Structural Analysis & Design summary: part III 2017-2018 extended edition

Based on *Aircraft Structures for Engineering Students* by T.H.G. Megson

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November 2016 - January 2017

Version 9.0

Contents

Preface

Unfortunately, the content regarding buckling and Castigliano's theorem is rather different this year, which kinda sucks because half of the stuff I wrote last year no longer applies, and I need to add a bunch of stuff that's totally new for me. Furthermore, I kinda rushed writing Castigliano's theorem as last year I didn't really have time to finish it properly (it was the last topic treated in the course, and I kinda had to start writing it 48 hours before the exam), so I want to make that a bit clearer as well. I'll do that during the Christmas break, probably at the beginning but please bear in mind that it is a considerable amount of stuff I need to update. **If you want to use this summary, my best advise is to enjoy the first days of your Christmas break a bit and wait till I update the summary**, cause I'm probably just going to delete everything after section 8.3 and start over again due to the large changes with respect to last year.

This summary is fully finished now.

Other than that, I've published two versions of this summary: the summary containing only the chapters on buckling and Castigliano, and the summary that contains all of the chapters we've discussed so far, including buckling and Castigliano. Idk which one is easier to use for you guys, but for me this small summary is easier to work in as I don't have 5000 lines of LaTeX-code to go through, and compilation is a bit faster so that's why I'm including this one as well.

Changelog

Version 9.0

- Finished chapter 5.

Version 8.5

- Added chapters 5.3 and part of 5.4. The summary is almost finished, I still want to add two examples (one of which I can copy almost literally from last year) and that's it.

Version 8.4.1

- Added chapters 5.1 and 5.2. It's a lot of mathematics currently though so don't worry about it too much yet. Only the boxes are important basically.

Version 8.4

- Added chapter 4. Chapter 5 is coming asap, and apologies for the delay; but I had to basically work out the entire dynamics assignment on my own so I prioritized that.

Version 8.3

- Added section 9.3.

Version 8.2

- Added section 9.2.

Version 8.1.1

- Added section 9.1.

Version 8.1

- Finished chapter 8, and removed section 8.5 since it's not part of the exam. Note that I still need to add the whole of chapter 9 since we did not do that at all last year.

16 Bending of open and closed, thin-walled beams

16.1 Symmetrical bending

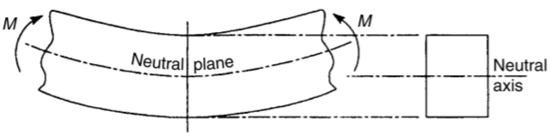


Figure 16.1: Beam subjected to a pure sagging bending moment.

Suppose we have a nice beam and subject it to a pure sagging bending moment[1], as depicted in figure 16.1. Rather obviously, this will lead to the fibers on the top to be shortened, whereas the fibers on the bottom are elongated. However, there is one fiber, somewhere in between, which isn't elongated at all. The plane containing these fibers is called the **neutral plane** and the line of intersection of the neutral plane and any cross-section of the beam is termed the **neutral axis**.

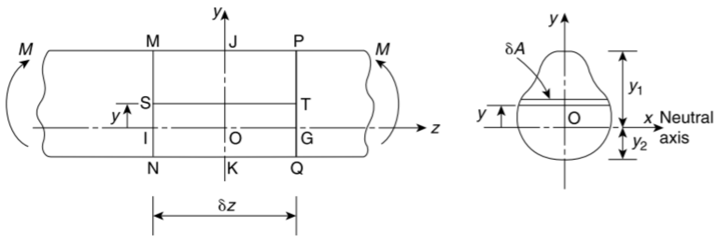


Figure 16.2: Bending of a symmetrical section beam.

Now, suppose we have a beam of arbitrary shape, but with a vertical axis of symmetry, as shown in figure 16.2. Under a pure bending moment, this deforms as shown in figure 16.3.

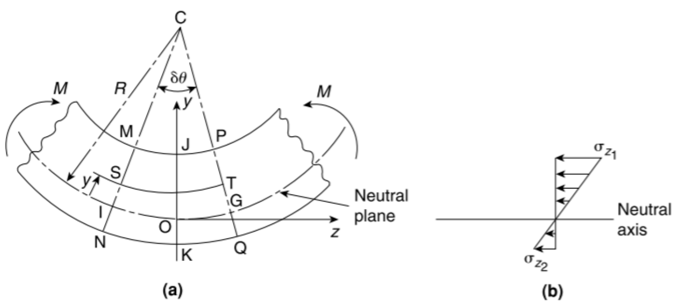


Figure 16.3: Length of beam subjected to a pure bending moment.

Now, please note that the neutral axis is *not* the middle fiber, as clearly visible from figure 16.3. The reason for this is rather simply that the bottom of the cross-section is "fatter" than the top, moving the neutral axis downward. Now, we can derive the following. Looking at fiber ST, which is located a distance y (which is

[1]Sagging means that the middle of the beam goes down. Had it gone upwards, then it'd be called hogging.

positive as defined by the coordinate system), we have that the elongation equals:

$$\begin{aligned} e_z &= \frac{\text{change in length}}{\text{original length}} = \frac{(R-y)\,\delta\theta - R\delta\theta}{R\delta\theta} = -\frac{y}{R} \\ \sigma_z &= -E\frac{y}{R} \end{aligned}$$

Now, R is not something that you can rather easily measure, obviously, as typically the radius will be rather large if it does not bend too much. So, we must do it in another way: t force produced by an infinitesimal area dA (that is, an area with height dy and thickness t (thickness into the page)) will simply be $\sigma_z dA$. However, as we have applied a bending moment M, we know that the following integral,

$$\int_A y\,\sigma_z\,dA = -\frac{E}{R}\int_A y^2\,dA$$

Must be equal to the applied bending moment. However, you may remember that $\int_A y^2\,dA$ is equal to the second moment of area, I. Hence, we have $M = -\frac{E}{R}I$, and hence we can write

$$\frac{M}{I} = -\frac{E}{R} = \frac{\sigma_z}{y}$$

And hence we arrive at the formula you already knew:

FORMULA

$$\sigma_z = \frac{My}{I} \tag{16.1}$$

Now, let's do some examples:

Example 1

The cross-section of a beam has the dimensions shown in figure 16.4. The beam is subjected to a negative bending moment of 100 kNm around the x-axis. Determine the distribution of the direct stress through the depth of the section.

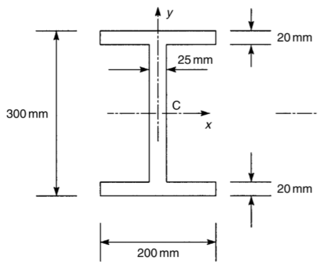


Figure 16.4: Beam of example.

So, if you use the right-hand rule, with your thumb pointing towards the left (as it points in negative x-direction), you see that in the top of the beam, the beam will be compressed, whereas in the bottom, it'll be in tension. Now, we must first find the second moment of area around the x-axis, I_{xx}. Now, you can use Steiner, but you can also realize that this shape is similar to a rectangle of 200 mm by 300 mm, where two rectangles of 87.5 mm by 260 mm have been taken out. As these two rectangles together have their centroid at the same place as the total shape, you don't have any Steiner terms:

$$I_{xx} = \frac{200 \cdot 300^3}{12} - \frac{175 \cdot 260^3}{12} = 193.7 \times 10^6\,\text{mm}^4$$

We then rather simply have that the distribution of the direct stress σ_z at any location y is given by

$$\sigma_z = -\frac{100 \cdot 10^6}{193.7 \cdot 10^6}y = -0.52y$$

Hence, the direct stress varies linearly, from $-0.52 \cdot 150 = -78\,\mathrm{N/mm^2}$ at the top to $-0.52 \cdot -150 = 78\,\mathrm{N/mm^2}$ at the bottom.

This shouldn't be too difficult at all. However, we can complicate it, unfortunately. First of all, we can have a bending moment around y rather than x; however, this should be fairly obvious. However, stuff gets more complicated if we have a moment around x and y at the same time, and they can confuse you even more by posing the question as:

Example 2

The beam section of the previous example is subjected to a bending moment of 100 kNm applied in a plane parallel to the longitudinal axis of the beam but inclined at 30° to the left of vertical. The sense of the bending moment is clockwise when viewed from the left-hand edge of the beam section. Determine the distribution of direction stress.

Now, it'll probably confuse you what the direction of the bending moment is when they talk about it like this, and figuring this out is indeed practically the only true difficulty regarding this question: remember that when a moment is applied, it's in fact nothing more than a couple consisting of two forces, working in a plane, but in opposite directions. This created a moment around the axis *perpendicular* to the plane the couple is working on. In this example, it is thus described that the force couple works in a plane which is inclined 30° to the left of the vertical (it's helpful to sketch this). Hence, we know that the moment itself points along an axis that is inclined 30° counterclockwise to the horizontal axis. Now, we must only figure out whether it then points towards the upper right or towards the lower left. This can be figured out by the sentence "clockwise when viewed from the left-hand edge": step into the paper and walk towards the left of the beam. Then, look towards the right, where you'll see a gigantic plane passing through the centroid. This plane is inclined 30° to the left of the vertical, and on it, two forces are drawn by God to indicate that he wanted to create a couple moment there. Apparently, these forces seem to create a clockwise moment. Applying your right-hand rule, you can realize that this means that the moment will be pointing towards the top right rather than to the bottom left of the front view.
Now that we know all of this, we are simply allowed to decompose the moment:

$$\begin{aligned} M_x &= 100\cos 30^\circ = 86.6\,\mathrm{kNm} \\ M_y &= 100\sin 30^\circ = 50.0\,\mathrm{kNm} \end{aligned}$$

Both pointing in positive directions. However, M_x will create a positive stress (tension) for positive y-values (as apparent from the right-hand rule), whereas M_y will create a negative stress (compression) for positive x-values. Hence, the correct formula for σ_z will be

$$\sigma_z = \frac{M_x}{I_{xx}}y - \frac{M_y}{I_{yy}}x$$

So, please, please do not automatically that if the moments are in positive direction, you can just plusses everywhere. Sign conventions can prevent some confusion (will discuss them in the next section), but even then, it all depends on how you've established your coordinate system. Hence, as a tip, every time signs start to get important, wonder yourself whether it should be positive or negative; don't automatically assume anything.
Anyway, calculation of I_{yy} gives $I_{yy} = 27.0 \times 10^6\,\mathrm{mm^4}$, and hence our stress formula becomes

$$\sigma_z = \frac{86.6 \cdot 10^6}{193.7 \cdot 10^6}y - \frac{50.0 \cdot 10^6}{27.0 \cdot 10^6}x = 0.45y - 1.85x$$

So, we again clearly see linearity, which is depicted in figure 16.5.

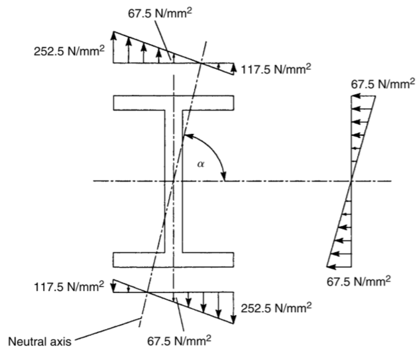


Figure 16.5: Direct stress distribution in beam of example.

Now, you may have been wondering for the last example, what happened to the neutral axis? The neutral axis is not the axis around which the moment is applied (definitely not), but it's the line where $\sigma_z = 0$. In other words, we have to solve:

$$\begin{aligned} 0.45y - 1.85x &= 0 \\ \frac{y}{x} &= \frac{1.85}{0.45} = 4.11 = \tan\alpha \end{aligned}$$

Or $\alpha = 76.3°$. Note that more generally, for symmetric beams, that $\tan\alpha = \frac{M_y I_{xx}}{M_x I_{yy}}$.

FINDING THE DIRECT STRESS FOR BENDING OF SYMMETRIC BEAMS

1. Establish a coordinate system.
2. Find the coordinates of the centroid.
3. Find the second moments of area.
4. Decompose the moment in a moment around the x-axis and y-axis.
5. Find the stress at any location by applying

$$\sigma_z = \frac{M_x}{I_{xx}} y + \frac{M_y}{I_{yy}} x \tag{16.2}$$

However, be absolutely careful regarding the signs: M_x should be positive if it causes tension in the positive y-direction, and M_y should be positive if it causes tension in the positive x-direction. If they cause compression in the positive direction, there should be a minus sign.

In the next section, we will look at a sign convention to prevent confusion with the minus signs.

16.2 Unsymmetrical bending

16.2.1 Sign conventions and notation

The sign convention the book uses is depicted in figure 16.6. It's pretty logical actually: all (distributed) forces and displacements are simply positive in the positive direction; T is simply the right hand rule applied to the z-axis. Now, M_x and M_y are the only ones that may be confusing: both of them are positive such that they create a tension force in the positive xy quadrant of the cross-section. We see that in the way they are drawn in figure 16.6, this is indeed the case: in the part above the x-axis and to the right of the y-axis (the positive xy quadrant), M_x and M_y both seem to "pull" the beam material out of the paper, causing tension.

If you strictly follow this sign convention at *all* times, you won't have any problems with forgetting minus signs etc. However, you may use *any* coordinate system you want (and you may establish it yourself, as long as it's right-handed). It's far more important to check regularly during your calculations whether your signs are still correct. Making sketches every now and then isn't a bad idea either.

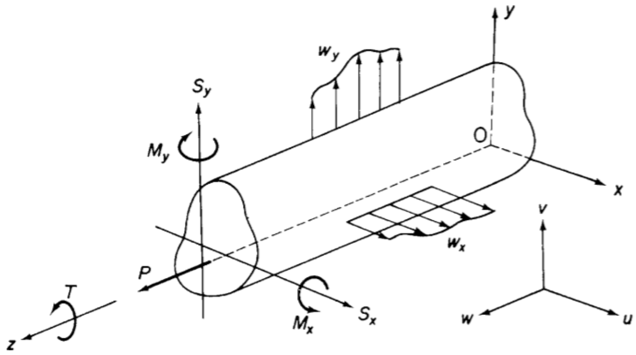


Figure 16.6: Notation and sign convention forces for external forces, moments and displacements.

For internal forces, we have the sign convention depicted in figure 16.7, which should be rather clear, I think. Please note that in the book (and I will too), they will pretty much always look at the *dashed* surface to calculate the internal moments and forces. So, you look at a surface that has its normal pointing in positive z-direction.

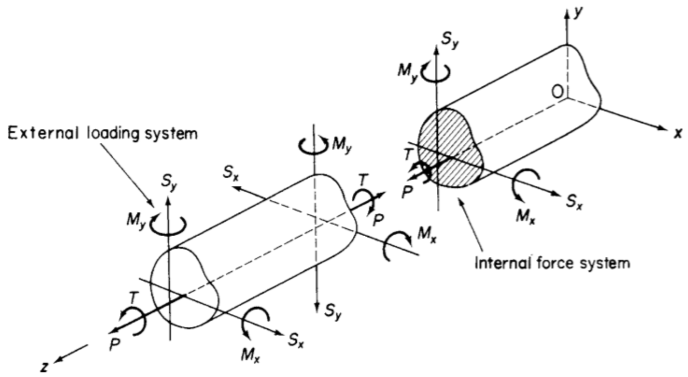


Figure 16.7: Notation and sign convention forces for internal forces and moments.

16.2.2 *Direct stress distribution due to bending*

Now, back to business. You may have wondered, when is a beam said to be symmetrical, and how does symmetry affect anything at all? Symmetrical beams are shown in figure 16.8. Clearly, a symmetrical beam is simply a beam of which the cross-section contains *at least* one axis of symmetry (it does not necessarily have to have two axes of symmetry).

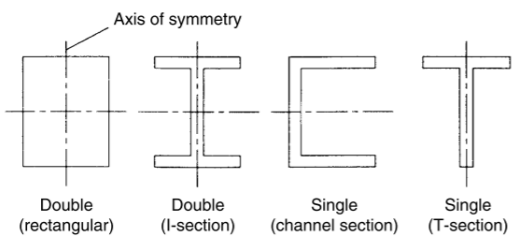


Figure 16.8: Symmetrical section beams.

For unsymmetrical beams, there are simply no axes of symmetry at all. This complicates calculations a bit, because now, using the coordinate system previously determined, looking at figure 16.9, the stress at any point (x, y), where the origin of the coordinate system is located at the centroid, is given by

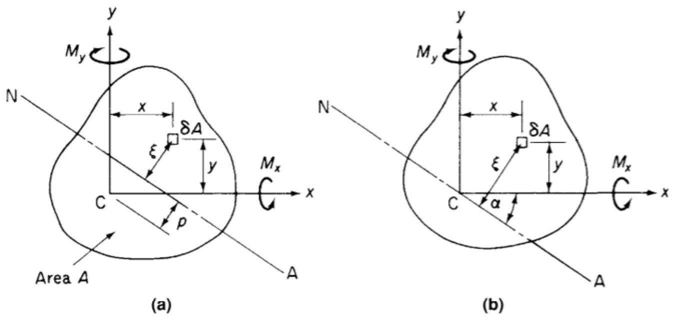


Figure 16.9: Unsymmetrical section beams.

FORMULA

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \tag{16.3}$$

Where I_{xy} is the product second moment of area[2],

$$I_{xy} = \int_A xy dA$$

You can look up the derivation of this formula for the direct stress yourself, do I don't recommend doing so. It is important to note that this formula is derived for the general case, where there is no axis of symmetry. However, when there is an axis of symmetry, $I_{xy} = 0$ automatically, which reduces the formula to simply

$$\sigma_z = \frac{M_y}{I_{yy}}x + \frac{M_x}{I_{xx}}y$$

which we saw was the formula for symmetrical beams. Hence, the symmetrical beam is simply a special case of the unsymmetrical beam.

