Fast Methods for PDE & IE

Github.com/mitmath/18336 – find everything here!

Gradescope – submit assignments here

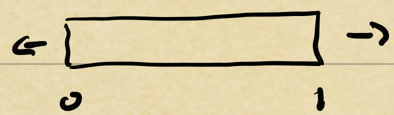
Piazza – ask questions here (and announcements)

$\Rightarrow$ Links in 1st Announcement on Canvas

Why solve PDEs & IEs?

Heat flow in a rod

$$\overset{\text{temp.}}{\frac{\partial u}{\partial t}} = \overset{\text{thermal diff.}}{\alpha} \frac{\partial^2 u}{\partial x^2} + \overset{\text{volumetric heat source}}{f(x)}$$



$u(0) = 0,\ u(1) = 1$

Modeling $\Longleftrightarrow$ Simulation

Detection/Inference (inverse problems)

Design (optimization)

Modern applications often involve very large simulations, that require solving complex PDEs many times in an "inner loop."

$\Rightarrow$ Weather / Climate simulations

$\Rightarrow$ Electronic Structure (material design)

$\Rightarrow$ Photonic design

$\Rightarrow$ Fluids / multiphysics

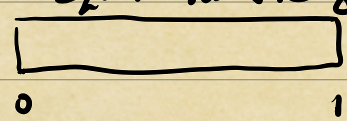
$\Rightarrow$ ... Many more!

Numerical Methods for PDE

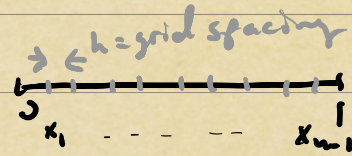
$$\frac{d^2u}{dx^2} = f(x) \quad (*)$$ Stationary PDE (Laplace) 1D

$u(0) = u(1) = 0$

Equilibrium temp



Step 1: Discretize



"solution vector"

$\underline{u}_n = [u_1, u_2, \ldots, u_{n-1}]$ $\quad \underline{f} = [f_1, \ldots, f_{n-1}]$

matrix $(n-1)\times(n-1)$

PDE becomes $\Rightarrow$ $A_n \underline{u}_n = \underline{f}_n$

Step 2: Solve

numerical linear algebra

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$A_n \qquad \underline{u}_n \qquad \underline{f}_n$$

Solve $(n-1)\times(n-1)$ linear sys. with NLA routines.

Step 3: Approximate/Analyze

How close is $\underline{u}_n = [u_1, \ldots, u_{n-1}]$ to the truth $\underline{u}_n^* = [u(x_1), \ldots, u(x_n)]$?

What is $\|\underline{u}_n - \underline{u}_n^*\|$?

Usually want to understand how

$\|\underline{u}_n - \underline{u}_n^*\|$ scales as $n \to \infty$.

This depends on how we "discretize" the problem in step 1.

How should we think about "cost" of solving this stationary PDE (Laplace Eq.)?

1) How much time to compute to a desired tolerance $\|\underline{u}_n - \underline{u}_n^*\| \le \varepsilon_*$?

2) How many FLOPS, required to compute $\underline{u}_n$ so that $\|\underline{u}_n - \underline{u}_n^*\| \le \varepsilon_*$?

(2) is traditional and often useful, but does not always correlate with (1)!

$\Rightarrow$ memory

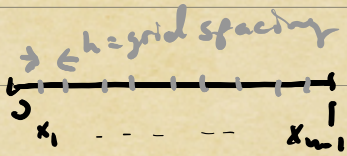
$\Rightarrow$ communication

$\Rightarrow$ Distributed, asynch, etc.

Finite Difference Approximations

$$\frac{du}{dx} = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}$$

$$\frac{d^2u}{dx^2} = \lim_{h\to 0} \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$



$\Rightarrow$ Pick finite $h > 0$ for "FD" approx

PDE at j^{th} Gridpt:

$$\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} = f_j \qquad \text{for } j=1,\dots,n-1$$

all together $\Rightarrow$
$$\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

tridiag! $\Rightarrow$ $\mathcal{O}(n)$ solve

Gaussian Elim
$$-\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

sparse/banded LU factorization!

What is the accuracy of u_n?

$$\frac{d^2u}{dx^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2)$$

from Taylor series

term $\leq Ch^2$ for suff. small $h>0$.

as $h \to 0$ (as $n \to \infty$)

$h \sim 1/n$

Can we improve the scaling with n?

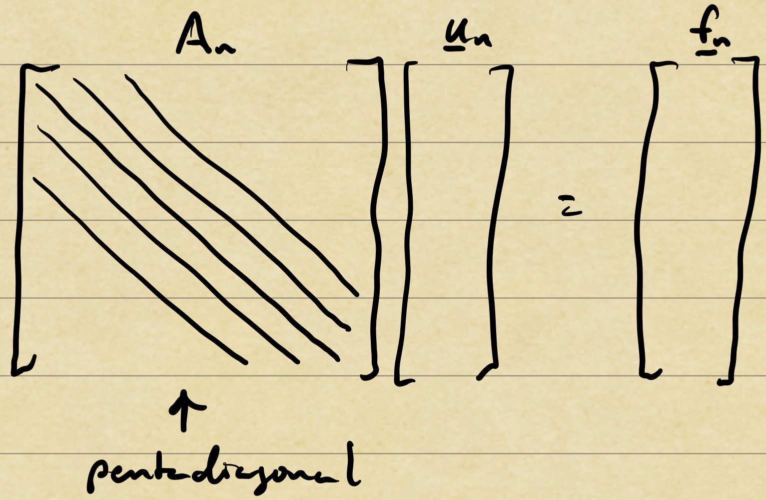
$\Rightarrow$ trade off smoothness!

Higher Order FD

$$\frac{d^2u}{dx^2} = \frac{-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h)}{12h^2} + O(h^4)$$

Error decreases faster with h!

But bandwidth increases



$\Rightarrow$ Trade-off between cost of linear solve and size of linear system required to achieve error tolerance $\varepsilon > 0$?