Massachusetts Institute of Technology

Inhomogeneous PDEs revisited

The inhomogeneous heat and wave equations

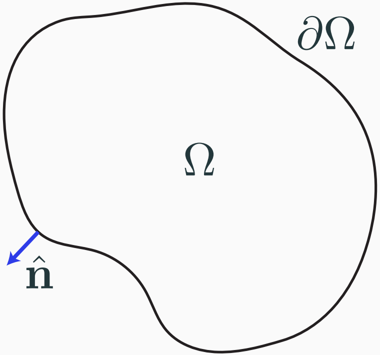
18.303 Linear Partial Differential Equations: Analysis and Numerics

Basic Notions

Boundary value problems look basically like this:

$$\mathcal{L}u(\mathrm{x}) = f(\mathrm{x}),\ \mathrm{x} \in \omega;$$
$$\mathcal{G}u(\mathrm{x}) = g(\mathrm{x}),\ \mathrm{x} \in \partial\omega.$$

Now, if $\mathbf{f} = 0$, we say that the boundary value problem (the differential equation) is homogeneous. Otherwise the problem is said to be inhomogeneous.



An example of a 2d domain and its boundary.

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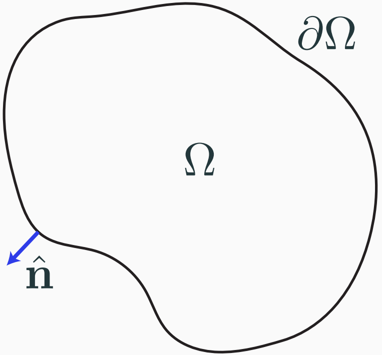

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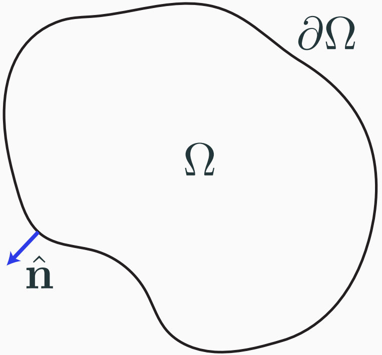
If the differential operator is $\hat{\mathsf{n}} \cdot \nabla = \frac{\partial}{\partial \hat{\mathsf{n}}}$ the boundary condition is called Neumann type or flux boundary condition. This denotes the derivative of function u in the normal direction of the boundary given by the normal vector $\hat{\mathsf{n}}(\mathsf{x})$ (note that generally it depends on the point $x \in \partial\omega$).



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Now, consider the function $u(\mathsf{x}) = v(\mathsf{x}) + u_0(\mathsf{x})$. We have

$$\mathcal{L}u(\mathsf{x}) = \mathcal{L}v(\mathsf{x}) + \underbrace{\mathcal{L}u_0(\mathsf{x})}_{=0} = f(\mathsf{x}).$$

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We notice that it suffices to require that

$$\mathcal{G}v(\mathsf{x}) = 0, \; \mathsf{x} \in \partial\omega$$

because this will make sure that $\mathcal{G}u(\mathsf{x}) = g(\mathsf{x})$, when $\mathsf{x} \in \partial\omega$.

Now we have a boundary value problem

$$\mathcal{L}v(\mathbf{x}) = f(\mathbf{x}), \ \mathbf{x} \in \omega;$$
$$\mathcal{G}v(\mathbf{x}) = 0, \ \mathbf{x} \in \partial\omega.$$

This shows us that the general solution can be sought as
the solution to the homogeneous problem with inhomogeneous boundaries + the solution to the inhomogeneous problem with homogeneous boundaries.

Heat equation

We have

$$\frac{\partial}{\partial t} u(t, \mathsf{x}) = \Delta u(t, \mathsf{x}) + f(t, \mathsf{x})$$

with some spatial boundary conditions for all t and in addition we are given

$$u(0, \mathsf{x}) = u^{(0)}(\mathsf{x}).$$

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We express u in the eigenbasis $\{\phi_\mathsf{n}\}$ of the Laplacian Δ giving

$$u(t, \mathsf{x}) = \sum_\mathsf{n} \hat{u}_\mathsf{n}(t) \phi_\mathsf{n}(\mathsf{x}).$$

Similarly,

$$f(t, \mathsf{x}) = \sum_\mathsf{n} \hat{f}_\mathsf{n}(t) \phi_\mathsf{n}(\mathsf{x}).$$

