

High Temperature Limit of Beta Ensembles and MOPS

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December 10, 2025

Outline

1 Beta-Laguerre Ensemble in the High Temperature Limit

- Background
- Moments for Beta-Laguerre

2 Relationship between γ -cumulants and Moments for General Distributions

3 Multivariate Orthogonal Polynomials Symbolically

High Temperature Limit

- β -ensemble: $\prod_{i < j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n w(\lambda_i)$
- Regime: $\beta \rightarrow 0, N \rightarrow \infty$ with $\beta N \rightarrow 2\gamma \in (0, \infty)$
- γ -cumulants introduced in [BGCG22] to be the analog of classical/free cumulants for the high-temperature limit.
 - γ -convolution \boxplus_γ as analog to classical/free convolution for γ -addition
 - $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$ give classical and free cumulants, respectively, after normalization.
 - γ -cumulants convertible to moments
- Notation: $(\gamma)_n = \gamma \cdot (\gamma + 1) \cdots (\gamma + n - 1)$

Generating Functions for γ -cumulants and Moments

- $g(z) := \sum_{l=1}^{\infty} \kappa_l z^{l-1}$
- Converting $\{\kappa_{\ell}^{(\gamma)}\}_{\ell} \leftrightarrow \{m_k\}_k$ via

$$m_k = [z^0] (\partial + \gamma d + *_g)^{k-1} g(z)$$

Theorem (γ -cumulants to moments, [BGC22])

Suppose $\{\kappa_l\}_{l \geq 1}$ are the γ -cumulants and $\{m_k\}_{k \geq 1}$ are the moments, then for any k , we have

$$m_k = \sum_{\pi \in \mathcal{P}(k)} W(\pi) \prod_{B \in \pi} \kappa_{|B|} \tag{1}$$

where $\mathcal{P}(k)$ is the collection of all set partitions of $[k]$ and $W(\pi)$ is the weight function of partitions defined in equation (3.4) of [BGC22].

Generating Functions for γ -cumulants and Moments

Theorem (Relating γ -CGF and MGF, Theorem 3.11 of [BGC22])

Let $\{m_k\}_{k \geq 1}$ and $\{\kappa_l\}_{l \geq 1}$ be the moments of γ -cumulants, respectively. Then

$$\exp\left(\sum_{l=1}^{\infty} \frac{\kappa_l y^l}{l}\right) = [z^0] \left\{ \sum_{n=0}^{\infty} \frac{(yz)^n}{(\gamma)_n} \cdot \exp\left(\gamma \sum_{k=1}^{\infty} \frac{m_k}{k} z^{-k}\right) \right\} \quad (2)$$

Equivalently, (2) can be restated through an auxiliary sequence $\{c_n\}_{n \geq 0}$

$$\begin{aligned} \exp\left(\sum_{l=1}^{\infty} \frac{\kappa_l}{l} z^l\right) &= \sum_{n=0}^{\infty} \frac{c_n}{(\gamma)_n} z^n \\ \exp\left(\gamma \sum_{k=1}^{\infty} \frac{m_k}{k} z^k\right) &= \sum_{n=0}^{\infty} c_n z^n \end{aligned} \quad (3)$$

Relation to Bell Polynomials

Proposition

In the high-temperature limit with $\beta N \rightarrow 2\gamma \in (0, \infty)$, the β -Laguerre ensemble empirical measure converges to a measure ν_λ^γ , with moments given by

