Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Fourier's basis. In the Fourier basis, a 2-periodic function $f(x)$ on $[-1,1)$ is written as

$$f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\pi k x}, \qquad \text{where} \qquad \hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi k x} f(x)\, dx.$$

(a) Compute the Fourier coordinates of $f(x) = \sin^3(\pi x)$, $g(x) = |x|$, and $h(x) = |\sin(\pi x)|^3$. Plot the magnitude of the Fourier coefficients $-250 \le k \le 250$ on a logarithmic scale. Based on the coefficient plots, roughly what accuracy do you expect if you approximate g and h by truncating their Fourier series, discarding terms with $|k| > 250$?

Solution: We can calculate each function's Fourier coordinates explicitly.

(i) The function $f(x) = \sin^3(\pi x)$ can be represented as a finite combination of four Fourier basis functions

$$\sin^3(\pi x) = \frac{1}{(2i)^3}\left(e^{i\pi x} - e^{-i\pi x}\right)^3 = \frac{i}{8}\left(e^{3i\pi x} - 3e^{i\pi x} + 3e^{-i\pi x} - e^{-3i\pi x}\right).$$

We then use the orthonormality of the basis functions to calculate that

$$\begin{aligned}
\hat{f}_k &= \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi k x}\left[\frac{i}{8}\left(e^{3i\pi x} - 3e^{i\pi x} + 3e^{-i\pi x} - e^{-3i\pi x}\right)\right] dx \\
&= \frac{i}{4\sqrt{2}}\left(\delta_{3,k} - 3\delta_{1,k} + 3\delta_{-1,k} - \delta_{-3,k}\right).
\end{aligned}$$

Here, $\delta_{j,k}$ is 1 when $j = k$ and 0 otherwise, so the only nonzero Fourier coordinates have $k = \pm 1, \pm 3$. Therefore, $f(x)$ can be represented exactly with just four terms from the Fourier series.

(ii) The function $g(x) = |x|$ cannot be represented as a finite combination of four Fourier basis functions. Instead, we split the interval of integration and calculate

$$\hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{0} e^{-i\pi k x}(-x)\, dx + \frac{1}{\sqrt{2}} \int_{0}^{1} e^{-i\pi k x} x\, dx.$$

When $k = 0$, we integrate $-x$ and x over the respective intervals and calculate that $\hat{g}_0 = 1/\sqrt{2}$. To evaluate these integrals for $k \neq 0$, we integrate-by-parts once to reduce the integrand to an exponential. The first integral is (using the fact that $e^{i\pi k} = (-1)^k$)

$$\begin{aligned}
\frac{1}{\sqrt{2}} \int_{-1}^{0} e^{-i\pi k x}(-x)\, dx &= \frac{-1}{\sqrt{2}}\left(\left[\frac{x e^{-i\pi k x}}{-i\pi k}\right]_{x=-1}^{x=0} - \int_{-1}^{0} \frac{e^{-i\pi k x}}{-i\pi k}\, dx\right) \\
&= \frac{-1}{\sqrt{2}}\left(\left[\frac{e^{i\pi k}}{-i\pi k}\right] - \left[\frac{e^{-i\pi k x}}{(-i\pi k)^2}\right]_{x=-1}^{x=0}\right) \\
&= \frac{-1}{\sqrt{2}}\left(\left[\frac{(-1)^k}{-i\pi k}\right] - \left[\frac{1}{(-i\pi k)^2}\right] + \left[\frac{(-1)^k}{(-i\pi k)^2}\right]\right).
\end{aligned}$$

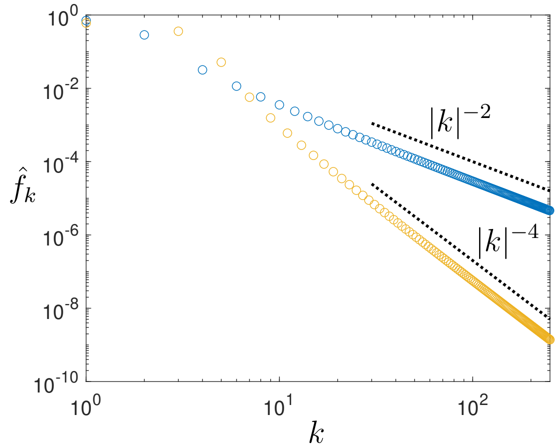

Figure 1: The first 251 Fourier coefficients of $g(x) = |x|$ (blue circles) and $h(x) = |\sin(\pi x)|^3$ (yellow circles) on a log-log plot decay algebraically with rates $|k|^{-2}$ and $|k|^{-4}$ (dotted lines).

