

# THE WEINGARTEN CALCULUS FOR HAAR-DISTRIBUTED UNITARY MATRICES

KYUHYEON CHOI

**ABSTRACT.** We review the Weingarten calculus for integrating polynomial functions of the matrix entries of a Haar-distributed unitary matrix. The central statement is the explicit expansion of mixed moments in terms of permutations and the Weingarten function. We emphasize the representation-theoretic mechanism behind the formula via Schur–Weyl duality and the group algebra  $\mathbb{C}[S_n]$ , and conclude with a brief discussion of large-dimension asymptotics.

## 1. INTRODUCTION

Integrals over the unitary group with respect to Haar measure arise naturally in a variety of physical problems, particularly in quantum scattering and transport. In the statistical theory of nuclear reactions, Mello and Seligman introduced a maximum-entropy ensemble of scattering matrices subject to unitarity and symmetry, which reduces the computation of correlation functions to integrals over  $U(d)$  [4]. Related unitary-group integrals also appear in the evaluation of partition functions in random-matrix models, where Gaudin and Mello used character expansions to compute expressions of the form  $\int_{U(d)} \exp(-\text{Re} \operatorname{Tr} \beta U) dU$  [3]. In the context of quantum transport through mesoscopic systems, Brouwer and Beenakker showed that the same unitary-group averages admit a diagrammatic expansion organized by permutations, providing a systematic way to compute non-Gaussian corrections beyond the leading approximation [1].

A common feature of these problems is the need to compute joint moments of matrix entries of a Haar-distributed unitary matrix. This problem was first systematically addressed in the seminal work of Weingarten [6]. Let  $U(d)$  denote the unitary group equipped with its normalized Haar probability measure. A basic question is how to evaluate integrals of the form

$$\int_{U(d)} u_{i_1 j_1} \cdots u_{i_n j_n} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_n j'_n} dU,$$

which encode correlations between matrix coefficients and play the role of building blocks for more complicated observables.

The Weingarten calculus provides an explicit answer to this problem. It shows that such Haar integrals can be written as finite sums over permutations, with coefficients given by the *unitary Weingarten function*. In its standard form, one has the moment formula

$$\int_{U(d)} u_{i_1 j_1} \cdots u_{i_n j_n} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_n j'_n} dU = \sum_{\sigma, \tau \in S_n} \delta_{i_{\tau(1)} i'_1} \cdots \delta_{i_{\tau(n)} i'_n} \delta_{j_{\sigma(1)} j'_1} \cdots \delta_{j_{\sigma(n)} j'_n} \operatorname{Wg}(d, \tau \sigma^{-1}). \quad (1.1)$$

The purpose of this note is to explain a conceptual route to (1.1). Rather than deriving the formula by direct combinatorial arguments, we reinterpret the integral using tensor powers of  $V = \mathbb{C}^d$  and

the averaging operator

$$T \mapsto \int U^{\otimes n} T (U^{-1})^{\otimes n} dU.$$

Within this framework, the unitary Weingarten function appears as the inverse of a Gram element associated with permutation operators. We conclude with a brief application concerning the leading large- $d$  asymptotics of  $\text{Wg}(d, \cdot)$ .

## 2. TENSOR PRODUCTS

In order to rigorously define the tensor product of linear operators, we first need to define the tensor product of vector spaces. Suppose that  $V_1, V_2, \dots, V_\ell$  are  $\mathbb{C}$ -vector spaces. The tensor product of  $V_1, V_2, \dots, V_\ell$  is the vector space  $\bigotimes_{i=1}^\ell V_i$ , equipped with an “additional structure” in the following sense:

- (1) As a vector space,  $\bigotimes_{i=1}^\ell V_i$  is  $\prod_{i=1}^\ell \dim V_i$ -dimensional.
- (2) We have a multilinear embedding  $\otimes : V_1 \times V_2 \times \cdots \times V_\ell \rightarrow \bigotimes_{i=1}^\ell V_i$ .
- (3) Taking an arbitrary basis  $e_1^i, e_2^i, \dots, e_{d_i}^i$  of  $V_i$  for each  $i$ , the set

$$\{e_{a_1}^1 \otimes e_{a_2}^2 \otimes \cdots \otimes e_{a_\ell}^\ell ; 1 \leq a_i \leq d_i, 1 \leq i \leq \ell\}$$

forms a basis of  $\bigotimes_{i=1}^\ell V_i$ .

