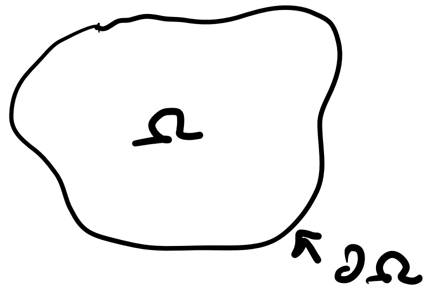
The Heat Eqn. (Pt. 3)

$$\partial_t u = \Delta u$$



$u|_{t=0} = g$, $u|_{\partial\Omega} = f$

initial condition — boundary condition (Dirichlet) $\rightarrow$ alt. Neumann, Robin, mixed

Step 1) Solve $\Delta e_k = \lambda_k e_k$ w/ homogeneous B.C.s
replace $f \to 0$

$\Rightarrow$ homogeneous soln is $u_h = \underbrace{\sum_k e^{\lambda_k t} \langle e_k, g - u_* \rangle e_k}_{= e^{\Delta t}(g - u_*)}$

Step 2) Solve $\Delta u_* = 0$ w/ inhomogeneous B.C.s

$\Rightarrow$ equilibrium soln is u_*

Step 3) Full soln to Heat Eq. w/ inhomogeneous B.C.s

$$u = u_* + \sum_k e^{\lambda_k t} \langle e_k, g - u_* \rangle e_k$$

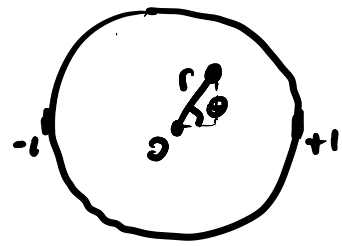
B.C.s: $u|_{\partial\Omega} = u_*|_{\partial\Omega} = f \;+\; u_h|_{\partial\Omega} = 0$ ✓

I.C.: $u|_{t=0} = u_* + g - u_* = g$ ✓

Eqn.: $\partial_t u = \underbrace{\cancel{\partial_t u_*}}_{=0} + \partial_t u_h = \Delta u_h$
$= \Delta u_h + \underbrace{\cancel{\Delta u_*}}_{=0} = \Delta u$ ✓

Diffusion in a disk

$$\partial_t u = \Delta u$$
$$u|_{t=0} = g \qquad u|_{\partial\Omega} = f$$



Step 1: Solve $\Delta u_* = 0$ with $u_*|_{\partial\Omega} = f$

see lecture 7

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}_k \, r^{|k|} e^{ik\theta}$$
$$\hookrightarrow \hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ik\theta} f(\theta)\, d\theta$$

Step 2: Solve $\Delta e_k = \lambda_k e_k$ with $e_k|_{\partial\Omega} = 0$.

$$\downarrow$$

$$\Rightarrow \quad \partial_r^2 e_k + r^{-1}\partial_r e_k + r^{-2}\partial_\theta^2 e_k = \lambda_k e_k$$

separation of variables

$$e_k = R(r)\,\Theta(\theta)$$

(drop k for brevity)

$$\Rightarrow \quad \underbrace{\frac{1}{R}\left[r^2\partial_r^2 R + r\,\partial_r R - r^2\lambda R\right]}_{\alpha} + \underbrace{\frac{1}{\Theta}\partial_\theta^2 \Theta}_{-\alpha} = 0$$

ODE for Θ: $\quad \partial_\theta^2 \Theta = -\alpha\,\Theta \qquad \Theta$ is 2π-periodic

$$\Rightarrow \quad \Theta_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta} \qquad -\alpha = -k^2$$

ODE for R:

$$(*) \qquad r^2 \partial_r^2 R + r \partial_r R - (r^2 \lambda + k^2) R = 0$$

This ODE eigenvalue problem (remember, we need to find both λ and R!) is closely related to a famous ODE called Bessel's equation, and it's solutions are related to Bessel functions $J_k(x)$.

With $\tilde{r} = \sqrt{-\lambda}\, r$ and $R(r) = R(\tilde{r}/\sqrt{-\lambda}) = \tilde{R}(\tilde{r})$

$$(*) \implies \tilde{r}\, \partial_{\tilde{r}}^2 \tilde{R} + \tilde{r}\, \partial_{\tilde{r}} \tilde{R} + (\tilde{r}^2 - k^2) \tilde{R} = 0$$

Bounded solution in disk is first-kind Bessel function of order k,

$$J_k(\tilde{r}) = \frac{1}{\pi} \int_0^{\pi} \cos(k\tau - \tilde{r} \sin \tau)\, d\tau$$

and our solution is $J_k(\tilde{r}) = \tilde{R}(\tilde{r}) = R(r)$

$$R(r) = J_k(\sqrt{-\lambda}\, r)$$

What about λ? To find λ, apply B.C.'s

$$u|_{\partial\Omega} = 0 \quad \Rightarrow \quad 0 = R(1) = J_k(\sqrt{-\lambda})$$

must be a root

So our eigenvalues λ are determined by zeros of Bessel functions! For each $k = 0, \pm 1, \pm 2, \ldots$ we get a countable set of distinct positive roots:

$$0 \leq s_{k,1} < s_{k,2} < \cdots < s_{k,m} < \cdots$$

The eigenfunctions & eigenvalues are therefore

$$R_{k,m}(r) = J_k(s_{k,m} r) \quad \text{and} \quad \lambda_{k,m} = -s_{k,m}^2.$$

Putting together the results with $k = 0, \pm 1, \pm 2, \ldots$
$m = 1, 2, 3, \ldots$

$$e_{k,m}(r,\theta) = R_{k,m}(r)\,\Theta_k(\theta)$$

$$= J_k(s_{k,m} r)\, e^{ik\theta}$$

$$\lambda_{k,m} = -s_{k,m}^2$$

The operator exponential $e^{\Delta t}$ on the disk is

$$u_h(r,\theta,t) = \sum_{k=-\infty}^{+\infty} \sum_{m=1}^{\infty} \tilde{g}_{k,m}\, e^{-(s_{k,m})^2 t} J_k(s_{k,m} r)\, e^{ik\theta}$$

Here, expansion coeffs $\hat{f}_{k,m}$ are such that

$$f(r,\theta) - u_*(r,\theta) = \sum_{k=-\infty}^{\infty} \sum_{m=1}^{\infty} \tilde{g}_{k,m} J_k(s_{k,m} r)\, e^{ik\theta}$$

so that $u_h(r,\theta,0) = g(r,\theta) - u_*(r,\theta)$ as required.

<u>Step 3</u> Finally, we can stitch together the full solution from steps 1 & 2.

$$u(r,\theta,t) = u_*(r,\theta) + u_h(r,\theta,t)$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}_k r^{|k|} e^{ik\theta}$$

$$+ \sum_{k=-\infty}^{\infty} \sum_{m=1}^{\infty} \tilde{g}_{k,m} e^{-(s_{k,m})^2 t} J_k(s_{k,m} r)\, e^{ik\theta}$$

$$= \sum_{k=-\infty}^{\infty} \left[\hat{f}_k r^{|k|} + \sum_{m=1}^{\infty} \tilde{g}_{k,m} e^{-(s_{k,m})^2 t} J_k(s_{k,m} r) \right] e^{ik\theta}$$