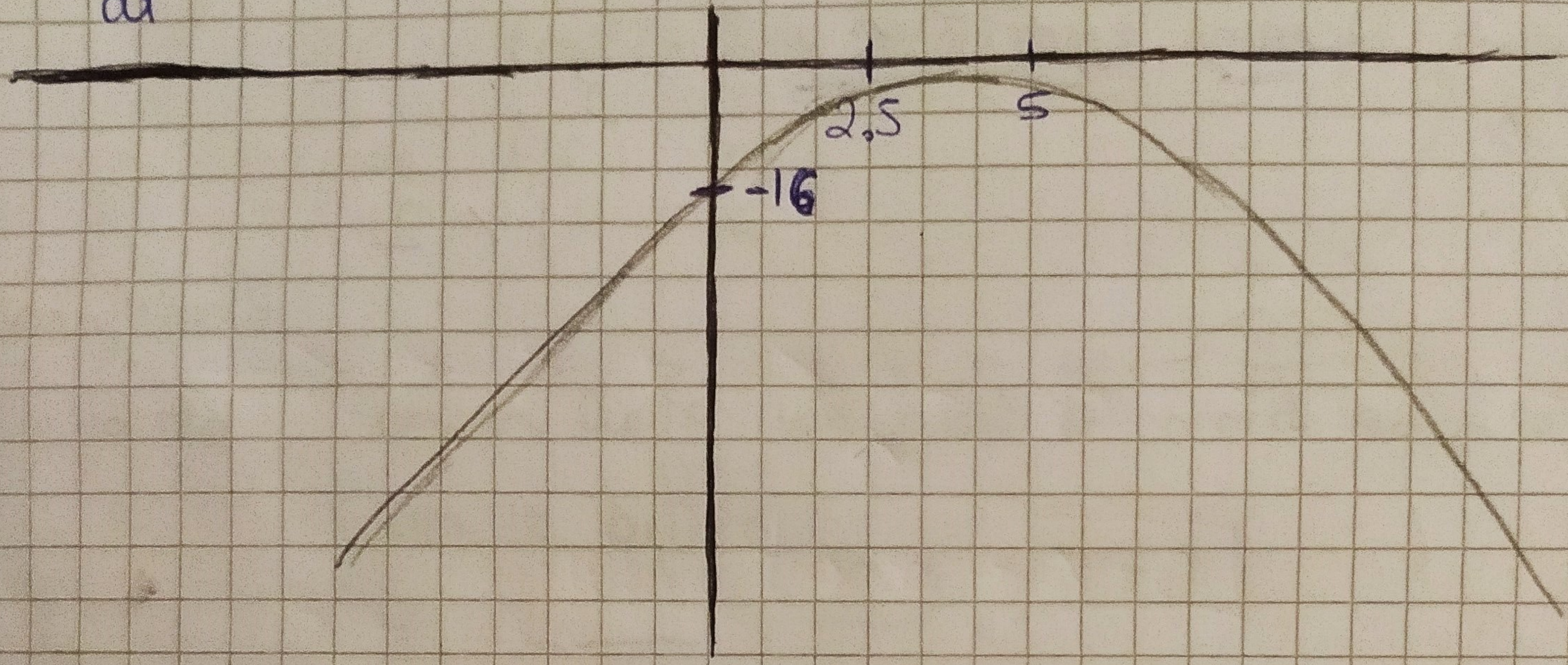
AF

Assignment 1 - Dynamic systems

DS1.

a) $\frac{dx}{dt} = -x^2 + 7x - 16$ - Phase portrait



a) $\frac{dx}{dt} = -x^2 + 7x - 16$

Equilibrium points

$\frac{dx}{dt} = 0 \Rightarrow -x^2 + 7x - 16 = 0 \Rightarrow x = \frac{-7 \pm \sqrt{49-64}}{-2} \Rightarrow x_1 = \frac{-7 + \sqrt{-15}}{-2}$

