THE DRIVEN PENDULUM

DERIVING THE EQUATION OF MOTION FOR THE VERTICALLY DRIVEN PENDULUM

The vertically driven pendulum is an inverted simple pendulum whose pivot oscillates up and down with amplitude A and frequency v. As with the simple pendulum, the driven pendulum only has one degree of freedom, and so its position at any time t can be described just with the angle θ that the pendulum makes with the vertical reference line.

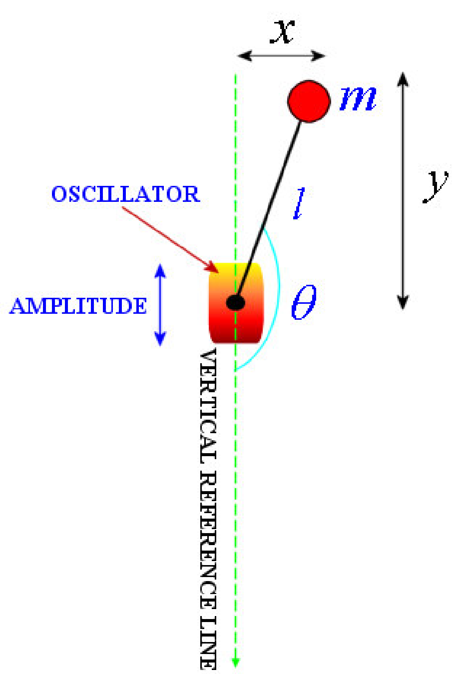

Figure 1: The Vertically Driven Pendulum

José and Saletan (2002) state the vertical position of the pivot (p_v) as:

$$p_V = A\cos(vt) \qquad \textbf{(1)}$$

Using equation **(1)**, we can see from **Figure 1** that:

$$x = l\sin(\theta)$$
$$y = l\cos(\theta) + A\cos(vt)$$

In order to find the Lagrangian function for this system, once again we can start by finding the kinetic energy:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

Where:

$$\dot{x} = l\dot{\theta}\cos(\theta)$$
$$\dot{y} = -l\dot{\theta}\sin(\theta) - Av\sin(vt) = -(l\dot{\theta}\sin(\theta) + Av\sin(vt))$$

So therefore:

$$T = \frac{m}{2}(\dot{\theta}^2 l^2 \cos^2(\theta) + l^2\dot{\theta}^2 \sin^2(\theta) + A^2 v^2 \sin^2(vt) + 2l\dot{\theta}Av\sin(\theta)\sin(vt))$$

Simplifying we get:

$$T = \frac{m}{2}(l^2\dot{\theta}^2 + A^2 v^2 \sin^2(vt) + 2l\dot{\theta}Av\sin(\theta)\sin(vt))$$

Next we find the potential energy:

$$U = -mgy$$
$$U = -mg(l\cos(\theta) + A\cos(vt))$$

And remember that:

$$L = T - U$$

So:

$$L = \frac{m}{2}(l^2\dot{\theta}^2 + A^2 v^2 \sin^2(vt) + 2l\dot{\theta}Av\sin(\theta)\sin(vt)) + mg(l\cos(\theta) + A\cos(vt))$$

Using the property **(1)** from the documentation for the double pendulum:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Where:

$$\frac{\partial L}{\partial \theta} = ml\dot{\theta}Av\cos(\theta)\sin(vt) - mgl\sin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + mlAv\sin(\theta)\sin(vt)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mlAv\dot{\theta}\cos(\theta)\sin(vt) + mlAv^2\sin(\theta)\cos(vt)$$

We arrive at:

$$\begin{pmatrix} ml^2\ddot{\theta} + mlAv\dot{\theta}\cos(\theta)\sin(vt) + mlAv^2\sin(\theta)\cos(vt) - ml\dot{\theta}Av\cos(\theta)\sin(vt) \\ + mgl\sin(\theta) \end{pmatrix} = 0$$

Rearranging and simplifying this, we find the equation of motion for the vertically driven pendulum to be:

$$\ddot{\theta}_V = \frac{-(g + Av^2\cos(vt))\sin(\theta)}{l}$$

Where $\ddot{\theta}_V$ denotes the angle θ describing the position of the vertically driven pendulum differentiated twice with respect to time.

DERIVING THE EQUATION OF MOTION FOR THE HORIZONTALLY DRIVEN PENDULUM

The horizontally driven pendulum is the same as the vertically driven pendulum, but the pivot oscillates horizontally instead of vertically.

