An analysis of Bessel generating functions

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Root systems and Dunkl operators

For $\alpha \in \mathbb{R}^N \backslash \{0\}$, r_α is the reflection defined by

$$r_\alpha : x \mapsto x - 2 \langle x, \alpha \rangle \|\alpha\|_2^{-2} \alpha.$$

Suppose $N \geq 2$ and $\mathcal{R} \subset \mathbb{R}^N$ is a finite crystallographic root system. The group $H(\mathcal{R})$ is the group of reflections generated by $\mathcal{R}$.

Let $\mathcal{R}^+$ be a set of positive roots in $\mathcal{R}$ and let $\theta(\mathcal{R})$ denote the set of multiplicity functions over $\mathcal{R}$. For $\theta \in \theta(\mathcal{R})$, we define the *Dunkl operator* [Dun89] by

$$\mathcal{D}_i(\mathcal{R}(\theta)) \triangleq \partial_i + \sum_{r \in \mathcal{R}^+} \theta(r) \frac{1 - r_\alpha}{\langle x, \alpha \rangle} \alpha_i.$$

Bessel function

Definition 1 ([Opd93])

Suppose $\theta \in \theta(\mathcal{R})$ is holomorphic. The holomorphic function $J_a^{\mathcal{R}(\theta)}(x)$ over $(a, x) \in \mathbb{C}^N \times \mathbb{C}^N$ is the unique eigenfunction that satisfies $J_a^{\mathcal{R}(\theta)} \in \mathbb{C}^{H(\mathcal{R}(\theta))}[[x_1, \ldots, x_N]]$,

$$f(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))J_a^{\mathcal{R}(\theta)}(x) = f(a)J_a^{\mathcal{R}(\theta)}(x)$$

for all $f \in \mathbb{C}^{H(\mathcal{R}(\theta))}[x_1, \ldots, X_N]$, and $J_a^{\mathcal{R}(\theta)}(0) = 1$. The function $J_a^{\mathcal{R}(\theta)}$ is referred to as the *Bessel function* and satisfies

$$J_a^{\mathcal{R}(\theta)}(x) = \frac{1}{|H(\mathcal{R})|} \sum_{h \in H(\mathcal{R})} E_a^{\mathcal{R}(\theta)}(hx).$$

Bessel generating function

To obtain the *Bessel generating function*, we set a to be a random variable and compute the average of $J_a^{\mathcal{R}(\theta)}$.

In particular, for a Borel probability measure μ over $\mathbb{C}^N$, the Bessel generating function $G_\mu^{\mathcal{R}(\theta)}(x)$ is defined as

$$G_\mu^{\mathcal{R}(\theta)}(x) \triangleq \mathbb{E}_{a\sim\mu}[J_a^\theta(x)].$$

- In order for the Bessel generating function to be holomorphic in a neighborhood of the origin, we require μ to be *exponentially decaying*.
- If we are given that there exists a distribution μ with a certain Bessel generating function without knowing μ, then the function is similar to the β-ghosts [Ede09], since we cannot sample from μ.

Irreducible root systems

We only consider the irreducible root systems $A^{N-1}(\theta)$, $BC^N(\theta_0, \theta_1)$, and $D^N(\theta)$.

For $\theta \in \mathbb{C}$, the Dunkl operator associated to $A^{N-1}(\theta)$ is

$$\mathcal{D}_i(A^{N-1}(\theta)) \triangleq \partial_i + \theta \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j}$$

for $i \in [N]$, where s_{ij} switches e_i and e_j.

For $\theta_0, \theta_1 \in \mathbb{C}$, the Dunkl operator associated to $BC^N(\theta_0, \theta_1)$ is

$$\mathcal{D}_i(BC^N(\theta_0, \theta_1)) \triangleq \partial_i + \theta_1 \frac{1 - \tau_i}{x_i} + \theta_0 \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j} + \frac{1 - \tau_i \tau_j s_{ij}}{x_i + x_j}$$

for $i \in [N]$, where τ_i switches e_i and $-e_i$.

Irreducible root systems

For $\theta \in \mathbb{C}$, the Dunkl operator associated to $D^N(\theta)$ is

$$\mathcal{D}_i(D^N(\theta)) \triangleq \partial_i + \theta \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j} + \frac{1 - \tau_i \tau_j s_{ij}}{x_i + x_j}$$

for $i \in [N]$.

