FFT-based Fast Solvers

Key step: transform problem to Fourier domain where "translation invariant" operators are diagonal!

diff. norm's possible

$$
\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}
= \frac{1}{\sqrt{n}}
\begin{bmatrix}
\omega_n^0 & \omega_n^0 & \omega_n^0 & \cdots & \omega_n^0 \\
\omega_n^0 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\
\omega_n^0 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
\omega_n^0 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)}
\end{bmatrix}
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}
$$

output coefficients $\underline{\hat{f}}_n$

$V_n = n \times n$ DFT

$\omega_n = e^{2\pi i/n}$

input values $\underline{f}_n$

Radix-2 FFT: $n = 2^\ell$ (power of 2)

$$
\underline{\tilde{f}}_n = V_n \underline{f}_n = \begin{bmatrix} I_{n/2} & I_{n/2} \\ I_{n/2} & -I_{n/2} \end{bmatrix} \begin{bmatrix} V_{n/2}\, \underline{f}_e^{(1)} \\ \Omega\, V_{n/2}\, \underline{f}_o^{(1)} \end{bmatrix}
$$

even index coeffs in $\underline{f}_n$

odd index coeffs in $\underline{f}_n$

"twiddle factors" = diagonal scaling

$\underline{\hat{f}}_n = V_n \underline{f}_n$
- $V_{n/2} f_e^{(1)}$
  - $V_{n/4} f_e^{(2)}$
  - $V_{n/4} f_o^{(2)}$
- $V_{n/2} f_o^{(1)}$
  - $V_{n/4} f_e^{(3)}$ ...
  - $V_{n/4} f_o^{(3)}$

$\log_2(n)$ levels $\Rightarrow O(n \log n)$

What about, e.g. $n = 3^\ell$ (power of 3)?

Fact 1: FFT is polynomial evaluation, $\underline{\hat{f}}_n = p(\underline{z}_n)$ ← n^{th} roots of unity

Fact 2: $p(z) = f_0 z^0 + f_1 z^1 + f_2 z^2 + f_3 z^3 + f_4 z^4 + f_5 z^5 + \ldots$

$$= p_0(z^3) + z p_1(z^3) + z^2 p_2(z^3)$$

↑ deg $\frac{n}{3} - 1$ poly's

Fact 3: $z_j = \omega_n^j$

a) $z_j^3 = (e^{-2\pi i j/n})^3 = e^{-2\pi i j/(n/3)} = (\omega_{n/3})^j$

b) $z_j^3 = z_{j-n/3}^3$ (aliasing)

c) $z_j = e^{-2\pi i/3} z_{j-n/3}$

$$\underline{\hat{f}}_n = \begin{bmatrix} p(z_0) \\ \vdots \\ \\ p(z_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} I & I & I \\ I & \omega_3 I & \omega_3^2 I \\ I & \omega_3^2 I & \omega_3^4 I \end{bmatrix}}_{\mathrm{DFT}(3) \otimes I_{n/3}} \begin{bmatrix} p_0(z_0^3) \\ \vdots \\ p_0(z_{\frac{n}{3}-1}^3) \\ z_0 p_1(z_0^3) \\ \vdots \\ z_{\frac{n}{3}-1} p_1(z_{n/3-1}^3) \\ z_0^2 p_2(z_0^3) \\ \vdots \\ z_{n/3-1}^2 p_2(z_{n/3-1}^3) \end{bmatrix}$$

$n/3 \times \frac{n}{3}$ identity

3 DFTs of size $\frac{n}{3} \times \frac{n}{3}$

In general, split based on prime factors of n.

E.g. $n = 12 = 3 \cdot 4 = 3 \cdot 2 \cdot 2$

$$
\mathrm{DFT}(12) \begin{cases} \mathrm{DFT}(4) < \begin{matrix}\mathrm{DFT}(2)\\ \mathrm{DFT}(2)\end{matrix} \\ \mathrm{DFT}(4) < \begin{matrix}\mathrm{DFT}(2)\\ \mathrm{DFT}(2)\end{matrix} \\ \mathrm{DFT}(4) < \begin{matrix}\mathrm{DFT}(2)\\ \mathrm{DFT}(2)\end{matrix} \end{cases}
$$

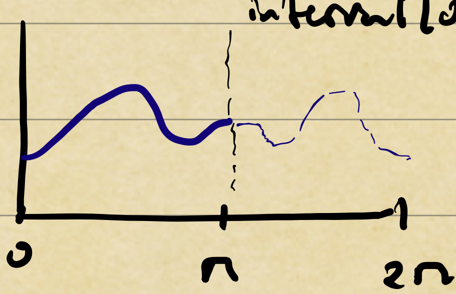
$\Rightarrow$ For large prime factors $\Rightarrow$ Rader's Algorithm
and others exploit special structure in V_n, n = prime

$\Rightarrow$ Essentially, scaling is always $\mathcal{O}(n \log n)$ but prefactors (constants in $\mathcal{O}(\ )$ notation) depend on prime factorization of n.