So:

$$x = l\sin(\theta) + p_H$$
$$y = l\cos(\theta)$$

Where p_H denotes the horizontal position of the pivot, and is given by:

$$p_H = A\cos(vt)$$

Thus:

$$x = l\sin(\theta) + A\cos(vt)$$

Again, in order to find the Lagrangian for this system, we start by finding the kinetic energy:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

Where:

$$\dot{x} = l\dot{\theta}\cos(\theta) - Av\sin(vt)$$
$$\dot{y} = -l\dot{\theta}\sin(\theta)$$

Therefore:

$$T = \frac{m}{2}(l^2\dot{\theta}^2\cos^2(\theta) - 2l\dot{\theta}Av\sin(vt)\cos(\theta) + A^2v^2\sin^2(vt) + l^2\dot{\theta}^2\sin^2(\theta))$$

Simplifying we get:

$$T = \frac{m}{2}(l^2\dot{\theta}^2 - 2l\dot{\theta}Av\sin(vt)\cos(\theta) + A^2v^2\sin^2(vt))$$

Next we find the potential energy:

$$U = -mgy$$

So:

$$U = -mgl\cos(\theta)$$

Again, remember that:

$$L = T - U$$

Thus the Lagrangian function of this system is:

$$L = \frac{m}{2}(l^2\dot{\theta}^2 - 2l\dot{\theta}Av\sin(vt)\cos(\theta) + A^2v^2\sin^2(vt)) + mgl\cos(\theta)$$

Using the property **(1)** from the documentation from the double pendulum, we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Where:

$$\frac{\partial L}{\partial \theta} = -mgl\sin(\theta) + ml\dot{\theta}Av\sin(vt)\sin(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - mlAv\sin(vt)\cos(\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - mlAv^2\cos(vt)\cos(\theta) + mlAv\sin(vt)\dot{\theta}\sin(\theta)$$

So:

$$ml^2\ddot{\theta} - mlAv^2\cos(vt)\cos(\theta) + mlAv\sin(vt)\dot{\theta}\sin(\theta) + mgl\sin(\theta) - ml\dot{\theta}Av\sin(vt)\sin(\theta) = 0$$

Rearranging and simplifying this, we find the equation of motion for the horizontally driven pendulum to be:

$$\ddot{\theta}_H = \frac{-g\sin(\theta) + Av^2\cos(vt)\cos(\theta)}{l}$$

Where $\ddot{\theta}_H$ denotes the angle θ describing the position of the horizontally driven pendulum differentiated twice with respect to time.

PROGRAMMING THE JAVA APPLET

I realised that graphically, both the vertically and the horizontally driven pendulums would be almost identical; and both would be very similar to the simple pendulum. With this in mind, I programmed a basic applet based heavily on that of the simple pendulum, but with a radio-box 'switch' to alternate between the vertically and horizontally driven pendulums.

The difference between this applet and that for the simple pendulum is the oscillating pivot.

As mentioned in the derivations for the equations of motion for this applet, the position of the pivot at time t is given by:

$$p_i = A\cos(vt)$$

Where p_i is either the vertical or horizontal position of the pivot, dependent on whether it oscillates vertically or horizontally.

It is obvious that the position of the pivot represents the reference point from which the coordinates of the rest of the pendulum are calculated, and so a function needed to be created that changes the position of the pivot/reference point over time, but with respect to the original reference point (when the amplitude of the oscillations $A = 0$).

Clearly, when the pivot oscillates vertically, its x-coordinate remains constant, but its y-coordinate changes with respect to time, where:

$$refy' = refy + \frac{l' A \cos(vt)}{l} \qquad \textbf{(2)}$$

Note that $refy'$ represents the new, dynamically changing y-coordinate of the pivot/reference point; $refy$ denotes the original, static y-coordinate of the pivot/reference point; and the $\frac{l'}{l}$ term scales the amplitude of the oscillations to the same as that scale which is used to draw the rest of the pendulum.

When the pivot oscillates horizontally, its y-coordinate remains constant, but its x-coordinate changes with respect to time, where:

$$refx' = refx + \frac{l' A \cos(vt)}{l}$$

Note that similar notation has been used in this equation to describe both the original and the new x-coordinates, as was used for the y-coordinates in equation **(2)**.

In order to draw attention to the fact that the pivot is oscillating I then plotted a green circle over the new, moving pivot point and a red crosshair centred over the original, static reference point. This makes it easier for the user to visualise the pivot oscillating. Finally I added a checkbox to give the user the option of turning the oscillations on and off.

I noticed the animation appeared to be more abrupt with this pendulum than with those investigated previously, so I decreased the time-step used in the geometric integration, and thus decreased the time delay between each frame. This meant more frames were produced per second, and therefore the animation was smoother. I then applied the same method to make the animation smoother in the previous applets.