18.369: Mathematical Methods in Nanophotonics

overview lecture slides

(don't get used to it: most lectures are blackboard)

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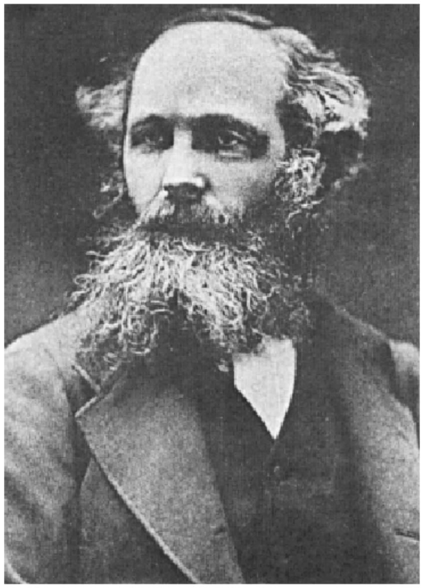


James Clerk Maxwell
1864

Maxwell's Equations

Gauss:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

Ampere:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Faraday:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

constitutive relations:

$$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$$

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$$

electromagnetic fields:

E = electric field
D = displacement field
H = magnetic field / induction
B = magnetic field / flux density

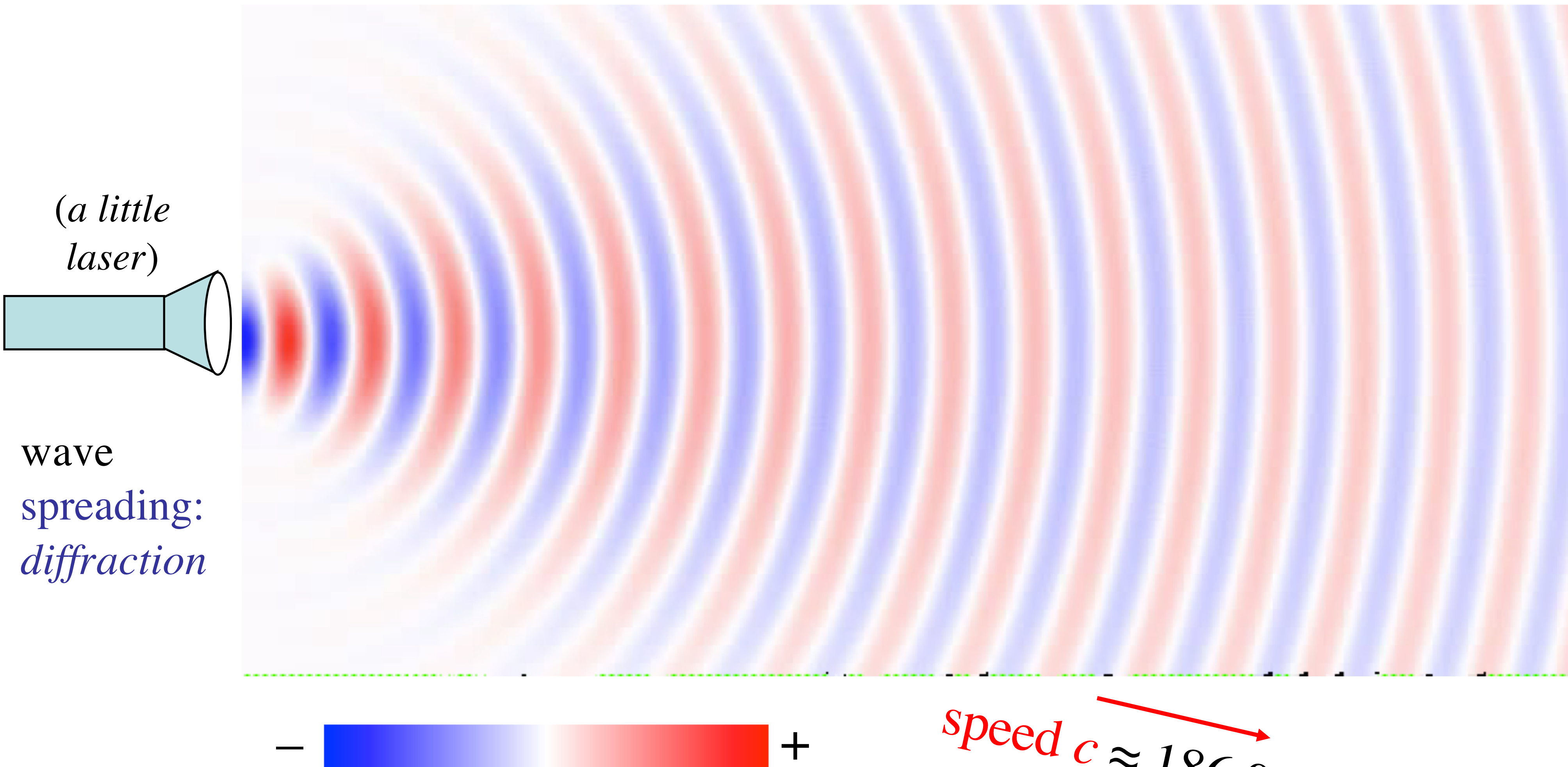
constants: ε_0, μ_0 = vacuum permittivity/permeability
c = vacuum speed of light = $(\varepsilon_0 \mu_0)^{-1/2}$

sources: **J** = current density
ρ = charge density

material response to fields:
P = polarization density
M = magnetization density

Light is a Wave (and a quantum particle)