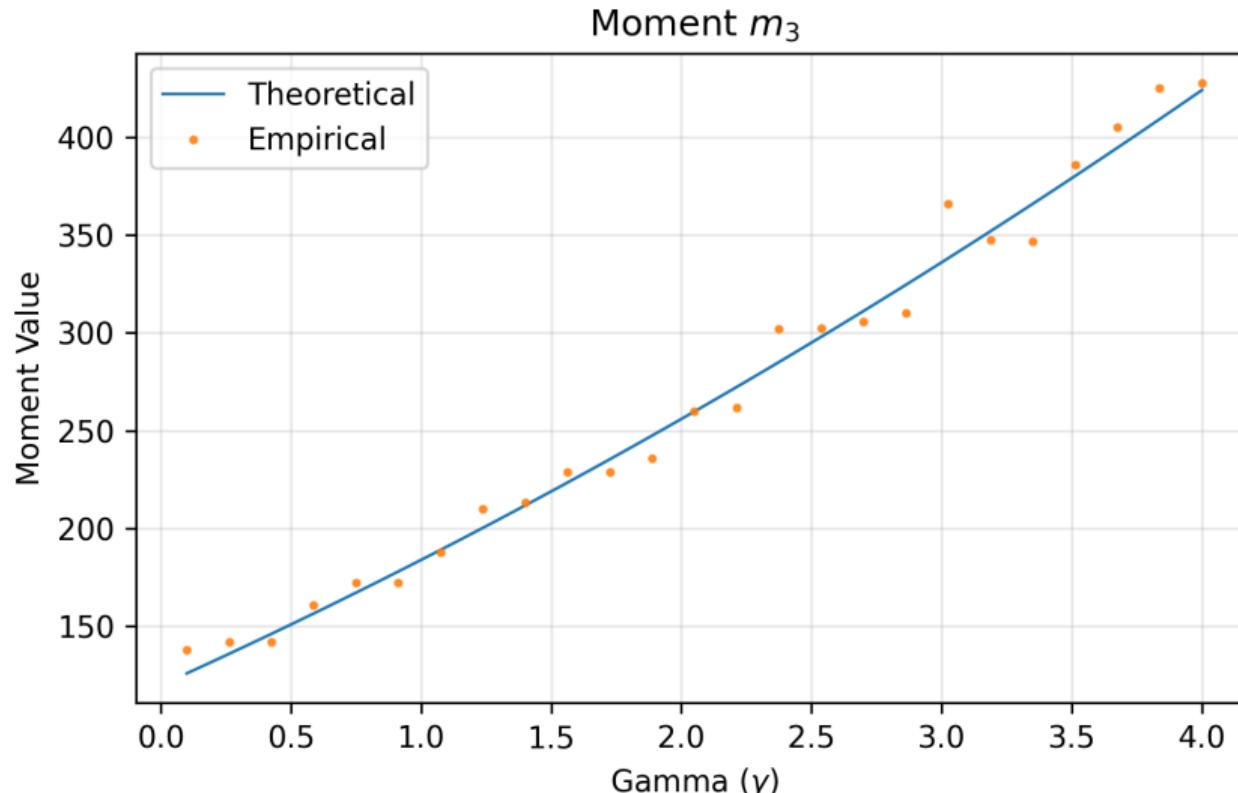
$$m_k = \frac{k}{\gamma} \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \dots, c_{k-j+1}) \quad (4)$$

where $\hat{B}_{k,j}$ are the partial ordinary Bell polynomials given by

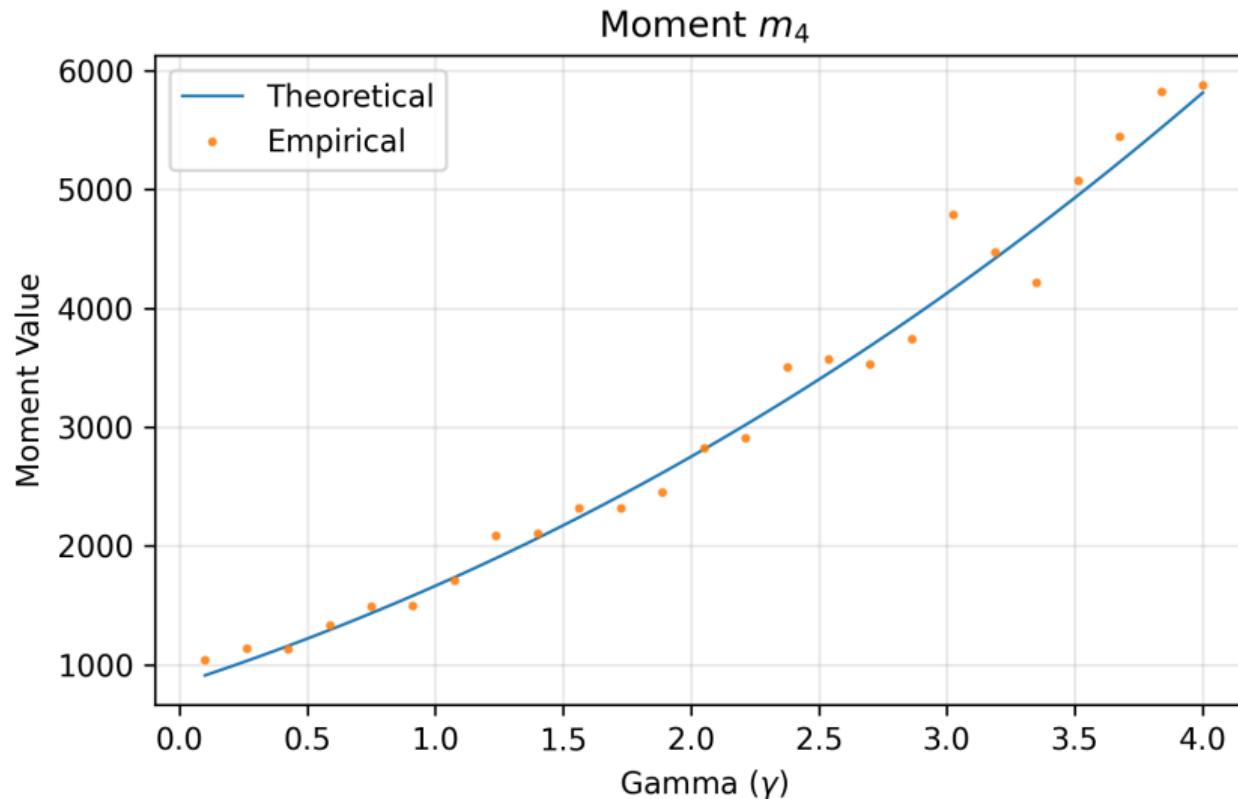
$$\hat{B}_{k,j}(a_1, \dots, a_{k-j+1}) = \sum_{\substack{n_1, \dots, n_{k-j+1} \geq 0 \\ \sum_{r=1}^{k-j+1} n_r = j, \sum_{r=1}^{k-j+1} r n_r = k}} \frac{j!}{\prod_{r=1}^{k-j+1} n_r!} \prod_{r=1}^{k-j+1} a_r^{n_r} \quad (5)$$

