AN ANALYSIS OF BESSEL GENERATING FUNCTIONS

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1. Introduction

First, we introduce the Dunkl operators from [Dun89] and afterwards we introduce the Bessel functions from [Opd93]. For $\alpha \in \mathbb{R}^N \backslash \{0\}$, r_α is the reflection defined by $r_\alpha : x \mapsto x - 2\langle x, \alpha\rangle \|\alpha\|_2^{-2} \alpha$. Suppose $N \geq 2$ and $\mathcal{R} \subset \mathbb{R}^N$ is a finite crystallographic root system. Recall that $\mathcal{R}$ is defined by the following properties:

(1) The elements of $\mathcal{R}$ span $\mathbb{R}^N$.
(2) If $\alpha \in \mathcal{R}$, then $-\alpha \in \mathcal{R}$. Furthermore, if $\alpha, c\alpha \in \mathcal{R}$ for some $c \in \mathbb{R}$, then $c \in \{-1, 1\}$.
(3) For all $\alpha \in \mathcal{R}$, $\mathcal{R}$ is fixed under the action of r_α.
(4) If $\alpha, \beta \in \mathcal{R}$, then $2\langle \alpha, \beta\rangle \in \|\alpha\|_2^2 \mathbb{Z}$.

Let $H(\mathcal{R})$ denote the finite reflection group generated by r_α for $\alpha \in \mathcal{R}$; $H(\mathcal{R})$ is known as a Coxeter group. We let $\mathbb{C}^{H(\mathcal{R})}[[x_1, \ldots, x_N]]$ denote the set of $f \in \mathbb{C}[[x_1, \ldots, x_N]]$ which are invariant under the action of $H(\mathcal{R})$.

Let $\mathcal{R}^+$ denote a set of positive roots of $\mathcal{R}$. There always exists a hyperplane that does not contain any roots such that $\mathcal{R}^+$ consists of the roots lying on a fixed side of the hyperplane. However, $\mathcal{R}^+$ is not unique; for example we may replace $\mathcal{R}^+$ with its complement. Let $\theta(\mathcal{R})$ denote the set of multiplicity functions $\theta : \mathcal{R} \to \mathbb{C}$ such that $\theta(\alpha_1) = \theta(\alpha_2)$ if r_{α_1} and r_{α_2} are conjugates in $H(\mathcal{R})$.

For $\theta \in \theta(\mathcal{R})$, we define the Dunkl operator, which operates over $\mathbb{C}[[x_1, \ldots, x_N]]$, by

$$\mathcal{D}_i(\mathcal{R}(\theta)) \triangleq \partial_i + \sum_{r \in \mathcal{R}^+} \theta(r) \frac{1 - r_\alpha}{\langle x, \alpha\rangle} \alpha_i$$

for $i \in [N]$. Note that the Dunkl operators do not depend on the choice of $\mathcal{R}^+$, see [dJ93]. Furthermore, the Dunkl operators are commutative; in other words, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$. Another useful property of the Dunkl operator is equivariance with respect to the action of $H(\mathcal{R})$.

Lemma 1.1 ([Dun89])**.** *For $h \in H(\mathcal{R})$ and $p, q \in \mathbb{C}^N[x_1, \ldots, x_N]$, $hp(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))q(x) = (hp)(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))(hq)(x)$.*

Before introducing the Bessel function, we discuss the notion of invertibility, which has been studied previously in [Opd93, DdJO94, Yao25a].

Definition 1.2. The multiplicity function $\theta \in \theta(\mathcal{R})$ is *invertible* if there does not exist a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ such that $g(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))f = 0$ for all homogeneous polynomials $g \in \mathbb{C}[x_1, \ldots, x_N]$ of the same degree as f. The set $\Theta(\mathcal{R})$ denotes the set of invertible elements of $\theta(\mathcal{R})$.

The paper [DdJO94] computes the set $\Theta(\mathcal{R})$ exactly. Furthermore, we can state many equivalent definitions of invertibility for a general set of degree lowering operators, see [Yao25a, Theorem 4.5]. Next, we define the eigenfunctions of the Dunkl operators.

Definition 1.3 ([Opd93])**.** Suppose $\theta \in \Theta(\mathcal{R})$. The holomorphic function $E_a^{\mathcal{R}(\theta)}(x)$ over $(a, x) \in \mathbb{C}^N \times \mathbb{C}^N$ is the unique eigenfunction that satisfies

$$f(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))E_a^{\mathcal{R}(\theta)}(x) = f(a)E_a^{\mathcal{R}(\theta)}(x)$$

for all $f \in \mathbb{C}[x_1, \ldots, x_N]$ and $E_a^{\mathcal{R}(\theta)}(0) = 1$.

The holomorphic function $J_a^{\mathcal{R}(\theta)}(x)$ over $(a, x) \in \mathbb{C}^N \times \mathbb{C}^N$ is the unique eigenfunction that satisfies $J_a^{\mathcal{R}(\theta)} \in \mathbb{C}^{H(\mathcal{R}(\theta))}[[x_1, \ldots, x_N]]$,

$$f(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))J_a^{\mathcal{R}(\theta)}(x) = f(a)J_a^{\mathcal{R}(\theta)}(x)$$

for all $f \in \mathbb{C}^{H(\mathcal{R}(\theta))}[x_1, \ldots, X_N]$, and $J_a^{\mathcal{R}(\theta)}(0) = 1$. The function $J_a^{\mathcal{R}(\theta)}$ is referred to as the *Bessel function* and satisfies

$$J_a^{\mathcal{R}(\theta)}(x) = \frac{1}{|H(\mathcal{R}(\theta))|} \sum_{h \in H(\mathcal{R}(\theta))} E_a^{\mathcal{R}(\theta)}(hx).$$

We list some well known properties of the eigenfunction $E_a^{\mathcal{R}(\theta)}$.

Lemma 1.4 ([dJ93])**.** *Suppose $\theta \in \Theta(\mathcal{R})$. The following are true:*

(a) *For $a, x \in \mathbb{C}^N$ and $c \in \mathbb{C}$, $E_a^{\mathcal{R}(\theta)}(cx) = E_{ca}^{\mathcal{R}(\theta)}(x)$*
(b) *For $a, x \in \mathbb{C}^N$, $E_a^{\mathcal{R}(\theta)}(x) = E_x^{\mathcal{R}(\theta)}(a)$.*
(c) *If $Re(\theta(r)) \geq 0$ for all $r \in \mathcal{R}$, then for all $a, x \in \mathbb{C}^N$, $|E_a^{\mathcal{R}(\theta)}(x)| \leq \sqrt{|H(\mathcal{R})|}$ $\exp(\max_{h \in H(\mathcal{R})}(Re(\langle ha, x\rangle)))$.*

The following result gives an exponential expression for the nonsymmetric eigenfunctions and therefore the Bessel function.

Theorem 1.5 ([Rös99])**.** *Assume that $\theta \in \theta(\mathcal{R})$ is nonnegative. Suppose $x \in \mathbb{R}^N$. There exists a Borel probability measure $\mu_x^{\mathcal{R}(\theta)}$ that is supported over the convex hull of $H(\mathcal{R})x$ such that*

$$E_a^{\mathcal{R}(\theta)}(x) = \int_{\mathbb{R}^N} e^{\langle a, y\rangle} d\mu_x^{\mathcal{R}(\theta)}(y)$$

for all $a \in \mathbb{C}^N$.

We analyze *Bessel generating functions* in this paper, which have been previously studied in [GS22, BGCG22, GXZ24, Yao25b, Xu25, Yao25a]. To obtain the Bessel generating function, we set a to be a random variable and compute the average of $J_a^{\mathcal{R}(\theta)}$. In particular, for a Borel probability measure μ over $\mathbb{C}^N$, the Bessel generating function $G_\mu^{\mathcal{R}(\theta)}(x)$ is defined as

$$G_\mu^{\mathcal{R}(\theta)}(x) \triangleq \mathbb{E}_{a \sim \mu}[J_a^\theta(x)].$$

However, it is not always true that the integral defining $G_\mu^\theta(x)$ converges for all $x \in \mathbb{C}^N$. We define an *exponentially decaying probability measure* to analyze the convergence over a neighborhood of the origin, see Lemma 1.7.

