Bell Polynomials and β-Ensemble Moments in the High-Temperature Limit

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1 Background [BGCG22]

We give a brief introduction to the high temperature limit considered in [BGCG22], focusing on the tools relevant to moment calculations. For a given probability density $w(\lambda)$ on $\mathbb{R}$, the β-ensemble of w is the probability distribution on $\mathbb{R}^N$ given by

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N w(\lambda_i)$$

normalized appropriately. With w being the Gaussian, Gamma, or Beta distributions, the β-ensemble becomes the joint eigenvalue distributions of the GβE, β-Laguerre, and β-Jacobi matrix ensembles, respectively. In [BGCG22], the authors study the high temperature limit, where the inverse temperature $\beta \to 0$ while the dimension $N \to \infty$ such that $\beta N \to 2\gamma \in (0, \infty)$. [BGCG22] also introduces tools to describe matrix addition for β-ensemble in the high-temperature limit. In particular, they introduced the γ-cumulants to be the analog of classical or free cumulants in the high-temperature regime (See Definition 3.7 of [BGCG22]). A main result of the paper is the following theorem relating a moment generating function to a γ-cumulant generating function.

Theorem 1.1 (Theorem 3.11 of [BGCG22]). *Fix $\gamma > 0$. Let $\{m_k\}_{k\geq 1}$ and $\{\kappa_\ell\}_{\ell \geq 1}$ be the sequences of moments and γ-cumulants, respectively. Then*

$$\exp\left(\sum_{l=1}^{\infty} \frac{\kappa_l y^l}{l}\right) = [z^0]\left\{\sum_{n=0}^{\infty} \frac{(yz)^n}{(\gamma)_n} \cdot \exp\left(\gamma \sum_{k=1}^{\infty} \frac{m_k}{k} z^{-k}\right)\right\} \tag{1}$$

where $(x)_q = x(x+1)\cdots(x+q-1)$ is the Pochhammer symbol. Equivalently, (1) can be rewritten as a combination of two identities involving an auxiliary sequence $\{c_n\}_{n\geq 0}$ through

$$\begin{aligned} \exp\left(\sum_{\ell=1}^{\infty} \kappa_\ell \frac{z^\ell}{\ell}\right) &= \sum_{n=0}^{\infty} \frac{c_n}{(\gamma)_n} z^n \\ \exp\left(\gamma \sum_{k=1}^{\infty} m_k \frac{z^k}{k}\right) &= \sum_{n=0}^{\infty} c_n z^n \end{aligned} \tag{2}$$

1.1 Connection to Classical and Free Cumulants

Classical cumulants can be retrieved by letting $\gamma \to 0$. In particular, suppose a distribution has γ-cumulants $\{\kappa_\ell^{(\gamma)}\}_{\ell \geq 0}$ for all $\gamma > 0$. Let $\kappa_\ell^{(0)} = \lim_{\gamma \to 0} \kappa_\ell^{(\gamma)}$. Then the classical cumulants are given by

$$\kappa_\ell = (\ell - 1)!\kappa_\ell^{(0)}$$

(See Theorem 8.2 of [BGCG22]).
Free cumulants can be retrieved by taking the $\gamma \to \infty$ limit. In particular, suppose a distribution has γ-cumulants $\{\kappa_\ell^{(\gamma)}\}_{\ell \geq 0}$ for all $\gamma \in (0, \infty)$. Then the free cumulants are given by

$$r_\ell = \lim_{\gamma \to \infty} \gamma^{\ell-1} \kappa_\ell^{(\gamma)}$$

(See Theorem 8.7 of [BGCG22]).

2 β-Laguerre Ensemble and Bell Polynomials

Bell polynomials are multivariable polynomials that give the nth moment in terms of the first n classical cumulants. Since γ-cumulants are defined to be analog of classical cumulants for the high-temperature regime, we can expect that the relationship between moments and γ-cumulants can be expressed via Bell polynomials as well. We begin with the case study of the limiting distribution of the β-Laguerre ensemble.

