Massachusetts
Institute of
Technology

Some important basis functions

We also solve some cool problems

18.303 Linear Partial Differential Equations: Analysis and Numerics

Laplace equation in spherical coordinates

We solve for $\Delta f = 0$ in spherical coordinates.

Laplace equation in spherical coordinates

We solve for $\Delta f = 0$ in spherical coordinates.

We have the coordinate chart between the Cartesian and spherical coordinates:

$$
\begin{aligned}
x &= r\sin(\theta)\cos(\varphi)\\
y &= r\sin(\theta)\sin(\varphi)\\
z &= r\cos(\theta),
\end{aligned}
$$

where $r \in \mathbb{R}_+$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$.

Laplace equation in spherical coordinates

We solve for $\Delta f = 0$ in spherical coordinates.

We have the coordinate chart between the Cartesian and spherical coordinates:

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi) \\ y &= r\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta), \end{aligned}$$

where $r \in \mathbb{R}_+$, $\theta \in [0,\pi]$, and $\varphi \in [0, 2\pi)$.

It requires a bit of work to calculate the change of coordinates but ultimately we get

$$\Delta f(r,\theta,\varphi) = \left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right) + \frac{1}{r^2\sin^2(\theta)}\partial_\varphi^2\right] f(r,\theta,\varphi) = 0.$$

Laplace equation in spherical coordinates

We solve for $\Delta f = 0$ in spherical coordinates.

We have the coordinate chart between the Cartesian and spherical coordinates:

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi) \\ y &= r\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta), \end{aligned}$$

where $r \in \mathbb{R}_+$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$.

It requires a bit of work to calculate the change of coordinates but ultimately we get

$$\Delta f(r,\theta,\varphi) = \left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right) + \frac{1}{r^2\sin^2(\theta)}\partial_\varphi^2\right] f(r,\theta,\varphi) = 0.$$

We use separation by variables and write $f(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$.

Multiplying by r^2 and reorganizing gives

$$\frac{\partial_r\left(r^2 R'(r)\right)}{R(r)} + \frac{\sin^{-1}(\theta)\partial_\theta\left(\sin(\theta)\Theta'(\theta)\right)}{\Theta(\theta)} + \frac{1}{\sin^2(\theta)}\frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0.$$

Multiplying by r^2 and reorganizing gives

$$\frac{\partial_r \left(r^2 R'(r)\right)}{R(r)} + \frac{\sin^{-1}(\theta)\partial_\theta \left(\sin(\theta)\Theta'(\theta)\right)}{\Theta(\theta)} + \frac{1}{\sin^2(\theta)} \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0.$$

This has to hold for all r, θ, and φ so some parts need to be constants. Probably the easiest is

$$\Phi''_m(\varphi) = m^2 \Phi_m(\varphi).$$

Multiplying by r^2 and reorganizing gives

$$\frac{\partial_r\left(r^2 R'(r)\right)}{R(r)} + \frac{\sin^{-1}(\theta)\partial_\theta\left(\sin(\theta)\Theta'(\theta)\right)}{\Theta(\theta)} + \frac{1}{\sin^2(\theta)}\frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0.$$

This has to hold for all r, θ, and φ so some parts need to be constants. Probably the easiest is

$$\Phi_m''(\varphi) = m^2\Phi_m(\varphi).$$

This gives the azimuthal part of the function and Φ is periodic. It is solved by

$$\Phi_m(\varphi) = e^{im\varphi},$$

where m is an integer.

Now we have

$$\partial_r \left(r^2 \partial_r R_\ell(r)\right) = \lambda_\ell R_\ell(r),$$
$$\frac{1}{\sin(\theta)}\partial_\theta \left(\sin(\theta)\partial_\theta \Theta_\ell^m(\theta)\right) - \frac{m^2}{\sin^2(\theta)}\Theta_\ell^m = -\lambda_\ell \Theta_\ell^m(\theta).$$

Now we have

$$\partial_r \left(r^2 \partial_r R_\ell(r)\right) = \lambda_\ell R_\ell(r),$$

$$\frac{1}{\sin(\theta)}\partial_\theta \left(\sin(\theta)\partial_\theta \Theta_\ell^m(\theta)\right) - \frac{m^2}{\sin^2(\theta)}\Theta_\ell^m = -\lambda_\ell \Theta_\ell^m(\theta).$$

Since $\theta \in [0, \pi]$, we can make a change of variables $t = \cos(\theta) \in [-1, 1]$ ($\cos$ is a bijection on this interval). We say $\Theta_\ell^m(\theta) = P_\ell^m(\cos(\theta)) = P_\ell^m(t)$.