and $c_n = \frac{(\lambda)_n (\gamma)_n}{n!}$

Numerical Verification



Numerical Verification



Recurrence Relation

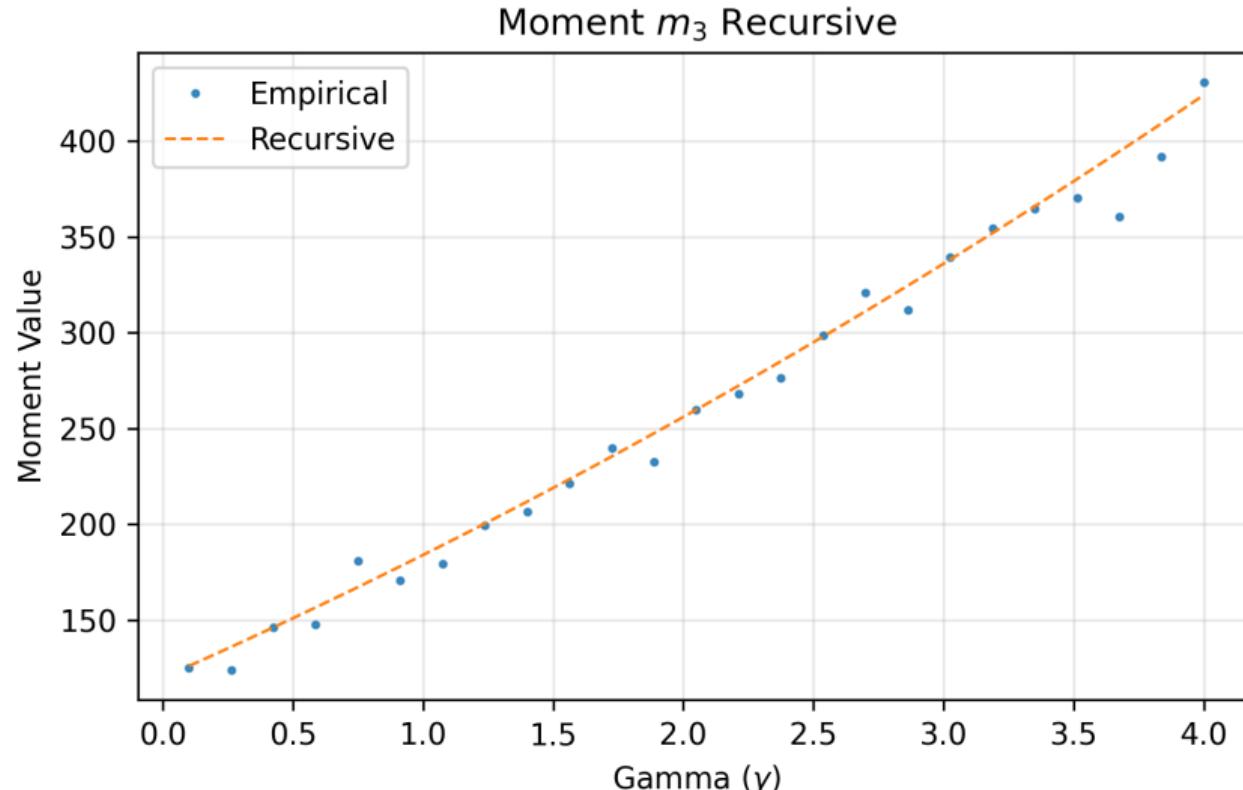
Proposition

The measure ν_{λ}^{γ} above has moments given by the recurrence relation

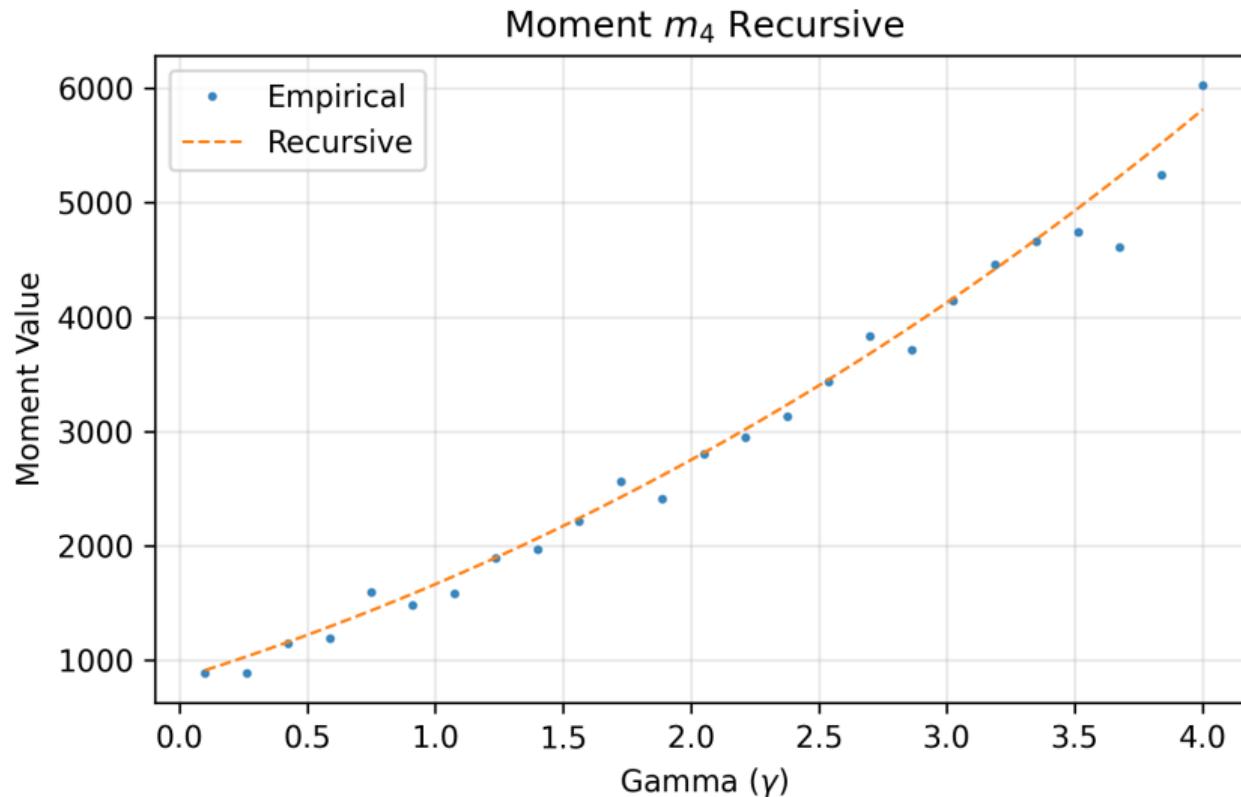
$$m_n = \frac{n}{\gamma} c_n - \sum_{k=1}^{n-1} m_k c_{n-k}$$

where $c_n = \frac{(\lambda)_n (\gamma)_n}{n!}$

Numerical Verification



Numerical Verification



General Moments from γ -cumulants

Theorem

Fix $\gamma > 0$ and a sequence of γ -cumulants $\{\kappa_\ell\}_{\ell \geq 1}$. For each ℓ, n , define

$$\alpha_\ell = (\ell - 1)! \kappa_\ell, \quad c_n = \frac{(\gamma)_n}{n!} B_n(\alpha_1, \dots, \alpha_n) \quad (6)$$

where B_n are the complete exponential Bell polynomials. Then the moments are given by

$$m_k = \frac{k}{\gamma} \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \dots, c_{k-j+1}) \quad (7)$$

Symbolic Computation of MOPs

- In [DES07], an algorithm to compute MOPS is proposed.
- Maple implementation has a bug in the computation of Jacobi polynomials.
- An error has been detected in the computation of generalized binomial coefficients.

References

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