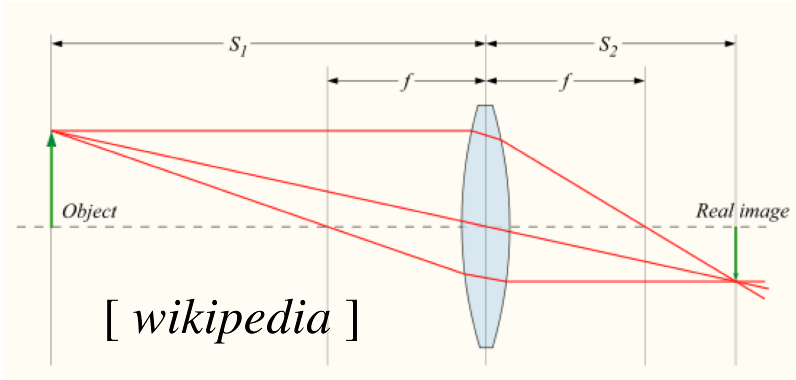
color ~ wavelength $\lambda \approx$ 500nm for visible light



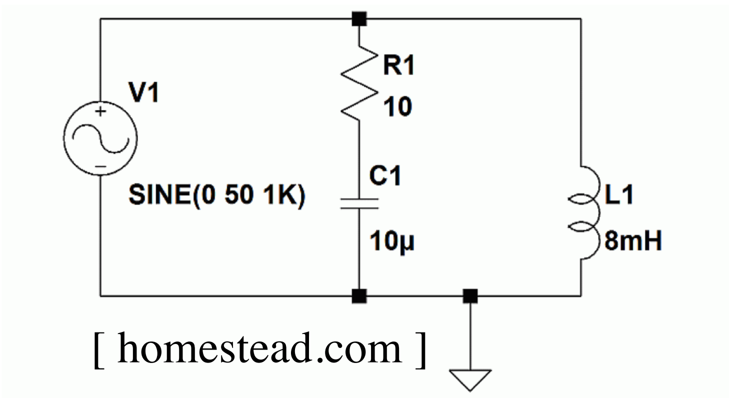
speed $c \approx$ 186,000 miles/sec

When can we solve this mess?

- Very small wavelengths: ray optics

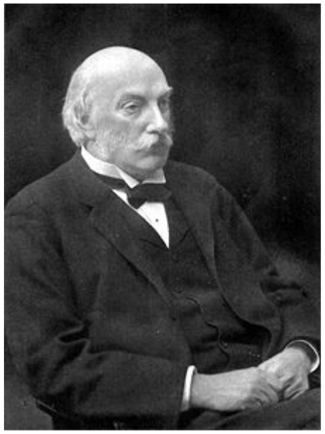


- Very large wavelengths:
  quasistatics (8.02)
  & lumped circuit models (6.002)



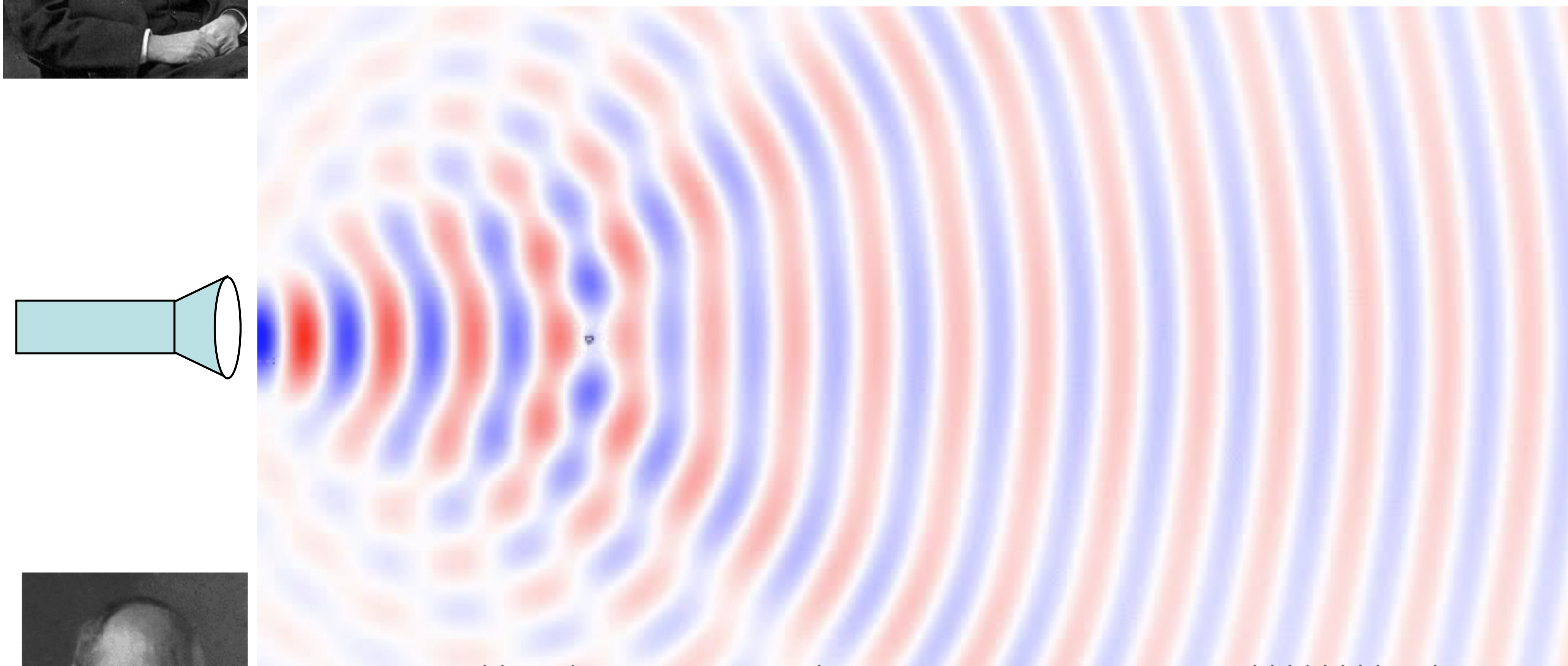
- Wavelengths comparable to geometry?
  — handful of cases can be ~solved analytically:
  planes, spheres, cylinders, empty space (8.07, 8.311)
  — everything else just a mess for computer…?

