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[J

In [1]:

1 Introduction to the Fourier representation

We begin by a simple example which shows that the addition of some sine waves, with special coefficients, converges construct signal can be expressed as a sum of sine waves. This is the notion of Fourier series. After an illustration (denoising of a corrupte filtering in the frequency domain, we show how the Fourier representation can be extended to aperiodic signals.

- Simple examples
- Decomposition on basis - scalar producs
- Decomposition of periodic functions -- Fourier series
- Complex Fourier series
- Computer experiment
- Towards Fourier transform

1.1 Simple examples

> Read the script below, execute (CTRL-Enter), experiment with the parameters.

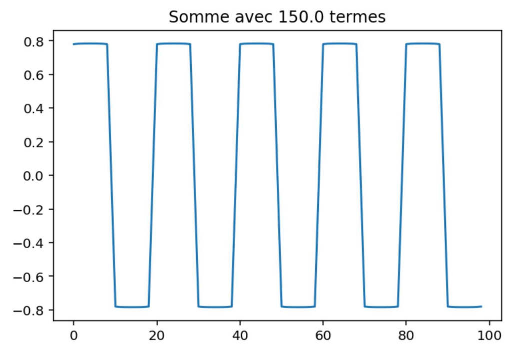
In [2]:

```
N = 100
L = 20
s = np.zeros(N - 1)

for k in np.arange(1, 300, 2):
    s = s + 1 / float(k) * sin(2 * pi * k / L * np.arange(1, N, 1))
plt.plot(s)
plt.title("Somme avec " + str((k - 1) / 2 + 1) + " termes")
```

Out[2]:

```
Text(0.5, 1.0, 'Somme avec 150.0 termes')
```



The next example is more involved in that it sums sin a cos of different frequencies and with different amplitudes. We also add w more easily with the program.

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In [3]:

```
@out.capture(clear_output=True, wait=True)
def sfou_exp(Km):
    #clear_output(wait=True)
    Kmax = int(Km)
    L = 400
    N = 1000
    k = 0
    s = np.zeros(N - 1)
    #plt.clf()
    for k in np.arange(1, Kmax):
        ak = 0
        bk = 1.0 / k if (k % 2) == 1 else 0  # k odd

        # ak=0 #if (k % 2) == 1 else -2.0/(pi*k**2)
        # bk=-1.0/k if (k % 2) == 1 else 1.0/k #

        s = s + ak * cos(2 * pi * k / L * np.arange(1, N, 1)) + bk * sin(
            2 * pi * k / L * np.arange(1, N, 1))
    ax = plt.axes(xlim=(0, N), ylim=(-2, 2))
    ax.plot(s)
    plt.title("Sum with {} terms".format(k + 1))
    plt.show()


### ------------------------------------------------
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-3-67dbb1757edc> in <module>()
----> 1 @out.capture(clear_output=True, wait=True)
      2 def sfou_exp(Km):
      3     #clear_output(wait=True)
      4     Kmax = int(Km)
      5     L = 400

NameError: name 'out' is not defined
```