Definition 1.6 ([Yao25b, Definition 1.2])**.** The Borel probability measure μ over $\mathbb{C}^N$ is *exponentially decaying* at rate $R > 0$ if $\mathbb{E}_{a\sim\mu}[\exp(R\,\|a\|_2)] < \infty$.

Lemma 1.7 ((A) is from [Yao25b] and (B) is from [BGCG22])**.** *Suppose the Borel probability measure μ over $\mathbb{C}^N$ is exponentially decaying at rate $R > 0$. Furthermore, assume that $\theta \in \theta(\mathcal{R})$ has nonnegative real part.*

(A) *The Bessel generating function $G_\mu^{\mathcal{R}(\theta)}(x)$ converges over the closed ball of radius R centered at the origin and is holomorphic over the interior of this domain.*
(B) *For all $f \in \mathbb{C}^{H(\mathcal{R})}[x_1, \ldots, x_N]$,*

$$f(\mathcal{D}_1(\mathcal{R}(\theta)), \ldots, \mathcal{D}_N(\mathcal{R}(\theta)))G_\mu^{\mathcal{R}(\theta)}(0) = \mathbb{E}_{a\sim\mu}[f(a)].$$

A conjecture that is related to the Bessel generating function is that if $\theta \in \theta(\mathcal{R})$ is nonnegative, then for $a, b \in \mathbb{R}^N$, there exists a Borel probability measure μ over $\mathbb{R}^N$ such that $J_a^{\mathcal{R}(\theta)}(x)J_b^{\mathcal{R}(\theta)}(x)$ equals $G_\mu^{\mathcal{R}(\theta)}(x)$ for some Borel probability measure μ over $\mathbb{R}^N$, see [Yao25a, Conjecture 1.6]. In [Tri02], there exists a signed measure μ supported over $B(0, \|a\|_2 + \|b\|_2)$ that satisfies this condition. Furthermore, [Rös03a] show that the conjecture is true if we replace $J_a^{\mathcal{R}(\theta)}$ with its spherically symmetric version. However, [TX05] shows that the conjecture is not true if we consider the nonsymmetric eigenfunctions rather than the Bessel functions.

The conjecture is also related to the β-ghosts concept introduced in [Ede09], where $\beta = 2\theta$. The reasoning for this is that if the conjecture is true, then we can define a probability measure based on its Bessel generating function $J_a^{\mathcal{R}(\theta)}(x)J_b^{\mathcal{R}(\theta)}(x)$ although we are unable to sample from the measure. Similarly, Theorem 1.5 defines the probability measure $\mu_x^{\mathcal{R}(\theta)}$ for $x \in \mathbb{R}^N$ although we are unable to sample from it. Note that although we are unable to sample from the measure μ given that $G_\mu^{\mathcal{R}(\theta)} = J_{a(N)}^{\mathcal{R}(\theta)}J_{b(N)}^{\mathcal{R}(\theta)}$ for finite values of N, we can sample from it asymptotically if we assume that $a(N)$ and $b(N)$ converge in terms of moments, see [Yao25a, Corollaries 1.7 and 1.8]. In addition, [Ede09, Section 4] discusses how the analogous conjecture for the Jack polynomials is related to β-ghosts. This conjecture implies the Bessel function conjecture, see [GM18].

However, we do not consider all choices for $\mathcal{R}$. We only consider when $\mathcal{R}$ is one of the irreducible root systems A^{N-1}, B^N, C^N, or D^N for $N \geq 2$. We define the Dunkl operators associated with these root systems.

For $\theta \in \mathbb{C}$, the Dunkl operator associated to $A^{N-1}(\theta)$ is

$$\mathcal{D}_i(A^{N-1}(\theta)) \triangleq \partial_i + \theta \sum_{j\neq i} \frac{1 - s_{ij}}{x_i - x_j}$$

for $i \in [N]$, where s_{ij} switches e_i and e_j for distinct $i, j \in [N]$. The group $H(A^{N-1})$ consists of the permutation reflections over $\mathbb{R}^N$.

For $\theta_0, \theta_1 \in \mathbb{C}$, the Dunkl operator associated to $BC^N(\theta_0, \theta_1)$ is

$$\mathcal{D}_i(BC^N(\theta_0, \theta_1)) \triangleq \partial_i + \theta_1 \frac{1 - \tau_i}{x_i} + \theta_0 \sum_{j\neq i} \frac{1 - s_{ij}}{x_i - x_j} + \frac{1 - \tau_i\tau_j s_{ij}}{x_i + x_j}$$

for $i \in [N]$, where τ_i switches e_i and $-e_i$ for $i \in [N]$. The group $H(BC^N)$ is generated by the permutation and sign flip reflections over $\mathbb{R}^N$.

For $\theta \in \mathbb{C}$, the Dunkl operator associated to $D^N(\theta)$ is

$$\mathcal{D}_i(D^N(\theta)) \triangleq \partial_i + \theta \sum_{j\neq i} \frac{1-s_{ij}}{x_i - x_j} + \frac{1-\tau_i\tau_j s_{ij}}{x_i + x_j}$$

for $i \in [N]$. The group $H(D^N)$ is generated by the permutation reflections and the reflections composed of an even number of sign flips over $\mathbb{R}^N$.

In the next section, we discuss how we may compute Bessel generating functions over the root systems A^{N-1}, BC^N, and D^N. In particular, we provide straightforward computations of these functions for the β-Hermite ensemble and the Dyson Brownian motion. Afterwards, we discuss a generalization of the results of [Yao25b], see Theorem 3.3, that allows for deducing a law of large numbers for multiple observations of the DBM.

Notation for partitions. Let Γ denote the set of nonzero partitions and let Γ_{even} denote the set of nonzero partitions with all even parts. Suppose $\nu \in \Gamma$ and $\nu = (\nu_1 \geq \cdots \geq \nu_m)$. Then, $|\nu|$ equals $\sum_{i=1}^m \nu_i$ and $\ell(\nu)$ equals m. Furthermore, $P(\nu)$ denotes the number of distinct permutations of $(\nu_1, \ldots, \nu_{\ell(\nu)})$ and for $\nu \in \Gamma$, $p_\nu(x_1, \ldots, x_N) \triangleq \prod_{i=1}^{\ell(\nu)} (\sum_{j=1}^N x_j^{\nu_i})$.

For $k \geq 1$, $NC(k)$ is the set of noncrossing partitions of $[k]$ and $NC^{\text{even}}(k)$ is the set of elements of $NC(k)$ that have all even block sizes.

2. Computations of Bessel generating functions

Next, we discuss a straightforward method to compute Bessel generating functions. For this direction, we first discuss the *Dunkl bilinear form* introduced in [Dun91]. Suppose $p, q \in \mathbb{C}[x_1, \ldots, x_N]$. Then, $[p,q]_{\mathcal{R}(\theta)} \triangleq [1]p(\mathcal{D}(\mathcal{R}(\theta)))q$. The bilinear form is symmetric, so $[1]p(\mathcal{D}(\mathcal{R}(\theta)))q = [1]q(\mathcal{D}(\mathcal{R}(\theta)))p$. If p and q are both homogeneous but have different degrees, then it is clear that $[p,q]_{\mathcal{R}(\theta)} = [q,p]_{\mathcal{R}(\theta)} = 0$.

Define the *Dunkl Laplacian* as

$$\Delta_{\mathcal{R}(\theta)} \triangleq \sum_{i=1}^N \mathcal{D}_i(\mathcal{R}(\theta))^2.$$

We present an integral formula for evaluating the Dunkl bilinear form when the multiplicity function is nonnegative.

Theorem 2.1 ([Dun91, Theorem 3.10])**.** *Suppose $\theta \in \theta(\mathcal{R})$ is nonnegative. For $p, q \in \mathbb{C}[x_1, \ldots, x_N]$,*

$$[p,q]_{\mathcal{R}(\theta)} = c_N^{-1} \int_{\mathbb{R}^N} (e^{-\Delta_{\mathcal{R}(\theta)}/2}p)(e^{-\Delta_{\mathcal{R}(\theta)}/2}q) \prod_{r\in\mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx,$$

where $c_N \triangleq \int_{\mathbb{R}^N} \prod_{r\in\mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx$.