Here, multilinearity means that the tensor product map satisfies

$$v_1 \otimes \cdots \otimes (v_k + cv'_k) \otimes \cdots \otimes v_\ell = v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_\ell + cv_1 \otimes \cdots \otimes v'_k \otimes \cdots \otimes v_\ell$$

for all  $v_i, v'_i \in V_i$  and  $c \in \mathbb{C}$ .

If we only impose the first two conditions, the map  $\otimes$  can be chosen in a highly non-unique way. For example, if we define a map  $0 : V_1 \times V_2 \times \cdots \times V_\ell \rightarrow \bigotimes_{i=1}^\ell V_i$  by  $0(v_1, v_2, \dots, v_\ell) = 0$ , then this map is multilinear, but this map carries no information relating the spaces  $V_i$  to  $\bigotimes_{i=1}^\ell V_i$ . The third condition ensures that every element of  $\bigotimes_{i=1}^\ell V_i$  can be expressed as a finite linear combination of elementary tensors.

As a simple example, when  $V_1 = \mathbb{C}^d$  and  $V_2 = \mathbb{C}^e$ , we have  $\bigotimes_{i=1}^2 V_i \cong \mathbb{C}^{de}$ . If we identify  $v_1 \otimes v_2 \in \mathbb{C}^{de}$  with the rank-one matrix  $v_1 v_2^\top$ , then our embedding becomes the inclusion of rank-one matrices into the space of all matrices. For the case  $\ell \geq 3$ , one may think of higher-order tensors and embeddings of “general rank-one tensors”. This comes from the fact that

Any tensor can be expressed as a finite linear combination of rank-one tensors.

Now we define the tensor product of linear operators  $T_1 : V_1 \rightarrow W_1, T_2 : V_2 \rightarrow W_2, \dots, T_\ell : V_\ell \rightarrow W_\ell$  as the linear operator  $T_1 \otimes T_2 \otimes \cdots \otimes T_\ell : \bigotimes_{i=1}^\ell V_i \rightarrow \bigotimes_{i=1}^\ell W_i$  such that

$$(T_1 \otimes T_2 \otimes \cdots \otimes T_\ell)(v_1 \otimes v_2 \otimes \cdots \otimes v_\ell) = T_1(v_1) \otimes T_2(v_2) \otimes \cdots \otimes T_\ell(v_\ell)$$

for all  $v_i \in V_i$ . It is straightforward to check that this definition is well defined.

We go back to our original problem (1.1). We define the multi-indices  $I = (i_1, i_2, \dots, i_n)$ ,  $I' = (i'_1, i'_2, \dots, i'_n)$ ,  $J = (j_1, j_2, \dots, j_n)$ , and  $J' = (j'_1, j'_2, \dots, j'_n)$ , which specify the indices of the matrix entries of  $U$  and its complex conjugate  $\bar{U}$ , respectively. Let  $e_I$  be the basis element of  $(\mathbb{C}^d)^{\otimes n}$  corresponding to the multi-index  $I$ . Let  $e_{IJ}$  denote the linear operator on  $(\mathbb{C}^d)^{\otimes n}$  defined

by

$$e_{IJ}(e_K) = e_I \delta_{J,K}$$

for all multi-indices  $I, J, K$ .

For two multi-indices  $I, J$ , we define  $\delta_{I,J}$  to be the Kronecker delta function, i.e.

$$\delta_{I,J} = \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}.$$

Then, we can rewrite (1.1) as

$$\int_{U(d)} \text{Tr} \left( U^{\otimes n} e_{JJ'} U^{-1 \otimes n} e_{I'I} \right) dU = \sum_{\sigma, \tau \in S_n} \delta_{I \circ \tau, I'} \delta_{J \circ \sigma, J'} \text{Wg}(d, \tau \sigma^{-1}). \quad (2.1)$$

### 3. REPRESENTATIONS OF $(\mathbb{C}^d)^{\otimes n}$

The space  $(\mathbb{C}^d)^{\otimes n}$  carries two well-known representations. The first is the representation arising from the natural action of  $GL(d)$  on this space. More precisely, for  $g \in GL(d)$ , the representation  $\rho(g)$  is defined by

$$\rho(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n.$$

Thus, this representation is the  $n$ -fold tensor product of the standard representation of  $GL(d)$  on  $\mathbb{C}^d$ . Restricting this representation to the subgroup  $U(d)$  yields a representation  $\rho_{U(d)}$  on the same space.