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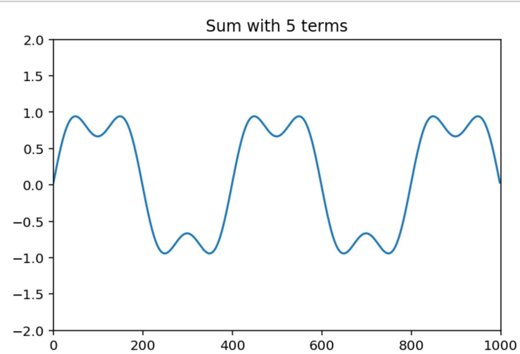
In [4]:

```
out = widgets.Output()

fig = plt.figure()
ax = plt.axes(xlim=(0, 100), ylim=(-2, 2))

# ---- Widgets -----------------------
# slider=widgets.FloatSlider(max=100,min=0,step=1,value=1)
slide = widgets.IntSlider(max=100, min=0, step=1, value=5)
val = widgets.IntText(value='1')


#----- Callbacks des widgets -------------
@out.capture(clear_output=True, wait=True)
def sfou_exp(Km):
    #clear_output(wait=True)
    Kmax = int(Km)
    L = 400
    N = 1000
    k = 0
    s = np.zeros(N - 1)
    #plt.clf()
    for k in np.arange(1, Kmax):
        ak = 0
        bk = 1.0 / k if (k % 2) == 1 else 0  # k odd

        # ak=0 #if (k % 2) == 1 else -2.0/(pi*k**2)
        # bk=-1.0/k if (k % 2) == 1 else 1.0/k #

        s = s + ak * cos(2 * pi * k / L * np.arange(1, N, 1)) + bk * sin(
            2 * pi * k / L * np.arange(1, N, 1))
    ax = plt.axes(xlim=(0, N), ylim=(-2, 2))
    ax.plot(s)
    plt.title("Sum with {} terms".format(k + 1))
    plt.show()


### --------------------------------------------------


#@out.capture(clear_output=True, wait=True)
def sfou1_Km(param):
    Km = param['new']
    val.value = str(Km)
    sfou_exp(Km)


#@out.capture(clear_output=True, wait=True)
def sfou2_Km(param):
    Km = param.new
    slide.value = Km
    #sfou_exp(Km.value)

# ---- Display ----------------
#display(slide)
#display(val)


slide.observe(sfou1_Km, names=['value'])
sfou_exp(5)
#val.observe(sfou2_Km,names='value')
display(widgets.VBox([slide, out]))
```




```
Widget Javascript not detected.  It may not be installed or enabled properly.
```

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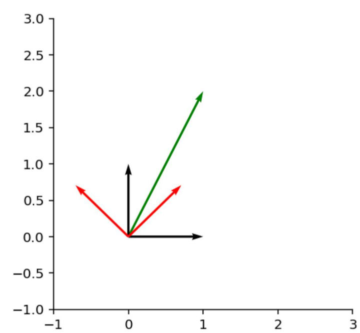
1.1.1 Decomposition on basis - scalar producs

We recall here that any vector can be expressed on a orthonormal basis, and that the coordinates are the scalar product of the v

In [5]:
```
z = array([1, 2])
u = array([0, 1])
v = array([1, 0])
u1 = array([1, 1]) / sqrt(2)
v1 = array([-1, 1]) / sqrt(2)

f, ax = subplots(1, 1, figsize=(4, 4))
ax.set_xlim([-1, 3])
ax.set_ylim([-1, 3])
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
#ax.spines['bottom'].set_position('center')
ax.quiver(
    0, 0, z[0], z[1], angles='xy', scale_units='xy', scale=1, color='green')
ax.quiver(
    0, 0, u[0], u[1], angles='xy', scale_units='xy', scale=1, color='black')
ax.quiver(
    0, 0, v[0], v[1], angles='xy', scale_units='xy', scale=1, color='black')
ax.quiver(
    0, 0, u1[0], u1[1], angles='xy', scale_units='xy', scale=1, color='red')
ax.quiver(
    0, 0, v1[0], v1[1], angles='xy', scale_units='xy', scale=1, color='red')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')
```



From a coordinate system to another: Take a vector (in green in the illustration). Its coordinates in the system (u, v) are [1,2]. In new system (O, u_1, v_1), we have to project the vector on u_1 and u_2. This is done by the scalar products:

In [6]:
```
x = z.dot(u1)
y = z.dot(v1)
print('New coordinates: ', x, y)
```

```
New coordinates:  2.1213203435596424 0.7071067811865475
```

1.2 Decomposition of periodic functions -- Fourier series

This idea can be extended to (periodic) functions. Consider the set of all even periodic functions, with a given period, say L. The multiple or the _fundamental_ frequency $1/L$ constitute a basis of even periodic functions with period T. Let us check that thes with each other.

