Basic Setup

Nonlinear Case

$g(x,p)$ $\leftrightarrow$ Cost/Loss $\quad x \in \mathbb{R}^M$, $p \in \mathbb{R}^P$, g scalar valued

$f(x,p) = 0_M \leftrightarrow$ Constraint $\quad f$ values in $\mathbb{R}^M$

$\cancel{f_x \in \mathbb{R}^{M,M}}$

Jacobians: $g_x \in \mathbb{R}^{1,M}$ $\quad f_x \in \mathbb{R}^{M,M}$ (non-singular)

$g_p \in \mathbb{R}^{1,P}$ $\quad f_p \in \mathbb{R}^{P,M}$

$$\boxed{\begin{pmatrix} \nabla_x g \\ \nabla_p g \end{pmatrix} = \begin{pmatrix} g_x^T \\ g_p^T \end{pmatrix} = \begin{pmatrix} f_x^T \\ f_p^T \end{pmatrix} \lambda + \begin{pmatrix} 0 \\ \text{gradient} \end{pmatrix}}$$

where $\quad \text{gradient} = \left(\frac{dg}{dp}\right)^T = \nabla_p \, g(x(p), p)$

$\lambda \in \mathbb{R}^M$ known as the Lagrange Multipliers

Equivalently

$$\boxed{\begin{aligned} g_x &= \lambda^T f_x \\ g_p - \lambda^T f_p &= (\nabla_p \, g(x(p), p))^T \end{aligned}}$$

Differential Equation

$u(t),\ t \in I = [t_0, t_f]$ serves as x
$u(t) \in \mathbb{R}^M$ (M element vector function of time)

The indices of x are $i = 1, \dots, M$
The "indices" of u are $\{1, \dots, M\} \times I$

G cost/loss $\qquad G(u, p) = \int_I g(u(t), p)\, du$
F constraint $\qquad u' - f(u, p, t) = 0$
Λ lagrange-multipliers $\qquad \lambda(t) \in \mathbb{R}^M$

$\cancel{\Lambda \ \lambda(t) \to \mathbb{R}}$

Remember before g_x was a row vector that describes small changes to x
so G_u must be a linear functional

$$G_u : s(t) \in \mathbb{R}^m \longrightarrow \int_I g_u\, s\, dt$$

Before f_x was a matrix that describe small changes to x
So F_u must be a linear operator

$$F_u : s \rightarrow s' - f_u s$$

$$\Lambda^T F_u s = \int_I \lambda^T (s' - f_u s)\, dt$$

$$\Lambda^T D s = \int_I \lambda^T s'\, dt$$

$$= (s^T D^T \Lambda) = -\int_I s^T \lambda'\, dt$$

(we can make boundary terms 0 $\boxed{\lambda(t_f) = 0}$)

$$\Rightarrow \Lambda^T F_u s = -\int_I \left[(\lambda')^T + \lambda^T f_u \right] s \, dt$$

Set $G_u = \lambda^T F_u$

$$\boxed{g_u = (\lambda')^T + \lambda^T f_u}$$

or $\boxed{\lambda' = -f_u^T \lambda + g_u^T}$

Finally

$$\text{gradient}^T = G_p - \lambda^T F_p$$

$$= \boxed{\int_I g_p \, dt - \int_I \lambda^T f_p \, dt}$$