Similarly, we can calculate the second integral directly directly to obtain

$$\begin{aligned}
\frac{1}{\sqrt{2}}\int_0^1 e^{-i\pi kx}x\,dx &= \frac{1}{\sqrt{2}}\left(\left[\frac{xe^{-i\pi kx}}{-i\pi k}\right]_{x=0}^{x=1} - \int_0^1 \frac{e^{-i\pi kx}}{-i\pi k}\,dx\right) \\
&= \frac{1}{\sqrt{2}}\left(\left[\frac{e^{-i\pi k}}{-i\pi k}\right] - \left[\frac{e^{-i\pi kx}}{(-i\pi k)^2}\right]_{x=0}^{x=1}\right) \\
&= \frac{1}{\sqrt{2}}\left(\left[\frac{(-1)^k}{-i\pi k}\right] - \left[\frac{(-1)^k}{(-i\pi k)^2}\right] + \left[\frac{1}{(-i\pi k)^2}\right]\right).
\end{aligned}$$

Adding the contributions from each half-interval, we arrive at the result

$$\begin{aligned}
\hat{g}_k &= \frac{2}{\sqrt{2}}\left(\left[\frac{1}{(-i\pi k)^2}\right] - \left[\frac{(-1)^k}{(-i\pi k)^2}\right]\right) \\
&= \frac{-\sqrt{2}}{(\pi k)^2}\left(1-(-1)^k\right) \\
&= \begin{cases} \frac{-2\sqrt{2}}{(\pi k)^2}, & k = \text{odd}, \\ 1/\sqrt{2}, & k = 0, \\ 0, & k = \text{even}(\neq 0). \end{cases}
\end{aligned}$$

The coefficients decay proportional to k^{-2}, as shown in Figure 1. If we truncate the Fourier series by keeping only terms with $|k| \leq 250$, we will make an error proportional to $\sum_{|k|=\text{odd}>250} \frac{2\sqrt{2}}{(\pi k)^2}$, whose magnitude is on the approximate order of $1/(251\pi) \approx 10^{-3}$.

(iii) For the final function $h(x) = |\sin(\pi x)|^3$, we split the integral into two pieces again:

$$\hat{h}_k = \frac{1}{\sqrt{2}}\left(\int_{-1}^0 e^{-i\pi kx}h(x)\,dx + \int_0^1 e^{-i\pi kx}h(x)\,dx\right).$$

We could evaluate the integrals directly over both half-periods as before, but this time let's take advantage of the fact that $h(x)$ is an even function and save ourselves some computation. By changing variables $x \to -x$ in the first integral from $x = -1$ to $x = 0$ and using $h(x) = h(-x)$, we calculate that

$$\begin{aligned}
\hat{h}_k &= \frac{1}{\sqrt{2}} \left(\int_{-1}^{0} e^{-i\pi k x} h(x)\, dx + \int_0^1 e^{-i\pi k x} h(x)\, dx \right) \\
&= \frac{1}{\sqrt{2}} \left(\int_{1}^{0} e^{i\pi k x} h(-x)\, (-dx) + \int_0^1 e^{-i\pi k x} h(x)\, dx \right) \\
&= \frac{1}{\sqrt{2}} \left(\int_{0}^{1} e^{i\pi k x} h(x)\, dx + \int_0^1 e^{-i\pi k x} h(x)\, dx \right) \\
&= \frac{1}{\sqrt{2}} \left(\int_{0}^{1} (e^{i\pi k x} + e^{-i\pi k x}) h(x)\, dx \right).
\end{aligned}$$

Now, $\sin(\pi x)^3$ is non-negative between $x = 0$ and $x = 1$, so we can replace $|\sin(\pi x)|^3$ with $\sin^3(\pi x)$ and plug in the Fourier series from the first function to get

$$\hat{h}_k = \frac{i}{8\sqrt{2}} \left(\int_0^1 (e^{i\pi k x} + e^{-i\pi k x})(e^{3i\pi x} - 3e^{i\pi x} + 3e^{-i\pi x} - e^{-3i\pi x})\, dx \right)$$

