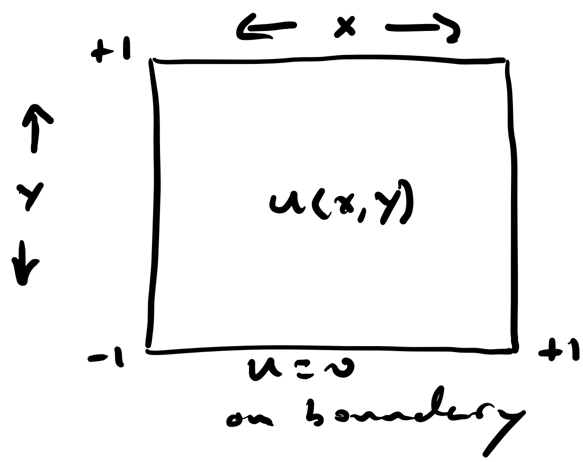
Diagonalizing Diff Ops (Part 2)

Poisson

$$\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\substack{\text{Laplacian} \\ \Delta}} = f(x,y)$$



Diagonalize

$$\Delta u_k = \lambda_k u_k \qquad \text{(eigenvals/funcs)}$$

$\Rightarrow$ solve w/ separation of variables

$$u(x) = X(x)\,Y(y) \quad \Rightarrow \quad \underbrace{\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}}_{=\lambda_x} + \underbrace{\frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2}}_{=\lambda_y} = \lambda \quad \text{const.}$$

$\Rightarrow$ Solve 1D eigenvalue problems

$$\frac{\partial^2 X}{\partial x^2} = \lambda_x X \qquad \text{and} \qquad \frac{\partial^2 Y}{\partial y^2} = \lambda_y Y$$

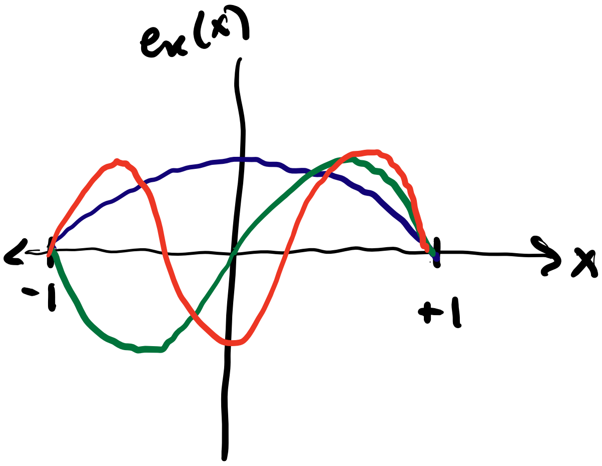
s.t. $X(\pm 1) = 0$ s.t. $Y(\pm 1) = 0$

$\uparrow$ build boundary conditions $\uparrow$
into eigenfunctions

$$u(\pm 1, y) = X(\pm 1)\,Y(y) = 0 = X(x)\,Y(\pm 1) = u(x, \pm 1) \quad \checkmark$$

For both X and Y, need to diagonalize $\frac{d^2}{dx^2}$, but now with zero boundary conditions at ± 1.

$$\frac{d^2}{dx^2} e_k = \lambda_k e_k \quad \text{s.t.} \quad e_k(\pm 1) = 0$$



$$e_k(x) = \begin{cases} \cos \frac{k\pi x}{2} & k = 1, 3, 5, \ldots \\ \sin \frac{k\pi x}{2} & k = 2, 4, 6, \ldots \end{cases}$$

$$\lambda_k = -\left(\frac{k\pi}{2}\right)^2 \qquad k = 1, 2, 3, \ldots$$

With homogeneous Dirichlet B.C.'s $e_k(\pm 1) = 0$, the eigenvalues & eigenfunctions of $-\frac{d^2}{dx^2}$ are not the same as $\frac{d^2}{dx^2}$ with periodic B.C.'s.