$\Rightarrow x_1 = 3.5 - \frac{\sqrt{15}}{2} i$

$x_2 = 3.5 + \frac{\sqrt{15}}{2} i$

Complex eq. points 2

Stability of eq. points

$f'(x) = -2x + 7$

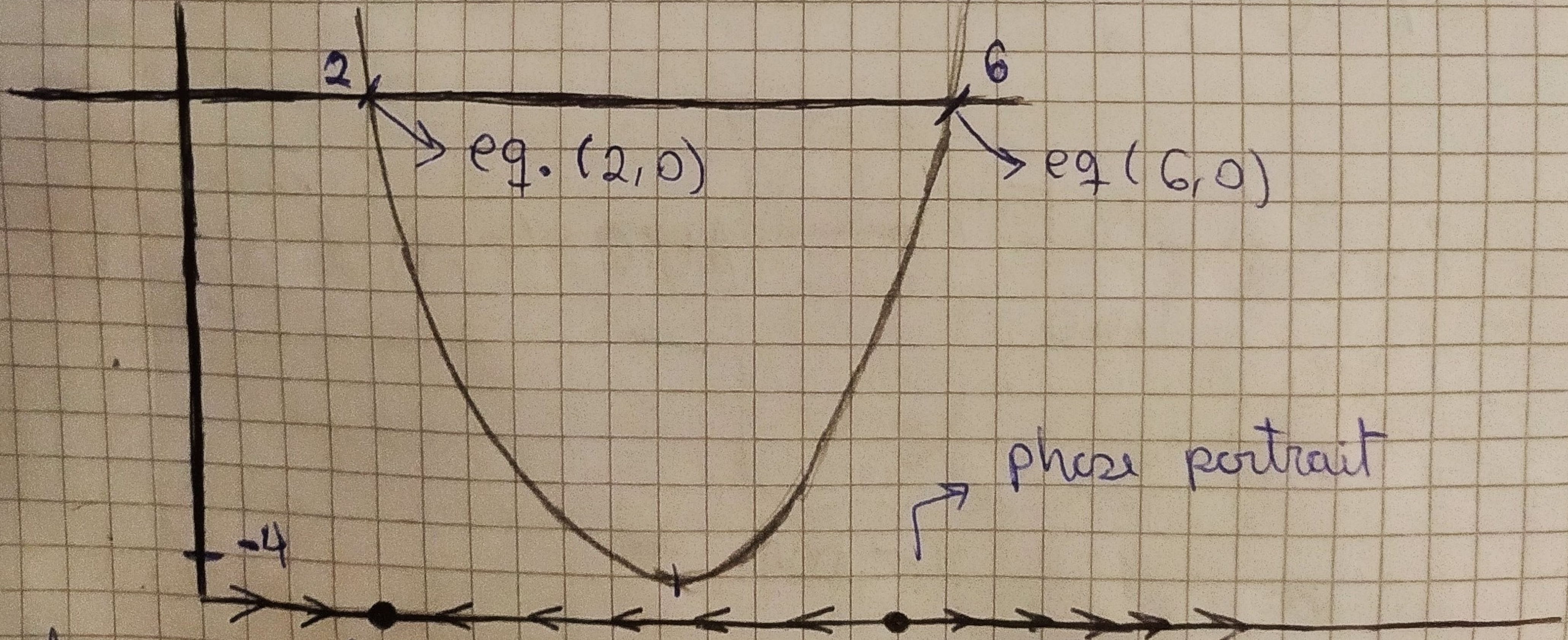
$f'(3.5 - \frac{\sqrt{15}}{2} i) = -7 + \sqrt{15} i + 7 = \sqrt{15} i > 0$; unstable at $(\sqrt{15} i, 0)$

$f'(3.5 + \frac{\sqrt{15}}{2} i) = -7 - \sqrt{15} i + 7 = -\sqrt{15} i < 0$; stable at $(-\sqrt{15} i, 0)$

when $t \to \infty$: the value of x goes to $-\infty$ in $\mathbb{R}$

There are no basins of attraction or points of attraction in $\mathbb{R}$

b) $\frac{dx}{dt} = x^2 - 8x + 12$ - Equilibria points $\rightarrow 2$



$$\frac{dx}{dt} = 0 \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow x = \frac{8 \pm \sqrt{64 - 48}}{2}$$

$$\Rightarrow x_1 = \frac{8+4}{2} \Rightarrow \left\{ \begin{array}{l} x_1 = 6 \ ; f(x_1) = 0 \\ x_2 = 2 \ ; f(x_2) = 0 \end{array} \right\}$$

$$x_2 = \frac{8-4}{2} \Rightarrow$$

b) Stability of eq. points

$f'(x) = 2x - 8$ $\Rightarrow$

$f'(2) = -4 < 0$; stable at $(2,0)$

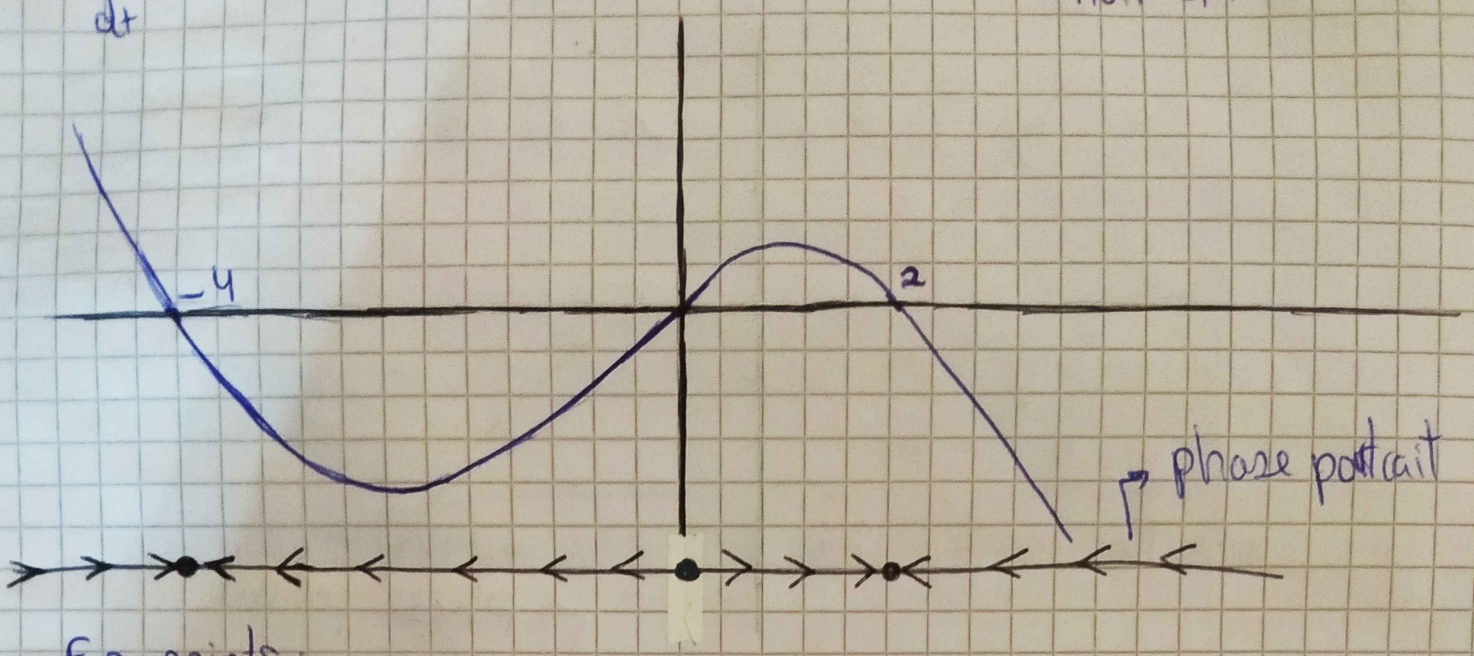
$f'(6) = 4 > 0$; unstable at $(6,0)$

when $t \to \infty$:

- If the starting point $\to x: (-\infty, 6] \to$ Basin of attraction of the final value of x is $\to x = 2$ attractor at point $(2,0)$
- If the starting point $\to x: [6, +\infty) \to$ Basin of attraction of attractor the final value of x is $\to x = 6$ at point $(6,0)$

d)

c) $\frac{dx}{dt} = -x^3 - 2x^2 + 8x$ - For each eq. point < stable / non stable



Eq points:

$f(x) = 0 \Rightarrow \frac{dx}{dt} = 0 \Rightarrow -x^3 - 2x^2 + 8x = 0 \Rightarrow x(-x^2 - 2x + 8) = 0$

$x_1 = 0 \Rightarrow$ 1 eq point at $(0,0)$

$-x^2 - 2x + 8 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4+32}}{-2} \Rightarrow x_1 = \frac{2+4}{-2} \Rightarrow \boxed{x_1 = -3} \Rightarrow$ eq. at $(-3,0)$

$x_2 = \frac{2-4}{-2} \Rightarrow \boxed{x_2 = 1}$ ↳ eq. at $(1,0)$

An equilibrium point x^* ($f(x^*) = 0$) is stable if $f'(x^*) < 0$ and unstable if $f'(x^*) > 0$

$f'(x) = -3x^2 - 4x + 8$

$f'(0) = 8 > 0$; unstable at $(0,0)$

$f'(-3) = -27 + 12 + 8 = -7 < 0$; stable at $(-3,0)$

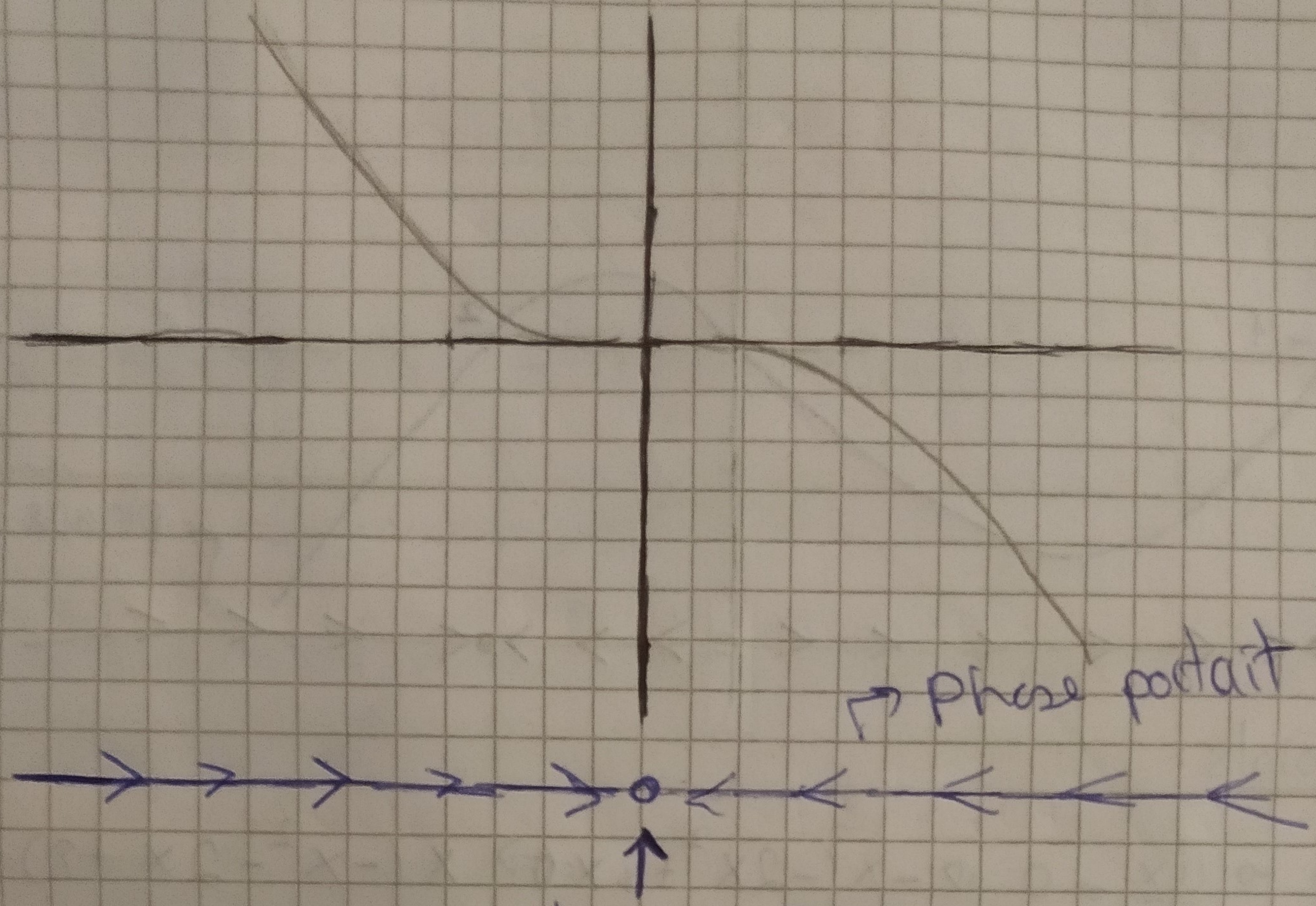
$f'(1) = -3 - 4 + 8 = 1 > 0$; unstable at $(1,0)$

