Pokerbots 2020

Lecture 3: Game Theory

Weekly Tournament: Tonight @ 11:59pm EST

Engine Updates!

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$ git pull
```

C++ Skeleton Fix Coming Soon

Games

First game

- Submit an integer between 0-1000 inclusive
- Winner is the closest guess to 2/3 of the average submission (a tie is broken randomly)
- Prize: today’s giveaway

pkr.bot/beauty

Second game

- Entry fee: wager 20 scrimmage server ELO points
- Parse the last 4 digits of your MIT ID as an integer 0-9999
- Winner is the highest integer (a tie is broken randomly)
- Prize: all wagered ELO points

pkr.bot/lemons

Agenda

- What is a game?
- Pure and mixed strategies
- Nash equilibria
- Applications to poker

What is a game?

Definition

We generally only consider two-player zero-sum games; we will refer to these as simply *games*.

Examples: RPS, tic-tac-toe, (chess? poker??)

For Blotto Hold'em, the game is technically non zero-sum, so there are a few more specific considerations.

Formalized

A *game* between players 1 and 2 consists of a pair of strategy sets S_1 and S_2, and a utility function $u : S_1 \times S_2 \longrightarrow \mathbf{R}$. Players submit strategies simultaneously. Player 1 seeks to maximize u, and player 2 seeks to minimize u.

Think of utility as chips. This means players want to submit actions that have the highest payout.

Player 2 is minimizing since we're talking about zero-sum games.

Pure and mixed strategies

Pure and mixed strategies

- There are only 3 pure strategies in RPS, but infinitely many mixed strategies

- A mixed strategy is described by a probability distribution over pure strategies:
    - Example:
    - P(rock) = 0.4
    - P(paper) = 0.3
    - P(scissors) = 0.3

- Mixed strategies show us the power of randomness

Matrix form games

Suppose the strategy sets S_1 and S_2 consist of probability distributions over a finite list of pure strategies. Also, suppose that the utility function is linear. Then, we can write our game in *matrix form*, with utility function shown.

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0

Revisiting the first game

The Keynesian beauty contest

- No rational player would play above $2/3 \cdot 1000 = 667$
- Hence, no rational player would play above $2/3 \cdot 667 = 445$
- What happens if we continue this logic?
- What would "rational" players play?

Dominance

- We say that strategy A *dominates* strategy B if playing A is always a better idea
- u(A, O) ≥ u(B, O) for all player 2 strategies O
- Second-order dominance: replace “is always a better idea” with “is always a better idea, if our opponent does not play dominated strategies”

Dominance Example

	C (c)	D (d)
A (a)	(+3, -3)	(+2, -2)
B (b)	(+1, -1)	(-1, 1)

	C (c)	D (d)
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What does this mean?

Nash equilibria

An *equilibrium* is a set of strategies, one for each player, such that nobody has an incentive to switch.

Nash equilibrium examples

- 0 is an equilibrium for the Keynesian beauty contest

- 1/3 rock, 1/3 paper, 1/3 scissors is an equilibrium for RPS

Solving the equilibrium for RPS

$r + p + s = 1$; $r, p, s \geq 0$

Expected value of opponent playing each case

	Rock (r)	Paper (p)	Scissors (s)
Rock (r)	0	-1	+1
Paper (p)	+1	0	-1
Scissors (s)	-1	+1	0
EV | Opponent	p-s	s-r	r-p

Solve $p - s = s - r = r - p$ to get $r = p = s = 1/3$. This guarantees us at least 0

Why p - s = s - r = r - p?

- Suppose opponent plays the optimal counterstrategy against our r, p, s strategy

- Opponent minimizes the utility function amongst their three options:

$$\min_{\text{args}}(p - s, s - r, r - p)$$

- We play the strategy that maximizes the utility function *even against the optimal counterstrategy*:

$$\max_{r,\,p,\,s}(\min_{\text{args}}(p - s, s - r, r - p))$$

Asymmetric game

Toy game for illustration: I am choosing battle plans against the enemy

	Full Defense	Defend in shifts
Charge	0	+3
Sneak attack	+1	-1

We notice two things: the game isn't symmetric, and that we see a lot of positive values.

