Please submit your solutions to the following problems for **extra credit** on Gradescope by **11pm** on the due date. You may collaborate, but please write up your solutions individually.

**1) Method of characteristics.** Consider the first-order linear transport PDE, given by

$$\partial_t u - x\,\partial_x u + u = 0, \qquad \text{where} \qquad u(x,0) = \cos(\pi x).$$

(a) Find the characteristic curves and plot them in the $(x,t)$-plane for $x > 0$, $t > 0$.

(b) Write down a formula for the solution $u(x,t)$ that is valid for any $x > 0$, $t > 0$

(c) What is the maximum value of the solution, $\sup_{x>0} |u(x,t)|$, at time $t = 10$? What happens to the maximum value of the solution as $t \to \infty$?

**2) Laplace's equation.** Solve the Laplace boundary-value problem on the unit disk,

$$\Delta u = 0, \qquad \text{where} \qquad \partial_r u(1,\theta) = \cos(\theta).$$

(a) Write down the general solution to the Laplace equation in the disk.

(b) Write down a particular solution that satisfies the boundary condition. Is it unique?

(c) Identify the maximum and minimum values of the solution and where they are located.

**3) Poisson's equation.** Solve the Poisson boundary-value problem on the unit square,

$$\Delta u = f, \qquad \text{where} \qquad u(\pm 1, y) = u(x, \pm 1) = 0.$$

(a) Write down the homogeneous Dirichlet eigenvalues/eigenfunctions of $\Delta$.

(b) Write down a series solution and give a formula for the series coefficients.

(c) Compute the coefficients explicitly in the case $f(x,y) = \sin(\pi x)\sin(\pi y)$.

**4) The operator exponential.** Consider the initial boundary value problem of the form

$$\partial_t u = Au, \qquad \text{where} \qquad u(\pm 1, t) = 0, \quad \text{and} \quad u(x,0) = g(x),$$

where $A$ is a self-adjoint differential operator on $L^2(-1,1)$ with a complete set of orthogonal eigenfunctions and its eigenvalues have real part $\leq M < \infty$.

(a) Find a solution using only the initial data and the eigenvalues/eigenfunctions of $A$.

(b) If the eigenvalues of $A$ are all less than $-1$, what can you say about $u(x,t)$? Explain.

(c) If all eigenvalues of $A$ are purely imaginary, what can you say about $u(x,t)$? Explain.