Green's Functions

Consider the boundary value problem

$(*) \quad -u''(x) = f(x) \quad \text{s.t.} \quad u(\pm 1) = 0.$

What is the "inverse" of the diff. op.

$$Lu = -u'' \quad \text{with} \quad u(\pm 1) = 0 \,?$$

$\Rightarrow$ Look for an operator L^{-1} that maps any $f \in L^2(-1,1)$ to the soln. u of $(*)$.

Matrix Inverse Review

$\overset{n\times n \text{ invertible}}{A}x = b \quad \Rightarrow$ If A is diagonalizable

$\Downarrow \qquad A = V\Lambda V^{-1} \Rightarrow A^{-1} = V\Lambda^{-1}V^{-1}$

$x = A^{-1}b \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \Lambda^{-1} = \begin{bmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & 1/\lambda_n \end{bmatrix}$

Construct A columnwise: $\quad Au_1 = e_1, \ \ldots, \ Au_n = e_n$

where $e_1 = (1, 0, \ldots, 0)^T, \ldots, e_n = (0, \ldots, 0, 1)^T$, then

$$A^{-1} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$$ columns of A^{-1} are $u_1, \ldots, u_n$

$$x = A^{-1} b = \underbrace{\sum_{k=1}^{n} b_k u_k}_{\text{solution to } Ax=b \text{ is superposition}}$$

Operator Inverse

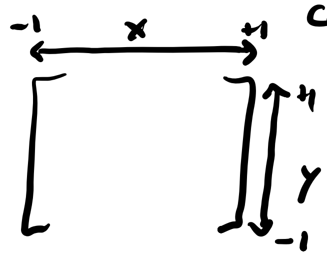
Idea: construct L^{-1} by solving, for each $y \in (-1, 1)$

$$[L u_y](x) = \delta(x-y), \quad \text{where} \quad u_y(\pm 1) = 0.$$

$\delta(x-y)$ is analogous to $e_{kj} = \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$

$$\int_{-1}^{1} \delta(x-y) f(y)\, dy = f(x) \quad \Longleftrightarrow \quad e_k^T v = \sum_{j=1}^{n} \delta_{kj} v_j = v_k$$

k^{th} component of v

Green's function: $G(x,y) = u_y(x) =$ 

$$u(x) = \int_{-1}^{1} G(x,y) f(y)\, dy \quad = \text{superposition}$$

Check that $[Lu](x) = f(x)$ and $u(\pm 1) = 0$

$$\Rightarrow \quad [Lu](x) = L\int_{-1}^{1} G(x,y) f(y)\,dy = \int_{-1}^{1} L u_y(x) f(y)\,dy$$

$$= \int_{-1}^{1} \delta(x-y) f(y)\,dy = f(x) \quad ✓$$

$$\Rightarrow \quad u(\pm 1) = \int_{-1}^{1} G(\pm 1, y) f(y)\,dy = \int_{-1}^{1} \overset{=0}{\cancel{u_y(\pm 1)}}\, f(y)\,dy = 0 \quad ✓$$

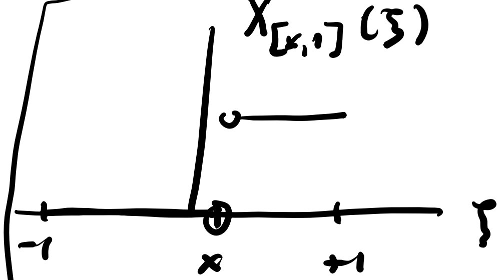
Example

$$-u_y''(x) = \delta(x-y) \quad \text{s.t.} \quad u(\pm 1) = 0$$

$$-\int_x^1 u_y''(\xi)\,d\xi = \int_x^1 \delta(\xi - y)\,d\xi$$

$\downarrow$

$$-u_y'(1) + u_y'(x) = \chi_{(-1,y)}(x)$$



$$\int_x^1 \delta(\xi - y)\,d\xi$$

$$= \int_{-1}^{1} \chi_{[x,1]}(\xi)\,\delta(\xi - y)\,dy$$

$$= \begin{cases} 1 & x<y \\ 0 & y<x \end{cases}$$

$$= \chi_{(-1,y)}(x)$$

$$\int_{-1}^{x} \left(-u_y'(1) + u_y'(\xi)\right) d\xi = \int_{-1}^{x} \chi_{(-1,y)}(\xi)\,d\xi$$

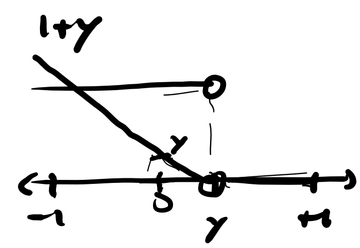
$\downarrow$

(1) $$-u_y'(1)(x+1) + u_y(x) - \overset{B.C.=0}{\cancel{u_y(-1)}} = \begin{cases} x+1 & x<y \\ y+1 & y<x \end{cases}$$

$$\int_x^1 \left(-u_y'(1) + u_y'(\xi)\right) d\xi = \int_x^1 \chi_{(-1,y)}(\xi)\,d\xi$$

$\downarrow$

(2) $$-u_y'(1)(1-x) + \overset{B.C.=0}{\cancel{u_y(1)}} - u_y(x) = \begin{cases} y-x & x<y \\ 0 & y<x \end{cases}$$



Add (1) + (2) to solve for unknown $u_y'(1)$:

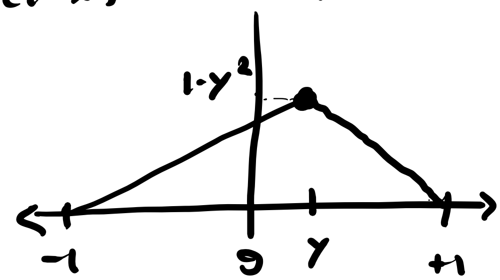
$$-2u_y'(1) = \begin{cases} \cancel{x}+1+y-\cancel{x} & x<y \\ y+1 & y<x \end{cases}$$

$$= 1+y$$

solve $\Rightarrow$ $u_y'(1) = -\frac{1}{2}(1+y)$

from (2) $\Rightarrow$ $-u_y(x) + \frac{1}{2}(1+y)(1-x) = \begin{cases} y-x & x<y \\ 0 & y<x \end{cases}$

$$G(x,y) = u_y(x) = \begin{cases} x-y+\frac{1}{2}(1+y)(1-x) & x<y \\ \frac{1}{2}(1+y)(1-x) & y<x \end{cases}$$



Note that $G(\pm 1, y) = 0$

and $-\partial_x^2 G(x,y) = 0$ $(y \neq x)$

as required