The second is the action of the symmetric group  $S_n$  on this space. Specifically, for any  $\sigma \in S_n$ , the representation  $\rho_{S_n}(\sigma)$  is defined by

$$\rho_{S_n}(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Here we use the notation  $\rho_{S_n}$  to distinguish it from the representation  $\rho$  of  $GL(d)$ .

To state the celebrated Schur–Weyl duality theorem for these representations, we first introduce the notion of the commutant of a representation. Suppose we have a representation  $\rho : G \rightarrow GL(V)$  of a group  $G$ . The commutant of the  $G$ -action on  $V$  is the set of all linear operators  $T : V \rightarrow V$  such that  $T\rho(g) = \rho(g)T$  for all  $g \in G$ . We denote this commutant by  $\text{End}_G(V)$ .

Finally, for a representation  $\rho : G \rightarrow GL(V)$ , let  $\langle \rho(G) \rangle_{\mathbb{C}}$  denote the  $\mathbb{C}$ -linear span of the image of  $\rho$  (viewed as a subset of  $\text{End}(V)$ ). This is a subalgebra of  $\text{End}(V)$ , called the algebra generated by the image of  $\rho$ .

**Theorem 3.1** (Schur–Weyl duality for  $U(d)$ ). *Let  $V = \mathbb{C}^d$  and consider the commuting actions of  $U(d)$  and  $S_n$  on  $V^{\otimes n}$  given above (the diagonal action  $\rho_{U(d)}$  of  $U(d)$  and the place-permutation action  $\rho_{S_n}$  of  $S_n$ ). Then each action is the full commutant of the other:*

$$\text{End}_{U(d)}(V^{\otimes n}) = \langle \rho_{S_n}(S_n) \rangle_{\mathbb{C}}, \quad \text{End}_{S_n}(V^{\otimes n}) = \langle \rho_{U(d)}(U(d)) \rangle_{\mathbb{C}}.$$

At first glance, this theorem appears to be unrelated to our original problem of computing Haar moments. However, it will turn out to be a cornerstone of our approach.

#### 4. GROUP ALGEBRA OF $S_n$

In Schur–Weyl duality, the commutant  $\text{End}_{GL(d)}((\mathbb{C}^d)^{\otimes n})$  is described as the  $\mathbb{C}$ -linear span of the place-permutation operators  $\rho_{S_n}(\sigma)$ . It is therefore natural to package these operators into a single algebraic object: the group algebra of  $S_n$ .

**Definition 4.1** (Group algebra). *The group algebra  $\mathbb{C}[S_n]$  is the  $\mathbb{C}$ -vector space with basis  $\{[\sigma] : \sigma \in S_n\}$ , equipped with multiplication determined by  $[\sigma] \cdot [\tau] = [\sigma\tau]$ , extended  $\mathbb{C}$ -bilinearly.*

**Example 4.2.** We give an explicit example in the group algebra of  $S_3$ . A permutation  $\sigma \in S_3$  is written in cycle notation as follows:

- $(ij)$  denotes the transposition exchanging  $i$  and  $j$ ,
- $(ijk)$  denotes the 3-cycle sending  $i \mapsto j$ ,  $j \mapsto k$ , and  $k \mapsto i$ .

We compute the product of the following two elements in  $\mathbb{C}[S_3]$ :

$$\begin{aligned} ([\langle 12 \rangle] + [\langle 123 \rangle]) \cdot ([\langle 23 \rangle] + [\langle 132 \rangle]) &= [\langle 12 \rangle \langle 23 \rangle] + [\langle 12 \rangle \langle 132 \rangle] + [\langle 123 \rangle \langle 23 \rangle] + [\langle 123 \rangle \langle 132 \rangle] \\ &= [\langle 123 \rangle] + [\langle 23 \rangle] + [\langle 12 \rangle] + [e]. \end{aligned}$$

Given the representation  $\rho_{S_n} : S_n \rightarrow GL(V^{\otimes n})$ , we extend it  $\mathbb{C}$ -linearly to an algebra homomorphism

$$\tilde{\rho}_{S_n} : \mathbb{C}[S_n] \rightarrow \text{End}(V^{\otimes n}), \quad \tilde{\rho}_{S_n}\left(\sum_{\sigma \in S_n} a_{\sigma} [\sigma]\right) = \sum_{\sigma \in S_n} a_{\sigma} \rho_{S_n}(\sigma).$$

By construction,  $\text{Im}(\tilde{\rho}_{S_n}) = \langle \rho_{S_n}(S_n) \rangle_{\mathbb{C}}$ .

