Diagonalizing Diff. Ops

Idea: choose an orthonormal basis of functions in which differentiation (or differential operator) is diagonal.

$$u(x) = \sum_k \underbrace{u_k}_{\text{coordinates of } u} e_k(x) \quad \overset{Lu=f}{\Longrightarrow} \quad f(x) = \sum_k \underbrace{\lambda_k u_k}_{\substack{\text{coordinates of} \\ f = Lu}} e_k(x)$$

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}$$

Solving $Lu = f$ becomes easy!

$\Rightarrow$ $f_k = \langle e_k, f \rangle$ b/c $e_1, e_2, \ldots$ orthonormal

IFF L^{-1} exists

$\Rightarrow$ $u_k = f_k / \lambda_k$ if $\lambda_k \neq 0$

$\Rightarrow$ $u(x) = \sum_k u_k e_k(x)$

To find such a special basis $e_1, e_2, \ldots,$
we need to solve the eigenvalue problem:

$$Lu = \lambda u$$

Warning! Not every differential operator has an orthonormal basis of eigenfunctions.

Fourier's Basis

Fourier's basis diagonalizes $\frac{d}{dx}$ and plays an important role in the analysis & numerical solution of many PDEs, like Poisson, Heat, Wave, ...

$$\frac{d}{dx} u(x) = \lambda u(x) \qquad x \in [-1, 1]$$

Solutions: $u(x) = C e^{\lambda x}$ for <u>any</u> complex λ

any scalar multiple is also eigfun.

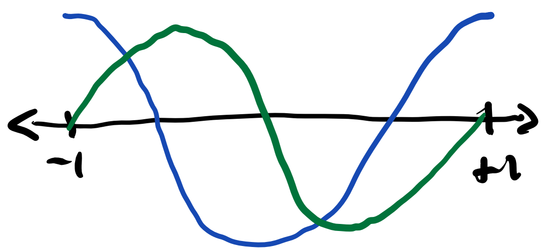
$\Rightarrow$ Too <u>many</u> eigenfunctions/values!

$\Rightarrow$ Impose periodic B.C.'s $\quad u(-1) = u(1)$

periodic solns: $u(x) = C e^{i\omega x}$ ← real #

$\cos\omega x + i\sin\omega x$

s.t. full wavelength "fit" in $[-1,1]$



$\omega = k\pi$

↑ integer = "how many wavelengths"

Fourier Basis: $e_k(x) = \frac{1}{\sqrt{2}} e^{i\pi k x}$ $\quad k = 0, \pm 1, \pm 2, \ldots$

We can check that this basis is orthonormal:

$k=j$

$$\|e_k\| = \sqrt{\langle e_k, e_k\rangle} = \left[\int_{-1}^{1} \frac{1}{\sqrt{2}} |e^{i\pi k x}|^2 dx\right]^{1/2}$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos^2 \pi k x + \sin^2 \pi k x)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} dx = 1$$ ✓ normalized

$k \neq j$

$$\langle e_k, e_j\rangle = \int_{-1}^{1} \overline{e_k(x)}\, e_j(x)\, dx$$

$$= \int_{-1}^{1} e^{-i\pi k x} e^{i\pi j x} dx = \int_{-1}^{1} e^{i\pi(j-k)x} dx$$

integrate sin or cos over full period = 0

$$= \int_{-1}^{1} \cos\pi(j-k)x\, dx + i\int_{-1}^{1} \sin\pi(j-k)x\, dx = 0$$ ✓ orthogonal

The Fourier Basis has $n=\infty$! But, it is complete in the sense that every function with $\|f\| < \infty$ is uniquely identified by it's Fourier coefficients $f_k = \langle e_k, f\rangle$ and that

$$\left\| f - \sum_{k=-N}^{N} f_k e_k \right\| \to 0 \quad \text{as} \quad N \to \infty.$$

Note: Convergence in $\|\cdot\|$ does not mean

$$\sum_{k=-N}^{N} f_k e_k(x) \to f(x) \text{ \underline{for every} } x \in [-1,1]!$$

But, this is true for all suitably <u>smooth functions</u> with $\|f\| < \infty$, and we will be working primarily w/ these smooth functions in 18.303.

Note that the basis $\{e_k\}_{k=-\infty}^{+\infty}$ also diagonalizes every "power" of $\frac{d}{dx}$, e.g.,

$$\frac{d}{dx} e_k = \underbrace{i\pi k}_{\lambda_k} e_k, \quad \frac{d^2}{dx^2} e_k = \underbrace{(i\pi k)^2}_{= -\pi^2 k^2} e_k, \quad \ldots, \quad \frac{d^n}{dx^n} e_k = (i\pi k)^n e_k, \ldots$$

eigenvalues are purely imaginary

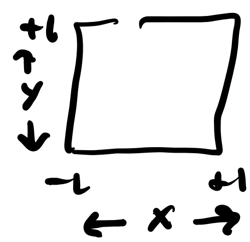
eigenvalues are real and ≤ 0

"skew-adjoint"

"self-adjoint" + "negative semidefinite"

Poisson's Eq.

Find $u(x,t)$ s.t. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$



subject to auxillary conditions.

First, what auxillary conditions ensure unique soln?

Nullspace

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} = 0$$ "Laplace's Equation"

Solution's are analytic functions, which are very smooth and determined by their boundary values. For a unique soln, we therefore need to specify values of $u(x,y)$ when $x, y = \pm 1$.

