

# High Temperature Limit of Beta Ensembles and MOPS

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# High Temperature Limit

- $\beta$ -ensemble:  $\prod_{i < j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n w(\lambda_i)$
- Regime:  $\beta \rightarrow 0, N \rightarrow \infty$  with  $\beta N \rightarrow 2\gamma \in (0, \infty)$
- $\gamma$ -cumulants introduced in [BGCG22] to be the analog of classical/free cumulants for the high-temperature limit.
  - $\gamma$ -convolution  $\boxplus_\gamma$  as analog to classical/free convolution for  $\gamma$ -addition
  - $\gamma \rightarrow 0$  and  $\gamma \rightarrow \infty$  give classical and free cumulants, respectively, after normalization.
  - $\gamma$ -cumulants convertible to moments
- Notation:  $(\gamma)_n = \gamma \cdot (\gamma + 1) \cdots (\gamma + n - 1)$

# Generating Functions for $\gamma$ -cumulants and Moments

- $g(z) := \sum_{l=1}^{\infty} \kappa_l z^{l-1}$
- Converting  $\{\kappa_\ell^{(\gamma)}\}_\ell \leftrightarrow \{m_k\}_k$  via

$$m_k = [z^0](\partial + \gamma d + *_g)^{k-1} g(z)$$

## Theorem ( $\gamma$ -cumulants to moments, [BGCG22])

*Suppose  $\{\kappa_l\}_{l \geq 1}$  are the  $\gamma$ -cumulants and  $\{m_k\}_{k \geq 1}$  are the moments, then for any  $k$ , we have*

$$m_k = \sum_{\pi \in \mathcal{P}(k)} W(\pi) \prod_{B \in \pi} \kappa_{|B|} \quad (1)$$

*where  $\mathcal{P}(k)$  is the collection of all set partitions of  $[k]$  and  $W(\pi)$  is the weight function of partitions defined in equation (3.4) of [BGCG22].*

# Generating Functions for $\gamma$ -cumulants and Moments

Theorem (Relating  $\gamma$ -CGF and MGF, Theorem 3.11 of [BGCG22])

Let  $\{m_k\}_{k \geq 1}$  and  $\{\kappa_l\}_{l \geq 1}$  be the moments of  $\gamma$ -cumulants, respectively. Then

$$\exp\left(\sum_{l=1}^{\infty} \frac{\kappa_l y^l}{l}\right) = [z^0] \left\{ \sum_{n=0}^{\infty} \frac{(yz)^n}{(\gamma)_n} \cdot \exp\left(\gamma \sum_{k=1}^{\infty} \frac{m_k}{k} z^{-k}\right) \right\} \quad (2)$$

Equivalently, (2) can be restated through an auxiliary sequence  $\{c_n\}_{n \geq 0}$

$$\begin{aligned} \exp\left(\sum_{l=1}^{\infty} \frac{\kappa_l}{l} z^l\right) &= \sum_{n=0}^{\infty} \frac{c_n}{(\gamma)_n} z^n \\ \exp\left(\gamma \sum_{k=1}^{\infty} \frac{m_k}{k} z^k\right) &= \sum_{n=0}^{\infty} c_n z^n \end{aligned} \quad (3)$$

# Relation to Bell Polynomials

## Proposition

*In the high-temperature limit with  $\beta N \rightarrow 2\gamma \in (0, \infty)$ , the  $\beta$ -Laguerre ensemble empirical measure converges to a measure  $\nu_\lambda^\gamma$ , with moments given by*