Waves Can Scatter

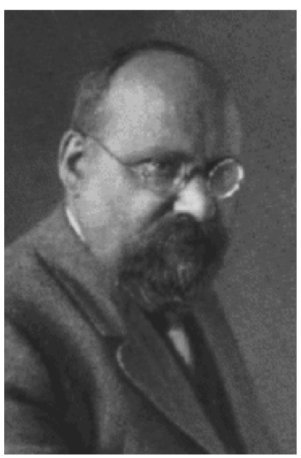


small particles:
Lord Rayleigh (1871)
why the sky is blue

here: a little circular speck of silicon



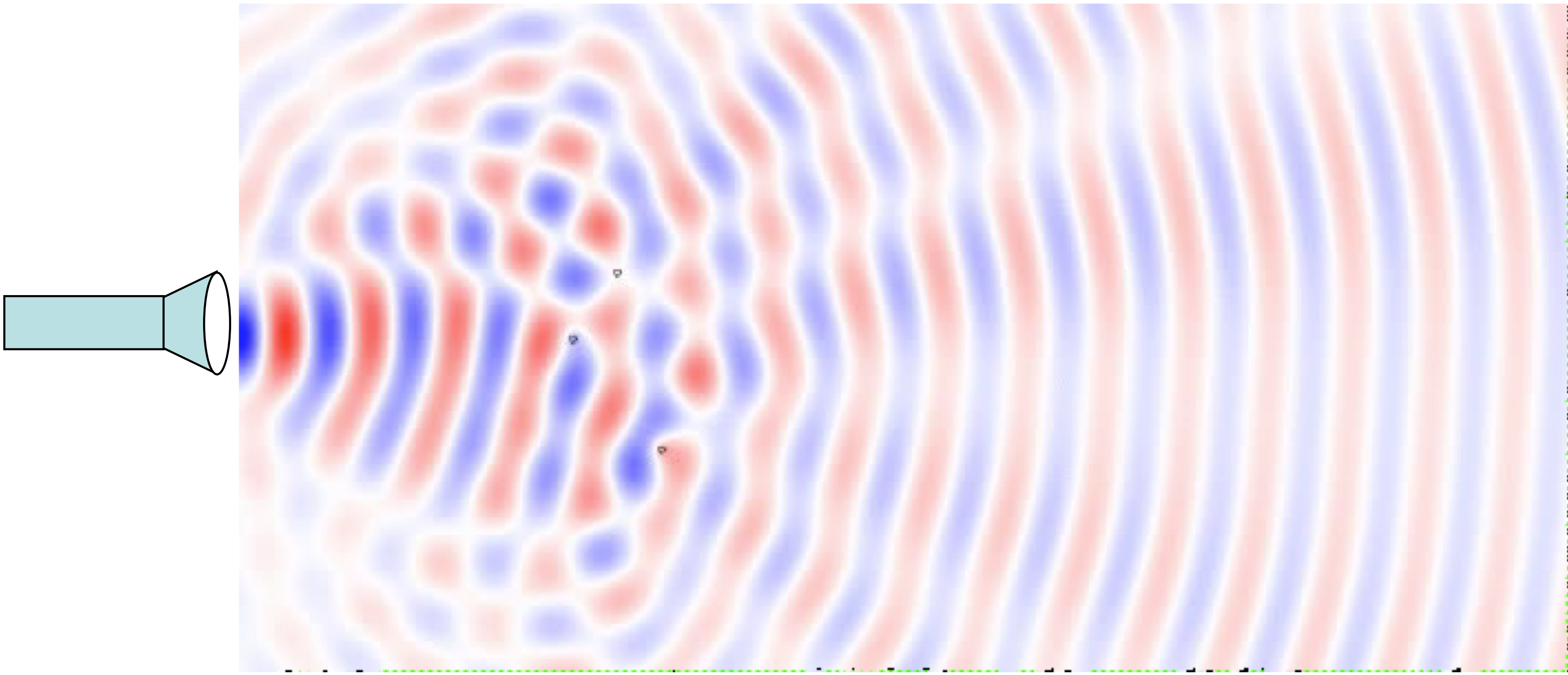
checkerboard pattern: interference of waves traveling in different directions



scattering by spheres:
solved by Gustave Mie (1908)

