Wave equation and resonances

How radio works etc...

18.303 Linear Partial Differential Equations: Analysis and Numerics

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The natural tool for studying these sort of systems e.g. the time evolution of the wave equation is the Laplace transform.

Laplace transform

We define a linear transformation $\mathcal{L} : V \to W$, where V and W are suitable spaces of functions. We write

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Integrating by part gives

$$e^{-st} f(t) \Big|_{t=0}^{\infty} + s \int_0^\infty e^{-st} f(t) \mathrm{d}t = sF(s) - f(0^-).$$

Here $f(0^-) = \lim_{t \to 0^-} f(0)$ is the left limit at zero. For continuous functions it is just $f(0)$.

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Using the above formula repeatedly gives

$$\mathcal{L}[f^{(n)}](s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

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We see immediately that this is perfect for initial value problems where derivatives at 0 are given.

Connection with Fourier transform

We see that

$$F(2\pi i\xi) = \int_0^{\infty} e^{-2\pi i\xi t} f(t)\mathrm{d}t = \hat{f}(\xi)$$

for functions with $f(t) = 0$ if $t < 0$.

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Strengths of Laplace transform:

- The transform is real. This is sometimes very handy.
- The transform exists for a larger class of functions since the Laplace integral converges very strongly.
- Can be evaluated with efficient algorithms for Fourier transform.

For example, let us calculate the Laplace transform for a function that is 0 for $t < 0$ and $\sin(\omega t)$ for $t > 0$:

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Casting the sin function in the exponential form gives

$$\begin{aligned}
\int_0^\infty e^{-st} \frac{1}{2i} \left(e^{i\omega t} - e^{-i\omega t} \right) \mathrm{d}t &= \frac{1}{2i} \int_0^\infty e^{(i\omega - s)t} - e^{-(i\omega + s)t} \mathrm{d}t \\
&= \frac{1}{2i} \left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega} \right) \\
&= \frac{\omega}{s^2 + \omega^2}.
\end{aligned}$$

We can try doing the same for the Fourier transform:

$$\begin{aligned}\mathcal{F}[\sin(\omega t)](\xi) &= \int_0^\infty e^{-2\pi i \xi t} \sin(\omega t) \mathrm{d}t \\ &= \frac{1}{2i} \int_0^\infty e^{(\omega - 2\pi\xi)it} - e^{-(\omega+2\pi\xi)it} \mathrm{d}t.\end{aligned}$$

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$$\mathcal{F}[\sin(\omega t)](\xi) = \frac{1}{2i}\left(\frac{1}{2\pi i\xi - i\omega} - \frac{1}{2\pi i\xi + i\omega}\right) = \frac{\omega}{\omega^2 - (2\pi\xi)^2}.$$

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We can recognize this as the Laplace transform evaluated at $s = 2\pi i\xi$ just as it should be.

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- Higher dimensional generalizations are not so useful since we hardly need the positive quadrant of $\mathbb{R}^n$.
- Inverse transform requires extending the Laplace transform $F(s)$ to the complex plane and sometimes requires some regularization magic.

Inverse Laplace transform

Let's use the relationship between Fourier and Laplace transforms

$$F(2\pi i\xi) = \int_0^\infty e^{-2\pi i\xi t} f(t)\mathrm{d}t = \hat{f}(\xi)$$

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We see that the integral has to be evaluated on the imaginary axis of the complex plane.

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Making the change of variable $s = i\omega$ gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \frac{\omega_0}{\omega_0^2 - \omega^2} \mathrm{d}\omega = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \frac{\omega_0}{(\omega - \omega_0)(\omega + \omega_0)} \mathrm{d}\omega.$$

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We notice that the integral has two singularities at $\omega = \pm\omega_0$ and thus can't be in general calculated.

The answer comes from regularization. If we had instead

$$-\frac{1}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{i\omega t}\frac{\omega_0}{(\omega-\omega_0)(\omega+\omega_0)}\mathrm{d}\omega$$

we wouldn't be integrating over the singularities on complex plane.

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This integral is evaluated by the residues i.e.

$$f(t) = -\frac{1}{2\pi}\oint_{\gamma} g(w)\mathrm{d}\omega = \frac{1}{i}\left[\mathrm{Res}(g,\omega_0) + \mathrm{Res}(g,-\omega_0)\right]$$

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Evaluating this gives

$$f(t) = \frac{\omega_0}{i}\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega t}}{2\omega_0}\right) = \sin(\omega_0 t).$$

If $t < 0$ it turns out that this integral gives 0, as it should.

Inverse Laplace transform

The formula we obtained can be modified to work for these difficult functions. In general we have

$$\mathcal{L}^{-1}[F](t) = \frac{1}{2\pi i} \int_{-i\infty+\Delta}^{i\infty+\Delta} e^{st} F(s) \mathrm{d}s,$$

where Δ is a real number s.t. it is larger than any real parts of singularities of $F(s)$ on the complex plane.

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So, in short, inverse Laplace transforms are a bit more complicated. However, if F is an entire function on $\mathbb{C}$ without singularities, we can set $\Delta = 0$.

Resonances

Consider the ODE

$$u''(t) + \lambda^2 u(t) = \sin(\omega_0 t),$$

with initial conditions $u'(0) = u(0) = 0$. With different initial conditions we would have to be concerned with the solution to the homogeneous equation, which we know very well how to do at this point.

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In general this is solved by an ansatz

$$u(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t).$$

Plugging this in gives

$$(\lambda^2 - \omega_0^2)(A\sin(\omega_0 t) + B\cos(\omega_0 t)) = \sin(\omega_0 t).$$

From this we can solve

$$A = \frac{1}{\lambda^2 - \omega_0^2},\ B = 0,$$

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$u(t) = At\sin(\omega_0 t) + Bt\cos(\omega_0 t) = t(A\sin(\lambda t) + B\cos(\lambda t))$.

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Equating this with the RHS gives

$$A = 0, B = -\frac{1}{2\lambda},$$

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The initial condition $u(0) = 0$ works well but calculating

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What to do?

We can always add terms that solve the homogeneous equation. These are terms $C\sin(\lambda t) + D\cos(\lambda t)$. $D = 0$ because $u(0) = 0$. Fixing the boundary condition gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} C\sin(\lambda t)\right)_{t=0} = C\lambda = \frac{1}{2\lambda}.$$

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From this we solve $C = 1/(2\lambda^2)$, which gives the solution

$$u(t) = \frac{1}{2\lambda^2}\sin(\lambda t) - \frac{t}{2\lambda}\cos(\lambda t).$$

Notice how the amplitude of this solution grows linearly in time. This is called a resonant frequency.

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Imagine that we would have a wave equation (we'll do this in detail later) with eigenvalues λ_{n}. If we have an input that is oscillating at a frequency $f = \lambda_{\mathsf{n}}/(2\pi)$ the system starts to vibrate uncontrollably.

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Radio is an example of a system where this phenomenon is made useful. Radio circuit is basically an electronic oscillator whose eigenmodes can be modified. By tuning the eigenmode to match the external frequency the oscillations increase greatly and we can listen to a radio channel at a given frequency.

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Of course nothing blows up in nature. Next time we will talk about softening of resonances due to some sort of damping in the system. We will also derive the results above using the Laplace transform. It is recommended to revise complex analysis for this.