Pokerbots 2024

Lecture 3: Game Theory

Sponsors

Announcements

- NO LECTURE Next Monday (MLK Day). Lecture will resume Wednesday 1/17.

- Poker Afternoon was great! Another one happening Tuesday 1/23.

- Last lecture's resume raffle winner: Leo Yao

- Team Forming

Weekly Tournament:
Bots Tonight @ 11:59pm EST

Prizes:
Winner: $1000
Biggest Upset: $500

Giveaway Games!

First game

- Submit an integer between 0-1000 inclusive

- Winner is the closest guess to 2/3 of the average submission (a tie is broken randomly)

- Prize: GTO Wizard Elite

pkr.bot/beauty

Second game

- 'Entry fee': your kerb gets blacklisted from winning raffles for the next week

- Parse the last 4 digits of your MIT ID as an integer 0-9999 (we can check for lying)

- Winner is the highest integer (a tie is broken randomly)

- Prize: Beats Studio Pro

pkr.bot/lemons

Agenda

- What is a game?

- Pure and mixed strategies

- Nash equilibria

- Applications to poker

What is a game?

Definition

We generally only consider two-player zero-sum games; we will refer to these as simply *games*.

Examples: RPS, tic-tac-toe, (chess? poker??)

Formalized

A *game* between players 1 and 2 consists of a pair of strategy sets S_1 and S_2, and a utility function $u : S_1 \times S_2 \to \mathbf{R}$. Players submit strategies simultaneously. Player 1 seeks to maximize u, and player 2 seeks to minimize u.

Think of utility as chips. This means players want to submit actions that have the highest payout.

Player 2 is minimizing since we're talking about zero-sum games.

Pure and mixed strategies

Pure and mixed strategies

- There are only 3 pure strategies in RPS, but infinitely many mixed strategies

- A mixed strategy is described by a probability distribution over pure strategies:
  - Example:
  - P(rock) = 0.4
  - P(paper) = 0.3
  - P(scissors) = 0.3

- Mixed strategies show us the power of randomness

Matrix form games

Suppose the strategy sets S_1 and S_2 consist of probability distributions over a finite list of pure strategies. Also, suppose that the utility function is linear. Then, we can write our game in *matrix form*, with utility function shown.

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0

Revisiting the first game

The Keynesian beauty contest

- No rational player would play above $2/3 \cdot 1000 = 667$

- Hence, no rational player would play above $2/3 \cdot 667 = 445$

- What happens if we continue this logic?

- What would "rational" players play?

Dominance

- We say that strategy A *dominates* strategy B if playing A is always a better idea

- $u(A, O) \geq u(B, O)$ for all player 2 strategies O

- Second-order dominance: replace "is always a better idea" with "is always a better idea, if our opponent does not play dominated strategies"

Dominance Example

	C (c)	D (d)
A (a)	(+3, -3)	(+2, -2)
B (b)	(+1, -1)	(-1, 1)

	C (c)	D (d)
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What does this mean?

Nash equilibria

An *equilibrium* is a set of strategies, one for each player, such that nobody has an incentive to switch.

Nash equilibrium examples

- 0 is an equilibrium for the Keynesian beauty contest

- 1/3 rock, 1/3 paper, 1/3 scissors is an equilibrium for RPS

Solving the equilibrium for RPS

r + p + s = 1; r, p, s ≥ 0

Expected utility given each case of opponent's possible play

	Rock (r')	Paper (p')	Scissors (s')
Rock (r)	0	-1	+1
Paper (p)	+1	0	-1
Scissors (s)	-1	+1	0
EV	Opponent	p-s	s-r	r-p

Solve p - s = s - r = r - p to get r = p = s = 1/3. This guarantees us at least 0

Why p − s = s − r = r − p?

- Suppose opponent plays the optimal r', p', s' counterstrategy against our r, p, s strategy

- Opponent minimizes the utility function amongst their three options:

$$\min_{r',\, p',\, s'}(p - s, s - r, r - p)$$

- We play the strategy that maximizes the utility function *even against the optimal counterstrategy*:

$$\max_{r,\, p,\, s}(\min_{r',\, p',\, s'}(p - s, s - r, r - p))$$

Asymmetric game

Toy game for illustration: I am choosing battle plans against the enemy

	Full Defense	Defend in shifts
Charge	0	+3
Sneak attack	+1	-1

We notice two things: the game isn't symmetric, and that we see a lot of positive values.