Dunkl bilinear form

For $p, q \in \mathbb{C}[x_1, \ldots, x_N]$, the symmetric *Dunkl bilinear form* is defined by $[p, q]_{\mathcal{R}(\theta)} \triangleq [1]p(\mathcal{D}(\mathcal{R}(\theta)))q$.

Theorem 2 ([Dun91, Theorem 3.10])

Suppose $\theta \in \theta(\mathcal{R})$ is nonnegative. For $p, q \in \mathbb{C}[x_1, \ldots, x_N]$,

$$[p, q]_{\mathcal{R}(\theta)} = c_N^{-1} \int_{\mathbb{R}^N} (e^{-\Delta_{\mathcal{R}(\theta)}/2} p)(e^{-\Delta_{\mathcal{R}(\theta)}/2} q) \prod_{r \in \mathcal{R}^+} |\langle x, r \rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx,$$

where $c_N \triangleq \int_{\mathbb{R}^N} \prod_{r \in \mathcal{R}^+} |\langle x, r \rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx.$

Notation

- We let Γ denote the set of partitions and Γ_{even} denote the set of partitions with all even parts.
- For $k \geq 1$, $NC(k)$ is the set of noncrossing partitions of $[k]$ and $NC^{\text{even}}(k)$ is the set of elements of $NC(k)$ that have all even block sizes.
- For $\nu \in \Gamma$, $p_\nu(x_1, \ldots, x_N) \triangleq \prod_{i=1}^{\ell(\nu)} (\sum_{j=1}^{N} x_j^{\nu_i})$.

Computations of Bessel generating functions

Lemma 3

Suppose p is holomorphic over $\mathbb{C}^N$ and $p = \sum_{i=0}^{\infty} p_i$, where p_i is homogeneous of degree i for $i \geq 0$. Suppose $q : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is measurable, satisfies $\|\sum_{i=0}^{n} p_i(x)\|_2 \leq q(x)$ over $\mathbb{R}^N$ for all $n \geq 0$, and

$$\int_{\mathbb{R}^N} q(x) e^{R\|x\|_2} \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx < \infty$$

for some $R > 0$. Then, $e^{\Delta_{\mathcal{R}(\theta)}/2} p(a)$ equals

$$e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} p(x) E_a^{\mathcal{R}(\theta)}(x) \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx$$

over $\overline{B}(0, R)$ and is holomorphic over $B(0, R)$.

The Bessel generating function for the β-Hermite ensemble

For $\theta \geq 0$ and $t > 0$, define the probability distribution $H_{\theta,t}^N$ over $\mathbb{R}^N$ to have the density of a proportional to

$$\prod_{1 \leq i < j \leq N} |a_i - a_j|^{2\theta} \prod_{i=1}^{N} e^{-\frac{a_i^2}{2t}}.$$

Note that $\beta = 2\theta$ and we obtain $H_{\theta,t}^N$ after rescaling the β-Hermite ensemble. The paper [DE02] computes a tridiagonal matrix whose eigenvalues are given by $H_{\theta,t}^N$.

Lemma 4

Suppose $\theta \geq 0$ and $t > 0$. Then,

$$G_{H_{\theta,t}^N}^{A^{N-1}(\theta)}(x) = \exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right).$$

The Chiral ensemble

Suppose $M \geq N$ such that $M - N + 1 - \frac{1}{2\theta} \geq 0$. Let X denote a random $M \times N$ matrix whose independent entries are real, complex, or real quaternion numbers with Gaussian densities. The Chiral ensemble [For10, Section 3.1] is defined as the set of positive eigenvalues of

$$H = \begin{bmatrix} 0_{M\times M} & X \\ X^T & 0_{N\times N} \end{bmatrix}.$$

For all $\theta \geq 0$, define the probability distribution $C^N_{\theta,M,t}$ over $\mathbb{R}^N_{\geq 0}$ to have the density of a proportional to

$$C^N_{\theta,M,t} : \prod_{i=1}^{N} a_i^{2\theta(M-N+1)-1} e^{-\frac{a_i^2}{2t}} \prod_{1\leq i<j\leq N} |a_i^2 - a_j^2|^{2\theta}.$$

The Bessel generating function for the Chiral ensemble

After rescaling, the density function of the positive eigenvalues of H is given by the probability distribution $C^N_{\theta,M,t}$ when $\theta \in \{\frac{1}{2}, 1, 2\}$.

Furthermore, the Chiral ensemble has the same distribution as the square roots of the β-Laguerre ensemble.

The following lemma is a generalization of [Xu25, Proposition 5.21].

