The Heat Eqn. (Pt. 3)

$$\partial_t u = \Delta u$$
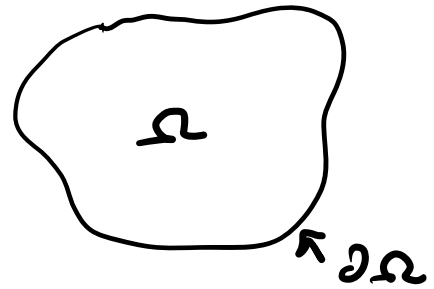
$$u|_{t=0} = g, \quad u|_{\partial\Omega} = f$$

initial condition   boundary condition (Dirichlet) $\rightarrow$ alt. Neumann, Robin, mixed

Ω $\nwarrow \partial\Omega$

__Step 1)__ Solve $\Delta e_\kappa = \lambda_\kappa e_\kappa$ w/ homogeneous B.C.s
     replace $f \rightarrow 0$

$\Rightarrow$ homogeneous soln is $\quad u_h = \underbrace{\sum_\kappa e^{\lambda_\kappa t} \langle e_\kappa, g - u_* \rangle}_{= e^{\Delta t}(g - u_*)}$

__Step 2)__ Solve $\Delta U_* = 0$ w/ inhomogeneous B.C.'s

$\Rightarrow$ equilibrium soln is $\quad u_*$

__Step 3)__ Full soln to Heat Eq. w/ inhomogeneous B.C.'s

$$u = u_* + \sum_\kappa e^{\lambda_\kappa t} \langle e_\kappa, g - u_* \rangle e_\kappa$$

B.C.'s $u|_{\partial\Omega} = u_*|_{\partial\Omega} = f \quad + \quad u_h|_{\partial\Omega} = 0 \quad \checkmark$

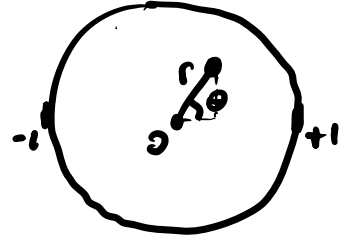I.C. $u|_{t=0} = u_* + g - u_* = g \quad \checkmark$

Eqn. $\partial_t u = \underbrace{\partial_t u_*}_{=0} + \partial_t u_h = \Delta u_h$

$$= \Delta u_h + \underbrace{\Delta u_*}_{=0} = \Delta u \quad \checkmark$$

Diffusion in a disk

$$\partial_t u = \Delta u$$

$$u|_{t=0} = g \qquad u|_{\partial\Omega} = f$$

Step 1: Solve $\Delta u_* = 0$ with $u_*|_{\partial\Omega} = f$

see lecture?

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}_k \, r^{|k|} e^{ik\theta}$$

$$\hookrightarrow \hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ik\theta} f(\theta)\, d\theta$$

Step 2: Solve $\Delta e_k = \lambda_k e_k$ with $e_k|_{\partial\Omega} = 0$.

$\downarrow$

Separation of Variables

$$\Rightarrow \quad \partial_r^2 e_k + r^{-1}\partial_r e_k + r^{-2}\partial_\theta^2 e_k = \lambda_k e_k$$

$$e_k = R(r)\,\Theta(\theta) \qquad \left(\begin{array}{c}\text{drop } k \\ \text{for brevity}\end{array}\right)$$

$$\Rightarrow \quad \frac{1}{R}\left[r^2 \partial_r^2 R + r\,\partial_r R - r^2 \lambda R \right] + \frac{1}{\Theta}\partial_\theta^2 \Theta = 0$$

$$\underbrace{\phantom{\frac{1}{R}\left[r^2 \partial_r^2 R + r\,\partial_r R - r^2 \lambda R \right]}}_{\alpha} \qquad \underbrace{\phantom{\frac{1}{\Theta}\partial_\theta^2}}_{-\alpha} \quad = 0$$

ODE for Θ:

$$\partial_\theta^2 \Theta = -\alpha\,\Theta \qquad \Theta \text{ is } 2\pi\text{-periodic}$$

$$\Rightarrow \quad \Theta_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta} \qquad -\alpha = -k^2$$

ODE for R:

(☆) $\qquad r^2 \partial_r^2 R + r \partial_r R - (r^2 \lambda + k^2) R = 0$

This ODE eigenvalue problem (remember, we need to find both λ and R!) is closely related to a famous ODE called Bessel's equation, and it's solutions are related to Bessel functions $J_n(x)$.

