Pokerbots 2024

Lecture 3: Game Theory

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Announcements

- NO LECTURE Next Monday (MLK Day). Lecture will resume Wednesday 1/17.
- Poker Afternoon was great! Another one happening Tuesday 1/23.
- Last lecture’s resume raffle winner: Leo Yao
- Team Forming

Weekly Tournament:
Bots Tonight @ 11:59pm EST

Prizes:
Winner: $1000
Biggest Upset: $500

Giveaway Games!

First game

- Submit an integer between 0-1000 inclusive
- Winner is the closest guess to 2/3 of the average submission (a tie is broken randomly)
- Prize: GTO Wizard Elite

pkr.bot/beauty

Second game

- ‘Entry fee’: your kerb gets blacklisted from winning raffles for the next week
- Parse the last 4 digits of your MIT ID as an integer 0-9999 (we can check for lying)
- Winner is the highest integer (a tie is broken randomly)
- Prize: Beats Studio Pro

pkr.bot/lemons

Agenda

- What is a game?
- Pure and mixed strategies
- Nash equilibria
- Applications to poker

What is a game?

Definition

We generally only consider two-player zero-sum games; we will refer to these as simply *games*.

Examples: RPS, tic-tac-toe, (chess? poker??)

Formalized

A *game* between players 1 and 2 consists of a pair of strategy sets S_1 and S_2, and a utility function $u : S_1 \times S_2 \longrightarrow \mathbf{R}$. Players submit strategies simultaneously. Player 1 seeks to maximize u, and player 2 seeks to minimize u.

Think of utility as chips. This means players want to submit actions that have the highest payout.

Player 2 is minimizing since we're talking about zero-sum games.

Pure and mixed strategies

Pure and mixed strategies

- There are only 3 pure strategies in RPS, but infinitely many mixed strategies

- A mixed strategy is described by a probability distribution over pure strategies:
    - Example:
    - P(rock) = 0.4
    - P(paper) = 0.3
    - P(scissors) = 0.3

- Mixed strategies show us the power of randomness

Matrix form games

Suppose the strategy sets S_1 and S_2 consist of probability distributions over a finite list of pure strategies. Also, suppose that the utility function is linear. Then, we can write our game in *matrix form*, with utility function shown.

	Rock	Paper	Scissors
Rock	0	-1	+1
Paper	+1	0	-1
Scissors	-1	+1	0

Revisiting the first game

The Keynesian beauty contest

- No rational player would play above $2/3 \cdot 1000 = 667$
- Hence, no rational player would play above $2/3 \cdot 667 = 445$
- What happens if we continue this logic?
- What would "rational" players play?

Dominance

- We say that strategy A *dominates* strategy B if playing A is always a better idea
- u(A, O) ≥ u(B, O) for all player 2 strategies O
- Second-order dominance: replace "is always a better idea" with "is always a better idea, if our opponent does not play dominated strategies"

Dominance Example

	C (c)	D (d)
A (a)	(+3, -3)	(+2, -2)
B (b)	(+1, -1)	(-1, 1)

	C (c)	D (d)
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What does this mean?

Nash equilibria

An *equilibrium* is a set of strategies, one for each player, such that nobody has an incentive to switch.

Nash equilibrium examples

- 0 is an equilibrium for the Keynesian beauty contest
- 1/3 rock, 1/3 paper, 1/3 scissors is an equilibrium for RPS

Solving the equilibrium for RPS

$r + p + s = 1; r, p, s \geq 0$

Expected utility given each case of opponent's possible play

	Rock (r')	Paper (p')	Scissors (s')
Rock (r)	0	-1	+1
Paper (p)	+1	0	-1
Scissors (s)	-1	+1	0
EV | Opponent	p-s	s-r	r-p

Solve $p - s = s - r = r - p$ to get $r = p = s = 1/3$. This guarantees us at least 0

Why p - s = s - r = r - p?

- Suppose opponent plays the optimal r', p', s' counterstrategy against our r, p, s strategy

- Opponent minimizes the utility function amongst their three options:

$$\min_{r', p', s'}(p - s, s - r, r - p)$$

- We play the strategy that maximizes the utility function *even against the optimal counterstrategy*:

$$\max_{r, p, s}(\min_{r', p', s'}(p - s, s - r, r - p))$$

Asymmetric game

Toy game for illustration: I am choosing battle plans against the enemy

	Full Defense	Defend in shifts
Charge	0	+3
Sneak attack	+1	-1

We notice two things: the game isn't symmetric, and that we see a lot of positive values.

