Numerical Painleve II and V σ-Forms with Julia and Their Random Matrix Interpretations

(18.338 Final Project)

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A Short History of Painleve Equations [Zhang, 2017]

- The Painleve equations possess the so-called Painleve property: all the solutions are free from movable branch points (See Board for Examples).
- Discovered by Painleve and his colleagues at the beginning of the 20th century while classifying all second-order ordinary differential equations

$$y'' = R(x, y, y')$$

  which possesses the Painleve property.
- The solutions of Painleve equations are called the Painleve transcendents.
- If you're interested in history of Painleve Equations, please check [Takasaki, 2000]. This is a good expository work in history.

General Form of Painleve Equations [Edelman and Rao, 2005]

(I) Painleve I

$$y'' = 6y^2 + t$$

(II) Painleve II

$$y'' = 2y^3 + ty + \alpha$$

(III) Painleve III

$$y'' = \frac{1}{y}(y')^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

General Form of Painleve Equations [Edelman and Rao, 2005]

(IV) Painleve IV

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

(V) Painleve V

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

(VI) Painleve VI

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(y-t)^2}\right]$$

Areas in Eigenvalue Distribution [Edelman and Rao, 2005]

- **Bulk**: Refers to the statistical properties of the *interior eigenvalues*. These describe the typical, central behaviour of the spectrum.
- **Edges**: Refers to the behaviour of the *extreme eigenvalues* — the largest and smallest ones. Universality here gives rise to special limit laws such as the Tracy–Widom distribution (soft edge) or Bessel-type distributions (hard edge).

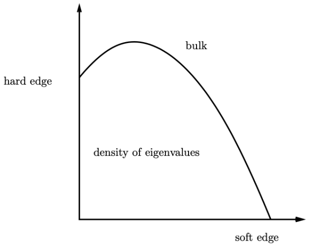


Figure 9.1. Regions corresponding to eigenvalue distributions that are of interest in random matrix theory.

Main Contributions

- Numerically verify that the appropriately normalized largest eigenvalues of Hermite/Laguerre ensembles match the Tracy–Widom distributions obtained from Painleve II σ form in Julia. [Soft Edge Universality]
- Implement a numerical solver for the Painleve V *sigma* and verify that its prediction for the spacing distribution agrees quantitatively with histograms from GUE (Hermitian ensemble with $\beta = 1$) simulations. [Bulk Universality]
- It is straightforward to verify the above results by comparing the figures in [Edelman and Persson, 2005]. However, the connections between random matrix theory and Painleve transcendents extend far beyond these classical examples. Many additional relationships remain to be fully explored, and several open questions are listed below.

Painleve II and Eigenvalue Distributions [Edelman and Rao, 2005]

Goal of the Project: Numerically verify that the appropriately normalized largest eigenvalues of Hermite/Laguerre ensembles match the Tracy–Widom distributions obtained from Painleve II σ form.

Painleve II equation

$$y''(x) = ty(x) + 2y(x)^3$$

Boundary condition (Hastings–McLeod solution):

$$y(x) \sim \mathrm{Ai}(x) \qquad x \to +\infty.$$

Painleve II and Eigenvalue Distributions [Edelman and Rao, 2005]

Connection to Random Matrices:

- Hermite ensembles (GOE/GUE/GSE), $\beta = 1, 2, 4$
- Laguerre ensembles, $\beta = 1, 2$
- Largest eigenvalue distribution *after proper scaling* equals Tracy–Widom F_β.

Numerical Tasks:

1. Solve Painlevé II using `DifferentialEquations.jl`.
2. Compute the Tracy–Widom CDF via the $y(x)$ solution.
3. Simulate Hermite/Laguerre matrices for $\beta = 1, 2, 4$.
4. Compare empirical CDF of largest eigenvalue with Painlevé prediction.

Hastings–McLeod Solution of Painleve II

The Hastings–McLeod solution $y_{\mathrm{HM}}(x)$ is a distinguished real solution of

$$y'' = 2y^3 + xy.$$

It is uniquely characterized by the asymptotic behavior

$$y_{\mathrm{HM}}(x) \sim \begin{cases} \mathrm{Ai}(x), & x \to +\infty, \\ \sqrt{-x/2}, & x \to -\infty. \end{cases}$$

Key properties:

- Real-valued and monotone.
- Connects Airy behavior to algebraic growth.
- Entire and *pole-free on the real axis*.
- Appears in the derivation of the Tracy–Widom distribution.

