Numerical Painleve σ-Forms with Julia: From ODE Solvers to Random Matrix Distributions

Final Project in 18.338

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Fall 2025

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1 Abstract

In this report, we investigate the role of Painleve equations in describing eigenvalue statistics of classical random matrix ensembles. Focusing on Hermitian, Laguerre, and Jacobi ensembles, we examine how Fredholm determinants associated with gap probabilities and extreme eigenvalue distributions are governed by Painleve transcendents and their corresponding σ-forms.

We numerically study the largest eigenvalue distributions of Hermitian ensembles for symmetry classes $\beta = 1, 2, 4$, verifying the Painleve II characterization of the soft-edge Tracy–Widom laws. For Laguerre ensembles with $\beta = 1, 2$, we study the largest eigenvalue distributions (soft-edge statistics) and compare numerical results with Painleve II based characterizations. In addition, we explore level spacing statistics via Painleve V equations and study the relationship between Painleve VI and the largest eigenvalue distribution of the Gaussian Unitary Ensemble (GUE).

We review the necessary background in random matrix theory and Painleve equations in Section 2. Section 3 presents the numerical implementation of Painleve-based methods and Monte Carlo simulations. In Section 4, we report numerical results for largest eigenvalue distributions and spacing statistics. Finally, Section 5 concludes with a summary of findings and a discussion of numerical limitations.

Our results demonstrate both the stability and numerical challenges inherent in solving Painleve equations and their σ-forms. By leveraging the connection between standard Painleve equations and σ-form representations, we propose a practical numerical procedure for computing eigenvalue statistics relevant to random matrix theory, highlighting the effectiveness and limitations of Painleve based approaches in finite-dimensional settings.

The Julia code for numerical results can be found at Github: `https://github.com/Strawberryshake-Ollie/18.338-Final-Project`. If you see anything wrong or unclear, let me know through email ziao0306@mit.edu!

2 Introduction

Random matrix theory (RMT) provides a rich framework for understanding universal statistical properties of eigenvalues arising in a wide range of applications [Ede25]. In many classical ensembles, quantities of interest such as gap probabilities, largest eigenvalue distributions, and level spacing statistics admit exact representations in terms of Fredholm determinants of integral operators. Remarkably, these Fredholm determinants are often governed by nonlinear ordinary differential equations belonging to the Painleve family [EP05, Zha17].

Historically, the Painleve equations were introduced at the turn of the twentieth century in the work of Painleve and his collaborators, who sought to classify second-order nonlinear ordinary differential equations with well-behaved analytic solutions [Tak00]. For several decades, these equations were primarily studied within the theory of special functions and integrable systems. Their relevance to random matrix theory emerged much later, beginning in the late twentieth century, when exact formulas for eigenvalue gap probabilities and extreme eigenvalue distributions were discovered [EP05, Zha17].

A major breakthrough occurred with the work of Tracy and Widom, who showed that the distribution of the largest eigenvalue in Gaussian random matrix ensembles can be expressed in terms of Painleve II transcendents [TW94, TW96b]. Subsequent developments revealed that Painleve equations and their σ-forms provide a unifying framework for describing universal spectral statistics across a wide range of random matrix ensembles and scaling regimes [For10]. This historical development highlights how Painleve equations serve as a natural bridge between classical analysis and modern probabilistic models, motivating their central role in the present study. To provide a foundation for the numerical investigations carried out in this report, we begin by reviewing the Painleve equations and their associated σ-form formulations.

2.1 The Painleve Equations and Their σ-Forms

In this section, we briefly review the analytic foundations of Painleve equations, emphasizing the Painleve property and the role of movable singularities. We then outline the general theory underlying the six Painleve equations before introducing the Jimbo–Miwa–Okamoto σ-forms, which play a central role in subsequent sections when connecting Painleve transcendents to eigenvalue distributions and universal scaling limits in random matrix theory.

2.1.1 Movable Pole and Painleve Property

A central concept in the theory of Painleve equations is the nature of singularities of solutions to nonlinear ordinary differential equations [EP05, Zha17]. Given an ordinary differential equation, a singularity of a solution is called fixed if its location is determined solely by the equation itself, and movable if its position depends on the choice of initial conditions.

Definition 2.1 (Movable/fixed Pole)**.** A pole x_0 of a solution $y(x)$ to an ordinary differential equation is called movable if its location depends on the initial conditions (integration constants), and fixed if it is determined solely by the differential equation itself.

With reference [ER05], there are two good examples to demonstrate the movable poles.

Example 2.2
Consider the first-order equation:

$$y' + y^2 = 0, \qquad y(0) = a$$

which has solution:

$$y(x) = \frac{a}{ax+1}$$

And it has a movable pole at $x = -\frac{1}{a}$ where the pole moves with the initial condition.

Another higher-order example is:

Example 2.3
Consider the second-order equation:

$$y'' + (y')^2 = 0, \qquad y(0) = a, \qquad y'(0) = b$$

which has solution:

$$y(x) = \log(1 + bx) + a$$

And is has a movable log singularity ($x = -1/b$).

Definition 2.4 (Painleve Property)**.** All the solutions are free from movable branch points.

Definition 2.5 (Painleve Equations)**.** Discovered by Painleve and his colleagues at the beginning of the 20th century while classifying all second-order ordinary differential equations:

$$y^{''} = R(x, y, y^{'})$$

which possesses the Painleve property and R is analytic in x and rational in y and $y^{'}$.