Asymmetric game mixed strategy equilibrium

Setting this up, we start with c + s = 1; c, s ≥ 0

	Full Defense (f)	Defend in shifts (d)	EV | Opponent
Charge (c)	0	+3	3d
Sneak attack (s)	+1	-1	f-d
EV | Opponent	s	3c-s	

Solving, we get c = 0.4, s = 0.6, f = 0.8, d = 0.2

"Value" of the game to us is 0.6

Existence of Nash equilibria

Theorem (Nash, 1951): Every finite game has a Nash equilibrium in mixed strategies.

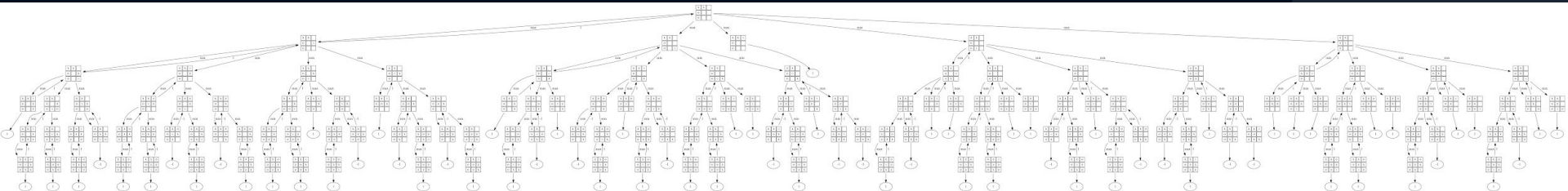Proof: Topology (uses Brouwer fixed-point theorem).
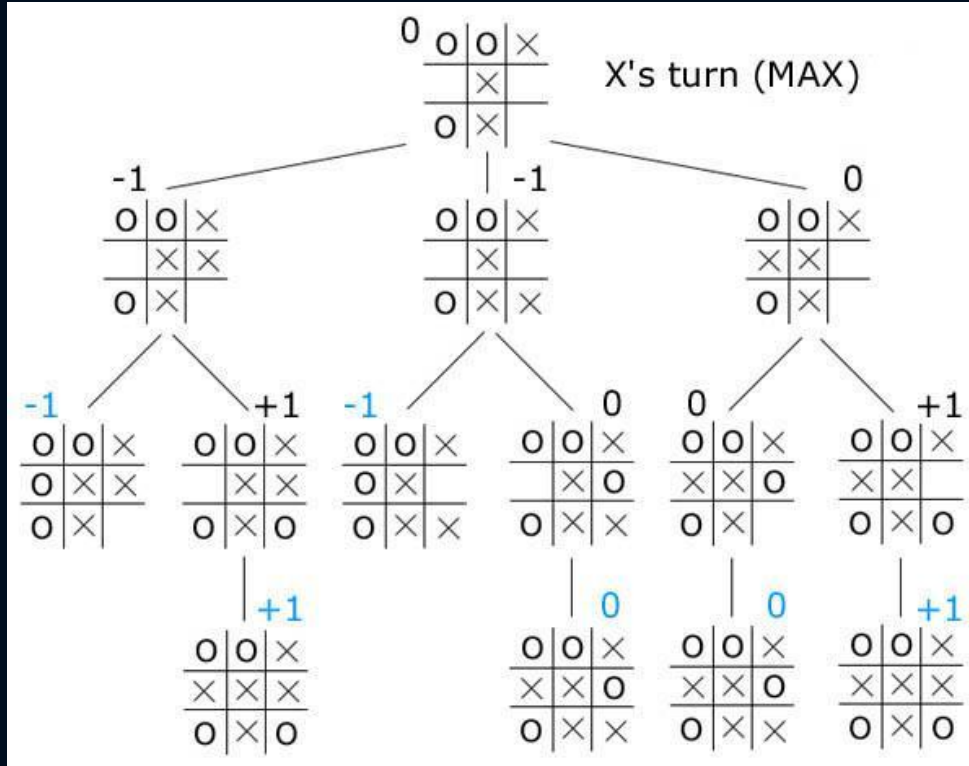
Applications to poker

