Please submit your solutions to the following problems on Gradescope by **10pm** on the due date. Collaboration is encouraged, however, you must write up your solutions individually.

1) Differential Operators. Consider the linear differential operator defined by

$$[Lu](x) = u'(x) + x\,u(x), \qquad u \in \mathcal{C}^1[-1,1].$$

(a) Describe the null-space of L, that is, find all solutions to $Lu = 0$.

Now, apply a boundary condition $u(-1) = 0$ and restrict the domain of L to the subspace of continuously differentiable functions that satisfy this boundary condition.

(b) Calculate the adjoint of L, that is, find $L^\dagger$ such that $\int_{-1}^{1} v(x)[Lu](x)\,dx = \int_{-1}^{1} u(x)[L^\dagger v](x)$ holds for any $u, v \in \mathcal{C}^1[-1,1]$ satisfying $u(-1) = 0$ and $v(1) = 0$. Does $L^\dagger L = LL^\dagger$?

(c) Calculate the inverse of L, that is, find an integral operator K such that $Lu = f$ if and only if $u = Kf$. (Hint: use the method of integrating factors from 18.03.)

2) Central Differences. The `hw1.jl` notebook on the course repository may be helpful for the computational components of this exercise (`https://github.com/mitmath/18303/`).

(a) Show that the centered difference formula (see Lecture 3 notes) approximates $u'(x)$ with accuracy proportional to h^2 if $u(x)$ has three continuous derivatives.

(b) Derive a fourth-order accurate centered difference formula to approximate $u'(x)$ from samples $u(x-2h), u(x-h), u(x), u(x+h), u(x+2h)$ with grid spacing $h > 0$.

(c) For parts (a) and (b), what are the corresponding difference matrices representing differentiation on a grid of n equispaced points (with spacing $h = 1/(n+1)$) on the periodic interval $[0,1)$? What can you say about the pattern of nonzero entries?

(d) Use the matrices in part (c) to approximate the derivatives of the functions $\sin(2\pi x)$, $\cos(\pi(x-0.5))$, and $\sqrt{(1+\cos(2\pi x))^3}$ on an equispaced grid of $n = 500$ points on the periodic interval $[0,1)$. Plot the error in your approximation of the derivative at each grid point and then plot the maximum absolute error on grids with $n = 100, 200, 300, \ldots, 10^4$ (use a logarithmic scale for both axes). Can you explain the behavior of the error for each function (e.g., why proportional to h^2, h^4, etc.)?

3) Method of Characteristics. Consider the first-order linear PDEs with form

$$\partial_t u(x,t) + b(x)\,\partial_x u(x,t) + c\,u(x,t) = 0, \qquad \text{where} \qquad u(x,0) = g(x).$$

(a) Find the characteristic curves for $b(x) = x^2$ and plot them in the (x,t)-plane.

(b) Given initial condition $g(x) = \exp(-100(x-0.5)^2)$, write down a solution $u(x,t)$ when $c = 0$. Is the solution unique? How does the solution change if $c = 1$?

(c) Use a forward Euler approximation in time and a second-order centered difference in space to approximate $u(x,t)$ on the periodic interval $x \in [0,1)$ from time $t = 0$ to $t = 1$. Use time step $h_t = 0.01$ and spatial grid of length 200. How does your numerical solution compare to the exact solution in part (b) for the case $c = 0$? How does it compare with the forward difference spatial discretization provided in `hw1.jl`?