Lemma 5

Suppose $\theta \geq 0$, $M \geq N$ *such that* $M - N + 1 - \frac{1}{2\theta} \geq 0$, *and* $t > 0$. *Then,*

$$G^{BC^N(\theta,\theta(M-N+1)-\frac{1}{2})}_{C^N_{\theta,M,t}}(x) = \exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right).$$

LLN for the Chiral ensemble

Using the results of [Yao25a], we obtain an LLN result for the Chiral ensemble in the $\theta N \to \infty$ regime.

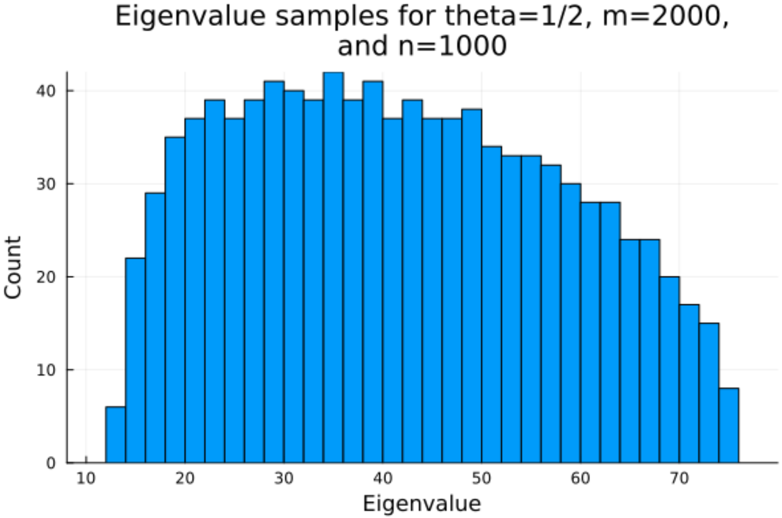
- For $k \geq 1$, let $\mathcal{D}_k$ denote the set of Dyck paths of length $2k$.
- For $p \in \mathcal{D}_k$, let $e(p)$ denote the number of descents located at p_i for even $i \in [2k]$; recall that $1 \leq e(p) \leq k$.

Theorem 6

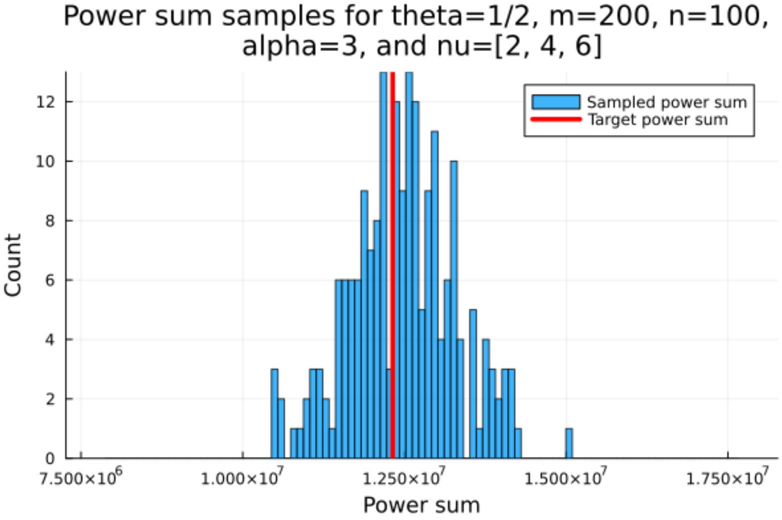
Assume that $\lim_{N\to\infty} \theta N = \infty$, $\lim_{N\to\infty} \frac{M}{N} = c \geq 1$, *and* $\lim_{N\to\infty} \frac{t}{\theta N} = \alpha \geq 0$. *Suppose* $\nu \in \Gamma_{even}$. *Then,*

$$\lim_{N\to\infty} \frac{\mathbb{E}_{a\sim C^N_{\theta,M,t}}[p_\nu(a)]}{(\theta N)^{|\nu|} N^{\ell(\nu)}} = \prod_{i=1}^{\ell(\nu)} \sum_{p\in\mathcal{D}_{\frac{\nu_i}{2}}} c^{e(p)}(2\alpha)^{\frac{\nu_i}{2}}$$