After expanding the product of the two exponential sums and simplifying using Euler's identity, we get (excluding the special cases $k = \pm 1, \pm 3$ for the moment)

$$\begin{aligned}
\hat{h}_k &= \frac{-1}{4\sqrt{2}} \int_0^1 (\sin((k+3)\pi x) - \sin((k-3)\pi x)) - 3\sin((k+1)\pi x) + 3\sin((k-1)\pi x)\, dx \\
&= \frac{-1}{4\sqrt{2}} \left(\frac{1-\cos((k+3)\pi)}{(k+3)\pi} - \frac{1-\cos((k-3)\pi)}{(k-3)\pi} - 3\frac{1-\cos((k+1)\pi)}{(k+1)\pi} + 3\frac{1-\cos((k-1)\pi)}{(k-1)\pi} \right) \\
&= \frac{-1}{4\sqrt{2}} \left(\frac{1-(-1)^{k+1}}{(k+3)\pi} - \frac{1-(-1)^{k+1}}{(k-3)\pi} - 3\frac{1-(-1)^{k+1}}{(k+1)\pi} + 3\frac{1-(-1)^{k+1}}{(k-1)\pi} \right) \\
&= \frac{-1+(-1)^{k+1}}{4\pi\sqrt{2}} \left(\frac{1}{(k+3)} - \frac{1}{(k-3)} - \frac{3}{(k+1)} + \frac{3}{(k-1)} \right) \\
&= \begin{cases} \frac{24}{\pi\sqrt{2}} \left(\frac{1}{(k^2-9)(k^2-1)} \right), & k = \text{even}, \\ 0, & k = \text{odd} \neq \pm 1, \pm 3. \end{cases}
\end{aligned}$$

In the second line, we integrated each $\sin(j\pi x)$ term directly and in the third line we used the fact that $\cos((k \pm 3)\pi) = \cos((k \pm 1)\pi) = (-1)^{k+1}$. Finally, when $k = \pm 1$ or ± 3, the corresponding sinusoid vanishes. The remaining terms vanish for $k = \pm 1, \pm 3$ since they vanish for all odd k, so we conclude that

$$\hat{h}_k = \begin{cases} \frac{24}{\pi\sqrt{2}} \left(\frac{1}{(k^2-9)(k^2-1)} \right), & k = \text{even}, \\ 0, & k = \text{odd}. \end{cases}$$

As $k \to \infty$, the coefficients decay at the algebraic rate $1/k^4$ as shown in Figure 1. Reasoning as we did for $g(x)$, the truncation error in the truncated Fourier series should be on the order of $1/(\pi\sqrt{2}(251)^3) \approx 10^{-8}$.

(b) Show that if f is n-times continuously differentiable with $|f^{(n)}(x)| \leq M$ on the periodic interval $[-1, 1)$, then $|\hat{f}_k| \leq \sqrt{2}M/(\pi k)^n$. (**Hint:** integrate by parts.) If $f(x)$ is approximated by the truncated series $f_N(x) = \sum_{k=-N}^{N} \hat{f}_k e^{i\pi kx}$, how do you expect the approximation error $E_N = \max_{-1\leq x\leq 1} |f(x) - f_N(x)|$ to scale as N is increased?

Solution: Since f and its first n derivatives are continuous on the periodic interval, we can integrate by parts n times without contributions from the endpoints:

$$\hat{f}_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi kx} f(x)\, dx = \frac{(-1)^n}{\sqrt{2}(-i\pi k)^n} \int_{-1}^{1} e^{-i\pi kx} f^{(n)}(x)\, dx.$$

Since $|f^{(n)}(x)| \leq M$ on $[-1, 1)$ and $|e^{-i\pi kx}| = 1$, we can bound the right-hand integral

$$\begin{aligned} |\hat{f}_k| &= \left| \frac{(-1)^n}{\sqrt{2}(-i\pi k)^n} \int_{-1}^{1} e^{-i\pi kx} f^{(n)}(x)\, dx \right| \\ &\leq \frac{M}{\sqrt{2}(\pi k)^n} \int_{-1}^{1} dx = \frac{\sqrt{2}M}{(\pi k)^n}. \end{aligned}$$

Therefore, the Fourier coefficients of an n-times continuously differentiable periodic function decay at least as fast as $1/k^n$. If we sum the terms discarded from the N-truncated Fourier series, the error will be roughly on the order of $1/N^{n-1}$. In fact, one can show $1/N^{n+1}$ decay (compare with the decay rates from part (a)) and $1/N^n$ truncation error with slightly different assumptions, but this is a more involved exercise. The main point is the algebraic decay of the Fourier coefficients of an n-times differentiable function is governed by the "number" and "size" of its derivatives.