16.2.3 *Position of the neutral axis*

As the neutral axis is the axis where $\sigma_z = 0$, we can use equation (16.3) to rewrite to find the neutral axis:

$$\frac{y}{x} = -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}$$

However, as α is positive when x and y are of opposite sign (see the right figure of figure 16.9), we have

FORMULA

$$\tan\alpha = \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}} \tag{16.4}$$

16.2.4 *Resolution of bending moments*

If we have a bending moment applied in any longitudinal plane parallel to the z-axis, we can decompose it in M_x and M_y, as depicted in figure 16.10. Note that θ is the angle between the positive x-axis and the plane in which M is applied:

$$M_x = M\sin\theta \tag{16.5}$$
$$M_y = M\cos\theta \tag{16.6}$$

[2] In section 16.4, we will take a closer look at calculating this.

Note how this would have made the second example of section 16.1 much easier: apparently, $\theta = 90+30 = 120°$. Plugging in this value of θ would have directly given you the right value and sign for M_x and M_y so that you could apply $\sigma_z = \frac{M_y}{I_{yy}}x + \frac{M_x}{I_{xx}}y$.

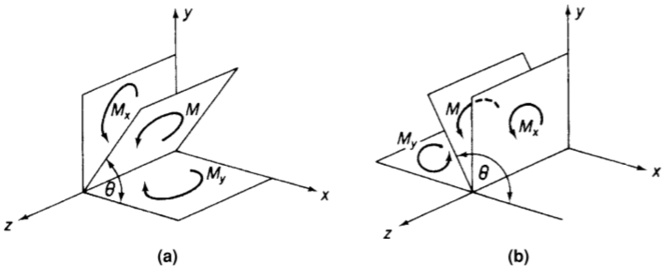


Figure 16.10: Resolution of bending moments. In the left figure, $\theta < 90°$, in the right figure, $\theta > 90°$.

16.2.5 Load intensity, shear force and bending moment relation ships, general case

Reducing figure 16.7 to a 2D yz plane, we end up at figure 16.11.

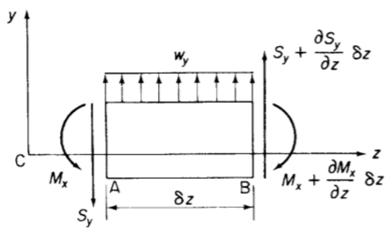


Figure 16.11: Equilibrium of beam element supporting a general force system in the yz plane.

For equilibrium of forces in y-direction, we end up at

$$-S_y + w_y\delta z + S_y + \frac{\partial S_y}{\partial z}\delta z = 0$$

Which results in $w_y = -\frac{\partial S_y}{\partial z}$. Similarly, for the moments around A (note that x should be pointing into the paper, hence clockwise is positive), we get

$$M_x + \frac{\partial M_x}{\partial z}\delta z - \left(S_y + \frac{\partial S_y}{\partial z}\delta z\right)\delta z - w_y\frac{(\delta z)^2}{2} - M_x = 0$$

When neglecting second-order terms, this reduces to $S_y = \frac{\partial M_x}{\partial z}$. This can be combined with the previous result and summarized to

FORMULAS

$$w_y = -\frac{\partial S_y}{\partial z} = -\frac{\partial^2 M_x}{\partial z^2} \tag{16.7}$$

$$w_x = -\frac{\partial S_x}{\partial z} = -\frac{\partial^2 M_y}{\partial z^2} \tag{16.8}$$

So, in the absolute worst case scenario, they will give you a nice beam, on which several forces, distributed loads and moments are working, and you have to calculate the direct stress distribution due to the bending moment as a function of distance z. Please note that I'll show a few examples when we've done section 16.4 which discusses calculating the second moment of area.

FINDING THE DIRECT STRESS DISTRIBUTION FOR BENDING OF ANY BEAM AS A FUNCTION OF z

1. Establish a coordinate system, preferably as described in subsection 16.2.1.
2. Analyse M_x by looking at w_y and S_y and finding a function for M_x as function of z. Note that it may be that this function consists of multiple parts, for example if a force only acts halfway through the beam, and not at $z = 0$.
3. Do the same for M_y by looking at w_x and S_x.
4. Calculate the centroid of the cross-section.
5. Calculate the second moments of area, I_{xx}, I_{xy} and I_{yy}, of the cross-section.
6. Find the stress at any location by applying

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \tag{16.9}$$

7. If asked, calculate the clockwise angle between the neutral axis and the positive x-axis α using

$$\tan\alpha = \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}} \tag{16.10}$$

16.4 Calculation of section properties

16.4.1 Parallel axes theorem

See figure 16.12. You know Steiner, he's a nice guy. We have that the second moment of area, I_n, about a parallel axis a distance b from the centroidal axis is given by

$$I_n = I_C + Ab^2 \tag{16.11}$$

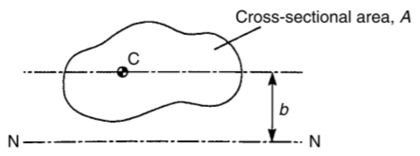


Figure 16.12: Parallel axes theorem.

16.4.2 Theorem of perpendicular axes

If x and y are perpendicular axes with second moments of area I_{xx} and I_{yy}, then the second moment of area about an axis through O perpendicular to the plane of the section (i.e. the z-axis), called the polar second moment of area is given by

$$I_o = I_{xx} + I_{yy}$$

16.4.3 Second moment of area of standard sections

For a rectangle with width w (in x-direction) and height h (in y-direction):

FORMULAS: SECOND MOMENT OF AREA FOR A RECTANGLE

$$I_{xx} = \frac{wh^3}{12} \tag{16.12}$$

$$I_{yy} = \frac{hw^3}{12} \tag{16.13}$$

For a circle with diameter d:

FORMULA: SECOND MOMENT OF AREA FOR A CIRCLE

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64} \tag{16.14}$$

16.4.4 Product second moment of area

The product second moment of area is given by

$$I_{xy} = \int_A xy dA$$

Please note that I first have to explain subsection 16.4.5 before I can really show some examples. Also, when there's symmetry, then *always* $I_{xy} = 0$. The reason for this is depicted in figure 16.13.

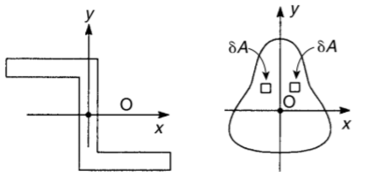


Figure 16.13: Product second moment of area.

If there's an axis of symmetry, then for each element of area δA, having the product of its coordinates positive, there's an identical element for which the product of its coordinates is negative.

16.4.5 Approximations for thin-walled sections

For thin-walled sections, calculations simplify a bit, because we will neglect all terms involving t^2 and higher.

Example 1

Take a look at the cross-section shown in figure 16.14.

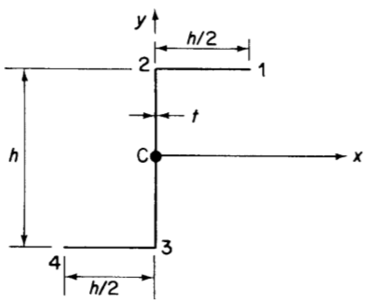


Figure 16.14: Z-section beam.

For this cross-section, let's first focus on the I_{xx}: for the vertical part, we have that the second moment of area equals $\frac{th^3}{12}$ so we cannot neglect that. We do not have any Steiner term for this part. For the two horizontal parts, which are located the same distance from the centroid, we do not have to take into account the second moment of area, as that equals $\frac{h}{2}\frac{t^3}{12}$ and hence can be neglected. However, we do have to take into account the Steiner terms twice, as for each of the two parts, the Steiner term equals

$\frac{th}{2} \cdot \left(\frac{h}{2}\right)^2 = \frac{th^3}{8}$. Hence:

$$I_{xx} = \frac{th^3}{12} + 2 \cdot \frac{th^3}{8} = \frac{th^3}{3}$$

Now, note two important things, which may have been confusing:

- You do not have to deal with the corners that connect two separate parts. For example, you may have thought that for the vertical part, you'd need to take $\frac{t(h-2t)^3}{12}$ as second moment of area. If you'd work this out, and then neglect all terms with t^2 or higher, you end up with $\frac{th^3}{12}$ anyway, so you can leave the $-2t$ out.
- In a similar fashion, for the distance of the horizontal parts to the x-axis, you do not take $\left(\frac{h}{2} - \frac{t}{2}\right)$. When you square it and multiply it with A, then neglect all terms involving t^2 or higher, you again end up with simply $\frac{h}{2}$ for the distance.

Also, for a circle, the second moment of area becomes $I_{xx} = I_{yy} = \pi r^3 t$. For a semi-circular shape, it becomes $I_{xx} = I_{yy} = \pi r^3 t/2$. In general, if there is an arc, where the starting point makes an angle of θ_1 with the positive x-axis and the ending point an angle of θ_2 with the positive x-axis, then the second moment of area is given by

FORMULA: SECOND MOMENT OF AREA FOR AN ARC

$$I_{xx} = \int_{\theta_1}^{\theta_2} t\,(r\sin\theta)^2\, r\, d\theta \quad (16.15)$$

$$I_{yy} = \int_{\theta_1}^{\theta_2} t\,(r\cos\theta)^2\, r\, d\theta \quad (16.16)$$

Please note that these are the second moments of area passing through the *center* of the arc, not the *centroid* of the arc. Hence, it is as if you have already added a Steiner term to the outcome of this: you went from the centroidal axes to different axes. Therefore, you need to *subtract* the Steiner term associated with this (do not add) by first calculating the location of the centroid.

Annoyingly, we sometimes have inclined sections for thin-walled cross-sections, as for example depicted in figure 16.15.

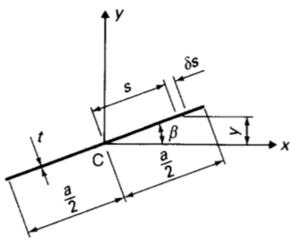


Figure 16.15: Second moments of area for an inclined thin section.

For these, it can be derived that

FORMULAS: SECOND MOMENT OF AREA FOR INCLINED THIN SECTIONS

$$I_{xx} = \frac{a^3 t \sin^2\beta}{12} \quad (16.17)$$

$$I_{yy} = \frac{a^3 t \cos^2\beta}{12} \quad (16.18)$$

$$I_{xy} = \frac{a^3 t \sin 2\beta}{24} \quad (16.19)$$

Remember that $\sin 2\beta = 2\sin\beta\cos\beta$. Furthermore, note what this means for I_{xy}: if a rectangular shape is either fully horizontal ($\beta = 0°$ or $180°$), then automatically $I_{xy} = 0$. Similarly, if the bar is completely vertical ($\beta = 90°$), then $I_{xy} = 0$ as well. However, you *still* have to include the Steiner terms. which for the product second moment of area is given by $A \cdot \Delta x \cdot \Delta y$, where Δx and Δy are the x- and y-distances between the centroid of the part and the centroid of the complete cross-section. Note that signs are important here: if the centroid is located in positive direction, then the distance is positive; if it is located in negative direction, it is negative.

Example 2

Take a look at figure 16.16. First, we take point C as the origin to calculate the location of the centroid (the "corner" near BE is taken to be part of the horizontal part):

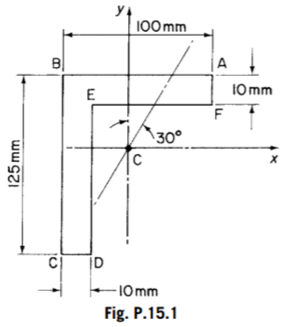


Figure 16.16: Beam cross-section.

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{10\cdot(125-10)\cdot\frac{10}{2} + 100\cdot 10\cdot\frac{100}{2}}{10\cdot(125-10)+100\cdot 10} = 25.93\,\text{mm}$$
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{10\cdot(125-10)\cdot\frac{125-10}{2} + 100\cdot 10\cdot\left(115+\frac{10}{2}\right)}{10\cdot(125-10)+100\cdot 10} = 86.57\,\text{mm}$$

Furthermore, note that the two bars are completely vertical/horizontal, and hence have no product second moment of area around their own centroids.

$$\begin{aligned} I_{xy} &= \sum \tilde{x}_i \tilde{y}_i A_i \\ &= \left(\frac{10}{2} - 25.93\right)\left(\frac{125-10}{2} - 86.57\right)\cdot 10\cdot(125-10) + \\ &\quad \left(\frac{100}{2} - 25.93\right)\left(115 + \frac{10}{2} - 86.57\right)\cdot 100\cdot 10 = 1.50\times 10^6\,\text{mm}^4 \end{aligned}$$

As you can see, it's not particularly difficult, but it can be very helpful (especially for more complicated cross-sections) to make a quick sketch where you indicate the centroids of the individual parts that make up the cross-section.

16.2 *Unsymmetrical bending: examples*

Now, let's do some examples regarding the blue problem solving guide in section 16.2.

Example 1

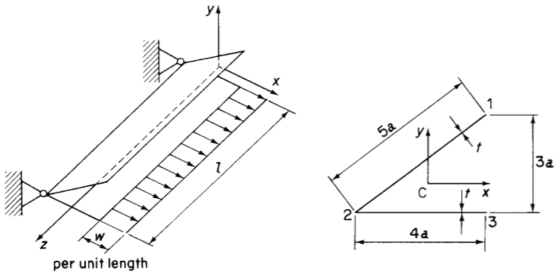


Figure 16.17: Beam discussed in first example.

A beam, simply supported at each end, has a thin-walled cross section shown in figure 16.17. If a uniformly distributed loading of intensity w/unit length acts on the beam in the plane of the lower, horizontal flange, calculate the maximum direct stress due to bending of the beam and show diagrammatically the distribution of the stress at the section where the maximum occurs. The thickness t is to be taken as small in comparison with the other cross-sectional dimensions in calculating the section properties I_{xx}, I_{yy}, and I_{xy}.

Note that a coordinate system has already been established, hence I'll be using this coordinate system as well. The reaction forces on the wall must be calculated first. Due to symmetry, the reaction forces in x-direction are both equal to $\frac{-wL}{2}$ (thus pointing in negative x-direction). In y- and z-direction, they are undetermined, so we'll just set them equal to 0. If we make a cut near (not at) the origin in order to determine the moments (so we look at a surface which has its normal pointing in positive z-direction), we realize that there is no moment around x. However, around y, we do have a moment. Remember from the sign convention discussed, that a positive internal moment around y would point downward if you look at a surface with its normal pointing in positive z-direction. Hence, if we look down on the xz-plane, we realize that, by virtue of the right-hand-rule, the internal moment M_y should be assumed to be clockwise. It's advisable to draw a sketch yourself to determine the following equilibrium equation (clockwise positive):

$$\begin{aligned} \sum M_y = 0 &= wz \cdot \frac{z}{2} - \frac{wL}{2} \cdot z + M_y \\ M_y &= \frac{wz}{2}(L - z) \end{aligned}$$

So there is a positive moment around y (pointing downward) of magnitude $\frac{wz}{2}(L-z)$. We can also calculate the section properties, starting with the centroid. If we establish our coordinate system for calculating this at point 2 in the right figure 16.17 (so ignore the currently drawn coordinate system), then:

$$\begin{aligned} \bar{x} &= \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{2a \cdot 4at + 2a \cdot 5at}{4at + 5at} = 2a \\ \bar{y} &= \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{0 \cdot 4at + \frac{3a}{2} \cdot 5at}{4at + 5at} = \frac{5}{6}a \end{aligned}$$

Now, calculating the second moments of area; noting that $\sin\beta = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$ and using the thin-walled approximation:

$$\begin{aligned} I_{xx} &= 4at \cdot \left(\frac{5}{6}a\right)^2 + \frac{(5a)^3 t \cdot \left(\frac{3}{5}\right)^2}{12} + 5at\left(\frac{3}{2}a - \frac{5}{6}a\right)^2 = \frac{35}{4}a^3 t \\ I_{yy} &= \frac{t\,(4a)^3}{12} + \frac{(5a)^3 t \cdot \left(\frac{4}{5}\right)^2}{12} = 12a^3 t \\ I_{xy} &= 0 + 0 \cdot \frac{-5}{6}a \cdot 4at + \frac{(5a)^3 t \cdot 2 \cdot \frac{3}{5} \cdot \frac{4}{5}}{24} + 0 \cdot \left(\frac{3}{2}a - \frac{5}{6}a\right) \cdot 5at = 5a^3 t \end{aligned}$$

Note that for I_{xy}, the horizontal bar obviously has no I_{xy} itself because it's completely horizontal. Furthermore, as its x-coordinate has the same value as the x-coordinate of the centroid, the Steiner term reduces to zero. The inclined bar has a finite I_{xy}, but as its x-coordinate has the same value as the x-coordinate of the centroid, the Steiner term also reduces to zero. With $M_x = 0$ and $M_y = \frac{wz}{2}(L-z)$:

$$\begin{aligned}
\sigma_z &= \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y = \frac{\frac{35}{4}a^3t \cdot \frac{wz}{2}(L-z) - 5a^3t \cdot 0}{\frac{35}{4}a^3t \cdot 12a^3t - \left(5a^3t\right)^2}x + \\
&\frac{12a^3t \cdot 0 - 5a^3t \cdot \frac{wz}{2}(L-z)}{\frac{35}{4}a^3t \cdot 12a^3t - \left(5a^3t\right)^2}y = \frac{\frac{35}{4}a^3t \cdot \frac{wz}{2}(L-z)}{80\left(a^3t\right)^2}x + \frac{-5a^3t \cdot \frac{wz}{2}(L-z)}{80\left(a^3t\right)^2}y \\
&= \frac{wz(L-z)}{160a^3t}\left(\frac{35}{4}x - 5y\right)
\end{aligned}$$

It can thus be seen that the direct stress, for constant z, varies linearly with x and y, and hence varies linearly along each flange. Hence, the maximum direct stress will be located at one of the corners 1, 2 or 3:

$$\text{At 1 where } x = 2a \quad y = 3a - \frac{5}{6}a = \frac{13}{6}a \qquad \sigma_{z,1} = \frac{wz(L-z)}{160a^3t}\left(\frac{35}{4} \cdot 2a - 5 \cdot \frac{13}{6}a\right) = \frac{wz(L-z)}{24a^2t}$$

$$\text{At 2 where } x = -2a \quad y = \frac{-5}{6}a \qquad \sigma_{z,2} = \frac{wz(L-z)}{160a^3t}\left(\frac{35}{4} \cdot -2a - 5 \cdot \frac{-5}{6}a\right) = \frac{-wz(L-z)}{12a^2t}$$

$$\text{At 3 where } x = 2a \quad y = \frac{-5}{6}a \qquad \sigma_{z,3} = \frac{wz(L-z)}{160a^3t}\left(\frac{35}{4} \cdot 2a - 5 \cdot \frac{-5}{6}a\right) = \frac{13wz(L-z)}{96a^2t}$$

Hence, it is visible that the stress will be maximum at point 3. Furthermore, the maximum stress occurs when $z = \frac{L}{2}$, and hence the maximum stress occurs at 3 and equals $\frac{13wL^2}{384a^2t}$.
Please note that you could have substituted $z = \frac{L}{2}$ long before, which'd have simplified your equations a bit. However, I only did it at the end to show that you can, in fact, end up with a function dependent on x, y and z. Furthermore, in some questions, it'll be easier to end up with this, so that you can directly input several values for x, y and z. Finally, the question may seem a lot of work (which is true), but note that it is mostly just a matter of working neatly.