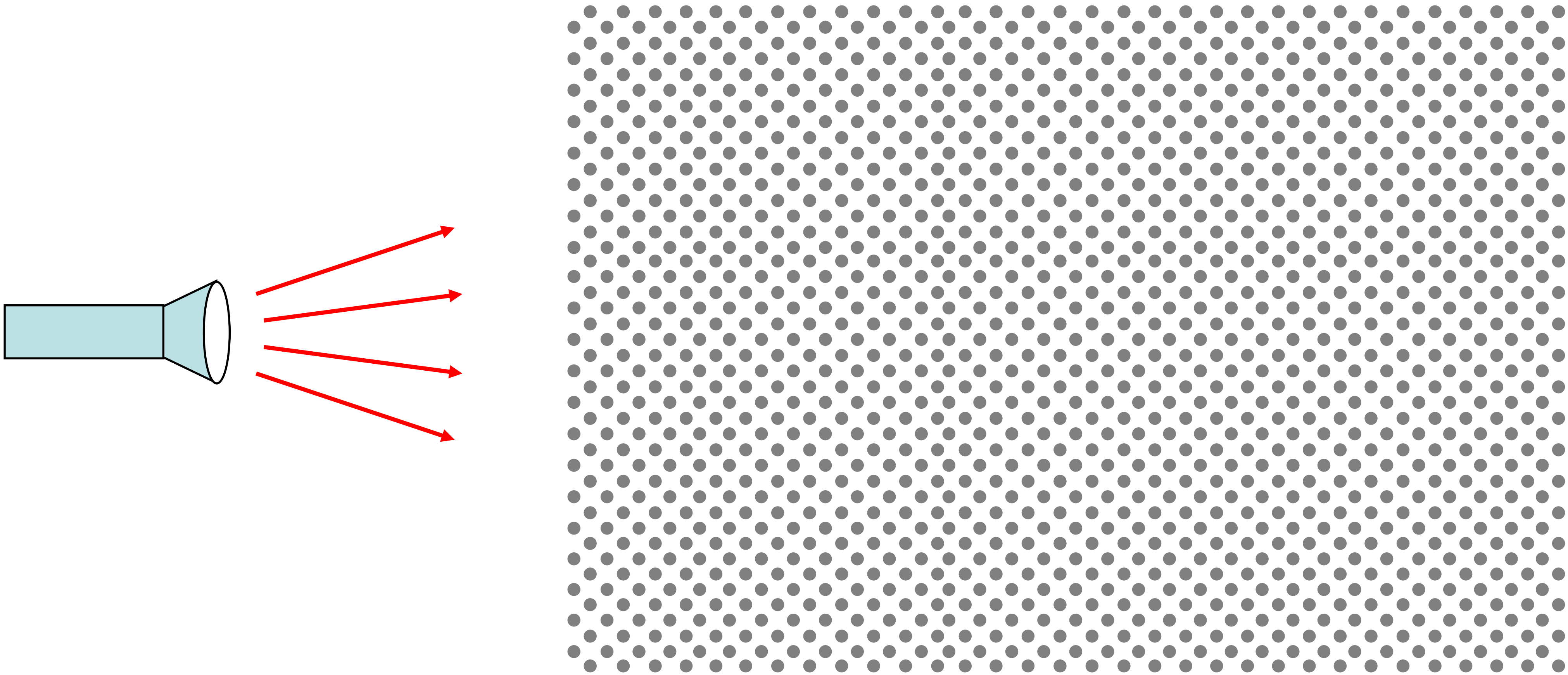
Multiple Scattering is Just Messier?

here: scattering off three specks of silicon



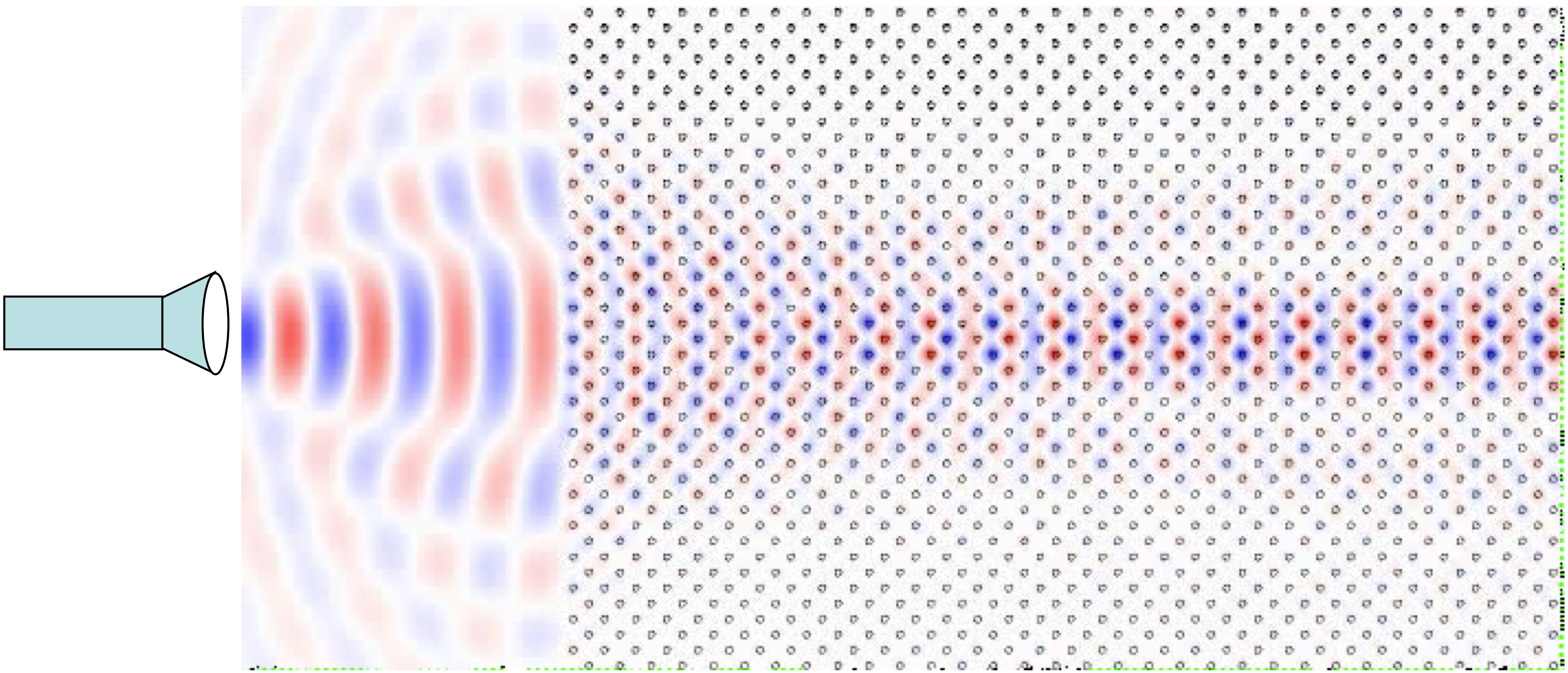
can be solved on a computer, but not terribly interesting…

An even bigger mess?
zillons of scatterers



Blech, light will just scatter like crazy
and go all over the place … how boring!

