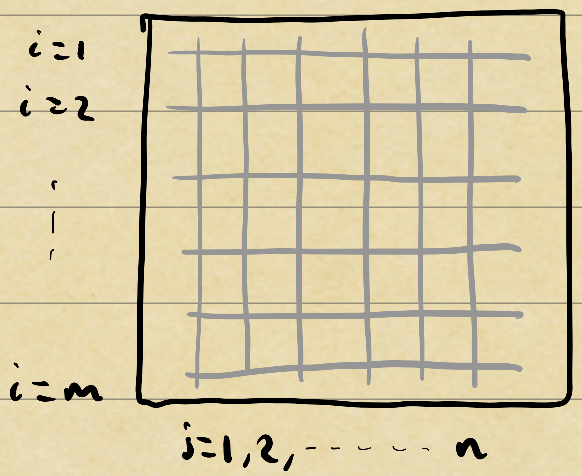
Low-Rank Methods

Many functions that appear in applied math have a hidden property. They are approx. low-rank.



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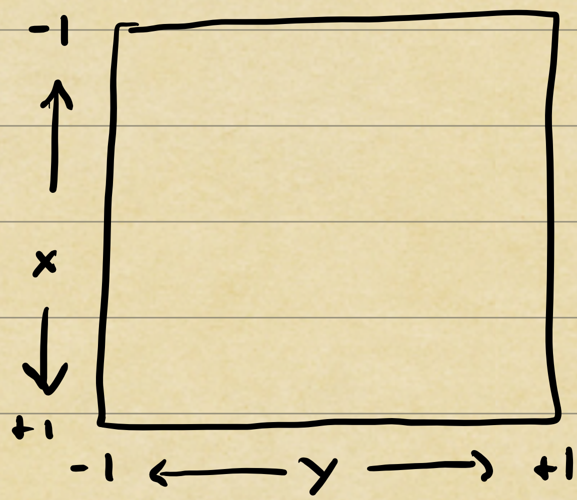

f(x,y)

Matrix

$$\underset{m\times n}{A} = \underset{m\times r}{C}\,\underset{r\times n}{V^T} = \sum_{j=1}^{r} c_j v_j^T$$

$r \leq \min(m,n)$

Function

$$f(x,y) = \sum_{j=1}^{r} c_j(x)\, r_j(y)$$

"Low-rank" means that $r \ll \min(m,n)$.

For functions we can think of (m, n) as the DOFs used to discretize in x and y, resp.

Storage costs for low-rank matrix/function are reduced from $O(mn)$ to $O((m+n)r)$!

Matrix-vector products

$x \mapsto Ax$ usually costs $O(mn)$

$\Rightarrow$ m dot products of length n

But if we use the low-rank structure:

$$x \mapsto Ax = U(V^T x)$$

$x \to V^T x = y$ $\qquad$ $y \to Uy = Ax$

$\Rightarrow$ r dot products of length n

$\Rightarrow$ m dot products of length r

Cost is reduced to $O((m+n)r)$ flops.

What about functions?

Consider $u(x) \mapsto [Fu](x)$, where

$$[Fu](x) = \int_{-1}^{+1} f(x,y)\,u(y)\,dy.$$

If we discretize with an n-pt quadrature rule and sample $[Fu](x)$ at m interpolation points, this costs $\mathcal{O}(mn)$ flops to execute.

If F has a "low-rank" kernel, then

$$[Fu](x) = \sum_{j=1}^{r} c_j(x) \int_{-1}^{+1} r_j(y)\,u(y)\,dy$$

$\Rightarrow$ r inner products w/ a n-pt. quad rule.
$\Rightarrow$ m length r sums to evaluate $[Fu](x)$ at m interpolation points.

The cost is reduced to $\mathcal{O}((m+n)r)$ flops.

Most of the functions we will encounter are not exactly low-rank, but can be approximated accurately by a low-rank function.

$$f(x,y) \approx \sum_{j=1}^{r} c_j(x) r_j(y) + \underbrace{e(x,y)}_{\text{"small"}}$$

Approximately low-rank functions can appear in many settings & for many reasons:

=> Smoothness (multi-variate Taylor Approx)

=> Scale Separation (multi-scale expansions)

=> Separability, axis-alignment, and other symmetries

=> What else? ...

We will consider a few ways to construct low-rank approximations to functions.

We will also want to consider $f(x,y)$ which are approximately low-rank only in certain regions of their domain. E.g., Green's functions of diff ops.