Lemma 2.2. *Suppose $p \in \mathbb{C}[x_1, \ldots, x_N]$. Then, for $a \in \mathbb{C}^N$,*

$$p(a) = e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} (e^{-\Delta_{\mathcal{R}(\theta)}/2}p) E_a^{\mathcal{R}(\theta)}(x) \prod_{r\in\mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx.$$

Proof. We follow the proof of [Rös03b, Proposition 2.37] and in particular we comment on the application of the dominated convergence theorem. Suppose

$$E_a^{\mathcal{R}(\theta)}(x) = \sum_{j=0}^{\infty} E_a^j(x),$$

where E_a^j is homogeneous of degree j for $j \geq 0$. Using Theorem 1.5, we have that if $x \in \mathbb{R}^N$, then

$$E_a^j(x) = \int_{\mathbb{R}^N} \frac{\langle a, y\rangle^j}{j!} d\mu_x^{\mathcal{R}(\theta)}(y)$$

for $a \in \mathbb{C}^N$, where $\mu_x^{\mathcal{R}(\theta)}$ is supported over the convex hull of $H(\mathcal{R})x$. Hence, Jensen's inequality implies that

$$\left\|E_a^j(x)\right\|_2 \leq \int_{\mathbb{R}^N} \frac{\|a\|_2^j \|y\|_2^j}{j!} d\mu_x^{\mathcal{R}(\theta)}(y) \leq \frac{\|a\|_2^j \|x\|_2^j}{j!} \Rightarrow \left\|E_a^{\mathcal{R}(\theta)}(x)\right\|_2 \leq e^{\|a\|_2\|x\|_2}. \tag{1}$$

Furthermore,

$$e^{-\Delta_{\mathcal{R}(\theta)}/2} \sum_{j=0}^{n} E_a^j(x) = \sum_{j=0}^{n} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-\frac{\|a\|_2^2}{2})^k}{k!} E_a^{j-2k}(x) = \sum_{j=0}^{n} \sum_{k=0}^{\lfloor \frac{n-j}{2} \rfloor} \frac{(-\frac{\|a\|_2^2}{2})^k}{k!} E_a^j(x).$$

It follows that

$$\left\| e^{-\Delta_{\mathcal{R}(\theta)}/2} \sum_{j=0}^{n} E_a^j(x) \right\|_2 \leq e^{\frac{\|a\|_2^2}{2}} e^{\|a\|_2\|x\|_2}. \tag{2}$$

By Theorem 2.1,

$$\begin{aligned} p(a) &= \left[p, E_a^{\mathcal{R}(\theta)}(x)\right]_{\mathcal{R}(\theta)} = \lim_{n\to\infty} \left[p, \sum_{j=0}^{n} E_a^j(x)\right]_{\mathcal{R}(\theta)} \\ &= \lim_{n\to\infty} c_N^{-1} \int_{\mathbb{R}^N} \left(e^{-\Delta_{\mathcal{R}(\theta)}/2} p\right) \left(e^{-\Delta_{\mathcal{R}(\theta)}/2} \sum_{j=0}^{n} E_a^j(x)\right) \prod_{r\in\mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx. \end{aligned}$$

Since

$$e^{-\frac{a_1^2+\cdots+a_N^2}{2}} E_a^{\mathcal{R}(\theta)}(x) = e^{-\Delta_{\mathcal{R}(\theta)}/2} E_a^{\mathcal{R}(\theta)}(x) = \lim_{n\to\infty} e^{-\Delta_{\mathcal{R}(\theta)}/2} \sum_{j=0}^{n} E_a^j(x),$$

for all $x \in \mathbb{C}^N$, it suffices to show that the integrand is dominated by an absolutely integrable function so that we can apply the dominated convergence theorem. However, this is clear by (2). □

Corollary 2.3. *Assume that $p \in \mathbb{R}[x_1, \ldots, x_N]$ is nonnegative over $\mathbb{R}^N$. Then, $e^{\Delta_{\mathcal{R}(\theta)}/2} p \in \mathbb{R}[x_1, \ldots, x_N]$ is nonnegative over $\mathbb{R}^N$.*

Proof. We can set p to be $e^{\Delta_{\mathcal{R}(\theta)}/2} p$ in Lemma 2.2. □

In fact, we may generalize this result as follows.

Lemma 2.4. *Assume that $p \in \mathbb{R}[x_1, \ldots, x_N]$ is nonnegative over $\mathbb{R}^N$ and $t \geq 0$. Then, $e^{t\Delta_{\mathcal{R}(\theta)}} p \in \mathbb{R}[x_1, \ldots, x_N]$ is nonnegative over $\mathbb{R}^N$.*

Proof. This follows from [Rös98, Lemma 4.1] and [Rös99, Theorem 2.3]. $\square$

Using Lemma 2.2, we can compute the well-known type A Bessel generating function for the β-Hermite ensemble, see [Cue21, BGCG22] where it has been computed previously. This approach is used in [Xu25] to compute the type BC Bessel generating function for the eigenvalues of the Chiral ensemble, see Lemma 2.6. For $\theta \geq 0$ and $t > 0$, define the probability distribution $H_{\theta,t}^N$ over $\mathbb{R}^N$ to have the density of a proportional to

$$\prod_{1 \leq i < j \leq N} |a_i - a_j|^{2\theta} \prod_{i=1}^{N} e^{-\frac{a_i^2}{2t}}.$$

The paper [DE02] computes a tridiagonal matrix whose eigenvalues are given by $H_{\theta,t}^N$.

Lemma 2.5. *Suppose $\theta \geq 0$ and $t > 0$. Then,*

$$G_{H_{\theta,t}^N}^{A^{N-1}(\theta)}(x) = \exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right).$$

Proof. First, set $t = 1$. Then, by Lemma 2.2 with $p = 1$ and using the formula for $J_a^{A^{N-1}(\theta)}$ given in Definition 1.3, we have that

$$G_{H_{\theta,1}^N}^{A^{N-1}(\theta)}(x) = \exp\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2\right).$$

Suppose $t > 0$. Note that $H_{\theta,t}^N$ is the push-forward of $H_{\theta,1}^N$ with respect to the function $x \mapsto \sqrt{t}x$. Thus,

$$\begin{aligned} G_{H_{\theta,t}^N}^{A^{N-1}(\theta)}(x) &= \mathbb{E}_{a \sim H_{\theta,t}^N}[J_a^{A^{N-1}(\theta)}(x)] = \mathbb{E}_{a \sim H_{\theta,1}^N}[J_{\sqrt{t}a}^{A^{N-1}(\theta)}(x)] \\ &= \mathbb{E}_{a \sim H_{\theta,1}^N}[J_a^{A^{N-1}(\theta)}(\sqrt{t}x)] = \exp\left(\frac{t}{2}\sum_{i=1}^{N} x_i^2\right). \end{aligned}$$

$\square$

Another example of an application of this method is the type BC Bessel generating function computed for the Chiral ensemble, which has the same distribution as the square roots of the β-Laguerre ensemble. We discuss its construction for when $\theta \in \{\frac{1}{2}, 1, 2\}$ and its definition for general values of $\theta \geq 0$ from [For10, Section 3.1]. Suppose $M \geq N$ such that $M - N + 1 - \frac{1}{2\theta} \geq 0$. Let X denote a random $M \times N$ matrix whose independent entries are real, complex, or real quaternion numbers with densities:

- $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ if $\theta = \frac{1}{2}$,
- $\frac{1}{\pi} e^{-|x|^2}$ if $\theta = 1$,
- $\frac{4}{\pi^2} e^{-2|x|^2 - 2|y|^2}$ if $\theta = 2$. Note that we parameterize a quaternion as $(x, y) \in \mathbb{C}^2$.

