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[Jupyte

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In [1]:

2 From Fourier Series to Fourier transforms

In this section, we go from the Fourier series to the Fourier transform for discrete signal. So doing, we also introduce the notion o
will study in more details later. For now, we focus on the representations in the frequency domain, detail and experiment with so

2.1 Introduction and definitions

Suppose that we only have an observation of length N. So no periodic signal, but a signal of size N. We do not know if there w
do not know if there were data after sample N. What to do? Facing to such situation, we can still

- imagine that the data are periodic outside of the observation interval, with a period N. Then the formulas for the Fourier ser
interval. Actually there is no problem with that. The resulting transformation is called the *Discrete Fourier Transform* . The c

\begin{equation} \eqboxc{

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{j2\pi\frac{kn}{N}}$$

$$\text{with } X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{kn}{N}}$$

} \label{eq:DFT} \end{equation}

- we may also consider that there is nothing --that is zeros, outside of the observation interval. In such condition, we can still i
but with an infinite period. Since the separation of two harmonics in the Fourier series is Δf=1/period, we see that $\Delta f \rightarrow$
becomes continuous. This is illustrated below.

In [2]:

```
# compute the coeffs ck
def coeffck(x, L, k):
    assert np.size(x) == L, "input must be of length L"
    karray = []
    res = []
    if isinstance(k, int):
        karray.append(k)
    else:
        karray = np.array(k)

    for k in karray:
        res.append(np.vdot(exp(1j * 2 * pi / L * k * np.arange(0, L)), x))
    return 1 / L * np.array(res)


#test: coeffck(x[0:L],L,[-12,1,7])
# --> array([  1.51702135e-02 +4.60742555e-17j,
#     -1.31708229e-05 -1.31708229e-05j,   1.37224241e-05 -1.37224241e-05j])
```

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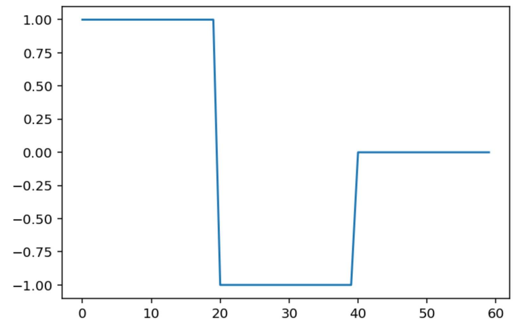
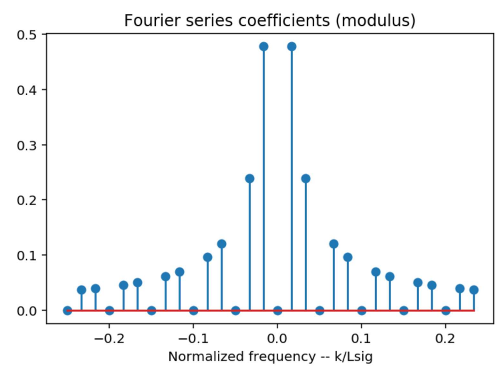
In [3]:
```python
Lpad = 20  # then 200, then 2000
# define a rectangular pulse
rect = np.concatenate((np.ones(20), -np.ones(20)))
# Add zeros after:
rect_zeropadded = np.concatenate((rect, np.zeros(Lpad)))
sig = rect_zeropadded
plt.plot(sig)
# compute the Fourier series for |k/Lsig|<1/4
Lsig = np.size(sig)
fmax = int(Lsig / 4)
kk = np.arange(-fmax, fmax)
c = coeffck(sig[0:Lsig], Lsig, kk)
# plot it
plt.figure()
plt.stem(kk / Lsig, np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("Normalized frequency -- k/Lsig")
```

Out[3]: `Text(0.5, 0, 'Normalized frequency -- k/Lsig')`





Hence we obtain a formula where the discrete sum for reconstructing the time-series $x(n)$ becomes a continuous sum, since f

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi\frac{kn}{N}} = \sum_{k/N=0}^{1-1/N} N X(k) e^{j2\pi\frac{kn}{N}} \frac{1}{N}$$

$$\rightarrow x(n) = \int_0^1 X(f) e^{j2\pi f n} \mathrm{d}f$$

\end{equation} Finally, we end with what is called the **Discrete-time Fourier transform** : \begin{equation}

\boxed{

$$x(n) = \int_0^1 X(f) e^{j2\pi f n} \mathrm{d}f$$

$$\text{with } X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

} \label{eq:DiscreteTimeFT} \end{equation}

Even before exploring the numerous properties of the Fourier transform, it is important to stress that

The Fourier transform of a discrete signal is periodic with period one.