Asymmetric game mixed strategy equilibrium

Setting this up, we start with $c + s = 1$; $c, s \geq 0$

	Full Defense (f)	Defend in shifts (d)	EV | Opponent
Charge (c)	0	+3	3d
Sneak attack (s)	+1	-1	f-d
EV | Opponent	s	3c-s	:)

Solving, we get $c = 0.4$, $s = 0.6$, $f = 0.8$, $d = 0.2$

"Value" of the game to us is 0.6

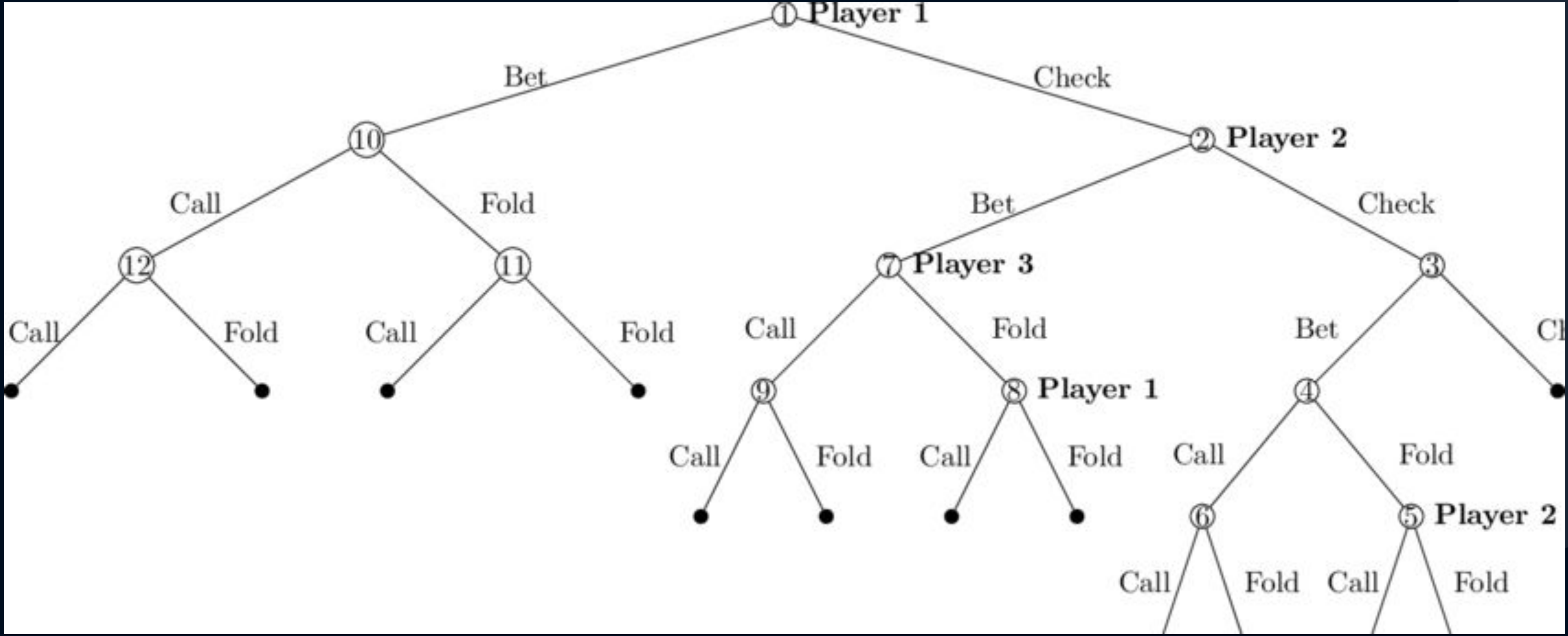
Existence of Nash equilibria

Theorem (Nash, 1951): Every finite game has a Nash equilibrium in mixed strategies.

Proof: Topology (uses fixed point theorem).