Let me provide one additional problem solving guide regarding finding the internal moments using the coordinate system used by the book:

FINDING THE INTERNAL MOMENTS

1. Make a cut somewhere in the beam.
2. Decide which side is easier to evaluate (probably the one with the fewest reaction forces).
3. Use sign convention of figure 16.6, that is:
4. If you want to evaluate the side in the negative z-direction:
   - Then $-M_x$ (note the minus sign!) is given by multiplying all forces in y-direction with the distances and summing them. Minus signs are added as follows:
     - If the force acts in negative direction, a minus sign should be added[a].
   - $-M_y$ (note the minus sign!) is found in exactly the same way.
5. If you want to evaluate the side in the positive z-direction:
   - Then M_x is given by multiplying all forces in y-direction with the distances and summing them. Minus signs are added as follows:
     - If the force acts in negative direction, a minus sign should be added.
     - As you have to travel *from* the point where the force is applied *towards* the point where you evaluate the moments in negative z-direction, you have to add another minus sign in front of each term designating the distance. If there is no distance in play (e.g. when there's a couple working on your beam), then you do not add a minus sign.

     Two minus signs obviously cancel out.
   - M_y is found in exactly the same way.

[a] Though don't be a stupid kid: if you already had to include a minus sign in the expression of the force due to the fact that it acts in negative direction, then you don't have to add another minus sign now obviously.

Note how we also could have used this in the previous example. Suppose we would evaluate M_y by making a cut somewhere along the beam and evaluating the side to the right of the cut (which is in negative z-direction). With a reaction force of $\frac{wL}{2}$ pointing in negative x-direction and a distributed load of w acting in positive x-direction, we get

$$\begin{aligned} -M_y &= -\frac{wL}{2}z + wz\frac{z}{2} = \frac{wz}{2}(z-L) \\ M_y &= \frac{wz}{2}(L-z) \end{aligned}$$

Had we evaluated the side to the left of the cut (which is in positive z-direction), we'd have gotten

$$M_y = -\frac{wL}{2} \cdot -(L-z) + w(L-z) \cdot \frac{-(L-z)}{2} = \frac{wL^2}{2} - \frac{wL}{2}z - w\frac{L^2}{2} = \frac{wL}{2}(L-z) - \frac{w(L-z)^2}{2} = \frac{wz}{2}(L-z)$$

so they match.

Example 2

A thin-walled cantilever has a constant cross section of uniform thickness with the dimensions shown in 16.18. It is subjected to a system of point loads acting in the planes of the walls of the section in the directions shown. Calculate the direct stresses according to the basic theory of bending at the points 1, 2, and 3 of the cross section along the beam. The thickness is to be taken as small in comparison with the other cross-sectional dimensions in calculating the section properties I_{xx}, I_{xy}, and so on.

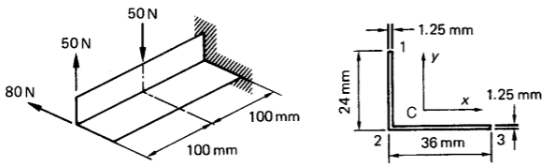


Figure 16.18: Beam discussed in second example.

Let me call the cross-section at the built-in end cross-section A, the cross-section halfway the beam B and the cross-section at the tip D. Furthermore, define a coordinate system in the same way as done in figure 16.7. We then have that between B and D, the moments are given by

$$\begin{aligned} M_x &= -50 \cdot (200-z) && 100 \leq z \leq 200 \\ M_y &= 80 \cdot (200-z) && 100 \leq z \leq 200 \end{aligned}$$

Don't immediately see why this is? Look at figure 16.18: this time, we make a cut somewhere between B and D and evaluate the left-side of this cut (so the side of the tip, not the side containing the wall). Using the problem solving guide, you can directly write down

$$\begin{aligned} M_x &= 50 \cdot -(200-z) = -50(200-z) \\ M_y &= -80 \cdot -(200-z) = 80(200-z) \end{aligned}$$

Where do the minus signs come from? We have that the 50 N points in positive direction, but from the point where the force is applied to the point where you make the cut to evaluate the internal moments, you travel in *negative* z-direction, hence $200 - z$ gets a minus sign in front of it. Similarly, the 80 N points in negative direction, but again you move in negative direction so they cancel out. Please note that via this method, you *immediately* end up at M_x and M_y (no need to set up equilibrium equations any more).
Anyway, we can do the same for the section AB, where we get that

$$\begin{aligned} M_x = -50 \cdot (200-z) + 50 \cdot (100-z) && 0 \leq z < 100 \\ M_y = \quad 80 \cdot (200-z) && 0 \leq z < 100 \end{aligned}$$

So, now we've found the functions describing the moments, and now we can calculate the section properties, taking the left-bottom corner as the origin:

$$\begin{aligned}\bar{x} &= \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{24\cdot 1.25\cdot 0 + 36\cdot 1.25\cdot \frac{36}{2}}{24\cdot 1.25 + 36\cdot 1.25} = 10.8\,\text{mm}\\ \bar{y} &= \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{24\cdot 1.25\cdot \frac{24}{2} + 36\cdot 1.25\cdot 0}{24\cdot 1.25 + 36\cdot 1.25} = 4.8\,\text{mm}\end{aligned}$$

Do note that this seems rather counterintuitive (and just plainly wrong) perhaps, but do note that this is simply the thin-walled approximation. Don't try to make it more accurate than this yourself (by for example taking the right locations of the centroids for the subcomponents). Now, onto the moments of inertia:

$$\begin{aligned}I_{xx} &= \frac{1.25\cdot 24^3}{12} + 1.25\cdot 24\cdot\left(\frac{24}{2} - 4.8\right)^2 + 36\cdot 1.25\cdot 4.8^2 = 4032\,\text{mm}^4\\ I_{yy} &= \frac{1.25\cdot 36^3}{12} + 1.25\cdot 24\cdot 10.8^2 + 36\cdot 1.25\cdot\left(\frac{36}{2} - 10.8\right)^2 = 10\,692\,\text{mm}^4\\ I_{xy} &= 0 + (-10.8)\left(\frac{24}{2} - 4.8\right)(1.25\cdot 24) + 0 + \left(\frac{36}{2} - 10.8\right)(-4.8)(36\cdot 1.25) = -3888\,\text{mm}^4\end{aligned}$$

This leads to

$$\begin{aligned}\sigma_z &= \frac{4032\cdot[80(200-z)] - (-3888)\cdot[-50(200-z)]}{4032\cdot 10692 - (-3888)^2}x\\ &\quad+\frac{10692\cdot[-50(200-z)] - (-3888)\cdot[80(200-z)]}{4032\cdot 10692 - (-3888)^2}y\\ &= \frac{25632000 - 128160z}{27993600}x + \frac{223560z - 4471200}{27993600}y \qquad 100 \le z \le 200\\ \sigma_z &= \frac{4032\cdot[80(200-z)] - (-3888)\cdot[-50(200-z) + 50(100-z)]}{4032\cdot 10692 - (-3888)^2}x\\ &\quad+\frac{10692\cdot[-50(200-z) + 50(100-z)] - (-3888)\cdot[80(200-z)]}{4032\cdot 10692 - (-3888)^2}y\\ &= \frac{45072000 - 322560z}{27993600}x + \frac{8748000 - 311040z}{27993600}y \qquad 0 \le z < 100\end{aligned}$$

With units N/mm^2. Again, note that I generalized a bit more than strictly necessary; the exercise in the book actually asked for the stress distributions at specific sections (sections A and B, to be precise), so you could have just plugged in the values of the moments at those points. However, if you were able to do the example I showed, then you could also do the easier version in the book.

Example 3

Suppose you have the beam shown in figure 16.19 with the pressure acting on it as shown. Calculate the normal stress at the top-left corner at the wall (in the top), point A.

Please note that during the lecture last year, a different coordinate system was chosen which messed up a lot; hence I've chosen the coordinate system the book would have used here. First, we have that the pressure acting on an element dz equals $(5a\,dz)\,p$. In horizontal direction, we then have a force of $(5a\,dz)\,p\frac{3}{5} = (3a\,dz)\,p$. The same can be done for the vertical direction, and hence the force per unit length (the distributed load) equals

$$w_x = 3pa, \qquad w_y = 4pa$$

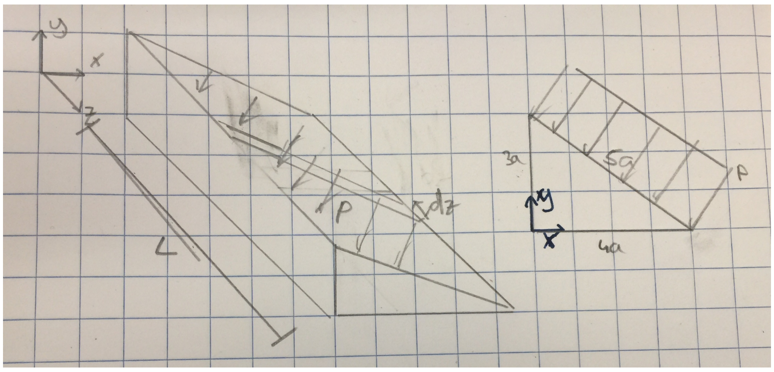


Figure 16.19: Beam discussed in third example.

Both in negative direction, quite clearly. In other words, there is a distributed load of $3pa$ working in negative x-direction and a distributed load of $4pa$ working in negative y-direction (both along the length of the beam). We have that the moments at the wall due to the distributed loads equal

$$\begin{aligned} M_x &= -w_y \cdot L \cdot -\frac{L}{2} = 2paL^2 \\ M_y &= -w_x \cdot L \cdot -\frac{L}{2} = \frac{3}{2}paL^2 \end{aligned}$$

Note that the signs make sense: M_x causes tension in positive y-direction; M_y causes tension in positive x-direction as well. Furthermore, note how brilliant the coordinate system was that is used by the book here. Again, if you make a cut (this time at the wall) to evaluate the moments: w_x and w_y both act in negative direction. However, as we again need to go in negative z-direction to get from the center of the distributed load to the cut, we need to multiply the distance, $\frac{L}{2}$ by -1.
Now, calculating the section properties:

$$\begin{aligned} \bar{x} &= \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{0 \cdot 3at + (2a) \cdot 4at + (2a) \cdot 5at}{3at + 4at + 5at} = \frac{3}{2}a \\ \bar{y} &= \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{\left(\frac{3}{2}a\right) \cdot 3at + 0 \cdot 4at + \left(\frac{3}{2}a\right) 5at}{3at + 4at + 5at} = a \end{aligned}$$

The second moments of inertia equals:

$$\begin{aligned} I_{xx} &= \frac{t\,(3a)^3}{12} + 3at \cdot \left(\frac{3}{2}a - a\right)^2 + 0 + 4at \cdot (0 - a)^2 + \frac{(5a)^3\, t \left(\frac{3}{5}\right)^2}{12} + 5at \cdot \left(\frac{3}{2}a - a\right)^2 = 12a^3 t \\ I_{yy} &= 0 + 3at \cdot \left(0 - \frac{3a}{2}\right)^2 + \frac{(4a)^3\, t}{12} + 4at \cdot \left(2a - \frac{3a}{2}\right)^2 + \frac{(5a)^3\, t \left(\frac{4}{5}\right)^2}{12} + 5at \cdot \left(2a - \frac{3a}{2}\right)^2 = 21a^3 t \\ I_{xy} &= 0 + 3at \cdot \left(0 - \frac{3a}{2}\right)\left(\frac{3a}{2} - a\right) + 0 + 4at \cdot \left(2a - \frac{3a}{2}\right)(0 - a) + \frac{(5a)^3\, t \cdot 2 \left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)}{24} \\ &\quad + 5at \cdot \left(2a - \frac{3}{2}a\right)\left(\frac{3}{2}a - a\right) = -8a^3 t \end{aligned}$$

Note how the I_{xy} of the inclined beam itself was calculated and where the part $\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)$ was calculated: take the middle of the beam as origin of your local coordinate system, as done in figure 16.15: we have the terms $\sin(2\beta) = 2\sin\beta\cos\beta$ appearing in our formula, but we see that β is really, really big (between 90° and 180°). However, for the sine, we still have that at this point, the vertical distance is positive (as it's in positive y-direction, so $\sin\beta = \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{5}$. However, for the cos, where we need to take the horizontal distance, we see that the sign of the horizontal distance has become a minus sign (as it's in negative direction), hence $\cos\beta = \frac{-4}{5}$. We now have everything we want to calculate the stress distribution at the wall:

$$\begin{aligned}\sigma_z &= \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \\ &= \frac{12a^3t \cdot \frac{3}{2}paL^2 - -8a^3t \cdot 2paL^2}{12a^3t \cdot 21a^3t - \left(-8a^3t\right)^2}x + \frac{21a^3t \cdot 2paL^2 - -8a^3t \cdot \frac{3}{2}paL^2}{12a^3t \cdot 21a^3t - \left(-8a^3t\right)^2}y = \frac{17}{94}\frac{a^4tpL^2}{a^6t^2}x + \frac{27}{94}\frac{a^4tpL^2}{a^6t^2}y\end{aligned}$$

Point A is located at $x = -\frac{3}{2}a$ and $y = 2a$ with respect to the centroid, hence

$$\sigma_z = \frac{17}{94}\frac{pL^2}{a^2t}\frac{-3a}{2} + \frac{27}{94}\frac{pL^2}{a^2t}2a = \frac{57}{188}\frac{pL^2}{at}$$

Example 4

Now, onto one final example. Dr. Abdalla mentioned this in the lecture in 2016-2017, but there's no example nor exercise in the book related to this, so I figured it'd be good to include it here. Suppose we have the same beam as in the previous example, but now we no longer have a pressure force, but an *axial* force working as shown in figure 16.20.

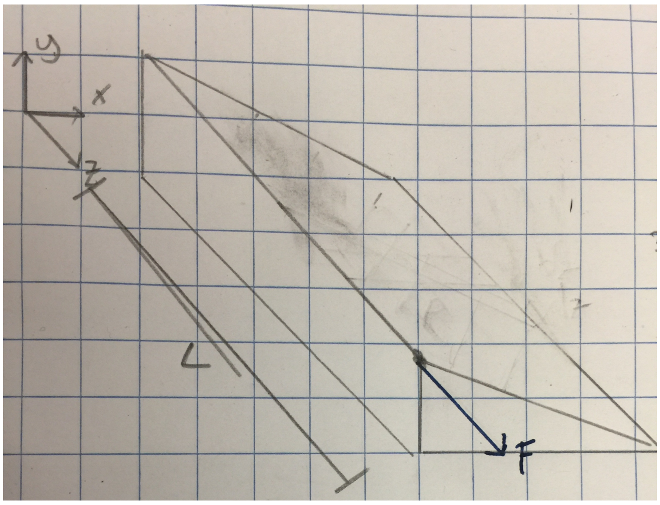


Figure 16.20: Beam discussed in fourth example.

How does it change anything at all? Two important things change:

- You may be inclined to think, yeah bitches no bending moment, but you'd be horribly wrong. Note that now in your cross-section, you have a force pointing out of the paper at the top-left corner. This causes a bending moment around both x and y, as there is a distance to the centroid. Using the coordinate system the book uses; that is, in x-direction, a positive moment points in positive direction, in y-direction, it points in negative direction, you can easily calculate that $M_x = 2aF$ and $M_y = -\frac{3}{2}aF$. Note that these moments are constant throughout the beam; they do not change with z. The normal stress due to bending thus becomes

$$\begin{aligned}\sigma_z &= \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \\ &= \frac{12a^3t \cdot \frac{-3}{2}aF - -8a^3t \cdot 2aF}{12a^3t \cdot 21a^3t - \left(-8a^3t\right)^2}x + \frac{21a^3t \cdot 2aF - -8a^3t \cdot \frac{-3}{2}aF}{12a^3t \cdot 21a^3t - \left(-8a^3t\right)^2}y = \frac{-1}{94}\frac{a^4tF}{a^6t^2}x + \frac{15}{94}\frac{a^4tF}{a^6t^2}y\end{aligned}$$

- The "total" normal stress, however, also includes the normal stress due to the axial force itself.