**Proposition 4.3.** *If  $d \geq n$ , then  $\tilde{\rho}_{S_n}$  is injective. In particular,  $\langle \rho_{S_n}(S_n) \rangle_{\mathbb{C}}$  is isomorphic (as a  $\mathbb{C}$ -algebra) to the group algebra  $\mathbb{C}[S_n]$ .*

*Proof.* Let  $e_1, \dots, e_d$  be the standard basis of  $V = \mathbb{C}^d$  and set  $v := e_1 \otimes \dots \otimes e_n \in V^{\otimes n}$  (which is well defined since  $d \geq n$ ). For  $\sigma \in S_n$  we have

$$\rho_{S_n}(\sigma)(v) = e_{\sigma^{-1}(1)} \otimes \dots \otimes e_{\sigma^{-1}(n)}.$$

These tensors are distinct for distinct  $\sigma$ , hence the operators  $\rho_{S_n}(\sigma)$  are linearly independent in  $\text{End}(V^{\otimes n})$ . Therefore the linear extension  $\tilde{\rho}_{S_n}$  has trivial kernel, i.e. it is injective.  $\square$

Using the notion of the group algebra, we can restate a part of Schur–Weyl duality as

$$\text{End}_{U(d)}((\mathbb{C}^d)^{\otimes n}) = \tilde{\rho}_{S_n}(\mathbb{C}[S_n]).$$

We return to the equation (2.1). We define the operator  $\mathbb{E} : \text{End}((\mathbb{C}^d)^{\otimes n}) \rightarrow \text{End}((\mathbb{C}^d)^{\otimes n})$  by

$$\mathbb{E}(T) = \int_{U(d)} U^{\otimes n} T U^{-1 \otimes n} dU.$$

**Lemma 4.4.** *Let  $V = \mathbb{C}^d$ . For any  $T \in \text{End}(V^{\otimes n})$ , the operator  $\mathbb{E}(T)$  lies in  $\text{End}_{U(d)}(V^{\otimes n})$ . Therefore, its codomain may be restricted, and we view  $\mathbb{E}$  as a map*

$$\mathbb{E} : \text{End}(V^{\otimes n}) \rightarrow \tilde{\rho}_{S_n}(\mathbb{C}[S_n]).$$

*Proof.* Fix  $A \in U(d)$ . Using  $\rho_{U(d)}(A)U^{\otimes n} = (AU)^{\otimes n}$  and the left-invariance of Haar measure, we compute

$$\rho_{U(d)}(A)\mathbb{E}(T)\rho_{U(d)}(A)^{-1} = \int_{U(d)} (AU)^{\otimes n} T (AU)^{-1\otimes n} dU = \int_{U(d)} U^{\otimes n} T U^{-1\otimes n} dU = \mathbb{E}(T).$$

Thus  $\mathbb{E}(T) \in \text{End}_{U(d)}(V^{\otimes n})$ . If  $d \geq n$ , then by Schur–Weyl duality we have  $\text{End}_{U(d)}(V^{\otimes n}) = \tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ , so  $\mathbb{E}(T)$  lies in  $\tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ .  $\square$

Using the operator  $\mathbb{E}$ , we can rewrite (2.1) as

$$\text{Tr}(A\mathbb{E}(B)) = \sum_{\sigma, \tau \in S_n} \delta_{I \circ \tau, I'} \delta_{J \circ \sigma, J'} \text{Wg}(d, \tau \sigma^{-1}), \quad (4.1)$$

where  $A = e_{JJ'}$  and  $B = e_{I'I'}$ .

Observing the left-hand side of (4.1), we may infer that studying Haar moments is deeply related to the study of the operator  $\mathbb{E}$ , as well as the group algebra  $\mathbb{C}[S_n]$ .

## 5. WEINGARTEN CALCULUS

We now give the rigorous definition of the function  $\text{Wg}(d, \cdot)$  appearing in (1.1). Assume that (1.1) holds. In order to guess the definition of  $\text{Wg}(d, \cdot)$ , we plug

$$\begin{aligned} I &= (1, \dots, n) \\ J &= (1, \dots, n) \\ I' &= (1, \dots, n) \\ J' &= (\sigma^{-1}(1), \dots, \sigma^{-1}(n)) \end{aligned}$$

Then we get

$$\int_{U(d)} u_{11} \cdots u_{nn} \bar{u}_{1\sigma^{-1}(1)} \cdots \bar{u}_{n\sigma^{-1}(n)} dU = \text{Wg}(d, \sigma). \quad (5.1)$$

**Definition 5.1.** *The function  $\text{Wg}(d, \cdot) : S_n \rightarrow \mathbb{C}$  is defined by the equation (5.1).*

However, this definition is not given in closed form, so we need to investigate an alternative definition of  $\text{Wg}(d, \cdot)$ . To do so, we need to define one more operator  $\Phi : \text{End}((\mathbb{C}^d)^{\otimes n}) \rightarrow \mathbb{C}[S_n]$ .