Definition 2.6 (Painleve Transcendents)**.** The solutions of Painleve equations are called the Painleve transcendents.

2.1.2 General Theory of Painleve Equations

As referenced from [EP05, Ede25], the following is the general forms of the Painleve family:

$$\begin{aligned}
&\text{(I)} && y'' = 6y^2 + t \\
&\text{(II)} && y'' = 2y^3 + ty + \alpha \\
&\text{(III)} && y'' = \frac{1}{y}(y')^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y} \\
&\text{(IV)} && y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
&\text{(V)} && y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1} \\
&\text{(VI)} && y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& && \quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(y-t)^2}\right]
\end{aligned}$$

2.1.3 Jimbo–Miwa–Okamoto σ-Forms

In the early 1980s, an alternative but equivalent formulation of the Painlevé equations, known as the σ-form, was introduced independently by Okamoto [Oka80] and by Jimbo and Miwa [JM81].This formulation is therefore commonly referred to as the Jimbo–Miwa–Okamoto σ-form of the Painleve equations. The σ-forms arise naturally from auxiliary Hamiltonian structures associated with monodromy preserving deformations of the underlying linear systems. The following is the corresponding σ-forms [Ede25] where the σ-form of Painleve I is indeed itself:

Painleve II: $$(\sigma'')^2 + 4\sigma'(\sigma^2 - t\sigma' + \sigma) - a^2 = 0 \tag{PII-σ}$$

Painleve III: $$(t\sigma'')^2 - v_1 v_2 (\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t\sigma') - \frac{1}{4^3}(v_1 - v_2)^2 = 0 \tag{PIII-σ}$$

Painleve IV: $$(\sigma'')^2 - 4(t\sigma' - \sigma)^2 + 4\sigma'(\sigma' + 2\alpha_1)(\sigma' - 2\alpha_2) = 0 \tag{PIV-σ}$$

Painleve V: $$(t\sigma'')^2 - \Big(\sigma - t\sigma' + 2(\sigma')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma'\Big)^2 + 4\prod_{i=0}^{3}(\sigma' + \nu_i) = 0 \tag{PV-σ}$$

Painleve VI: $$\sigma'\big((t - t^2)\sigma''\big)^2 + \Big(\sigma'(2\sigma - (2t - 1)\sigma') + \nu_1\nu_2\nu_3\nu_4\Big)^2 = \prod_{i=1}^{4}(\sigma' + \nu_i^2) \tag{PVI-σ}$$

2.2 Background in Random Matrix Theory

Random matrix theory (RMT) provides a universal framework for describing spectral statistics of large complex systems. In this project, we focus on the classical invariant ensembles whose eigenvalue distributions admit determinantal and whose limiting statistics can be characterized by Painleve equations and their σ-forms. We briefly review the three primary families relevant to our study: Hermitian, Laguerre, and Jacobi ensembles. More details are in [Ede25].

2.2.1 Hermitian Ensembles

Hermitian ensembles consist of random Hermitian matrices whose probability density is invariant under unitary conjugation. The most prominent example is the Gaussian Unitary Ensemble (GUE), whose probability density is given by

$$\mathbb{P}(H) \propto \exp\left(-\frac{\beta}{2}\mathrm{Tr}(H^2)\right) \qquad \beta = 2$$

with analogous definitions for the Gaussian Orthogonal and Symplectic Ensembles (GOE, $\beta = 1$) and (GSE, $\beta = 4$). The joint probability density function of the eigenvalues $\lambda_1, \ldots, \lambda_n$ takes the form

$$P(\lambda_1, \ldots, \lambda_n) \propto \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^{n} e^{-\beta\lambda_k^2/2}$$

which reveals a logarithmic repulsion between eigenvalues. In the large-n limit, the global eigenvalue density converges to the Wigner semicircle law, while local statistics exhibit universal behavior. In particular, the distribution of the largest eigenvalue converges, under appropriate soft-edge scaling, to the Tracy–Widom distribution. This distribution can be expressed in terms of the Painleve II transcendent and its associated σ-form. At finite matrix size, the cumulative distribution function of the largest GUE eigenvalue admits a representation in terms of the Painleve IV σ-form.

2.2.2 Laguerre Ensembles

Laguerre ensembles, also known as Wishart ensembles, arise naturally in multivariate statistics and sample covariance analysis. They are defined by random matrices of the form

$$L = X^\dagger X$$

where X is a rectangular matrix with independent Gaussian entries. The corresponding eigenvalue density is supported on the positive real axis. The joint eigenvalue distribution of the Laguerre β-ensemble is given by

$$P(\lambda_1, \ldots, \lambda_n) \propto \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^{n} \lambda_k^\alpha e^{-\beta\lambda_k/2} \qquad \lambda_k > 0$$

where the parameter α depends on the rectangularity of X. The Laguerre ensembles exhibit a hard edge at the origin. After appropriate scaling, the smallest eigenvalue and gap probabilities near zero are governed by the Bessel kernel. These hard-edge statistics can be expressed in terms of Painleve III transcendents and the Jimbo–Miwa–Okamoto σ-form of Painleve III.