Now we have

$$\begin{aligned}\partial_r\left(r^2\partial_r R_\ell(r)\right) &= \lambda_\ell R_\ell(r),\\ \frac{1}{\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\Theta_\ell^m(\theta)\right) - \frac{m^2}{\sin^2(\theta)}\Theta_\ell^m &= -\lambda_\ell\Theta_\ell^m(\theta).\end{aligned}$$

Since $\theta \in [0, \pi]$, we can make a change of variables $t = \cos(\theta) \in [-1, 1]$ ($\cos$ is a bijection on this interval). We say $\Theta_\ell^m(\theta) = P_\ell^m(\cos(\theta)) = P_\ell^m(t)$.

The derivative with respect to θ transforms as

$$\partial_\theta \to \frac{\partial t}{\partial\theta}\partial_t = -\sin(\theta)\partial_t.$$

Now we have

$$\partial_r \left(r^2 \partial_r R_\ell(r)\right) = \lambda_\ell R_\ell(r),$$
$$\frac{1}{\sin(\theta)}\partial_\theta \left(\sin(\theta)\partial_\theta \Theta_\ell^m(\theta)\right) - \frac{m^2}{\sin^2(\theta)}\Theta_\ell^m = -\lambda_\ell \Theta_\ell^m(\theta).$$

Since $\theta \in [0,\pi]$, we can make a change of variables $t = \cos(\theta) \in [-1,1]$ ($\cos$ is a bijection on this interval). We say $\Theta_\ell^m(\theta) = P_\ell^m(\cos(\theta)) = P_\ell^m(t)$.

The derivative with respect to θ transforms as

$$\partial_\theta \to \frac{\partial t}{\partial \theta}\partial_t = -\sin(\theta)\partial_t.$$

Now the equation for the latitudinal part becomes

$$\partial_t \left(\sin^2(\theta)\partial_t P_\ell^m\right) - \frac{m^2}{\sin^2(\theta)}P_\ell^m = -\lambda_\ell P_\ell^m(t).$$

Writing $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - t^2$ gives

$$\partial_t \left((1-t^2)\partial_t P_\ell^m\right) - \frac{m^2}{1-t^2} P_\ell^m = -\lambda_\ell P_\ell^m(t).$$

Writing $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - t^2$ gives

$$\partial_t \left((1 - t^2) \partial_t P_\ell^m \right) - \frac{m^2}{1 - t^2} P_\ell^m = -\lambda_\ell P_\ell^m(t).$$

The only way $P_\ell^m(\pm 1)$ is finite is if $\lambda_\ell = \ell(\ell + 1)$, where ℓ is a non-negative integer (this is a long story why this happens). Because the differential operator to the left is negative semi-definite, we have to have $|m| \leq \ell$.

Writing $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - t^2$ gives

$$\partial_t\left((1-t^2)\partial_t P_\ell^m\right) - \frac{m^2}{1-t^2}P_\ell^m = -\lambda_\ell P_\ell^m(t).$$

The only way $P_\ell^m(\pm 1)$ is finite is if $\lambda_\ell = \ell(\ell+1)$, where ℓ is a non-negative integer (this is a long story why this happens). Because the differential operator to the left is negative semi-definite, we have to have $|m| \leq \ell$.

The solution to this DE is given by the Associated Legendre polynomials. The usual Legendre polynomials P_ℓ are given for $m = 0$. Now $\Theta_\ell^m(\theta) = P_\ell^m(\cos(\theta))$.

Writing $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - t^2$ gives

$$\partial_t \left((1-t^2)\partial_t P_\ell^m\right) - \frac{m^2}{1-t^2} P_\ell^m = -\lambda_\ell P_\ell^m(t).$$

The only way $P_\ell^m(\pm 1)$ is finite is if $\lambda_\ell = \ell(\ell+1)$, where ℓ is a non-negative integer (this is a long story why this happens). Because the differential operator to the left is negative semi-definite, we have to have $|m| \leq \ell$.