=> This fixes a unique solution to $\Delta u_n = 0$

=> Full solution is then $u = u_n + u_p$

in null-space particular soln

Eigendecomp

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda$$

Solve by seperation of variables: $u(x,y) = X(x)\,Y(y)$

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{\text{function of } x} + \underbrace{\frac{1}{Y(x)}\frac{\partial^2 Y(y)}{\partial y^2}}_{\text{function of } y} = \lambda$$

Can only be true if $\lambda_1 + \lambda_2 = \lambda$ and

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \lambda_1 \quad \text{and} \quad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \lambda_2 .$$

i.e.,
$$\frac{\partial^2 X(x)}{\partial x^2} = \lambda_1 X(x) \quad \text{and} \quad \frac{\partial^2 Y(y)}{\partial y^2} = \lambda_2 Y(y).$$

The solutions are just eigenfunctions of 2nd derivative operator with eigenvalues!

eigenfunctions: $$e_{k_1,k_2}(x,y) = \underbrace{\left[\frac{1}{\sqrt{2}} e^{i\pi k_1 x}\right]}_{X_{k_1}(x)}\underbrace{\left[\frac{1}{\sqrt{2}} e^{i\pi k_2 y}\right]}_{Y_{k_1}(y)}$$
$$= \frac{1}{2} e^{i\pi(k_1 x + k_2 x)}$$

eigenvalues: $$\lambda_{k_1,k_2} = \underbrace{-\pi^2 k_1^2}_{\lambda_{k_1}} \underbrace{- \pi^2 k_2^2}_{\lambda_{k_2}}$$

To solve Poisson's eqn. w/ framework on pg. 1

we just need to calculate

$$f_{k_1,k_2} = \langle e_{k_1,k_2}, f \rangle = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\, e^{-i\pi(k_1 x + k_2 x)}\, dx dy$$

$\Rightarrow$ Fourier Coeffs of $f(x,y)$.

We also need to consider boundary conditions:

A) If periodic boundary conditions,

$$u(x) = \sum_{k_1,k_2} f_{k_1,k_2}\, e_{k_1,k_2}(x,y)$$

already satisfies them b/c $e_{k_1,k_2}(x,y)$ and derivatives are already satisfy periodic boundary conditions

B) Alternatively, we can enforce the boundary conditions via auxillary conditions on the Fourier Coeffs u_{k_1,k_2} of $u(x,y)$.

More on B.C.'s next-time ...

Adjoints

The diagonalization of $\frac{d}{dx}, \frac{d^2}{dx^2}, \dots$ by orthogonal basis is a special case of the Spectral Theorems for skew-adjoint and self-adjoint operators on Hilbert Spaces of functions.

The adjoint of a differential operator satisfies

$$\langle f, Lg \rangle = \langle \overset{\text{"adjoint" of } L}{L^*} f, g \rangle$$

E.g.

$L = \frac{d}{dx}$ $\quad$ $\langle f, Lg \rangle = \int_{-1}^{1} f(x) \frac{dg}{dx} dx$

integrate-by-parts $\quad$ $= f(1)g(1) - f(-1)g(-1) - \underbrace{\int_{-1}^{1} \frac{df}{dx} g(x)\, dx}_{= \langle Lf, g \rangle}$

If L acts on space of periodic functions with $f(1) = f(-1)$, then $f(1)g(1) - f(-1)g(-1) = 0$, provided that L also acts on space of periodic functions so that $g(-1) = g(1)$. In this case

$$\langle f, Lg \rangle = - \langle Lf, g \rangle$$

and we say that L is skew-adjoint.

E.g.:

Similarly, consider $L = \frac{d^2}{dx^2}$ with periodic boundary conditions so that L acts on functions with

$$f(-1) = f(1) \quad \text{and} \quad \frac{df}{dx}(-1) = \frac{df}{dx}(1)$$

Then we calculate that

$$\langle f, Lg \rangle = \int_{-1}^{1} f \frac{d^2 g}{dx^2} \, dx$$

$$= \underbrace{f(1) \frac{dg}{dx}(1) - f(-1) \frac{dg}{dx}(-1)}_{=0}$$

2x Integrate-by-parts

$$- \int_{-1}^{1} \frac{df}{dx} \frac{dg}{dx} \, dx$$

$$= \frac{df}{dx}(1) g(1) - \frac{df}{dx}(-1) g(-1) + \int_{-1}^{1} \frac{d^2 f}{dx^2} g \, dx$$

$$= \langle Lf, g \rangle$$

In this case $\langle f, Lg \rangle = \langle Lf, g \rangle$ and we say that $L = \frac{d^2}{dx^2}$ is self-adjoint.

$\Rightarrow$ Note that auxillary/boundary conditions play a key role in the invertibility, diagonalizability, and in the adjoint behavior of differential operators.

$\Rightarrow$ The spectrum (including eigenvalues) of skew-adjoint operators is purely imaginary.

$\Rightarrow$ The spectrum (including eigenvalues) of self-adjoint operators is purely real.

$\Rightarrow$ Unlike matrices, skew-symmetric & self-adjoint diff ops are not always diagonalized by an orthonormal basis of eigenfunctions. For a counterexample consider $\frac{d}{dx}$ on $L^2(\mathbb{R})$ (Fourier Transform).