin(\sqrt{\alpha}\tau - \varphi),$$

where A and φ can be solved from the initial condition. This is described by an elliptical time evolution around the fixed point on a fixed trajectory (just as the conserved energy suggests).

Fixed points of general non-linear systems

Assume we have a more general system

$$\dot{x} = f^{(x)}(x, y),$$
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We can repeat the calculation we did for fixed points $f^{(x)}(x_p, y_p) = f^{(y)}(x_p, y_p) = 0$. We expand around the fixed point. We get

$$\delta\dot{\mathbf{r}} = J\delta\mathbf{r},$$

where

$$J = \begin{pmatrix} f_x^{(x)}(\mathbf{r}_p) & f_y^{(x)}(\mathbf{r}_p) \\ f_x^{(y)}(\mathbf{r}_p) & f_y^{(y)}(\mathbf{r}_p) \end{pmatrix}$$

is the *Jacobian* of the system and $\mathbf{r}_p + \delta\mathbf{r} = (x, y)^T$.

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$$\dot{\phi} = \Lambda\phi$$

in the eigenbasis of J. Here Λ is a diagonal matrix with the corresponding eigenvalues on the diagonal.

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The behavior of the system near the fixed point is determined by the eigenvalues of J.

- If the eigenvalues are complex, they come in complex conjugate pairs (J is real)
    - For such systems its faith is determined by the sign of the real part of the eigenvalues: if it's negative, the solution decays to the fixed point, if it's positive the amplitudes of the oscillations grow exponentially.
    - For the limiting case where the real part is zero we get a harmonic oscillator (the latter case for our predator-prey system)
- If the eigenvalues are real:
    - If both of the eigenvalues are negative, the solution decays exponentially to the fixed point
    - If either one of the eigenvalues is positive, the perturbation grows exponentially (the first case for the Lotka-Volterra system)

Categorization of fixed points

Fixed points can be attractive or repulsive. The first case corresponds to negative eigenvalues of J and the latter to positive.

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For complex eigenvalues the solution can be oscillatory without a change in the oscillation amplitude. There's also the case when J has a degenerate eigenvalue. This is a bit more complicated but generally the behavior will still depend on the sign of the real part of the eigenvalue. However, in this case it is possible that even systems with a purely imaginary eigenvalue decay.