**Definition 5.2.** *The operator  $\Phi : \text{End}((\mathbb{C}^d)^{\otimes n}) \rightarrow \mathbb{C}[S_n]$  is defined by*

$$\Phi(A) = \sum_{\sigma \in S_n} \text{Tr}(A \rho_{S_n}(\sigma^{-1})) \sigma.$$

**Definition 5.3.** *For  $d \geq n$ ,  $\text{Wg}(d, \cdot)$  is defined as the inverse of  $\Phi(Id)$  in  $\mathbb{C}[S_n]$ , i.e.*

$$\Phi(Id) \left( \sum_{\sigma \in S_n} \text{Wg}(d, \sigma) \sigma \right) = 1.$$

**Remark 5.4.** *In the previous definition, we assume that  $d \geq n$ , since this is needed to ensure that there exists an inverse of  $\Phi(Id)$  in  $\mathbb{C}[S_n]$  (see Lemma 5.6(3)). To see this type of closed form definition for the general  $d$ , see [2, Section 2].*

**Example 5.5.** We compute  $\text{Wg}(d, \cdot)$  for  $S_3$ . Write  $S_3 = \{(1)(2)(3), (12)(3), (13)(2), (23)(1), (123), (132)\}$ . Then,

$$\Phi(Id) = \sum_{\sigma \in S_3} \text{Tr}(\rho_{S_3}(\sigma^{-1}))\sigma.$$

We can easily compute that

$$\text{Tr}(\rho_{S_3}(\sigma^{-1})) = \begin{cases} d^3 & \text{if } \sigma = (1)(2)(3), \\ d^2 & \text{if } \sigma = (12)(3), \text{ or } (13)(2), \text{ or } (23)(1), \\ d & \text{if } \sigma = (123), \text{ or } (132). \end{cases}$$

Therefore,

$$\Phi(Id) = d^3(1)(2)(3) + d^2(12)(3) + d^2(13)(2) + d^2(23)(1) + d(123) + d(132).$$

From this, we have

$$\begin{aligned} \text{Wg}(d, (1)(2)(3)) &= \frac{d^2 - 2}{d(d^2 - 1)(d^2 - 4)}, \\ \text{Wg}(d, (12)(3)) = \text{Wg}(d, (13)(2)) = \text{Wg}(d, (23)(1)) &= \frac{-1}{(d^2 - 1)(d^2 - 4)}, \\ \text{Wg}(d, (123)) = \text{Wg}(d, (132)) &= \frac{2}{d(d^2 - 1)(d^2 - 4)}. \end{aligned}$$

To prove that these two definitions are equivalent, we need to prove (5.1) holds, after choosing Definition 5.3. We need the following lemma from [2, Proposition 2.3].

**Lemma 5.6.** Let  $d \geq n$ .  $\Phi$  fulfills the following properties:

(1)  $\Phi$  is a  $\mathbb{C}[S_n]$  bimodule morphism in the sense that

$$\begin{aligned} \Phi(A\rho_{S_n}(\sigma)) &= \Phi(A)\sigma \\ \Phi(\rho_{S_n}(\sigma)A) &= \sigma\Phi(A) \end{aligned}$$

(2) the relation between  $\Phi$  and  $\mathbb{E}$  is explicitly given by

$$\Phi(A) = \tilde{\rho}_{S_n}^{-1}(\mathbb{E}(A))\Phi(Id).$$

(3)  $\Phi(Id)$  is invertible in  $\mathbb{C}[S_n]$ .