2.2.3 Jacobi Ensembles

Jacobi ensembles describe random matrices with eigenvalues supported on a finite interval, typically $[0, 1]$. They arise in problems involving ratios of Wishart matrices, canonical correlation analysis, and random projections. The joint eigenvalue density of the Jacobi β-ensemble takes the form

$$P(\lambda_1, \ldots, \lambda_n) \propto \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^{n} \lambda_k^\alpha (1 - \lambda_k)^\gamma \qquad 0 < \lambda_k < 1$$

Due to the presence of two finite endpoints, Jacobi ensembles exhibit both soft-edge and hard-edge behavior depending on parameter regimes. Gap probabilities on finite intervals and extreme eigenvalue statistics can be expressed in terms of Painleve VI σ-forms. Moreover, Painleve VI serves as the most general member of the Painleve hierarchy, with Painleve V, III, and II arising as degenerations under suitable scaling limits.

2.3 Main Contributions

The main contribution of this project is a systematic numerical study in Julia of the connections between Painleve equations (in particular their Jimbo–Miwa–Okamoto σ-forms) and eigenvalue statistics of classical random matrix ensembles. Rather than focusing on formal derivations alone, we emphasize numerical verification of Painleve-based representations for finite-size and asymptotic random matrix distributions.

2.3.1 Painleve II and Soft-Edge Statistics

The soft-edge scaling limit describes the behavior of eigenvalues near the spectral edge where the limiting density vanishes smoothly [EP05].

Theorem 2.7 ([TW96a, EP05])
Let $\lambda_{\max}$ denote the largest eigenvalue of the β-Hermite ensemble $G_\beta(n, n)$ for $\beta = 1, 2, 4$. Define the normalized largest eigenvalue by

$$\lambda'_{\max} = n^{1/6}\big(\lambda_{\max} - 2\sqrt{n}\big).$$

Then, as $n \to \infty$,

$$\lambda'_{\max} \xrightarrow{\mathcal{D}} F_\beta(s),$$

where F_β denotes the Tracy–Widom distribution of index β.

For Hermitian ensembles such as GOE, GUE, and GSE, the properly rescaled largest eigenvalue converges in distribution to the Tracy–Widom law. This distribution admits a representation in terms of the Painleve II transcendent. More precisely, the cumulative distribution function of the largest eigenvalue can be expressed via the Jimbo–Miwa–Okamoto σ-form of Painleve II, which arises from the Fredholm determinant of the Airy kernel. In this project, we numerically solved the Painleve II equation using high-precision ODE solvers and reconstructed the associated σ-form. We then compared the resulting Tracy–Widom distributions with empirical distributions obtained from Monte Carlo simulations of Gaussian Hermitian matrices for $\beta = 1, 2, 4$.

Beyond Gaussian Hermitian ensembles, we also investigated the largest eigenvalue of Laguerre ensembles for $\beta = 1, 2$. Although Laguerre ensembles are supported on $\mathbb{R}_{>0}$ and possess a hard edge at the origin, their upper spectral edge is a soft edge. Consequently, after appropriate centering and soft-edge scaling, the largest eigenvalue fluctuations are again governed by the Tracy–Widom law and hence by Painleve II. Some useful theorems are as following:

Theorem 2.8 ([Joh01, Ede25])
Let $\lambda_{\max}$ denote the largest eigenvalue of $W_1(m,n)$, the real Laguerre ensemble ($\beta = 1$). Define the normalized largest eigenvalue by

$$\lambda'_{\max} = \frac{\lambda_{\max} - \mu_{mn}}{\sigma_{mn}}$$

where

$$\mu_{mn} = \left(\sqrt{m-1} + \sqrt{n}\right)^2 \qquad \sigma_{mn} = \left(\sqrt{m-1} + \sqrt{n}\right)\left(\frac{1}{\sqrt{m-1}} + \frac{1}{n}\right)^{1/3}$$

If $m/n \to \gamma \geq 1$ as $n \to \infty$, then

$$\lambda'_{\max} \xrightarrow{\mathcal{D}} F_1(s)$$

where F_1 denotes the Tracy–Widom distribution of index $\beta = 1$.

Theorem 2.9 ([Joh00, Ede25])
Let $\lambda_{\max}$ denote the largest eigenvalue of $W_2(m,n)$, the complex Laguerre ensemble ($\beta = 2$). Define the normalized largest eigenvalue by

$$\lambda'_{\max} = \frac{\lambda_{\max} - \mu_{mn}}{\sigma_{mn}}$$

where

$$\mu_{mn} = \left(\sqrt{m} + \sqrt{n}\right)^2 \qquad \sigma_{mn} = \left(\sqrt{m} + \sqrt{n}\right)\left(\frac{1}{\sqrt{m}} + \frac{1}{n}\right)^{1/3}$$

If $m/n \to \gamma \geq 1$ as $n \to \infty$, then

$$\lambda'_{\max} \xrightarrow{\mathcal{D}} F_2(s)$$

where F_2 denotes the Tracy–Widom distribution of index $\beta = 2$.

2.3.2 Painleve IV and Soft-Edge Statistics

While Painleve II governs the universal soft-edge limit, finite-size corrections to the largest eigenvalue distribution in the Gaussian Unitary Ensemble can be described by Painleve IV. In particular, the cumulative distribution function of the largest eigenvalue of an $n \times n$ GUE matrix admits an exact representation in terms of the Painleve IV σ-form. We numerically evaluated the Painleve IV σ-form and compared it with finite-n eigenvalue distributions obtained from direct diagonalization of GUE matrices.