Game Tree
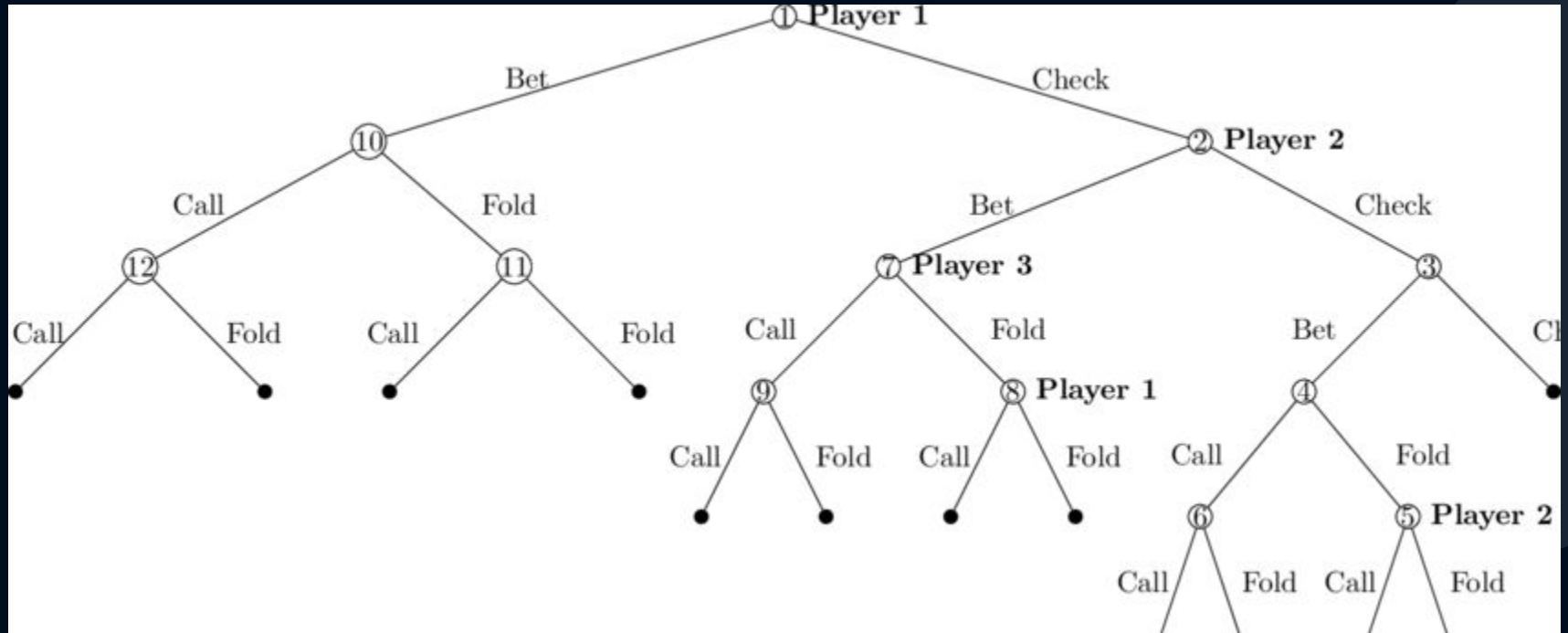
Extensive form games
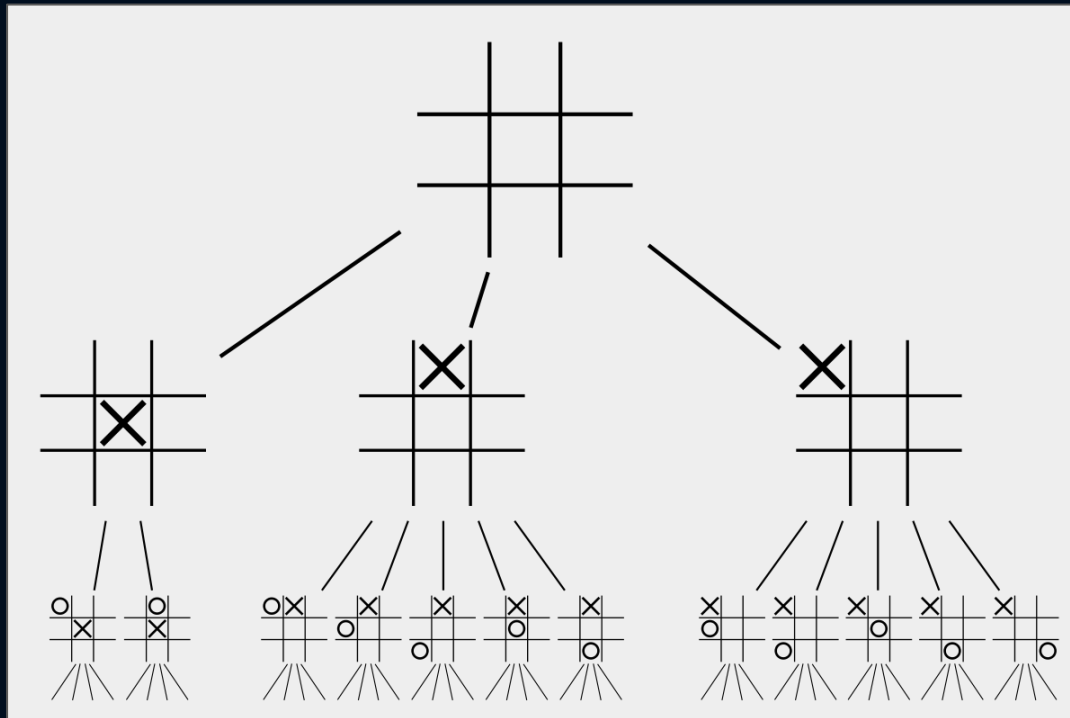