Asymmetric game mixed strategy equilibrium

Setting this up, we start with $c + s = 1; c, s \geq 0$

	Full Defense (f)	Defend in shifts (d)	EV | Opponent
Charge (c)	0	+3	3d
Sneak attack (s)	+1	-1	f-d
EV | Opponent	s	3c-s	

Solving, we get $c = 0.4, s = 0.6, f = 0.8, d = 0.2$

"Value" of the game to us is 0.6

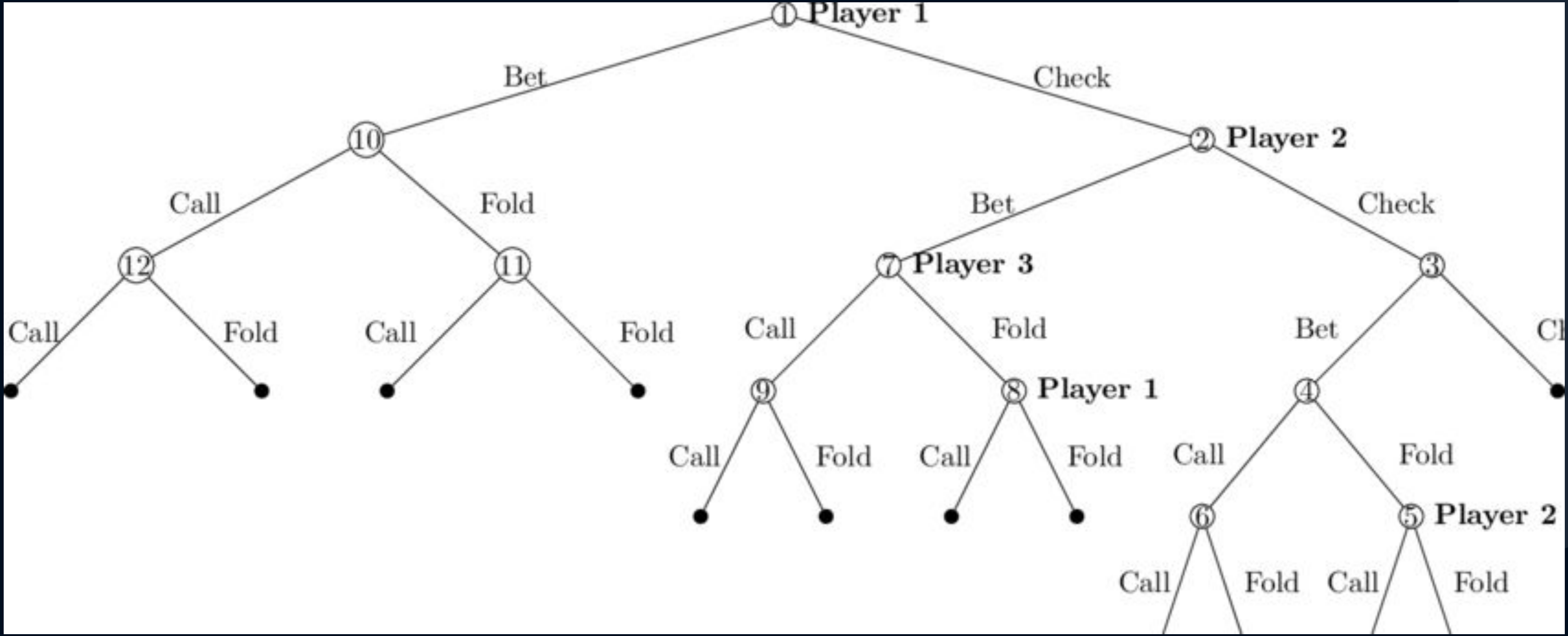
Existence of Nash equilibria

Theorem (Nash, 1951): Every finite game has a Nash equilibrium in mixed strategies.

Proof: Topology (uses Brouwer fixed-point theorem).