2.3.3 Painleve V and Bulk Statistics

In contrast to edge behavior, bulk statistics describe eigenvalue fluctuations in the interior of the spectrum. A central object in this regime is the level spacing distribution, which exhibits universal behavior independent of the underlying matrix ensemble. For unitary ensembles, bulk gap probabilities and spacing distributions can be expressed in terms of the Fredholm determinant of the sine kernel. This determinant satisfies the Jimbo–Miwa–Okamoto σ-form of Painleve V. We numerically solved the Painleve V σ-form and used it to compute bulk gap probabilities. These results were compared with level spacing distributions obtained from large GUE matrices.

2.4 Additional Numerical Experiments and Open Questions

While this project focuses on the numerical verification of Painleve-based representations for classical random matrix ensembles, a number of natural extensions and open questions remain. We briefly outline several directions that are both computationally feasible and mathematically interesting.

Painleve III [For10]

- Arises at the hard edge of Laguerre/Wishart β-ensembles, describing the statistics of the smallest eigenvalue near the origin.
- The hard-edge gap probability, governed by the Bessel kernel, admits a Fredholm determinant representation whose logarithmic derivative satisfies the Jimbo–Miwa–Okamoto σ-form of Painleve III.

More broadly, there remain many open questions at the interface between random matrix theory and Painleve equations. These include the development of robust numerical methods for Painleve equations in their σ-forms, a deeper understanding of finite-size corrections beyond the leading asymptotic limits, and the extension of Painleve-based representations to general β-ensembles with $\beta \neq 1, 2, 4$. Addressing these questions would further clarify the role of Painleve equations as a unifying framework for universal spectral statistics.

3 Numerical Implementation

In this section, we describe the numerical methods used to solve Painleve equations and to compute the associated random matrix statistics. Our primary goal is to numerically evaluate distribution functions arising from Painleve transcendents and to verify their agreement with finite-size random matrix simulations.

3.1 Painleve II

The distribution of the appropriately normalized largest eigenvalues of the Hermite ensembles ($\beta = 1, 2, 4$) and Laguerre ensembles ($\beta = 1, 2$) at the soft edge can be computed from the solution of the Painlevé II equation

$$q''(s) = sq(s) + 2q(s)^3 \tag{1}$$

subject to the Hastings–McLeod boundary condition

$$q(s) \sim \mathrm{Ai}(s) \qquad s \to +\infty \tag{2}$$

The resulting distributions are the well-known Tracy–Widom laws. For $\beta = 2$, the probability density function is given by

$$f_2(s) = \frac{d}{ds} F_2(s) \tag{3}$$

where the cumulative distribution function admits the representation

$$F_2(s) = \exp\left(-\int_s^\infty (x - s)q(x)^2\, dx\right) \tag{4}$$

The distributions $F_1(s)$ and $F_4(s)$ for $\beta = 1$ and $\beta = 4$ are related to $F_2(s)$ through

$$F_1(s)^2 = F_2(s) \exp\left(-\int_s^\infty q(x)\, dx\right) \tag{5}$$

and

$$F_4\left(\frac{s}{2^{2/3}}\right)^2 = F_2(s) \cosh^2\left(\int_s^\infty q(x)\, dx\right) \tag{6}$$

To compute these distributions numerically, we rewrite (1) as a first-order system

$$\frac{d}{ds}\begin{pmatrix} q \\ q' \end{pmatrix} = \begin{pmatrix} q' \\ sq + 2q^3 \end{pmatrix} \tag{7}$$

The boundary condition (2) is imposed at a sufficiently large value $s = s_0 \gg 1$, where the Airy function provides an accurate approximation. We therefore set

$$q(s_0) = \mathrm{Ai}(s_0) \qquad q'(s_0) = \mathrm{Ai}'(s_0) \tag{8}$$

and integrate the system backwards along the s-axis. In practice, this initial value problem can be efficiently solved using standard Runge–Kutta methods. In our numerical experiments, we employ `DifferentialEquations.jl` in Julia.

3.1.1 Hermitian Ensembles

In our numerical experiments, we sample eigenvalues from the tridiagonal representations of the classical Gaussian β-ensembles and compute empirical statistics of the largest eigenvalue. We observe that sampling from finite-size tridiagonal matrices leads to nontrivial centering and scaling constants for the largest eigenvalue, which depend on both the matrix size and the symmetry parameter β. These normalization constants are determined empirically by examining the mean behavior of the rescaled eigenvalue distributions.

As reviewed in class, the Gaussian β-ensemble admits an explicit tridiagonal matrix representation due to Dumitriu and Edelman. Specifically, the tridiagonal matrix

$$H_n^{(\beta)} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & N(0,2) \end{pmatrix} \tag{9}$$

has the same eigenvalue distribution as the full Gaussian β-ensemble, while allowing for efficient numerical sampling.

