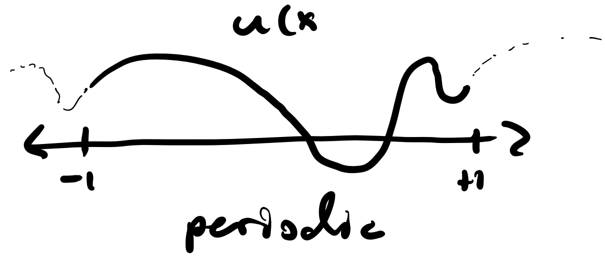
The Wave Eqn. (pt. 2)

$$\partial_t^2 u = c^2 \partial_x^2 u$$

$c>0$, "wave speed"



Initial Condition:

displacement $\Rightarrow u(x,0) = g(x)$

velocity $\Rightarrow \partial_t u(x,0) = h(x)$

Fourier Soln: $\partial_x^2 e_k = \lambda_k e_k$ (Diagonalize)

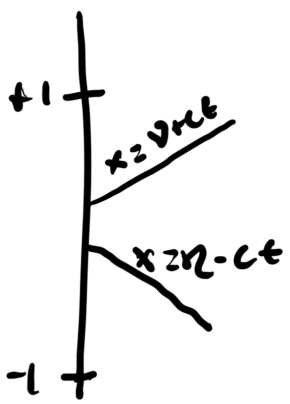
$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \Big[\hat{g}_k \cos(\overbrace{c\pi k t}^{i\sqrt{\lambda_k} t}) + \frac{\hat{h}_k}{c\pi k} \sin(\overbrace{c\pi k t}^{i\sqrt{\lambda_k} t}) \Big] \overbrace{e^{i\pi k x}}^{e_k(x)}$$

$\langle e_k, g \rangle = \hat{g}_k$ $\qquad$ $\frac{\hat{h}_k}{c\pi k} = \frac{\langle e_k, h \rangle}{i\sqrt{\lambda_k}}$

$$= \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_1^{(k)} e^{i\pi k(ct+x)} + \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_2^{(k)} e^{i\pi k(-ct+x)}$$

$$= F_1(\underbrace{x+ct}_{\eta}) + F_2(\underbrace{x-ct}_{\nu})$$

characteristic coordinates select which characteristic curve we are on.



d'Alembert's Formula

Apply initial conditions to $u(x,t) = F_1(\eta) + F_2(\nu)$:

$$u(x,0) = F_1(x) + F_2(x) = g(x)$$

$$\partial_t u(x,0) = cF_1'(x) - cF_2'(x) = h(x)$$

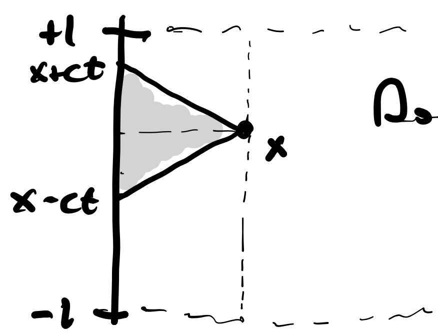
$$\Rightarrow \quad 2F_1'(x) = g'(x) + \frac{1}{c} h(x)$$

$$\Rightarrow \quad F_1(x) = \frac{1}{2} g(x) + \frac{1}{2c}\int_0^x h(x')\,dx' + \text{const.}$$

$$F_2(x) = \frac{1}{2} g(x) - \frac{1}{2c}\int_0^x h(x')\,dx' - \text{const.}$$

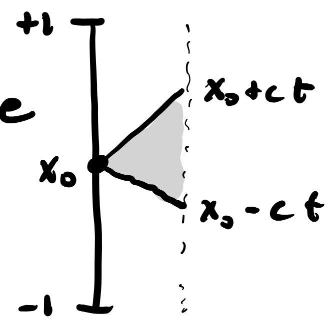
$$\Rightarrow \quad u(x,t) = F_1(x+ct) + F_2(x-ct)$$

$$= \frac{1}{2}\left[g(x+ct) + g(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} h(x')\,dx'\right]$$



Domain of dependence

$\Rightarrow$ causality! $\Rightarrow$



What happens for diff. initial conditions?

$\Rightarrow h = 0$ $\Rightarrow h \neq 0$

As speed $c \geq 0$ increases, slope of $x_0 \pm ct$ increases and displacement/velocity info at x_0 "travels faster" to influence the displacement $u(x,t)$ nearby.

Example 2.16 and 2.17 Textbook

Forced Waves

$$\partial_t^2 u = c^2 \partial_x^2 u + f(x,t)$$

Forcing term

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx'$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,x')\,dx'\,ds$$