The Chiral ensemble is defined as the positive eigenvalues of

$$H = \begin{bmatrix} 0_{M\times M} & X \\ X^H & 0_{N\times N} \end{bmatrix}.$$

This matrix almost surely has N negative eigenvalues, $M-N$ zero eigenvalues, and N positive eigenvalues.

Define the probability distribution $C_{\theta,M,t}^N$ over $\mathbb{R}_{\geq 0}^N$ to have the density of a proportional to

$$\prod_{i=1}^N a_i^{2\theta(M-N+1)-1} e^{-\frac{a_i^2}{2t}} \prod_{1\leq i<j\leq N} |a_i^2 - a_j^2|^{2\theta}.$$

Then, $C_{\theta,M,t}^N$ is the density function of the positive eigenvalues of H after they are rescaled when $\theta \in \{\frac{1}{2}, 1, 2\}$. The following lemma is a generalization of the result [Xu25, Proposition 5.21] and we can prove the lemma using the same method that this result is proved with, which is also the method we use to prove Lemma 2.5.

Lemma 2.6. *Suppose $\theta \geq 0$, $M \geq N$ such that $M-N+1-\frac{1}{2\theta} \geq 0$, and $t>0$. Then,*

$$G_{C_{\theta,M,t}^N}^{BC^N(\theta,\theta(M-N+1)-\frac{1}{2})}(x) = \exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right).$$

Remark 2.7. Note the probability distribution $C_{\theta,M,t}^N$ is defined over $\mathbb{R}_{\geq 0}^N$ but it is straightforward to extend the probability distribution symmetrically to the probability distribution $\tilde{C}_{\theta,M,t}^N$ over $\mathbb{R}^N$ so that we can apply Lemma 2.2.

Using the results of [Yao25a], we can obtain a law of large numbers result for the Chiral ensemble in the $\theta N \to \infty$ regime. For $k \geq 1$, let $\mathcal{D}_k$ denote the set of Dyck paths of length $2k$ and for $p \in \mathcal{D}_k$, let $e(p)$ denote the number of descents located at the ith position for even $i \in [2k]$; recall that $1 \leq e(p) \leq k$.

Theorem 2.8. *Assume that $\lim_{N\to\infty} \theta N = \infty$, $\lim_{N\to\infty} \frac{M}{N} = c \geq 1$, and $\lim_{N\to\infty} \frac{t}{\theta N} = \alpha \geq 0$. Suppose $\nu \in \Gamma_{even}$. Then,*

$$\lim_{N\to\infty} \frac{\mathbb{E}_{a\sim C_{\theta,M,t}^N}[p_\nu(a)]}{(\theta N)^{|\nu|} N^{\ell(\nu)}} = \prod_{i=1}^{\ell(\nu)} \sum_{p\in\mathcal{D}_{\frac{\nu_i}{2}}} c^{e(p)} (2\alpha)^{\frac{\nu_i}{2}}$$

Proof. We can apply the analogue of Corollary 10.18 of the paper [Yao25a] for type BC root systems as well as Lemma 2.6 to obtain that

$$\lim_{N\to\infty} \frac{\mathbb{E}_{a\sim C_{\theta,M,t}^N}[p_\nu(a)]}{(\theta N)^{|\nu|} N^{\ell(\nu)}} = \prod_{i=1}^{\ell(\nu)} \sum_{\pi\in NC^{\text{even}}(\nu_i)} c^{o(\pi)} \prod_{B\in\pi} \mathbf{1}\{|B|=2\} 2\alpha.$$

Note that the analgoue of Corollary 10.18 of the paper can be determined from part (B) of Theorem 1.1 of the paper.

For $k \geq 1$ and $\pi \in NC(k)$, $o(\pi)$ is the number of $i \in [k]$ such that i is not the start of a block of π and the number of elements of $\{i, \ldots, k\}$ appearing in the block of i or in a block that starts earlier than the block of i is odd. From the proof of Corollary 1.8 in Subsection 10.1 of [Yao25a], if k is even and $\pi \in NC^{\text{even}}(k)$, then $o(\pi)$ equals the number of even $j \in [k]$ that are not the start of a block. In the case where the size of all blocks is two, $\pi \in NC^{\text{even}}(k)$ corresponds to a Dyck path $p \in \mathcal{D}_{\frac{k}{2}}$ and $o(\pi)$ equals $e(p)$. $\square$

Based on this result, we can prove the following corollary, which implies that if t is fixed then the distribution $C_{\theta,M,t}^N$ has order $\sqrt{\theta N}$.

Corollary 2.9. *Assume that* $\lim_{N\to\infty}\theta N = \infty$*,* $\lim_{N\to\infty}\frac{M}{N} = c \geq 1$*, and* $\lim_{N\to\infty} t = \alpha \geq 0$*. Suppose* $\nu \in \Gamma_{even}$*. Then,*

$$\lim_{N\to\infty} \frac{\mathbb{E}_{a\sim C^N_{\theta,M,t}}[p_\nu(a)]}{(\theta N)^{\frac{|\nu|}{2}} N^{\ell(\nu)}} = \prod_{i=1}^{\ell(\nu)} \sum_{p\in\mathcal{D}_{\frac{\nu_i}{2}}} c^{e(p)}(2\alpha)^{\frac{\nu_i}{2}}$$

Proof. Observe that $C^N_{\theta,M,\theta N t}$ is the image of $C^N_{\theta,M,t}$ under the map $x \mapsto \sqrt{\theta N}x$. Then, we can apply Theorem 2.8. $\square$

Next, we discuss the β-Laguerre ensemble, which has the same distribution as the squares of the eigenvalues of the Chiral ensemble. The paper [DE02] computes a tridiagonal matrix whose eigenvalues are distributed as the β-Laguerre ensemble. Suppose $M \geq N$ such that $M - N + 1 - \frac{1}{\theta} \geq 0$. Define the probability distribution $L^N_{\theta,M,t}$ over $\mathbb{R}^N_{\geq 0}$ to have the density of a proportional to

$$\prod_{i=1}^{N} a_i^{\theta(M-N+1)-1} e^{-\frac{a_i}{2t}} \prod_{1\leq i<j\leq N} |a_i - a_j|^\theta.$$

Then, $L^N_{\theta,M,t}$ is the image of $C^N_{\theta,M,t}$ under the push-forward map $x \mapsto x^2$. Despite this relation, it remains challenging to compute the Bessel generating function for the β-Laguerre ensemble. We obtain the following result, which is analogous to Corollary 2.9. Note that the result appears in [TT21].

Corollary 2.10 ([TT21, Theorem 1.1])**.** *Assume that* $\lim_{N\to\infty}\theta N = \infty$*,* $\lim_{N\to\infty}\frac{M}{N} = c \geq 1$*, and* $\lim_{N\to\infty} t = \alpha \geq 0$*. Suppose* $\nu \in \Gamma$*. Then,*

$$\lim_{N\to\infty} \frac{\mathbb{E}_{a\sim L^N_{\theta,M,t}}[p_\nu(a)]}{(\theta N)^{|\nu|} N^{\ell(\nu)}} = \prod_{i=1}^{\ell(\nu)} \sum_{p\in\mathcal{D}_{\nu_i}} c^{e(p)}(2\alpha)^{\nu_i}.$$

Proof. Use Corollary 2.9 and the fact that $L^N_{\theta,M,t}$ is the image of $C^N_{\theta,M,t}$ under the push-forward map $x \mapsto x^2$. $\square$

The result [TT21, Theorem 1.1] states that the β-Laguerre ensemble weakly converges to the Marchenko-Pastur law. It is straightforward to verify that the moments appearing in Corollary 2.10 are the same as the moments of the Marchenko-Pastur law, since it is well known that the number of Dyck paths with a fixed number of descents at even positions is given by the Narayana numbers.

The following result allows for the computation of the Bessel generating functions for a general set of exponentially decaying probability measures that satisfy an additional regularity condition.

