Fourier Spectral Methods (Part 1)

Solve $-\partial_x^2 u = f$
w/ periodic B.C.s
on $x \in [0, 2\pi]$

Idea: expand $u(x)$, $f(x)$ in eigenmodes of $-\partial_x^2$ (linear op.).

$$-\partial_x^2 \underbrace{e^{inx}}_{\text{eigen-mode } \varphi_n} = \overbrace{n^2}^{\text{eigenvalue } \lambda_n} e^{inx} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The functions $\varphi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ form an orthonormal basis for the Hilbert space $L^2(\Omega)$, with $\Omega = [0, 2\pi]$

$$L^2(\Omega) = \left\{ u \text{ s.t. } \int_\Omega |u(x)|^2 dx \right\}$$

This is a vector space of functions* with

inner product $\qquad \langle f, g \rangle = \int_\Omega \overline{f(x)} g(x) dx$

norm $\qquad \|f\| = \int_\Omega |f(x)|^2 dx = \sqrt{\langle f, f \rangle}$

* Technically, of "equivalence classes of functions that agree except on sets of measure zero."

Every function $u \in L^2$ has an expansion

$$u(x) = \sum_{n=-\infty}^{+\infty} \hat{u}_n \varphi_n(x) \quad \text{with} \quad \hat{u}_n = \langle \varphi_n, u \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{u}_n e^{inx} \qquad \qquad = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} u(x)\,dx$$

i.e., the usual Fourier series.

Caution! For general $u \in L^2$, the Fourier series converges in the norm $\|\cdot\|$, but may fail to converge on a set of points with measure zero! In the context of spectral methods, our functions will typically be smooth enough that the series converges rapidly at every point in Ω.

Goal: Compute Fourier coeffs of u

$\Rightarrow$ In principle, we can reduce the Poisson problem to an infinite-dimensional, diagonal linear system for $\hat{u}_n$, $n = 0, \pm 1, \pm 2, \ldots$ because $\{\varphi_n\}$ diagonalize $-\partial_x^2$.

$\Rightarrow$ In practice, we can only solve finitely many equations for finitely many unknowns on the computer.

Enforce PDE holds on subspace $\operatorname{span}\{\varphi_n\}_{n=-N}^{N}$ (often called "test basis")

$$\langle \varphi_n, -\partial_x^2 u \rangle = \langle \varphi_n, f \rangle \qquad n=0,\pm 1,\pm 2,\dots$$

Since $\{\varphi_n\}$ are eigenfunctions and an ONB, we

get $\Rightarrow$
$$\lambda_n \hat{u}_n = \hat{f}_n \qquad n=0,\pm1,\pm2,\dots$$

or
$$\hat{u}_n = \frac{\hat{f}_n}{\lambda_n} \qquad n=\pm1,\pm2,\dots$$

with a guage condition replacing the $n=0$ condition (because $\lambda_0=0$) so the problem is well-posed: $\hat{u}_0 =$ user-specified guage $= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} u(x)\,dx$.

We can then approximate $u(x)$ anywhere on the domain $\Omega=[0,2\pi]$ by computing

$$u(x) \approx \sum_{n=-N}^{N} \hat{u}_n \varphi_n(x).$$

Q1: How do we get the coeffs $\hat{f}_n$?

Q2: How small is the truncation error

$$E_N = \| u - \sum_{n=-N}^{N} \hat{u}_n \varphi_n \| ?$$

coeffs

Use quadrature to approximate

quad weights

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)\,dx \approx \sum_{n=-N}^{N} w_n e^{-inx_n} f(x_n)$$

quad nodes

If we choose $x_n = \frac{2\pi n}{N}$ and $w_n = \frac{2\pi}{N}$, this

(a) is essentially a DFT of the vector

$$\underline{f} = [f(x_{-N})\ f(x_{-N+1})\ \cdots\ f(x_N)]^T$$

(b) is exact for the unique trigonometric polynomial interpolant of f on the grid $\{x_n\}_{n=-N}^{N}$, which only differs from the truncated

Fourier series $\sum_{n=-N}^{N} \hat{f}_n \phi_n$ by aliasing errors. These, as a rule of thumb, are of roughly the same magnitude as the truncation error itself.

$\Rightarrow$ How do we understand truncation error? How fast does $\sum_{n=-N}^{N} \hat{f}_n \phi_n \to f$?

Truncation Error

The truncation error depends on the smoothness of f.

If f has k continuous derivatives and $f^{(k+1)}$ is integrable on the periodic interval Ω:

$$E_N = \| f - \sum_{n=-N}^{N} \hat{f}_n \phi_n \| \leq \underset{\uparrow \text{const.}}{C_k} N^{-k}$$

$\Rightarrow$ Algebraic convergence of order k.

If f has infinitely many continuous derivatives then $E_N \to 0$ faster than any power of N.

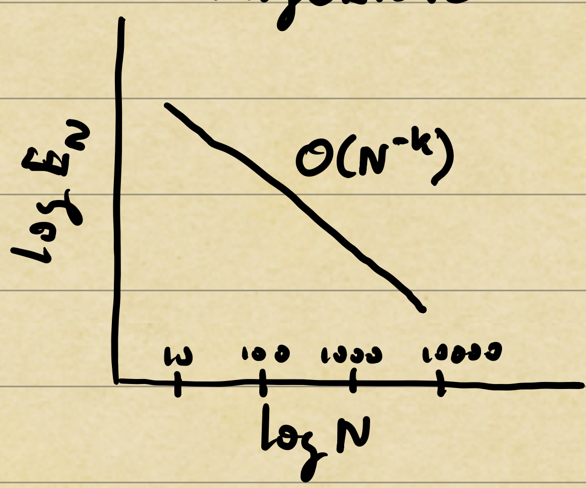
If f is analytic in a complex strip with width $> \rho$, then

$$E_N = \| f - \sum_{n=-N}^{N} \hat{f}_n \varphi_n \| \leq \overset{\text{const}}{C} e^{-\rho N}$$

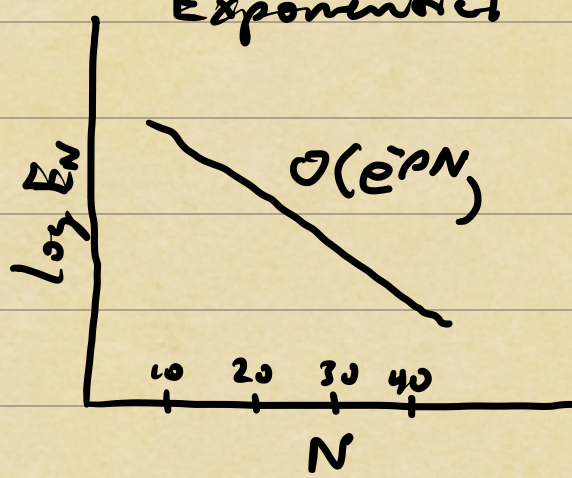
$\Rightarrow$ Exponential convergence with rate $\rho > 0$.

If f is an entire function, then $E_N \to 0$ faster than exponentially in N!

Algebraic



Exponential



So when f is very smooth, errors in $\hat{f}_n$ decrease extremely fast as N grows!

For elliptic problems like Poisson's equation on domains w/smooth boundary

smooth RHS f $\Rightarrow$ Smooth solution u

This is the notion of elliptic regularity.

$\Rightarrow$ Spectral methods can lead to extremely efficient schemes for elliptic problems on smooth domains: sphere, disk, cylinder, etc.

$\Rightarrow$ They leverage smoothness in the problem to build rapidly convergent approximations.