Linear Partial Differential Eqn's

(Analysis + Numerics)

Course information on Github repository

github.com/mitmath/18303

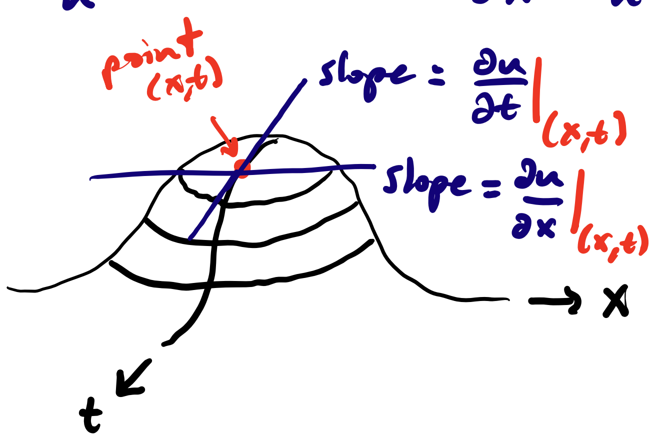
What is a partial differential equation (PDE)?

Find a function $u(x,t)$ that solves

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$$

PDE relates partial derivatives of $u(x,t)$

$$\frac{\partial u}{\partial t} = \lim_{h \to 0} \frac{u(x,t+h) - u(x,t)}{h}, \quad \frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h}$$



18.303 focuses on <u>linear</u> PDEs, like these:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$ Poisson Eq.

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$$ Heat/Diffusion Eq.

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0$$ Wave Eq.

On the left-hand side of each equation, only linear operations are applied to u.

$$\frac{\partial}{\partial x}\left(u(x,y) + \alpha v(x,y)\right) = \frac{\partial u}{\partial x} + \alpha \frac{\partial v}{\partial x}$$

Linear combo of inputs $\rightarrow$ Linear combo of outputs.

<u>Key Idea 1</u>: The linear structure of these eqn's allows us to adapt and apply many ideas and tools from linear algebra to analyze and solve linear PDEs.

vectors

$$Ax = b$$

matrix

functions

$$Lu = f$$

linear differential operator

Why Study Linear PDEs?

⇒ Linear PDE's have been used to model rich array of physical systems & phenomena since (at least) the 1700's!

⇒ Nonlinear systems and Engineering Design tasks often involve linear PDEs in an "inner loop."

⇒ Techniques used to analyze and solve linear PDEs are foundational in applied math and are often imported into other domains in the quantitative sciences! (Even if PDEs do not appear explicitly.)

How do we solve linear PDEs?

200+ years of techniques to treat a wide variety of linear PDE. We'll examine some of the most powerful techniques to solve PDEs on paper & on the computer, highlighting common themes & principles as we go.

Key Idea 2: Most techniques in this course construct solutions (or approximate solutions) by reducing the PDE to systems of simpler algebraic or ordinary differential equations that are easier for humans and/or computers to solve.

Example

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \qquad \text{"transport" equation}$$

Reduce to ODE by looking along characteristic curves $x = x(s)$, $y = y(s)$, s = parameter.

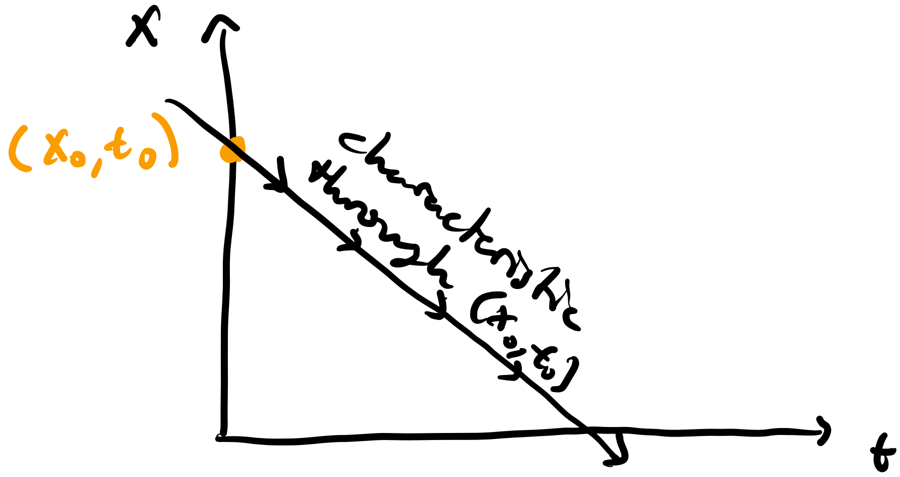
$u(x(s), y(s))$ is constant along curves:

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = 0$$

IF $\quad \frac{dx}{ds} = 1 \qquad \frac{dt}{ds} = -1.$

characteristic curves are $\quad x = s + x_0 \qquad t = -s + t_0$

So $\quad u(s+x_0, s+t_0) = u(x_0, t_0)$



This describes a family of solutions, functions constant along charteristic curves = lines w/slope of -1. For a unique solution, we can specify initial condition

$$u(x,0) = g(x).$$

choose $t_0 = 0$ to apply initial condition

Then, $\quad u(x,t) = u(s+x_0, s+\cancel{t_0}^{0})$

$\qquad\qquad = u(x_0, \cancel{t_0}^{0})$

$\qquad\qquad = g(x+t) \Leftarrow$

b/c $\quad x = s+x_0$
$\quad t = -s+t_0$

and $t_0 = 0$ implies $t = -s$ and therefore $x_0 = x - s = x+t$

Solutions to transport eqn. simply carry initial data at $t=0$ along the characteristic curves $x = -t + x_0$.