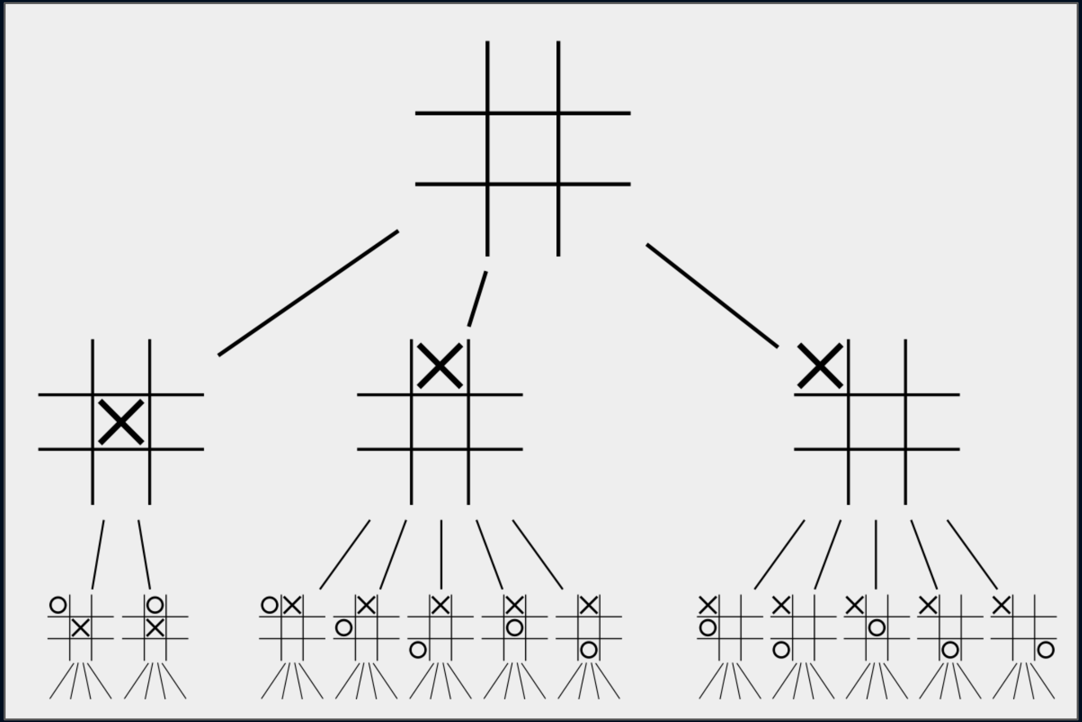
Applications to poker

Game Tree

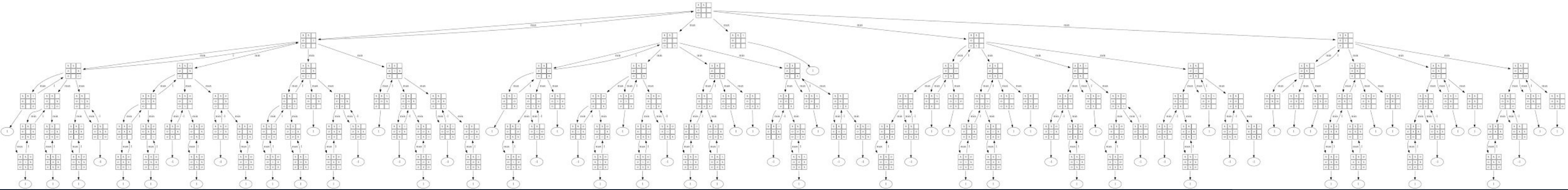


1 Player 1
Bet
Check
10
2 Player 2
Call
Fold
Bet
Check
12
11
7 Player 3
3
Call
Fold
Call
Fold
Call
Fold
Bet
8 Player 1
9
4
Call
Fold
Call
Fold
Call
Fold
6
5 Player 2
Call
Fold
Call
Fold