Not so messy, not so boring…



the light seems to form several *coherent beams*
that propagate *without scattering*
… and almost *without diffraction* (*supercollimation*)

…the magic of symmetry…

[Emmy Noether, 1915]

Noether's theorem:
symmetry = conservation laws

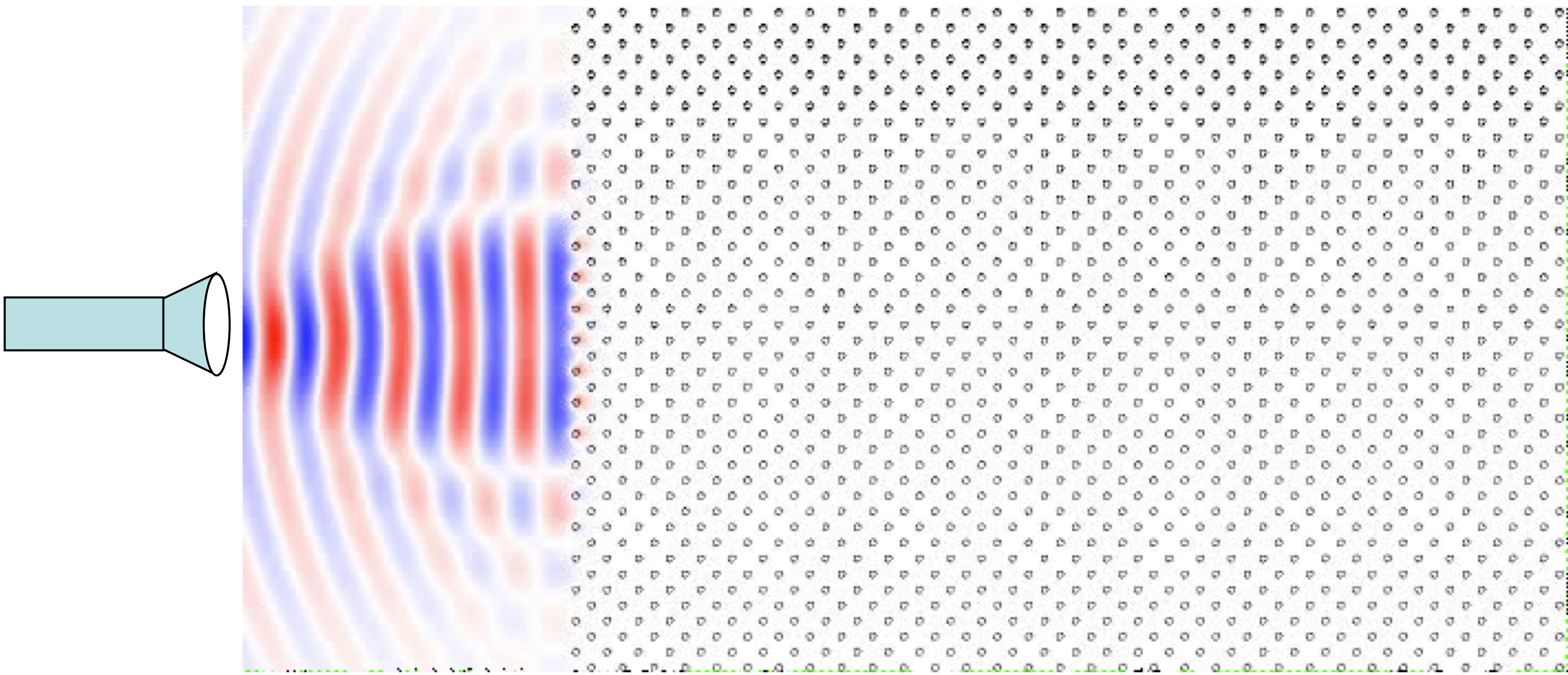
In this case, periodicity
= conserved "momentum"
= wave solutions without scattering
[Bloch waves]

Felix Bloch
(1928)

Mathematically, use *structure* of the equations, not explicit solution:
linear algebra, group theory, functional analysis, …

A slight change? Shrink λ by 20%

an "optical insulator" (photonic bandgap)



light cannot penetrate the structure at this wavelength!