When $t \to \infty$

i) starting point is between $(-\infty, 0]$: Basin of attraction of attractor at point $(-4,0)$
final x is -4

ii) starting point is between $[0, +\infty)$; Basin of attraction of attractor at point $(2,0)$
final x is 2

d) $\frac{dx}{dt} = -x^3 - 3x$



when $t \to \infty \to \boxed{x = 0}$

d)

<u>eq. points</u>

$f(x)=0 \Rightarrow -x^3-3x=0 \Rightarrow x(-x^2-3)=0 \Rightarrow \boxed{x_1=0}$ → Real 2 eq. points

$\boxed{x_2=\sqrt{-3}=\sqrt{+3}\,i}$ → complex

<u>stability</u>

$f'(x) = -3x^2-3=0 \Rightarrow 3x^2+3=0$

$f'(0)=3>$ → Unstable at $(0,0)$

$f'(\sqrt{3}i) = -9+3=-6<0$ → Stable at $(\sqrt{3}i, 0)$

<u>When $t \to \infty$:</u> $x=0$

<u>basins of attraction</u> → $\left.\begin{array}{l}(-\infty,0] \\ [0,+\infty)\end{array}\right\}$ both end in $x=0$

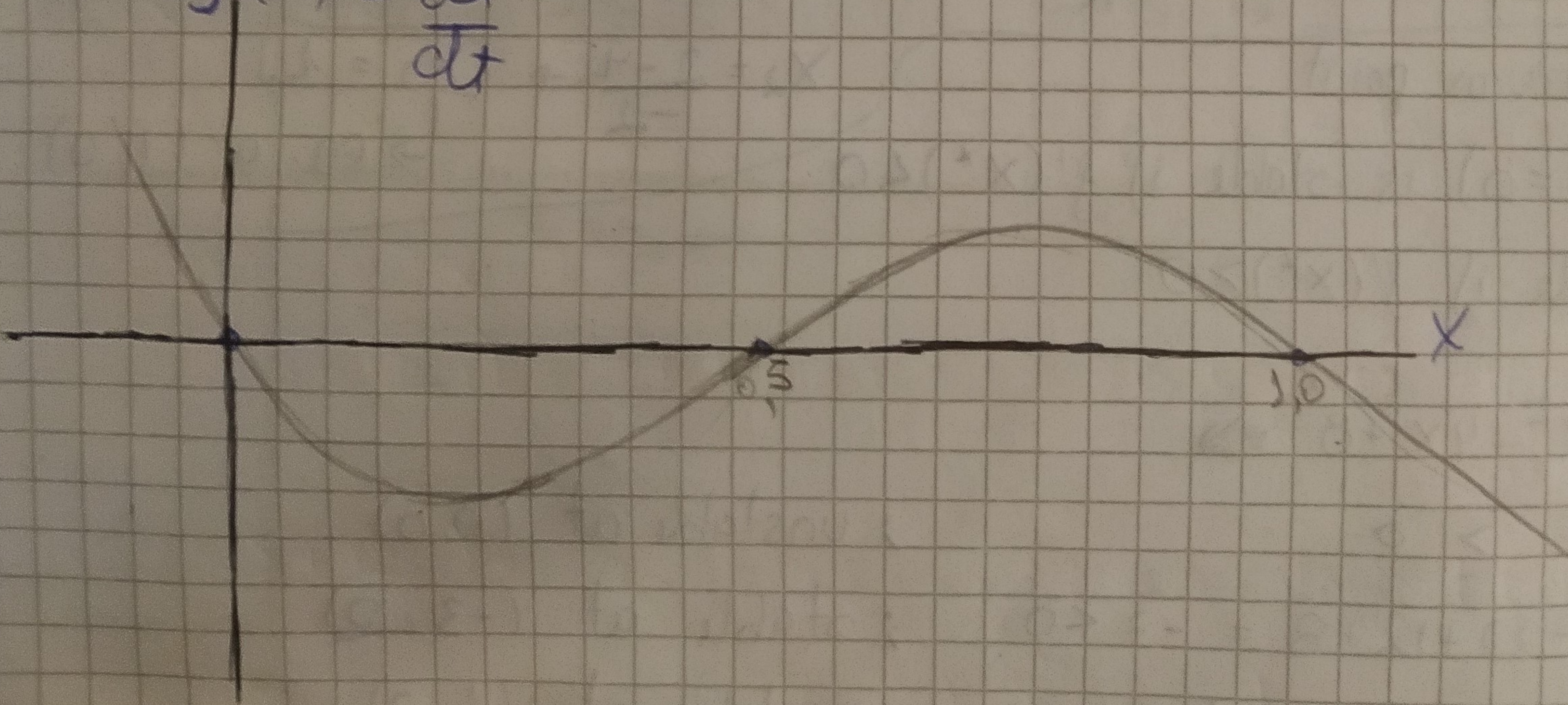
↓

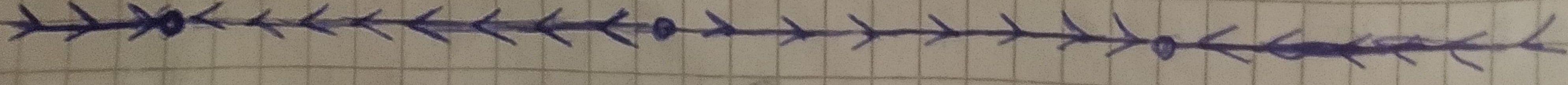
<u>attraction point</u>

e) $\frac{dx}{dt} = \frac{12x^2}{2+4x^2} - 2x$

$f(x) = \frac{dx}{dt}$

+ $>$ $-$ $<$





$f(x) = 0$

$Eq \rightarrow f(x) = 0 \longrightarrow \left.\begin{matrix}(0,0)\\(1,0)\\(0.5,0)\end{matrix}\right\}$ 3 eq. points

$$f(x) = \frac{dx}{dt} = \frac{12x^2}{2+4x^2} - 2x = 0 \Rightarrow$$

$$12x^2 = 4x + 8x^3 \Rightarrow 8x^3 - 12x^2 + 4x = 0$$

$$\Rightarrow x(8x^2 - 12x + 4) = 0$$

$$\Rightarrow x_1 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 128}}{16} \Rightarrow \boxed{x_2 = 1} \quad \boxed{x_3 = 0.5}$$

