Radiation [Equations marked with (*) were covered in class]

I. Green's function "review"

Related to the fields produced by a current:

(*) $$\left(\nabla\times\nabla\times - \epsilon(r,\omega)\,\omega^2/c^2\right)E(r,\omega) = i\omega\mu_0 J(r,\omega)$$

Useful representation of the solution: Green's functions. Basic concept:

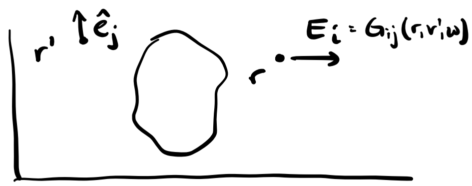
(*) $$J(r,\omega) = \int dr'\; J(r',\omega)\,\delta(r-r')$$

(*) By linearity: $$E(r,\omega) = i\omega\mu_0 \int dr'\; \underbrace{G(r,r',\omega)}_{\text{Green's function}}\, J(r',\omega),$$ where

(*) $$\left[\nabla\times\nabla\times - \epsilon(r,\omega)\frac{\omega^2}{c^2}\right]\underbrace{G(r,r',\omega)}_{3\times 3,\ \text{indices}} = \delta(r-r')\, \underset{\uparrow\ 3\times3\ \text{identity}}{I}$$

$G_{ij}(r,r',\omega)$

field direction (i), source direction (j), field point (r), source point (r')



Major utility: allows us to generalize beyond the lossless case.

Modal representation of Green's functions:

For simplicity: isotropic, non-dispersive, non-magnetic medium (then we can use completeness relations)

Completeness:

$$\hat{e}_j\,\delta(r-r') = \sum_n C_n E_n(r) \qquad (*)$$

$$\int \epsilon(r) E_m^*(r)\cdot \hat{e}_i\,\delta(r-r') = \epsilon(r') E_{m,j}^*(r')$$

$$= C_m$$

$$\Rightarrow \hat{e}_j\,\delta(r-r') = \sum_n \epsilon(r') E_{n,j}^*(r') E_n(r). \qquad (*)$$

From $$\left(\nabla\times\nabla\times - \epsilon(r)\frac{\omega^2}{c^2}\right)G_j(r,\omega) = \delta(r-r')\,\hat{e}_j$$

$$= \sum_n \epsilon(r') E_{n,j}^*(r') E_n(r)$$

and $$G_{ij}(r,\omega) = \sum_n G_n E_{n,i}(r),$$

we get $\sum_n \varepsilon(r)(\omega_n^2/c^2 - \omega^2/c^2)\, G_n\, E_{n,i}(r) = \sum_n \varepsilon(r')\, E_{n,j}^*(r')\, E_{n,i}(r)$

$$\Rightarrow \quad G_n = \frac{\varepsilon(r')\, E_{n,j}^*(r')}{\varepsilon(r)(\omega_n^2 - \omega^2)/c^2}$$

$$\Rightarrow \boxed{G_{ij}(r, r', \omega) = \sum_n \frac{\varepsilon(r')\, E_{n,j}^*(r')\, E_{n,i}(r)}{\varepsilon(r)\,(\omega_n^2 - \omega^2)/c^2}} \qquad (*)$$

Boundary condition: no incoming fields $\Rightarrow E \to 0$ as $r \to \infty$, enforced via $\omega_n \to \omega_n - i\eta$ ($\eta > 0$, infinitesimal absorption).

$$\frac{1}{(\omega_n - i\eta)^2 - \omega^2} \longrightarrow \frac{1}{(\omega_n - \omega - i\eta)(\omega_n + \omega - i\eta)}$$

Near a pole, say $\omega \approx \omega_n$, approx

$$\frac{1}{2\omega_n}\frac{1}{\omega_n - \omega - i\eta} = \frac{1}{2\omega_n}\left[P\left(\frac{1}{\omega_n - \omega}\right) + i\pi\delta(\omega - \omega_n)\right]$$

$(*)$

II. Energy dissipated by a source

Reminder: work per unit time by a current $J(r,t)$

Time-domain: Power $= \int dr\, J(r,t)\cdot E(r,t)$, energy $= \int dt\,[\text{power}]$.
In frequency-domain, the energy is given by:

$$(*) \qquad \mathcal{U} = \int_0^\infty \frac{d\omega}{2\pi}\, u(\omega) \quad , \quad u(\omega) = \frac{1}{2}\,\mathrm{Re}\left[\int dr\; J^*(r,\omega)\cdot E(r,\omega)\right]$$