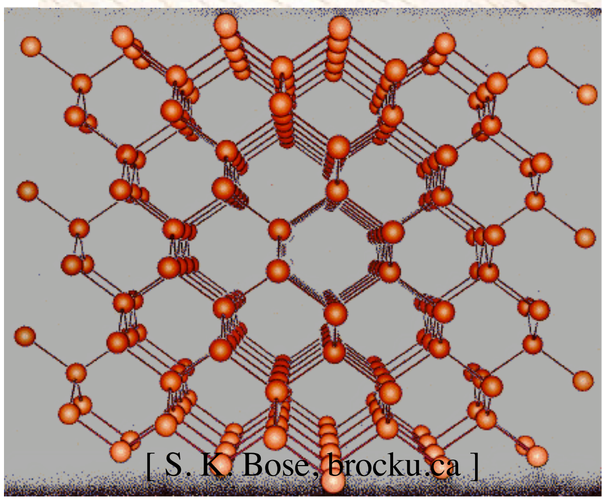
all of the scattering destructively interferes

Photonic Crystals:
periodic structures for light

crystalline minerals:
periodic atoms

silicon:

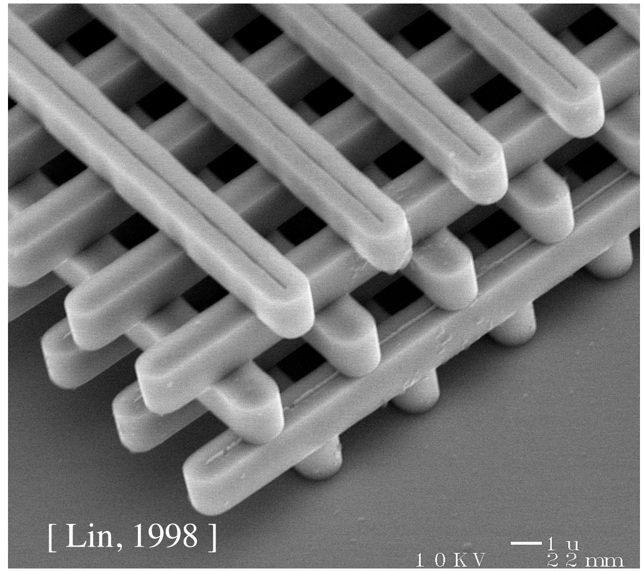
[wikipedia.org]



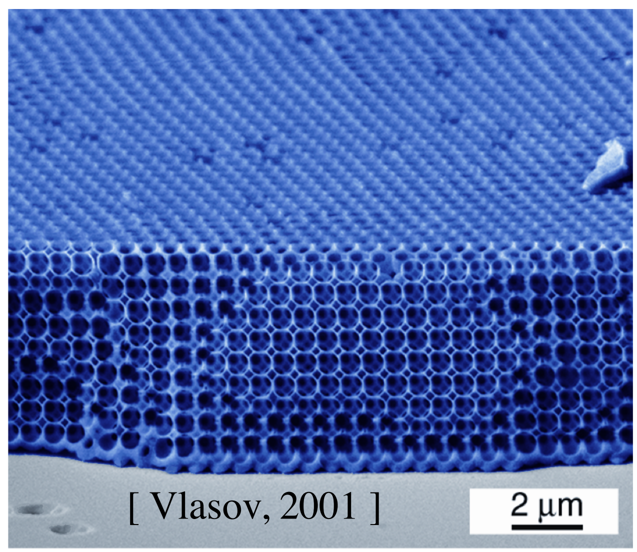
[S. K. Bose, brocku.ca]

bandgaps = insulators & semiconductors

synthetic microstructures



[Lin, 1998]



[Vlasov, 2001]

Some nice math to *understand* this…

…linear algebra:

$$\nabla \times \frac{1}{\varepsilon} \nabla \times \vec{H} = \left(\frac{\omega}{c}\right)^2 \vec{H}$$

(eigenproblem)