Check it as an exercise! Begin with the formula for $X(f)$ an compute $X(f+1)$. use the fact that n is an integer and that exp(

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2.2 Examples

Exercise 1 .

- Compute the *Fourier transform* of a rectangular window given on N points. The result is called a (discrete) cardinal sine (c plot, and study the behaviour of this function with N.
- Experiment numerically... See below the provided functions.
- Compute the Fourier transform of a sine wave $\sin(2\pi f_0 n)$ given on N points.
- Examine what happens when the N and f_0 vary.

2.2.1 The Fourier transform of a rectangular window

The derivation of the formula will be done in class. Let us see the experimental part.

For the numerical experiments, import the fft (Fast Fourier Transform) function,

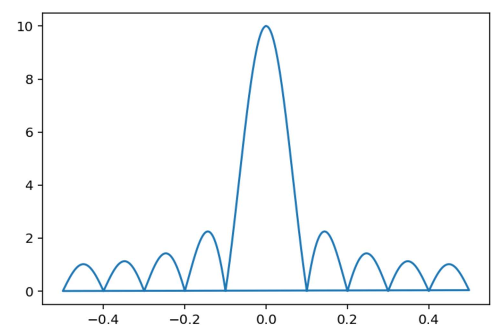
```
from numpy.fft import fft, ifft
```

define a sine wave, complute and plot its Fourier transform. As the FFT is actually an implementation of a discrete Fourier transf the true Fourier transform by using zero-padding (check that a parameter in the fft enables to do this zero-padding).

In [4]:
```
from numpy.fft import fft, ifft
```

In [5]:
```
#Define a rectangular window, of length L
#on N points, zeropad to NN=1000
# take eg L=100, N=500
NN = 1000
L = 10  # 10, then 6, then 20, then 50, then 100...
r = np.ones(L)
Rf = fft(r, NN)
f = fftfreq(NN)
plt.plot(f, np.abs(Rf))
```

Out[5]:
```
[<matplotlib.lines.Line2D at 0x7f5d27641d68>]
```



It remain to compare this to a discrete cardinal sinus. First we define a function and then compare the results.

In [6]:
```
def dsinc(x, L):
    if isinstance(x, (int, float)): x = [x]
    x = np.array(x)
    out = np.ones(np.shape(x))
    I = np.where(x != 0)
    out[I] = np.sin(x[I]) / (L * np.sin(x[I] / L))
    return out
```

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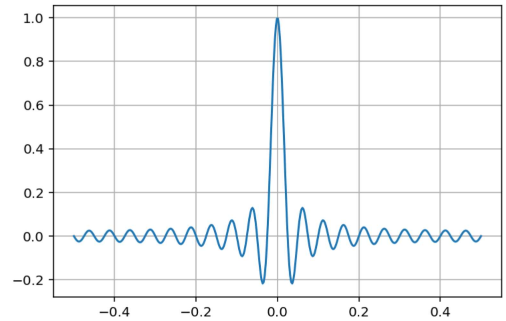
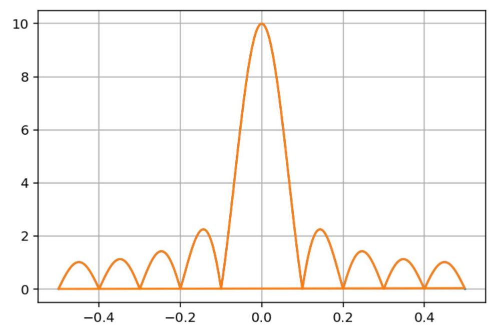
In [7]:
```
N = 1000
L = 40
f = np.linspace(-0.5, 0.5, 400)
plt.plot(f, dsinc(pi * L * f, L))
plt.grid(b=True)
```



Comparison of the Fourier transform of a rectangle and a cardinal sine:

In [8]:
```
NN = 1000
L = 10  # 10, then 6, then 20, then 50, then 100...
r = np.ones(L)
Rf = fft(r, NN)

N = 1000
f = np.linspace(-0.5, 0.5, 400)
plt.plot(f, L * np.abs(dsinc(pi * L * f, L)))
f = fftfreq(NN)
plt.plot(f, np.abs(Rf))
plt.grid(b=True)
```



Interactive versions...