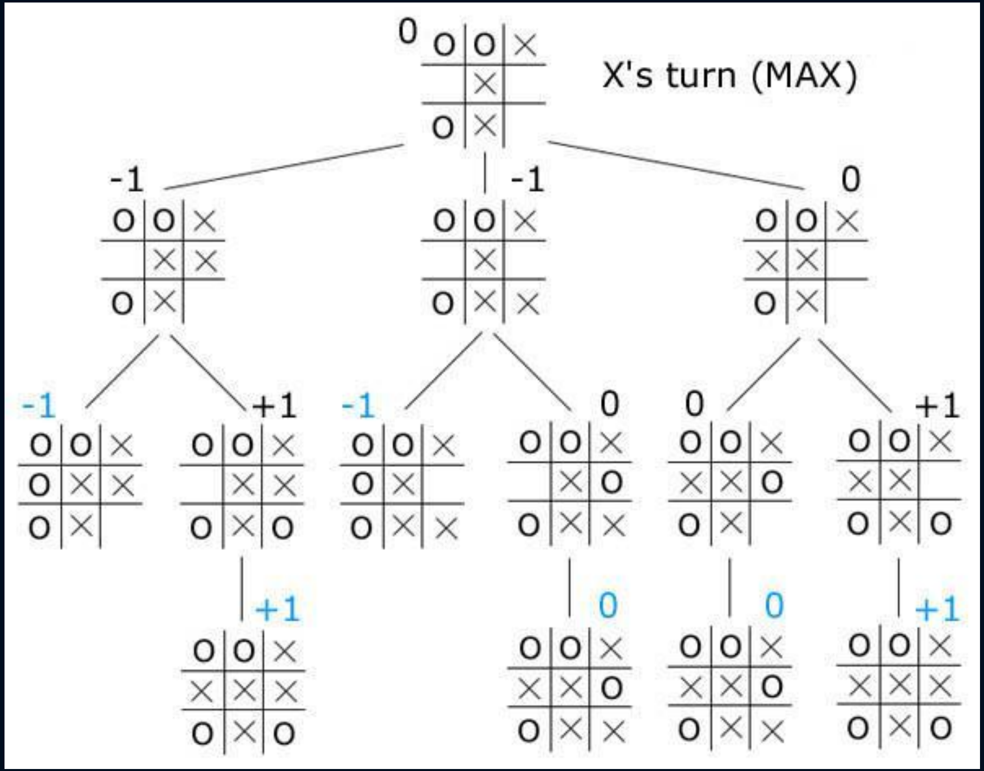
Extensive form games



Tic Tac Toe Full Extensive Form



Backwards induction

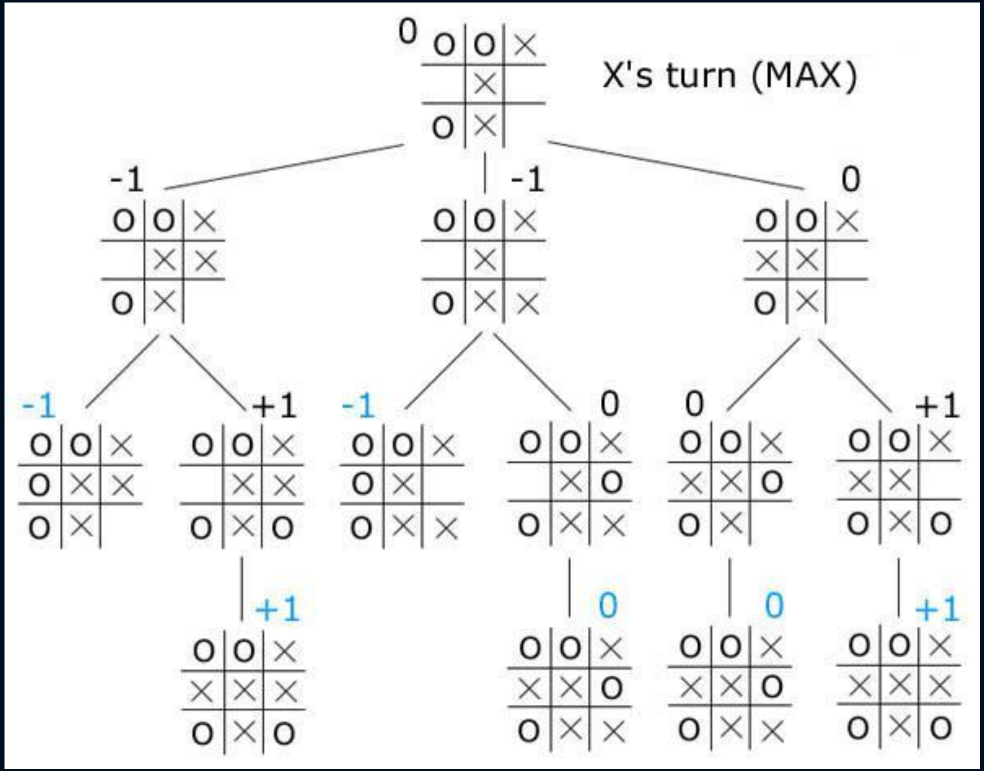


Black are the determined payouts based on players acting rationally

Blue are the leaves - the terminal states of the game

Follow the game tree up (backwards) to find the state's value

Backwards induction



Looking at the leaf states first, I see -1s in the left 2 branches, leaving their parents as -1.