Tic Tac Toe Full Extensive Form
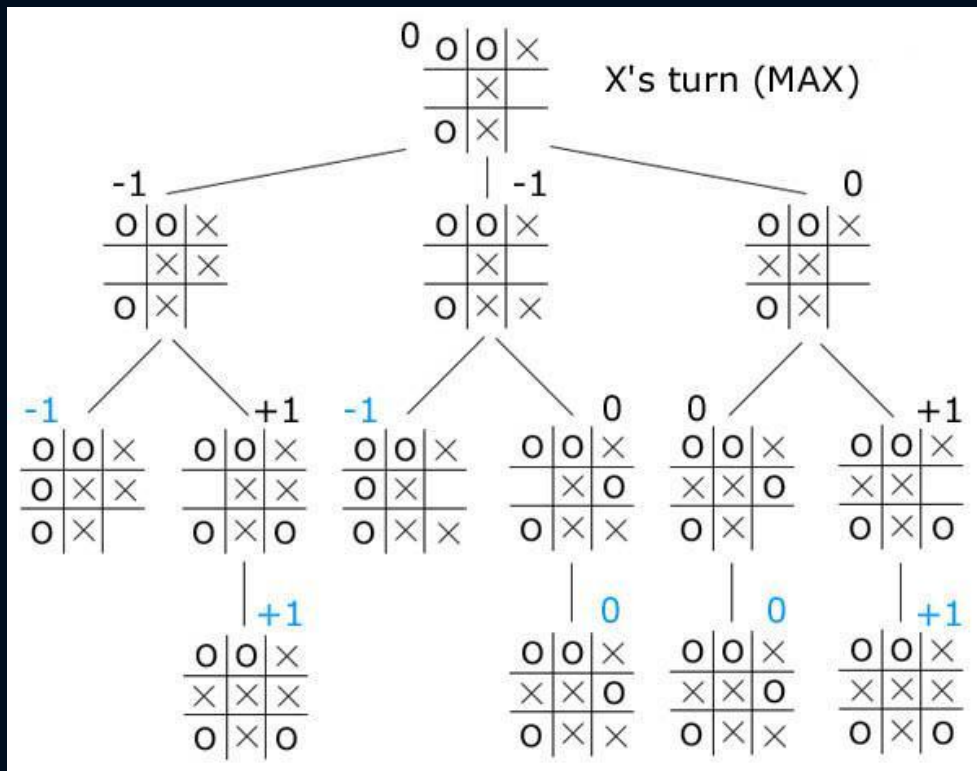
Backwards induction

