Chebyshev Spectral Methods (pt. 1)

For generic nonperiodic problems, convergence of Fourier spectral methods may suffer due to singularities in odd/even extensions of RHS.

Polynomial spectral methods provide a powerful tool for such nonperiodic problems.

Orthogonal Basis $\quad \{\varphi_n\}_{n=0}^{\infty}$ with $\varphi_n \in P_n =$ degree n polynomials

$$\langle \varphi_n, \varphi_m \rangle_w = \int_{\Omega \subset \mathbb{R}} \varphi_n(x)\varphi_m(x) \underset{\text{weight} > 0}{\underset{\uparrow}{w(x)}}\, dx = \begin{cases} 0 & n \neq m \\ C_n & n = m \end{cases}$$

$\quad$ ↳ nonzero

$\Rightarrow$ orthogonal w.r.t. weighted inner product

Expansions $\quad u(x) = \sum_{n=0}^{\infty} \hat{u}_n \varphi_n(x) \qquad (*)$

$$\hat{u}_n = \frac{1}{\sqrt{C_n}} \langle \varphi_n, u \rangle_w = \frac{1}{C_n} \int_{\Omega} \varphi_n(x) u(x) w(x)\, dx$$

with $(*)$ holding in $\|\cdot\|_w$ for all u with $\|u\|_w < \infty$.

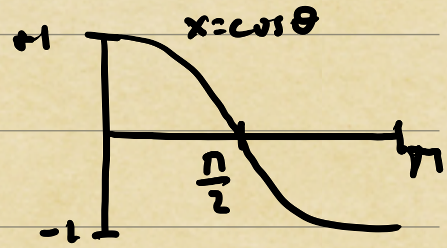
Chebyshev Polynomials

$$T_k(x) = \underbrace{\cos(k\cos^{-1}(x))}_{\text{yes, this is a degree } k \text{ polynomial!}}, \quad x \in [-1,1], \quad \text{for} \quad k = 0, 1, 2, \dots$$

$\Rightarrow$ Orthogonal w.r.t. weight $w(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$\langle T_n, T_m \rangle_w = \int_{-1}^{+1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}}$$

$x = \cos\theta$
$dx = -\sin\theta \, d\theta$



$$= \int_0^\pi \cos(n\theta)\cos(m\theta)\, d\theta$$

$$= \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \end{cases}$$

Note that these are orthogonal but unnormalized.

Closely related to cosine series through map

$$x = \cos\theta \quad \longleftrightarrow \quad \theta = \cos^{-1} x$$

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x) \quad \longleftrightarrow \quad \underbrace{u(\cos\theta)}_{\tilde{u}(\theta)} = \sum_{k=0}^{\infty} \hat{u}_k \cos(k\theta)$$

Convergence rate determined by cosine series for $\tilde{u}(\theta)$! I.e., by smoothness of even periodic extension of $u(\cos\theta)$.

<u>Even derivatives</u>

$$\tilde{u}(\theta) = u(\cos\theta)$$

$$\tilde{u}''(\theta) = \sin^2\theta\, u''(\cos\theta) + \cos\theta\, u'(\cos\theta)$$

$$\vdots$$

<u>Odd derivatives</u>

$$\tilde{u}'(\theta) = -\sin\theta\, u'(\cos\theta)$$

$$\tilde{u}'''(\theta) = 2\sin\theta\cos\theta\, u''(\cos\theta) - \sin^3\theta\, u'''(\cos\theta) - \sin\theta\cos\theta\, u''(\cos\theta) - \sin\theta\, u'(\cos\theta)$$

$$\vdots$$

When $u^{(k)}(x)$ is continuous, so is $\tilde{u}^{(k)}(\theta)$ and $\tilde{u}^{(k)}(\theta)$ vanishes at $x=0,\pi$ when k is odd.

$\Rightarrow$ $u(\cos\theta)$ has a smooth periodic extension to $[0,2\pi]$ when $u(x)$ is smooth on $[0,\pi]$.

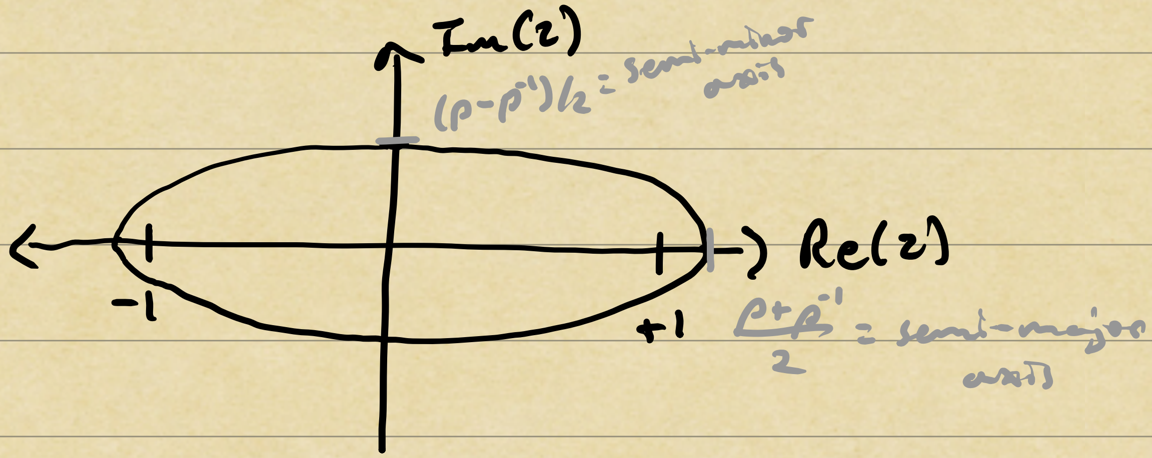
$\Rightarrow$ Chebyshev convergence rates for smooth nonperiodic functions are similar to Fourier convergence rates for smooth periodic functions.