The rightmost branch has payouts 0 and 1. Following this up the tree, we get 0 for the black value.

Finally, all together X should try to play for 0 (tie), and does so by putting x in left center.

Conversion to normal form

- Every extensive form is also a normal form (matrix) game

Conversion to normal form

- Every extensive form is also a normal form (matrix) game
- Mapping: each pure strategy (row/column) is a complete set of instructions for what to do at each node
- Example: in poker, each pure strategy would be a dictionary which maps the game_state to a unique action

Implications

- Poker has a Nash equilibrium - it's a finite game that satisfies the conditions

- Can we use backwards induction to solve for it? Can we still use the same type of game tree?

- How is poker different from Chess? tic-tac-toe? RPS?

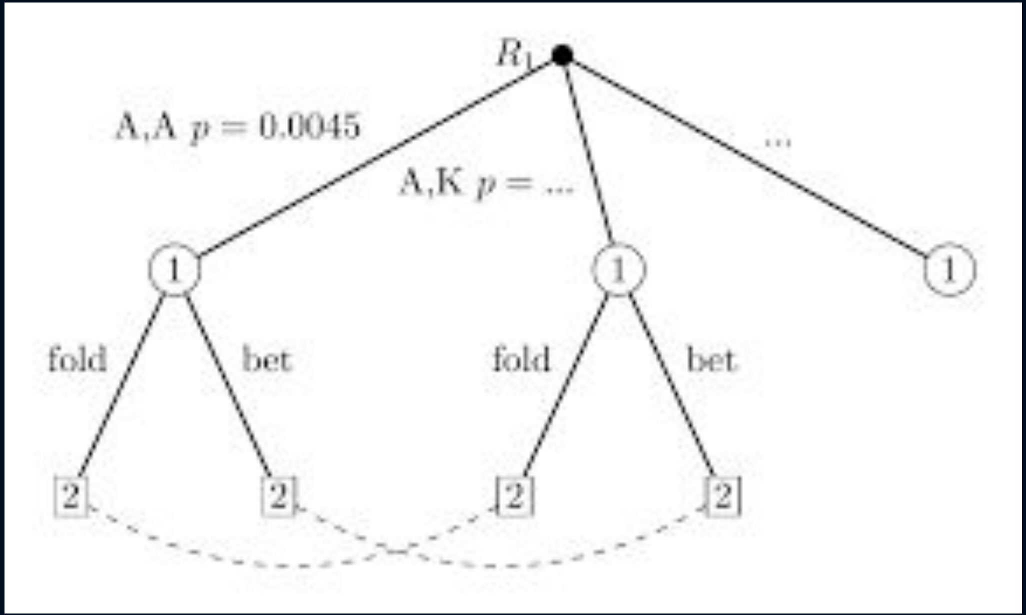
Implications

- Poker has a Nash equilibrium - it's a finite game that satisfies the conditions

- Can we use backwards induction to solve for it? Can we still use the same type of game tree?

- How is poker different from Chess? tic-tac-toe? RPS?
  - Size
  - Imperfect Information
  - Randomness

Adding in imperfect information and randomness



Imperfect Information and Randomness

- Imperfect: Don't know what node you're currently at since your opponent's hole cards are hidden (information sets)

- Randomness: Don't know where in the game tree you're moving next since future cards to be dealt are unknown

- Together, this leads to a really nice property of poker. The general game tree movements (bet/fold/call) are known, but the definite game state isn't. Imperfect information makes games fun!