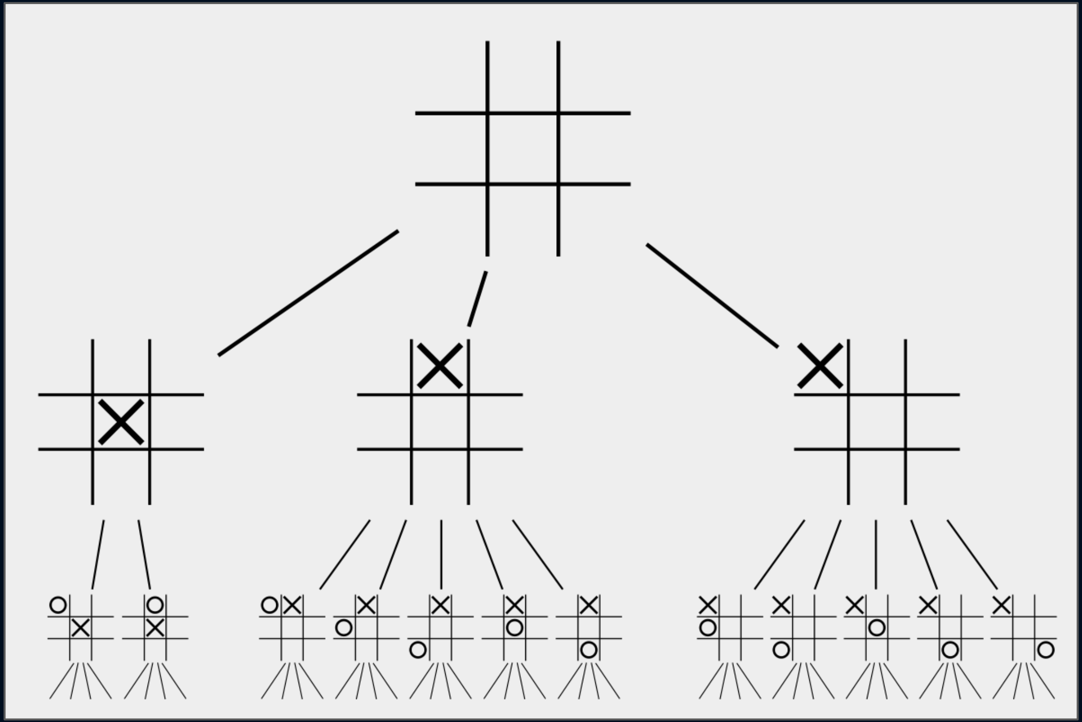
Applications to poker

Game Tree

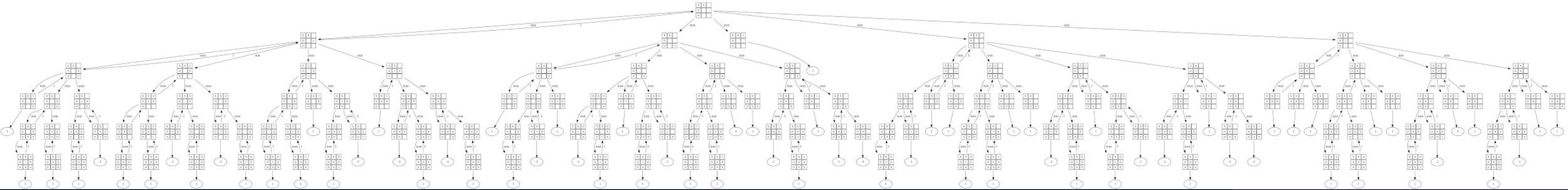


1 Player 1
Bet
Check
10
2 Player 2
Call
Fold
Bet
Check
12
11
7 Player 3
3
Call
Fold
Call
Fold
Call
Fold
Bet
9
8 Player 1
4
Call
Fold
Call
Fold
Call
Fold
6
5 Player 2
Call
Fold
Call
Fold

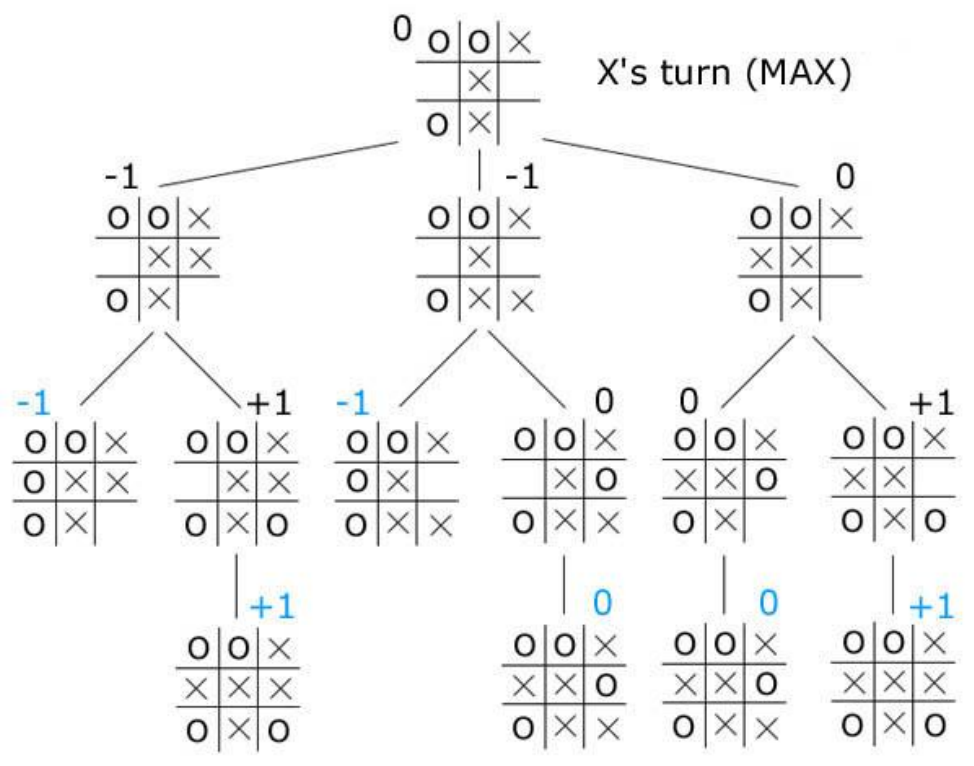
Extensive form games