Black are the determined payouts based on players acting rationally

Blue are the leaves - the terminal states of the game

Follow the game tree up (backwards) to find the state's value

Backwards induction

Looking at the leaf states first, I see -1s in the left 2 branches, leaving their parents as -1.

The rightmost branch has payouts 0 and 1. Following this up the tree, we get 0 for the black value.

Finally, all together X should try to play for 0 (tie), and does so by putting x in left center.

Conversion to normal form

- Every extensive form is also a normal form (matrix) game

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- Mapping: each pure strategy (row/column) is a complete set of instructions for what to do at each node

- Example: in poker, each pure strategy would be a dictionary which maps the game_state to a unique action

Implications

- Poker has a Nash equilibrium - it's a finite game that satisfies the conditions

- Can we use backwards induction to solve for it? Can we still use the same type of game tree?

- How is poker different from Chess? tic-tac-toe? RPS?

Implications

- Poker has a Nash equilibrium - it's a finite game that satisfies the conditions

- Can we use backwards induction to solve for it? Can we still use the same type of game tree?

- How is poker different from Chess? tic-tac-toe? RPS?
  - Size
  - Imperfect Information
  - Randomness

Adding in imperfect information and randomness
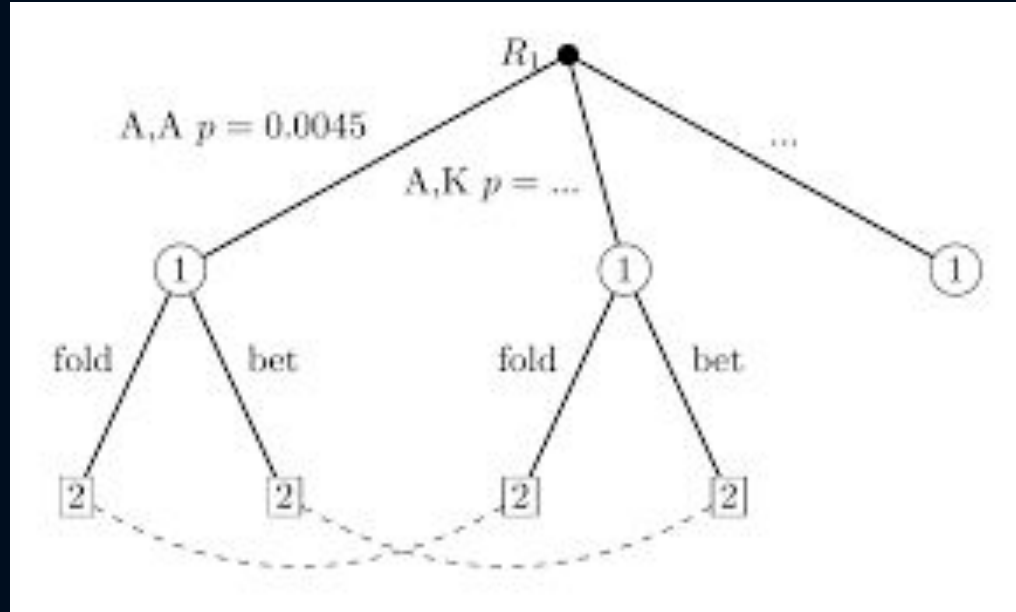
Imperfect Information and Randomness

- Imperfect: Don't know what node you're currently at since your opponent's hole cards are hidden (information sets)

- Randomness: Don't know where in the game tree you're moving next since future cards to be dealt are unknown

- Together, this leads to a really nice property of poker. The general game tree movements (bet/fold/call) are known, but the definite game state isn't. Imperfect information makes games fun!