From the Green's function, we have

$$u(\omega) = -\frac{1}{2}\,\mathrm{Re}\left[\int dr\; J^*(r,\omega)\int dr'\; i\omega\mu_0\, G(r,r',\omega)\cdot J(r',\omega)\right]$$

$$= \boxed{\frac{\omega\mu_0}{2}\,\mathrm{Im}\left[\int dr\, dr'\; J^*(r,\omega)\cdot G(r,r',\omega)\cdot J(r',\omega)\right]} \qquad (*)$$

This holds for an arbitrary current and an arbitrary medium. We now cover three important cases.

(1) A dipole point source.
(2) A moving point charge.
(3) A fluctuating current source.

(1) Point dipole [time-harmonic]

$$\underbrace{P(r,t)}_{\text{pol. density}} = \underbrace{p_0}_{\text{dipole moment}} \delta(r-r_0)\, e^{-i\omega_0 t} \qquad (*)$$

$$J = \partial_t \overset{(*)}{P} = -i\omega p_0 \delta(r-r_0) e^{-i\omega_0 t} \Rightarrow J(r,\omega) = -2\pi i \omega p_0 \delta(r-r_0)\delta(\omega-\omega_0)$$

$$\Rightarrow u(\omega) = \frac{1}{2}(2\pi\omega)^2 \delta(\omega-\omega_0)^2 \omega\mu_0 \,\mathrm{Im}\left[p_0^* \cdot G(r_0,r_0,\omega)\cdot p_0\right]$$

$$= \frac{1}{2}(2\pi\omega)^2 \omega\mu_0 \delta(\omega-\omega_0) \frac{T}{2\pi} \mathrm{Im}\left[p_0^* \cdot G(r_0,r_0,\omega)\cdot p_0\right]$$

$$u = \int \frac{d\omega}{2\pi} \cdot \frac{1}{2} \cdot 2\pi\omega^3 \mu_0 T \,\mathrm{Im}\left[p_0^* \cdot G(r_0,r_0,\omega_0)\cdot p_0\right] \delta(\omega-\omega_0)$$

$$= \frac{1}{2}\mu_0 \omega_0^3 \,\mathrm{Im}\left[p_0^* \cdot G(r_0,r_0,\omega_0)\cdot p_0\right] T$$

δ-fn regularization

$$2\pi\delta(\omega) = \int dt\, e^{-i\omega t} = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, e^{-i\omega t}$$

$$\omega = 0 \Rightarrow 2\pi\delta(0) = \lim_{T\to\infty} T$$

Power, $$\boxed{u/T = \frac{1}{2}\mu_0 \omega_0^3 \,\mathrm{Im}\, p_0^* \cdot G(r_0,r_0,\omega_0)\cdot p_0} \quad (*)$$

For a dipole along some orientation $\hat{e}_i$, get:

$$\boxed{u/T = \frac{1}{2}\mu_0 \omega_0^3 |p_0|^2 \,\mathrm{Im}\, G_{ii}(r_0,r_0,\omega)}$$

<u>Modal expression:</u>

At positive frequencies,

$$\mathrm{Im}\, G_{ii}(r,r,\omega) = \sum_n \frac{\pi c^2}{2\omega_n} |E_{n,i}(r)|^2 \delta(\omega_0-\omega_n)$$

$$\Rightarrow u/T = \frac{1}{2}\mu_0 \omega^3 |p_0|^2 \sum_n \frac{\pi c^2}{2\omega_0} |E_{n,i}(r)|^2 \delta(\omega_0-\omega_n)$$

* allows calculating emission in terms of a set of modes, and to identify which modes are emitted into.

example: in free space, get $\mathrm{Im}\, G_{ii}(r_0,r_0,\omega) = \frac{\omega_0}{6\pi c} I$. (*)

$$\Rightarrow u/T = \frac{\mu_0 \omega_0^4}{12\pi c}|p_0|^2. \quad (*)$$

Example: Purcell effect in a resonator

Consider a two-level system interacting with a localized resonance with a mode function $E_n(r)$. In the vicinity of the resonance, we can write its contribution to the Green's function as

$$G(r, r', \omega) \approx \frac{\varepsilon(r')\, E_n^*(r')\, E_n(r)}{\varepsilon(r)(\omega_n^2 - \omega^2)/c^2}$$

