Chebyshev Spectral Methods (pt. 2)

Truncated Chebyshev series

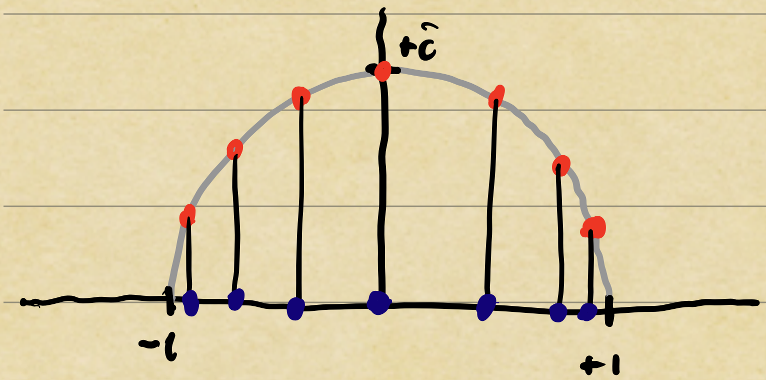
$$u(x) \approx \frac{\hat{u}_0}{2} + \sum_{k=1}^{N-1} \hat{u}_k T_k(x)$$

where

$$\hat{u}_k = \frac{1}{\pi}\int_{-1}^{+1} T_k(x)\, u(x) \frac{dx}{\sqrt{1-x^2}} = \langle T_k, u \rangle_w$$

$$\underline{u} = [u(x_0) \cdots u(x_{N-1})] \underset{\text{Transform}}{\overset{\text{DCT-based}}{\longleftrightarrow}} \underline{\hat{u}} = [\hat{u}_0 \cdots \hat{u}_{N-1}]$$

Chebyshev grid — Chebyshev coeffs



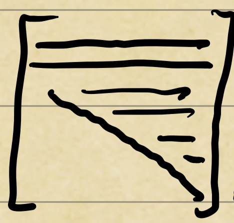
Chebyshev "roots" grid

$$x_n = \cos\left(\frac{\pi}{N}(n+1)\right)$$

Nonperiodic Analogue of Fourier Spectral Method?

$$-\partial_x^2 u = f \qquad u(\pm 1) = 0$$

$$\langle T_m, -T_n'' \rangle = \langle T_m, f \rangle \longrightarrow \quad n, m = 0, \ldots, N-1$$



$$\begin{bmatrix} \hat{u}_0 \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}$$

upper triangular

⇒ Dense diff matrizes, but good convergence properties.

⇒ Construct mats from fast recurrence relations.

Spectral Collocation

Idea: Interpolate from values on grid and enforce differential equation at gridpts.

Given N distinct pts $x_0, \ldots, x_{N-1} \in [-1,1]$, there is a unique interpolating polynomial of degree $N-1$:

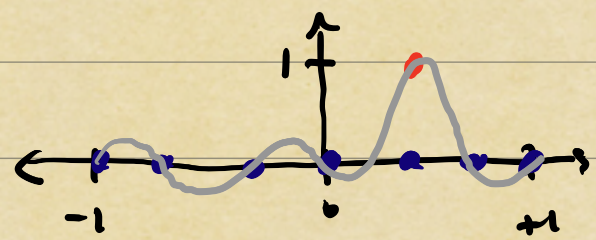
$$p \in \Pi^{N-1} \quad \text{s.t.} \quad p(x_n) = u(x_n) \quad \text{for} \quad n = 0, \ldots, N-1.$$

How to discretize and what is the error?

Lagrange Basis (for Π^{N-1})

$$\ell_n = \prod_{m \neq n} \frac{x - x_m}{x_n - x_m} \quad \Rightarrow \quad \ell_n(x_\ell) = \delta_{n,\ell}$$

$\Rightarrow$ degree $N-1$ poly interpolating $\delta_n(x_\ell) = \begin{cases} 0 & \ell \neq n \\ 1 & \ell = n \end{cases}$



Interpolant of $f(x)$ on grid has form

$$p_{N-1}(x) = \sum_{n=0}^{N-1} u(x) \, l_n(x)$$

"Lagrange" form of unique interpolant $p_N \in \Pi_n^{N-1}$

Lebesgue Constants

When grid is a "good" set of points, polynomial interpolation is close to the best possible polynomial approximant of f, in particular, it can't be much worse than the truncated Chebyshev approx.

Let $p_* = \underset{p \in \Pi^{N-1}}{\operatorname{argmin}} \|u - p\| =$ "best $\overset{\text{degree } N-1}{\text{polynomial}}$ approximant"

and $P_{N-1}(u) = \sum_{n=0}^{N-1} u(x_n) \, l_n(x) \in \Pi^{N-1}$ (interpolant).

$$\begin{aligned}
\Rightarrow \quad \|u - P_{N-1}(u)\| &\leq \|u - p_*\| + \|p_* - P_{N-1}(u)\| \\
&= \|u - p_*\| + \|P_{N-1}(p_* - u)\| \\
&\leq (1 + \|P_{N-1}\|)\|u - p_*\|
\end{aligned}$$

$\Rightarrow \|P_{N-1}\| = \Lambda_N$ is the "Lebesgue Constant" associated w/ interpolation grid $\{x_0, \ldots, x_{N-1}\}$.

$\Rightarrow$ When Λ_N grows modestly with N, $p_{N-1} \to u$ rapidly (e.g., spectral accuracy) as $N \to \infty$.

For equally spaced points, Λ_N grows exponentially.

For Chebyshev points, Λ_N grows only logarithmically.

Chebyshev "Roots Grid"

$$\Lambda_N \leq 1 + \frac{2}{\pi} \log(N)$$