Theorem 2.1. *In the high-temperature limit with $\beta \to 0$ and $N \to \infty$, with $\beta N \to 2\gamma \in (0, \infty)$, the β-Laguerre ensemble, with aspect ratio parameter λ, empirical measure converges weakly, in probability to a measure ν_λ^γ, with moments given by*

$$m_k = \frac{k}{\gamma} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \ldots, c_{k-j+1})$$

where $\hat{B}_{k,j}$ are the partial ordinary Bell polynomials given by

$$\hat{B}_{k,j}(a_1, \ldots, a_{k-j+1}) = \sum_{\substack{n_1, \ldots, n_{k-j+1} \geq 0 \\ \sum_{r=1}^{k-j+1} n_r = j \\ \sum_{r=1}^{k-j+1} r n_r = k}} \frac{j!}{\prod_{r=1}^{k-j+1} n_r!} \prod_{r=1}^{k-j+1} a_r^{n_r}$$

and

$$c_n = \frac{(\lambda)_n (\gamma)_n}{n!}$$

is the auxiliary sequence used in Theorem 3.11 in [BGCG22].

2.1 Proof of Theorem 2.1

Convergence of the empirical measure of the β-Laguerre ensemble to ν_λ^γ is shown in [BGCG22] (Example 4.9) and [TT21] (Lemma 2.1). We need only to calculate its moments.

For the β-Laguerre ensemble, the spectral distribution converges to ν_λ^γ, where $\beta N \to 2\gamma$ and the aspect ratio parameter is λ. From [BGCG22] and [TT21], the γ-cumulants of ν_λ^γ are given by

$$\kappa_l = \lambda, \quad \forall\, l \geq 1$$

We see that

$$\sum_{\ell=1}^{\infty} \lambda \frac{z^\ell}{\ell} = -\lambda \log(1-z)$$

Therefore, we have

$$\begin{aligned} \exp\left(\sum_{\ell=1}^{\infty} \lambda \frac{z^\ell}{\ell}\right) &= \exp\Big(-\lambda \log(1-z)\Big) \\ &= (1-z)^{-\lambda} \\ &= \sum_{n=0}^{\infty} \binom{n+\lambda-1}{n} z^n \end{aligned}$$

We can switch to Pochhammer symbol: $\binom{n+\lambda-1}{n} = \frac{(\lambda)_n}{n!}$:

$$\begin{aligned}(1-z)^{-\lambda} &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{c_n}{(\gamma)_n} z^n\end{aligned}$$

The last equality comes from the first identity in (2). We have the auxiliary sequence $\{c_n\}_{n\geq 0}$ given by

$$c_n = \frac{(\lambda)_n(\gamma)_n}{n!} \tag{3}$$

Now we use the second identity in (2) to get

$$\exp\left(\gamma \sum_{k=1}^{\infty} m_k \frac{z^k}{k}\right) = \sum_{n=0}^{\infty} \frac{(\lambda)_n(\gamma)_n}{n!} z^n =: A(z)$$

We let

$$S(z) := A(z) - 1 = \sum_{n=1}^{\infty} c_n z^n \tag{4}$$

We use the Taylor series for $\log(1+x)$

$$\begin{aligned}\log A(z) &= \log\big(1 + S(z)\big) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} S(z)^j\end{aligned}$$

From (2)'s second identity, we can read off the moments of ν_λ^γ from the series coefficients. Note that $S(z)$ does not have a constant term, so the lowest degree of z in $S(z)^j$ is j. In particular, we have

$$\begin{aligned}m_k &= \frac{k}{\gamma}[z^k] \log\big(1 + S(z)\big) \\ &= \frac{k}{\gamma} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} [z^k] S(z)^j \\ &= \frac{k}{\gamma} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} [z^k] S(z)^j\end{aligned}$$

