Problem Set 3 MIT 18.303 – Spring 2021

The DL for the Pset is at 6 pm on Monday 4/26.

For this Pset, please use the following definition for the Fourier transform:

$$\hat{f}(k) = \mathcal{F}[f](x) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x,$$

whose inverse transform is given by

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\mathrm{d}x.$$

1 Delta distribution and Fourier transform

(1) Evaluate the following integrals:

$$\text{(a)} \int_0^4 \delta(x-1)\mathrm{e}^{2x}\mathrm{d}x \qquad\qquad \text{(b)} \int_{-1}^{1} \frac{\delta(x+2)}{1+3x^2}\mathrm{d}x$$

(2) Find the Fourier Transform of the following functions:

$$\text{(a) } f(x) = \mathrm{e}^{-\alpha^2 x^2},\ \alpha > 0 \qquad\qquad \text{(b) } f(x) = \mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} \mathrm{e}^{-y^2}\mathrm{d}y$$

(3) Use the Fourier Transform to find a bounded solution ($|u(x)|$ is bounded by some finite real number for all $x \in \mathbb{R}$) to

$$\frac{\mathrm{d}^4 u(x)}{\mathrm{d}x^4} + u(x) = \mathrm{e}^{-2|x|}$$

2 Heat equation on the real line

Let $u_t(t,x) = u_{xx}(t,x)$ with $x \in \mathbb{R}$ and the initial condition $u(t=0,x) = \delta(x)$.
Using Fourier Transform, find $u(t,x)$.

3 Convolution and integral transforms

Let's define the convolution of two functions f and g from $\mathbb{R} \to \mathbb{R}$ as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y.$$

(1) Show that if $\hat{f}$ and $\hat{g}$ are the Fourier transforms of f and g respectively, we have

$$\mathcal{F}[f*g] = \hat{f}\hat{g}.$$

(2) Derive the convolution rule for the Laplace transform for functions f and g that are zero for $x < 0$.