UNDERSTANDING THE LIMA BEAN

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1. Introduction

In the recent paper [2], the spectral behavior of discretized random walks on a matrix group $G \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$, defined by

$$B_k = (I + A_1)\,(I + A_2)\cdots(I + A_k)\,, \quad A_1, \ldots, A_k \in G$$

for $k \in \mathbb{N}$ is investigated. It is known that a properly scaled sequence of such random walks will converge to Brownian motion on the respective group when letting the discretization size go to 0. However, the authors were interested in a more involved question. They study the eigenvalue distribution first as the matrix dimension goes to infinity and then let the discretization size go to 0. This double limit happens to behave well enough for it to converge to a certain probabilistic object. The natural candidate for the limit distribution of the spectral couting measure of the random walk should then be some kind of measure that characterizes the spectrum of the limit object. Of course the limit object is infinite-dimensional, and since matrix groups generally do not consist of normal matrices, one should not expect that the limiting operators be normal. In particular, one needs a proper notion of what a spectral measure for a non-normal operator means. As they show in the paper, the so-called *Brown measure* of the limit operator is the proper object to consider as the limit.

2. Goals of the Project

Due to the size of the paper (109 pages) it seems reasonable to set some goals for the final report and some goals for further investigation. I propose the following goals, where I use (*) for more ambitious goals:

- **Understand the mathematical prerequisites:** The first step of this project will be to understand the mathematics that is used to prove the Lima Bean Theorem. Namely, to learn about free probability, Brown measures and random walks on matrix groups as well as their interactions.
- **Numerical verification of the results:** The next goal is then to verify some of the findings of the paper numerically using Julia. In particular, I would like to understand the dynamics of the distribution of the (pseudo)spectrum of the matrix random walk over time. Ideally, the numerical studies will provide the most content for the final report. In fact, it would probably already be interesting to see what happens when we fix the matrix dimension $n \in \mathbb{N}$ and a time $t > 0$ and compute the distribution of the eigenvalues from the matrix Brownian motion that we obtain.
- (*) **Investigate extensions to more matrix groups:** Only a handful of matrix groups are covered by the theorem in the paper.
- (*) **Examine any interesting phenomena that are encountered:** If any unexpected interesting phenomena appear they should of course be investigated.

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- (**) **Numerical computation of Brown measures of random walks:** It would be interesting to find out how to numerically compute Brown measures along the way. According to [1, §1.3], the general numerical approximation of Brown measures is an open problem. However, it might be possible to exploit the structure of the random walk in this case to find an explicit numerical algorithm to approximate the Brown measure. This is in particular interesting since the Brown measure of the limit is explicitly known and would thus provide a good basis for verification of such an algorithm. A simple first step here could be to consider random symmetric tridiagonal matrices, for which there are explicit results known about the spectrum, see [1, §1.4].

References

[1] Matthew J. Colbrook. "Computing Spectral Measures and Spectral Types". In: *Communications in Mathematical Physics* 384.1 (May 2021), pp. 433–501. ISSN: 1432-0916. DOI: 10.1007/s00220-021-04072-4. URL: https://doi.org/10.1007/s00220-021-04072-4.

[2] Bruce K. Driver et al. *Matrix Random Walks and the Lima Bean Law*. 2025. arXiv: 2510.10712 [math.PR]. URL: https://arxiv.org/abs/2510.10712.