With $\tilde{r} = \sqrt{-\lambda}\, r$ and $R(r) = R(\tilde{r}/\sqrt{-\lambda}) = \tilde{R}(\tilde{r})$

(☆) $\Rightarrow$ $\qquad \tilde{r}\, \partial_{\tilde{r}}^2 \tilde{R} + \tilde{r}\, \partial_{\tilde{r}} \tilde{R} + (\tilde{r}^2 - k^2) \tilde{R} = 0$

Bounded solution in disk is first-kind Bessel function of order $n = k^2$,

$n = k^2$ $\qquad J_n(\tilde{r}) = \frac{1}{\pi} \int_0^\pi \cos(n z - \tilde{r} \sin z)\, dz$

and our solution is $\qquad J_n(\tilde{r}) = \tilde{R}(\tilde{r}) = R(r)$

$$R(r) = J_n(\sqrt{-\lambda}\, r) \qquad \text{for } n = k^2$$

What about λ? To find λ, apply B.C.'s

$$u|_{\partial \Omega = 0} \implies 0 = R(1) = \overline{J_n}(\sqrt{-\lambda})$$

$\overset{\displaystyle\frown}{}$ λ must be a root

So our eigenvalues λ are zeros of Bessel functions! For each $n = k^2$, we get a countable set of distinct positive roots:

$$0 \leq S_{n,1} < S_{n,2} \cdots < S_{n,m} < \cdots$$

The eigenfunctions & eigenvalues are therefore

$$R_{n,m}(r) = \overline{J_n}(S_{n,m} r) \quad \text{and} \quad \lambda_{n,m} = -S_{n,m}^2.$$

Putting together the results with $\quad K = 0, \pm 1, \pm 2, \cdots$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m = 1, 2, 3, \cdots$

$$e_{k,m}(r,\theta) = R_{k^2,m}(r) \, \Theta_k(\theta)$$

$$= \overline{J_{k^2}}(S_{k^2,m} r) \, e^{ik\theta}$$

$$\lambda_{k,m} = -S_{k^2,m}^2$$

The operator exponential $e^{\Delta t}$ on the disk is

$$U_h(r,\theta,t) = \sum_{k=-\infty}^{+\infty} \sum_{m\geq 1}^{\infty} \tilde{f}_{k,m} \, e^{-(s_{k,m})^2 t} \, \overline{J_{k^2}(s_{k,m} r)} \, e^{ik\theta}$$

Here, expansion coeffs $\tilde{f}_{k,m}$ are such that

$$f(r,\theta) - U_x(r,\theta) = \sum_{k=-\infty}^{\infty} \sum_{m\geq 1}^{\infty} \tilde{f}_{k,m} \, J_{k^2}(s_{k,m} r) \, e^{ik\theta}$$

So that $U_h(r,\theta,0) = f(r,\theta) - U_x(r,\theta)$ as required.

Step 3 Finally, we can stitch together the full solution from steps 1 & 2.

$$U(r,\theta,t) = U_x(r,\theta) + U_h(r,\theta,t)$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}_k \, r^{|k|} \, e^{ik\theta}$$

$$+ \sum_{k=-\infty}^{\infty} \sum_{m\geq 1}^{\infty} \tilde{f}_{k,m} \, e^{-(s_{k,m})^2 t} \, J_{k^2}(s_{k,m} r) \, e^{ik\theta}$$

$$= \sum_{k=-\infty}^{\infty} \left[\hat{f}_k \, r^{|k|} + \sum_{m\geq 1}^{\infty} \tilde{f}_{k,m} \, e^{-(-s_{k,m})^2 t} \, J_{k^2}(s_{k,m} r) \right] e^{ik\theta}$$