In [7]:
```
L = 200
k = 8
l = 3
sk = sqrt(2 / L) * cos(2 * pi / L * k * np.arange(0, L))
sl = sqrt(2 / L) * cos(2 * pi / L * l * np.arange(0, L))
```

In [8]:
```
sl.dot(sl)
```

Out[8]:
```
1.0000000000000004
```

Except in the case $l = 0$ where a factor 2 entails

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In [9]:
```
l = 0
sl = sqrt(2 / L) * cos(2 * pi / L * l * np.arange(0, L))
sl.dot(sl)
```

Out[9]:
```
1.9999999999999998
```

Therefore, the decomposition of any even periodic function $x(n)$ with period L on the basis of cosines expresses as

$$x(n) = \sqrt{\frac{2}{L}}\left(\frac{a_0}{2} + \sum_{k=1}^{L-1} a_k \cos(2\pi k/Ln)\right)$$

with

$$a_k = \sqrt{\frac{2}{L}} \sum_{n\in[L]} x(n)\cos(2\pi k/Ln).$$

Regrouping the factors, the series can also be expressed as

$$x_{\text{even}}(n) = \left(\frac{a_0}{2} + \sum_{k=1}^{L-1} a_k \cos(2\pi k/Ln)\right)$$

with

$$a_k = \frac{2}{L} \sum_{n\in[L]} x(n)\cos(2\pi k/Ln),$$

where the notation $n \in [L]$ indicates that the sum has to be done on any length-L interval. The very same reasoning can be dc decomposition into sine waves:

$$x_{\text{odd}}(n) = \sum_{k=0}^{L-1} b_k \sin(2\pi k/Ln)$$

with

$$b_k = \frac{2}{L} \sum_{n\in[L]} x(n)\sin(2\pi k/Ln),$$

Since any function can be decomposed into an odd + even part

$$x(n) = x_{\text{even}}(n) + x_{\text{odd}}(n) = \frac{x(n) + x(-n)}{2} + \frac{x(n) - x(-n)}{2},$$

we have the sum of the decompositions:

$$\boxed{x(n) = \frac{a_0}{2} + \sum_{k=1}^{L-1} a_k \cos(2\pi k/Ln) + \sum_{k=1}^{+\infty} b_k \sin(2\pi k/Ln)}$$

with \begin{equation} \boxed{

$$\begin{cases} a_k = \frac{2}{L}\sum_{n\in[L]} x(n)\cos(2\pi k/Ln), \\ b_k = \frac{2}{L}\sum_{n\in[L]} x(n)\sin(2\pi k/Ln), \end{cases}$$

} \end{equation} This is the definition of the Fourier series, and this is no more compicated than that... A remaining question is th the series converge to the true function? The short answer is Yes: the equality in the series expansion is a true equality, not an a for this course, but you may have a look at [this article (http://en.wikipedia.org/wiki/Convergence_of_Fourier_series)](http://en.wikipedia.org/wiki/Convergence_of_Fourier_series).

There of course exists a continuous version, valid for time-continuous dignals.

1.3 Complex Fourier series

1.3.1 Introduction

Another series expansion can be defined for complex valued signals. In such case, the trigonometric functions will be replaced b $\exp(j2\pi k/Ln)$. Let us check that they indeed form a basis of signals:

In [10]:
```
L = 200
k = 8
l = 3
sk = sqrt(1 / L) * exp(1j * 2 * pi / L * k * np.arange(0, L))
sl = sqrt(1 / L) * exp(1j * 2 * pi / L * l * np.arange(0, L))
print("scalar product between sk and sl: ", np.vdot(sk, sl))
print("scalar product between sk and sk (i.e. norm of sk): ", np.vdot(sk, sk))
```

```
scalar product between sk and sl:  (-1.9932252838852267e-17+5.886840063482993e-17j)
scalar product between sk and sk (i.e. norm of sk):  (1+0j)
```

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It is thus possible to decompose a signal as follows:

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{L-1} c_k e^{j2\pi\frac{kn}{L}} \\ \text{with } c_k &= \frac{1}{L}\sum_{n\in[L]} x(n) e^{-j2\pi\frac{kn}{L}} \end{aligned}}$$

where c_k is the dot product between $x(n)$ and $\exp(j2\pi k/Ln)$, i.e. the 'coordinate' of x with respect to the 'vector' $\exp(j2\pi k/$ the `complex Fourier series` .