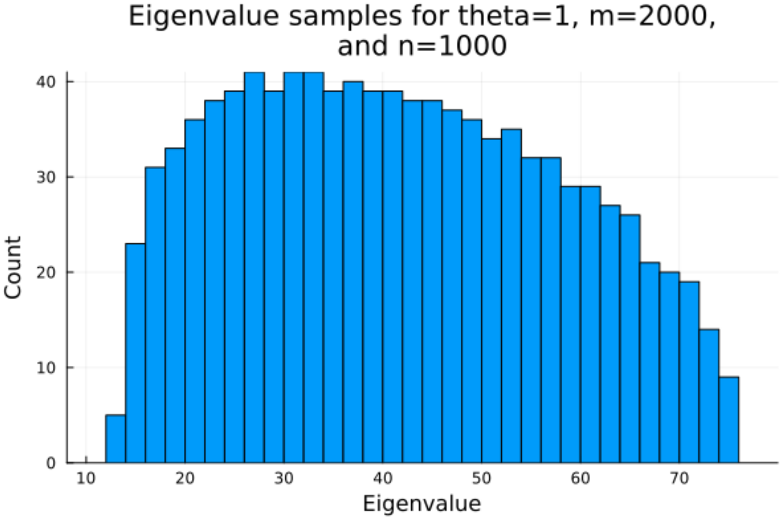
Plot of Chiral ensemble eigenvalues for $\theta = \frac{1}{2}$



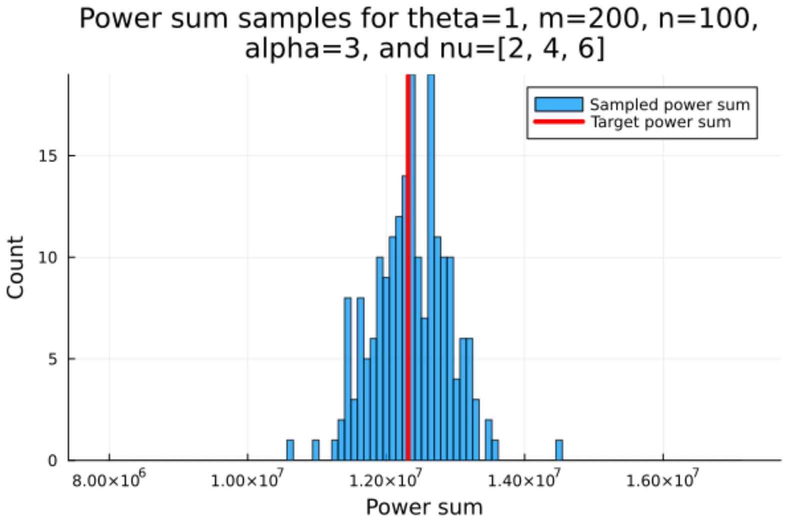
Plot for Chiral ensemble LLN for $\theta = \frac{1}{2}$



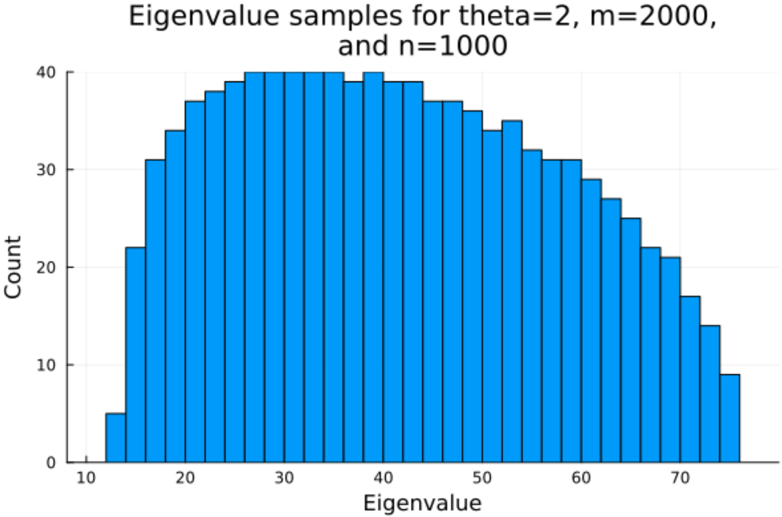
Plot of Chiral ensemble eigenvalues for $\theta = 1$



