

# Multidimensional Spectral Methods

We can couple sparse, spectrally accurate ultra-spherical discretizations w/ other techniques to build fast solvers for multidimensional problems.

Some care is required to achieve optimal complex and maintain spectral accuracy or similar.

## Nonlinear IVP

$$\partial_t u + Lu = f(u)$$

$\uparrow$   
linear diff. op.  
 $\downarrow$   
nonlinear function

Implicit  $\Downarrow$  Explicit

## Time-Stepping Scheme

From initial  $u_0 = u(0)$

- 1) Compute RHS w/ $u_n$
- 2) Solve BVP for  $u_{n+1}$

$$(I - \Delta t L) u_{n+1} = u_n + \Delta t f(u_n)$$

$\uparrow$   
timestep  $\Delta t > 0$

Given RHS, we can solve the BVP at a time-step in  $\mathcal{O}(N)$  complexity using ultraspherical discretizations.

However, for typical nonlinear functions (like  $f(u) = |u|^{1/2}u$ ), multiplication via Cheb coeffs of  $u_n$  requires  $\mathcal{O}(N^2)$  flops. It's faster to transform coeffs  $\rightarrow$  values and update RHS on Cheb grid, where multiplication is diagonal.

### "Pseudospectral Method"

Given  $u_0 \doteq u(0)$

Repeat for each time step:

- $\Rightarrow$  Compute RHS values on Cheb Grid.
- $\Rightarrow$  Fast Cheb Transform for Cheb Coeffs of RHS.
- $\Rightarrow$  Solve BVP using ultras method.
- $\Rightarrow$  Fast Cheb Transform for soln. values on grid.

Each iteration costs  $\mathcal{O}(N \log N)$  flops.

### Spectral Methods in IVPs

Note that timestepping scheme is not high-order accurate. Why/When to use spectrally accurate spectral disc.?

$\Rightarrow$  May be able to use smaller spectral discretizations, saving computational cost for solve at each iteration.

$\Rightarrow$  Spectral methods often have better dispersion properties than low-order spectral discretizations, so some physics may be better captured w/ spectral method.

Spectrally accurate discretizations in space may also be used w/ spectrally accurate methods in time.

$\Rightarrow$  Global-in-time spectral methods

- use global basis for time-dependence.

$\Rightarrow$  Contour-Integral Methods

$$\cdot e^{it} u_0 = \frac{1}{2\pi i} \left[ \int_{\Gamma} e^{zt} (z-L)^{-1} dz \right] u_0$$

$$\approx \sum_{k=1}^m u_k e^{z_k t} (z_k - L)^{-1} u_0$$

$\Rightarrow$  Other exponential integrators

## Systems of Eqn's

Suppose we have a system of coupled ODEs:

$$\begin{bmatrix} \partial_x^2 - \omega_1^2 - k & k \\ k & \partial_x^2 - \omega_2^2 - k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(coupled oscillators)

If we discretize w/ltors on  $[-1, 1]$ , we get

$$\left[ \begin{array}{cc} \text{4 almost-banded nonzero blocks} & \end{array} \right] \begin{bmatrix} \hat{u}_{11} \\ \hat{u}_{1N} \\ \hat{u}_{21} \\ \hat{u}_{2N} \end{bmatrix} = \left[ \begin{array}{cc} \text{2 banded nonzero blocks} & \end{array} \right] \begin{bmatrix} \hat{f}_{11} \\ \hat{f}_{1N} \\ \hat{f}_{21} \\ \hat{f}_{2N} \end{bmatrix}$$

Notice that bandwidth is of order  $N$  and standard sparse solvers may be slow due to fill in if applied naively.

Instead, we can interleave unknown coeffs:  
Only "nearby" nodes interact between  $u_1$  and  $u_2$ .

Now have almost-banded structure composed of dense  $2 \times 2$  blocks from interactions among coupled nodes of  $u_1$  and  $u_2$ .

$\Rightarrow \mathcal{O}(N)$  solver via Woodbury, etc.

