Hierarchical Low-Rank Approx of Green's Functions

$$\phi(y) = \int G(x-y)\rho(x)dx$$

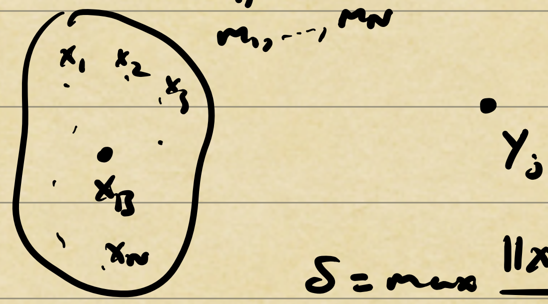
targets $y_1, \dots, y_M$

sources $x_1, \dots, x_N$

$m_1, \dots, m_N$

Poisson Kernel

$$G(x-y) = \frac{1}{2\pi}\log(\|x-y\|)$$



$$\delta = \max_{x_i} \frac{\|x_i - x_B\|}{\|y_j - x_B\|}$$

"separation"

Low-rank Approx.

Represent 2D vec by complex #'s

$$G(x_i - y_j) = G(x_B - y_j) - \frac{1}{2\pi}\sum_{p=1}^{k}\frac{1}{p}\left(\frac{x_i - x_B}{y_j - x_B}\right)^p + O(\delta^{k+1})$$

↖ separable terms from Taylor series

"Multipole expansion"

$$\Rightarrow \phi(y_j) = \sum_{i=1}^{N} m_i G(x_i - y_j)$$

$$= m_B G(x_B - y_j) - \frac{1}{2\pi}\sum_{p=1}^{k}\frac{1}{(y_j - x_B)^p}\underbrace{\sum_{i=1}^{N}\frac{m_i}{p}(x_i - x_B)^p}_{k \text{ length } N \text{ sums } = a_k} + O(\delta^{k+1})$$

↖ Error

$\Rightarrow$ Compute a_k $O(Nk)$

$\Rightarrow$ Compute pot at each y_j $O(Mk)$

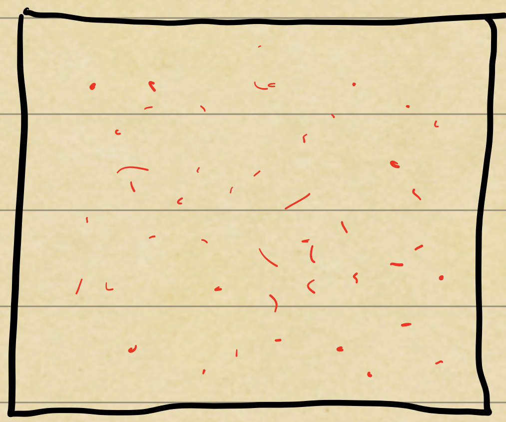
$\Rightarrow O((N+M)k)$

⇒ Similar expansions for 3D Poisson and other translation invariant "additive" or "multiplicative" kernels.

⇒ Only effective for $\delta \ll 1$ so that $k \ll \min(M,N)$.

<u>Idea:</u> Hierarchical subdivision to maintain separation between sources & targets.