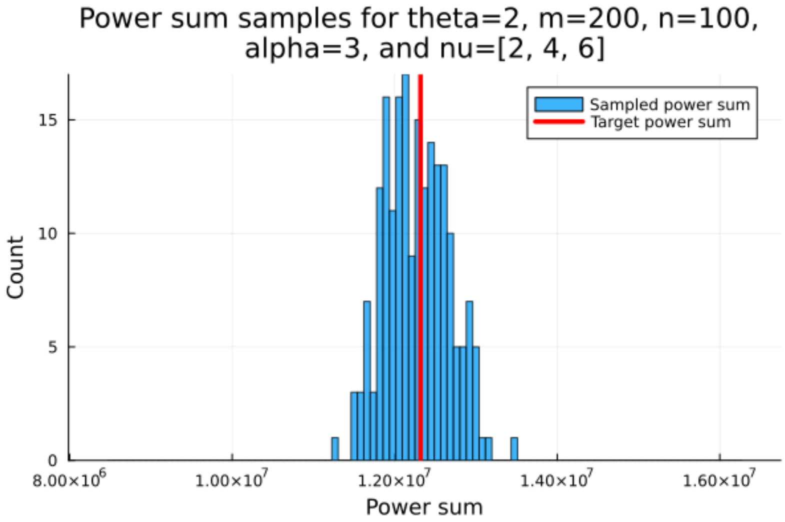
Plot for Chiral ensemble LLN for $\theta = 1$



Plot of Chiral ensemble eigenvalues for $\theta = 2$



Plot for Chiral ensemble LLN for $\theta = 2$



Dyson Brownian motion

The Dyson Brownian motion (DBM) $Y^N(t) \triangleq (Y_i(t))_{1\le i\le N}$ is the unique strong solution to

$$dY_i(t) = \theta \sum_{j\in[N]\setminus\{i\}} \frac{1}{Y_i(t) - Y_j(t)} dt + dB_i(t),$$

where the initial value $(Y_i(0))_{1\le i\le N}$ is fixed and the standard Brownian motions B_i for $1 \le i \le N$ are independent.

The transition formula for the DBM given in [GXZ24, (23)] is that if $y \triangleq (y_1, \ldots, y_N)$ has the same ordering as $z \triangleq (z_1, \ldots, z_N)$, then:

$$\Pr[Y^N(t) = y | Y^N(0) = z] \propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right)$$

$$\prod_{1\le i<j\le N} |y_i - y_j|^{2\theta} J^{A^{N-1}(\theta)}_{\frac{z}{\sqrt{t}}}\left(\frac{y}{\sqrt{t}}\right).$$

The Bessel generating function for the DBM

Lemma 7 ([GXZ24, Lemma 3.8])

Suppose $t \geq 0$. The type A Bessel generating function of $Y^N(t)$ if the initial value is fixed at $(Y_i(0))_{1\leq i\leq N}$ is

$$G_{Y^N(t)}^{A^{N-1}(\theta)}(x) = J_{(Y_i(0))_{1\leq i\leq N}}^{A^{N-1}(\theta)}(x)\exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right)$$

over $\mathbb{C}^N$.

Proof. Consider the transition formula and the identity

$$c_N^{-1}\int_{\mathbb{R}^N} E_b^{\mathcal{R}(\theta)}(x)E_a^{\mathcal{R}(\theta)}(x)\prod_{r\in\mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)}\frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}}dx$$
$$= e^{\frac{a_1^2+\cdots+a_N^2+b_1^2+\cdots+b_N^2}{2}}E_b^{\mathcal{R}(\theta)}(a).$$

The Bessel generating function for the DBM with random initialization

Lemma 8 ([Yao25a, Lemma 10.22])

Assume that μ is a Borel probability measure over $\mathbb{R}^N$ that exponentially decays at rate $R > 0$. Then, for all $t \geq 0$, $Y_\mu^N(t)$ exponentially decays at any rate less than R and

$$G_{Y_\mu^N(t)}^{A^{N-1}(\theta)}(x) = G_\mu^{A^{N-1}(\theta)}(x) \exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right)$$

over $\overline{B}(0, R)$.

Modifications of the DBM (asymmetric type A)

Assume that $y \triangleq (y_1, \ldots, y_N)$ has the same ordering as $z \triangleq (z_1, \ldots, z_N)$. Asymmetric type A DBM:

$$\Pr[X_t = y | X_0 = z]$$
$$\propto \exp\left(-\frac{\sum_{i=1}^{N} y_i^2 + z_i^2}{2t}\right) \prod_{1 \le i < j \le N} |y_i - y_j|^{2\theta} E_{\frac{z}{\sqrt{t}}}^{A^{N-1}(\theta)}\left(\frac{y}{\sqrt{t}}\right).$$

Recall that the symmetric type A DBM solves a continuous stochastic PDE. Due to this, we assume y has the same ordering as z so that the transition probability is continuous.