3.1.2 Laguerre Ensembles

In the Laguerre case, the β-ensemble admits a tridiagonal formulation through an associated bidiagonal matrix representation [Ede25]. Specifically, the Laguerre β-ensemble can be written as

$$L_n^{(\beta)} = B_n^{(\beta)}\left(B_n^{(\beta)}\right)^\top \tag{10}$$

where the bidiagonal matrix $B_n^{(\beta)}$ has independent χ-distributed entries of the form

$$B_n^{(\beta)} \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(n-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(n-1)} \end{pmatrix}. \tag{11}$$

Here $n \in \mathbb{N}$ and $a \in \mathbb{R}$ satisfies the constraint

$$a > \frac{\beta}{2}(n-1) \tag{12}$$

which ensures the positivity of all χ-distribution parameters.

3.2 Painleve IV

In contrast to the soft-edge scaling limit governed by Painleve II, the distribution of the largest eigenvalue of a finite-dimensional Gaussian Unitary Ensemble (GUE) can be characterized exactly in terms of Painleve IV transcendents. Let $\lambda_{\max}$ denote the largest eigenvalue of an $n \times n$ GUE matrix. The cumulative distribution function

$$F_n(s) = \mathbb{P}\big(\lambda_{\max} \le s\big) \tag{13}$$

admits a representation in terms of a Painleve IV σ-function. Specifically, $F_n(s)$ can be written as

$$F_n(s) = \exp\bigg(-\int_s^{\infty} \sigma_n(x)\,dx\bigg) \tag{14}$$

where $\sigma_n(s)$ satisfies the Jimbo–Miwa–Okamoto σ-form of the Painleve IV equation,

$$\big(\sigma_n''\big)^2 - 4\big(s\sigma_n' - \sigma_n\big)^2 + 4\sigma_n'\big(\sigma_n' + 2n\big)\big(\sigma_n' - 2n\big) = 0 \tag{15}$$

For numerical purposes, we solve the Painleve IV equation in its σ-form as an initial value problem. Appropriate initial conditions are imposed at a sufficiently large value of s, where the asymptotic behavior of $\sigma_n(s)$ is known. The resulting numerical solution is then integrated to recover the cumulative distribution function $F_n(s)$. We compare the Painleve IV prediction with empirical distributions obtained from finite-size GUE simulations using tridiagonal matrix sampling.

3.3 Painleve V

Another fundamental spectral statistic in random matrix theory is the distribution of eigenvalue spacings in the bulk of the spectrum. For the Gaussian Unitary Ensemble (GUE), this quantity describes the local correlations between neighboring eigenvalues away from the spectral edges and is universal in the large-n limit.

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ denote the ordered eigenvalues of an $n \times n$ GUE matrix. Following standard normalization, the bulk-scaled eigenvalue spacings are defined by

$$\delta_k' = \frac{\lambda_{k+1} - \lambda_k}{\pi\beta}\sqrt{2\beta n - \lambda_k^2} \qquad k \approx \frac{n}{2} \tag{16}$$

where only eigenvalues near the center of the spectrum are retained, and a fixed fraction of eigenvalues near each spectral edge is discarded to eliminate boundary effects.

For $\beta = 2$, the probability density function $p(s)$ of the normalized level spacings can be computed through the solution of a Painlevé V nonlinear differential equation in σ-form. Specifically, the function $\sigma(t)$ satisfies

$$\big(t\sigma''\big)^2 + 4\big(t\sigma' - \sigma\big)\big(t\sigma' - \sigma + (\sigma')^2\big) = 0 \tag{17}$$

with the boundary condition

$$\sigma(t) \sim -\frac{t}{\pi} - \left(\frac{t}{\pi}\right)^2 \qquad t \to 0^+ \tag{18}$$

The spacing distribution is obtained from the gap probability

$$E(s) = \exp\left(\int_0^{\pi s} \frac{\sigma(t)}{t}\, dt\right) \tag{19}$$

via

$$p(s) = \frac{d^2}{ds^2} E(s) \tag{20}$$

An explicit expression for the density is given by

$$p(s) = \frac{1}{s^2}\Big(\pi s \sigma'(\pi s) - \sigma(\pi s) + \sigma(\pi s)^2\Big) E(s) \tag{21}$$

For numerical implementation, the second-order Painleve V equation is rewritten as a first-order system and solved as an initial value problem. The integration is initiated at a small positive value $t = t_0$ to avoid the singularity at $t = 0$, with initial conditions determined from the asymptotic boundary behavior,

$$\sigma(t_0) = -\frac{t_0}{\pi} - \left(\frac{t_0}{\pi}\right)^2 \qquad \sigma'(t_0) = -\frac{1}{\pi} - \frac{2t_0}{\pi^2} \tag{22}$$

The resulting numerical solution yields the bulk level spacing distribution, which is compared with empirical histograms obtained from finite-size GUE simulations.