Stability of eq. points

$$f'(x) = \frac{(24x)\cdot(2+4x^2) - 12x^2\cdot(8x)}{(2+4x^2)^2} - 2$$

$$= \frac{48x + \cancel{96x^3} - \cancel{96x^3}}{4 + 16x^4 + 16x^2} - 2$$

Point $(0,0)$

$f'(0) = -2 < 0 \rightarrow$ Stable

Point $(1,0)$

$$f'(1) = \frac{48 + \cancel{96} - \cancel{96}}{36} - 2 = \frac{16}{12} - 2 = \frac{8}{6} - 2 = \boxed{\frac{4}{3} - 2}$$

$$= -\frac{2}{3} \approx -0.\hat{6} < 0 \rightarrow \text{Stable}$$

Point $(\frac{1}{2}, 0)$

$f'(0.5) = 0.\hat{6} > 0 \rightarrow$ Unstable

When $t \to \infty$:

(i) the starting point → $x: (-\infty, 0.5]$ → Basin of attraction of
the final value of x is → $x = 0$ ⟶ Attaractor at point $(0, 0)$

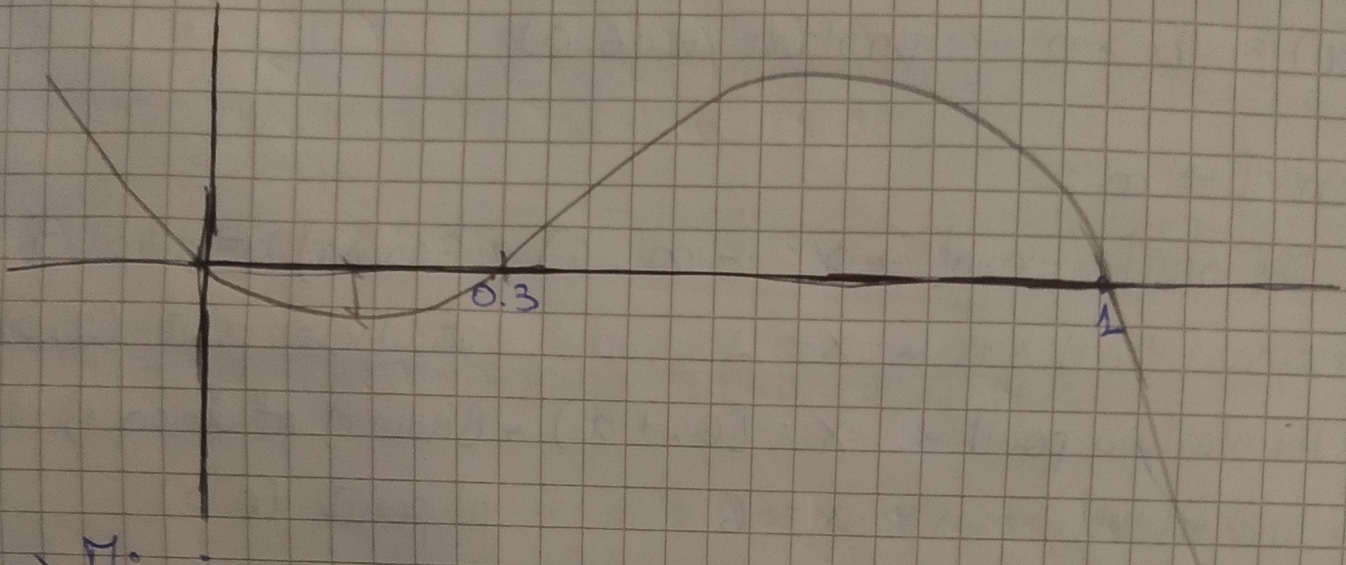
(II) the starting point → $x: [0.5, +\infty)$ Basing attraction of
the final value of x is → $x = 1$ ⟶ Atractor at point $(1, 0)$

a) $\frac{dx}{dt} = -x^2 + 7x - 16$ Equilibrium points

DS2.

(a) X and C

(b) graph



(c) Minimum

$f(x) = -x(x-0.3)(x-1) = (-x^2+0.3x)(x-1) = -x^3+x^2+0.3x^2-0.3x$

$= -x^3 + 1.3x^2 - 0.3x = x^3 - 1.3x^2 + 0.3x$

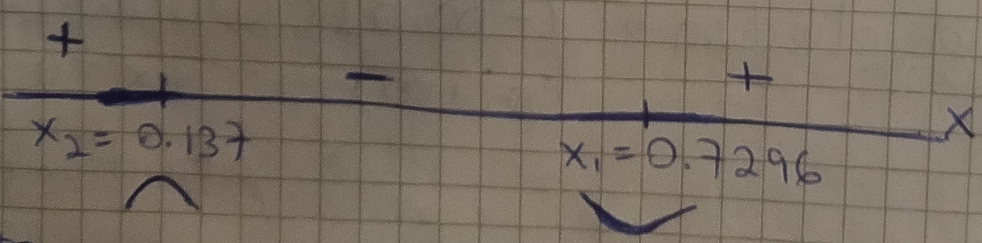
First, I need to find the critical points $\Rightarrow f'(x)=0$

