Fourier Spectral Methods (pt. 2)

$-u_{xx} = f$

periodic B.C.s

on $[0, 2\pi]_{per}$

w/gauge $g = \frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx$

ONB $\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$

for $n = 0, \pm 1, \pm 2, \ldots$

Soln $u(x) \approx \sum_{|n|\leq N} \hat{u}_n \varphi_n(x)$

RHS $f(x) \approx \sum_{|n|\leq N} \hat{f}_n \varphi_n(x)$

Algorithm

① Compute $\underbrace{\hat{\underline{f}}}_{\text{Fourier coeffs}} = \overset{\text{FFT}}{F} \underbrace{\underline{f}}_{\text{values on grid } x_n = \frac{hn}{2\pi}}$

② Solve $\hat{u}_m = \hat{f}_m / \lambda_m$ ← eigenvalues of $-\partial_x^2$ on $[0, 2\pi]_{per}$ $\quad m = \pm 1, \pm 2, \ldots, \pm N$

and $\hat{u}_0 = g$ (gauge)

③ Compute $\underline{u} = \underset{\text{IFFT}}{F^{-1}} \hat{\underline{u}}$

Note: identical to fast FD solver except we use eigenvalues of $-\partial_x^2$ (n^2) instead of $D = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$.

$\Rightarrow$ fast 2^{nd} order Δ FD solver is only 2^{nd} order accurate because

$$2-2(1-(\tfrac{2\pi m}{N})^2+(\tfrac{2\pi m}{N})^4\ldots)$$

$$\lambda_m^{FD} = \left(\frac{2\pi}{N}\right)^2\left[2-2\cos\left(\frac{2\pi m}{N}\right)\right] = m^2\left(1+O\left(\left(\frac{m}{N}\right)^2\right)\right)$$

Taylor expand

Only the low lying modes are accurate ($m \ll N$), leading to the low-order accuracy of the FD scheme.

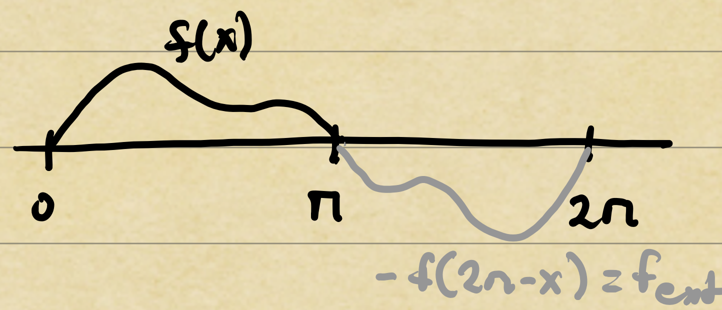
Non-periodic problems

Odd periodic extension

$-u_{xx} = f$

$u(0) = u(\pi) = 0$

(Dirichlet B.C.s)



We can solve using a sine series, BUT convergence rates will depend on smoothness of the odd-periodic extension of $f(x)$.

Algorithm

ONB $\quad \phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$

soln $\quad u(x) = \sum_{n=1}^{N} \hat{u}_n \phi_n(x)$

RHS $\quad f(x) = \sum_{n=1}^{N} \hat{f}_n \phi_n(x)$

1. $\hat{\underline{f}} = DST1(\underline{f})$
2. $\hat{u}_n = \hat{f}_n / n^2$
3. $\underline{u} = IDST1(\hat{\underline{u}})$

Note that for, e.g., super-algebraic convergence we need for $k = 0, 1, 2, \ldots$

smooth interior $\Rightarrow$ $f^{(k)}(x)$ continuous on $[0, 2\pi]$

smooth extension $\Rightarrow$ $f^{(k)}(0) = f^{(k)}(2\pi)$

$$f_{ext}(x) = -f(2\pi - x) \Rightarrow f_{ext}^{(n)}(x) = (-1)^{n+1} f^{(n)}(2\pi - x)$$

for continuity at $x = \pi$

$$\lim_{x \to \pi^-} f^{(n)}(x) = \lim_{x \to \pi^+} (-1)^{n+1} f^{(n)}(2\pi - x)$$

for odd n $\Rightarrow$ $f^{(n)}(\pi) = f^{(n)}(\pi)$

for even n $\Rightarrow$ $f^{(n)}(\pi) = -f^{(n)}(\pi)$

The odd n condition is always satisfied but the even n condition can only be true when $f^{(n)}(\pi) = 0$. Similar calculation at $x = 0$ leads to