Exercise -- Show that c_k is periodic with period L; i.e. $c_k = c_{k+L}$.

Since c_k is periodic in k of period L, we see that in term or the _"normalized frequency"_ k/L, it is periodic with period 1.

1.3.1.1 Relation of the complex Fourier Series with the standard Fourier Series

It is easy to find a relation between this complex Fourier series and the classical Fourier series. The series can be rewritten as

$$x(n) = c_0 + \sum_{k=1}^{+\infty} c_k e^{j2\pi k/Ln} + c_{-k} e^{-j2\pi k/Ln}.$$

By using the Euler formulas (http://en.wikipedia.org/wiki/Euler's_formula), developping and rearranging, we get

$$\begin{aligned} x(n) = c_0 + \sum_{k=1}^{+\infty} & \mathcal{R}\left\{c_k + c_{-k}\right\}\cos(2\pi k/Ln) + \mathcal{I}\left\{c_{-k} - c_k\right\}\sin(2\pi k/Ln) \\ & + j\left(\mathcal{R}\left\{c_k - c_{-k}\right\}\sin(2\pi k/Ln) + \mathcal{I}\left\{c_k + c_{-k}\right\}\cos(2\pi k/Ln)\right). \end{aligned}$$

Suppose that $x(n)$ is real valued. Then by direct identification, we have \begin{equation}

$$\begin{cases} a_k = \mathcal{R}\left\{c_k + c_{-k}\right\} \\ b_k = \mathcal{I}\left\{c_{-k} - c_k\right\} \end{cases}$$

\end{equation} and, by the cancellation of the imaginary part, the following symmetry relationships for real signals: \begin{equati \I{c_k}=-\I{c_{-k}}. \end{cases}

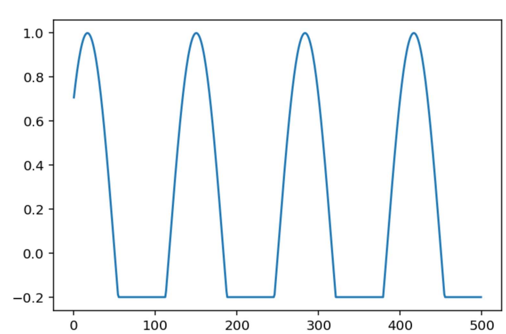
\end{equation} This symmetry is called `Hermitian symmetry'.

1.3.2 Computer experiment

> Experiment. Given a signal, computes its decomposition and then reconstruct the signal from its individual components.

In [11]:
```
%matplotlib inline
L = 400
N = 500
t = np.arange(N)
s = sin(2 * pi * 3 * t / L + pi / 4)
x = [ss if ss > -0.2 else -0.2 for ss in s]
plt.plot(t, x)
```

Out[11]:
```
[<matplotlib.lines.Line2D at 0x7f17d7e863c8>]
```



A function for computing the Fourier series coefficients

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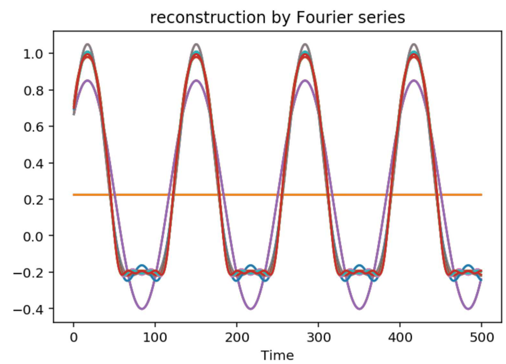
In [12]:
```
# compute the coeffs ck
def coeffck(x, L, k):
    assert np.size(x) == L, "input must be of length L"
    karray = []
    res = []
    if isinstance(k, int):
        karray.append(k)
    else:
        karray = np.array(k)

    for k in karray:
        res.append(np.vdot(exp(1j * 2 * pi / L * k * np.arange(0, L)), x))
    return 1 / L * np.array(res)


#test: coeffck(x[0:L],L,[-12,1,7])
# --> array([  1.51702135e-02 +4.60742555e-17j,
#     -1.31708229e-05 -1.31708229e-05j,   1.37224241e-05 -1.37224241e-05j])
```