We relate this to the Bell polynomials by the following claim

Claim 2.2. *Suppose $S(z)$ is a formal series defined by*

$$S(z) = \sum_{n=1}^{\infty} c_n z^n$$

for some sequence $\{c_n\}_{n\geq 0}$. Then

$$[z^k] S(z)^j = \hat{B}_{k,j}(c_1, \ldots, c_{k-j+1})$$

where $\hat{B}_{k,j}$ are the partial ordinary Bell polynomials given by

$$\hat{B}_{k,j}(a_1, \ldots, a_{k-j+1}) = \sum_{\substack{n_1,\ldots,n_{k-j+1}\geq 0 \\ \sum_{r=1}^{k-j+1} n_r = j \\ \sum_{r=1}^{k-j+1} r n_r = k}} \frac{j!}{\prod_{r=1}^{k-j+1} n_r!} \prod_{r=1}^{k-j+1} a_r^{n_r}$$

Remark 1. *This claim does not rely on the constant γ-cumulant assumption and works for general sequence $\{c_n\}_{n\geq 0}$.*

Proof. We have using the fact that $S(z)$ as defined in (4) does not have a constant term

$$S(z)^j = \left(\sum_{n=1}^{\infty} c_n z^n\right)^j = \sum_{\ell \geq j} \sum_{\substack{i_1,\ldots,i_j \geq 1 \\ i_1+\cdots+i_j=\ell}} c_{i_1}\cdots c_{i_j} z^\ell \tag{5}$$

Therefore, we have

$$[z^k]S(z)^j = \sum_{\substack{i_1,\ldots,i_j \geq 1 \\ i_1+\cdots+i_j=k}} c_{i_1}\cdots c_{i_j}$$

Fix k. Suppose $(i_1,\ldots,i_j)$ is a tuple satisfying the sum second sum condition in (5), $i_t \geq 1$ and $\sum i_t = k$, must satisfy $i_t \leq k-j+1$. Therefore, $[z^k]S(z)^j$ only depends on $c_1,\ldots,c_{k-j+1}$. Therefore, we can replace $S(z)$ with a truncation

$$S_{k,j}(z) = \sum_{n=1}^{k-j+1} c_n z^n$$

and still have $[z^k]S(z)^j = [z^k]S_{k,j}(z)^j$ for all k. Therefore, it suffices to show that

$$[z^k]S_{k,j}(z)^j = \hat{B}_{k,j}(c_1,\ldots,c_{k-j+1})$$

Now we have from multinomial theorem

$$\begin{aligned} S_{k,j}(z)^j &= \left(\sum_{r=1}^{k-j+1} c_r z^r\right)^j \\ &= \sum_{\substack{n_1,\ldots,n_{k-j+1}\geq 0 \\ n_1+\cdots+n_{k-j+1}=j}} \binom{j}{n_1,\ldots,n_{k-j+1}} \prod_{r=1}^{k-j+1} (c_r z^r)^{n_r} \end{aligned}$$

We can group together the c_r coefficients and take the z out.

$$S_{k,j}(z)^j = \sum_{\substack{n_1,\ldots,n_{k-j+1}\geq 0 \\ n_1+\cdots+n_{k-j+1}=j}} \binom{j}{n_1,\ldots,n_{k-j+1}} \left(\prod_{r=1}^{k-j+1} c_r^{n_r}\right) z^{\sum_{r=1}^{k-j+1} r n_r}$$

Since we wish to take $[z^k]S_{k,j}(z)^j$, we restrict the sum to tuples such that the degree of z is k. So we want $\sum_{r=1}^{k-j+1} r n_r = k$. In particular,

$$\begin{aligned} [z^k]S_{k,j}^j &= \sum_{\substack{n_1,\ldots,n_{k-j+1}\geq 0 \\ \sum_{r=1}^{k-j+1} n_r=j \\ \sum_{r=1}^{k-j+1} r n_r=k}} \binom{j}{n_1,\ldots,n_{k-j+1}} \left(\prod_{r=1}^{k-j+1} c_r^{n_r}\right) \\ &= \hat{B}_{k,j}(c_1,\ldots,c_{k-j+1}) \end{aligned}$$