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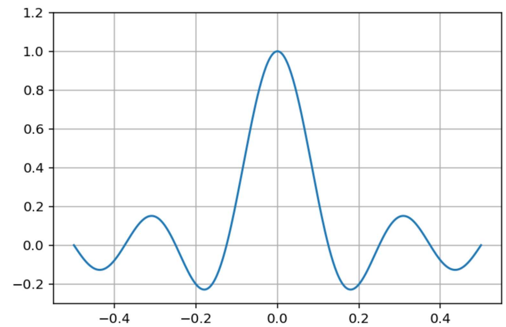
In [9]:

```
# using %matplotlib use a backend that allows external figures
# using %matplotlib inline plots the results in the notebook
%matplotlib inline
slider = widgets.FloatSlider(min=0.1, max=100, step=0.1, value=8)
display(slider)


#----- Callbacks des widgets -------------
def pltsinc(change):
    L = change['new']
    plt.clf()
    clear_output(wait=True)
    #val.value=str(f)
    f = np.linspace(-0.5, 0.5, 400)
    plt.plot(f, dsinc(pi * L * f, L))
    plt.ylim([-0.3, 1.2])
    plt.grid(b=True)


pltsinc({'new': 8})
slider.observe(pltsinc, names=['value'])
```



This is an example with matplotlib widgets interactivity, (instead of html widgets). The docs can be found [here (http://nbviewer.ipython.org/github/jakevdp/matplotlib_pydata2013/blob/master/notebooks/03_Widgets.html)](http://nbviewer.ipython.org/github/jakevdp/matplotlib_pydata2013/blob/master/notebooks/03_Widgets.html)

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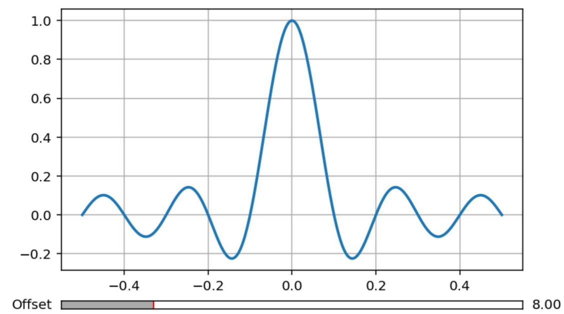
In [10]:

```
%matplotlib
from matplotlib.widgets import Slider

fig, ax = plt.subplots()
fig.subplots_adjust(bottom=0.2, left=0.1)

slider_ax = plt.axes([0.1, 0.1, 0.8, 0.02])
slider = Slider(slider_ax, "Offset", 0, 40, valinit=8, color='#AAAAAA')
L = 10
f = np.linspace(-0.5, 0.5, 400)

line, = ax.plot(f, dsinc(pi * L * f, L), lw=2)
#line2, = ax.plot(f,sinc(pi*L*f), lw=2)
#line2 is in order to compare with the "true" sinc
ax.grid(b=True)


def on_change(L):
    line.set_ydata(dsinc(pi * L * f, L))


#    line2.set_ydata(sinc(pi*L*f))

slider.on_changed(on_change)
```

```
Using matplotlib backend: TkAgg
```

Out[10]: 0



2.2.2 Fourier transform of a sine wave

Again, the derivation will be done in class.

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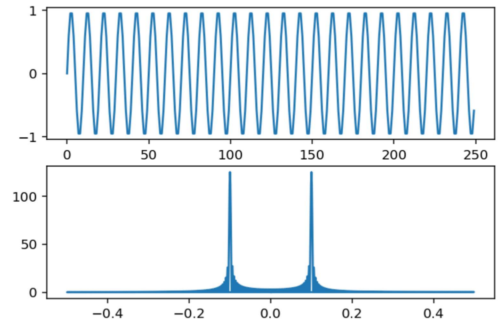
In [11]:
```
%matplotlib inline

from numpy.fft import fft, ifft
N = 250
f0 = 0.1
NN = 1000
fig, ax = plt.subplots(2, 1)


def plot_sin_and_transform(N, f0, ax):
    t = np.arange(N)
    s = np.sin(2 * pi * f0 * t)
    Sf = fft(s, NN)
    ax[0].plot(t, s)
    f = np.fft.fftfreq(NN)
    ax[1].plot(f, np.abs(Sf))


plot_sin_and_transform(N, f0, ax)
```