Now let us compute the coeffs for actual signal

In [13]:
```
# compute the coeffs for actual signal
c1 = coeffck(x[0:L], L, np.arange(0, 100))
c2 = coeffck(x[0:L], L, np.arange(0, -100, -1))
s = c1[0] * np.ones((N))
for k in np.arange(1, 25):
    s = s + c1[k] * exp(1j * 2 * pi / L * k * np.arange(0, N)) + c2[k] * exp(
        -1j * 2 * pi / L * k * np.arange(0, N))
    plt.plot(t, np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")
```

Out[13]: `Text(0.5, 0, 'Time')`



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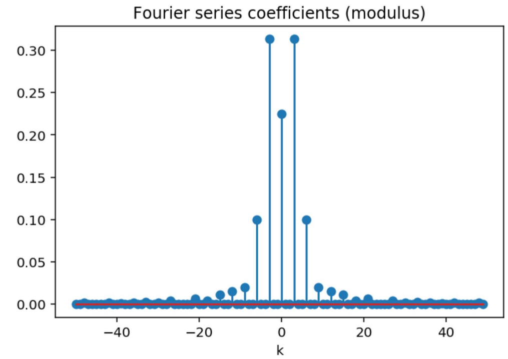
In [14]:
```
plt.figure()
kk = np.arange(-50, 50)
c = coeffck(x[0:L], L, kk)
plt.stem(kk, np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
msg = """In the frequency representation, the x axis corresponds to the frequencies k/L
of the complex exponentials.
Therefore, if a signal is periodic of period M, the corresponding fundamental frequency
is 1/M. This frequency then appears at index ko=L/M (if this ratio is an integer).
Harmonics will appear at multiples of ko."""
print(msg)
```

```
In the frequency representation, the x axis corresponds to the frequencies k/L
of the complex exponentials.
Therefore, if a signal is periodic of period M, the corresponding fundamental frequency
is 1/M. This frequency then appears at index ko=L/M (if this ratio is an integer).
Harmonics will appear at multiples of ko.
```



A pulse train corrupts our original signal

In [15]:
```
L = 400


# define a pulse train which will corrupt our original signal
def sign(x):
    if isinstance(x, (int, float)):
        return 1 if x >= 0 else -1
    else:
        return np.array([1 if u >= 0 else -1 for u in x])


#test: sign([2, 1, -0.2, 0])


def repeat(x, n):
    if isinstance(x, (np.ndarray, list, int, float)):
        return np.array([list(x) * n]).flatten()
    else:
        raise ('input must be an array,list,or float/int')


#t=np.arange(N)
#sig=sign(sin(2*pi*10*t/L))
```

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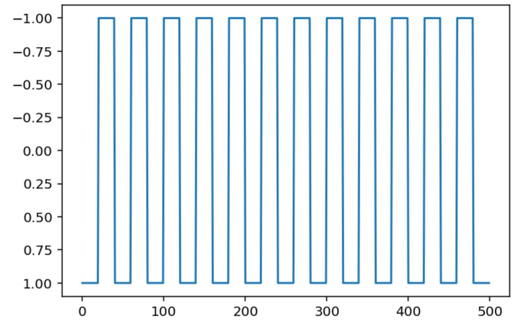
In [16]:
```
rect = np.concatenate((np.ones(20), -np.ones(20)))
#[1,1,1,1,1,-1,-1,-1,-1,-1]

sig = repeat(rect, 15)
sig = sig[0:N]
plt.plot(sig)
plt.ylim({-1.1, 1.1})
```

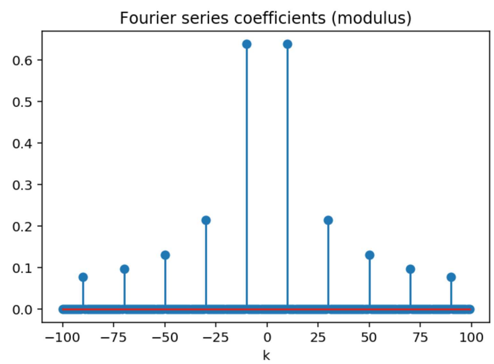
Out[16]: (1.1, -1.1)



Compute and represent the Fourier coeffs of the pulse train

In [17]:
```
kk = np.arange(-100, 100)
c = coeffck(sig[0:L], L, kk)
plt.figure()
plt.stem(kk, np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
```

Out[17]: Text(0.5, 0, 'k')



The fundamental frequency of the pulse train is 1 over the length of the pulse, that is 1/40 here. Since The Fourier series is com appear every 10 samples (ie at indexes k multiples of 10).