$f'(x) = 3x^2 - 2.6x + 0.3 = 0 \Rightarrow$

$X = \dfrac{2.6 \pm \sqrt{6.76 - 3.6}}{6} \rightarrow X_1 = 0.7296$

$X_2 = 0.137$

Now let's evaluate the critical points



local Minimum at $\boxed{X_1 = 0.137}$

$x^* = 0.2 \begin{matrix} > X_2 \\ < X_1 \end{matrix} \Big\{ f'(0.2) = -0.1 < 0$

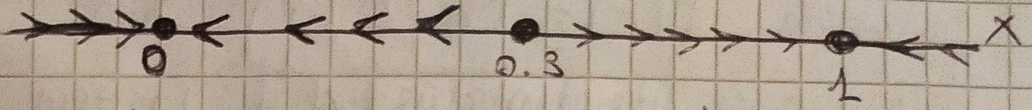
$x^* = 0.1 \begin{matrix} < X_2 \\ < X_1 \end{matrix} \Big\{ f'(0.1) = 0.07 > 0$

$x^* = 0.8 \begin{matrix} > X_2 \\ > X_1 \end{matrix} \Big\{ f'(0.8) = 0.14 > 0$

d) $C_{max} = 0.005$

When $t \to \infty$, what is x?

we have C_{max}

phase portrait: 

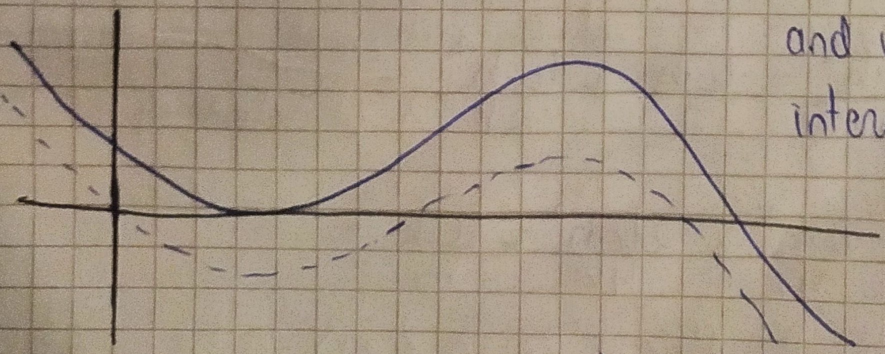
The value of X does not depend on C_{max} when $t \to \infty$

- If the starting point of X: $(-\infty, 0.3)$, then the final value is $X = 0$
- If the starting point of X: $[0.3, +\infty)$, then the final value is $X = 1$, therefore we need at least 0.3 concentration value for producing the product, which final concentration will be $X = 1$

e) Yes, it is the same. The final value of x $(t \to \infty)$ does not depend on c.

f) ~~If the initial state of $X = 0$, it will not change in time because it is ≤ 0.3, therefore:~~

If c increases, the function "moves" up ↑



and when the function $f(x)$ intersects $X = 0.137$ (minimum) the behavior of the system changes

If: $X = 0.137$ and $\frac{dx}{dt} = 0$ } Bifurcation point

$-(0.137)^3 + 1.3 - 0.3 \cdot 1.3 = C \Rightarrow \boxed{C = 0.91} \Rightarrow C_{max}$

DS3.

$H = h \cdot n$

$$\frac{dn}{dt} = rn\left(1-\frac{n}{k}\right) - hn = rn\left(1-\frac{n}{k}\right) - H$$

The maximal yied occurs if H equals to the maximum of the parabola $f(n) = rn\left(1-\frac{n}{k}\right)$. We find it by determining where its derivative is zero

$f'(n) = r - 2 \cdot \frac{r}{k} \cdot n$

$f'(n) = 0 \Rightarrow n = \frac{k}{2}$

Substituting in $f(n)$:

$$f\left(\frac{k}{2}\right) = r \cdot \frac{k}{2} - \frac{r}{k} \cdot \frac{k^2}{4} = \frac{r \cdot k}{2} - \frac{r \cdot k}{4}$$

$$= \frac{2 \cdot r \cdot k}{4} - \frac{r \cdot k}{4} = \frac{r \cdot k}{4}$$

$$\boxed{H_{max} = \frac{r \cdot k}{4} = hn}$$

DS4.

$$\begin{cases} dN/dt = aN - bNC \\ dC/dt = Q - \alpha C + \beta NC \end{cases}$$

$\underbrace{a, b, Q, \alpha, \beta}_{\text{positive constants}}$

$aN \rightarrow$ algae that grow
$bNC \rightarrow$ algae that dies

$Q \rightarrow$ quantity of chemical that's added
$\alpha C \rightarrow$ chemical that dissapears because of absence of algae
$\beta NC \rightarrow$ chemical that's eaten by the algae

a)