Kuhn Poker

Kuhn poker is a simplified form of poker developed by Harold W. Kuhn as a simple model zero-sum two-player imperfect-information game, amenable to a complete game-theoretic analysis. In Kuhn poker, the deck includes only three playing cards, for example a King, Queen, and Jack. One card is dealt to each player, which may place bets similarly to a standard poker. If both players bet or both players pass, the player with the higher card wins, otherwise, the betting player wins.

Game description [edit]

In conventional poker terms, a game of Kuhn poker proceeds as follows:

- Each player antes 1.
- Each player is dealt one of the three cards, and the third is put aside unseen.
- Player one can check or bet 1.
  - If player one checks then player two can check or bet 1.
    - If player two checks there is a showdown for the pot of 2 (i.e. the higher card wins 1 from the other player).
    - If player two bets then player one can fold or call.
      - If player one folds then player two takes the pot of 3 (i.e. winning 1 from player 1).
      - If player one calls there is a showdown for the pot of 4 (i.e. the higher card wins 2 from the other player).
  - If player one bets then player two can fold or call.
    - If player two folds then player one takes the pot of 3 (i.e. winning 1 from player 2).
    - If player two calls there is a showdown for the pot of 4 (i.e. the higher card wins 2 from the other player).

Kuhn Poker Game Tree
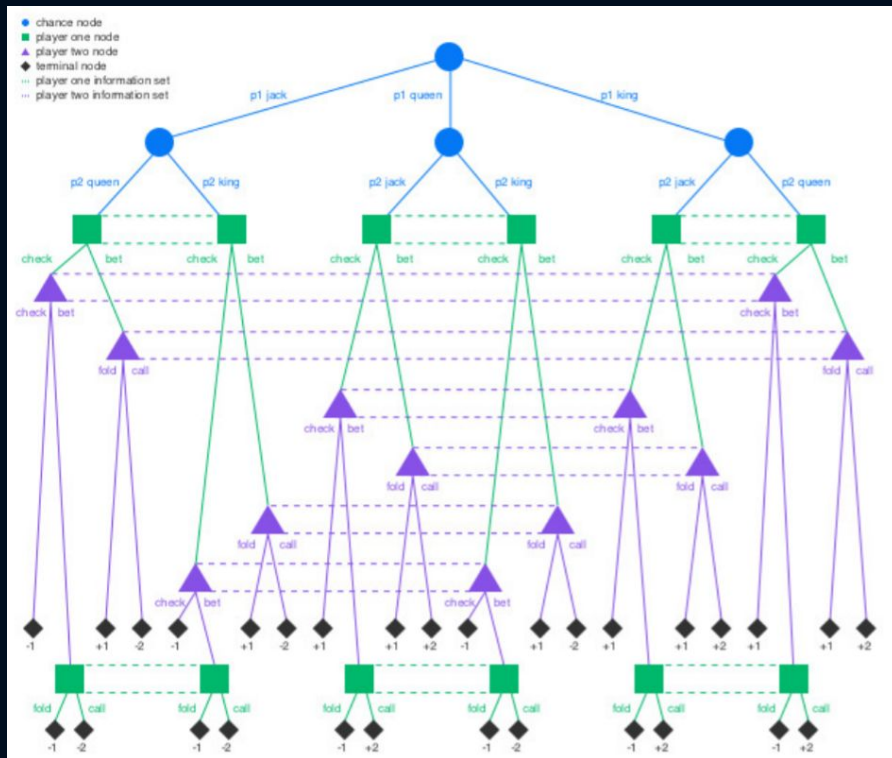
Poker Nash equilibrium and use

- Poker has a Nash equilibrium that has EV 0

- The matrix is doubly exponential in size - incomputable

- Does "play good poker" reduce to "play the equilibrium strategy"?

- Exploitative strategies as alternatives

Applied game theory

The MIT ID game

- Suppose rational players would only play if their number is ≥X
  - Students might submit only if ≥9000

- But if only ≥X is submitted, then X is the smallest value and shouldn't be submitted
  - Why submit 9000?

- Continue this logic (nth order dominance)
  - Why submit 9001? 9002?

Adverse selection

- Occurs anytime "buyer" and "seller" have asymmetric information

- Market for cars: suppose there are used cars with private value distributed uniformly between $1000 and $10000. What should the market price be?

- What would car owners do? Then what happens?