4 Numerical Results

4.1 Painleve II Transcendents and the Largest Eigenvalue Distribution

We pick num_trials = 10000 and $n = 10000$ (dimensions of random matrix) of Hermitian Ensembles. For Laguerre Ensembles, the computational cost is higher. We pick num_trials = 3000 and $n = 500$ instead to get Figure 5 and 6. And the numerical results are as following:

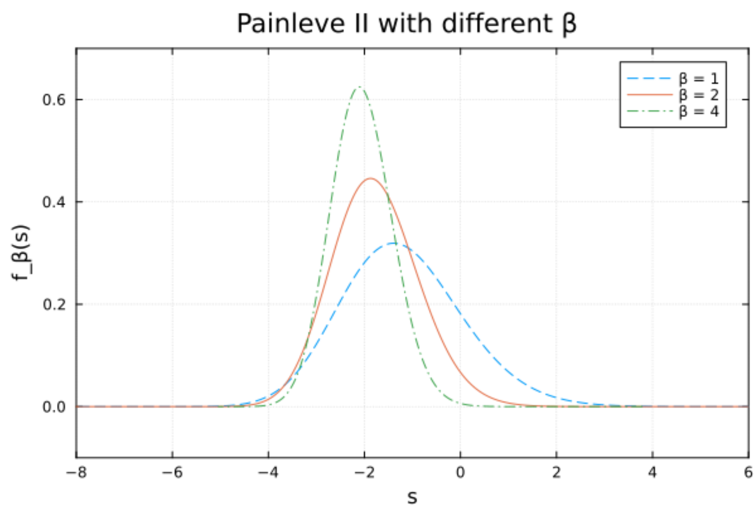

Figure 1: Probability density functions of the Tracy–Widom distributions $F'_\beta(s)$ for $\beta = 1, 2, 4$, computed from numerical solutions of the Painlevé II equation in Jimbo–Miwa–Okamoto σ-form.

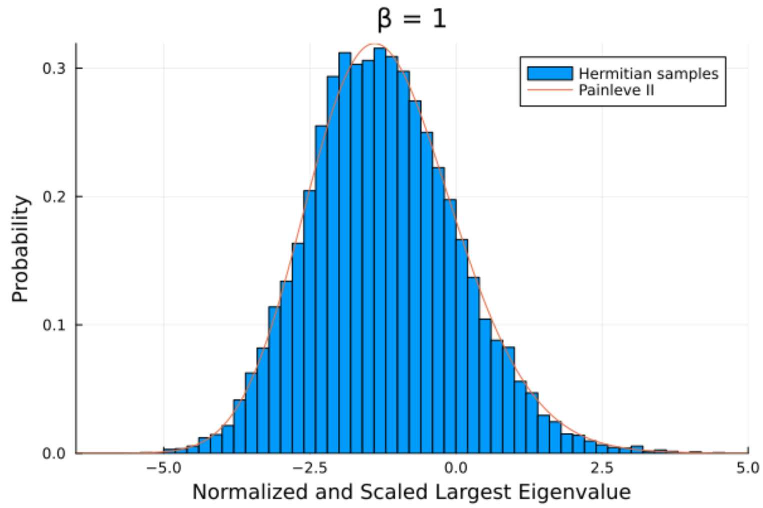

Figure 2: The empirical distribution of the largest eigenvalue of finite-size Gaussian Hermitian matrices ($\beta = 1$) and the Tracy–Widom law. The histogram is obtained from Monte Carlo simulations after soft-edge centering and scaling, while the solid curve is computed from the Painleve II representation of the Tracy–Widom distribution.

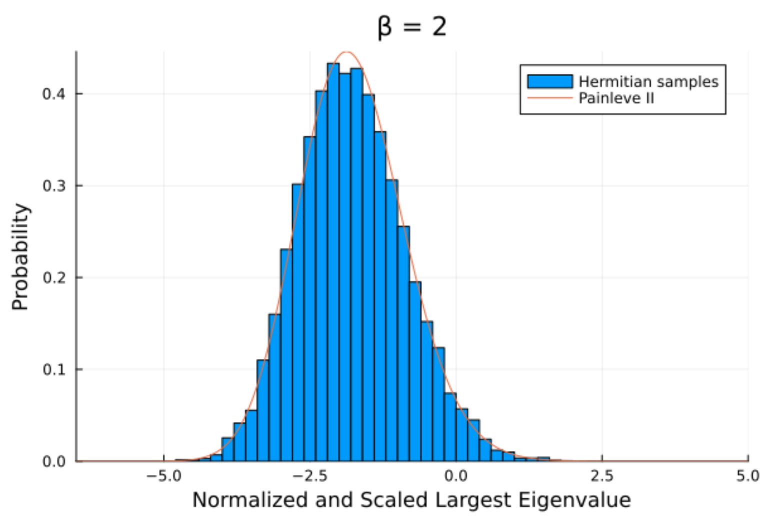

Figure 3: The empirical distribution of the largest eigenvalue of finite-size Gaussian Hermitian matrices ($\beta = 2$) and the Tracy–Widom law. The histogram is obtained from Monte Carlo simulations after soft-edge centering and scaling, while the solid curve is computed from the Painleve II representation of the Tracy–Widom distribution.