Lemma 2.11. *Suppose* p *is holomorphic over* $\mathbb{C}^N$ *and*

$$p = \sum_{i=0}^{\infty} p_i$$

where $p_i \in \mathbb{C}[x_1, \dots, x_N]$ is homogeneous of degree i for $i \geq 0$. Suppose $q : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is measurable, satisfies $\|\sum_{i=0}^n p_i(x)\|_2 \leq q(x)$ over $\mathbb{R}^N$ for all $n \geq 0$, and

$$\int_{\mathbb{R}^N} q(x) e^{R\|x\|_2} \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx < \infty \tag{3}$$

for some $R > 0$. Then, $e^{\Delta_{\mathcal{R}(\theta)}/2} p(a)$ equals

$$e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} p(x) E_a^{\mathcal{R}(\theta)}(x) \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx$$

over $\overline{B}(0, R)$ and is holomorphic over $B(0, R)$.

Proof. Since $\|p(x)\|_2 \leq q(x)$ over $\mathbb{R}^N$ and $\left\| E_a^{\mathcal{R}(\theta)}(x) \right\|_2 \leq e^{\|a\|_2\|x\|_2}$ for $x \in \mathbb{R}_N$ by (1), we can show that the integral $e^{-\frac{\sum_{i=1}^N a_i^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} p(x) E_a^{\mathcal{R}(\theta)}(x) |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx$ converges over $\overline{B}(0, R)$ and is holomorphic over $B(0, R)$ using the proof of Lemma 1.7, see [Yao25a, Proof of Lemma 10.16] for more details. Note that we assume that the multiplicity function is nonnegative whereas Lemma 1.7 assumes that it has nonnegative real part. It suffices to show that the integral equals $e^{\Delta_{\mathcal{R}(\theta)}/2} p(a)$ over $\overline{B}(0, R)$.

Suppose $p = \sum_{i=0}^\infty p_i$ where $p_i \in \mathbb{C}[x_1, \dots, x_N]$ is homogeneous of degree i for $i \geq 0$. Then, we have that for $a \in \mathbb{C}^N$,

$$e^{\Delta_{\mathcal{R}(\theta)}/2} p(a) \triangleq \lim_{n \to \infty} \sum_{i=0}^n e^{\Delta_{R(\theta)}/2} p_i(a),$$

assuming that the limit converges.

By Lemma 2.2, for $n \geq 0$,

$$e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} \sum_{i=0}^n p_i(x) E_a^{\mathcal{R}(\theta)}(x) \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx = \sum_{i=0}^n e^{\Delta_{R(\theta)}/2} p_i(a).$$

By the dominated convergence theorem,

$$\begin{aligned}
&\lim_{n \to \infty} e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} \sum_{i=0}^n p_i(x) E_a^{\mathcal{R}(\theta)}(x) \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx \\
&= e^{-\frac{a_1^2+\cdots+a_N^2}{2}} c_N^{-1} \int_{\mathbb{R}^N} p(x) E_a^{\mathcal{R}(\theta)}(x) \prod_{r \in \mathcal{R}^+} |\langle x, r\rangle|^{2\theta(r)} \frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}} dx
\end{aligned}$$

when $a \in \overline{B}(0, R)$. We can apply the dominated convergence theorem due to the fact that $\left\| E_a^{\mathcal{R}(\theta)}(x) \right\|_2 \leq e^{\|a\|_2\|x\|_2} \leq e^{R\|x\|_2}$ when $a \in \overline{B}(0, R)$ and due to the condition (3). This finishes the proof. $\square$

By setting the multiplicity function to be zero in the previous result, we obtain the following result that appears in [Mac82]:

Corollary 2.12. *Suppose p satisfies the conditions of Lemma 2.11 when $\theta \in \theta(\mathcal{R})$ equals zero. Then, for all $a \in \mathbb{C}^N$,*

$$e^{\Delta/2}p(a) = c_N^{-1}\int_{\mathbb{R}^N} p(x)\frac{e^{-\frac{\langle x-a,\, x-a\rangle}{2}}}{(2\pi)^{\frac{N}{2}}}dx.$$

A potential choice for $p(x)$ would be $E_b(x)$ for some $b \in \mathbb{C}^N$. By (1), the regularity condition (3) is satisfied for all $R > 0$ since we can set $q(x) \triangleq e^{\|b\|_2\|x\|_2}$. Then, Lemma 2.11 implies that

$$c_N^{-1}\int_{\mathbb{R}^N} E_b^{\mathcal{R}(\theta)}(x)E_a^{\mathcal{R}(\theta)}(x)\prod_{r\in\mathcal{R}^+}|\langle x, r\rangle|^{2\theta(r)}\frac{e^{-\frac{\|x\|_2^2}{2}}}{(2\pi)^{\frac{N}{2}}}dx = e^{\frac{a_1^2+\cdots+a_N^2+b_1^2+\cdots+b_N^2}{2}}E_b^{\mathcal{R}(\theta)}(a). \tag{4}$$

This identity also appears in [Rös03b, Proposition 2.37]. We can use it to deduce the Bessel generating function for the Dyson Brownian motion (DBM).

The DBM $Y^N(t) \triangleq (Y_i(t))_{1\le i\le N}$ is the unique strong solution to

$$dY_i(t) = \theta\sum_{j\in[N]\setminus\{i\}}\frac{1}{Y_i(t)-Y_j(t)}dt + dB_i(t),$$

where the initial value $(Y_i(0))_{1\le i\le N}$ is fixed and the standard Brownian motions B_i for $1 \le i \le N$ are independent.

Lemma 2.13 ([GXZ24, Lemma 3.8])**.** *Suppose $t \ge 0$. The type A Bessel generating function of $Y^N(t)$ if the initial value is fixed at $(Y_i(0))_{1\le i\le N}$ is*

$$G_{Y^N(t)}^{A^{N-1}(\theta)}(x) = J_{(Y_i(0))_{1\le i\le N}}^{A^{N-1}(\theta)}(x)\exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right)$$

over $\mathbb{C}^N$.

Proof. The result is clear when $t = 0$. Assume that $t > 0$. In the following proof, we utilize Lemma 1.4.

The transition formula for the DBM given in [GXZ24, (23)] is that if $y \triangleq (y_1, \dots, y_N)$ has the same ordering as $z \triangleq (z_1, \dots, z_N)$, then:

$$\Pr[Y^N(t) = y | Y^N(0) = z] \propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right)\prod_{1\le i<j\le N}|y_i - y_j|^{2\theta}J_{\frac{z}{\sqrt{t}}}^{A^{N-1}(\theta)}\left(\frac{y}{\sqrt{t}}\right). \tag{5}$$

Thus, after consider the symmetric version of this transition formula, using (4) gives that

$$\begin{aligned}G_{Y^N(t)}^{A^{N-1}(\theta)}(x) &\propto \int_{\mathbb{R}^N}\exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right)\prod_{1\le i<j\le N}|y_i - y_j|^{2\theta}\times \\ &\qquad J_{\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}}\right)}^{A^{N-1}(\theta)}\left(\frac{y_1}{\sqrt{t}},\dots,\frac{y_N}{\sqrt{t}}\right)J_{(\sqrt{t}x_1,\dots,\sqrt{t}x_N)}^{A^{N-1}(\theta)}\left(\frac{y_1}{\sqrt{t}},\dots,\frac{y_N}{\sqrt{t}}\right)dy_1\cdots dy_N \\ &\propto \exp\left(-\frac{\sum_{i=1}^N z_i^2}{2t}\right)\int_{\mathbb{R}^N}\exp\left(-\frac{\sum_{i=1}^N y_i^2}{2}\right)\prod_{1\le i<j\le N}|y_i - y_j|^{2\theta}\times\end{aligned}$$

$$\begin{aligned}
&J^{A^{N-1}(\theta)}_{\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}}\right)}(y_1,\dots,y_N)J^{A^{N-1}(\theta)}_{(\sqrt{t}x_1,\dots,\sqrt{t}x_N)}(y_1,\dots,y_N)dy_1\cdots dy_N\\
\propto&J^{A^{N-1}(\theta)}_{\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}}\right)}(\sqrt{t}x_1,\dots,\sqrt{t}x_N)\exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right)\\
=&J^{A^{N-1}(\theta)}_{(z_1,\dots,z_N)}(x_1,\dots,x_N)\exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right).
\end{aligned}$$

This implies that

$$G^{A^{N-1}(\theta)}_{Y^N(t)}(x)=cJ^{A^{N-1}(\theta)}_{(z_1,\dots,z_N)}(x_1,\dots,x_N)\exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right)$$

for some $c>0$. We deduce that $c=1$ from the fact that $G^{A^{N-1}(\theta)}_{Y^N(t)}(0)=1$. □

We can also consider when the initial distribution $\mu\in\mathbb{R}^N$ of the DBM is random rather than deterministic. For a Borel probability measure μ over $\mathbb{R}^N$, we let the random sequence $\{Y^N_\mu(t)\}_{t\geq 0}$ denote the DBM $\{Y^N(t)\}_{t\geq 0}$ after the starting point $(Y_i(0))_{1\leq i\leq N}$ is distributed as μ.