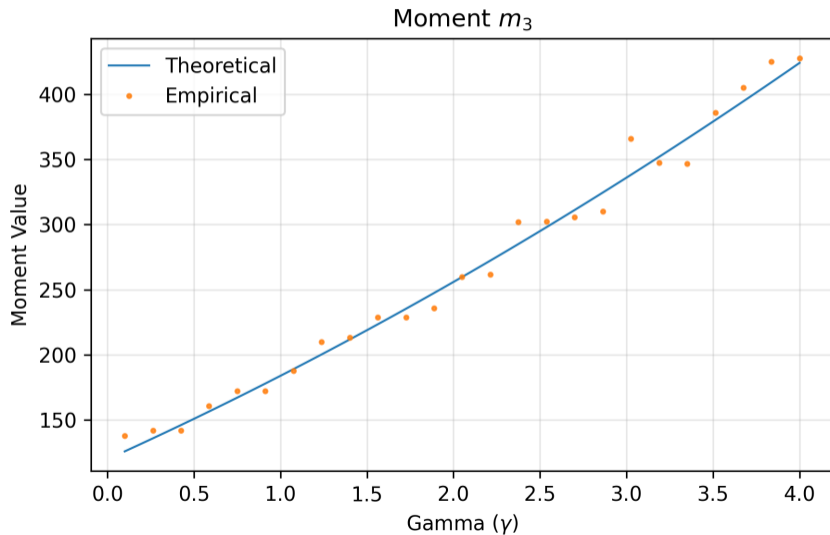
$$m_k = \frac{k}{\gamma} \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \dots, c_{k-j+1}) \quad (4)$$

where  $\hat{B}_{k,j}$  are the partial ordinary Bell polynomials given by

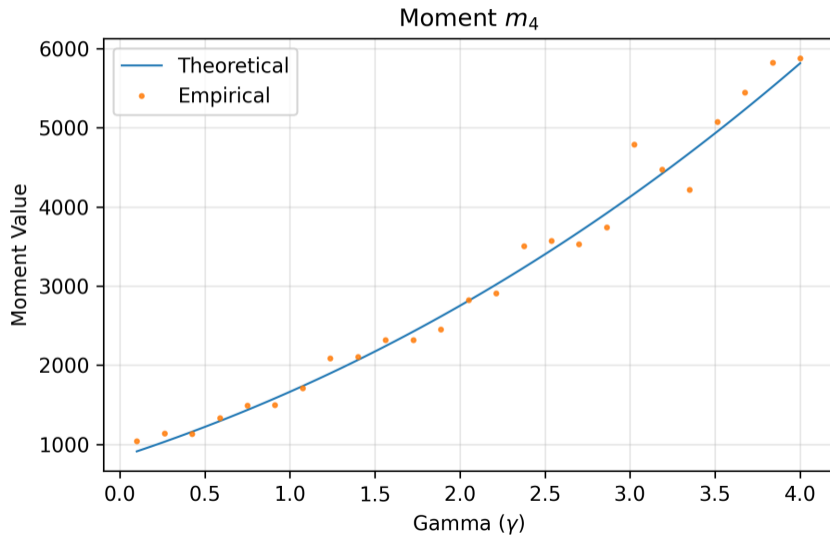
$$\hat{B}_{k,j}(a_1, \dots, a_{k-j+1}) = \sum_{\substack{n_1, \dots, n_{k-j+1} \geq 0 \\ \sum_{r=1}^{k-j+1} n_r = j, \sum_{r=1}^{k-j+1} r n_r = k}} \frac{j!}{\prod_{r=1}^{k-j+1} n_r!} \prod_{r=1}^{k-j+1} a_r^{n_r} \quad (5)$$