In particular, the truncation error for Cheb. series

$\Rightarrow$ is $\mathcal{O}(N^{-k})$ as $k \to \infty$ for $u(x)$ with k absolutely continuous derivatives on $[-1,1]$.

(Actually only need $u^{(k)}(x)$ w/ b'dd variation)

$\Rightarrow$ is $\mathcal{O}(\rho^{-N})$ for $u(x)$ analytic + b'dd in a "Bernstein ellipse" with radius $\rho > 1$.



The Bernstein ellipse is the image of the periodic strip, under map $x = \cos\theta$, used to determine Fourier convergence.

Many "good" families of orthogonal polynomials on b'dd intervals have similar approximation properties.

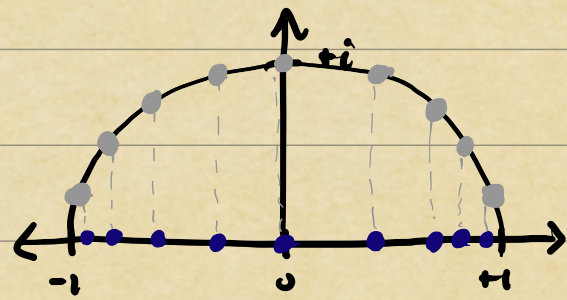
Fast Chebyshev Transforms

The connection w/ Cosine series allows us to evaluate Cheb. series & compute Cheb. coeffs fast.

To connect with cosine transform, take equispaced pts

in θ: $\quad \theta_n = \cos^{-1} x_n = \frac{\pi}{N}(n+\frac{1}{2}) \qquad n=0,1,\ldots,N-1$

Cheb pts: $\quad x_n = \cos\left(\frac{\pi}{N}(n+\frac{1}{2})\right) \qquad n=0,1,\ldots,N-1$



Chebyshev pts in $(-1,1)$

Evaluation:

$$U(x_n) = \sum_{k=0}^{N-1} \hat{u}_k T_k(x_n)$$

$$= \sum_{n=0}^{N-1} \hat{u}_k \underbrace{\cos\left(\frac{\pi}{N} k(n+1/2)\right)}_{\text{close to DCT III}}$$

For DCT III, need $\hat{u}_0 \to \hat{u}_0/2$ in above series.

Coefficients:

$$\hat{u}_k = \frac{1}{c_k}\int_{-1}^{1} T_k(x)\,u(x)\frac{dx}{\sqrt{1-x^2}} \overset{\substack{x=\cos\theta \\ dx=-\sin\theta\, d\theta}}{=} \frac{1}{c_k}\int_0^{\pi} \cos(k\theta)\,u(\cos\theta)\,d\theta$$

$$c_k = \begin{cases} \pi & k=0 \\ \pi/2 & k\geq 1 \end{cases}$$

$$\tilde{u}_k = \frac{1}{2c_k}\left[\int_{-\pi}^{\pi} \cos(k\theta)\, u(\cos\theta)\, d\theta\right]$$

since cos( ) is even

Now, approx by periodic trapezoidal rule

$$\approx \frac{1}{2c_k}\,\frac{2\pi}{N}\sum_{n=0}^{N-1} u(x_n)\cos\left(\frac{\pi}{N}k(n+\tfrac{1}{2})\right)$$

$$= \frac{2a_{k0}}{N}\sum_{n=0}^{N-1} u(x_n)\underbrace{\cos\left(\frac{\pi}{N}k(n+\tfrac{1}{2})\right)}_{\text{DCT II}}$$

$$a_{k0} = \begin{cases} 1 & k \geq 1 \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Transforming between Cheb coeffs and values on Cheb grid can be done fast w/DCT.

Just need to tweak $\hat{u}_0$ in series representation:

$$\hat{u}_0 \to \frac{\hat{u}_0}{2}$$

$$u(x) = \frac{\hat{u}_0}{2} + \sum_{k=1}^{N-1} \hat{u}_k T_k(x)$$

coeffs → vals

$\underline{u} = \mathrm{dct3}(\underline{\hat{u}})$

$$\hat{u}_k = \frac{2}{\pi}\int_{-1}^{+1} T_k(x)\, u(x)\, \frac{dx}{\sqrt{1-x^2}}$$

$\underline{\hat{u}} = \frac{2}{N}\mathrm{dct2}(\underline{u})$

vals → coeffs