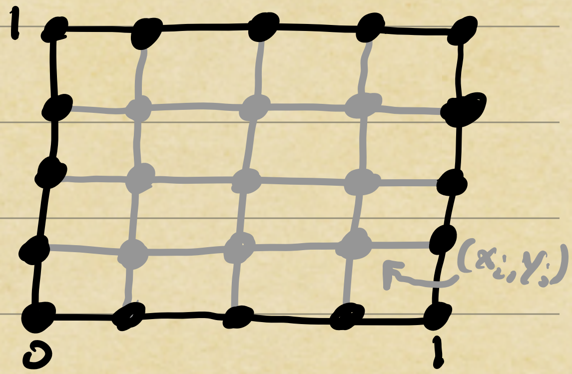
Fast FD Solvers in Multi Dim.'s

2D Poisson w/ Dirichlet B.C.'s

$\Omega = [0,1]^2$

$$-u_{xx} - u_{yy} = f(x,y)$$

$u|_{\partial\Omega} = g$ (Dirichlet B.C.'s)



FD discretization on $(n-1)\times(n-1)$ interior grid

$\Rightarrow \mathcal{O}(n^2)$ degrees of freedom

$\Rightarrow$ Aim for $\mathcal{O}(n^2 \log_2 n)$ solver?

$(n-1)\times(n-1)$ matrix U_{ij} = unknown value of $u(x,y)$ at $(i,j)^{th}$ grid point

$$-u_{xx}(x_i, y_j) \approx \tfrac{1}{h^2}(KU)_{ij}, \quad -u_{yy}(x_i, y_j) \approx \tfrac{1}{h^2}(UK)_{ij}$$

acts on cols of U

acts on rows of U

$$\frac{1}{h^2}\left(KU + UK^T\right) = F_h$$

$\hookrightarrow$ $f(x,y)$ modified on first and last col/row for D.B.C.'s

"Sylvester Equation"

Sylvester Matrix Eqn's

$\Rightarrow$ Solved in $\mathcal{O}(n^3)$ using Bartels-Stewart, relies on dense matrix factorizations

$\Rightarrow$ Iterative Methods like ADI use fast 1D solvers to build up approx 2D soln.

$\Rightarrow$ For us, Poisson Eq. discretization inherits Fourier eigenbasis (fast diagonalization) due to

(a) separable domain $[0\ 1]\times[0\ 1]$

(b) translation invariant in x and y (up to boundary conditions)

To exploit (a) & (b) in a way that generalizes well to higher dimensions, we'll use

Block Matrices, Vectorization, & Kronecker Products

Caution! Many Eq's below depend on row-major "vec" ↳ slight mod's needed for col-major

2x2 matrix $\quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \xrightarrow[\text{ordering}]{\text{row-major}} \mathrm{vec}(X) = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$

Apply $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ to rows/cols of X:

$$XA^T \quad \rightarrow \quad \underbrace{\overset{4 \times 4}{\begin{bmatrix} A & \\ & A \end{bmatrix}}}_{I_2 \otimes A} \overset{\text{length } 4}{\mathrm{vec}(X)}$$

acts on rows

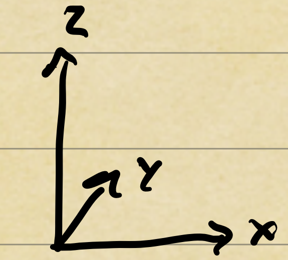
$$AX \quad \rightarrow \quad \underbrace{\overset{4 \times 4}{\left[\begin{array}{cc|cc} a_{11} & & a_{12} & \\ & a_{11} & & a_{12} \\ \hline a_{21} & & a_{22} & \\ & a_{21} & & a_{22} \end{array}\right]}}_{A \otimes I_2} \overset{\text{length } 4}{\mathrm{vec}(X)}$$

acts on cols

Kronecker Products transform linear operators acting along separate array axes into a single linear operator acting on vectorizations.

$$\overset{n \times m}{A} \otimes \overset{n' \times m'}{B} = \overset{nn' \times mm'}{\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & & a_{2m}B \\ \vdots & & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}}$$

Easily extends to higher dimensions



$\mathrm{vec}(X)$ stacks rows into vector n^3

Apply A to x-axis: $(A \otimes I \otimes I)\,\mathrm{vec}(X)$

y-axis: $(I \otimes A \otimes I)\,\mathrm{vec}(X)$

z-axis: $(I \otimes I \otimes A)\,\mathrm{vec}(X)$

Property 1 $\quad (AXB) = (A \otimes B^T)\,\mathrm{vec}(X)$

Property 2 $\quad (A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$

$\Rightarrow$ Systematic manipulation and derivations of highly structured matrices derived from multi-dimensional problems on separable dom.'s.