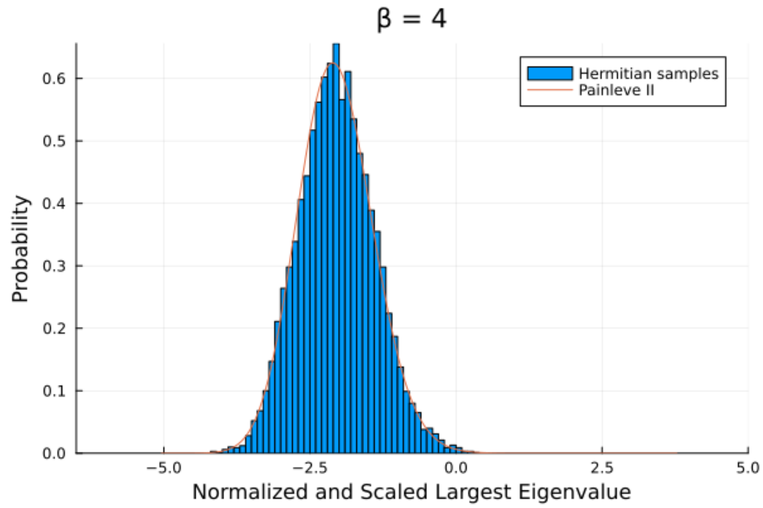

Figure 4: The empirical distribution of the largest eigenvalue of finite-size Gaussian Hermitian matrices ($\beta = 4$) and the Tracy–Widom law. The histogram is obtained from Monte Carlo simulations after soft-edge centering and scaling, while the solid curve is computed from the Painleve II representation of the Tracy–Widom distribution.

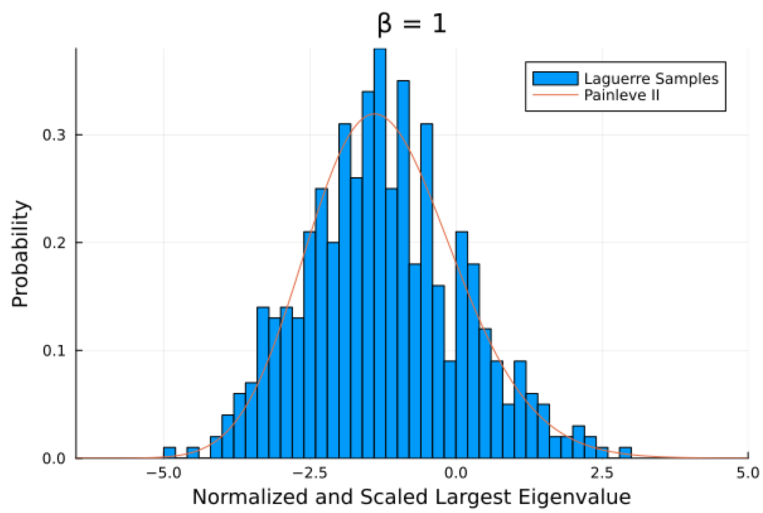

Figure 5: The empirical distribution of the largest eigenvalue of finite-size Laguerre matrices ($\beta = 1$) and the Tracy–Widom law. The histogram is obtained from Monte Carlo simulations after soft-edge centering and scaling, while the solid curve is computed from the Painleve II representation of the Tracy–Widom distribution.

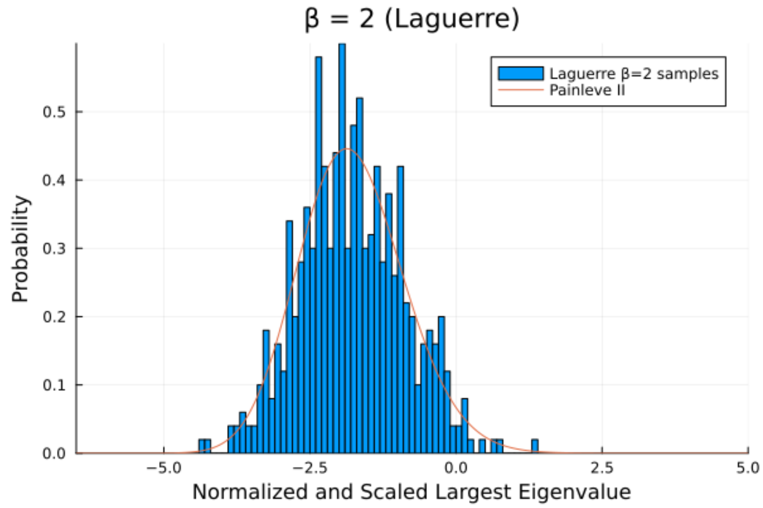

Figure 6: The empirical distribution of the largest eigenvalue of finite-size Laguerre matrices ($\beta = 2$) and the Tracy–Widom law. The histogram is obtained from Monte Carlo simulations after soft-edge centering and scaling, while the solid curve is computed from the Painleve II representation of the Tracy–Widom distribution.

4.2 Painleve IV Transcendents and Finite-n Largest Eigenvalue Distributions

We pick trials = 10000 in this case to get the following result:

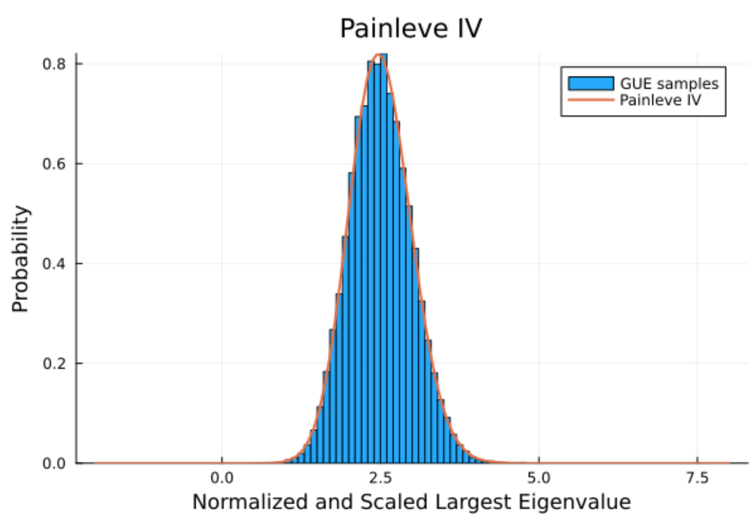

Figure 7: Finite-n distribution of the largest eigenvalue of the Gaussian Unitary Ensemble ($n = 6$) and Painleve-IV