But resonances decay. We can approximate the change to the GF as:

$$\omega_n \to \omega_n - i\gamma_n/2 .$$

Radiation is proportional to Im G $\Rightarrow$

$$\operatorname{Im} G(r, r, \omega_0) = \operatorname{Im} \frac{c^2 E_n^*(r) E_n(r)}{\omega_n^2 - \omega_0^2 - i\gamma\omega_n} \qquad (*)$$

$$= c^2 E_n^*(r) E_n(r) \frac{\gamma\omega_n}{(\omega_n^2-\omega_0^2)^2 + \gamma^2\omega_n^2}$$

Near the resonance $(\omega_0 \approx \omega_n)$, we have

$$\operatorname{Im} G(r, r, \omega_0) = \frac{1}{2} c^2 E_n^*(r) E_n(r) \frac{\gamma\omega_n}{\left[(\omega_n-\omega_0)^2 + \frac{\gamma^2}{4}\right]\omega_n^2}$$

The emission rate by a dipole of dipole moment p_0 is then:

$$\text{power} = \frac{1}{2}\mu_0 \omega^3 |p_0 \cdot E_n|^2 \cdot \left(\frac{1}{2}\frac{c^2}{\omega_n} \frac{\frac{\gamma}{2}}{(\omega_n-\omega_0)^2 + \gamma^2/4}\right)$$

Compared to free space, we can say (on resonance, $\omega_n = \omega_0$)

$$\frac{(\text{power})_{\text{resonance}}}{(\text{power})_{\text{free}}} \sim \frac{|p_0 \cdot E_n|^2 \cdot \frac{1}{\cancel{2}} \frac{c^2}{\omega_0} \frac{\cancel{\gamma}}{\cancel{2}} \frac{4}{\gamma^{\cancel{2}}}}{\frac{\omega_0}{6\pi c} |p_0|^2}$$

$$= \frac{|p_0 \cdot E_n|^2}{|p_0|^2} \underbrace{6\pi c \cdot \frac{c^2}{\omega_0^2 \gamma}}_{\frac{6\pi c^3 \omega_0}{\omega_0^3 \gamma}}$$

$$= \frac{|p_0 \cdot \hat{u}|^2}{|p_0|^2} \, \frac{1}{V} \, \underbrace{6\pi \left(\frac{\lambda}{2\pi}\right)^3}_{\frac{6\pi}{8\pi^2} = \frac{3}{4\pi^2}} Q$$

$$= \boxed{\frac{3}{4\pi^2} \, \frac{Q}{(V/\lambda^3)} \, (\text{polarization factor})} \qquad (*)$$

(2) Moving charge (rectilinear trajectory)

$$J = e v \, \delta(r - r(t)), \quad r(t) = r_0 + v t \qquad v = v_0 \hat{v}$$

$$J(r,\omega) = e v \int dt \, e^{i\omega t} \delta(r - r_0 - v t) \quad ; \; (r - r_0)\cdot\hat{v} = v_0 t$$

$$= e v_0 \underbrace{e^{i \frac{\omega}{v_0} [(r - r_0)\cdot \hat{v}]} \delta_\perp(\rho)}_{\text{space harmonic source with wavenumber } \omega/v_0.} \hat{v} \equiv J_0 e^{ik r\cdot\hat{v}} \delta_\perp(\rho - \rho_0) \hat{v} \qquad (*)$$

To understand radiation dynamics, can use conservation laws.

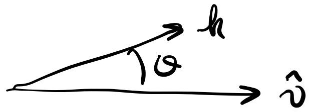
Recall: $\left(\nabla\times\nabla\times - \epsilon(r,\omega)\,\omega^2/c^2\right) E(r,\omega) = i\omega\mu_0 J(r,\omega)$

$\Longleftrightarrow \quad AE = b \quad \Longrightarrow \quad E = A^{-1} b.$

Consider a symmetry operation S such that $Sb = \lambda b$ and $[A,S] = 0$. Then $SAE = Sb = \lambda b \Rightarrow A(SE) = \lambda b$, $SE = \lambda A^{-1} b \Rightarrow$ if E is a soln. arising from a source, then it is an eigenvector of the symmetry operator S with the same eigenvalue. Let's apply this to translations:

$S = T_x$, $\quad T_x J(r,\omega) = e^{ik\cdot x} J(r,\omega) \Rightarrow \lambda = e^{ik\cdot x}$. Therefore, translation of E by x also yields $e^{ik\cdot x} \Rightarrow E$ is a plane wave of wavevector k.

