The Heat Equation

Thermal Diffusivity > 0

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

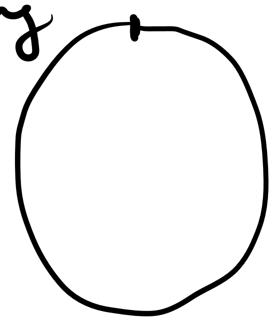
$u(x,t)$

-1 $\qquad$ $+1$

$\partial_x u(-1,t) = \partial_x u(1,t)$
$u(-1,t) = u(1,t)$

$u(x,0) = g(x)$
initial condition

periodic boundary
= "Heated Ring"



$\Rightarrow$ Heat Eqn models temperature profile $u(x,t)$ evolving in time over ring.

$\Rightarrow$ Other geometries & boundary conditions later

Derivation of Heat Eqn.

"Zoom in around x"

$x-h \qquad x \qquad x+h$

1) Conservation Law

$$\frac{d}{dt}\int_{x-h}^{x+h} h(x,t)\,dx = H(x+h) - H(x-h)$$

change in heat $h(x,t)$ in interval = heat flux at boundaries of interval

As $h \to 0$, $$\frac{\partial h}{\partial t} = \frac{\partial H}{\partial x}$$

This is the "infinitesimal form" of the Law.

2) Constitutive Law (material assumption).

$$h(x,t) = \underbrace{\rho\chi}_{\substack{\text{density} \\ \times \text{ specific} \\ \text{heat capacity}}} \overset{\text{temperature}}{u(x,t)}$$

Heat (energy density) is proportional to temperature in material.

3) Fourier's Law of Cooling.

$$H(x,t) = \underset{\text{thermal conductivity}}{k} \frac{\partial u}{\partial x}$$

Heat flux is from hot to cold with strength proportional to local temperature gradient.

Putting 1) - 3) together, we get

$$\frac{\partial u}{\partial t} = \underbrace{\frac{K}{\rho \chi}}_{= \gamma > 0} \frac{\partial^2 u}{\partial x^2}$$

<u>Boundary Conditions</u> (nonperiodic)

Dirichlet $\Rightarrow$ $u(-1,t) = T_1(t)$

"Fixed temperature"

Neumann $\Rightarrow$ $\frac{\partial}{\partial x} u(-1,t) = \mu_1(t)$

"Fixed heat flux"

Robin $\Rightarrow$ $\frac{\partial u}{\partial x}(-1,t) + \alpha(t) u(-1,t) = \tau(t)$

"Heat bath / thermal reservoir"

One such boundary condition prescribed at each end of bar.

The operator exponential

$$\frac{\partial u}{\partial t} = Au \qquad A = \frac{\partial^2}{\partial x^2} + \text{periodic B.C.'s}$$

is a differential op.

By analogy w/ linear ODEs, $u(x,t) = \underbrace{e^{At}}_{\text{what is this?!}} g(x)$?

To compute solution, we diagonalize A:

$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{u}_k(t)\, e^{i\pi kx}$$

↑ Fourier Basis

$$Au = \frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} -(\pi k)^2\, \hat{u}_k(t) e^{i\pi kx}$$

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{u}_k'(t)\, e^{i\pi kx}$$

Equating coeffs of each orthonormal basis function:

$$\left\langle \frac{1}{\sqrt{2}} e^{i\pi kx}, \frac{\partial u}{\partial t} \right\rangle = \left\langle \frac{1}{\sqrt{2}} e^{i\pi kx}, \frac{\partial^2 u}{\partial x^2} \right\rangle$$

$$\Rightarrow \quad \hat{u}_k'(t) = -(\pi k)^2 \hat{u}_k(t)$$

or

$$\frac{d}{dt}\begin{bmatrix} \vdots \\ \hat{u}_{-2} \\ \hat{u}_{-1} \\ \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & & \\ & -4\pi^2 & & & & & \\ & & -\pi^2 & & & & \\ & & & 0 & & & \\ & & & & \pi^2 & & \\ & & & & & 4\pi^2 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}_{-2} \\ \hat{u}_{-1} \\ \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \end{bmatrix}$$

Diagonal linear system of ODE's decouples spatial degrees of freedom.

$$\Rightarrow \quad \hat{u}_k(t) = e^{-(\pi k)^2 t}\, \hat{u}_k(0)$$

$$\Rightarrow \quad u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} e^{-(\pi k)^2 t} e^{i\pi k x}\, \hat{u}_k(0)$$

To satisfy initial condition, we set

$$u(x,0) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{u}_k(0)\, e^{i\pi k x}$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \Rightarrow \quad \hat{u}_k(0) = \hat{g}_k$$

$$g(0) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{g}_k\, e^{i\pi k x}$$

$$\hat{g}_k = \langle \tfrac{1}{\sqrt{2}} e^{i\pi k x}, g \rangle$$

Fourier coeffs of g

Solution: $$u(x,t) = \underbrace{\frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \tilde{g}_k \, e^{-(\pi k)^2 t} \, e^{i\pi k x}}_{e^{At} g(x)}$$

"operator exponential"