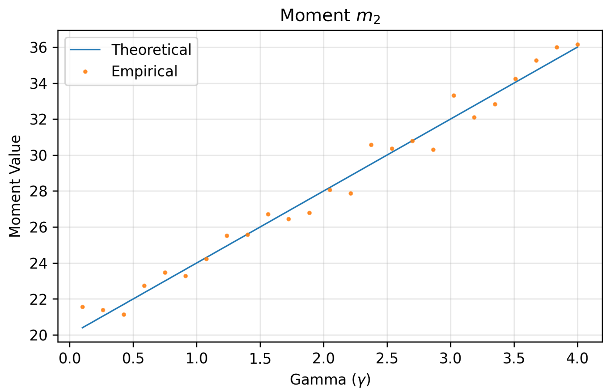
as desired. ◻

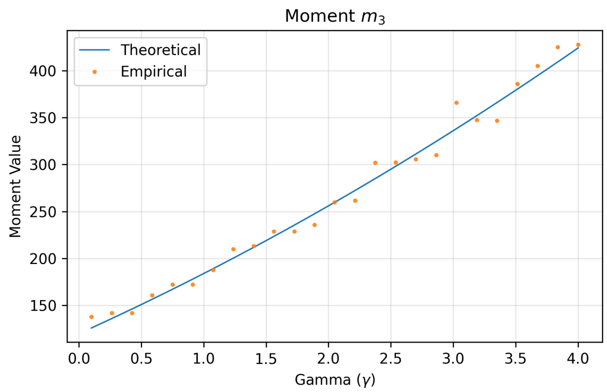
From Claim 2.2, we conclude

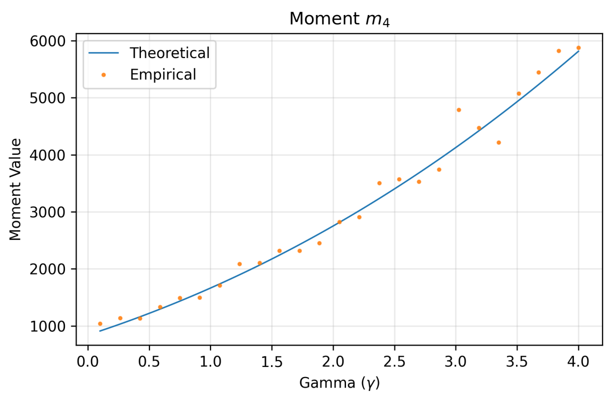
$$m_k = \frac{k}{\gamma}\sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1,\ldots,c_{k-j+1})$$

where c_i are defined in (3). Thus Theorem 2.1 is proven.

2.2 Numerical Verification







3 Recursion for Moments

Theorem 3.1. *In the high-temperature limit with $\beta \to 0$ and $N \to \infty$, with $\beta N \to 2\gamma \in (0, \infty)$, the β-Laguerre ensemble, with aspect ratio parameter λ, empirical measure converges weakly, in probability to a* *measure ν_λ^γ, with moments given by the recursive relation*

$$m_n = \frac{n}{\gamma}c_n - \sum_{k=1}^{n-1} m_k c_{n-k}$$

where

$$c_n = \frac{(\lambda)_n(\gamma)_n}{n!}$$

Proof. We let

$$M(z) = \sum_{k=1}^{\infty} \frac{m_k}{k} z^k, \quad C(z) = \sum_{n=0}^{\infty} c_n z^n$$

By the second identity in (2), we have

$$\begin{aligned}
\exp\Big(\gamma M(z)\Big) &= C(z) \\
\gamma M(z) &= \log C(z) \\
\gamma M'(z) &= \frac{C'(z)}{C(z)}
\end{aligned}$$

But

$$M'(z) = \sum_{k=1}^{\infty} m_k z^{k-1}, \quad C'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$$

In particular, we have

$$\begin{aligned}
\gamma \sum_{k=1}^{\infty} m_k z^{k-1} &= \frac{\sum_{n=1}^{\infty} n c_n z^{n-1}}{\sum_{n=0}^{\infty} c_n z^n} \\
\gamma\Big(\sum_{k=1}^{\infty} m_k z^{k-1}\Big)\Big(\sum_{n=0}^{\infty} c_n z^n\Big) &= \sum_{r=0}^{\infty}(r+1)c_{r+1}z^r \\
\gamma \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} m_k c_n z^{k-1+n} &= \sum_{r=0}^{\infty}(r+1)c_{r+1}z^r
\end{aligned}$$