$\Rightarrow$ Eigenfunctions/values of Δ w/ $u(\pm 1, y) = u(x, \pm 1) = 0$

$$e_{k_1,k_2} = \begin{cases} \cos\left(\frac{k_1\pi x}{2}\right)\cos\left(\frac{k_2\pi x}{2}\right) & k_1, k_2 \text{ odd} \\ \cos\left(\frac{k_1\pi x}{2}\right)\sin\left(\frac{k_2\pi x}{2}\right) & k_1 \text{ odd}, k_2 \text{ even} \\ \sin\left(\frac{k_1\pi x}{2}\right)\cos\left(\frac{k_2\pi x}{2}\right) & k_1 \text{ even}, k_2 \text{ odd} \\ \sin\left(\frac{k_1\pi x}{2}\right)\sin\left(\frac{k_2\pi x}{2}\right) & k_1, k_2 \text{ even} \end{cases}$$

$$\lambda_{k_1,k_2} = -\left(\frac{k_1\pi}{2}\right)^2 - \left(\frac{k_2\pi}{2}\right)^2$$

Transform

Now calculate coeffs of $f(x,y)$ in $\{e_{k_1,k_2}\}$ basis.

$$f_{k_1,k_2} = \langle e_{k_1,k_2}, f \rangle = \int_{-1}^{+1}\int_{-1}^{+1} e_{k_1}(x) e_{k_2}(x) f(x,y)\, dx\, dy$$

Solve

Then, $u(x,y) = \sum_{k_1,k_2} u_{k_1,k_2} e_{k_1}(x) e_{k_2}(y)$ and

coeffs are just $u_{k_1,k_2} = (\lambda_{k_1,k_2})^{-1} f_{k_1,k_2}$ ✓

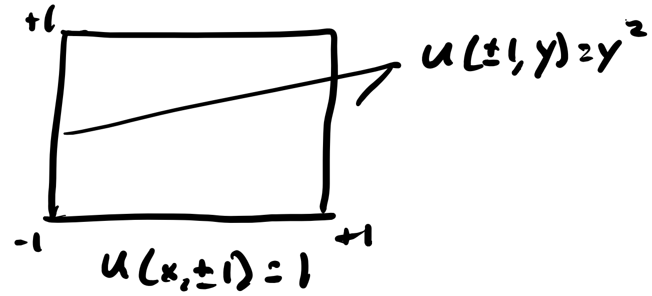
Solution via diagonalization gives us a series representation of the solution, which can be used for further computation & analysis.

Note: we constructed $u(x,y)$ to satisfy b.c.'s b/c each eigenfunction $\{e_{k_1}(x) e_{k_2}(y)\}$ satisfies the b.c.'s, and $u(x,y)$ is a linear combo of the eigenfunctions $\{e_{k_1}(x) e_{k_2}(y)\}$.

What if b.c.'s are non-homogeneous?

E.g., $u(\pm 1, y) = y^2$ and $u(x, \pm 1) = 1$

Inhomogeneous B.C.s



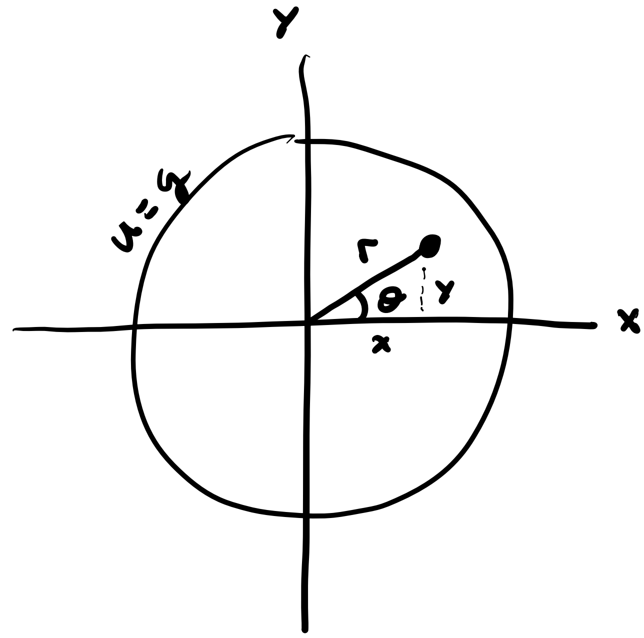
Poisson $\Delta u = f$

$$u = u_h + u_p$$

solution to $\Delta u = 0$ w/ inhom. b.c.'s + solution to $\Delta u = f$ w/ homo. b.c.'s = u satisfies $\Delta u = f$ w/ inhom. b.c.'s!

We can solve for u_p by diagonalizing again. The problem is to solve $\Delta u = 0$ (Laplace) w/ the inhomogeneous b.c.'s: choose the unique function in the nullspace that satisfies the B.C.'s!

Solving Laplace's Eqn.



$$\Delta u = 0$$

s.t. $u\big|_{r=1} = g(\theta)$