Because the basis is linearly independent, we get

$$\frac{\partial}{\partial t}\hat{u}_{\mathsf{n}}(t) = -\lambda_{\mathsf{n}}^2 \hat{u}_{\mathsf{n}}(t) + \hat{f}_{\mathsf{n}}(t).$$

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Multiplying both sides by $\exp(\lambda_{\mathsf{n}}^2 t)$ and reorganizing gives

$$\frac{\partial}{\partial t}\hat{u}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t} + \lambda_{\mathsf{n}}^2\hat{u}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t} = \hat{f}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t}.$$

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$$\frac{\partial}{\partial t}\left(\hat{u}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t}\right) = \hat{f}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t},$$

which we can integrate from 0 to t giving

$$\hat{u}_{\mathsf{n}}(t)e^{\lambda_{\mathsf{n}}^2 t} - \hat{u}_{\mathsf{n}}(0) = \int_0^t \hat{f}_{\mathsf{n}}(t')e^{\lambda_{\mathsf{n}}^2 t'}\,\mathrm{d}t'$$

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What if f is independent of time?

This gives

$$
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&= \hat{u}_{\mathsf{n}}^{(0)} e^{-\lambda_{\mathsf{n}}^2 t} + \frac{\hat{f}_{\mathsf{n}}}{\lambda_{\mathsf{n}}^2} \left(e^{-\lambda_{\mathsf{n}}^2 (t-t')} \right)_{t'=0}^{t} \\
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What if $t \to \infty$?

We obtain the solution to the Poisson equation

$$\hat{u}_{\mathsf{n}}(t) = \frac{\hat{f}_{\mathsf{n}}}{\lambda_{\mathsf{n}}^2}.$$

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Just assuming that we will reach a steady state with $\partial_t u = 0$ gives

$$-\Delta u(t,\mathsf{x}) = f(t,\mathsf{x}).$$

Wave equation

Now

$$\frac{\partial^2}{\partial t^2} u(t, \mathsf{x}) = \Delta u(t, \mathsf{x}) + f(t, \mathsf{x})$$

with some appropriate boundary conditions and initial conditions

$$\begin{aligned} u(0, \mathsf{x}) &= u^{(0)}(\mathsf{x}), \\ \left(\frac{\partial}{\partial t} u(t, \mathsf{x}) \right)_{t=0} &= v^{(0)}(\mathsf{x}). \end{aligned}$$

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We proceed in a same way by writing the equation for the coefficients $\hat{u}_{\mathsf{n}}(t)$ as

$$\frac{\partial^2}{\partial t^2} \hat{u}_{\mathsf{n}}(t) = -\lambda_{\mathsf{n}}^2 \hat{u}_{\mathsf{n}}(t) + \hat{f}_{\mathsf{n}}(t).$$

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This is an ordinary differential equation (ODE) but it can be a bit harder to solve.

Recall that the complete solution is the solution to the homogeneous problem with the necessary initial (boundary) conditions + the solution to the inhomogeneous equation with zero boundaries.

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We have already covered the solution to the homogeneous problem (here we write $h(t, \mathsf{x})$) and the coefficients are given by

$$\hat{h}_{\mathsf{n}}(t) = \alpha_{\mathsf{n}} \cos(\lambda_{\mathsf{n}} t) + \beta_{\mathsf{n}} \sin(\lambda_{\mathsf{n}} t),$$

where the coefficients α_{n} and β_{n} can be solved from the initial conditions. (For complex equations we have $\exp(\pm i\lambda_{\mathsf{n}} t)$ instead of $\sin$ and $\cos$).

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We can seek the particular solution to the inhomogeneous problem for the coefficients in the form

$$\hat{p}_{\mathrm{n}}(t) = \hat{\xi}_{\mathrm{n}}(t)\hat{h}_{\mathrm{n}}(t).$$

Substituting this into the inhomogeneous equation for the coefficients gives

$$\begin{aligned}\hat{f}_{\mathsf{n}}(t) &= \frac{\partial^2}{\partial t^2}\left(\hat{\xi}_{\mathsf{n}}(t)\hat{h}_{\mathsf{n}}(t)\right) + \lambda_{\mathsf{n}}^2\hat{\xi}_{\mathsf{n}}(t)\hat{h}_{\mathsf{n}}(t) \\ &= \hat{\xi}_{\mathsf{n}}''(t)\hat{h}_{\mathsf{n}} + 2\hat{\xi}_{\mathsf{n}}'(t)\hat{h}_{\mathsf{n}}'(t) + \hat{h}_{\mathsf{n}}''(t)\hat{\xi}_{\mathsf{n}}(t) + \lambda_{\mathsf{n}}(t)^2\hat{h}_{\mathsf{n}}(t)\hat{\xi}_{\mathsf{n}}(t).\end{aligned}$$