Tic Tac Toe Full Extensive Form



Backwards induction

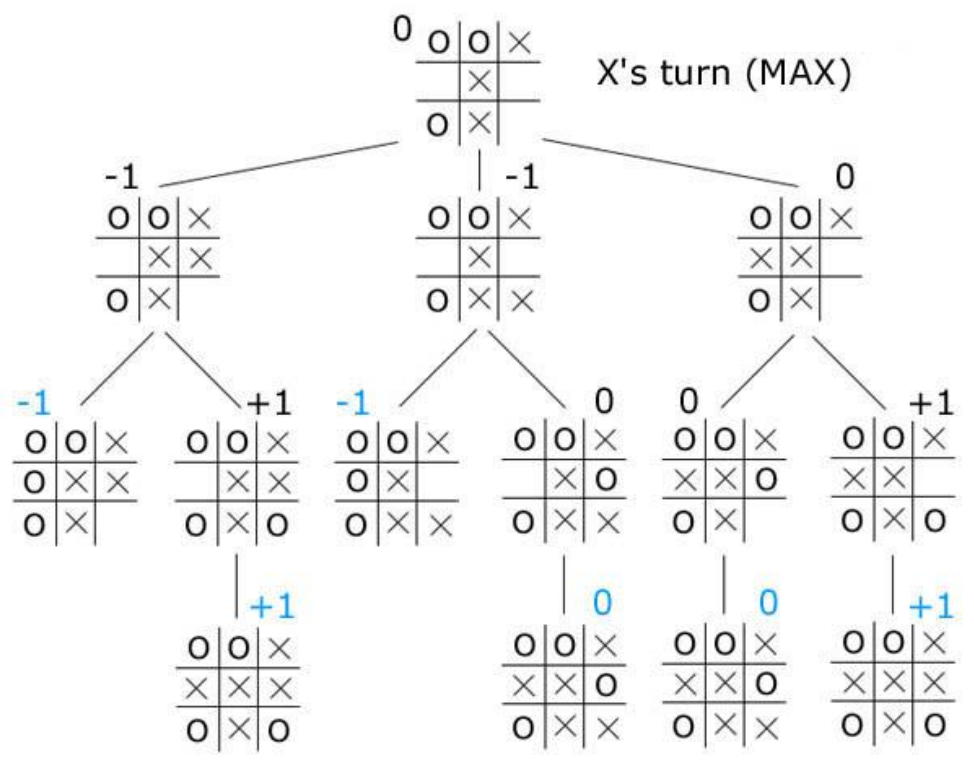


Black are the determined payouts based on players acting rationally

Blue are the leaves - the terminal states of the game

Follow the game tree up (backwards) to find the state's value

Backwards induction



Looking at the leaf states first, I see -1s in the left 2 branches, leaving their parents as -1.