Consider the coefficient of the term z^r on each hand side, we have $r = k-1+n \iff n = r+1-k$. Therefore, the part of the LHS sum that will contribute to the coefficient of z^r is over k such that $1 \leq k \leq r+1$. We thus have

$$[z^r]\gamma \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} m_k c_n z^{k-1+n} = \gamma \sum_{k=1}^{r+1} m_k c_{r+1-k}$$

But on the RHS, the coefficient of z^r is just $(r+1)c_{r+1}$. Therefore, we have for any $n \geq 1$,

$$\begin{aligned}
\gamma \sum_{k=1}^{n} m_k c_{n-k} &= n c_n \\
\gamma\Big(m_n + \sum_{k=1}^{n-1} m_k c_{n-k}\Big) &= n c_n \\
\implies m_n &= \frac{n}{\gamma}c_n - \sum_{k=1}^{n-1} m_k c_{n-k}
\end{aligned}$$

$\square$

3.1 Numerical Verification

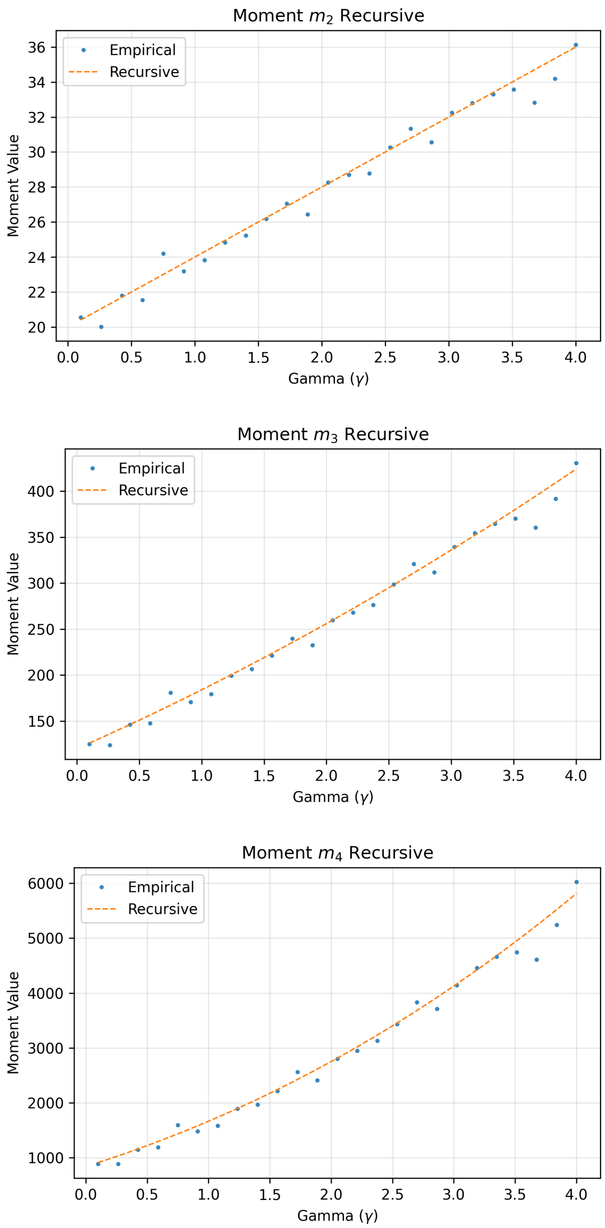


Figure 1: Theoretical and Empirical Moments

4 Moments from γ-cumulants via Bell Polynomials

We will generalize the connection between γ-cumulants and moments via Bell polynomials beyond the limiting distribution of the β-Laguerre ensemble. Since Bell polynomials connect classical cumulants to moments, it is natural to expect for there to be connections with γ-cumulants for a general distribution.