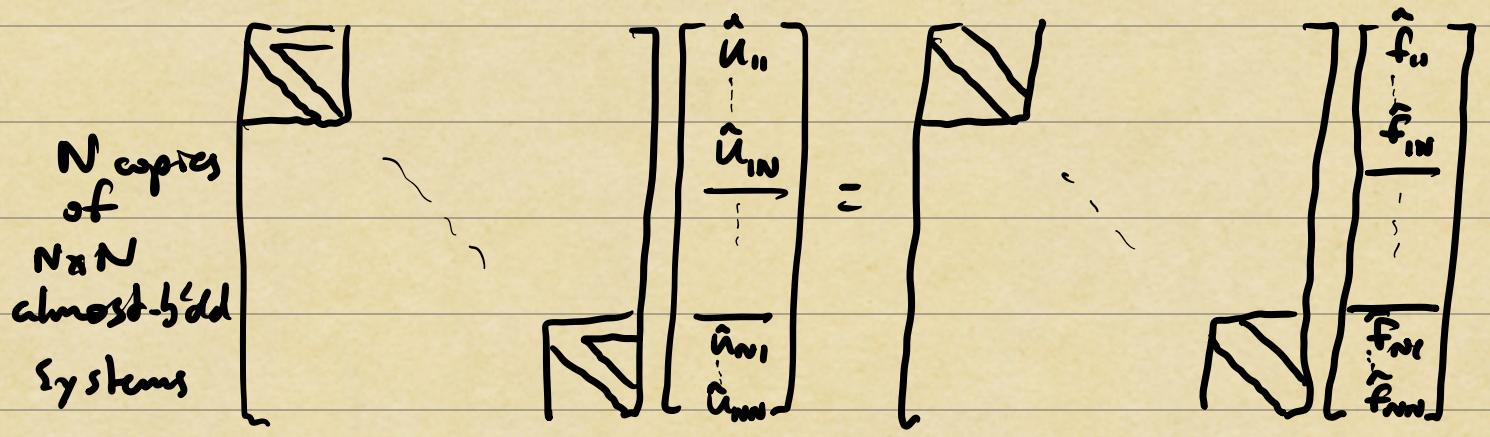
### One nonperiodic direction

Similarly, for multidim. problems w/one nonperiodic direction, we can get sparse almost-banded discretizations by interleaving

E.g. 
$$-\partial_x^2 u - \partial_y^2 u = f(x, y) \quad \begin{array}{l} y = \text{periodic} \\ x = \text{nonperiodic} \end{array}$$

$k^{\text{th}}$   
Fourier  
Mode  $\Rightarrow -\partial_x^2 \hat{u}_k - k^2 \hat{u}_k = \hat{f}_k(x)$

Now, discretize in  $x$  to get block diagonal system w/almost-banded blocks.



Similar decoupling even when multiple periodic directions are present.

### Multiple nonperiodic directions

Optimal Complexity spectral methods for multiple nonperiodic directions are more challenging and a current area of research.

A recent line of work develops fast Poisson solvers from fast 1D spectral methods using the ADI method.

Poisson soln. is steady-state soln

$$\partial_t u = \overbrace{\partial_{xx} u + \partial_{yy} u - f(x,y)}^{\text{Poisson soln. is steady-state soln}} \\ \sim \text{PBCs on } [-1,1]^2$$

Stony  
Splitting

$$e^{\alpha t(-\partial_{xx} - \partial_{yy})} \approx e^{-\alpha t \partial_{xx}} e^{\alpha t \partial_{yy}} \quad (\omega(\Delta t^2))$$

↑                      ↑  
propagated in x    propagated in y

Take

$N \times N$

$K$  = discretization of  $\partial_{xx}$

$K^T$  = discretization of  $\partial_{yy}$

ADI iterates:

$$(I - \frac{\alpha t}{2} K) X^{(n+1/2)} = X^{(n)} (I + \frac{\alpha t}{2} K^T) + F$$
$$X^{(n+1)} (I - \frac{\alpha t}{2} K^T) = (I + \frac{\alpha t}{2} K) X^{(n+1/2)} + F$$

Under certain conditions on  $K$ , ADI can take large "time-steps" and finds a steady-state solution (i.e., solves Poisson's eqn) fast.

$\Rightarrow K$  has  $N$  orthogonal eigenvectors

$\Rightarrow$  Eigenvalues are contained in  $[N, -1]$

In this case, special "time-steps" can be chosen for which ADI converges to Poisson steady-state exponentially fast!

(up to log factors)

Since solves w/K are  $\mathcal{O}(N)$ , this is optimal.