Lemma 2.14 ([Yao25a, Lemma 10.22])**.** *Assume that μ is a Borel probability measure over $\mathbb{R}^N$ that exponentially decays at rate $R>0$. Then, for all $t\geq 0$, $Y^N_\mu(t)$ exponentially decays at any rate less than R and*

$$G^{A^{N-1}(\theta)}_{Y^N_\mu(t)}(x)=G^{A^{N-1}(\theta)}_{\mu}(x)\exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right)$$

over $\overline{B}(0,R)$.

Remark 2.15. Note that we prove that $Y^N_\mu(t)$ exponentially decays at any rate less than R but compute its Bessel generating function over $\overline{B}(0,R)$. Based on Lemma 1.7, we would expect to be able to compute its Bessel generating function over $B(0,R)$ rather than $\overline{B}(0,R)$. For the proof of the fact that $Y^N_\mu(t)$ exponentially decays at any rate less than R, see the proof of [Yao25a, Corollary 10.23].

An obvious modification that we can make to (5) is to set $J^{A^{N-1}(\theta)}_{(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}})}$ to be $E^{\mathcal{R}(\theta)}_{(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}})}$ or $J^{\mathcal{R}(\theta)}_{(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}})}$ for any choices of $\mathcal{R}$ and nonnegative multiplicity function θ. However, it is not always straightforward to define the transition probability if we assume that the process is continuous. For example, recall that in (5), we assume that $(y_1,\dots,y_N)$ has the same ordering as $(z_1,\dots,z_N)$. Therefore, we do not define the transition probability for all choices of $\mathcal{R}$.

Assume that $(y_1,\dots,y_N)$ has the same ordering as $(z_1,\dots,z_N)$. The definition of the asymmetric type A DBM has transition probability given by

$$\begin{aligned}
&\Pr[X_t=(y_1,\dots,y_N)|X_0=(z_1,\dots,z_N)]\\
\propto&\exp\left(-\frac{\sum_{i=1}^N y_i^2+z_i^2}{2t}\right)\prod_{1\leq i<j\leq N}|y_i-y_j|^{2\theta}E^{A^{N-1}(\theta)}_{\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_N}{\sqrt{t}}\right)}\left(\frac{y_1}{\sqrt{t}},\dots,\frac{y_N}{\sqrt{t}}\right).
\end{aligned}$$

In addition, assume that $(y_1^2, \dots, y_N^2)$ has the same ordering as $(z_1^2, \dots, z_N^2)$. For $\theta_0, \theta_1 \geq 0$, the definitions of the asymmetric and symmetric type BC DBMs are given by

$$\begin{aligned}&\Pr[X_t = (y_1, \dots, y_N) | X_0 = (z_1, \dots, z_N)] \\ &\propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right) \prod_{i=1}^N |y_i|^{2\theta_1} \prod_{1\leq i<j\leq N} |y_i^2 - y_j^2|^{2\theta_0} E^{BC^N(\theta_0,\theta_1)}_{\left(\frac{z_1}{\sqrt{t}}, \dots, \frac{z_N}{\sqrt{t}}\right)}\left(\frac{y_1}{\sqrt{t}}, \dots, \frac{y_N}{\sqrt{t}}\right)\end{aligned}$$

and

$$\begin{aligned}&\Pr[X_t = (y_1, \dots, y_N) | X_0 = (z_1, \dots, z_N)] \\ &\propto \exp\left(-\frac{\sum_{i=1}^N y_i^2 + z_i^2}{2t}\right) \prod_{i=1}^N |y_i|^{2\theta_1} \prod_{1\leq i<j\leq N} |y_i^2 - y_j^2|^{2\theta_0} J^{BC^N(\theta_0,\theta_1)}_{\left(\frac{z_1}{\sqrt{t}}, \dots, \frac{z_N}{\sqrt{t}}\right)}\left(\frac{y_1}{\sqrt{t}}, \dots, \frac{y_N}{\sqrt{t}}\right),\end{aligned}$$

respectively.

3. The law of large numbers for multiple observations of the DBM

The goal of this section is to prove the following result about multiple observations of the DBM.

Theorem 3.1. *Suppose $m \geq 1$ and $0 \leq \alpha_1 < \cdots < \alpha_m$. Assume that μ_N is an exponentially decaying Borel probability measure over $\mathbb{R}^N$ for all $N \geq 1$ and that for all $\nu \in \Gamma$, a complex number c_ν exists such that*

$$\lim_{N\to\infty} \frac{1}{\theta N}[1] \prod_{j=1}^{\ell(\nu)} \partial_j^{\nu_j} G_{\mu_N}^{A^{N-1}(\theta)}(x_1, \dots, x_N) = \frac{|\nu|! c_\nu}{P(\nu)}.$$

Then, if $\theta N \to \infty$, for any $\nu_1, \dots, \nu_m \in \Gamma$,

$$\begin{aligned}&\lim_{N\to\infty} \frac{\mathbb{E}\left[\prod_{i=1}^m p_{\nu_i}(Y_{\mu_N}^N(\alpha_i \theta N))\right]}{(\theta N)^{|\nu_1+\cdots+\nu_m|} N^{\ell(\nu_1+\cdots+\nu_m)}} \\ &= \prod_{i=1}^m \prod_{j=1}^{\ell(\nu_i)} \sum_{\pi\in NC(\nu_{ij})} \prod_{B\in\pi} \left(\sum_{\nu\in\Gamma,\, |\nu|=|B|} (-1)^{\ell(\nu)-1} \frac{|\nu| P(\nu)}{\ell(\nu)} c_\nu + \mathbf{1}\{|B|=2\}\alpha_i \right).\end{aligned}$$

Proof of Theorem 3.1. This follows from Lemma 3.2 and Theorem 3.3. $\square$

Note that the $m = 1$ version of the previous result is proven in [Yao25a, Corollary 10.23] for a more general set of sequences $\{\mu_N\}_{N\geq 1}$. The most significant contribution of the previous result is analyzing when $m > 1$. First, we prove the following lemma which relates the moments of multiple observations of the DBM to the Bessel generating function of the process. The lemma generalizes [GXZ24, Theorem 3.4].

Lemma 3.2. *Suppose $N \geq 1$. Suppose μ is an exponentially decaying distribution over $\mathbb{R}^N$. For all $m \geq 1$, $0 \leq \tau_1 < \cdots < \tau_m$, and $\nu_i \in \Gamma$ for $1 \leq i \leq m$,*

$$\mathbb{E}\left[\prod_{i=1}^m p_{\nu_i}(Y_\mu^N(\tau_i))\right] = [1] p_{\nu_m}(\mathcal{D}(A^{N-1}(\theta))) \exp\left(\frac{\tau_m - \tau_{m-1}}{2} p_{(2)}(x)\right)$$

$$\cdots p_{\nu_2}(\mathcal{D}(A^{N-1}(\theta)))\exp\left(\frac{\tau_2-\tau_1}{2}p_{(2)}(x)\right)p_{\nu_1}(\mathcal{D}(A^{N-1}(\theta)))\exp\left(\frac{\tau_1}{2}p_{(2)}(x)\right)G_\mu^{A^{N-1}(\theta)}(x).$$