In a bulk medium, we have $\hat{v}\cdot k = n(\omega)\,\omega/c = \frac{\omega}{v_0}$



$$k v_0 \cos\theta = n(\omega)\,\omega/c$$

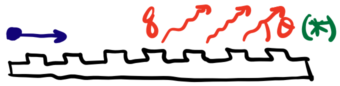
$\downarrow$ non-dispersive case

$$\Rightarrow \boxed{v_0 \cos\theta = c/n} \qquad (*)$$

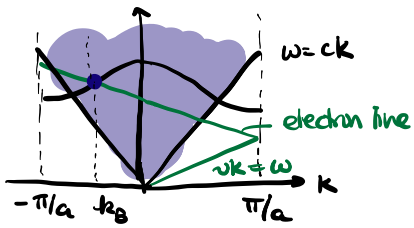
$$\boxed{v_0 \geq c/n} \quad (\text{Cherenkov effect})$$

Discrete translation symmetry: consider a system with DTS, such that A commutes with T_R, where R is a lattice vector. Now consider translating our current source. $T_R J = e^{i(\omega/v_0)\hat{v}\cdot R} J$, so $\lambda = e^{i(\frac{\omega}{v_0})[R\cdot\hat{v}]}$. Since T_R commutes with A, we require $T_R E = e^{i\frac{\omega}{v_0}R\cdot\hat{v}} E = e^{i(\frac{\omega}{v_0}\hat{v}+G)\cdot R} = e^{ik_B R}$, where k_B is in the FBZ. Therefore, $k_B = \frac{\omega}{v_0}\hat{v} + G$.

1D case: $R = a\hat{x} \Rightarrow \frac{\omega}{v_0}[\hat{v}\cdot\hat{x}] + \frac{2\pi m}{a} = k_B$ (*)



Some harmonic above light line.



$$q = \omega/c = \sqrt{(k_B+G')^2 + k_\perp^2}$$

$$= \sqrt{(k_B+G')^2 + \left(\frac{\omega}{c}\right)^2 \sin^2\theta}$$

$$\Rightarrow \left(\frac{\omega}{c}\right)^2 \cos^2\theta = (k_B + G')^2$$

$$\frac{2\pi}{\lambda}\cos\theta = k_B + G$$

$$= \frac{\omega}{v_0}[\hat{v}\cdot\hat{x}] + G''$$

$$\Rightarrow \frac{2\pi}{\lambda}\left(\cos\theta - \frac{c}{v_0}[\hat{v}\cdot\hat{x}]\right) = \frac{2\pi\ell}{a}$$

$$\Rightarrow \lambda = \frac{a}{\ell}\left(\cos\theta - \frac{1}{\beta}\right) \xrightarrow[\text{incidence}]{\text{grazing}} \lambda = \frac{a}{\ell}\left(\cos\theta - \frac{1}{\beta}\right). \quad (*)$$

$\beta \equiv v/c < 1$

This is called the Smith-Purcell effect. Enables light emission by slower electrons.

(3) Fluctuating currents: spontaneous emission, thermal emission

Thus far, I've discussed "deterministic" currents, $J(r,t)$ is some known function of time. In reality: most light sources have: $\langle J(r,\omega)\rangle = 0$.

Yet, radiation is produced because $\langle J(r,\omega)\, J^*(r',\omega)\rangle \neq 0$. From the energy-loss spectrum, we have

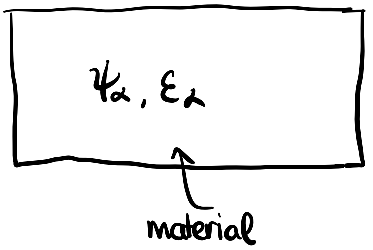
$$U(\omega) = \frac{\omega\mu_0}{2}\,\mathrm{Im}\left[\int dr\,dr'\, J^*(r,\omega)\cdot G(r,r',\omega)\cdot J(r',\omega)\right]$$

Ensemble averaging $\Rightarrow$

$$\boxed{U(\omega) = \frac{1}{2}\omega\mu_0 \int dr\,dr'\, G_{ij}(r,r',\omega)\left\langle J_i^*(r,\omega) J_j(r',\omega)\right\rangle} \quad (*)$$

(*) I use repeated index notation:
$a\cdot b = a_i b_i$
$u^T M v = u_i M_{ij} v_j$, etc.