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The last two terms solve the homogeneous problem

$$\hat{\xi}_{\mathsf{n}}(t)\underbrace{\left(\hat{h}_{\mathsf{n}}''(t) + \lambda_{\mathsf{n}}(t)^2\hat{h}_{\mathsf{n}}(t)\right)}_{=0}.$$

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Multiplying by $\hat{h}_n$ gives

$$\hat{\xi}_n''(t)\hat{h}_n(t)^2 + 2\hat{\xi}_n'(t)\hat{h}_n'(t)\hat{h}_n(t) = \hat{f}_n(t)\hat{h}_n(t),$$

which we can write as

$$\frac{\partial}{\partial t}\left(\hat{\xi}_n'(t)\hat{h}_n(t)^2\right) = \hat{f}_n(t)\hat{h}_n(t).$$

Now this can be integrated from 0 to t giving

$$\hat{\xi}'_{\mathsf{n}}(t)\hat{h}_{\mathsf{n}}(t)^2 - \hat{\xi}'_{\mathsf{n}}(0)\hat{h}_{\mathsf{n}}(0)^2 = \int_0^t \hat{f}_{\mathsf{n}}(t')\hat{h}_{\mathsf{n}}(t')\mathrm{d}t'.$$

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and the complete solution is

$$\hat{u}_{\mathsf{n}}(t) = \hat{h}_{\mathsf{n}}(t) + \hat{p}_{\mathsf{n}}(t).$$

The other way

That all seemed somewhat complicated so let's do something else. For the coefficients of the particular solution we have

$$\frac{\partial^2}{\partial t^2}\hat{p}_n(t) + \lambda_n^2 \hat{p}_n(t) = \hat{f}_n(t).$$

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It turns out that the calculation of the Fourier coefficients can be extended to infinity and the function $\hat{p}_{\mathsf{n}}$ can be expressed as

$$\hat{p}_{\mathsf{n}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{P}_{\mathsf{n}}(\omega) e^{i\omega t} \mathrm{d}\omega.$$

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$$\hat{P}_{\mathsf{n}} = \int_{-\infty}^{\infty} \hat{p}_{\mathsf{n}}(t)e^{-i\omega t}\mathrm{d}t.$$

This is the actual Fourier transform and here we have also defined the inverse transform.

We will write the coefficients using the Fourier expression giving the equation

$$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega+\int_{-\infty}^{\infty}\lambda_{\mathsf{n}}^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega=\int_{-\infty}^{\infty}\hat{F}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega.$$

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Taking the derivative inside the integral gives

$$-\int_{-\infty}^{\infty}\omega^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega+\int_{-\infty}^{\infty}\lambda_{\mathsf{n}}^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega=\int_{-\infty}^{\infty}\hat{F}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega.$$

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$$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega+\int_{-\infty}^{\infty}\lambda_{\mathsf{n}}^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega=\int_{-\infty}^{\infty}\hat{F}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega.$$

Taking the derivative inside the integral gives

$$-\int_{-\infty}^{\infty}\omega^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega+\int_{-\infty}^{\infty}\lambda_{\mathsf{n}}^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega=\int_{-\infty}^{\infty}\hat{F}_{\mathsf{n}}(\omega)e^{i\omega t}\mathrm{d}\omega.$$

It turns out that the functions $\exp(i\omega t)$ form an orthogonal basis implying that the coefficients have to be the same. This gives

$$-\omega^2\hat{P}_{\mathsf{n}}(\omega)+\lambda_{\mathsf{n}}^2\hat{P}_{\mathsf{n}}(\omega)e^{i\omega t}=\hat{F}_{\mathsf{n}}(\omega).$$

Solving for $\hat{P}_{\mathsf{n}}$ gives

$$\hat{P}_{\mathsf{n}}(\omega) = \frac{\hat{F}_{\mathsf{n}}(\omega)}{\lambda_{\mathsf{n}}^2 - \omega^2}.$$

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That is something that is called a resonance but that will be a story for another day...