Sources of adverse selection in poker

- First action when betting (differential check to the preflop aggressor)

- Multiple bets in a row (what types of hands do this?)

- Being in the later stages of the round (why would are they still in the pot?)

- Others?

When I bet, am I happy with my bet *conditioned on my opponent calling*?

Adverse selection and determinism

- Suppose I have the following deterministic betting strategy:

- All-in when I have a top X% hand

- How can my opponent exploit me?

Second-Price Auctions

Second-Price (Vickrey) Auction

- Two players submit sealed bids b_1, b_2 for an object that they value at s_1, s_2. The bidder with the highest bid is awarded the object at the second-highest price.

- It can be shown that it is a weakly dominant strategy for each player to bid according to how much they value the object: $b_i = s_i$

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  - Let's assume we're Player 1

  - If $b_2 > s_1$, then winning is not worth it because we'd be paying more than the object's worth. So the best we can do is lose the auction, and bidding $b_1 = s_1$ achieves this.

  - If $b_2 <= s_1$, then winning is worth it, and the price we pay is fixed regardless of our bid. In this case, bidding $b_1 = s_1$ always wins us the auction.

Auction Hold'em is not exactly the same

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- We care about how much the opponent pays if they win (more in the pot)

- Value of extra card is not well defined

- When we bid (or take any action in general), we're sending information

- So what do we do?

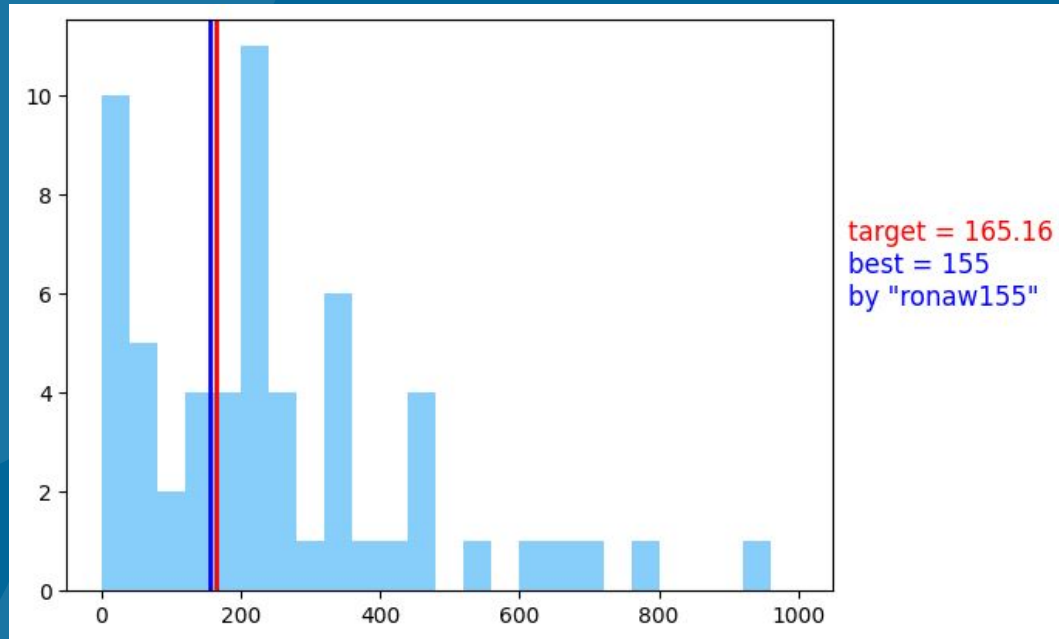# Auction Hold'em is not exactly the same

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- So what do we do?

  - Exploitative strategies

  - Can still bid on a value-based metric (ref bot 2)

  - Incorporate randomness to hide information

Game Theory Takeaways

- Try to be one step ahead of your opponent (dominance)

- Equilibrium strategies can be good but hard to find

- Exploitative strategies can be very powerful - think about what you're trying to optimize for when making a pokerbot

- Be careful of adverse selection

Beauty Contest Winner:

target = 165.16
best = 155
by "ronaw155"

MIT ID Contest Winner

best = 9848
by "vincy"