(c) If $a(x) = \sin^3(\pi x)$ and $f(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\pi kx}$, then what are the Fourier coefficients of $a(x)f(x)$? Write down the (infinite) matrix representing "multiplication-by-$a(x)$" in the Fourier basis. How many nonzero entries are there in each row?

Solution: Plugging the Fourier series for $a(x) = \sin^3(\pi x)$ from part (a) into the formula for the Fourier coefficients of the product $m(x) = a(x)f(x)$, we calculate

$$\begin{aligned} \hat{m}_k &= \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-i\pi kx} a(x) f(x)\, dx \\ &= \frac{i}{8\sqrt{2}} \int_{-1}^{1} e^{-i\pi kx} \left(e^{3i\pi x} - 3e^{i\pi x} + 3e^{-i\pi x} - e^{-3i\pi x}\right) f(x)\, dx \\ &= \frac{i}{8\sqrt{2}} \int_{-1}^{1} \left(e^{-i\pi(k-3)x} - 3e^{-i\pi(k-1)x} + 3e^{-i\pi(k+1)x} - e^{-i\pi(k+3)x}\right) f(x)\, dx \\ &= \frac{i}{8} \left(\hat{f}_{k-3} - 3\hat{f}_{k-1} + 3\hat{f}_{k+1} - \hat{f}_{k+3}\right). \end{aligned}$$

Therefore, each Fourier coefficient of the product $m(x) = a(x)f(x)$ is a finite linear combination of the neighboring odd Fourier coefficients of $f(x)$. We can assemble this transformation into a linear matrix with four nonzero diagonals, with

$$\begin{pmatrix} \vdots \\ \hat{m}_{-4} \\ \hat{m}_{-3} \\ m_{-2} \\ \hat{m}_{-1} \\ \hat{m}_0 \\ \hat{m}_1 \\ \hat{m}_2 \\ \hat{m}_3 \\ \hat{m}_4 \\ \vdots \end{pmatrix} = \frac{i}{8} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & -3 & 0 & 3 & 0 & -1 & & & \\ & & 1 & 0 & -3 & 0 & 3 & 0 & -1 & & \\ & & & 1 & 0 & -3 & 0 & 3 & 0 & -1 & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \hat{f}_{-4} \\ \hat{f}_{-3} \\ f_{-2} \\ \hat{f}_{-1} \\ \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \\ \vdots \end{pmatrix}$$

A matrix of this form, with entries that do not change along each diagonal, is called a Toeplitz matrix. They play a central role in the theory and practice of discrete convolution and are related to a number of fast algorithms in numerical linear algebra.

2) Finite differences in 2D. Consider Poisson's equation on the unit square:

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = f(x,y), \qquad \text{where} \qquad u(\pm 1, y) = 1 - y^2, \quad \text{and} \quad u(x, \pm 1) = 1.$$

The `poissonFD.ipynb` notebook accompanying Lecture 8 may be helpful in parts (a)-(d).

(a) Using centered second-order finite differences in x and y on an $N \times N$ grid, discretize the PDE (without boundary conditions) to obtain a matrix equation $D_2 U + U D_2 = F$.

Solution: Following the notes from lecture 8, we can discretize the square into a grid of $N + 2$ equally-spaced points, (x_j, y_k), where $0 \leq j, k \leq N + 1$ with

$$x_j = -1 + \frac{2j}{N+1}, \qquad \text{and} \qquad y_k = -1 + \frac{2k}{N+1}.$$

If we represent the solution on the $N \times N$ interior of the grid (where $1 \leq j, k \leq N$) using an $N \times N$ matrix U, the right-hand side by $F_{j,k} = f(x_j, y_k)$, and discretize using second-order central differences, we get the equations

$$\frac{1}{h^2}\left(U_{j+1,k} - 2U_{j,k} + U_{j-1,k}\right) + \frac{1}{h^2}\left(U_{j,k+1} - 2U_{j,k} + U_{j,k-1}\right) = F_{j,k}.$$

If we set the boundary values to zero for the moment, this corresponds to the matrix equation $D_2 U + U D_2 = F$, where D_2 is the second central difference matrix

$$D_2 = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

(b) Modify the right-hand side, F, of the matrix equation in part (a) to enforce the non-homogeneous boundary conditions $u(\pm 1, y) = 1 - y^2$ and $u(x, \pm 1) = 1$.