*Proof.* (1) We prove the right-module property; the left-module property is similar. Using the definition of  $\Phi$  and the fact that  $\rho_{S_n}$  is a representation,

$$\begin{aligned} \Phi(A\rho_{S_n}(\sigma)) &= \sum_{\tau \in S_n} \text{Tr}(A\rho_{S_n}(\sigma)\rho_{S_n}(\tau^{-1}))\tau \\ &= \sum_{\tau \in S_n} \text{Tr}(A\rho_{S_n}(\sigma\tau^{-1}))\tau = \sum_{\tau \in S_n} \text{Tr}(A\rho_{S_n}((\tau\sigma^{-1})^{-1}))\tau. \end{aligned}$$

Making the change of variables  $\tau' = \tau\sigma^{-1}$  (so  $\tau = \tau'\sigma$ ) gives

$$\Phi(A\rho_{S_n}(\sigma)) = \sum_{\tau' \in S_n} \text{Tr}(A\rho_{S_n}((\tau')^{-1}))(\tau'\sigma) = \Phi(A)\sigma.$$

For the left-module property, we use cyclicity of trace:

$$\begin{aligned}\Phi(\rho_{S_n}(\sigma)A) &= \sum_{\tau \in S_n} \text{Tr}(\rho_{S_n}(\sigma)A\rho_{S_n}(\tau^{-1})) \tau \\ &= \sum_{\tau \in S_n} \text{Tr}(A\rho_{S_n}(\tau^{-1})\rho_{S_n}(\sigma)) \tau = \sum_{\tau \in S_n} \text{Tr}(A\rho_{S_n}((\sigma^{-1}\tau)^{-1})) \tau \\ &= \sum_{\tau' \in S_n} \text{Tr}(A\rho_{S_n}((\tau')^{-1})) (\sigma\tau') = \sigma \Phi(A),\end{aligned}$$

where we changed variables  $\tau' = \sigma^{-1}\tau$  in the last line.

(2) Fix  $\sigma \in S_n$ . Since the diagonal action  $\rho_{U(d)}$  and the place-permutation action  $\rho_{S_n}$  commute, we have  $U^{\otimes n} \rho_{S_n}(\sigma^{-1}) = \rho_{S_n}(\sigma^{-1}) U^{\otimes n}$  for all  $U \in U(d)$ . Therefore,

$$\begin{aligned}\text{Tr}(\mathbb{E}(A)\rho_{S_n}(\sigma^{-1})) &= \int_{U(d)} \text{Tr}(U^{\otimes n} A U^{-1 \otimes n} \rho_{S_n}(\sigma^{-1})) dU \\ &= \int_{U(d)} \text{Tr}(A U^{-1 \otimes n} \rho_{S_n}(\sigma^{-1}) U^{\otimes n}) dU \\ &= \int_{U(d)} \text{Tr}(A \rho_{S_n}(\sigma^{-1})) dU = \text{Tr}(A \rho_{S_n}(\sigma^{-1})).\end{aligned}$$

Comparing coefficients in the definition of  $\Phi$ , this shows  $\Phi(A) = \Phi(\mathbb{E}(A))$ . Since  $\mathbb{E}(A) \in \tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ , we may write  $\mathbb{E}(A) = \tilde{\rho}_{S_n}(x)$  with  $x = \tilde{\rho}_{S_n}^{-1}(\mathbb{E}(A)) \in \mathbb{C}[S_n]$ . Writing  $x = \sum_{\tau \in S_n} a_\tau \tau$ , we get

$$\Phi(\mathbb{E}(A)) = \sum_{\tau \in S_n} a_\tau \Phi(\rho_{S_n}(\tau)) = \sum_{\tau \in S_n} a_\tau \tau \Phi(Id) = x \Phi(Id),$$

where we used part (1) with  $A = Id$  to conclude  $\Phi(\rho_{S_n}(\tau)) = \tau \Phi(Id)$ . Hence  $\Phi(A) = \tilde{\rho}_{S_n}^{-1}(\mathbb{E}(A)) \Phi(Id)$ , as claimed.

(3) Consider the linear map

$$\Theta := \Phi \circ \tilde{\rho}_{S_n} : \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n].$$

By the right-module property in (1) with  $A = Id$ , we have

$$\Theta(x) = \Phi(\tilde{\rho}_{S_n}(x)) = x \Phi(Id),$$

so  $\Theta$  is right multiplication by  $\Phi(Id)$ . We claim that  $\Theta$  is injective (hence bijective, since  $\mathbb{C}[S_n]$  is finite-dimensional). If  $\Theta(x) = 0$ , then for every  $\sigma \in S_n$ ,

$$0 = [\sigma]\Theta(x) = [\sigma]\Phi(\tilde{\rho}_{S_n}(x)) = \text{Tr}(\tilde{\rho}_{S_n}(x) \rho_{S_n}(\sigma^{-1})),$$