$a = 1$ $\alpha = 1$
$b = 1$ $\beta = 1$

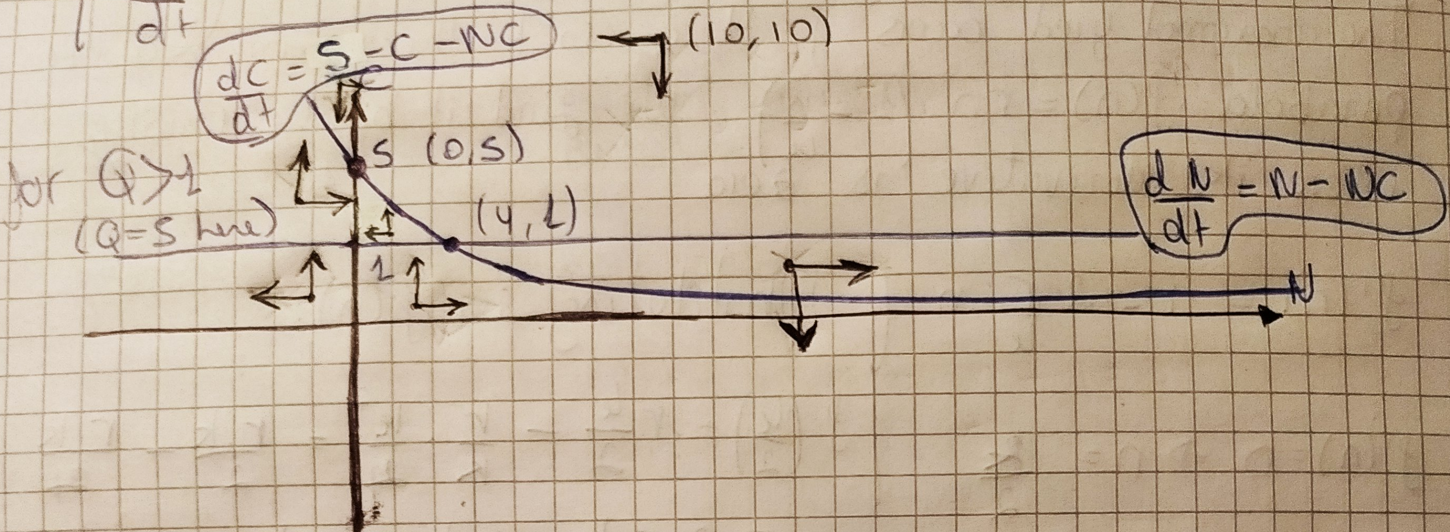
$f'(N) = a - bC$
$f'(C) = -\alpha + \beta N$

$$\begin{cases} \frac{dN}{dt} = N - NC \\ \frac{dC}{dt} = Q - C - NC \end{cases}$$

If: $Q = 5$; $\frac{dN}{dt} = 0$; $\frac{dC}{dt} = 0$

$\cancel{N} = \cancel{N}C \Rightarrow C = 1$

$N = 5 - 1 \Rightarrow N = 4$



$N - NC = 0$; $-NC - C + 5 = 0$ $\Big\}$ $\underbrace{C=1,\ N=4 \ \Big|\ C=5,\ N=0}$

Solutions are the non-equilibric points, where the both functions intersect and $\frac{dC}{dt}$ intersects with zero

- $(10, 10)$

$\frac{dN}{dt} = 10 - 100 = -90 < 0$

$\frac{dC}{dt} = 5 - 10 - 100 = -105 < 0$

P_1 : $(0,5)$ Stable

The concentration of the chemical is zero, and the change on the algae population is zero. It is the point of extinction of the algae

P_2 : $(4,1)$ Unstable → Saddle point ⊛

The area formed by P_1, P_2 and $(N=0, C=1)$ is the area where any point that starts there is going to be atracted by P_1 → extinction of the algae

This area can be inverted increasing a

(*)

$$J = \begin{pmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial C} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial C} \end{pmatrix} = \begin{bmatrix} 1-C & -N \\ -C & -1-N \end{bmatrix}$$

P (4, 1)

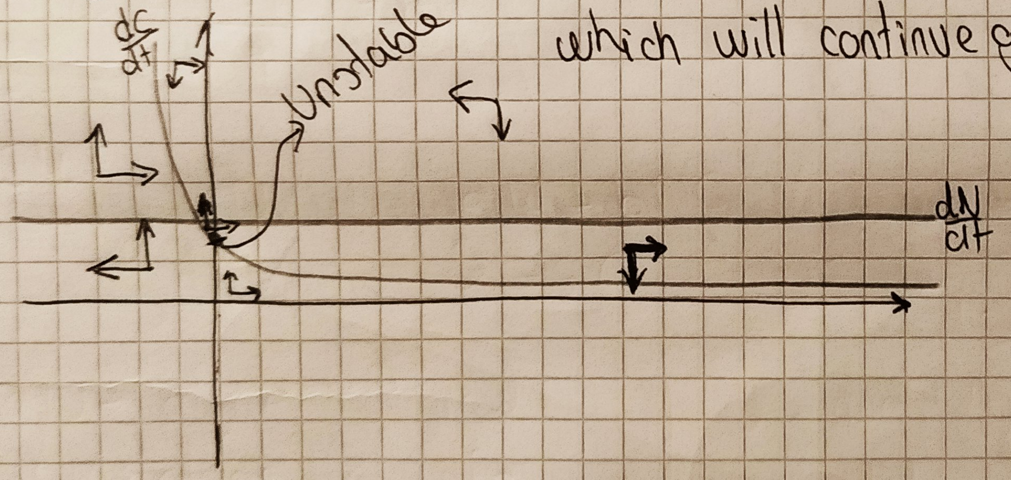
Eigenvalues:

$$\begin{bmatrix} 0 & -4 \\ -1 & -5 \end{bmatrix} \quad (0-\lambda)\cdot(-5-\lambda) - 4 = 0$$

$$5\lambda + \lambda^2 - 4 = 0 \Rightarrow \lambda = \frac{-5 \pm \sqrt{25+16}}{2} \Rightarrow$$

$\lambda_1 = 0.7$; $\lambda_2 = -5.7$ => (4,1) is a saddle point

b) If $Q < 1$ → the chemical is not going to kill the algae, which will continue growing



. (10, 10)

$\frac{dN}{dt} = -90 < 0$

$\frac{dC}{dt} = -101 < 0$

c) $\frac{dN}{dt} = aN - bNC = 0$

$N(a-bC) = 0$ $\qquad N=0$ or $C = \frac{a}{b}$

two lins

$\frac{dC}{dt} = Q - \alpha C - \beta NC = 0$ $\qquad$ one line

- for $N=0$ $\qquad Q - \alpha C = 0$

$C = \frac{Q}{\alpha} \rightarrow P_1 = \left(N=0, C = \frac{Q}{\alpha}\right)$

- for $C = \frac{a}{b}$: $Q - \alpha \cdot \frac{a}{b} - \beta N \frac{a}{b} = 0$