…symmetry ⇒ group representation theory

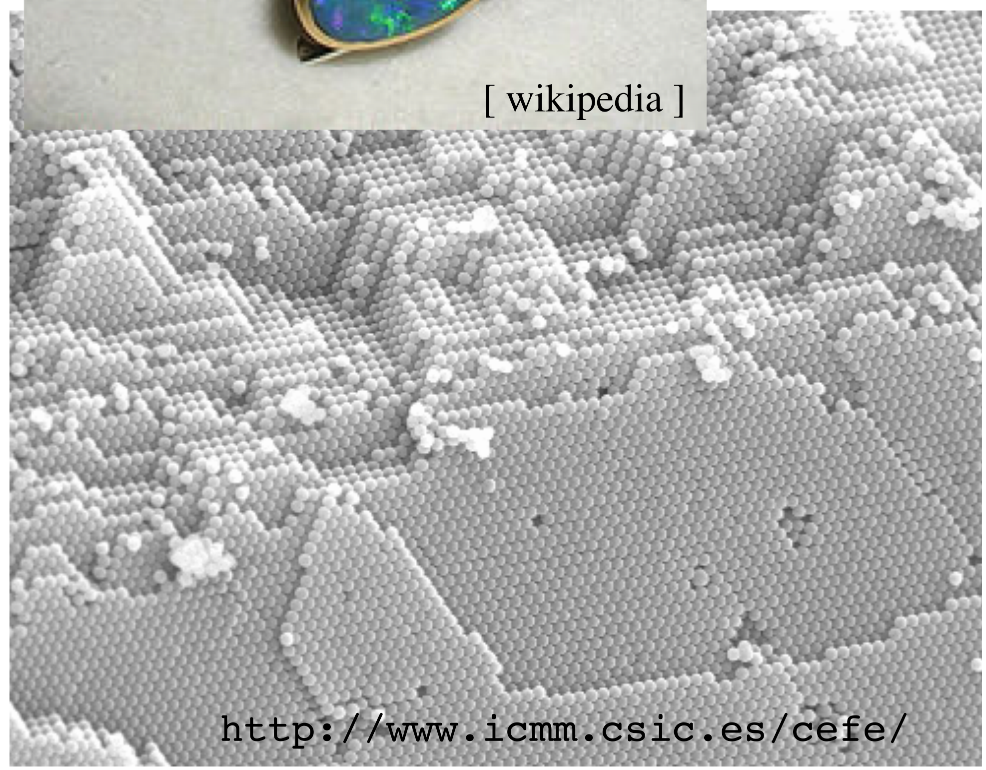
…computational methods…

…many open questions…

Structural Color in Nature

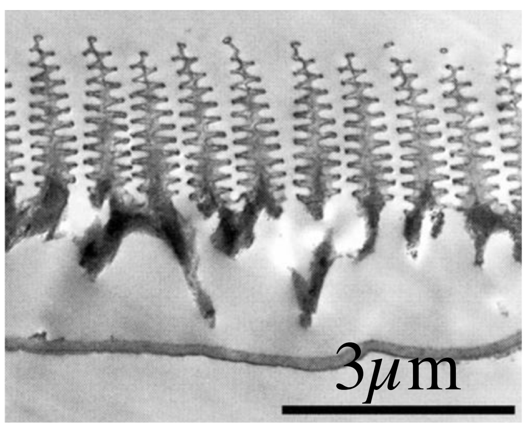
bandgap = wavelength-selective mirror = bright iridescent colors

opals



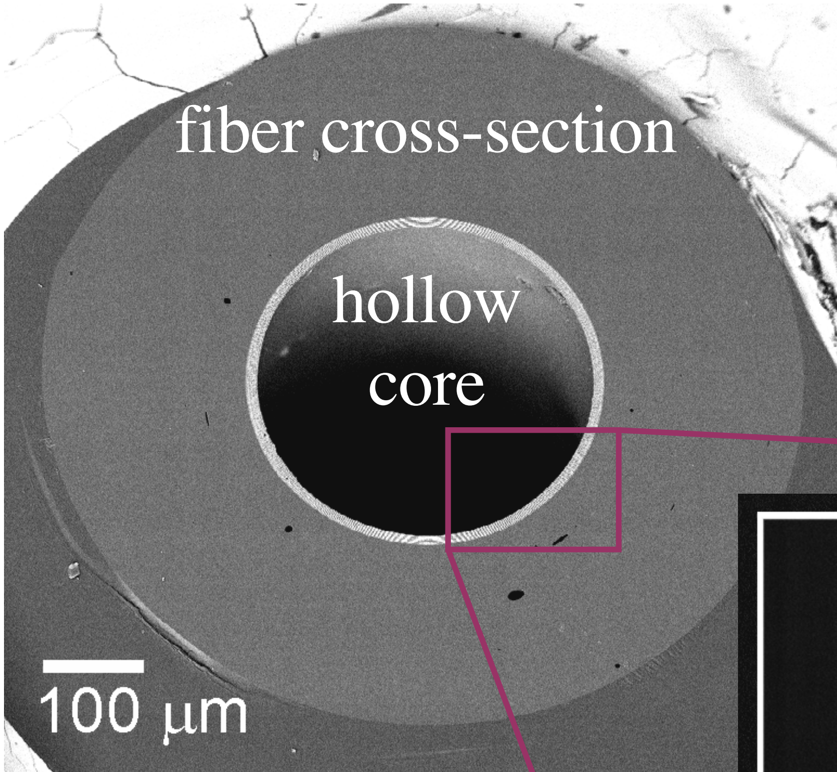
wing scale:

[P. Vukosic (1999)]