$$\text{and } c_n = \frac{(\lambda)_n (\gamma)_n}{n!}$$

# Numerical Verification



# Numerical Verification



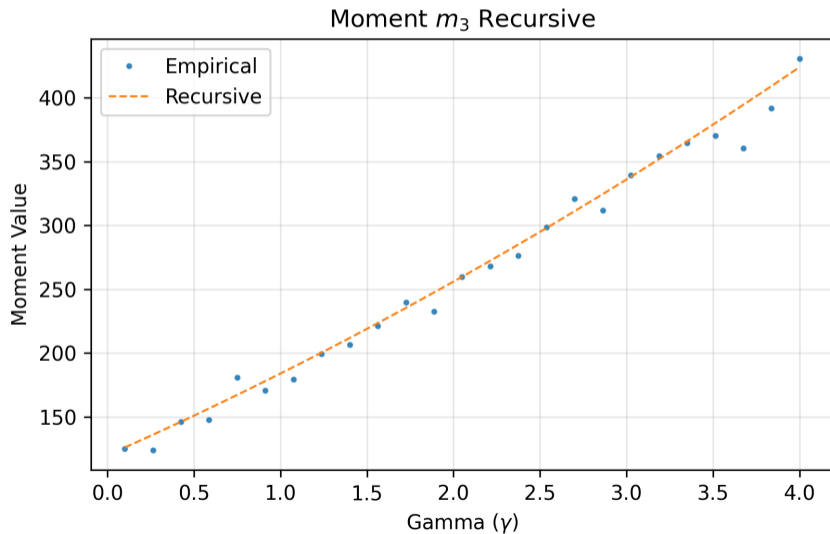
# Recurrence Relation

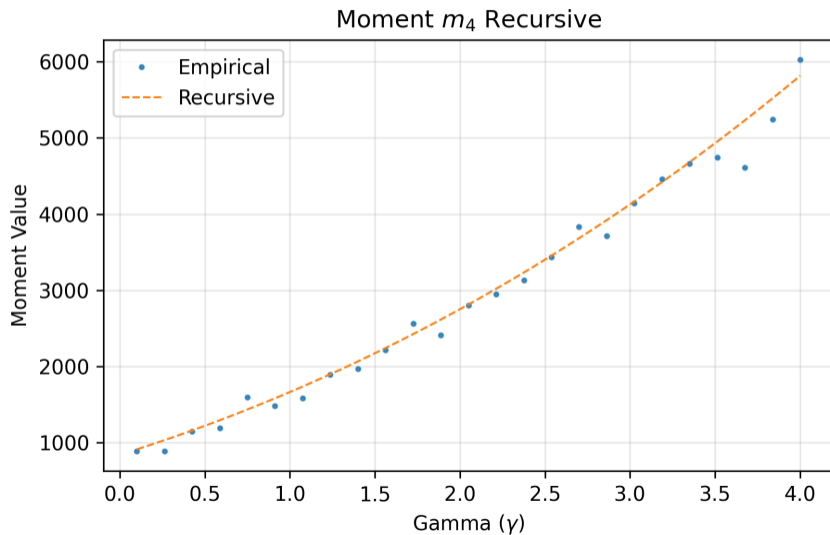
## Proposition

*The measure  $\nu_\lambda^\gamma$  above has moments given by the recurrence relation*

$$m_n = \frac{n}{\gamma} c_n - \sum_{k=1}^{n-1} m_k c_{n-k}$$

*where  $c_n = \frac{(\lambda)_n (\gamma)_n}{n!}$*





# General Moments from $\gamma$ -cumulants

## Theorem



Fix  $\gamma > 0$  and a sequence of  $\gamma$ -cumulants  $\{\kappa_\ell\}_{\ell \geq 1}$ . For each  $\ell, n$ , define

$$\alpha_\ell = (\ell - 1)! \kappa_\ell, \quad c_n = \frac{(\gamma)_n}{n!} B_n(\alpha_1, \dots, \alpha_n) \quad (6)$$

where  $B_n$  are the complete exponential Bell polynomials. Then the moments are given by

$$m_k = \frac{k}{\gamma} \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \hat{B}_{k,j}(c_1, \dots, c_{k-j+1}) \quad (7)$$

- In [DES07], an algorithm to compute MOPS is proposed.
- Maple implementation has a bug in the computation of Jacobi polynomials.
- An error has been detected in the computation of generalized binomial coefficients.

-  Florent Benaych-Georges, Cesar Cuenca, and Vadim Gorin.  
Matrix addition and the dunkl transform at high temperature.  
*arXiv.org*, 2022.
-  Ioana Dumitriu, Alan Edelman, and Gene Shuman.  
Mops: Multivariate orthogonal polynomials (symbolically).  
*Journal of symbolic computation*, 42(6):587–620, 2007.