i.e.  $\tilde{\rho}_{S_n}(x)$  is orthogonal (for the Hilbert–Schmidt pairing) to every  $\rho_{S_n}(\sigma)$ , hence to the whole subspace  $\tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ . But  $\tilde{\rho}_{S_n}(x) \in \tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ , so it must be zero:  $\tilde{\rho}_{S_n}(x) = 0$ . Since  $d \geq n$ ,  $\tilde{\rho}_{S_n}$  is injective by Proposition 4.3, hence  $x = 0$ . Thus  $\Theta$  is injective, hence bijective. Therefore right multiplication by  $\Phi(Id)$  is bijective, so there exists  $y \in \mathbb{C}[S_n]$  such that  $y \Phi(Id) = 1$ . Similarly, using the left-module property in (1) one sees that left multiplication by  $\Phi(Id)$  is bijective, hence there exists  $z \in \mathbb{C}[S_n]$  with  $\Phi(Id) z = 1$ . It follows that  $\Phi(Id)$  is invertible in  $\mathbb{C}[S_n]$ .  $\square$

We finally prove (5.1).

*Proof of (5.1).* Fix  $d \geq n$  so that  $\Phi(Id)$  is invertible in  $\mathbb{C}[S_n]$  by Definition 5.3, and write

$$\Phi(Id)^{-1} = \sum_{\pi \in S_n} \text{Wg}(d, \pi) \pi.$$

We first prove the identity

$$\Phi(A\mathbb{E}(B)) = \Phi(A)\Phi(B)\Phi(Id)^{-1}.$$

Since  $\mathbb{E}(B) \in \tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ , we may write  $\mathbb{E}(B) = \tilde{\rho}_{S_n}(x)$  with  $x = \tilde{\rho}_{S_n}^{-1}(\mathbb{E}(B)) \in \mathbb{C}[S_n]$ . By the right  $\mathbb{C}[S_n]$ -module property of  $\Phi$  (Lemma 5.4(1)),

$$\Phi(A\mathbb{E}(B)) = \Phi(A\tilde{\rho}_{S_n}(x)) = \Phi(A)x.$$

On the other hand, Lemma 5.4(2) gives  $\Phi(B) = x\Phi(Id)$ , hence  $x = \Phi(B)\Phi(Id)^{-1}$ . Substituting this into the previous display yields

$$\Phi(A\mathbb{E}(B)) = \Phi(A)\Phi(B)\Phi(Id)^{-1}. \quad (5.2)$$

For  $y = \sum_{\sigma \in S_n} c_\sigma \sigma \in \mathbb{C}[S_n]$ , write  $[e]y := c_e$  for the coefficient of the identity permutation  $e$ . By definition of  $\Phi$ ,

$$[e]\Phi(T) = \text{Tr}(T\rho_{S_n}(e^{-1})) = \text{Tr}(T).$$

Applying  $[e]$  to the identity in (5.2) therefore gives

$$\text{Tr}(A\mathbb{E}(B)) = [e]\left(\Phi(A)\Phi(B)\Phi(Id)^{-1}\right). \quad (5.3)$$

Expanding  $\Phi(A)$  and  $\Phi(B)$  and using  $\Phi(Id)^{-1} = \sum_{\pi} \text{Wg}(d, \pi) \pi$ , the right-hand side of (5.3) is the coefficient of  $e$  in

$$\left( \sum_{\sigma \in S_n} \text{Tr}(A\rho_{S_n}(\sigma^{-1})) \sigma \right) \left( \sum_{\tau \in S_n} \text{Tr}(B\rho_{S_n}(\tau^{-1})) \tau \right) \left( \sum_{\pi \in S_n} \text{Wg}(d, \pi) \pi \right).$$

Equivalently,

$$\text{Tr}(A\mathbb{E}(B)) = \sum_{\sigma, \tau, \pi \in S_n} \text{Tr}(A\rho_{S_n}(\sigma^{-1})) \text{Tr}(B\rho_{S_n}(\tau^{-1})) \text{Wg}(d, \pi) \delta_{\sigma\tau\pi, e}. \quad (5.4)$$

