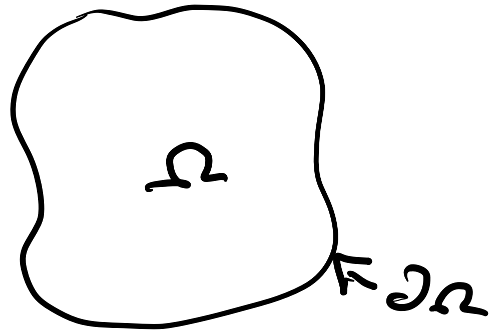
Operator Exponentials

$$\partial_t u = Au$$



$$u|_{t=0} = g, \quad u|_{\partial\Omega} = 0$$

If eigenfunctions of A form orthonormal basis:
(for our usual $H = \{u \text{ s.t. } \int_\Omega |u|^2 dx < \infty\}$)

$$Ae_k = \lambda_k e_k, \qquad e_k|_{\partial\Omega} = 0.$$

and eigenvalues have $\mathrm{Re}(\lambda_k) \leq M$ for some $M < \infty$.

Then, solution to the initial boundary value problem:

$$(*) \qquad u(x,t) = \sum_{k=1}^{\infty} \underbrace{e^{\lambda_k t} \langle e_k, g \rangle}_{\tilde{u}_k(t)} e_k(x).$$

Warning: have to be a little careful about initial condition $u(x,0) = g(x)$ satisfied at every $x \in \Omega$. In general, we have

$$\lim_{t\to\infty} \|u(t) - g\| = \lim_{t\to\infty} \int |u(x,t) - g(x)|^2 dx = 0.$$

From $(*)$, we can say a few things:

$\Rightarrow$ If $\operatorname{Re}(\lambda_k) < 0$, then $|u_k(t)| \to 0$ as $t \to \infty$

$\Rightarrow$ If $\operatorname{Re}(\lambda_k) = 0$, then $|u_k(t)| = \text{const}$ as $t \to \infty$

In general $|u_k(t)| = |\langle e_k, g\rangle| e^{(\operatorname{Re}\lambda_k)t}$.

What does this imply about $u(x,t)$?

<u>Parseval's Identity</u>

For any ONB $\{e_k\}_{k=1}^{\infty}$ for H and $f \in H$,

$$\|f\|^2 = \sum_{k=1}^{\infty} |\langle e_k, f\rangle|^2$$

This is an inf-dim analogue of Pythagoreas!

$$\Rightarrow \quad \|f\|^2 = \langle f, f \rangle \quad \text{and} \quad f = \sum_{k=1}^{\infty} \langle e_k, f \rangle e_k$$

$$\downarrow$$

$$= \left\langle \sum_{k=1}^{\infty} \langle e_k, f \rangle e_k, f \right\rangle$$

$$= \sum_{k=1}^{\infty} \overline{\langle e_k, f \rangle} \langle e_k, f \rangle$$

conjugate linear in first argument

$$= \sum_{k=1}^{\infty} |\langle e_k, f \rangle|^2 \quad \checkmark$$

Therefore, the behavior of $\hat{u}_k(t) = \langle e_k, u \rangle$ tells us about $\|u(\cdot, t)\|$. Since $|\hat{u}_k|$ depends on the spectrum of A, the eigenvalues of A tell us about the behavior of $\|u(\cdot, t)\|$ as $t \to \infty$.

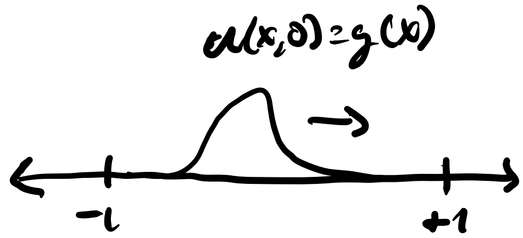
If $Re(\lambda_k) < -\delta$ for all k and some $\delta > 0$, $\Rightarrow \quad \|u(\cdot, t)\| \to 0$ as $t \to \infty$

If $Re(\lambda_k) = 0$ for all k $\Rightarrow \quad \|u(\cdot, t)\| = \text{constant}$

"Wave"-like equations

Example 1: Advection

$$\partial_t u = \partial_x u$$
$$u(x,0) = g(x)$$



Operator exponential soln:

$$k = 0, \pm 1, \pm 2, \dots$$

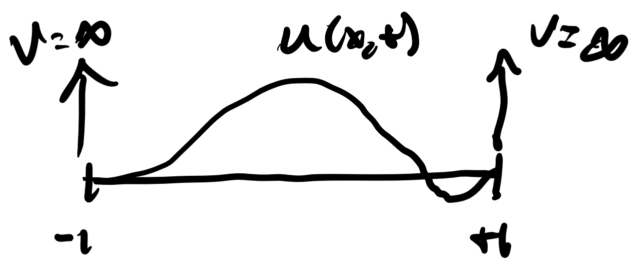
$$\partial_x e_k = \lambda_k e_k \quad \Rightarrow \quad e_k = \frac{1}{\sqrt{2}} e^{i\pi k x}, \quad \lambda_k = i\pi k$$

$$u(x,t) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{g}_k \, e^{i\pi k t} e^{i\pi k x}$$

↳ Fourier coeffs: $\hat{g}_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi k x} g(x)\,dx$

Coeffs $\hat{u}_k(t)$ oscillate with no damping b/c
$$\mathrm{Re}(\lambda_k) = 0, \quad k = 0, \pm 1, \pm 2, \dots$$

Example 2: "Particle-in-a-box"



$$\partial_t^2 u = i \partial_x^2 u$$
$$u(-1,t) = u(1,t) = 0$$

$u(x,0) = g(x)$

$$\partial_x^2 e_k = \lambda_k e_k \quad \Rightarrow \quad e_k(x) = \begin{cases} \cos\frac{k\pi x}{2} & k = 1, 3, \ldots \\ \sin\frac{k\pi x}{2} & k = 2, 4, 6 \end{cases}$$

$$\lambda_k = -i\left(\frac{k\pi}{2}\right)^2 \quad k = 1, 2, 3, \ldots$$

$$u(x,t) = \sum \hat{a}_k e^{-i\left(\frac{k\pi}{2}\right)^2 t} \cos\frac{k\pi x}{2} + \sum \hat{b}_k e^{-i\left(\frac{k\pi}{2}\right)^2 t} \sin\frac{k\pi x}{2}$$

Instead of decaying (heat eq), we get oscillation!