4.3 Painleve V Transcendents and Bulk Level Spacing Distributions

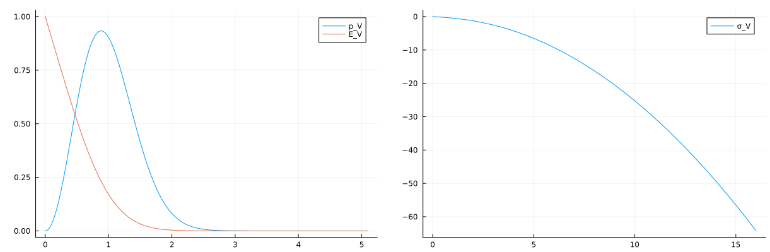

Figure 8: Painleve V (left), E(s) and p(s) (right)

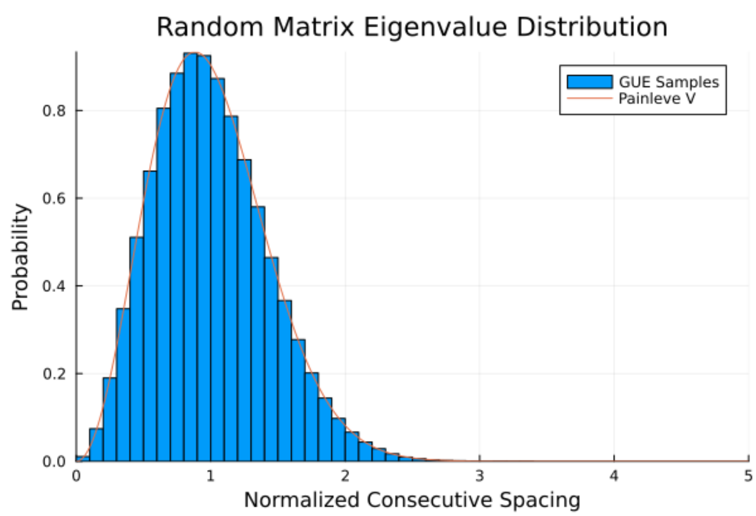

Figure 9: Probability distribution of consecutive spacings of random matrix eigenvalues

5 Conclusion

In this project, we conducted a systematic numerical investigation of the connections between Painleve equations and eigenvalue statistics of classical random matrix ensembles. Focusing on the Jimbo–Miwa–Okamoto σ-form formulation, we demonstrated how nonlinear Painleve transcendents arise naturally in the description of gap probabilities, extreme eigenvalue distributions, and bulk level spacing statistics.

At the soft edge of the spectrum, we numerically verified the Painleve II characterization of the Tracy–Widom distributions for Gaussian Hermitian ensembles with symmetry classes $\beta = 1, 2, 4$. By solving the Painleve II equation under the Hastings–McLeod boundary condition and reconstructing the associated σ-form, we obtained probability distributions that show excellent agreement with Monte Carlo simulations of finite-size Gaussian Hermitian and Laguerre ensembles after appropriate centering and scaling. These results confirm the universality of the Tracy–Widom law and illustrate the numerical stability of Painleve II-based methods in the soft-edge regime.

Beyond the asymptotic limit, we explored finite-size corrections to the largest eigenvalue distribution of the Gaussian Unitary Ensemble. In this setting, the cumulative distribution function admits an exact representation in terms of the Painleve IV σ-form. Our numerical experiments demonstrate that, after careful normalization consistent with the underlying Hermite weight, the finite-n Painleve IV prediction matches empirical distributions obtained from direct diagonalization of GUE matrices. This provides a concrete numerical validation of the role of Painleve IV in describing finite-dimensional random matrix effects beyond universal scaling limits.

In the bulk of the spectrum, we studied level spacing statistics governed by the sine kernel. By numerically solving the Painleve V σ-form and reconstructing the associated gap probability and spacing density, we obtained bulk spacing distributions that agree closely with histograms generated from large GUE simulations. This completes a unified numerical pipeline connecting Painleve V transcendents, Fredholm determinants, and universal bulk statistics.

Overall, our results highlight both the power and the limitations of Painleve- based numerical approaches. While the σ-form formulation provides a conceptually elegant and unifying framework, numerical implementations are sensitive to boundary conditions, normalization conventions, and finite-size effects. Nevertheless, when these issues are handled carefully, Painlevé equations offer an effective computational tool for accessing eigenvalue statistics that are otherwise difficult to compute directly.

Several natural directions for future work remain. These include extending the numerical framework to Painleve III at the hard edge of Laguerre ensembles, investigating higher-order finite-size corrections, and developing more robust numerical solvers for Painleve equations in their σ-forms. More broadly, the interplay between integrable systems and random matrix theory continues to provide a rich setting in which analytical structure and numerical computation can fruitfully inform one another.

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