- Therefore, we do not analogously define the transition probability for all choices of $\mathcal{R}$.

Modifications of the DBM (type BC)

Assume that $(y_1, \ldots, y_N)$ has the same ordering as $(z_1, \ldots, z_N)$ and that $(y_1^2, \ldots, y_N^2)$ has the same ordering as $(z_1^2, \ldots, z_N^2)$.

Asymmetric type BC DBM:

$$\Pr[X_t = y | X_0 = z] \propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right)$$

$$\prod_{i=1}^N |y_i|^{2\theta_1} \prod_{1 \le i < j \le N} |y_i^2 - y_j^2|^{2\theta_0} E_{\frac{z}{\sqrt{t}}}^{BC^N(\theta_0,\theta_1)}\left(\frac{y}{\sqrt{t}}\right)$$

Symmetric type BC DBM:

$$\Pr[X_t = y | X_0 = z] \propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right)$$

$$\prod_{i=1}^N |y_i|^{2\theta_1} \prod_{1 \le i < j \le N} |y_i^2 - y_j^2|^{2\theta_0} J_{\frac{z}{\sqrt{t}}}^{BC^N(\theta_0,\theta_1)}\left(\frac{y}{\sqrt{t}}\right)$$

LLN for multiple observations of the DBM

Theorem 9

Assume that μ_N is an exponentially decaying Borel probability measure over $\mathbb{R}^N$ for all $N \geq 1$ and that for all $\nu \in \Gamma$, a complex number c_ν exists such that

$$\lim_{N\to\infty} \frac{1}{\theta N}[1] \prod_{j=1}^{\ell(\nu)} \partial_j^{\nu_j} G_{\mu_N}^{A^{N-1}(\theta)}(x_1, \ldots, x_N) = \frac{|\nu|! c_\nu}{P(\nu)}.$$

LLN for multiple observations of the DBM

Theorem 9 (Continued)

Suppose $m \geq 1$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m$. Then, if $\theta N \to \infty$, for all $\nu_1, \ldots, \nu_m \in \Gamma$,

$$\lim_{N\to\infty} \frac{\mathbb{E}[\prod_{i=1}^m p_{\nu_i}(Y_{\mu_N}^N(\alpha_i \theta N))]}{(\theta N)^{|\nu_1+\cdots+\nu_m|} N^{\ell(\nu_1+\cdots+\nu_m)}} = \prod_{i=1}^m \prod_{j=1}^{\ell(\nu_i)} \sum_{\pi \in NC(\nu_{ij})}$$

$$\prod_{B\in\pi} \left(\sum_{\nu\in\Gamma,\, |\nu|=|B|} (-1)^{\ell(\nu)-1} \frac{|\nu| P(\nu)}{\ell(\nu)} c_\nu + \mathbf{1}\{|B|=2\}\alpha_i \right).$$

We prove this theorem using the results of [Yao25b].

Bibliography I

[DE02] Ioana Dumitriu and Alan Edelman, *Matrix models for beta ensembles*, Journal of Mathematical Physics **43** (2002), no. 11, 5830–5847.

[Dun89] Charles F. Dunkl, *Differential-Difference Operators Associated to Reflection Groups*, Transactions of the American Mathematical Society **311** (1989), no. 1, 167–183.

[Dun91] ______, *Integral Kernels with Reflection Group Invariance*, Canadian Journal of Mathematics **43** (1991), no. 6, 1213–1227.

[Ede09] Alan Edelman, *The Random Matrix Technique of Ghosts and Shadows*, 2009.

[For10] P. J. Forrester, *Log-Gases and Random Matrices (LMS-34)*, Princeton University Press, 2010.

[GXZ24] Vadim Gorin, Jiaming Xu, and Lingfu Zhang, *Airy$_\beta$ line ensemble and its Laplace transform*, arXiv preprint arXiv:2411.10829 (2024).

[Opd93] E. M. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Compositio Mathematica **85** (1993), no. 3, 333–373.

Bibliography II

[Xu25] Jiaming Xu, *Rectangular matrix additions in low and high temperatures*, arXiv preprint arXiv:2303.13812 (2025).

[Yao25a] Andrew Yao, *Approximating the coefficients of the Bessel functions*, arXiv preprint arXiv:2510.10370 (2025).

[Yao25b] ______, *Limits of Probability Measures with General Coefficients*, arXiv preprint arXiv:2109.14052 (2025).