Proof. We follow the proof of [GXZ24, Theorem 3.4]. Suppose μ exponentially decays at rate $R>0$. We prove the result using induction on m. In particular, we prove two statements. The first statement is that the distribution with density

$$p_{\nu_1}(a_1)\cdots p_{\nu_{m-1}}(a_{m-1})dY_\mu^N(\tau_1,\dots,\tau_m)(a_1,\dots,a_m)$$

exponentially decays at any rate less than R for all $m\geq 1$, where this is viewed as a distribution for a_m. More precisely, we show that the distribution over $\mathbb{R}^N$ with density of a_m equal to

$$\int_{\mathbb{R}^{N(m-1)}}p_{\nu_1}(a_1)\cdots p_{\nu_{m-1}}(a_{m-1})dY_\mu^N(\tau_1,\dots,\tau_{m-1})(a_1,\dots,a_{m-1})dY_\mu^N(\tau_m|\tau_{m-1})(a_m|a_{m-1})$$

exponentially decays at any rate less than R. However, we state it in the abbreviated format that we mentioned previously. Note that it is clear that this is true when $m=1$. The second statement is that

$$\int_{\mathbb{R}^{Nm}}p_{\nu_1}(a_1)\cdots p_{\nu_m}(a_m)J_{a_m}^{A^{N-1}(\theta)}(x)dY_\mu^N(\tau_1,\dots,\tau_m)(a_1,\dots,a_m)=p_{\nu_m}(\mathcal{D}(A^{N-1}(\theta)))$$
$$\exp\left(\frac{\tau_m-\tau_{m-1}}{2}p_{(2)}\right)\cdots p_{\nu_1}(\mathcal{D}(A^{N-1}(\theta)))\exp\left(\frac{\tau_1}{2}p_{(2)}(x)\right)G_\mu^{A^{N-1}(\theta)}(x)$$

over $B_R(0)$. It is clear that this is true when $m=1$ after applying Lemma 2.14 and part (D) of [Yao25a, Lemma 10.16]. Note that the result that we need to prove is implied by the second statement.

Suppose $l\geq 1$. Assume that the two statements are true when $m=l$. We show that the two statements are true when $m=l+1$. First, we argue that

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)|dY_\mu^N(\tau_1,\dots,\tau_{l+1})(a_1,\dots,a_{m+l})$$

exponentially decays at any rate less than R. By induction,

$$|p_{\nu_1}(a_1)\cdots p_{\nu_{l-1}}(a_{l-1})|dY_\mu^N(\tau_1,\dots,\tau_l)(a_1,\dots,a_l)$$

exponentially decays at any rate less than R. Hence,

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)|dY_\mu^N(\tau_1,\dots,\tau_l)(a_1,\dots,a_l)$$

exponentially decays at any rate less than R. Note that

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)|dY_\mu^N(\tau_1,\dots,\tau_{l+1})(a_1,\dots,a_{l+1})$$

is equivalent to the time $\tau_{l+1}-\tau_l$ observations of a DBM with initial distribution

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)|dY_\mu^N(\tau_1,\dots,\tau_l)(a_1,\dots,a_l).$$

So, by the proof of [Yao25a, Corollary 10.23],

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)|dY_\mu^N(\tau_1,\dots,\tau_{l+1})(a_1,\dots,a_{l+1})$$

exponentially decays at any rate less than R.

Observe that over $B_R(0)$,

$$\int_{\mathbb{R}^{N(l+1)}}p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)J_{a_{l+1}}^{A^{N-1}(\theta)}(x)dY_\mu^N(\tau_1,\dots,\tau_{l+1})(a_1,\dots,a_{l+1})$$

$$= \int_{\mathbb{R}^{Nl}} p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l) \int_{\mathbb{R}^N} J_{a_{l+1}}^{A^{N-1}(\theta)}(x) dY_\mu^N(\tau_{l+1}|\tau_l)(a_{l+1}|a_l)\times$$
$$dY_\mu^N(\tau_1,\ldots,\tau_l)(a_1,\ldots,a_l)$$
$$= \int_{\mathbb{R}^{Nl}} p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l) \exp\left(\frac{\tau_{l+1}-\tau_l}{2} p_{(2)}(x)\right) J_{a_l}^{A^{N-1}(\theta)}(x)\times$$
$$dY_\mu^N(\tau_1,\ldots,\tau_l)(a_1,\ldots,a_l)$$
$$= \exp\left(\frac{\tau_{l+1}-\tau_l}{2} p_{(2)}(x)\right) p_{\nu_l}(\mathcal{D}(A^{N-1}(\theta)))\cdots p_{\nu_1}(\mathcal{D}(A^{N-1}(\theta)))\times$$
$$\exp\left(\frac{\tau_1}{2} p_{(2)}(x)\right) G_\mu^{A^{N-1}(\theta)}(x),$$

where we have applied the inductive hypothesis and Fubini's theorem since

$$\int_{\mathbb{R}^{N(l+1)}} |p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l) J_{a_{l+1}}^{A^{N-1}(\theta)}(x)| dY_\mu^N(\tau_1,\ldots,\tau_{m+1})(a_1,\ldots,a_{l+1})$$

is finite. We prove that this is the case. Suppose $x \in B_R(0)$. Then, for some $\epsilon > 0$,

$$|J_{a_{l+1}}^{A^{N-1}(\theta)}(x)| \le \exp((R-\epsilon)\left\|a_{l+1}\right\|_2)$$

by Theorem 1.5 so the integral is upper bounded by

$$\int_{\mathbb{R}^{N(l+1)}} |p_{\nu_1}(a_1)\cdots p_{\nu_m}(a_l)| \exp((R-\epsilon)\left\|a_{l+1}\right\|_2) dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1}).$$

This is finite because $|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)| dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1})$ exponentially decays at any rate less than R.

To complete the induction, we must prove that

$$\int_{\mathbb{R}^{N(l+1)}} p_{\nu_1}(a_1)\cdots p_{\nu_{l+1}}(a_{l+1}) J_{a_{l+1}}^{A^{N-1}(\theta)}(x) dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1})$$
$$= p_{\nu_{l+1}}(\mathcal{D}(A^{N-1}(\theta))) \exp\left(\frac{\tau_{l+1}-\tau_l}{2} p_{(2)}(x)\right)\cdots p_{\nu_1}(\mathcal{D}(A^{N-1}(\theta))) \exp\left(\frac{\tau_1}{2} p_{(2)}(x)\right)\times$$
$$G_\mu^{A^{N-1}(\theta)}(x)$$

for $x \in B_R(0)$. It suffices to prove that

$$\int_{\mathbb{R}^{N(l+1)}} p_{\nu_1}(a_1)\cdots p_{\nu_{l+1}}(a_{l+1}) J_{a_{l+1}}^{A^{N-1}(\theta)}(x) dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1})$$
$$= p_{\nu_{l+1}}(\mathcal{D}(A^{N-1}(\theta))) \int_{\mathbb{R}^{N(l+1)}} p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l) J_{a_{l+1}}^{A^{N-1}(\theta)}(x)\times$$
$$dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1})$$

for $x \in B_R(0)$. However, this follows from part (D) of [Yao25a, Lemma 10.16] because

$$|p_{\nu_1}(a_1)\cdots p_{\nu_l}(a_l)| dY_\mu^N(\tau_1,\ldots,\tau_{l+1})(a_1,\ldots,a_{l+1})$$

exponentially decays at any rate less than R. $\square$

To prove Theorem 3.1, the next step is to prove the following theorem using results from [Yao25b]. First, we introduce some notation.

Let $\mathcal{F}_i^N$ be the set of elements of $\mathbb{C}[[x_1,\ldots,x_N]]$ which are symmetric in all variables except for x_i. For $d \geq 0$ and $\nu \in \Gamma \cup \{0\}$, let $c_F^{d,\nu}$ denote the coefficient of $x_i^d M_\nu(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$ in $F \in \mathcal{F}_i^N$.