Hence, with a cross-sectional area of $12at$ and a tensional force, we arrive at

$$\sigma_z = \frac{F}{12at} - \frac{1}{94}\frac{F}{a^2t}x + \frac{15}{94}\frac{F}{a^2t}y$$

Now, you may be left wondering that with regards to the first bullet, doesn't a shear force such as the once depicted in figure 16.18 not also create two moments when it does not pass through the centroid? The answer is yes, but:

- A force S_x or S_y indeed creates a moment about the y- and x-axis respectively, but this is the exact moment you've been dealing with all along.
- A force S_x or S_y both create a moment around z if it does not pass through the *shear center* (which is not the centroid), namely the torque. However, we're not dealing with torques here (which don't create normal stresses anyway so you can just ignore them), so you don't have to deal with them. Most of the time it'll be stated that the forces act through the shear center or something.

16.6 Design problem

Like I said in the preface, this year there's also some focus on design in this course. Let me just immediately jump to an example because it's not really that difficult at all.

Example 1

Given a maximum bending moment of $M_{\text{max}} = 29\,430\,\text{Nm}$ and a height of the airfoil (see figure 16.21 of $h_{\text{max}} = 0.3\,\text{m}$. What shape/size/material should the design team use for this aircraft?

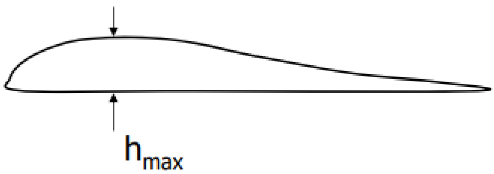


Figure 16.21: Airfoil.

First, regarding the shape: let's just go for a cylindrical shape. The reason why we don't go for a rectangular shape or an I-beam is that the airfoil is curved, so it wouldn't exactly fit nicely so we wouldn't be able to use that $h_{\text{max}} = 0.3\,\text{m}$ to the fullest. Furthermore, cylindrical shapes don't cause stress concentrations and they perform better under torsion (although we're not considering torsion here, it's something worth bearing in mind).
For a simply cylindrical shape, which is evidently symmetric, we simply have

$$\sigma_z = \frac{M_x y}{I_{xx}}$$

Even though we don't have our geometry yet, we will assume that our circular tube is thin-walled, although we should check at the very end whether our design is indeed thin-walled (otherwise our calculations were based on an invalid assumption so it'd be pretty wrong).
Now, we can just compute I_{xx} for a circular section (where R_o is the radius of the outer section and R_i the radius of the inner section, $R_i = R_o - t$)

$$I_{xx} = \frac{\pi}{4}\left(R_o^4 - R_i^4\right) = \frac{\pi}{4}\left(R_o^4 - \left(R_o - t\right)^4\right) = \frac{\pi}{4}\left(R_o^4 - \left(R_o^4 - 4R_0^3t + 6R_o^2t^2 - 4R_ot^3 + t^4\right)\right)$$

We neglect higher order terms of t and the R_o^4 cancels out, so we get

$$I_{xx} = \pi R_o^3 t$$

and yeah I have no idea why he works it out like this in the lecture since we already had this formula but anyway. Plugging this into $\sigma_z = M_x y / I_{xx}$ leads to

$$\sigma_z = \frac{M_x y}{I_{xx}} = \frac{M y}{\pi R_o^3 t}$$

The maximum stress occurs at $y = R_o$ (the maximum value of y):

$$\sigma_{z,\text{max}} = \frac{M R_o}{\pi R_o^3 t} = \frac{M}{\pi R_o^2 t}$$

We desire minimum stress, so we'll take $\sigma_{z,\text{max}} = \sigma_{\text{ult}}$:

$$\sigma_{\text{ult}} = \frac{M}{\pi R_o^2 t_{\text{min}}}$$

so that

$$t_{\text{min}} = \frac{M}{\pi \sigma_{\text{ult}} R_o^2}$$

Here, M, σ_{ult} and R_o are all known (σ_{ult} is given for a given material). Now, which material should we pick? Look at the table below.

Material	σ_{ult} (MPa)	Density (kg/m^3)	Material cost (euros/kg)
Steel	1262	7822	4.4
Aluminium	552	2801	6.6
Titanium	958	4438	22
Quasi-isotropic composite	483	1609	176

You're capable of plugging in these σ_{ult} and then compute t_{min}, the mass per unit length $m/l = 2\pi R_o t_{\text{min}}$ and cost per unit length $C/l = m \cdot$ material cost, which leads to:

Material	t_{min} (mm)	m/l (kg/m)	C/l
Steel	2.8	2.4	10.6
Aluminium	6.4	2.0	13.2
Titanium	3.7	1.8	39.6
Quasi-isotropic composite	7.3	1.3	229

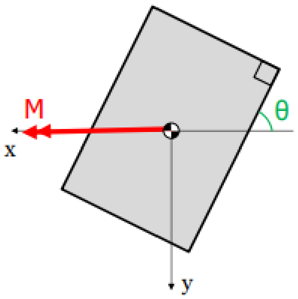
We see that steel leads to the thinnest thickness, but its weight is the worst, although it's also the cheapest. It's a bit of a trade-off which material you select: maybe you think minimum weight is super important and don't care about cost at all (then you'd pick the composite), maybe you think everything is reasonably important and then it's just a bit arbitrary which one you want (e.g. you could write that you want steel because it's only 20% heavier than aluminium, but it's the thinnest and cheapest. Or that you want aluminium because it's only marginally more expensive than steel but it's considerably lighter, etc. etc.). Finally, note that material costs do not include production costs: this would raise the costs of titanium and composites significantly as they are hard to produce.

In hindsight, although it may seem a bit different at first, most of the calculations are the same as before: you just use $\sigma_z = M_x y / I_{xx}$ and compute I_{xx} and then plug it in. However, in this particular example, there were two things different:

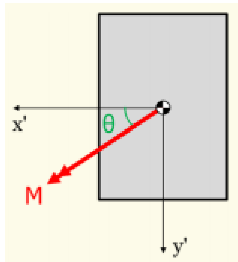
- Before you start the calculations, you decide on the basic shape of the cross-section yourself. Circular cross-sections are pretty good most of the time, but at the moment I've literally seen only this example discussed in class so I don't know when you'd be using different cross-sections. However, at all times, just try to keep it as simple as possible and don't make it dick-shaped or something because that makes the moment of inertia fucked up (and in general, no unsymmetric cross-sections pls).
- Once you've plugged in I_{xx}, you need to identify which of symbols in the formula you do know (in this case, you knew σ_{ult}, R_o and M), and then just rewrite it to an explicit expression for the remaining unknown. The remaining computations (computing t_{min}, m/l and C/l are then pretty logical, albeit it can take a bit of time since it's so repetitive.

16.7 Decomposing the moment along principal axes

This section is new this year, but the lecturer stressed it a bit so it seems important. The problem has to do with the following: consider the moments applied to the beams shown in figure 16.22.



(a) Moment applied to an inclinded beam.



(b) Inclined moment applied to a beam.

Figure 16.22: Inclined moments and beams.

The problem shown in figure 16.22a has the advantage that you only have M_x. However, I_{xx}, I_{yy} and I_{xy} are all fucked up as the section is inclined and you don't know how to do it and you probably never will (since the section is not thin walled). So, what you can do instead is rotate the figure yourself an angle θ to what is shown in figure 16.22b, and then solve that problem (with $M_{x'} = \cos(\theta)\, M$ and $M_{y'} = \sin(\theta)$).

In fact, there's some fancy name associated with this, namely *principal axes*.

DEFINITION: PRINCIPAL AXES

> The **principal axes** of a cross-section are the axes around which I_{xy} is zero.

For symmetric cross-sections, the principal axes are simply the axes of symmetry: as you've seen before, if you compute I_{xy} where either the x-axis or the y-axis is an axis of symmetry, then $I_{xy} = 0$. So that means that this axis is also a principal axis, as a principal axis also has $I_{xy} = 0$. If there's only one axis of symmetry, then one of the principal axes will be the same as that axis of symmetry; the other principal axis will be perpendicular to this axis, passing through the origin. It is is possible for a cross-section to have multiple axes of symmetry (think of a circle, which has infinitely many axes of symmetry); in this case, there will also be infinitely many principal axes as each symmetry axis is also a principal axis. Please also see the solution manual for the practice exam for some clarification on how to determine the principal axes. For now, see also figure 16.23, where the principal axes for a symmetric beam are shown.

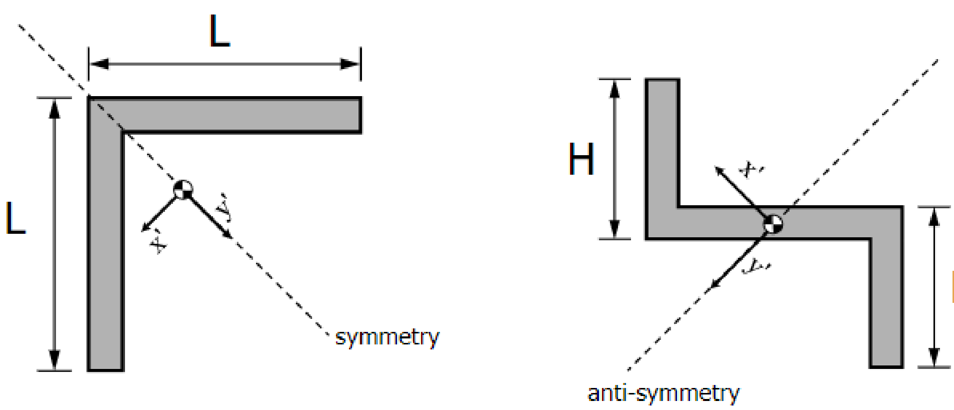


Figure 16.23: Principal axes for a symmetric cross-section on the left. The figure on the right is a mistake.

If a cross-section does not contain an axis of symmetry, it'll still have principal axes! However, you'd have to determine them using Mohr's circle and that's beyond the scope of this course fortunately. But any cross-section has at least one set of principal axes.

Furthermore, note that principal axes are a section property: whereas the orientation of the neutral axis depended on the applied moments M_x and M_y, this is not the case for principal axes. You only need to know the section geometry to determine them.

The use of principal axes is that if you define your coordinate system to be aligned with the principal axes, then you have $I_{xy} = 0$ so you don't have to calculate it. However, if you for example look at the left part of figure 16.23, then although it's nice that $I_{xy} = 0$, you do get inclined sections which aren't that nice imo. Furthermore, although you can write

$$\sigma_z = \frac{M_{x'}}{I_{x'x'}} y' + \frac{M_{y'}}{I_{y'y'}} x'$$

where x' and y' are the coordinates in the principal axis system, you now need to relate x' and y' to the original coordinates x and y (which are the coordinates if you'd use a regular coordinate system, i.e. x towards the right and y in upwards direction). This can be just as annoying so I wouldn't really recommend making the transformation unless incredibly obvious (e.g. in figure 16.22, you should see that it's much easier to do the right part than to do the left part). In general, although the lecturer specifically said he disliked people who just mindlessly plug in formulas, this is actually the best advice I have: once you've got the problem, just start working immediately as fast as you can, and don't spend time wasting, but what I'd do it like this? Or like this? Just work mindlessly and you get the job done in time.

16.8 *Some closing remarks*

You can do the example at the end of the lecture about bending; however, this question is comparatively easy in my opinion so you're better of doing exercises in the book that have slightly more complicated geometry (also the solution posted on brightspace is not the best solution ever as it contains some consistency errors imo). The only difficulty is that you have to manually think about whether there should be plusses or minuses. You have to remember: a moment is positive if it causes tension in the positive coordinate direction, and negative if it causes compression in the positive coordinate direction. Consider examples 1 and 2 of section 16.1 again if you're confused.

Furthermore, just in case you weren't sure, of course you don't need to bother with the running design problem and the numerical optimization of it. Just don't.

18 Torsion

Rather unfortunately, the lectures now literally jump from section 17.1, to 18.2, to 18.1. To make the summary slightly more logical in order, I'll discuss them all together in one chapter.

17.1 General stress, strain, and displacement relationships for open and single-cell closed section thin-walled beams

First of all, start by watching `https://www.youtube.com/watch?v=ovk3KHR_0QU`, the video made by Calvin Rans explaining the relation between shear stresses and bending moment stresses, to give you a nice visualization of what we're doing. It explains much better why varying bending moments induce shear stresses than I could possibly do. You can start watching from 3:26 if you want to skip the first minutes.

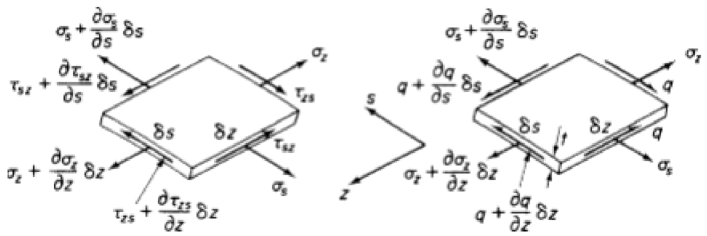


Figure 17.1: General state of stress.

We'll now expand on the concept Calvin explained, by generalizing a bit. Suppose you take a very small section of the cross-section, with thickness t, length δz (so in the axial direction of the beam) and depth (so along the cross-section) ds. If we then assume we have both bending stresses and shear stresses acting in all directions, we get the system shown in the left part of figure 17.1. If we introduce shear flow, which is simply

$$q = \tau t \tag{17.1}$$

then we get the system shown in the right part of figure 17.1. Note that this system is very general: σ_z is caused by bending stresses (axial forces are ignored here); σ_s is the hoop stress which usually equals 0, though internal pressure (in your fuselage) may cause it to exist. We have that for equilibrium in z-direction:

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\delta z\right) t\delta s - \sigma_z t\delta s + \left(q + \frac{\partial q}{\partial s}\delta s\right)\delta z - q\delta z = \frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

Similarly, for equilibrium in s-direction, we have:

$$\frac{\partial q}{\partial z} + t\frac{\partial \sigma_s}{\partial s} = 0$$

18.2 Torsion of open section beams

In this chapter, we're only dealing with pure torsion, that is, there are no other loads acting on the beam other than a pure torsion (so no bending or anything). Hence, in

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

$\frac{\partial \sigma_z}{\partial z}$ should be zero, in principle. Consequently, $\frac{\partial q}{\partial s}$ should be zero as well; in other words, q must be constant. Now, q is a flow so it should start and end somewhere. For an open section beam, this is kind of a problem. If the magnitude of q is constant, it means that it never starts nor ends. Hence, the flow must have an infinite pathway. Looking at an open section as shown in figure 18.2, we see that this is obviously not possible: it must start at for example point 4, and end at point 1 (or the opposite way around), which isn't allowed.

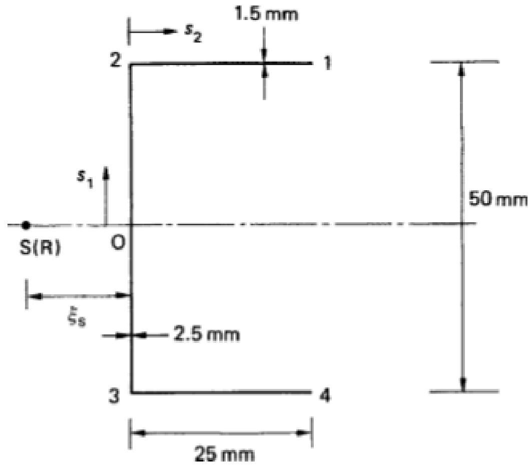


Figure 18.2: An open section beam.

Nevertheless, obviously, even if you have an open section, you can apply a torque to it without breaking the entire universe. In fact, what actually happens for an open section is that although the 'average' shear flow across the thickness at a certain point along the cross-section will be zero everywhere, the shear flow will vary linearly across the thickness.

To clarify a bit: consider first the rectangular section shown in figure 18.3. If you put it under torsion, you get the shear flow distribution as shown (but just ignore the gray left-top corner and the weird distribution near the right-bottom corner, they are not really important) (I hope you at least kinda remembers this vaguely from mechanics of materials). Now, we can make this rectangular section very thin, so that we get to a thin-walled cross-section. We see that we are able to create a torque using the shear flows, as shown in figure 18.4.

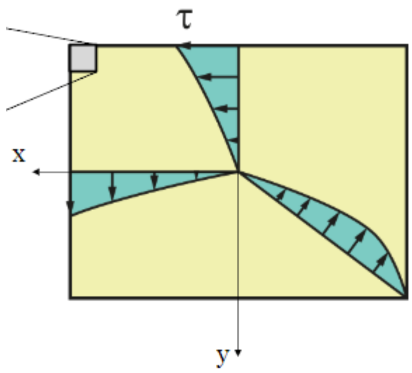


Figure 18.3: A rectangle subjected to torsion.

In fact, let's analyse the torque produced by figure 18.4. First, we analyse the contribution of the horizontal shear flows, as shown in figure 18.5. The torque around the origin (and around everywhere else, but around the origin it's easiest to compute) will simply be

$$T = \int_A y \, dF$$

It's the force dF created by the small area dA multiplied with the moment arm, equal to y, and then integrated.