Super-algebraic convergence rate $\iff$ $f^{(n)}\big|_{x=0,\pi} = 0$ for all even n

Similar scheme based on even extensions and DCT1 for Neumann boundary conditions.

Multi-dimensional Fourier Spectral Methods

Fourier, Sine, and Cosine bases diagonalize const coefficient differential operators. Can extend to multi-dimensions using Kronecker products analogous to multi-dim. finite difference schemes.

Example: 2D Poisson w/homogeneous Dirichlet BCs. on $[0,\pi]^2$

ONB/ $\quad \phi_{nm}(x,y) = \frac{2}{\pi}\sin(nx)\sin(my)$

w/inner product $\langle f,g\rangle = \int_0^\pi \int_0^\pi f(x,y)g(x,y)\,dx\,dy$

soln/ $\quad u(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{N} \hat{u}_{nm}\,\phi_{nm}(x,y)$

RHS/ $\quad f(x,y) = \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=1}^{N} \hat{f}_{nm}\,\phi_{nm}(x,y)$

Step 1: Compute $\quad \hat{f}_{nm} = \langle \phi_{nm}, f\rangle$

Iterated integral = Sine Transform in x, then y

$$\hat{f}_{nm} = \langle \phi_{nm}, f \rangle = \frac{2}{\pi} \int_0^\pi \int_0^\pi \sin(nx)\sin(my) f(x,y)\,dx\,dy$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin(my) \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\pi \sin(nx) f(x,y)\,dx}_{\substack{\text{1D sine transform} \\ \text{in } x \Rightarrow \tilde{f}_n(y)}} \,dy$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin(my)\ \tilde{f}_n(y)\,dy \qquad \text{1D transform in } y$$

Approximating the inner and outer integrals using a trapezoidal rule on an equi-spaced grid leads to a Discrete Sine Transform (Type-1) along both array axis of sample matrix

$$F = \begin{bmatrix} f(x_1, y_1) & \cdots & f(x_1, y_N) \\ \vdots & & \vdots \\ f(x_N, y_1) & \cdots & f(x_N, y_N) \end{bmatrix}$$

of values of $f(x,y)$ on 2D equispaced grid (x_n, y_m)

(N interior grid points)

$$y_1 = x_1 = \frac{\pi}{N+1},\ y_2 = x_2 = \frac{2\pi}{N+1},\ \ldots,\ y_N = x_N = \frac{N\pi}{N+1}$$

Step 2: Solve for $\hat{u}_{nm}$ (diagonal system)

$$\langle \phi_{nm}, -\Delta u \rangle = \underbrace{(n^2+m^2)}_{\text{eigenvals of } -\Delta \text{ w/ ZDBCs}} \hat{u}_{nm} \quad \text{and} \quad \langle \phi_{nm}, f \rangle = \hat{f}_{nm}$$

$$\Rightarrow \hat{u}_{nm} = \hat{F}_{nm} / (n^2+m^2) \qquad \text{for } n,m = 1,\ldots,N$$

Step 3: Compute $u(x,y)$ on grid (equispaced)

Evaluating the 2D sine series

$$u(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} \hat{u}_{nm} \phi_{nm}(x,y)$$

on the equispaced grid (x_n, y_m) above leads to an inverse Discrete Sine Transform (Type-I) along both array axes of the $N \times N$ matrix

$$\hat{U} = \begin{bmatrix} \hat{u}_{11} & \cdots & \hat{u}_{1N} \\ \vdots & & \vdots \\ \hat{u}_{N1} & \cdots & \hat{u}_{NN} \end{bmatrix}$$