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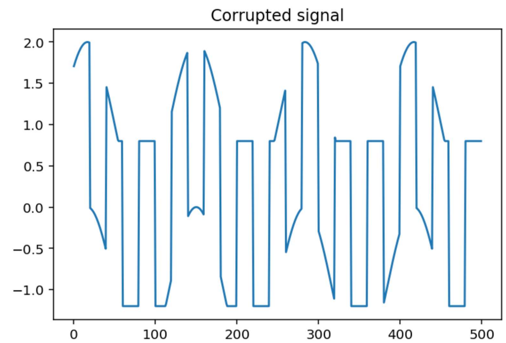
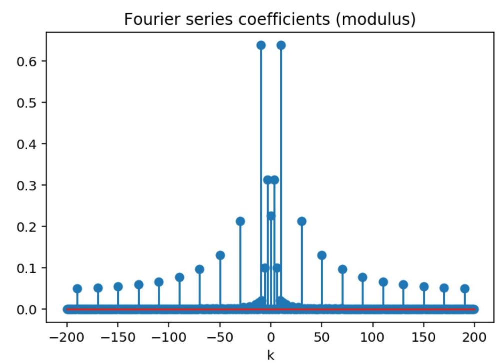
In [18]:
```
z = x + 1 * sig
plt.plot(z)
plt.title("Corrupted signal")

kk = np.arange(-200, 200)
cz = coeffck(z[0:L], L, kk)
plt.figure()
plt.stem(kk, np.abs(cz))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
```

Out[18]: `Text(0.5, 0, 'k')`





Now, we try to kill all the frequencies harmonics of 10 (the fundamental frequency of the pulse train), and reconstruct the resultin

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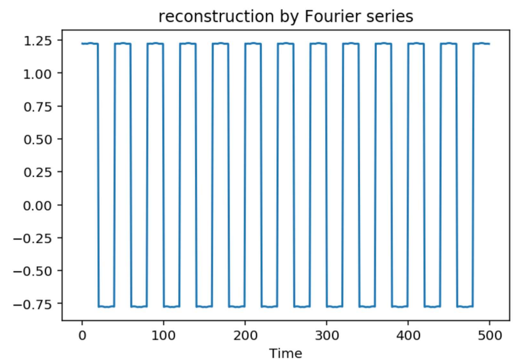
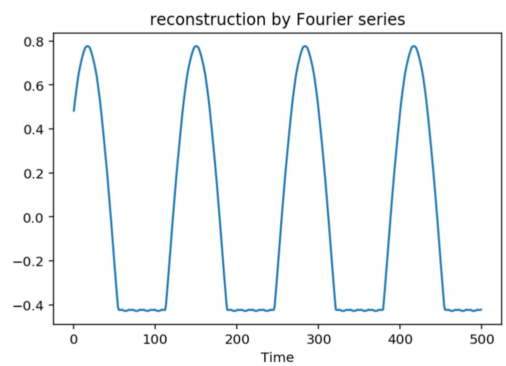
In [19]:
```
# kill frequencies harmonics of 10 (the fundamental frequency of the pulse train)
# and reconstruct the resulting signal

s = np.zeros(N)
kmin = np.min(kk)
for k in kk:
    if not k % 10:  #true if k is multiple of 10
        s = s + cz[k + kmin] * exp(1j * 2 * pi / L * k * np.arange(0, N))
plt.figure()
plt.plot(t, np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")

plt.figure()
plt.plot(t, z - np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")
```

Out[19]: `Text(0.5, 0, 'Time')`





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