Here, I will simply *state* the form of these current-current correlations, since they require quantum mechanics to derive. See the last section of these notes for a derivation if you're interested.

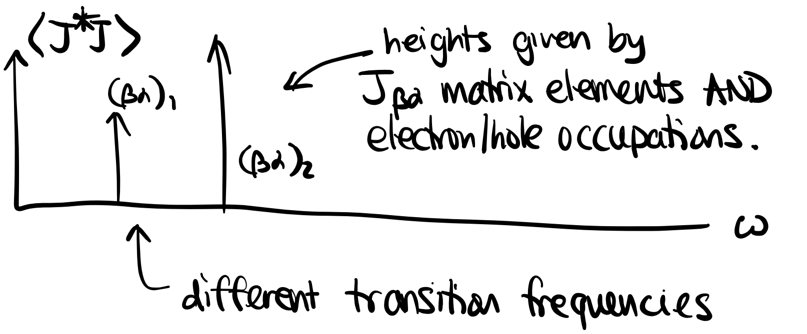


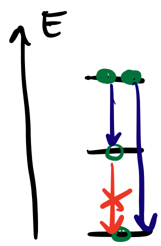
= quantum system, can solve Schrödinger equation for wavefunctions $\psi_\alpha(r)$ and energies E_α ($i\hbar\partial_t|\psi\rangle = H|\psi\rangle$) [Ignoring many-body effects, which is often fine].

$$\langle J_i^*(r,\omega) J(r',\omega')\rangle$$

$$(2\pi)^2\delta(\omega-\omega')\sum_{\alpha>\beta} J^*_{\beta\alpha,i}(r) J_{\beta\alpha,j}(r') f_\alpha(1-f_\beta)\,\delta(\omega-\omega_{\alpha\beta})$$

$$\Rightarrow \boxed{\langle J_i^*(r,\omega) J_j(r',\omega)\rangle = 2\pi T\delta(\omega-\omega')\sum_{\alpha>\beta} J^*_{\beta\alpha,i}(r) J_{\beta\alpha,j}(r') f_\alpha(1-f_\beta)\,\delta(\omega-\omega_{\alpha\beta})} \quad (*)$$





(B) Blackbody radiation.

For a system at temperature T, the correlations are given by the fluctuation dissipation theorem as:

$$\langle J_i^{(-)}(r,\omega) J_j^{(+)}(r',\omega)\rangle = \frac{\varepsilon_0\hbar\omega^2}{\pi}\frac{1}{e^{\hbar\omega/kT}-1}\underbrace{\mathrm{Im}\,\varepsilon_j(r,\omega)}_{\text{material absorption}}\delta(r-r') \quad (*)$$

In either case, the radiation at a given ω can be computed by summing the energy radiated by dipoles at different positions in a medium – "fluctuational electrodynamics".

Aside:

Typically, this cross-correlation requires quantum mechanics to calculate. For a quantum system with initial density matrix ρ, the relevant cross-correlation is computed as:

$$\langle J_i^{(-)}(r,\omega) J_j^{(+)}(r',\omega') \rangle$$

$$= \mathrm{tr}\left[\rho J_i^{(-)}(r,\omega) J_j^{(+)}(r',\omega')\right]$$

* Relevant background on second quantization & correlation functions in grad. solid-state physics, e.g. 8.512.

where $J^{(+)}(r,t) = \sum_{\alpha>\beta} J_{\beta\alpha}(r)\, c_\beta^+ c_\alpha\, e^{-i\omega_{\alpha\beta} t}$, $\omega_{\alpha\beta} = (E_\alpha - E_\beta)/\hbar > 0$,

and $J^{(-)} = [J^{(+)}]^\dagger$.

$$J^{(-)}(r,t) = \sum_{\alpha>\beta} J_{\beta\alpha}^*(r)\, c_\alpha^+ c_\beta\, e^{i\omega_{\alpha\beta} t}.$$

Here, α labels energy eigenstates of the Schrödinger equation. The matrix element $J_{\alpha\beta} = ie\hbar/m \int dr\, \psi_\alpha^*(r) \nabla \psi_\beta(r)$, where $\psi_\alpha(r)$ are single-particle eigenstates, and we assumed the current to be carried by electrons. $c^{(+)}$ is an electron annihilation (creation) operator.