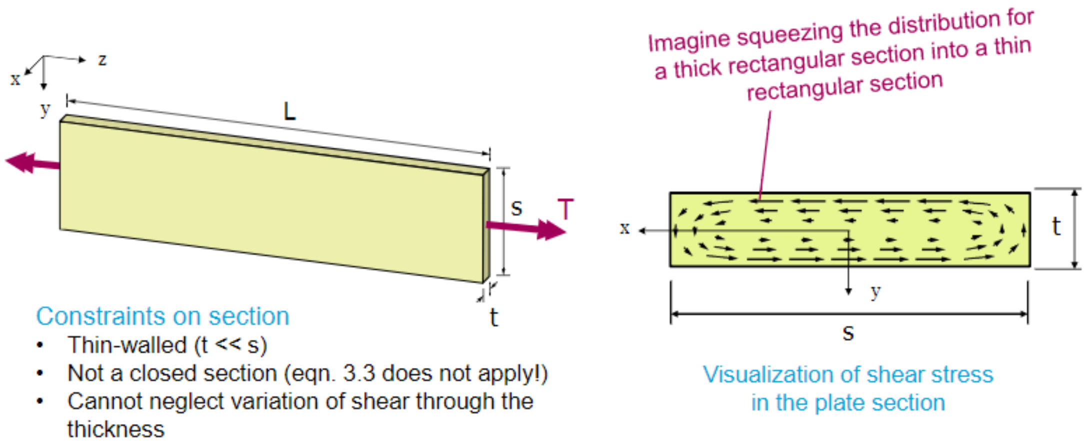


Figure 18.4: A thin strip subjected to torsion. Note that the coordinate systems are incorrect; in the left part, x should point downward (in the direction y is currently in) and y should point to the back (in the opposite direction of how x is currently drawn.

The force created by dA is simply equal to (with s being the length of the strip)[1]

$$dF = \frac{2\tau_{max}s}{t} \cdot y$$

so that

$$T = \int_A y\,dF = \frac{2\tau_{max}s}{t}\int_{-t/2}^{t/2} y^2\,dy = \left[\frac{2\tau_{max}s}{t}\frac{y^3}{3}\right]_{-t/2}^{t/2} = \frac{\tau_{max}st}{6}$$

You can do exactly the same for the vertical parts shown in figure 18.6, which results in an additional torque of $\tau_{max}st/6$, so that the torque created by a rectangular thin strip equals

MAXIMUM SHEAR STRESS FOR A THIN RECTANGULAR PLATE WITH CONSTANT THICKNESS

The maximum shear stress in a thin rectangular plate of thickness t and length s equals

$$\tau_{max} = \frac{3T}{st^2} \tag{18.2}$$

The corresponding rate of twist is

$$\frac{d\theta}{dz} = \frac{3T}{Gst^3} \tag{18.3}$$

However, please note that I'll modify these equations after the following two examples! The reason for this is that above equations assume constant thickness, which may not always be the case. The formula for non-constant thickness is also really not that much harder than above equations, so don't worry. I'll also only provide you with a problem-solving-guide after these two examples.

Let us now consider some examples.

Example 1

Consider the thin circular shape with a slit in it, as shown in figure 18.7. Compute the maximum shear stress and the rate of twist if a torque T is applied.

[1] Don't see why? The shear flow varies between $-\tau_{max}$ and τ_{max}, so that $\tau(y) = \tau_{max}y/(t/2)$, where we need to divide by $t/2$ as the maximum value of y is $y = t/2$. Multiplication by s then yields the total shear force generated by an area dA, so

$$dF = \tau_{max}y/(t/2) \cdot s = \frac{2\tau_{max}s}{t}y$$

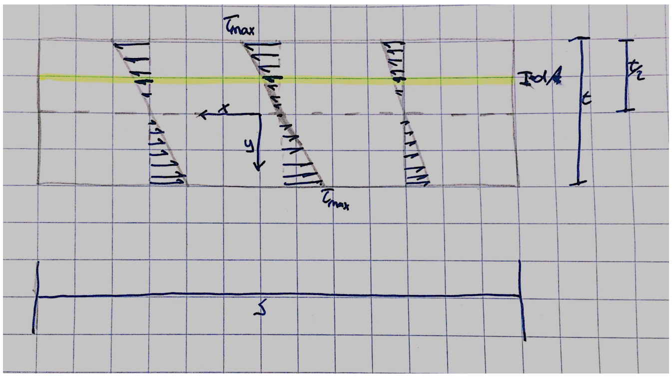


Figure 18.5: Sketch of force created by an area dA. Not the best of sketches, but hopefully it gets the point across. Otherwise, this derivation isn't really too important so feel free to ignore it if you don't get it.

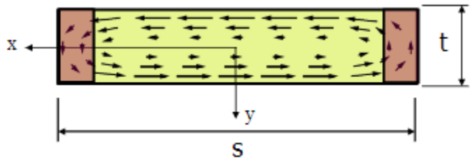


Figure 18.6: Vertical parts of the shear stresses are denoted in red.

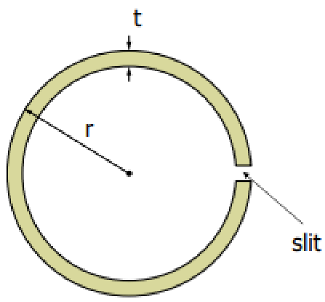


Figure 18.7: Example 1.

It's really, really simple. The thin circular shape can simply be unwrapped to become a thin rectangular strip with length $s = 2\pi r$ and thickness t. Thus, the maximum shear stress is simply

$$\tau_{max} = \frac{3T}{st^2} = \frac{3T}{(2\pi r)\,t^2}$$

Easy peasy lemon's squeezy. Similarly, we simply have for the rate of twist

$$\frac{d\theta}{dz} = \frac{3T}{Gst^3} = \frac{3T}{G\cdot(2\pi r)\cdot t^3}$$

Now let's do a slightly more difficult example. This was actually discussed in class, but for some reason, Calvin didn't show the easy method, but chose to do it differently. I'll use my own method for this question as it's way better. The way he did it is in my opinion really bad, as it needlessly complicates matters by introducing a system of equations that you have to solve, and since you don't have a graphical calculator, you should avoid having to solve matrices at all costs. Furthermore, I can easily modify the problem below slightly and it becomes impossible to solve with Calvin's method even though my own method would still give you the correct result. Therefore, just stick with my method (of course, you can use Calvin's method, but honestly why?).

Example 2

Consider the candy cane shaped beam shown in figure 18.8. Compute the maximum shear strength and the rate of twist of the beam, if a torque of $T = 10\,\text{Nm}$ is applied. The shear modulus of the material is $G = 28\,\text{GPa}$.

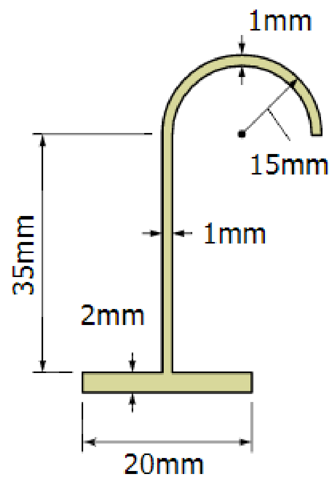


Figure 18.8: Example 2.

This one is slightly more fucked up. However, the trick is as follows: remember that I told you that the formulas in the previous red box were not 100% correct. Indeed, more correct (at least in my opinion) is to write

MAXIMUM SHEAR STRESS FOR A THIN RECTANGULAR PLATE WITH VARYING THICKNESS

The shear stress in a open section subject to torque is given by

$$\tau = \frac{3T}{st^2} = \frac{Tt}{J} \tag{18.4}$$

where t is the local thickness (on the right-side of the equation; for $3T/(st^2)$ t is awkwardly defined if the thickness is not constant everywhere. The rate of twist of the entire cross-section equals

$$\frac{d\theta}{dz} = \frac{T}{GJ} \tag{18.5}$$

where we define J, the **torsion constant**, to be

$$J = \sum \frac{st^3}{3} \tag{18.6}$$

where you sum the individual parts of the beam.

How can we apply this? Well, we first compute J: for the bottom horizontal part, we have $s = 20\,\text{mm}$ and $t = 2\,\text{mm}$, and for the vertical + circular part we have $s = 50 + 15\pi\text{mm}$ and $t = 1\,\text{mm}$. Thus, our combined torsion constant equals

$$J = \sum \frac{st^3}{3} = \frac{20 \cdot 2^3}{3} + \frac{(35 + 15\pi) \cdot 1^3}{3} = 80.71\,\text{mm}^4$$

Then, to compute the shear stresses, we have that for the horizontal part the thickness is $2\,\text{mm}$ and for the vertical + circular part that $t = 1\,\text{mm}$. Thus, we have

$$\begin{aligned}
\tau_{max_{\text{horizontal}}} &= \frac{Tt_{\text{horizontal}}}{J} = \frac{10 \cdot 10^3 \cdot 2}{80.71} = 247.8\,\text{N/mm}^2 = 247.8\,\text{MPa} \\
\tau_{max_{\text{vert+circ}}} &= \frac{Tt_{\text{vert+circ}}}{J} = \frac{10 \cdot 10^3 \cdot 1}{80.71} = 123.9\,\text{N/mm}^2 = 123.9\,\text{MPa}
\end{aligned}$$

For the rate of twist, which is the same for the entire cross-section, we get

$$\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{10 \cdot 10^3}{28 \cdot 10^3 \cdot 80.71} = 4.425 \times 10^{-3}\,\text{rad/mm} = 4.425\,\text{rad/m} = 253.5\,°/\text{m}$$

which is an insane amount of twist. Open sections are pretty worthless against torsion.

So, what should your plan of approach be?

FINDING THE SHEAR STRESS AND RATE OF TWIST FOR A OPEN SECTION

1. Find the torsion constant $J = \sum st^3/3$ by summing the $st^3/3$ of each member of the section.
2. Compute the maximum shear stress at a location p with local thickness t_p by $\tau_{max,p} = Tt_p/J$.
3. Compute the rate of twist by applying $d\theta/dz = T/(GJ)$ (note that $d\theta/dz$ holds for the entire cross-section).

Note that above guide also applies to cross-sections that only consist of one distinct member; then you simply only have one term in your summation after all. Honestly, this is as easy as structural analysis is gonna get. And I genuinely don't know why Rans didn't use this method in the lecture. This is just so much faster[2]. Let's do one final example.

Example 3

Consider the cross-section shown in figure 18.9. Assume the structure is thin-walled. Compute the maximum shear stress in the upper and lower flange, and in the web if a torque T is applied. Furthermore, compute the rate of twist if a torque T is applied. The shear modulus equals G.

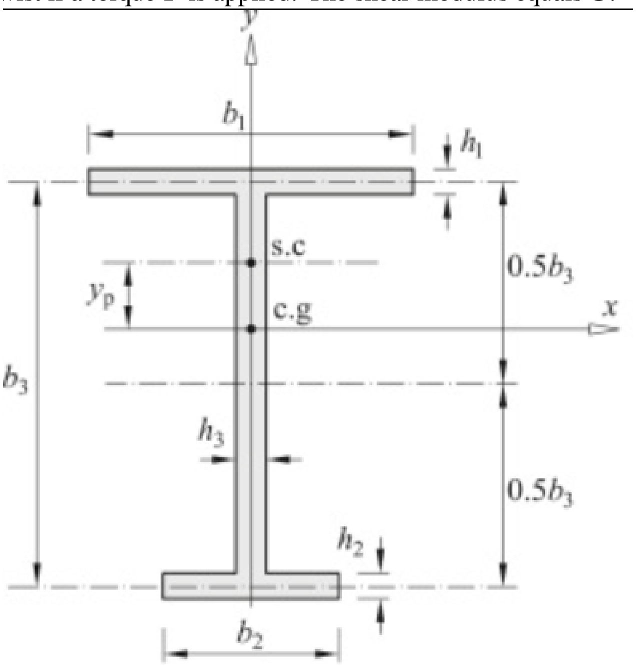


Figure 18.9: Example 3.

[2] Also in case you're wondering, my method is more easily extended to more complex structures where the thickness varies along the parts of the beam (so not only that the thickness is different for distinct parts of the beam, but where the thickness also varies along each distinct part itself, i.e. $t(s)$). In that case, we simply get

$$J = \sum \frac{\int t^3 ds}{3}$$

You see that when t is constant along s, then this simply becomes $J = \sum st^3/3$. However, when t is a function of s, i.e. $t = t(s)$, you'd need to perform the integration. After that, computations are just as easy as they were. However, this is beyond the scope of the course, as Rans' method is unable to deal with these varying thicknesses. So another reason why my method is better.

With Rans' method you'd have to solve a system of 3 equations with 3 unknowns; good luck with doing that by hand. With my method you simply get (although it sucks that it's in pure variable form) for the torsion constant:

$$J = \sum \frac{st^3}{3} = \frac{b_1 h_1^3}{3} + \frac{b_2 h_2^3}{3} + \frac{b_3 h_3^3}{3} = \frac{b_1 h_1^3 + b_2 h_2^3 + b_3 h_3^3}{3}$$

Then, the maximum shear stress in the upper flange is

$$\tau_{max,\text{upper flange}} = \frac{Th_1}{J} = \frac{3Th_1}{b_1 h_1^3 + b_2 h_2^3 + b_3 h_3^3}$$

For the lower flange it is

$$\tau_{max,\text{lower flange}} = \frac{Th_2}{J} = \frac{3Th_2}{b_1 h_1^3 + b_2 h_2^3 + b_3 h_3^3}$$

and for the web it is

$$\tau_{max,\text{web}} = \frac{Th_3}{J} = \frac{3Th_1}{b_1 h_1^3 + b_2 h_2^3 + b_3 h_3^3}$$

For the rate of twist, we simply have

$$\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{3T}{J\left(b_1 h_1^3 + b_2 h_2^3 + b_3 h_3^3\right)}$$

Eazy peazy lemon's squeezy. Comparing this with solving a 3×3 matrix using purely symbols, this is laughably easy.

18.2.1 Note regarding the method of solving

In case you went to the lecture or studied the official solution slides, you'll have seen a different way of solving above problem, where the beam was dissected into two rectangular strips and then the fact that the angle of twist had to be same for both strips was used to compute the solution. However, this is way, way more complicated than what I used.

18.1 Torsion of closed section beams

The problem of the previous section (that the shear flow had to begin somewhere and end somewhere) isn't a problem for closed sections obviously: in a closed section, it can just flow on freely as there is no end and no beginning for a closed section so it can flow on until eternity. Now, suppose we have the beam shown in figure 18.2[3]. Then, the force produced by a shear flow q acting on an infinitesimal length δs equals $q\delta s$, and thence, if the distance of the arm to the point O (which is arbitrary) equals p, then the moment caused by this equals $pq\delta s$, and the total torque will equal

$$T = \oint pq\, ds \tag{18.7}$$

However, you can derive nicely that $\oint p\, ds = 2A$, and with q being constant (as $\frac{\partial q}{\partial s}$ must be zero still), we have that

FORMULA: TORQUE IN CLOSED SECTION BEAMS

$$T = 2Aq \tag{18.8}$$

[3]Please note that this is a thin-walled beam: the black line is the entire thickness of the beam.

Please, please note that A is the *enclosed* area, not the cross-sectional area you'd get by multiplying the perimeter by the thickness.

18.1.1 *Displacements associated with the Bredt-Batho shear flow*

Now, there are some fun derivations you can do. They're not really specifically hard in the sense that you have to apply difficult mathematical operations, but they don't really aid you in understanding why the following formula is correct (and you'll also not understand other questions better, probably), so I'll just leave it out. Anyway, here's the formula you need to remember (or just look it up on the formula sheet):

FORMULA: RATE OF TWIST

$$\frac{d\theta}{dz} = \frac{q}{2A}\oint\frac{ds}{Gt} = \frac{T}{4A^2}\oint\frac{ds}{Gt} \tag{18.9}$$

FINDING THE MAXIMUM SHEAR STRESS DUE TO TORSION AND DISTRIBUTION OF TWIST: SINGLE CELL

1. Identity the magnitude of the maximum torque acting on the beam.
2. Find the value of the shear flow q by $q = \frac{T}{2A}$; remember that this is constant throughout the entire cross-section.
3. Calculate the maximum shear stress by dividing the shear flow by the minimum wall thickness of the section, i.e. $\tau_{max} = \frac{q}{t_{min}}$.
4. Find the formula for $\frac{d\theta}{dz}$ by plugging in $\frac{d\theta}{dz} = \frac{T}{4A^2}\oint\frac{ds}{Gt}$. Please note that T is *not* the maximum torque, but the formula expressing T as a function of z. Evaluate $\oint\frac{ds}{Gt}$ carefully, by taking the sum of the lengths of each part of the cross-section divided by Gt.
5. Integrate $\frac{d\theta}{dz}$ to find a function for θ as function of z. This will lead to one integration constant, which can easily be determined by applying a boundary condition.

Example 1

A uniform, thin-walled, cantilever beam of closed rectangular cross-section has the dimensions shown in figure 18.10. The shear modulus G of the top and bottom covers of the beam is 18 000 N/mm^2 while that of the vertical webs is 26 000 N/mm^2. The beam is subjected to a uniformly distributed torque of 20 Nm/mm along its length. Calculate the maximum shear stress according to the Bred-Batho theory of torsion. Calculate also, and sketch, the distribution of twist along the length of the cantilever, assuming that axial constraint effects are negligible.

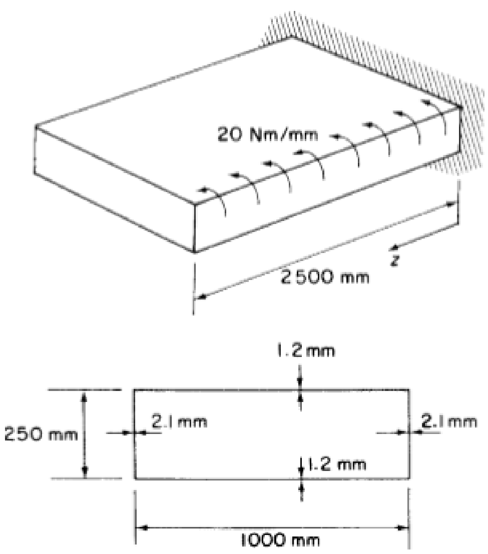


Figure 18.10: Example 1.