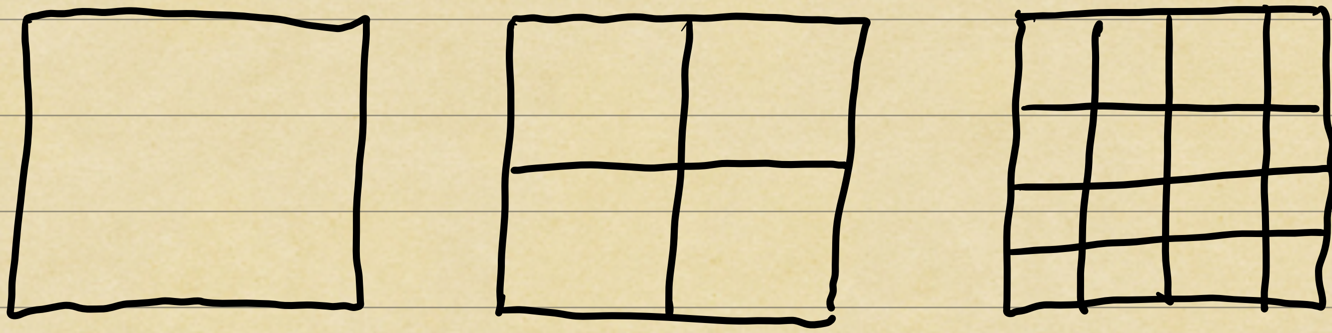
Suppose N "particles" are distributed in $[-1,1]^2$.



⇒ Calculate potential each particle feels due to others to target accuracy $\varepsilon > 0$.

Although the "sources" & "targets are not separate, we can iteratively partition domain into well separated regions. Recursively divide into 4s, until each box contains $\leq C$ particles ($C \leq 10$).

Total # levels $= L$

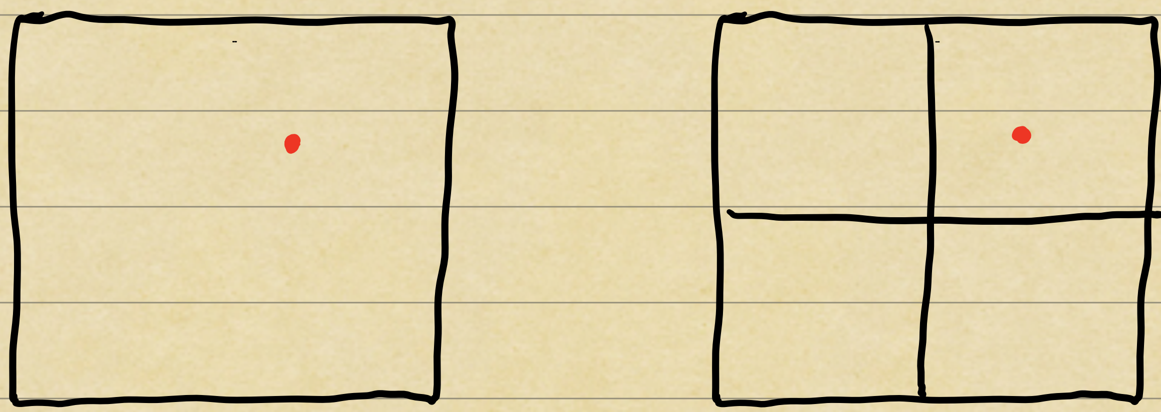


$l=0$ $l=1$ $l=2$

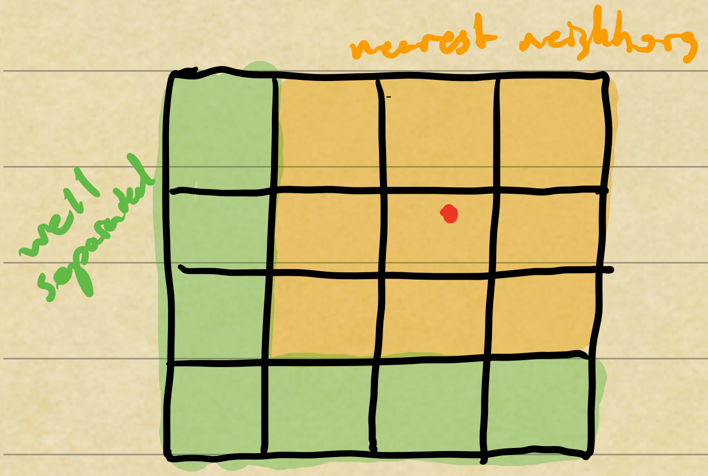
Need $N \approx 4^L c \Rightarrow L \approx \frac{1}{2}\log_2 \frac{N}{c} = O(\log N)$.