$\alpha \cdot \frac{a}{b} + \beta . N . \frac{a}{b} = Q$

$N = \frac{1}{\beta}\left(Q \cdot \frac{b}{a} - \alpha\right) \rightarrow P_2 = \left(N = \frac{1}{\beta}\left(Q \cdot \frac{a}{b} - \alpha\right), C = \frac{a}{b}\right)$

From these points and the phase portrait we have the dollowing relations:

$\boxed{C_0 > \frac{a}{b}}$ and $\boxed{N_0 < \frac{1}{\beta}\left(Q \cdot \frac{b}{a} - \alpha\right)}$

and $\boxed{\frac{Q}{\alpha} > \frac{a}{b}}$

DS4. Ch5-1

a) $\begin{cases} \frac{dx}{dt} = x - 3y \\ \frac{dy}{dt} = x + y \end{cases}$ $\quad A = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$

Aigenvalues of A

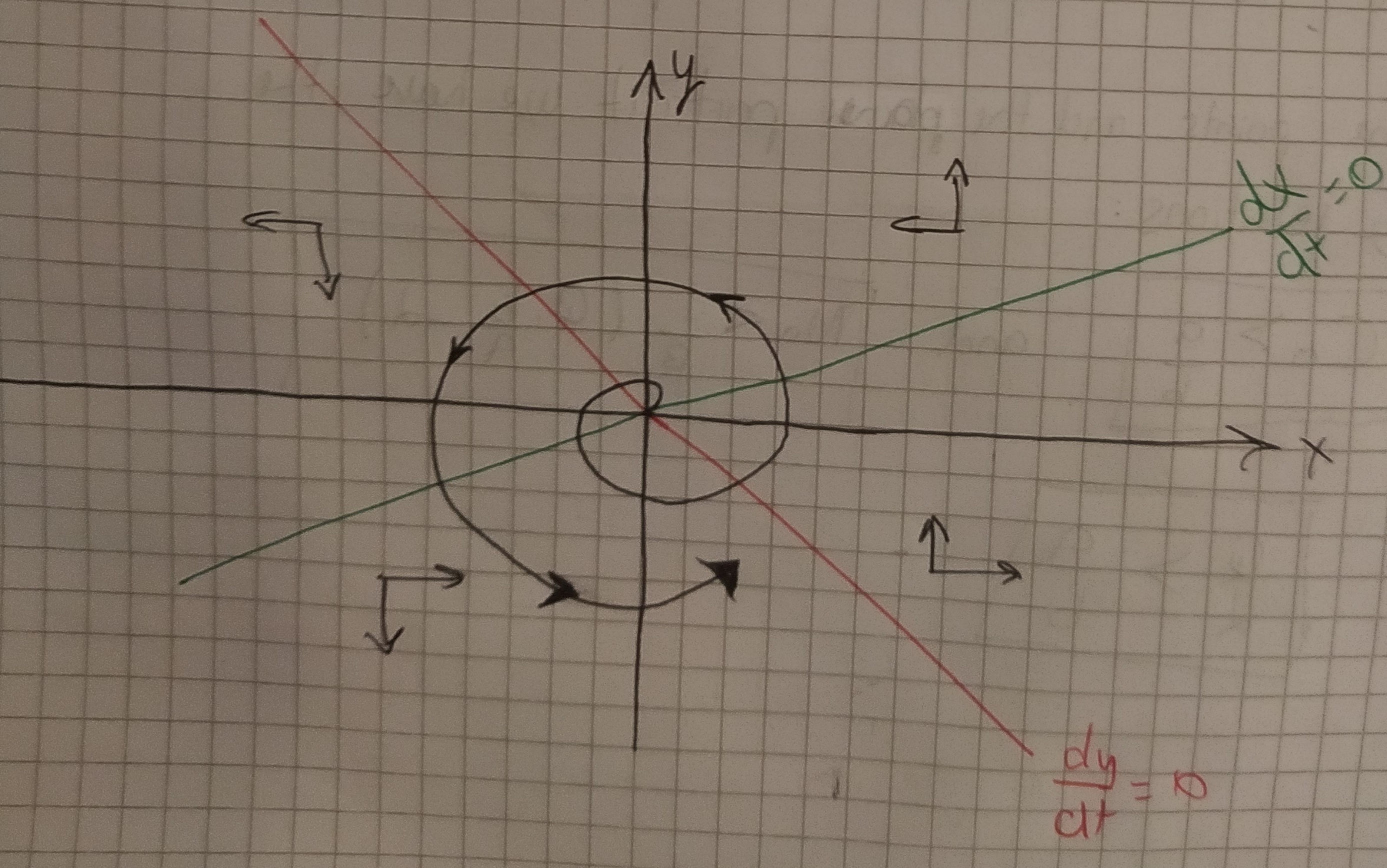
$$\det(A - I\lambda) = \begin{vmatrix} 1-\lambda & -3 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 3 = 1 + \lambda^2 - 2\lambda + 3$$
$$= \lambda^2 - 2\lambda + 4 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} \Rightarrow \begin{cases} \lambda_1 = 1 + \sqrt{12}\,i \\ \lambda_2 = 1 - \sqrt{12}\,i \end{cases} \quad \boxed{\alpha > 0}$$

$\frac{dx}{dt} = 0 = x - 3y$

$\frac{dy}{dt} = 0 = x + y$

The equilibrium point is a unstable spiral



b) $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Eigenvalues:

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ $\lambda_1 = +j$

$\lambda_2 = -j \rightarrow \boxed{\alpha = 0}$

$\dfrac{dx}{dt} = 0 = y$

The equilibrium is a Center Point

$\dfrac{dy}{dt} = 0 = -x$