$\Rightarrow$ Usually avoid explicitly forming $A \otimes B$ for computation, its expensive!

Structured Block Matrices

Block matrices from Kronecker products of PDE discretizations often retain important structure from 1D problems.

Example: Circulant Block Circulant

=> FD discretization of 2D Poisson w/periodic B.C.s

$$C = \begin{array}{c} \begin{array}{ccc} n\times n & n\times n & n\times n \end{array} \\ \begin{bmatrix} C_{11} & C_{31} & C_{21} \\ C_{21} & C_{11} & C_{31} \\ C_{31} & C_{21} & C_{11} \end{bmatrix} \end{array}$$

$N = m\cdot n$

w/ $m=3$

C_{11}, C_{21}, C_{31} circulant

$\Rightarrow C_{ij} = F_n^{-1} \Lambda_{ij} F_n$

↑ DFT matrix

Each block is circulant:

$$\underbrace{\begin{bmatrix} F_n & & \\ & F_n & \\ & & F_n \end{bmatrix}}_{I_m \otimes F_n} C \underbrace{\begin{bmatrix} F_n^{-1} & & \\ & F_n^{-1} & \\ & & F_n^{-1} \end{bmatrix}}_{I_m \otimes F_n^{-1}} = \begin{bmatrix} F_n C_{11} F_n^{-1} & \cdots \\ F_n C_{21} F_n^{-1} & \\ F_n C_{31} F_n^{-1} & \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda_{11} & \Lambda_{31} & \Lambda_{21} \\ \Lambda_{21} & \Lambda_{11} & \Lambda_{31} \\ \Lambda_{31} & \Lambda_{21} & \Lambda_{11} \end{bmatrix}$$

Circulant matrix of diagonal blocks! ↑ Λ_R

$$(F_m \otimes I_n)\Lambda_R(F_m^{-1} \otimes I_n) = \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix} = \underset{\text{diagonal}}{\Lambda_{RC}}$$

$$(F_m \otimes I_n)(I_m \otimes F_n)\, C\, (I_m \otimes F_n^{-1})(F_m^{-1} \otimes I_n)$$

Apply Property 2
$$= \underbrace{(F_m \otimes F_n)}\, C\, \underbrace{(F_m^{-1} \otimes F_n^{-1})}$$

$\uparrow$ 2D FFT/IFFT $\uparrow$

By Property (3),

$$(F_m \otimes F_n)\,\mathrm{vec}(X) = F_m X F_n$$
$$(F_m^{-1} \otimes F_n^{-1})\,\mathrm{vec}(X) = F_m^{-1} X F_n^{-1}$$

So we can interpret these as transforming separately along each array axis.

What is Λ_{RC}? From theory of cyclic permutations,

$$\Lambda_{RC} = (I_n \otimes \Lambda_{11}) + (P \otimes \Lambda_{21}) + (P^2 \otimes \Lambda_{31}) + \dots + (P^{m-1} \otimes \Lambda_{m1})$$

where $P = \begin{array}{c} m \times m \\ \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} \end{array}$ so that $P^m = I$

matrix
$m \times n$

Fast solver for $Cy = x$

1) $x' = (I_m \otimes F_n)x$ row FFT $\mathcal{O}(mn\log_2 n$
2) $x'' = (F_m \otimes I_n)x'$ col FFT $\mathcal{O}(nm\log_2 m)$
3) Compute $y'' = \Lambda_{RC}^{-1} x''$ $\mathcal{O}(mn)$
4) $y' = (F_m^{-1} \otimes I_n)y''$ col IFFT $\mathcal{O}(nm\log_2 m)$
5) $y = (I_m \otimes F_n^{-1})y'$ row IFFT $\mathcal{O}(mn\log_2 n)$

Total $\Rightarrow$ $\mathcal{O}(mn\log_2 n) + \mathcal{O}(nm\log_2 m)$
$= \mathcal{O}(mn\log_2 mn)$
$= \mathcal{O}(N\log_2 N)$ $N = mn =$ total deg. of freedom

Can you design a fast Poisson solver for

$$KU + UK^T = F_b$$

based on the DST-Type 1?