Then $J^{(+)}(r,\omega) = \sum_{\alpha>\beta} J_{\beta\alpha}(r)\, c_\beta^+ c_\alpha\, 2\pi\delta(\omega - \omega_{\alpha\beta})$,

and the relevant cross-correlation is given by:

$$\langle J_i^{(-)}(r,\omega) J^{(+)}(r',\omega') \rangle$$

$$= \sum_{\alpha>\beta} \sum_{\alpha'>\beta'} J_{\beta\alpha;i}^*(r)\, J_{\beta'\alpha';j}(r')\, (2\pi)^2 \delta(\omega - \omega_{\alpha\beta})\, \delta(\omega' - \omega_{\alpha'\beta'})\, \mathrm{tr}\left[\rho\, c_\alpha^+ c_\beta\, c_{\beta'}^+ c_{\alpha'}\right]$$

The following trace is evaluated as:

$$\mathrm{tr}\left[\rho\, c_\alpha^+ c_\beta c_{\beta'}^+ c_{\alpha'}\right]$$

$$= \mathrm{tr}\left[c_{\beta'}^+ c_{\alpha'}\, \rho\, c_\alpha^+ c_\beta\right]$$

For any state like $\sum_k p_k |k\rangle\langle k|$, get $\longrightarrow$ e.g. thermal state

$$\langle c^\dagger_{\beta'} c_{\alpha'} |k\rangle\langle k| c^\dagger_\alpha c_\beta\rangle = 0 \text{ unless } \alpha=\alpha',\ \beta=\beta' \Rightarrow$$

$$\langle J_i^{(-)}(r,\omega)\, J^{(+)}(r',\omega')\rangle$$

$$= \sum_{\alpha>\beta} J^*_{\beta\alpha,i}(r)\, J_{\beta\alpha,j}(r')\,(2\pi)^2\, \underbrace{\delta(\omega-\omega_{\alpha\beta})\,\delta(\omega'-\omega_{\alpha\beta})}_{=\delta(\omega-\omega')\,\delta(\omega-\omega_{\alpha\beta})}$$

$$\mathrm{tr}\left[\rho\, c^\dagger_\alpha c_\beta\, c^\dagger_\beta c_\alpha\right]$$

$\alpha=\beta$ is not so interesting, since then $\omega=0$ and we're looking at light generation, which implies finite frequency.

$$= \sum_{\alpha>\beta} J^*_{\beta\alpha,i}(r)\, J_{\beta\alpha,j}(r')\,(2\pi)^2\,\delta(\omega-\omega_{\alpha\beta})\,\delta(\omega-\omega')$$

$$\underbrace{\mathrm{tr}\left[\rho\, c^\dagger_\alpha c_\alpha\, c_\beta c^\dagger_\beta\right]}_{}$$

$$= f_\alpha\,(1-f_\beta)$$

$c_\beta c^\dagger_\beta + c^\dagger_\beta c_\beta = 1$

$\Rightarrow c_\beta c^\dagger_\beta = 1 - \underbrace{c^\dagger_\beta c_\beta}_{\text{fermion number operator}}$

$\langle c^\dagger c\rangle \in [0,1]$.

$$\Rightarrow \langle J_i^{(-)}(r,\omega)\, J^{(+)}(r',\omega')\rangle$$

$$= (2\pi)^2\delta(\omega-\omega')\sum_{\alpha>\beta} \underbrace{J^*_{\beta\alpha,i}(r)\, J_{\beta\alpha,j}(r')}_{}\, f_\alpha(1-f_\beta)\,\delta(\omega-\omega_{\alpha\beta})$$

typically, these matrix elements are very localized compared to the photon characteristic wavelength $\lambda_{\alpha\beta} = 2\pi c/\omega_{\alpha\beta}$ and so a typical value of $r-r'$ in the photon Green's function gives ≈ 0 for the current-current correlation. This is a local approximation.

Therefore, when evaluating e.g.

$$u(\omega) = \tfrac{1}{2}\omega\mu_0 \int dr\,dr'\; G_{ij}(r,r',\omega)\,\langle J^*_i(r,\omega)\, J_j(r',\omega)\rangle,$$

we can define a *local* correlator via

$$\langle J_i^{(-)}(r,\omega)\, J^{(+)}(\underset{\uparrow\, r'\to r}{r},\omega')\rangle = \int dr'\; \overbrace{\langle J_i^{(-)}(r,\omega)\, J^{(+)}(r',\omega')\rangle}^{\text{peaked about } r'=r}.$$