Interactive versions

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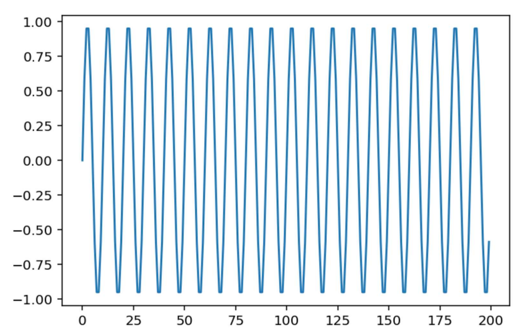
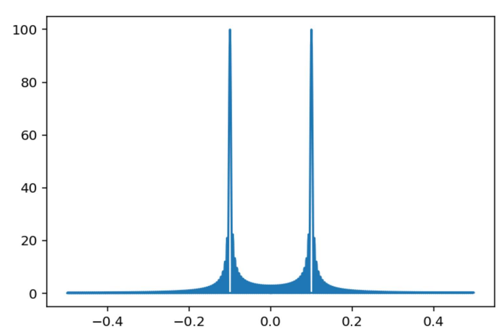
In [12]:
```python
# using %matplotlib use a backend that allows external figures
# using %matplotlib inline plots the results in the notebook
%matplotlib inline
sliderN = widgets.IntSlider(
    description="N", min=1, max=1000, step=1, value=200)
sliderf0 = widgets.FloatSlider(
    description="f0", min=0, max=0.5, step=0.01, value=0.1)
c1 = widgets.Checkbox(description="Display time signal", value=True)
c2 = widgets.Checkbox(description="Display frequency signal", value=True)

#display(sliderN)
#display(sliderf0)
N = 500
f0 = 0.1
t = np.arange(N)
s = np.sin(2 * pi * f0 * t)
Sf = fft(s, NN)
f = np.fft.fftfreq(NN)

out = widgets.Output()


#----- Callbacks des widgets -------------
@out.capture(clear_output=True, wait=True)
def pltsin(dummy):
    #clear_output(wait=True)
    N = sliderN.value
    f0 = sliderf0.value
    t = np.arange(N)
    s = np.sin(2 * pi * f0 * t)
    Sf = fft(s, NN)
    f = np.fft.fftfreq(NN)
    if c1.value:
        plt.figure(1)
        plt.clf()
        plt.plot(t, s)
    if c2.value:
        plt.figure(2)
        plt.clf()
        plt.plot(f, np.abs(Sf))
    plt.show()


pltsin(8)
sliderN.observe(pltsin, names='value')
sliderf0.observe(pltsin, names='value')
c1.observe(pltsin, names='value')
c2.observe(pltsin, names='value')
display(widgets.VBox([sliderN, sliderf0, c1, c2, out]))
```



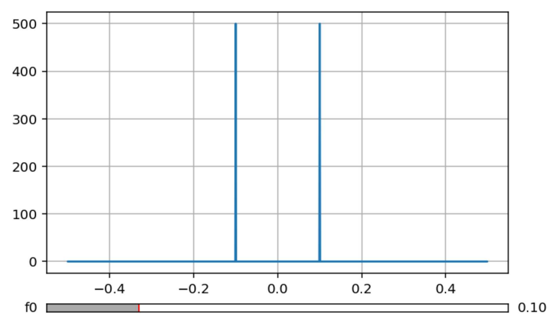



```
Widget Javascript not detected.  It may not be installed or enabled properly.
```

Contents

In [13]:
```
%matplotlib tk
from matplotlib.widgets import Slider

fig, ax = plt.subplots()
fig.subplots_adjust(bottom=0.2, left=0.1)

slider_ax = plt.axes([0.1, 0.1, 0.8, 0.02])
slider = Slider(slider_ax, "f0", 0, 0.5, valinit=0.1, color='#AAAAAA')
f = np.linspace(-0.5, 0.5, 400)
N = 1000
t = np.arange(N)
s = np.sin(2 * pi * f0 * t)
Sf = fft(s, NN)
f = np.fft.fftfreq(NN)
line, = ax.plot(f, np.abs(Sf))

ax.grid(b=True)


def on_change(f0):
    s = np.sin(2 * pi * f0 * t)
    Sf = fft(s, NN)
    line.set_ydata(np.abs(Sf))


#    line2.set_ydata(sinc(pi*L*f))

slider.on_changed(on_change)
```

Out[13]: 0



In [14]:
```
%matplotlib inline
```

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[Jupyter notebook] (Fourier_transform.ipynb)