Fast Sine & Cosine Transforms

DCT (Type 1)

For functions where extension to periodic interval $[0, 2\pi)$



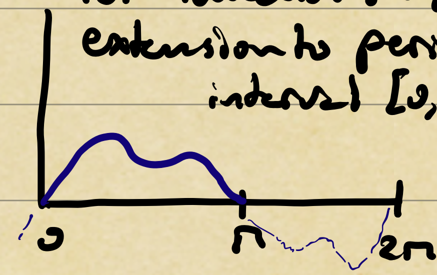
$$\hat{f}_j = \frac{2}{n} \sum_{k=0}^{n-1} f_k \cos\left(\frac{\pi j k}{n-1}\right)$$

$$= \mathrm{Re}\left[\frac{2}{n} \sum_{k=0}^{n-1} f_k e^{-2\pi i j k / 2(n-1)}\right]$$

FFT of length $2n-1$

DST (Type 1)

For functions w/odd extension to periodic interval $[0, 2\pi)$



$$\hat{f}_j = \underline{2} \sum_{k=0}^{n-1} f_k \sin\left(\frac{\pi(j+1)(k+1)}{n+1}\right)$$

$$= \mathrm{Im}\left[\frac{2}{n} \sum_{k=0}^{n-1} f_k e^{+2\pi i (j+1)(k+1)/2(n-1)}\right]$$

FFT of length $2n$ $(\hat{f}_0 = 0)$

$\Rightarrow$ Careful about normalization and index conventions!

$\Rightarrow$ DCT & DST have Type II, III variations for functions with other endpt behavior.

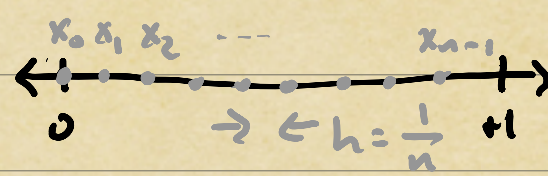
Fast 1D Poisson Solvers

$-\partial_x^2 u = f$ with $x \in [0,1)$

periodic B.C.'s $\quad u(0) = u(1)$, $\quad u'(0) = u'(1)$

2nd order FD Disc.

$$\frac{1}{h^2}\underbrace{\begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 2 \end{bmatrix}}_{A_n} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}}_{u_n} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}}_{f_n}$$



A_n is diagonalized by DFT matrix

$$\frac{1}{h^2}\begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}$$

where $\lambda_j = 2 - 2\cos\left(\frac{2\pi j}{n}\right)$ $\quad j = 0, 1, 2, \dots, n-1$

$\Rightarrow$ $\lambda_0 = 0$! A_n is not invertible.

$\Rightarrow$ The soln of the BVP w/periodic B.C.'s is not unique b/c u + const. is also a solution, for any const.

<u>Gauge Fixing</u> - we can determine a unique solution by specifying the mean of $u(x)$:

$$\int_0^1 u(x)\,dx = \underset{\uparrow}{g}^{\text{const. our choice of "gauge"}} \Rightarrow \underbrace{h\sum_{j=0}^{n-1} u_j}_{\text{discrete approx "trap. rule"}} = \sqrt{n}\,\hat{u}_0$$

Note that $\int_0^1 f(x)\,dx = -\int_0^1 \partial_x^2 u\,dx = -[\partial_x u]\Big|_0^1 \overset{\text{By periodic B.C.'s}}{=} 0$

Discrete version $\Rightarrow$ $h\sum_{j=0}^{n-1} f_j = \sqrt{n}\,\hat{f}_0 = 0$

So makes sense to replace 1^{st} Equation:

$$\frac{1}{h^2}\begin{bmatrix} 0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}\begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}$$

$\rightarrow$ system is consistent only if $\hat{f}_0 = 0$, but no useful info to determine $\hat{u}_0$.

$$\frac{1}{h^2}\begin{bmatrix} \sqrt{n}\,h & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}\begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{n-1} \end{bmatrix} = \begin{bmatrix} g \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}$$

$\rightarrow$ guage determines $\hat{u}_0$

All other solutions can be recovered by adjusting guage (adding constant to soln).