At each level we define the categories:

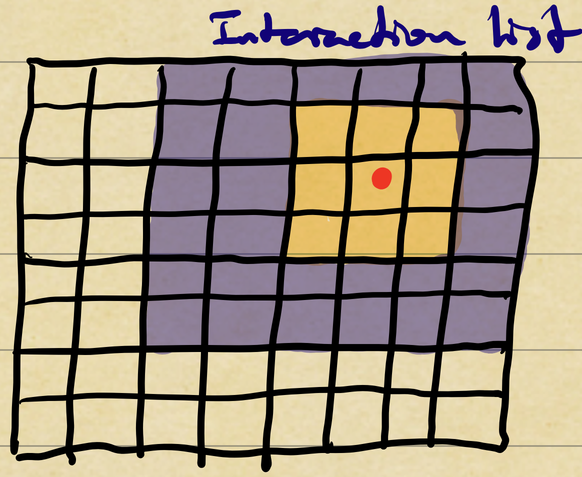
- $\Rightarrow$ Two boxes are "nearest neighbors" if touching.
- $\Rightarrow$ Two boxes are "well-separated" if not.
- $\Rightarrow$ Every box has an "interaction list," which are children of near-neighbors that are well-separated from current box at level.



Level 0 Level 1



Level 2



Level 3

At each level, a box has

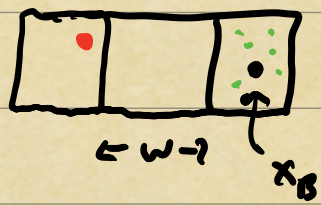
$\#$ nearest neighbors ≤ 9

$\#$ boxes interacting $\leq \underbrace{9 \cdot 4}_{\text{children of nearest neighbors}} - \underbrace{9}_{\text{not well separated}} = 27$

For each target point:

1) Compute potential from sources in nearest-neighbors list at lowest level
   $\sim 9c$ sources $\Rightarrow \mathcal{O}((9c)^2) = \mathcal{O}(1)$ (indep. of N)
2) For each level, compute potential from each box in interaction list using the multipole expansion about its center.

$\Rightarrow$ Boxes in interaction list are well-separated so that $\delta \le 1/2$.



$$\delta = \max_i \frac{\|x_B - x_i\|}{\|x_B - y_j\|} \le \frac{1}{2}$$

$$\delta \le \frac{\sqrt{2}\,\frac{w}{2}}{3\,\frac{w}{2}} = \frac{\sqrt{2}}{3} \approx 0.47$$

$\Rightarrow$ To achieve accuracy $\varepsilon > 0$, need

$$\varepsilon \sim \delta^{k+1} = \left(\tfrac{1}{2}\right)^{k+1} \Rightarrow k \sim 1 + \log_2 \varepsilon^{-1} = O(\log 1/\varepsilon)$$

At each level, costs $O(NK)$ to construct multipole expansions for each box.

For each target, it costs $O(27k)$ to evaluate potential contribution from each box in interaction list.

Total cost at all levels:

$$\begin{aligned} &\text{cost of multipole expansions} = O(kN\log_2 N) \\ +\ &\text{cost of potential at } N \text{ pts} = O(27kN\log_2 N) \\ \hline &= O(28 N \log_2 N \log_2 1/\varepsilon) \end{aligned}$$

<u>Idea:</u> Higherarchical subdivision to maintain separation between sources & targets.

⇒ The algorithm above is due to Barnes & Hut (1986)

⇒ It relies purely on "projection," compressing sources via multipole expansions and "broadcasting" to every target at each level.

⇒ The Fast Multipole Method (FMM) improves scaling to $O(N \log \frac{1}{\varepsilon})$ by carefully incorporating "interpolation," compressing targets via multipole expansions and carefully broadcasting to targets via "multipole-to-multiple" and "multipole-to-local" approximations.