The rightmost branch has payouts 0 and 1. Following this up the tree, we get 0 for the black value.

Finally, all together X should try to play for 0 (tie), and does so by putting x in left center.

Conversion to normal form

- Every extensive form is also a normal form (matrix) game

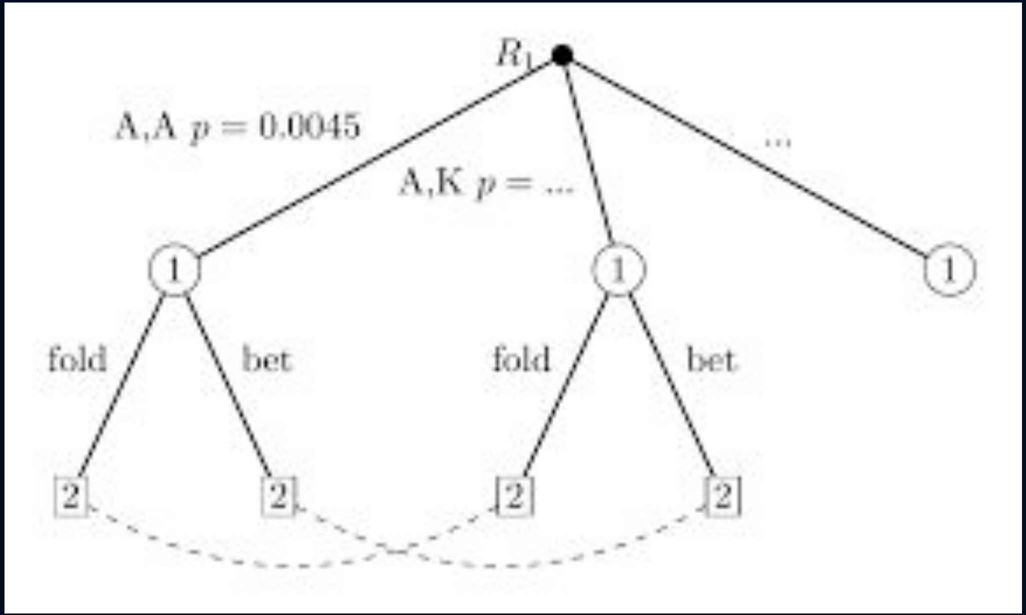
Conversion to normal form

- Every extensive form is also a normal form (matrix) game
- Mapping: each pure strategy (row/column) is a complete set of instructions for what to do at each node
- Example: in poker, each pure strategy would be a dictionary which maps the game_state to a unique action

Implications

- Poker has a Nash equilibrium - it's a finite game that satisfies the conditions
- Can we use backwards induction to solve for it? Can we still use the same type of game tree?
- How is poker different from tic-tac-toe? RPS? Chess?