First, we identify the maximum torque: at the wall, we will have a reaction torque of $2500 \cdot 20 = 50\,000\,\text{Nm} = 50 \times 10^6\,\text{Nmm}$, which obviously will also be the maximum torque. The shear flow then equals

$$q = \frac{T}{2A} = \frac{50 \cdot 10^6}{2 \cdot 1000 \cdot 250} = 100\,\text{N/mm}$$

The maximum shear stress will thus be, with a minimal thickness of 1.2 mm:

$$\tau_{\text{max}} = \frac{q}{t_{\text{min}}} = \frac{100}{1.2} = 83.3\,\text{N/mm}^2$$

Then, for the rate of twist, the torsion is related to z by $50 \cdot 10^6 - 20000z$ and we must also evaluate $\oint \frac{ds}{GT}$. Starting at the top-right corner and going counterclockwise:

$$\oint \frac{ds}{Gt} = \frac{1000}{18000 \cdot 1.2} + \frac{250}{26000 \cdot 2.1} + \frac{1000}{18000 \cdot 1.2} + \frac{250}{26000 \cdot 2.1} = 0.102\,\text{mm}^2/\text{N}$$

This leads to

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt} = \frac{50 \cdot 10^6 - 20000z}{4 \cdot (1000 \cdot 250)^2} \cdot 0.102 = 8.14 \cdot 10^{-9} \cdot (2500 - z) \quad [\text{rad/m}]$$

Integrating this leads to

$$\theta(z) = 8.14 \cdot 10^{-9} \cdot \left(2500z - \frac{z^2}{2}\right) + C$$

where C can be determined as $\theta(0) = 0$, hence $C = 0$ as well. So, the twist distribution is given by

$$\theta(z) = 8.14 \cdot 10^{-9} \cdot \left(2500z - \frac{z^2}{2}\right)$$

Example 2

A uniform thin-walled beam is circular in cross-section and has constant thickness of 2.5 mm. The beam is 2000 mm long, carrying end torques and, in the same sense, a distributed torque load of 1.0 Nm/mm. The loads are reacted by equal couples R at sections 500 mm distant from each end (figure 18.11). Calculate the maximum shear stress in the beam and sketch the distribution of twist along its length. Take $G = 30\,000\,\text{N/mm}^2$ and neglect axial constraint effects.

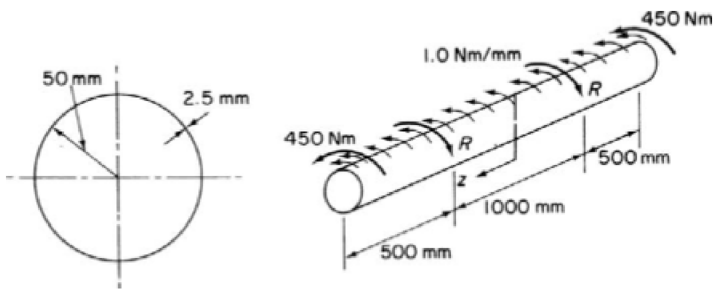


Figure 18.11: Example 2.

Now, identifying the torque distribution is a bit more complicated. However, we know that due to symmetry, both reaction torques will be equally large. With a total torque of $2 \cdot 450 + 2000 \cdot 1.0 = 2900\,\text{Nm}$ acting on the beam, this means that $R = 1450\,\text{Nm} = 1.45 \times 10^6\,\text{Nmm}$. Starting at the left end of the beam and using the drawn coordinate system (so use your right hand rule), we then have that for

500 mm $\leq z \leq$ 1000 mm we have that

$$T = 450000 + 1000\,(1000 - z) = 1.45 \cdot 10^6 - 1000z$$

For -500 mm $\leq z \leq$ 500 mm, this becomes

$$T = 450000 + 1000\,(1000 - z) - 1.45 \cdot 10^6 = -1000z$$

And for -1000 mm $\leq z \leq -500$ mm, this becomes

$$T = 450000 + 1000\,(1000 - z) - 1.45 \cdot 10^6 - 1.45 \cdot 10^6 = -1.45 \cdot 10^6 - 1000z$$

From this, we see that the maximum torque is 950 000 Nmm (note that it is not 1.45×10^6 Nmm as you cannot plug in $z = 0$ into the first equation). Now, everything else can be calculated:

$$q_{max} = \frac{T_{max}}{2A} = \frac{950000}{2 \cdot \pi 50^2} = 60.48\,\text{N/mm}$$

With a constant wall thickness of 2.5 mm, this leads to

$$\tau_{max} = \frac{q_{max}}{t_{min}} = \frac{60.48}{2.5} = 24.2\,\text{N/mm}^2$$

Then, onto the distribution of twist. We must evaluate this for each of the three sections of the beam separately, unfortunately. First of all, we have

$$\oint \frac{ds}{Gt} = \frac{2\pi 50}{30000 \cdot 2.5} = 0.004\,189\,\text{mm}^2/\text{N}$$

And hence, for -500 mm $\leq z \leq$ 500 mm we have

$$\frac{d\theta}{dz} = \frac{-1000z}{4 \cdot \left(\pi \cdot 50^2\right)^2} \cdot 0.004189 = -1.70 \cdot 10^{-8} z$$

Integration leads to

$$\theta\,(z) = -0.85 \cdot 10^{-8} z^2 + C$$

However, we don't have any boundary conditions to satisfy. Hence, we can only calculate the angle of twist relative to other points along the beam, for which we'll set a datum point at the middle of the beam, i.e. $\theta\,(0) = 0$. Thence, $C = 0$ as well. Now, we can determine the distribution function for 500 mm $\leq z \leq$ 1000 mm as well:

$$\frac{d\theta}{dz} = \frac{1.45 \cdot 10^6 - 1000z}{4 \cdot \left(\pi \cdot 50^2\right)^2} \cdot 0.004189 = -1.70 \cdot 10^{-8}\,(1450 - z)$$

leading to

$$\theta\,(z) = 1.7 \cdot 10^{-8} \left(1450z - \frac{z^2}{2}\right) + C$$

However, using the equation we found before for -500 mm $\leq z \leq$ 500 mm, we find that $\theta\,(500) = -0.85 \cdot 10^{-8} \cdot 500^2 = -2.13 \times 10^{-3}$ rad. Hence, we must have

$$\theta\,(500) = 1.7 \cdot 10^{-8} \left(1450 \cdot 500 - \frac{500^2}{2}\right) + C = -2.13 \times 10^{-3}\,\text{rad}$$

as well, leading to $C = -12.33 \cdot 10^{-3}$. Hence, for 500 mm $\leq z \leq$ 1000 mm, this leads to

$$\theta\,(z) = 1.7 \cdot 10^{-8} \left(1450z - \frac{z^2}{2}\right) - 12.33 \cdot 10^{-3}$$

The distribution function for -1000 mm $\leq z \leq -500$ mm can be found in an exactly analogous manner.

Example 3 (old-style)

Consider a thin-walled beam of rectangular cross-section as shown in figure 18.12. Under a pure torque $T = 101$ kNm, an engineer measures the shear strain $\gamma_{yz} = 0.008$ at the mid-point A of the right vertical web. The thickness t_1 of the two vertical webs is 0.1 mm. The thickness t_2 of the upper and lower skins is 2 mm. (a) Use the list of the materials given in table 18.1 to determine which material was used to make the beam. (b) If the beam were cut at the center of the bottom skin all along its span as shown in figure 18.13 and the torque T from part A were applied what shear strain would the engineer measure at point A?

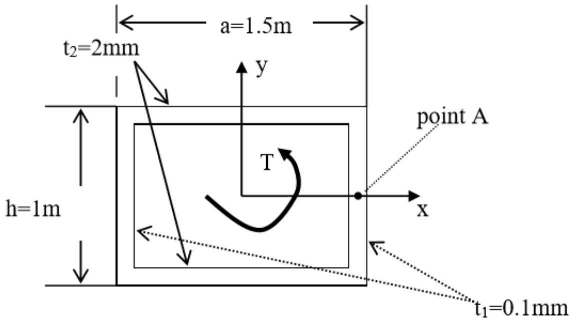


Figure 18.12: Example 3.

Table 18.1: Materials example 3.

Material	Shear modulus (GPa)
Aluminium	26.5
Steel	79.5
Titanium	16.6
Magnesium	42.1

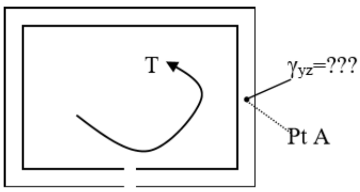


Figure 18.13: Example 3.

This is not an exercises for which you cannot apply the problem solving guide one to one. However, with some basic logical thinking, we can easily determine the answer for (a). First, let's think ahead and how we can relate G and γ. We have

$$\tau = \gamma G$$

or $G = \frac{\tau}{\gamma}$. We know how to easily compute τ from torsion at a point, and γ is given to us. We can then easily check the table for which value comes closest to see what material is used.
First, we need the shear flow however. We have

$$q = \frac{T}{2A} = \frac{101000}{2 \cdot 1.5 \cdot 1} = 33\,667\,\mathrm{N/m}.$$

With a thickness at point A of 0.1 mm, the shear stress becomes

$$\tau = \frac{33667}{0.1 \cdot 10^{-3}} = 336.67\,\mathrm{MPa}$$

And thus we can deduce that

$$G = \frac{336.67 \cdot 10^6}{0.0008} = 42.08\,\mathrm{GPa}$$

and the material used is magnesium.

For (b), remember what I explained about the open section: the shear flow/shear stress varies linearly throughout the thickness, reaching a maximum/minimum on the ends. In the mid-point, the shear stress needs to be zero, to make sure that the average shear stress is zero as well. The shear strain is thus zero as well at the mid-point.

Example 4 (old-style)

A thin-walled beam has square cross-section with side $a = 15\,\text{cm}$ and its length is $L = 1.75\,\text{m}$. It is kept from rotating at both ends (but the axial displacement is not constrained). A torque $T = 300\,\text{kNm}$ is applied at mid-span. The beam material has $E = 69\,\text{GPa}$, $G = 26\,\text{GPa}$, and yield stress in tension of 480 MPa. If the beam is not supposed to fail (yield) and its maximum rotation is not supposed to exceed 2.5 degrees, determine what its thickness should be. (Hint: Split the beam in two halves at its mid-point). See also figure 18.14.

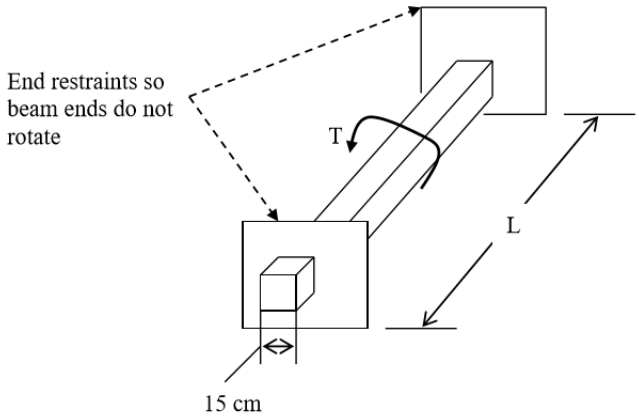


Figure 18.14: Example 4

Please note that the we have two requirements: it may not yield and it may not rotate more than a certain amount. We thus have to do two computations. Before we do anything at all, let's determine a coordinate system: the origin is at the end of the beam closes to us, z points along the beam towards the other end, y points upwards and x points to the left to make it right-handed. This means that the applied torque actually points in negative z-direction, according to the right hand rule. Now, let's start with the non-yielding condition:
Note that this is not something that you can solve completely directly from the problem solving guide, but requires a very small moderation: in step 3, we already know τ_{max} (sort of, at least), and we want to know t_{min}, i.e.

$$t_{\text{min}} = \frac{q}{\tau_{\text{yield}}}$$

Now, let's first focus on q and worry about τ_{yield} later. Using the problem solving guide, we need to identify the maximum torque on the beam. Both walls will have generate a reaction torque of 150 kNm; hence, the torque will be 150 kNm for $0 \leq z \leq 0.875$ and -150 kNm for $0.875 \leq z \leq 1.75$. Hence, the maximum magnitude of the torque is 150 kNm. To find q:

$$q = \frac{T}{2A} = \frac{150 \cdot 1000}{2 \cdot 0.15^2} = 3.33 \times 10^6\,\text{N/m}$$

That's easy. Now, τ_{yield} is a bit harder to find, depending on how well you've remembered last year's stuff (or studied chapter 1), or how well you are able to read a formula sheet: for yielding, we have that the Von Mises stress equals the yield stress Y:

$$Y = \sqrt{\frac{1}{2}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2\right] + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{xz}^2}$$

As we have only[a] τ_{xy}, and hence this reduces to

$$\begin{aligned} Y &= \sqrt{3\tau_{xy}^2} \\ \tau_{xy} &= \frac{Y}{\sqrt{3}} = \frac{480 \cdot 10^6}{\sqrt{3}} = 277.1\,\text{MPa} \end{aligned}$$

Hence, we have that the minimum required wall thickness, based on yielding, equals

$$t_{\text{min}} = \frac{3.33 \cdot 10^6}{277.1 \cdot 10^6} = 0.012\,\text{m}$$

Now, onto the requirement for twist. 2.5 degrees equal 0.0436 rad. Furthermore, from symmetry, we realize that the mid-point experiences the maximum twist, and thus we deduce that

$$\int_0^{0.875} \frac{d\theta}{dz} dz \leq 0.0436$$

Now, let's evaluate $\frac{d\theta}{dz}$, which is a tiny bit different from what the problem solving guide says as t is obviously unknown[b]. However, even you can do this still: first, we have that the function for T as a function of z for $0 \leq z \leq 0.875$ is given by $T = 150000$ (so it's not really a function at all), and hence we can compute:

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt} = \frac{150000}{4 \cdot \left(0.15^2\right)^2} \oint \frac{4 \cdot 0.15}{26 \cdot 10^9 \cdot t} = \frac{0.00171}{t}$$

Please note that the integral became very easy as G and t are both constant; had they varied over the cross-section, you would of course have had a more difficult integral to evaluate, similar to how it was done in other examples (e.g. example 1). Integration of this over z leads to:

$$\int_0^{0.875} \frac{d\theta}{dz} dz = \int_0^{0.875} \frac{0.00171}{t} dz = \frac{0.00150}{t} \leq 0.0436$$

And hence $t_{\text{min}} = 0.034\,\text{m}$. Ergo, the minimum thickness to meet both requirements is $t = 3.4\,\text{cm}$. Note that this violates our thin-walled beam quite clearly, but yeah what are you gonna do about that.

[a] Why? The coordinate system is defined as z pointing along the beam, y pointing upwards and x pointing to the left. Then consider for example the top plate of the beam: we there clearly have a shear stress pointing in positive x-direction towards the left; furthermore, the normal to the top surface points upwards in positive y-direction, so there's only τ_{xy}. For the left web, we have that the shear stress points downward ((negative) y-direction), and the normal then points to the left (in (positive) x-direction. So we clearly only have τ_{xy}.

[b] You may actually plug in the value of t you found before in here and check whether this meets the requirement. If it does, then good on you, you don't have to do anything extra. However, if it doesn't meet the requirement, you need to do this again. Personally, I wouldn't recommend doing this: you win only a very little amount of time by plugging in the value for t without making stuff that much easier; furthermore, there's a 50% chance that it's not enough and you have to redo your calculations, now with unknown t, so then it wasn't useful for anything at all.

Also, I forgot to include it before, but we also define a torsion constant J (we also defined this for the open section, but I didn't explicitly mention it here yet):

FORMULA: TORSION CONSTANT OF CLOSED SECTION BEAM

The torsion constant J of a closed section beam is given by

$$J = \frac{T}{G\frac{d\theta}{dz}} = \frac{4A^2}{\oint \frac{ds}{t}} \tag{18.10}$$

Note that the first fraction is simply equation (18.5) rewritten.

18.1.2 Comparison between torsion of open section and closed section

In example 1 of section 18.2 we discussed the maximum shear stress and rate of twist for a circular cross-section with a slit in it (we had $\tau_{max} = 3T/(2\pi r t^2)$ and $d\theta/dz = 3T/(G \cdot (2\pi r)\, t^3)$). Now let's compute the maximum shear stress and rate of twist for a closed circular cross-section. Evidently, we now have

$$\tau_{max} = \frac{q}{t} = \frac{T}{2At} = \frac{T}{\pi r^2 t}$$

so the ratio between the maximum shear stresses is

$$\frac{\tau_{max_{\text{open}}}}{\tau_{max_{\text{closed}}}} = \frac{\frac{3T}{2\pi r t^2}}{\frac{T}{\pi r^2 t}} = \frac{3r}{t}$$

Similarly, we now will have

$$\frac{d\theta}{dz} = \frac{T}{4A^2}\oint \frac{ds}{Gt} = \frac{T}{4\left(2\pi r^2\right)^2 Gt} \cdot \oint ds = \frac{T\,(2\pi r)}{4\left(\pi r^2\right)^2 Gt}$$

so that

$$\frac{(d\theta/dz)_{\text{open}}}{(d\theta/dz)_{\text{closed}}} = \frac{3r^2}{t^2}$$

For a thin walled structure, we have $r > 10t$. Thus, it is evident that the maximum shear stress and rate of twist are much, much higher for an open section. This is why we don't use open sections to design structures that are under torque.

18.2 Torsion for non-thin walled structures

Everything we've done in this chapter assumes *thin-walled* structures: this is the assumption we used to derive $\partial q/\partial z + t\partial d\sigma_s/\partial s = 0$. However, Rans also mentioned how to do torsion for non-thin walled structures during the lecture (in particular for solid circular shafts), so I'll repeat it here too.