Solution: At the boundary nodes, where j or k are equal to 0 or $N+1$, we can plug the boundary value into the difference equation and rearrange all known terms on the right-hand side. For example, at the boundary $x = -1$ corresponding to $j = 0$, we have $u(\pm 1, y) = 1 - y^2$, so $U_{1,k} = 1 - y_k^2$ and we get (for $1 \le k \le N$)

$$\frac{1}{h^2}\left(U_{2,k} - 2U_{1,k}\right) + \frac{1}{h^2}\left(U_{1,k+1} - 2U_{1,k} + U_{1,k-1}\right) = F_{1,k} - \frac{1}{h^2}(1 - y_k^2).$$

Similarly, at the boundary $x = 1$ corresponding to $j = N+1$, we have (for $1 \le k \le N$)

$$\frac{1}{h^2}\left(-2U_{N,k} + U_{N-1,k}\right) + \frac{1}{h^2}\left(U_{N,k+1} - 2U_{N,k} + U_{N,k-1}\right) = F_{N,k} - \frac{1}{h^2}(1 - y_k^2).$$

At the boundary $y = -1$, corresponding to $k = 0$, we have $u(x, \pm 1) = 1$, so that (for $0 \le j \le N+1$)

$$\frac{1}{h^2}\left(U_{j+1,1} - 2U_{j,1} + U_{j-1,1}\right) + \frac{1}{h^2}\left(U_{j,2} - 2U_{j,1}\right) = F_{j,1} - \frac{1}{h^2}.$$

Finally, at the boundary $y = 1$, corresponding to $k = N+1$, we have (for $0 \le j \le N+1$)

$$\frac{1}{h^2}\left(U_{j+1,N} - 2U_{j,N} + U_{j-1,N}\right) + \frac{1}{h^2}\left(-2U_{j,N} + U_{j,N-1}\right) = F_{j,N} - \frac{1}{h^2}.$$

If we modify the first and last columns and rows of F as indicated above, we enforce the non-homogeneous boundary conditions for the Poisson problem while the left-hand side of the matrix equation in part (a) remains unchanged.

(c) Use the Kronecker product to rewrite the matrix equation from (a) and (b) in the standard form $Ax = b$, where A is an $N^2 \times N^2$ matrix and b is an $N^2 \times 1$ vector.

Solution: Denoting the modified right-hand side from part (b) by $\tilde{F}$, we vectorized both sides of the equation $D_2U + UD_2 = \tilde{F}$ and apply the Kronecker identity $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$ to obtain (using $D_2^T = D_2$ in the last equality)

$$\begin{aligned}
\mathrm{vec}(\tilde{F}) &= \mathrm{vec}(D_2U + UD_2) \\
&= \mathrm{vec}(D_2UI) + \mathrm{vec}(IUD_2) \\
&= (I^T \otimes D_2)\mathrm{vec}(U) + (D_2^T \otimes I)\mathrm{vec}(U) \\
&= (I \otimes D_2 + D_2 \otimes I)\mathrm{vec}(U).
\end{aligned}$$

This is a standard linear system with $N^2 \times N^2$ matrix $A = I \otimes D_2 + D_2 \otimes I$, $N^2 \times 1$ right-hand side vector $b = \mathrm{vec}(\tilde{F})$, and $N^2 \times 1$ solution vector $x = \mathrm{vec}(U)$.

(d) Using the Gaussian right-hand side $f(x, y) = 5\exp(-10(x^2 + y^2))$, solve the discretized linear system in part (c) numerically and plot the solution on the $N \times N$ grid. Try increasing the value of N until the numerical solution appears to converge. Should the solution satisfy a maximum or minimum principle? Explain your reasoning.