A sequence $\{f_N\}_{N\geq i}$ such that $f_N \in \mathcal{F}_i^N$ for $N \geq i$ is *symmetric outside of i* if $\lim_{N\to\infty} c_{f_N}^{d,\nu}$ exists for all $d \geq 0$ and $\nu \in \Gamma \cup \{0\}$ and the *limiting sequence outside of i* of $\{f_N\}_{N\geq i}$ is $\{\lim_{N\to\infty} c_{f_N}^{d,\nu}\}_{d\geq 0,\nu\in\Gamma\cup\{0\}}$. Furthermore, for a sequence $g = \{c_g^{d,\nu}\}_{d\geq 0,\nu\in\Gamma\cup\{0\}}$ and $l \geq 1$, we define

$$c_l(g) \triangleq \sum_{\substack{\nu\in\Gamma,d\geq 0,\\ |\nu|+d=l-1}} (-1)^{\ell(\nu)} P(\nu) c^{d,\nu}.$$

Theorem 3.3. *For $1 \leq i \leq m$, suppose $G_N^i \in \mathbb{C}^{H(A^{N-1}(\theta))}[[x_1,\ldots,x_N]]$. Assume that $|\theta N| \to \infty$ and that for all $\nu \in \Gamma$, a complex number $c_\nu(i)$ exists such that*

$$\lim_{N\to\infty} \frac{1}{\theta N}[1]\prod_{j=1}^{\ell(\nu)} \partial_j^{\nu_j} G_N^i = \frac{|\nu|! c_\nu(i)}{P(\nu)}.$$

Then, for all $k_1,\ldots,k_m \geq 1$,

$$\lim_{N\to\infty} (\theta N)^{-|k_1|-\cdots-|k_m|}[1]\mathcal{D}_m^{k_m}(A^{N-1}(\theta))\exp(G_N^m - G_N^{m-1})\cdots\mathcal{D}_2^{k_2}(A^{N-1}(\theta))\exp(G_N^2 - G_N^1)$$
$$\mathcal{D}_1^{k_1}(A^{N-1}(\theta))\exp(G_N^1) = \prod_{i=1}^m \sum_{\pi\in NC(k_i)} \prod_{B\in\pi} \sum_{\nu\in\Gamma,\,|\nu|=|B|} (-1)^{\ell(\nu)-1}\frac{|\nu|P(\nu)}{\ell(\nu)} c_\nu(i).$$

Proof. We note that when applying the Dunkl operator $\mathcal{D}_i$ to the function $F_N\exp(G_N)$ for $F_N \in \mathcal{F}_i^N$ and symmetric G_N, we obtain $(\mathcal{D}_i F_N + \partial_i G_N)\exp(G_N)$, where $\mathcal{D}_i F_N + \partial_i G_N \in \mathcal{F}_i^N$. For the proof of this theorem, the main idea is that the exponential function changes as we apply the Dunkl operators: first, it is $\exp(G_N^1)$, then it is $\exp(G_N^2)$, and so forth.

In the setting of the paper [Yao25b], we view the constant term after applying a sequence of Dunkl operators to the Bessel generating function $\exp(G_N)$ as a polynomial in the coefficients of G_N, where each coefficient of G_N has order θN. Note that in the paper, we require θ to grow at the rate of N^{-c} for some $c < 1$ and set each coefficient of G_N to have order N^{1-c}, but it is straightforward to set each coefficient to have order θN and consider when $|\theta N| \to \infty$. The only difference now is that the functions G_N are changing as we apply the Dunkl operators so the set of coefficients of G_N are changing. However, each coefficient still has order θN, so many results of [Yao25b] are still true. In particular, to compute the leading order term of the expression, it suffices to consider the operators $\mathcal{Q}_i^N$.

For $1 \leq i \leq N$ and $f \in \mathcal{F}_i^N$, we define the operator $\mathcal{Q}_i^N(f)$ as

$$\mathcal{Q}_i^N(f) \triangleq \sum_{j\in[N]\setminus\{i\}} \frac{d_i - C_{ij}}{N} + f(x_1,\ldots,x_N),$$

which is an operator from $\mathcal{F}_i^N$ to $\mathcal{F}_i^N$. The operator d_i lowers the degree of x_i by one whereas the operator C_{ij} maps $f \in \mathbb{C}[[x_1,\ldots,x_N]]$ to $d_i s_{ij}(f|_{x_i=0})$.

From [Yao25b], the leading order term of

$$[1]\mathcal{D}_m^{k_m}(A^{N-1}(\theta))\cdots\mathcal{D}_1^{k_1}(A^{N-1}(\theta))\exp(G_N)$$

is

$$(\theta N)^{k_1+\cdots+k_m}[1]\left(\mathcal{Q}_{N-m+1}^{N-m+1}\left(\frac{\partial_{N-m+1}G_N|_{x_N=\cdots=x_{N-m+2}=0}}{\theta N}\right)^{k_m}\right)\Bigg|_{x_{N-m+1}=0}\cdots$$
$$\left(\mathcal{Q}_N^N\left(\frac{\partial_N G_N}{\theta N}\right)^{k_1}\right)\Bigg|_{x_N=0}(1);$$

recall that the coefficients of G_N have order θN. In this equation, we apply $\mathcal{Q}_{N-i+1}^{N-i+1}\left(\frac{\partial_i G_N|_{x_N=\cdots=x_{N-i+2}=0}}{\theta N}\right)^{k_i}$ and then set x_{N-i+1} as zero, since $x_N,\ldots,x_{N-i+2}$ are already set as zero, from $i=1$ to $i=m$. Note that $\mathcal{Q}_{N-i+1}^{N-i+1}\left(\frac{\partial_i G_N|_{x_N=\cdots=x_{N-i+2}=0}}{\theta N}\right)^{k_i}$ is an operator from $\mathcal{F}_{N-i+1}^{N-i+1}$ to $\mathcal{F}_{N-i+1}^{N-i+1}$ and setting $x_{N-i+1}=0$ in the output gives an element of $\mathcal{F}_{N-i}^{N-i}$.

The key observation is that there is no need to use the same function G_N for each value of i. We then obtain that the leading order term of

$$[1]\mathcal{D}_m^{k_m}(A^{N-1}(\theta))\exp(G_N^m-G_N^{m-1})\cdots\mathcal{D}_2^{k_2}(A^{N-1}(\theta))\exp(G_N^2-G_N^1)\mathcal{D}_1^{k_1}(A^{N-1}(\theta))\exp(G_N^1)$$

is

$$(\theta N)^{k_1+\cdots+k_m}[1]\left(\mathcal{Q}_{N-m+1}^{N-m+1}\left(\frac{\partial_{N-m+1}G_N^m|_{x_N=\cdots=x_{N-m+2}=0}}{\theta N}\right)^{k_m}\right)\Bigg|_{x_{N-m+1}=0}\cdots$$
$$\left(\mathcal{Q}_N^N\left(\frac{\partial_N G_N^1}{\theta N}\right)^{k_1}\right)\Bigg|_{x_N=0}(1).$$

Suppose the limiting sequence of $s_{1,N-i+1}\frac{\partial_{N-i+1}G_N^i|_{x_N=\cdots=x_{N-i+2}=0}}{\theta N}\in\mathcal{F}_1^{N-i+1}$ outside of 1 is g_i for $1\le i\le m$.

By [Yao25b, Theorem 2.13], the value of

$$\lim_{N\to\infty}[1]\left(\mathcal{Q}_{N-m+1}^{N-m+1}\left(\frac{\partial_{N-m+1}G_N^m|_{x_N=\cdots=x_{N-m+2}=0}}{\theta N}\right)^{k_m}\right)\Bigg|_{x_{N-m+1}=0}\cdots$$
$$\left(\mathcal{Q}_N^N\left(\frac{\partial_N G_N^1}{\theta N}\right)^{k_1}\right)\Bigg|_{x_N=0}(1)$$

is

$$\prod_{i=1}^m\sum_{\pi\in NC(k_i)}\prod_{B\in\pi}c_{|B|}(g_i).$$

Furthermore, by following the computations of [Yao25b, Subsection 3.6], we obtain that

$$c_l(g_i)=\sum_{\nu\in\Gamma,\,|\nu|=l}(-1)^{\ell(\nu)-1}\frac{|\nu|P(\nu)}{\ell(\nu)}c_\nu(i)$$

for $1\le i\le m$ and $l\ge 1$, which completes the proof. $\square$

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