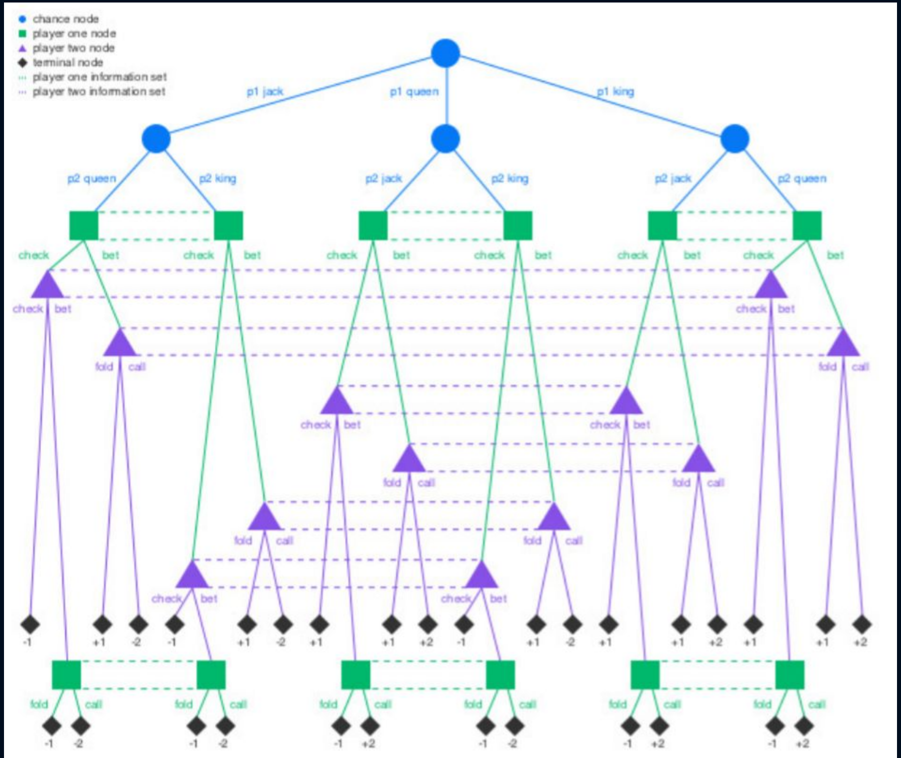
Adding in imperfect information and randomness



Imperfect Information and Randomness

- Randomness: Don't know where in the game tree you're moving next
- Imperfect: Don't know what node you're currently at since your opponent's hole cards are hidden (information sets)
- Together, this leads to a really nice property of poker. The general game tree movements (bet/fold/call) are known, but the definite game state isn't. Imperfect information makes games fun!

Kuhn Poker Game Tree



Poker Nash equilibrium and use

- Poker has a Nash equilibrium that has EV 0
- The matrix is doubly exponential in size - incomputable
- Does “play good poker” reduce to “play the equilibrium strategy”?
- Exploitative strategies as alternatives

Variant Specific Implications

Non Zero-Sum Game

This game is non zero sum. A very simplified matrix form is below.

	Fold	Check/Call	Raise
Fold	-1	-1	-1
Check/Call	x	0	-1.5
Raise	+6	+3	0

The main difference here is the larger potential gains on raises, especially when the opponent folds.

Applied game theory

Revisiting the second game

The MIT ID game

- Suppose rational players would only play if their number is ≥X
    - Students might submit only if ≥9000

- But if only ≥X is submitted, then X is the smallest value and shouldn't be submitted
    - Why submit 9000?

- Continue this logic (nth order dominance)
    - Why submit 9001? 9002?

Adverse selection

- Occurs anytime “buyer” and “seller” have asymmetric information
- Market for cars: suppose there are used cars with private value distributed uniformly between \$1000 and \$10000. What should the market price be?
- What would car owners do? Then what happens?

Sources of adverse selection in poker

- First action when betting (differential check to the preflop aggressor)
- Multiple bets in a row (what types of hands do this?)
- Being in the later stages of the round (why would are they still in the pot?)
- Others?

When I bet, am I happy with my bet *conditioned on my opponent calling?*

Adverse selection and determinism

- Suppose I have the following deterministic betting strategy:
- All-in when I have a top X% hand
- How can my opponent exploit me?

Coding Reference-3 Bot

Beauty Contest Winner

MIT ID Contest Winner