Kuhn Poker

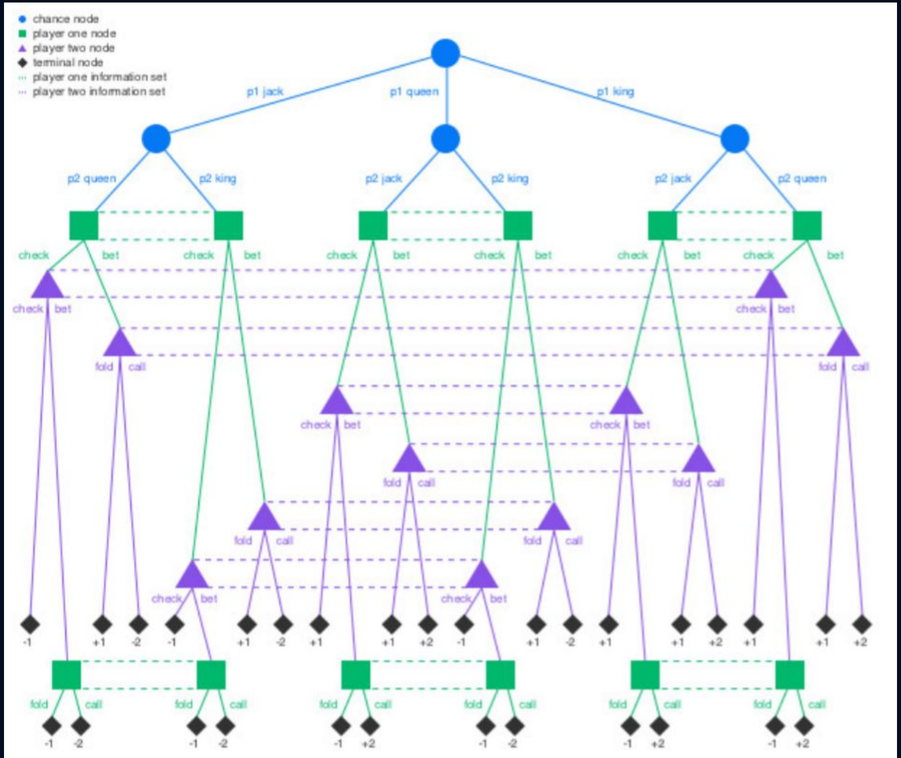
Kuhn poker is a simplified form of poker developed by Harold W. Kuhn as a simple model zero-sum two-player imperfect-information game, amenable to a complete game-theoretic analysis. In Kuhn poker, the deck includes only three playing cards, for example a King, Queen, and Jack. One card is dealt to each player, which may place bets similarly to a standard poker. If both players bet or both players pass, the player with the higher card wins, otherwise, the betting player wins.

Game description [edit]

In conventional poker terms, a game of Kuhn poker proceeds as follows:

- Each player antes 1.
- Each player is dealt one of the three cards, and the third is put aside unseen.
- Player one can check or bet 1.
    - If player one checks then player two can check or bet 1.
        - If player two checks there is a showdown for the pot of 2 (i.e. the higher card wins 1 from the other player).
        - If player two bets then player one can fold or call.
            - If player one folds then player two takes the pot of 3 (i.e. winning 1 from player 1).
            - If player one calls there is a showdown for the pot of 4 (i.e. the higher card wins 2 from the other player).
    - If player one bets then player two can fold or call.
        - If player two folds then player one takes the pot of 3 (i.e. winning 1 from player 2).
        - If player two calls there is a showdown for the pot of 4 (i.e. the higher card wins 2 from the other player).

Kuhn Poker Game Tree



Poker Nash equilibrium and use

- Poker has a Nash equilibrium that has EV 0
- The matrix is doubly exponential in size - incomputable
- Does “play good poker” reduce to “play the equilibrium strategy”?
- Exploitative strategies as alternatives

Applied game theory

The MIT ID game

- Suppose rational players would only play if their number is ≥X
  - Students might submit only if ≥9000
- But if only ≥X is submitted, then X is the smallest value and shouldn't be submitted
  - Why submit 9000?
- Continue this logic (nth order dominance)
  - Why submit 9001? 9002?

Adverse selection

- Occurs anytime “buyer” and “seller” have asymmetric information
- Market for cars: suppose there are used cars with private value distributed uniformly between \$1000 and \$10000. What should the market price be?
- What would car owners do? Then what happens?