Tracy–Widom Laws F_β and Densities f_β

We denote by $F_\beta(s)$ the CDF and by $f_\beta(s)$ the PDF of the Tracy–Widom law for $\beta = 1, 2, 4$.

$\beta = 2$ **(GUE)**

$$f_2(s) = \frac{d}{ds} F_2(s)$$

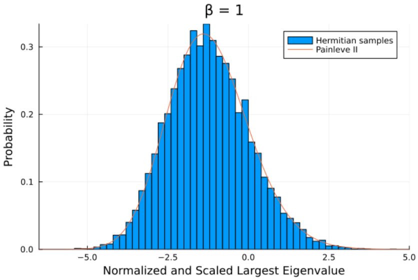
$$F_2(s) = \exp\left(- \int_s^\infty (x - s) q(x)^2 \, dx \right).$$

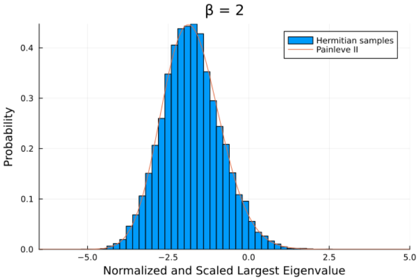
$\beta = 1, 4$ **(GOE, GSE):** The CDFs F_1 and F_4 are expressed in terms of F_2 and the same Painleve II solution q

$$F_1(s)^2 = F_2(s) \exp\left(- \int_s^\infty q(x) \, dx \right)$$

$$F_4\left(\frac{s}{2^{2/3}}\right)^2 = F_2(s) \left(\cosh \int_s^\infty q(x) \, dx \right)^2$$

Results to Soft Edge Universality [Painleve II]





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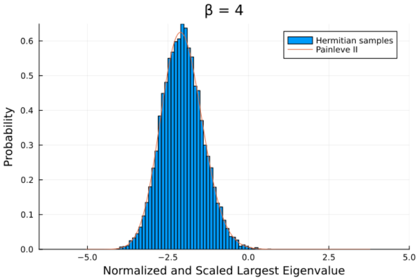

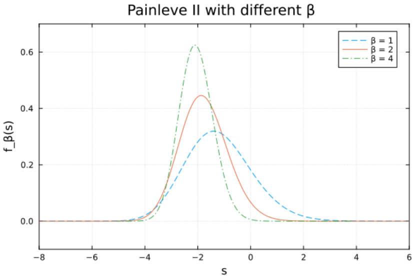

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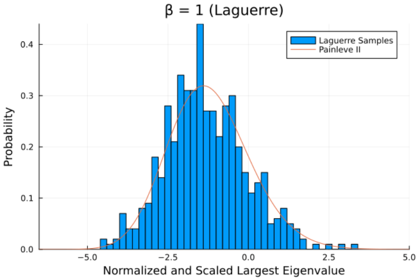
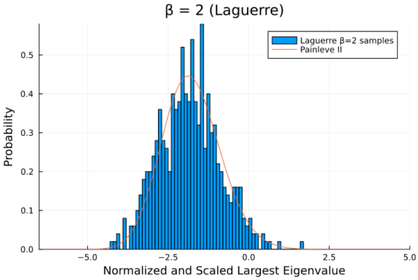
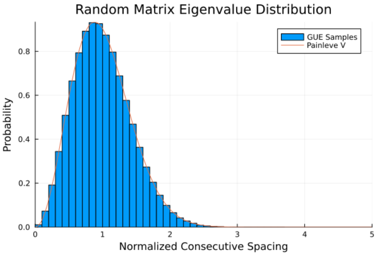




The number of trials is kept small due to the computational cost.

Results to Bulk Universality [Painleve V]



Open Questions

- **Painleve III [Forrester, 2010]**
  - Hard edge of Laguerre/Wishart β-ensembles (smallest eigenvalue near 0).
  - Gap probability at the hard edge (Bessel kernel) is a Fredholm determinant whose log-derivative satisfies a σ-form of III.
- **Painleve VI [Miller, 2011]**
  - Appears in more general deformations: unitary ensembles with external source, critical random matrix models, and analogues in number theory (L-functions).
  - Certain spacing and gap probabilities can be written in terms of VI tau-functions.

Any Questions?

Thank you for a great semester!

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