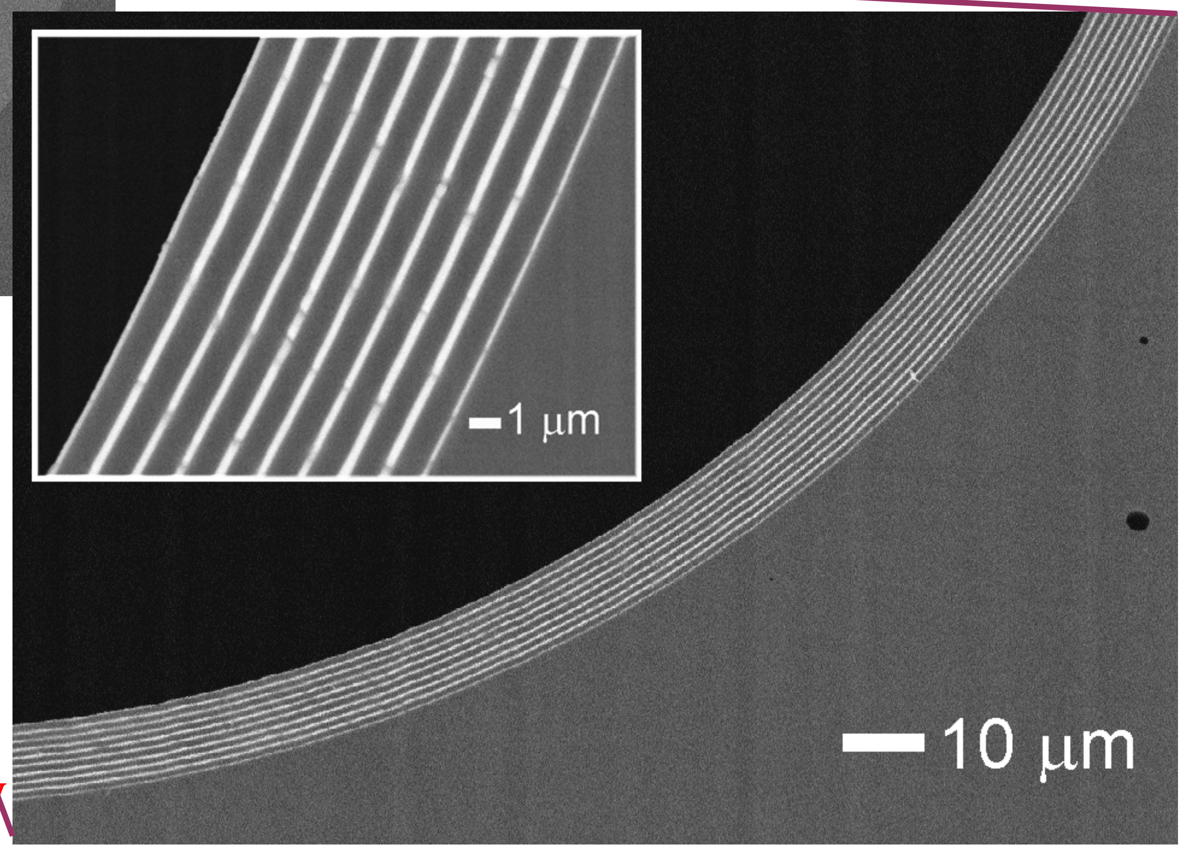


also peacocks, beetles, …

Light Confinement for Surgery



MIT startup: omni-guide.com



Periodic layering
confines laser light
that would vaporize glass
within a hollow core

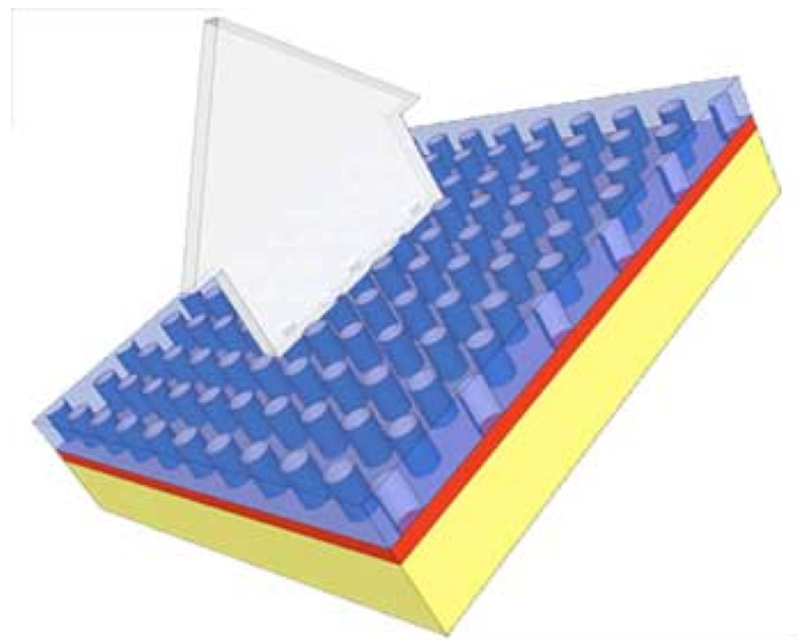
Used for endoscopic surgery

Molding Diffraction for Lighting

[another MIT startup (by a colleague): Luminus.com]

ultra-bright/efficient LEDs

periodic pattern gathers & redirects it in one direction



new projection TVs,
pocket projectors,
lighting applications,
…

Back to Maxwell, with some simplifications

- *source-free* equations (propagation of light, not creation): $\mathbf{J} = 0$, $\rho = 0$

- *Linear*, *dispersionless* (instantaneous response) materials:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \qquad \Rightarrow \qquad \mathbf{D} = \varepsilon_0 (1+\chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad \qquad \mathbf{B} = \mu_0 (1+\chi_m) \mathbf{H} = \mu_0 \mu_r \mathbf{H}$$

(nonlinearities very weak
in EM … we'll treat later)
(dispersion can be negligible
in narrow enough bandwidth)

where $\varepsilon_{\cancel{r}} = 1+\chi_e$ = relative permittivity
(drop r subscript) or dielectric constant
$\mu_{\cancel{r}} = 1+\chi_m$ = relative permeability

$\varepsilon\mu$ = (refractive index)^2

- *Isotropic* materials: ε, μ = scalars (not matrices)
- *Non-magnetic* materials: $\mu = 1$ (true at optical/infrared)
- *Lossless, transparent materials*: ε real, > 0 (< 0 for metals…bad at infrared)

Simplified Maxwell

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \cdot \varepsilon \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \varepsilon(\mathbf{x}) \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\mu_0 \mu \frac{\partial \mathbf{H}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

- Linear, time-invariant system:
  $\Rightarrow$ look for sinusoidal solutions $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$, $\mathbf{H}(\mathbf{x},t) = \mathbf{H}(\mathbf{x})e^{-i\omega t}$

  (i.e. Fourier transform)

$$\nabla \times \mathbf{H} = -i\omega\varepsilon_0 \varepsilon(\mathbf{x})\mathbf{E} \qquad \nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}$$

[note: real materials have
dispersion: ε depends on ω
= *non-instantaneous* response]

… these, we can work with

Dimensionless Maxwell

$$\nabla \times \mathbf{H} = -i\omega \cancel{\varepsilon_0} \varepsilon(\mathbf{x})\mathbf{E} \qquad \nabla \times \mathbf{E} = i\omega \cancel{\mu_0} \mathbf{H}$$

rescale the units so that ε_0 and μ_0 both = 1 ($c = 1$)

Scale-invariance:

if we rescale space by $\boldsymbol{x} \to s\boldsymbol{x}$ and $\omega \to \omega/s$

(equivalently, wavelength $\lambda = 2\pi c/\omega \to s\lambda$),

the Maxwell equations don't change

(assuming same ε and μ!)

This means we can set any convenient distance = 1.

Equivalently, pick a typical lengthscale a and write all distances (and λ) in units of a, and all frequencies ω in units of $2\pi c/a$.