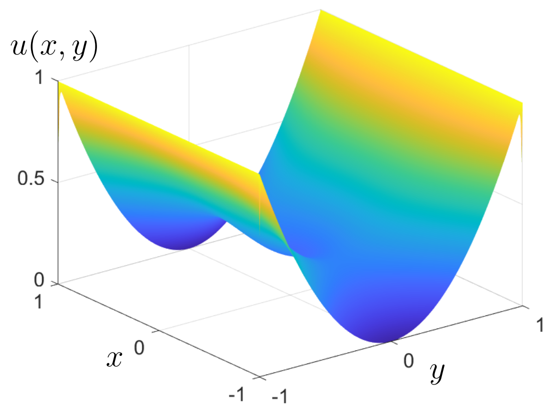

Figure 2: The numerical solution to the Poisson equation in part (d) on a 500×500 grid.

Solution: The numerical solution is plotted on a 500×500 grid in Figure 2. Since the right-hand side is strictly positive, we expect that the solution will satisfy the *maximum principle*, that is, the maximum of the solution will be located on the boundary. The numerical solution appears to achieve the maximum of $u(x, \pm 1) = 1$ on the left and right boundaries and has no local maxima. It does appear to have a local minimum near the center of the domain, which does not violate our expectation since the solution need not satisfy the minimum principle when $f(x, y)$ is positive.

3) Separation of variables. Consider the exterior Laplace problem in polar coordinates,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] u(r,\theta) = 0, \qquad \text{where} \qquad r \geq 1 \quad \text{and} \quad u(1,\theta) = |\sin(\theta)|^3.$$

Use separation of variables in polar coordinates to find a *bounded* solution, $|u(r,\theta)| \leq M$. Is your solution unique? Explain why or why not. If not, provide the general solution form.

Solution: In Lecture 7, we derived a general solution for the Laplace equation in polar coordinates, which had the form (recall that the sum is over all nonzero integers $k \neq 0$)

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + B_0 \log(r) + \sum_{k\neq 0}(A_k r^k + B_k r^{-k})e^{ik\theta}\right].$$

Any solution that is bounded in the exterior of the unit disk, $r \geq 1$, must have $A_k = 0$ for $k \geq 1$ and $B_k = 0$ for $k \leq 0$ to eliminate terms that grow as $r \to \infty$. We are left with

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + \sum_{k\geq 1} r^{-k}\left(A_{-k}e^{-ik\theta} + B_k e^{ik\theta}\right)\right].$$

To satisfy the boundary condition $u(1,\theta) = |\sin(\theta)|^3$, we must choose the remaining coefficients so that

$$u(1,\theta) = \frac{1}{\sqrt{2\pi}}\left[A_0 + \sum_{k\geq 1}\left(A_{-k}e^{-ik\theta} + B_k e^{ik\theta}\right)\right] = |\sin(\theta)|^3.$$

In other words, we must choose the remaining coefficients A_k $(k \leq 0)$ and B_k $(k \geq 1)$ to match the Fourier coefficients of $w(\theta) = |\sin(\theta)|^3$. With the substitution $\theta = \pi x$, these Fourier coefficients can be related to those of $h(x) = |\sin(\pi x)|^3$ on $[-1, 1)$ from Problem 1:

$$\begin{aligned}
\hat{w}_k &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ik\theta} |\sin(\theta)|^3 \, d\theta \\
&= \sqrt{\frac{\pi}{2}} \int_0^{2} e^{-i\pi kx} |\sin(\pi x)|^3 \, dx \\
&= \sqrt{\frac{\pi}{2}} \in_{-1}^{1} e^{-i\pi kx} |\sin(\pi x)|^3 \, dx \\
&= \sqrt{\pi} \hat{h}_k.
\end{aligned}$$

In the last line, we have used that the integrand is 2-periodic so that integrating from 0 to 2 is equivalent from integrating from -1 to 1. Therefore, we set $A_{-k} = \sqrt{\pi}\hat{h}_{-k}$ and $B_k = \sqrt{\pi}\hat{h}_k$. We can write the resulting series solution explicitly as

$$\begin{aligned}
u(r, \theta) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} r^{-|k|} (\sqrt{\pi}\hat{h}_k) e^{ik\theta} \\
&= \frac{12}{\pi} \sum_{k=\text{even}} \frac{r^{-|k|} e^{ik\theta}}{(k^2 - 9)(k^2 - 1)}
\end{aligned}$$

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