The solution to this DE is given by the Associated Legendre polynomials. The usual Legendre polynomials P_ℓ are given for $m = 0$. Now $\Theta_\ell^m(\theta) = P_\ell^m(\cos(\theta))$.

The polynomials P_ℓ are degree ℓ and are uniquely defined by $P_\ell(1) = 1$ and the orthogonality condition

$$\int_{-1}^{1} P_\ell(t) P_{\ell'}(t) \mathrm{d}t = 0$$

if $\ell \neq \ell'$. If $\ell = \ell'$, this integral gives $2/(2\ell+1)$.

The associated Legendre polynomials can be calculated from the Legendre polynomials through

$$P_\ell^m(t) = (-1)^m (1-t^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P_\ell(t).$$

The associated Legendre polynomials can be calculated from the Legendre polynomials through

$$P_\ell^m(t) = (-1)^m (1 - t^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P_\ell(t).$$

The (normal) Legendre polynomials can also be obtained as

$$P_\ell(t) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} (t^2 - 1)^\ell.$$

The associated Legendre polynomials can be calculated from the Legendre polynomials through

$$P_\ell^m(t) = (-1)^m (1-t^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P_\ell(t).$$

The (normal) Legendre polynomials can also be obtained as

$$P_\ell(t) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} (t^2-1)^\ell.$$

These polynomials have many nifty properties but we will not spend too much time on that today.

Finally we have the radial part

$$\partial_r(r^2 R_\ell'(r)) = \ell(\ell+1)R_\ell(r).$$

Finally we have the radial part

$$\partial_r(r^2 R'_\ell(r)) = \ell(\ell+1)R_\ell(r).$$

This is solved by an ansatz $R = r^\alpha$ giving

$$\alpha(\alpha+1) = \ell(\ell+1),$$

which implies $\alpha = \ell$ or $\alpha = -\ell - 1$. Now

$$R(r) = A_\ell r^{\ell r} + B_\ell r^{-\ell-1}.$$

Finally we have the radial part

$$\partial_r(r^2 R_\ell'(r)) = \ell(\ell+1)R_\ell(r).$$

This is solved by an ansatz $R = r^\alpha$ giving

$$\alpha(\alpha+1) = \ell(\ell+1),$$

which implies $\alpha = \ell$ or $\alpha = -\ell - 1$. Now

$$R(r) = A_\ell r^{\ell r} + B_\ell r^{-\ell-1}.$$

If we require that R is bounded at the origin, it implies that $B_\ell = 0$. Another usual boundary condition for R is that it is bounded at infinity. In that case $A_\ell = 0$. Alternatively you can add your favorite boundary condition at some fixed r_0 and solve for the coefficients.

Spherical harmonics

The functions

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

are the spherical harmonics that satisfy

$$r^2 \Delta Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

i.e. they are the eigenfunctions of the Laplacian on the surface of a sphere with a fixed radius r.

Spherical harmonics

The functions

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

are the spherical harmonics that satisfy

$$r^2 \Delta Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

i.e. they are the eigenfunctions of the Laplacian on the surface of a sphere with a fixed radius r.

They are also normalized satisfying

$$\int_{\partial\Omega} Y_\ell^m Y_{\ell'}^{m'} \, \mathrm{d}S = \int_0^{2\pi} \int_0^{\pi} \sin(\theta) Y_\ell^m Y_{\ell'}^{m'} \, \mathrm{d}\theta \mathrm{d}\varphi = \delta_\ell^{\ell'} \delta_m^{m'}.$$

Spherical harmonics

The functions

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\varphi}$$

are the spherical harmonics that satisfy

$$r^2\Delta Y_\ell^m = -\ell(\ell+1)Y_\ell^m$$

i.e. they are the eigenfunctions of the Laplacian on the surface of a sphere with a fixed radius r.

They are also normalized satisfying

$$\int_{\partial\Omega} Y_\ell^m Y_{\ell'}^{m'}\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi}\sin(\theta)Y_\ell^m Y_{\ell'}^{m'}\,\mathrm{d}\theta\mathrm{d}\varphi = \delta_\ell^{\ell'}\delta_m^{m'}.$$

They define a perfectly good basis for dealing with differential equations on a sphere.