Lemma 4.1. *Let B_n be the complete exponential Bell polynomial of degree n. Let $\{\kappa_\ell\}_{\ell\geq 1}$ be a sequence of γ-cumulants of a distribution and $\alpha_\ell := (\ell-1)!\kappa_\ell$. Then the auxiliary sequence $\{c_n\}_{n\geq 0}$ defined in Theorem 3.11 of [BGCG22] is given by*

$$c_n = \frac{(\gamma)_n}{n!} B_n(\alpha_1, \ldots, \alpha_n)$$

Proof. We have

$$\sum_{\ell=1}^{\infty} \kappa_\ell \frac{z^\ell}{\ell} = \sum_{\ell=1}^{\infty} \alpha_\ell \frac{z^\ell}{\ell!}$$

But we know that the B_n complete exponential Bell polynomial express the nth moment in terms of the first n cumulants. In particular, we have

$$\exp\left(\alpha_\ell \frac{z^\ell}{\ell!}\right) = \sum_{n=0}^{\infty} B_n(\alpha_1, \ldots, \alpha_n)\frac{z^n}{n!}$$

However, by (2), we have

$$\sum_{n=0}^{\infty} \frac{c_n}{(\gamma)_n} z^n = \exp\left(\sum_{\ell=1}^{\infty} \kappa_\ell \frac{z^\ell}{\ell}\right) = \exp\left(\sum_{\ell=1}^{\infty} \alpha_\ell \frac{z^\ell}{\ell!}\right) = \sum_{n=0}^{\infty} B_n(\alpha_1, \ldots, \alpha_n)\frac{z^n}{n!}$$

matching coefficients give desired result. $\square$

Lemma 4.2. *With $\{c_k\}_{k\geq 0}$ be the auxiliary sequence defined in Theorem 3.11 of [BGCG22] with respect to the γ-cumulants. Then the moments of the distribution is given by*

$$m_k = \frac{k}{\gamma} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \ldots, c_{k-j+1})$$

where $\hat{B}_{k,j}$ are the partial ordinary Bell polynomials.

Proof. Recall that we have from (2)

$$\exp\left(\gamma \sum_{k=1}^{\infty} m_k \frac{z^k}{k}\right) = \sum_{n=0}^{\infty} c_n z^n$$

Let

$$A(z) := \sum_{n=0}^{\infty} c_n z^n, \qquad S(z) = A(z) - 1 = \sum_{n=1}^{\infty} c_n z^n, \quad M(z) = \sum_{k=1}^{\infty} m_k \frac{z^k}{k}$$

Taking the log, we have

$$\begin{aligned}
\gamma M(z) &= \log(1 + S(z)) \\
M(z) &= \frac{1}{\gamma} \log(1 + S(z)) \\
&= \frac{1}{\gamma} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} S(z)^j
\end{aligned}$$

Therefore, we have

$$\frac{m_k}{k} = [z^k]M(z) = \frac{1}{\gamma} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} [z^k]S(z)^j$$

But by Claim 2.2, $[z^k]S(z)^j = \hat{B}_{k,j}(c_1, \ldots, c_{k-j+1})$. Thus the lemma is proven. $\square$

Theorem 4.3. *Fix $\gamma > 0$ and a sequence of γ-cumulants $\{\kappa_\ell\}_{\ell \geq 1}$. For each ℓ, n, define*

$$\alpha_\ell = (\ell - 1)!\kappa_\ell, \quad c_n = \frac{(\gamma)_n}{n!} B_n(\alpha_1, \ldots, \alpha_n)$$

where B_n are the complete exponential Bell polynomials. Then the moments are given by

$$m_k = \frac{k}{\gamma} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \ldots, c_{k-j+1})$$

Proof. Follows directly from Lemma 4.1 and Lemma 4.2. $\square$

References

[BGCG22] Florent Benaych-Georges, Cesar Cuenca, and Vadim Gorin. Matrix addition and the dunkl transform at high temperature. *arXiv.org*, 2022.

[TT21] Hoang Dung Trinh and Khanh Duy Trinh. Beta laguerre ensembles in global regime. *Osaka Journal of Mathematics*, 58(2):435–450, 2021.