Suppose we have a solid circular shaft, as shown in figure 18.15a. Then, for a cross-section of it, we get the figure shown in figure 18.15b, where r is the radius of the circle, and ρ the distance of the small area dA you are analyzing till the origin. We then found that

FORMULA: SHEAR STRESS AND ANGLE OF TWIST FOR SOLID CIRCULAR SHAFT

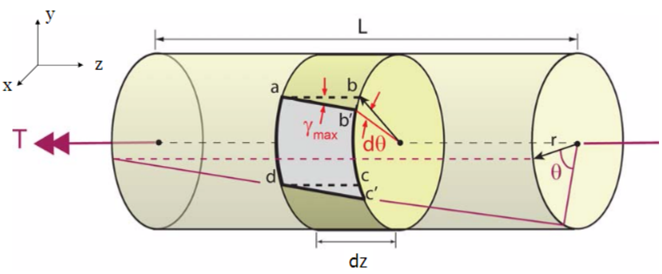
$$\tau = \frac{T\rho}{J} \tag{18.11}$$

$$\frac{d\theta}{dz} = \frac{T}{GJ} \tag{18.12}$$

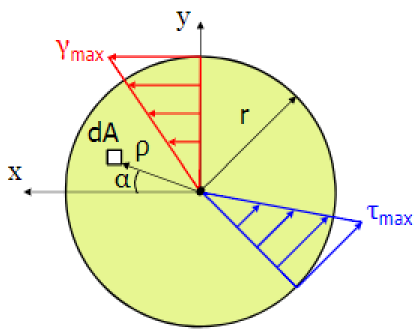
where J is the polar moment of inertia, defined as

$$J \equiv \int_A \rho^2 dA \tag{18.13}$$

which is nice enough. Doubt you'll get questions about it on the exam, but just in case. Note that since it's not thin walled, we don't have any shear flows (as they are purely a concept existing only for thin walled structures).



(a) Solid circular shaft.



(b) Cross-section of solid circular shaft.

Figure 18.15: Solid circular shaft.

23.3 Multicell torsion

Sometimes in life, you'll come across for example a beam that has been divided up in multiple sections, as shown in figure 23.16. This is called a **multicell section**.

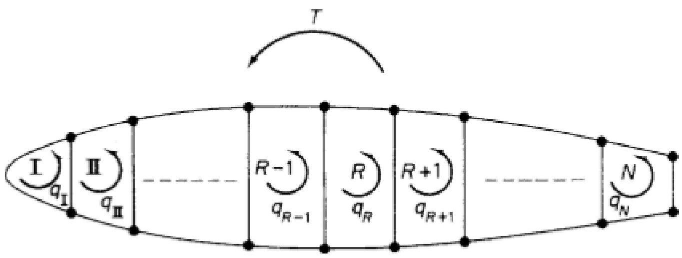


Figure 23.16: Multicell wing subject to torsion.

In this case, we can divide up the torsion into a torsion for each cell. The sum of these individual torsion must then be equal to the total torques, i.e.

$$T = \sum_{R=1}^{N} = 2A_r q_r \tag{23.14}$$

This leads to one equation with N unknowns (the shear flows). Note that for each cell you examine individually, q_r will be constant; only upon addition of the torques, you'll get different values of q along parts of the cell.

We must invoke more equations to solve problems regarding this, as we can't solve one equation with r unknowns. We can hence also use the compatibility equation: for each cell, $\frac{d\theta}{dz}$ must be equal. Now, last section, you learned that

$$\frac{d\theta}{dz} = \frac{q}{2AG} \oint \frac{ds}{t}$$

However, this was only true if q was constant everywhere along the entire cross-section. For a cell, q is *not constant*; at the vertical webs, there will clearly be both a contribution due to q_R and q_{R-1} or q_{R+1} so q will be different there. Hence, we must place it inside the integral, so that for each section, we get

$$\frac{d\theta}{dz} = \frac{1}{2A_r} \oint \frac{q\,ds}{Gt}$$

as the shear modulus also sometimes differs from wall to wall. It is vital that you perform this integration in a careful manner: for each wall (usually four walls), multiply the local shear flow with the length of the wall, and divide this by the shear modulus of the wall and the local thickness. Also be consistent with your direction: q should be positive when it points in counterclockwise direction for that cell.

FINDING THE SHEAR FLOW FOR EACH WALL FOR MULTICELL PROBLEMS

1. Note that the total torque over the entire cross-section is equal to the sum of the torques over each individual cell, leading to
$$T = 2A_1q_1 + 2A_2q_2 + ... + 2A_nq_n$$
2. Note that the angle of twist must be the same for each cell, i.e.
$$\frac{d\theta}{dz} = \frac{1}{2A_r}\oint\frac{q\,ds}{Gt}$$
Carefully perform this integration: for each cell, sketch what the shear flow is along each wall by taking into account the influence of surrounding cells. For each wall, multiply this shear flow with the length and divide it by the thickness and shear modulus, and then take the sum of this.
3. This will yield n equations with $n+1$ unknowns (the extra unknown being $\frac{d\theta}{dz}$). Combine this with the equation resulting from the torsion distribution, and you have a system of $n+1$ equations and $n+1$ unknowns.
4. Solve this system. Be smart and bring your graphical calculator so that you can use the matrix solver there. Do note that sometimes you have to manipulate the system slightly to make sure your graphical calculator can solve it (see example below).

23.3.1 *Torsional stiffness*

This is not mentioned explicitly in the book, but the torsional stiffness J is a value such that

FORMULA: TORSIONAL STIFFNESS

$$J = \frac{T}{G\frac{d\theta}{dz}} = \frac{4A^2}{\oint\frac{ds}{t}} \tag{23.15}$$

The second expression has two important limitations:

- It assumes that G is constant along all of the walls you're integrating $\oint\frac{ds}{t}$ over;
- More importantly, it assumes that q is constant along the walls you're integrating $\oint\frac{ds}{t}$ over.

So, my preference would be to use the first equation. Please note that you calculate the torsional stiffness for the *entire* cross-section, so the combination of all cells together. Furthermore, note that you will, if not always, compute an expression for $\frac{d\theta}{dz}$ before you can compute J, so you will almost never have to calculate $\frac{d\theta}{dz}$, but you can just use the value found in the first part of the exercise.

Example 1

Calculate the shear stress distribution in the walls of the three-cell wing section shown in figure 23.17, when it is subjected to an counterclockwise torque of 11.3 kNm. The properties of each wall are given in table 23.2.

Table 23.2: Data for example 1.

Wall	Length (mm)	Thickness (mm)	G(N/mm^2)	Cell area (mm^2)
12 outer	1650	1.22	24,000	$A_I = 258,000$
12 inner	508	2.03	27,600	$A_{II} = 355,000$
13, 24	775	1.22	24,200	$A_{III} = 161,000$
34	380	1.63	27,600	
35, 46	508	0.92	20,700	
56	254	0.92	20,700	

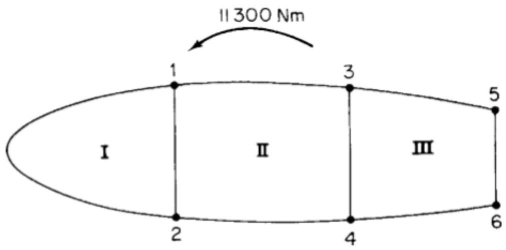


Figure 23.17: Example 1.

First, let's write down the equation for the torsion. We clearly have (note that the torque is 11.3 kNm = 11 300 000 Nmm):

$$11300000 = 2 \cdot 258000 q_1 + 2 \cdot 355000 q_2 + 2 \cdot 161000 q_3 = 516000 q_1 + 710000 q_2 + 322000 q_3$$

Now, let's start evaluating all of them integrals. First focussing on cell 1, starting with the right wall:

$$\begin{aligned}\frac{d\theta}{dz} &= \frac{1}{2A_r}\oint\frac{q\,ds}{Gt} = \frac{1}{2\cdot 258000}\left[\frac{(q_1-q_2)\cdot 508}{27600\cdot 2.03}+\frac{q_1\cdot 1650}{24000\cdot 1.22}\right]\\ &= 1.268\cdot 10^{-7}q_1 - 1.757\cdot 10^{-8}q_2\end{aligned}$$

For cell 2, starting at the right wall again:

$$\begin{aligned}\frac{d\theta}{dz} &= \frac{1}{2A_r}\oint\frac{q\,ds}{Gt} = \frac{1}{2\cdot 355000}\left[\frac{(q_2-q_3)\cdot 380}{27600\cdot 1.63}+\frac{q_2\cdot 775}{24200\cdot 1.22}+\frac{(q_2-q_1)\cdot 508}{27600\cdot 2.03}+\frac{q_2\cdot 775}{24200\cdot 1.22}\right]\\ &= -1.277\cdot 10^{-8}q_1 + 9.861\cdot 10^{-8}q_2 - 1.190\cdot 10^{-8}q_3\end{aligned}$$

For cell 3, starting at the right wall again:

$$\begin{aligned}\frac{d\theta}{dz} &= \frac{1}{2A_r}\oint\frac{q\,ds}{Gt} = \frac{1}{2\cdot 161000}\left[\frac{q_3\cdot 254}{20700\cdot 0.92}+\frac{q_3\cdot 508}{20700\cdot 0.92}+\frac{(q_3-q_2)\cdot 380}{27600\cdot 1.63}+\frac{q_3\cdot 508}{20700\cdot 0.92}\right]\\ &= -2.62\cdot 10^{-8}q_2 + 2.33\cdot 10^{-7}q_3\end{aligned}$$

This means that we have the following system of equations:

$$\begin{aligned}516000q_1 + 710000q_2 + 322000q_3 &= 11300000\\ 1.268\cdot 10^{-7}q_1 - 1.757\cdot 10^{-8}q_2 - \frac{d\theta}{dz} &= 0\\ -1.277\cdot 10^{-8}q_1 + 9.861\cdot 10^{-8}q_2 - 1.190\cdot 10^{-8}q_3 - \frac{d\theta}{dz} &= 0\\ -2.62\cdot 10^{-8}q_2 + 2.33\cdot 10^{-7}q_3 - \frac{d\theta}{dz} &= 0\end{aligned}$$

You can plug this into a matrix, let your graphical calculator row reduce it, to find $q_1 = 7.1$ N/mm, $q_2 = 8.9$ N/mm and $q_3 = 4.2$ N/mm. A small problem is that for example my calculator wasn't able to solve this directly[a]; I suppose it was due to the large differences in order of magnitudes between the first and other equations. A solution to this is to divide the first equation by 10^{12}: this gives roughly the same order of magnitude for the coefficients for all of the equations (except for $\frac{d\theta}{dz}$, which is in the order of magnitude of 10^0, so much higher than $O10^{-7}$). Doing this allowed my graphical calculator (TI-84 plus) to solve it, so if you happen to run into the same problem during the (final) exam, that your calculator clearly gives the wrong results, try doing this. You can even put for example 10^{-7} in front of $\frac{d\theta}{dz}$ to make that have the same order of magnitude in your calculator, but then remember that the result for $\frac{d\theta}{dz}$ your calculator gives should be multiplied by 10^{-7} in real life.

Finally, these results give the shear flows; the shear stress in any wall is obtained by dividing the shear flow by the actual wall thickness. Hence, the shear stress distribution is as shown in figure 23.18.

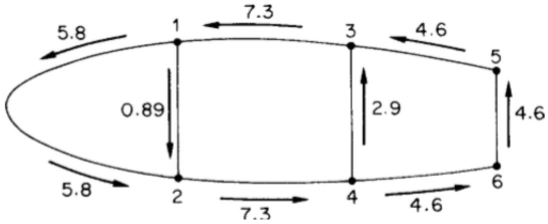


Figure 23.18: Example 1: shear stress distribution.

[a] It gave as solution $\frac{d\theta}{dz} = 0$ and $q_1 + \frac{355}{258}q_2 + \frac{161}{258}q_3 = \frac{2825}{129}$. This is merely the first equation divided by 516000, so it's basically useless.

Example 2

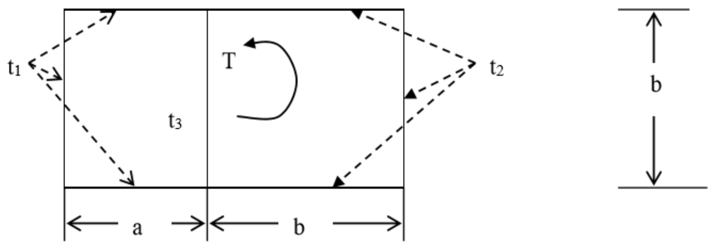


Figure 23.19: Example 2

The thin-walled cross-section shown in figure 23.19 is loaded by a pure torque T. It is measured that over 1 meter length of the beam, the angle of twist changes by 5 degrees. Determine the torque T applied and the torsion constant J if the material shear modulus is 27 GPa. The dimensions are $a = 400$ mm, $b = 500$ mm, $c = 2$ mm, $t_2 = 4$ mm and $t_3 = 3$ mm.

Let's first focus on calculating T. Once again, we have an example that may seem a bit strange new compared to the problem solving guides, as we now have to calculate T rather than it is given to us. However, we can just use the same method, but then in reverse so to say. Let the shear flow induced by the left cell in counterclockwise direction be denoted by q_1 and the shear flow induced by the right cell in counterclockwise direction be denoted by q_2. We know that $\frac{d\theta}{dz} = 5\,°/\text{m} = 0.0873\,\text{rad/m} = 8.73 \times 10^{-5}\,\text{rad/mm}$. We also know that for the left cell, starting at the right wall:

$$\begin{aligned} \frac{d\theta}{dz} &= \frac{1}{2A_1}\oint \frac{q\,ds}{Gt} = \frac{1}{2\cdot 400\cdot 500}\left[\frac{(q_1 - q_2)\cdot 500}{27000\cdot 3} + \frac{q_1\cdot 400}{27000\cdot 2} + \frac{q_1\cdot 500}{27000\cdot 2} + \frac{q_1\cdot 400}{27000\cdot 2}\right] \\ &= 7.562\cdot 10^{-8}q_1 - 1.543\cdot 10^{-8}q_2 = 8.73\times 10^{-5}\,\text{rad/mm} \end{aligned}$$

For the right cell, starting at the right wall:

$$\begin{aligned} \frac{d\theta}{dz} &= \frac{1}{2A_2}\oint \frac{q\,ds}{Gt} = \frac{1}{2\cdot 500\cdot 500}\left[\frac{q_2\cdot 500}{27000\cdot 4} + \frac{q_2\cdot 500}{27000\cdot 4} + \frac{(q_2 - q_1)\cdot 500}{27000\cdot 3} + \frac{q_2\cdot 500}{27000\cdot 4}\right] \\ &= -1.235\cdot 10^{-8}q_1 + 4.012\cdot 10^{-8}q_2 = 8.73\times 10^{-5}\,\text{rad/mm} \end{aligned}$$

Solving this system of equations leads to $q_1 = 1706\,\text{N/mm}$ and $q_2 = 2701\,\text{N/mm}$. Now the torsion is easily obtained:

$$T = 2A_1q_1 + 2A_2q_2 = 2\cdot 400\cdot 500\cdot 1706 + 2\cdot 500\cdot 500\cdot 2701 = 2.033\times 10^9\,\text{Nmm} = 2.033\times 10^6\,\text{Nm}$$

Beautiful. We can now also compute J relatively easily, as long as you remember the formula (or remember that you can also read a formula sheet):

$$J = \frac{T}{G\frac{d\theta}{dz}} = \frac{2.033 \cdot 10^6}{27 \cdot 10^9 \cdot 0.0873} = 8.625 \times 10^{-4}\,\mathrm{m}^4$$

Example 3

The cross section shown in figure 23.20 is square of side length a and is subjected to a torque T. The cross section is made of a single material and all sides have the same thickness t. The diagonal web has thickness $2t$. (a) Find the torsional rigidity of the cross section GJ. (b) If the tension yield stress of the material is $Y = 434\,\mathrm{MPa}$ and $G = 26.5\,\mathrm{GPa}$, determine the value of T at which the structure yields when $t = 1\,\mathrm{mm}$ and $a = 25\,\mathrm{cm}$. Use a safety factor of 1.5. (Hint: you will need to calculate the shear yield stress using the von Mises yield condition)

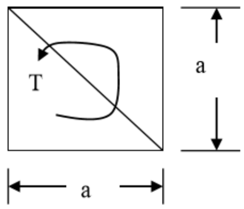


Figure 23.20: Example 3

Again, not entirely in line with what the problem solving guide is able to tell you, and once again we can use it in sort of reverse mode. First off, let the shear flow generated by the bottom left cell be denoted by q_1 and the one generated by the top right cell q_2 (both counterclockwise). Then we have for the left cell, starting at the diagonal wall, that

$$\frac{d\theta}{dz} = \frac{1}{2A_1}\oint\frac{q\,ds}{Gt} = \frac{1}{2\cdot\frac{a^2}{2}}\left[\frac{(q_1-q_2)\cdot\sqrt{2}a}{G2t} + \frac{q_1\cdot a}{Gt} + \frac{q_1\cdot a}{Gt}\right] = \frac{1}{aGt}\left[\left(\frac{1}{2}\sqrt{2}+2\right)q_1 - \frac{1}{2}\sqrt{2}q_2\right]$$

Similarly, for the right cell, starting at the diagonal wall, that

$$\frac{d\theta}{dz} = \frac{1}{2A_1}\oint\frac{q\,ds}{Gt} = \frac{1}{2\cdot\frac{a^2}{2}}\left[\frac{(q_2-q_1)\cdot\sqrt{2}a}{G2t} + \frac{q_2\cdot a}{Gt} + \frac{q_2\cdot a}{Gt}\right] = \frac{1}{aGt}\left[-\frac{1}{2}\sqrt{2}q_1 + \left(\frac{1}{2}\sqrt{2}+2\right)q_2\right]$$

