Chebyshev Spectral Methods (pt. 3)

Key Idea: Interpolate from values on grid and enforce differential equation at gridpts.

$\Rightarrow$ Given N distinct pts $x_0, \dots, x_{N-1} \in [-1,1]$, there is a unique interpolating polynomial of degree $N-1$:

p_N interpolates u on grid

$$p_N(x) = \sum_{n=0}^{N-1} u(x_n)\, l_n(x), \qquad l_n = \prod_{m \neq n} \frac{x - x_m}{x_n - x_m}$$

"Lagrange form" of p_N Lagrange basis

$\Rightarrow$ If $x_0, \dots, x_{N-1}$ is a "good" set of pts, p_N can't be much worse than the best possible deg. $N-1$ polynomial approximation of u.

interpolant in Π^{N-1} best approx in Π^{N-1}

$$\|u - p_N\| \leq (1 + \Lambda_N)\|u - p_*\|$$

"Lebesgue const."

Cheb Points ("Roots") $\Lambda_N \leq 1 + \frac{2}{\pi}\log(N)$

Differentiation Matrices:

$$\partial_x u(x) = \sum_{n=0}^{N-1} u(x_n)\, l_n'(x)$$

$$\begin{bmatrix} u'(x_0) \\ \vdots \\ u'(x_{N-1}) \end{bmatrix} = \begin{bmatrix} l_0'(x_0) & \cdots & l_{N-1}'(x_0) \\ \vdots & & \\ l_0'(x_{N-1}) & \cdots & l_{N-1}'(x_{N-1}) \end{bmatrix} \begin{bmatrix} u(x_0) \\ \vdots \\ u(x_{N-1}) \end{bmatrix}$$

$$\underline{u}' \qquad\qquad D \qquad\qquad \underline{u}$$

$\Rightarrow$ Result is exact when $u \in \Pi^{N-1}$

Multiplication Matrices:

$$a(x)\,u(x) = \sum_{n=0}^{N-1} a(x_n)\,u(x_n)\, l_n(x)$$

$$\begin{bmatrix} (a\cdot u)(x_0) \\ \vdots \\ (a\cdot u)(x_{N-1}) \end{bmatrix} = \begin{bmatrix} a(x_0) & & \\ & \ddots & \\ & & a(x_{N-1}) \end{bmatrix} \begin{bmatrix} u(x_0) \\ \vdots \\ u(x_{N-1}) \end{bmatrix}$$

$$\underline{a\cdot u} \qquad\qquad M_a \qquad\qquad \underline{u}$$

$\Rightarrow$ Even when $u \in \Pi^{N-1}$, result is not exact unless $au \in \Pi^{N-1}$ due to aliasing on grid.

Boundary Conditions

E.g. $\partial_x u + a u = f$ with $u(1) = 0$ on $x \in [-1, 1]$

Chebyshev pts "Extrema" $x_n = \cos\left(\frac{\pi n}{N-1}\right)$ $n = 0, \ldots, N-1$

"Boundary bordering" = replace eqn's at endpts

$$\begin{array}{l} \\ \\ \text{evaluate } u(1) \Rightarrow \end{array} \begin{bmatrix} & & \\ & D + M_a & \\ & \text{at } x_0, \ldots, x_{N-2} & \\ 0 & \cdots & 0 \quad 1 \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-2} \\ 0 \end{bmatrix} \Leftarrow \text{set } u(1) = 0$$

"Rectangular Projection" = Carefully resample on smaller grid to make room for B.C.s (see Lecture 14 "Further Reading")

$\Rightarrow$ Can handle general linear auxillary constraints accompanying ODE.

Nonlinear Problems

Example: $\partial_x u + u^2 = f$ s.t. $u(1) = 0$, $x \in [-1,1]$.

D.B.C.

nonlinear in u

Idea 1: "Fixed point" iteration

1. Guess u_0
2. Solve $\partial_x u_{n+1} + u_n u_{n+1} = f$ s.t. $u(1) = 0$

   linearization

   for $n = 0, 1, 2, \ldots,$ until convergence.

$\Rightarrow$ In general, may not converge (need fixed pt. operator to be contractive) or may converge slowly, depending on diff. op.

Idea 2: "Newton" iteration

Find u s.t. $F[u] = \partial_x u + u^2 - f = 0$

1. Initial guess w/ $u(1) = 0$
2. Solve $F_u[u_n] v_{n+1} = -F[u_n]$, s.t. $v_{n+1}(1) = 0$

   "Frechet" derivative
3. Update $u_{n+1} = u_n + v_{n+1}$

generalizes Newton's iter. from finite-dim. vector space to functions

Frechet Derivative

Generalizes Jacobian from finite-dim. vector spaces to inf-dim. spaces of functions.

Directional "Gateaux" derivative

$$F_u[u_n]v = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} F[u_n + \varepsilon v]$$

Example $F[u] = \partial_x u + u^2 - f$

$$\frac{\partial}{\partial \varepsilon} F[u_n + \varepsilon v] = \frac{\partial}{\partial \varepsilon}\left[\partial_x u_n + \varepsilon \partial_x v + (u_n + \varepsilon v)^2 - f\right]$$
$$= \partial_x v + 2(u_n + \varepsilon v)v$$

As $\varepsilon \to 0$, we get

$$F_u[u_n]v = \partial_x v + 2u_n v$$

linear diff. operator

So, we can write Newton's iteration as

1. Solve $\underbrace{\partial_x v_{n+1} + 2u_n v_{n+1}}_{F_u[u_n]v} = \underbrace{f - \partial_x u_n - u_n^2}_{-F[u_n]}$ s.t. $v_{n+1}(1) = 0$

2. Update $u_{n+1} = u_n + v_{n+1}$ (repeat until conv.)