Take  $A = e_{JJ'}$  and  $B = e_{I'I}$  as in (4.1). For  $\sigma \in S_n$  and a multi-index  $K = (k_1, \dots, k_n)$ , we write  $K \circ \sigma$  for the multi-index characterized by  $\rho_{S_n}(\sigma^{-1})(e_K) = e_{K \circ \sigma}$ . Using the defining property  $e_{IJ}(e_K) = e_I \delta_{J,K}$  and the basis  $\{e_K\}$ , we compute

$$\text{Tr}(e_{JJ'}\rho_{S_n}(\sigma^{-1})) = \sum_K \langle e_K, e_{JJ'}\rho_{S_n}(\sigma^{-1})(e_K) \rangle = \sum_K \langle e_K, e_{JJ'}(e_{K \circ \sigma}) \rangle = \sum_K \langle e_K, e_J \delta_{J', K \circ \sigma} \rangle = \delta_{J \circ \sigma, J'}.$$

The same argument gives

$$\text{Tr}(e_{I'I}\rho_{S_n}(\tau^{-1})) = \delta_{I \circ \tau, I'}.$$

Substituting these into (5.4) and reindexing the dummy summation variables gives precisely (4.1). Finally, choosing the special multi-indices used just before (5.1) yields (5.1).  $\square$

**Remark 5.7.** Conceptually, the proof of (5.1) is an instance of the “commutant” method: the Haar-averaging operator  $\mathbb{E}$  projects  $\text{End}(V^{\otimes n})$  onto the commutant  $\text{End}_{U(d)}(V^{\otimes n})$ , and Schur–Weyl duality identifies this commutant (for  $d \geq n$ ) with the permutation algebra  $\tilde{\rho}_{S_n}(\mathbb{C}[S_n])$ . The same mechanism works for other compact groups: one replaces  $\text{End}_{U(d)}(V^{\otimes n})$  by the appropriate centralizer algebra (e.g. the Brauer algebra for orthogonal and symplectic groups), and the corresponding Schur–Weyl-type duality leads to the analogue of (5.1). See [2, Section 3].

## 6. APPLICATIONS: ASYMPTOTICS OF WEINGARTEN FUNCTIONS

In this section, we investigate the asymptotics of  $\text{Wg}(d, \cdot)$  as  $d \rightarrow \infty$ .

**Definition 6.1.** For  $\sigma \in S_n$ , let  $|\sigma|$  be the minimal number of transpositions needed to write  $\sigma$  as a product of transpositions. Let  $\#\sigma$  be the number of cycles in the cycle decomposition of  $\sigma$ .

**Lemma 6.2.** For  $\sigma \in S_n$ , we have  $|\sigma| + \#\sigma = n$ .

We start from the definition of  $\Phi$ :

$$\Phi(Id) = \sum_{\sigma \in S_n} \text{Tr}(\rho_{S_n}(\sigma^{-1}))\sigma = \sum_{\sigma \in S_n} d^{\#\sigma}\sigma.$$

Let  $e$  be the identity permutation. This implies that

$$d^{-n}\Phi(Id) = e + \sum_{\sigma \neq e} d^{-|\sigma|}\sigma,$$

so we have

$$e - d^{-n}\Phi(Id) = - \sum_{\sigma \neq e} d^{-|\sigma|}\sigma.$$

Using the fact that  $1/(1-a) = 1 + a + a^2 + \dots$ , we have

$$d^n Wg = \sum_{i \geq 0} (- \sum_{\sigma \neq e} d^{-|\sigma|})^i = \sum_{\sigma \in S_n} O(d^{-|\sigma|})\sigma.$$

Therefore, we have

$$Wg(d, \sigma) = O(d^{-|\sigma|-n}).$$

For  $\sigma \in S_n$ , define

$$\text{Moeb}(\sigma) = \prod_{i=1}^k C_{|C_i|-1}(-1)^{|C_i|-1},$$

where  $C_1, \dots, C_k$  are the cycles in the cycle decomposition of  $\sigma$  and

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

is the Catalan number. Then, the asymptotics can be written explicitly as follows:

$$d^{n+|\sigma|} Wg(d, \sigma) = \text{Moeb}(\sigma) + O(d^{-2}).$$

From this asymptotic one can derive *asymptotic freeness*: if  $A^{(d)}$  and  $B^{(d)}$  are deterministic matrix families with a limiting  $*$ -distribution as  $d \rightarrow \infty$ , and  $U$  is Haar in  $U(d)$  independent of

them, then the conjugated family  $UA^{(d)}U^{-1}$  becomes asymptotically free from  $B^{(d)}$ . Concretely, mixed normalized traces of alternating centered words in  $UA^{(d)}U^{-1}$  and  $B^{(d)}$  tend to 0, and this is proved by expanding the moments using (1.1) and inserting the large- $d$  estimate above, which suppresses all but the leading (noncrossing) contributions. See [5, Lecture 23].

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