Problem Set 4 MIT 18.303 – Spring 2021

The DL for the Pset is at 6 pm on Friday 5/7.

For this Pset, please use the following definition for the Fourier transform:

$$\hat{f}(k) = \mathcal{F}[f](x) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x,$$

whose inverse transform is given by

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\mathrm{d}x.$$

1 Heat equation on the positive real line

Solve the heat equation

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x), \\ u_x(t,0) &= 0, \\ u(0,x) &= u_0(x). \end{aligned} \tag{1}$$

Here $x \in \mathbb{R}_+$ i.e. positive real numbers. *Hint: introduce a function U using u that is defined on the whole real line and that respects the boundary condition at 0. How should one define U in a way that at all times $U(t,x)\Big|_{x>0} = u(t,x)$ and $U_x(t,0) = 0$?*

What about the problem with the Dirichlet condition $u(t,0) = 0$?

2 Heat equation with non-constant heat conductance

Consider the equation

$$u_t(t,x) = \partial_x(c(x)u_x(t,x))$$

on the interval $x \in [0,1]$ with boundary conditions

$$u(t,0) = 1,\ u(t,1) = 0.$$

We have solved this problem for $c(x) = 1$ in which case we can use a multitude of analytical techniques we learned during this class. However, things become more tricky if c is non-constant.

Assume

$$c(x) = 1 - A\sin(\pi x).$$

1. Write a numerical solver for the time evolution of the heat equation and calculate a numerical solution. Here you can use $A = 0.9$.
2. Does the system reach a steady state?
3. What are the allowed values for the amplitude A?

Feel free to use your favorite numerical methods for this problem.