Sources of adverse selection in poker

- First action when betting (differential check to the preflop aggressor)
- Multiple bets in a row (what types of hands do this?)
- Being in the later stages of the round (why would are they still in the pot?)
- Others?

When I bet, am I happy with my bet *conditioned on my opponent calling?*

Adverse selection and determinism

- Suppose I have the following deterministic betting strategy:

- All-in when I have a top X% hand

- How can my opponent exploit me?

Second-Price Auctions

Second-Price (Vickrey) Auction

- Two players submit sealed bids b_1, b_2 for an object that they value at s_1, s_2. The bidder with the highest bid is awarded the object at the second-highest price.

- It can be shown that it is a weakly dominant strategy for each player to bid according to how much they value the object: $b_i = s_i$

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- It can be shown that it is a weakly dominant strategy for each player to bid according to how much they value the object: $b_i = s_i$
  - Let's assume we're Player 1
  - If $b_2 > s_1$, then winning is not worth it because we'd be paying more than the object's worth. So the best we can do is lose the auction, and bidding $b_1 = s_1$ achieves this.
  - If $b_2 <= s_1$, then winning is worth it, and the price we pay is fixed regardless of our bid. In this case, bidding $b_1 = s_1$ always wins us the auction.

Auction Hold'em is not exactly the same

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- We care about how much the opponent pays if they win (more in the pot)
- Value of extra card is not well defined
- When we bid (or take any action in general), we're sending information
- So what do we do?

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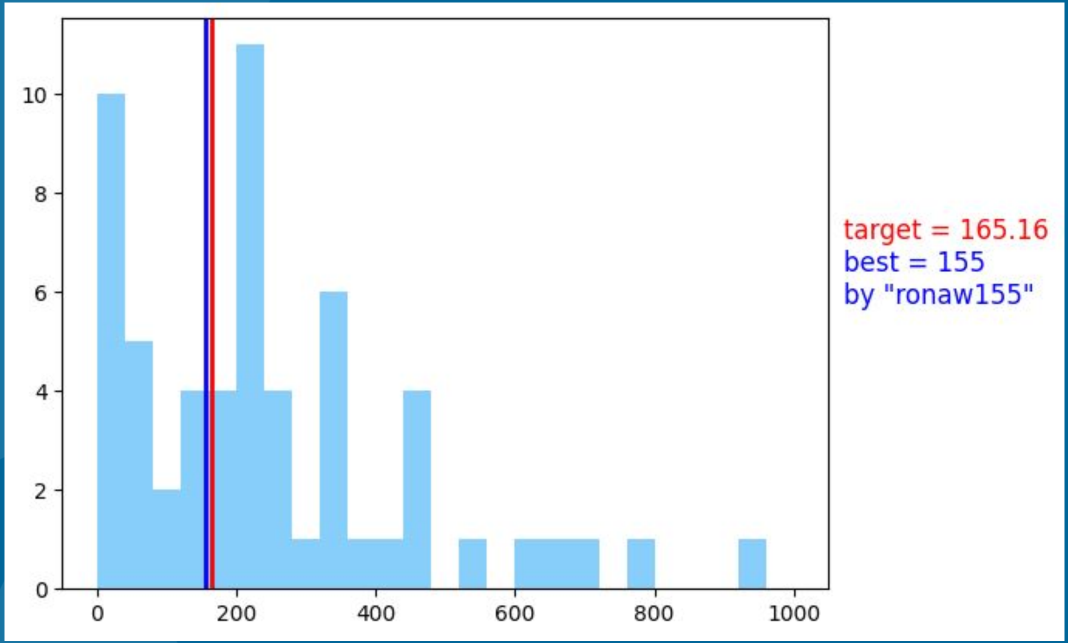
- We care about how much the opponent pays if they win (more in the pot)
- Value of extra card is not well defined
- When we bid (or take any action in general), we're sending information
- So what do we do?
  - Exploitative strategies
  - Can still bid on a value-based metric (ref bot 2)
  - Incorporate randomness to hide information

Game Theory Takeaways

- Try to be one step ahead of your opponent (dominance)
- Equilibrium strategies can be good but hard to find
- Exploitative strategies can be very powerful - think about what you’re trying to optimize for when making a pokerbot
- Be careful of adverse selection

Coding Reference-3 Bot

Beauty Contest Winner:



MIT ID Contest Winner