We realize that these two must be equal, hence $\left(\frac{1}{2}\sqrt{2}+2\right)q_1 - \frac{1}{2}\sqrt{2}q_2 = -\frac{1}{2}\sqrt{2}q_1 + \left(\frac{1}{2}\sqrt{2}+2\right)q_2$, which only has a nontrivial solution for $q_1 = q_2$. Hence, we have that

$$\frac{d\theta}{dz} = \frac{1}{aGt}\left[\left(\frac{1}{2}\sqrt{2}+2\right)q_1 - \frac{1}{2}\sqrt{2}q_2\right] = \frac{2q_1}{aGt}$$

Now, we were originally wanting to calculate GJ, which is related to T and $\frac{d\theta}{dz}$ by

$$T = GJ\frac{d\theta}{dz}$$

ergo $GJ = \frac{T}{\frac{d\theta}{dz}}$. Now, this may seem like we've headed down the wrong road all along, as we have now both q_1 and T unknown, but don't worry. We can relate q_1 to T, which will make both q_1 and T completely disappear. Let's relate q_1 to T. You may be inclined to do

$$q_1 = \frac{T}{2A} = \frac{T}{2\cdot\frac{a^2}{2}}$$

but this is completely wrong, as you've taken the wrong area: the left cell does not take up the entirety of T, but only part of it. Now, you can do two things:

- Explain that due to symmetry, the torsion is divided up equally over the two cells, i.e.

$$q_1 = \frac{\frac{T}{2}}{2 \cdot \frac{a^2}{2}} = \frac{T}{2a^2}$$

- Explain that due $q_1 = q_2$, we can also compute q_1 by seeing it as a single cell with q_1 all along the outer walls, i.e.

$$q_1 = \frac{T}{2a^2}$$

Thence, we have

$$GJ = \frac{T}{\frac{d\theta}{dz}} = \frac{T}{\frac{2q_1}{aGt}} = \frac{T}{\frac{2\frac{T}{2a^2}}{aGt}} = a^3 Gt$$

For part (b), it may be a bit unclearer at first what exactly you have to do, as it is very different indeed from any of the problem solving guides. However, we can do it anyway by just starting of calculating a bunch of shit. First, they give us the hint of calculating the von Mises yield condition, so let's do that, just to rack up some partial points even if you wouldn't be able to do the rest of the exercise. From the formula sheet,

$$Y = \sqrt{\frac{1}{2}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2\right] + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{xz}^2}$$

If we have only shear stress, this simply becomes

$$Y = \sqrt{3\tau_{xy}^2}$$

And hence the shear stress at which it yields equals

$$\tau_{xy} = \frac{Y}{\sqrt{3}} = \frac{434 \cdot 10^6}{\sqrt{3}} = 250.57\,\text{MPa}$$

However, as we have a safety factor of 1.5, we must determine T such that $\tau_{xy} = 167.05\,\text{MPa}$. Now, what can we do with this? At yielding, the shear stress somewhere in the cross-section will be 167.05 MPa. Where will the shear stress be maximum? That's where the quotient q/t will be maximum. As the diagonal web did not carry any shear stresses (see (a)), this will only occur at the outer walls, which have uniform thickness and shear flow. Hence, the most critical point is anywhere along the outer walls. The shear flow there may be $q = \tau t = 167.05 \cdot 1 = 167.05\,\text{N/mm}$. How can we determine what torque is associated with this? As the outer walls are the only walls to carry the shear flow, we simply have[a]

$$q = \frac{T}{2A}$$

or

$$T = q \cdot 2A = 167.05 \cdot 250^2 = 20.88 \times 10^6\,\text{Nmm} = 20\,881\,\text{Nm}$$

[a]Had the structure been more complex and there would have been innerwalls carrying shear stresses, you'd have to do the more complicated approach of setting up a system of equations via multiple equations for $\frac{d\theta}{dz}$, yielding n equations with $n+1$ unknowns initially: n shear flows and $\frac{d\theta}{dz}$ is also unknown. However, one shear flow is known, namely the shear flow at the most critical point in the structure, so you can substitute in that value, making it n equations with n unknowns. The solutions for all the shear flows can then be plugged in $T = \sum_{R=1}^{n} 2A_R q_R$. It's only because the outer walls are the only shear stress carrying walls with a uniform shear flow that this question is so easy.

17 Shear

The numbering system of this summary is now really getting fucked up, considering this chapter is about sections 17.2, 17.3 and 23.4 respectively, but yeah I can't really do much about it without fucking up the numbering of the previous chapters so you'll have to deal with it.

17.2 Shear of open section beams

We remember from the previous chapter that open sections were useless in dealing with torsion. However, fortunately, they can carry shear stresses, and calculations for open sections are easier than for closed sections. This is mainly due to the fact that for all questions regarding shear in open sections, the shear force will be acting through the shear center; this is a concept that will be explained later, but it's basically the point so that if a shear force acts through it, it does not cause any torsion.

Now, how can we calculate stuff for open sections? Remember the formula

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

Furthermore, we have from bending

$$\sigma_z = \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}y$$

and with $S_x = \frac{\partial M_y}{\partial z}$ and $S_y = \frac{\partial M_x}{\partial z}$, this means that

$$\frac{\partial \sigma_z}{\partial z} = \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}y$$

And thus we have

$$\frac{\partial q}{\partial s} = -t\frac{\partial \sigma_z}{\partial s} = -\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}tx - \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}ty$$

And thus upon integration, from some origin for s to any point around the cross-section, we have

$$\int_0^s \frac{\partial q}{\partial s}ds = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds$$

Thus, if we take the origin for s at a point where $q = q_0$, then the shear flow at any point located a distance s from this point is given by

FORMULA

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds + q_0 \tag{17.1}$$

Note that this formula simplifies quite a bit if you have only one shear force (so either S_x or S_y equals zero) or when there's symmetry.

Example 1

Determine the shear flow distribution in the thin-walled Z section shown in figure 17.1 due to a shear load S_y applied through the shear center of the section.

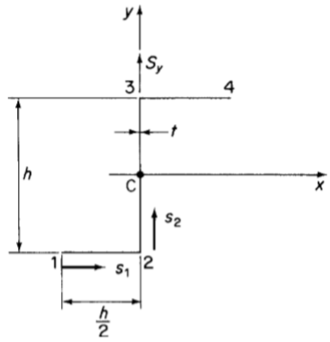


Figure 17.1: Shear loaded Z section.

Let's establish the origin of our coordinate system at the centroid to make calculations easier. Note that there's no line symmetry, so we have to use

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds$$

However, $S_x = 0$ so we can reduce this to

$$q_s = \frac{S_y}{I_{xx} I_{yy} - I_{xy}^2}\left(I_{xy}\int_0^s tx\,ds - I_{yy}\int_0^s ty\,ds\right)$$

Let's first calculate those section properties and then evaluate those integrals. We have

$$\begin{aligned}
I_{xx} &= \frac{th^3}{12} + 2\cdot\left(\frac{th}{2}\right)\cdot\left(\frac{h}{2}\right)^2 = \frac{h^3 t}{3} \\
I_{yy} &= 2\cdot\frac{t\left(\frac{h}{2}\right)^3}{12} + 2\cdot\left(\frac{th}{2}\right)\cdot\left(\frac{h}{4}\right)^2 = \frac{h^3 t}{12} \\
I_{xy} &= \left(\frac{th}{2}\right)\cdot -\frac{h}{4}\cdot -\frac{h}{2} + \left(\frac{th}{2}\right)\cdot -\frac{h}{4}\cdot -\frac{h}{2} = \frac{h^3 t}{8}
\end{aligned}$$

If you forgot how to do this, please refer to section 16.4.
Now, we can plug these numbers in to get

$$q_s = \frac{S_y}{\frac{1}{3}h^3 t \frac{1}{12}h^3 t - \left(\frac{1}{8}h^3 t\right)^2}\left(\frac{1}{8}h^3 t\int_0^s tx\,ds - \frac{1}{12}h^3 t\int_0^s ty\,ds\right) = \frac{S_y}{h^3}\int_0^s\left(\frac{72}{7}x - \frac{48}{7}y\right)ds$$

On the bottom flange 12, we have $x = -\frac{h}{2} + s_1$ and $y = -\frac{h}{2}$. Plugging this in yields (note that for mathematical reasons the limit change accordingly)

$$\begin{aligned}
q_{12} &= \frac{S_y}{h^3}\int_0^{s_1}\left(\frac{72}{7}s_1 - \frac{36}{7}h + \frac{24}{7}h\right)ds = \frac{S_y}{h^3}\int_0^{s_1}\left(\frac{72}{7}s_1 - \frac{12}{7}h\right)ds \\
&= \frac{S_y}{h^3}\left[\frac{36}{7}s_1^2 - \frac{12}{7}hs_1\right]_0^{s_1} = \frac{S_y}{h^3}\left(5.14s_1^2 - 1.72hs_1\right)
\end{aligned}$$

And thus at $s_1 = 0$ (point 1), $q_1 = 0$ (as expected) and at point 2, where $s_1 = \frac{h}{2}$, $q_2 = 0.42\frac{S_y}{h}$. Also, note that it varies quadratically along the flange, and that it has a change of sign at $s_1 = 0.336$. For values of $s_1 < 0.336h$, q_{12} is negative and therefore in the opposite direction to s_1. For the web 23, we have $x = 0$ and $y = -\frac{h}{2} + s_2$ with $0 \leq s_2 \leq h$. Thus, the integration becomes

$$q_{23} = \frac{S_y}{h^3} \int_0^{s_2} \left(\frac{72}{7} \cdot 0 - \frac{48}{7} \left(\frac{-h}{2} + s_2 \right) \right) ds_2 + q_2$$

Note the appearance of q_2 here: the integral itself assumes $q = 0$ when $s_2 = 0$, but obviously, when $s_2 = 0$, we have $q = q_2$. Performing the integration leads to

$$q_{23} = \frac{S_y}{h^3} \left[3.42hs_2 - 3.42s_2^2 \right]_0^{s_2} + 0.42\frac{S_y}{h} = \frac{S_y}{h^3} \left(0.42h^2 + 3.42hs_2 - 3.42s_2^2 \right)$$

Which is symmetrical about the x-axis passing through the centroid and has a maximum when $s_2 = \frac{h}{2}$; furthermore the shear flow is positive at all points in the web. The shear flow in the upper flange may be deduced from point symmetry, but I don't really like that because it's quite confusing regarding the signs, and thus I prefer just integrating once again (with $x = s_3$ and $y = \frac{h}{2}$):

$$\begin{aligned} q_{34} &= \frac{S_y}{h^3} \int_0^{s_3} \left(\frac{72}{7} s_3 - \frac{48}{7} \cdot \frac{h}{2} \right) ds_3 + q_3 = \frac{S_y}{h^3} \left[5.14s_3^2 - 3.43hs_3 \right]_0^{s_3} + 0.42\frac{S_y}{h} \\ &= \frac{S_y}{h^3} \left(0.42h^2 + 5.14s_3^2 - 3.43hs_3 \right) \end{aligned}$$

From this, it's a bit clearer why I don't like using point symmetry to deduce the shear flow distribution: though it is *somewhat* similar to q_{12}, it's clear that you'd need do some more stuff to end up at the same formula. Please note, later we'll see that line symmetry is easier to use. Anyway, using these shear flow distributions, we can draw the distribution as done in figure 17.2.

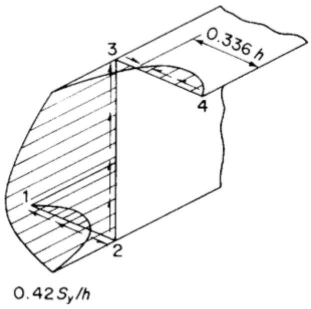


Figure 17.2: Shear flow distribution in Z section.

FINDING THE SHEAR FLOW DISTRIBUTION DUE TO SHEAR FOR AN OPEN SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}.
3. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ and $\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ as much as possible. If the expressions for I_{xx}, I_{yy} or I_{xy} are very complex, consider not plugging the values in just yet but let the fraction just stand there.
4. Start at an open edge. Perform the integration for the first geometric component, between points 1 and 2, where point 1 is the open edge and point 2 a discontinuity (typically a junction) as required by

$$q_{12} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^{s_1} tx \, ds_1 - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \int_0^{s_1} ty \, ds_1$$

by first finding the correct expression for x and y (note that these are *always* with respect to the centroid; you cannot make your life easier by picking a convenient coordinate system), relating them to s_1 if applicable. Note that the plugging in of integration limits is incredibly straightforward and that you should get a medal if you manage to fuck that up. Calculate the shear flow at point 2.

5. For the second component, between points 2 and 3, perform the integration as required by

$$q_{23} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^{s_2} tx \, ds_2 - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^{s_2} ty \, ds_2 + q_2$$

by first finding the correct expression for x and y, relating them to s_2 if applicable.

6. Do the same for the other components. As a final note; if q_{ij} is positive, then it points in the same direction as s_{ij}; if it's negative, then it points in the opposite direction.

17.2.1 *Shear center*

The shear center is the point in the cross-section through which shear loads produce no twisting. This point is the same point as the center of twist. When a cross-section has an axis of symmetry, the shear center must lie on that axis. So, if you have a doubly symmetrical cross-section, then it'll be really easy to locate. However, please don't think that open sections can be doubly symmetric; this is never the case. For example, if you look at figure 17.3, then that circle may seem doubly symmetric to you, so it may be surprising why the shear center is located to the left of the circle. However, note that the y-axis is *not* a line of symmetry; after all, to the right of it, there's a cut at the point where the circle crosses the x-axis, whereas it is not there to the left of it. Now, how can we compute the location of the shear center? Let's first do an example.

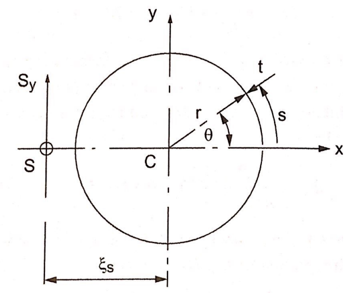


Figure 17.3: Circular beam section.

Example 2

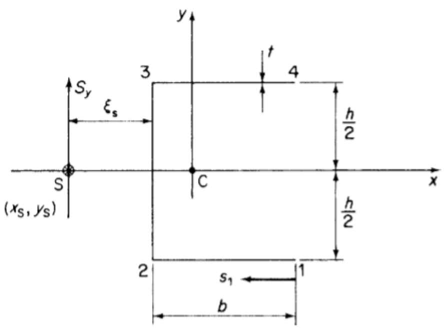


Figure 17.4: Determination of shear center position of channel section.

Calculate the position of the shear center of the thin-walled channel section shown in figure 17.4. The thickness t of the walls is constant.

If a shear force passes through the shear center, then the moment about any point in the cross-section produced by these shear flows is equal to the moment of the applied shear load[a]. Let's assume that the shear center is located a distance ξ_s from the web 23. A convenient point to compute the moments around is point 3: the shear flows in web 23 and flange 34 both go through this point, so they won't create a moment. The force 'built up' in flange 12 is simply $\int_0^b q_{12} ds_1$, so we need to compute q_{12} first. Using the problem solving guide shown before, we first look at which section properties we have to compute. As the section is symmetric, we do not have to compute I_{xy}, and as $S_x = 0$, we do not have to compute I_{yy}, and the formula for shear flow is reduced to

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty\, ds$$

Now, let's compute I_{xx}:

$$I_{xx} = \frac{th^3}{12} + 2 \cdot (bt) \cdot \left(\frac{h}{2}\right)^2 = \frac{h^3 t}{12}\left(1 + \frac{6b}{h}\right)$$

and thus we have (note the disappearing t)

$$q_s = \frac{-12S_y}{h^3\left(1 + \frac{6b}{h}\right)} \int_0^s y\, ds$$

For flange 12, we have $y = -\frac{h}{2}$ and thus

$$q_{12} = \frac{-12S_y}{h^3\left(1 + \frac{6b}{h}\right)} \int_0^{s_1} -\frac{h}{2} ds_1 = \frac{6S_y}{h^2\left(1 + \frac{6b}{h}\right)} s_1$$

Then, integrating this along s_1 from 0 to b yields

$$F_{12} = \int_0^b q_{12} ds_1 = \int_0^b \frac{6S_y}{h^2\left(1 + \frac{6b}{h}\right)} s_1 ds_1 = \left[\frac{3S_y}{h^2\left(1 + \frac{6b}{h}\right)} s_1^2\right]_0^b = \frac{3S_y}{h^2\left(1 + \frac{6b}{h}\right)} b^2$$

And thus the moment around point 3 is[b]

$$M = Fh = \frac{3S_y}{h^2\left(1 + \frac{6b}{h}\right)} b^2 \cdot h = \frac{3S_y}{h\left(1 + \frac{6b}{h}\right)} b^2$$

We must have that when a shear force acts through the shear center, a horizontal distance ξ_s from web 23, it causes the same moment, i.e.

$$S_y \xi_s = \frac{3S_y}{h\left(1 + \frac{6b}{h}\right)} b^2$$

and thus

$$\xi_s = \frac{3}{h\left(1 + \frac{6b}{h}\right)} b^2$$

[a] Before you start thinking, but wasn't the entire point of the shear center that it does not produce a moment, no it wasn't. When a load passes through the shear center, it produces no torque (as torque is equal to the shear force times the distance to the shear center), but it can produce a moment around an arbitrary point in the cross-section for rather obvious reasons.

[b] In clockwise direction, as q_{12} is positive for all positive values of s_1 and thus points in the same direction as s_1.

FINDING THE SHEAR CENTER FOR AN OPEN SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if you only need compute the horizontal coordinate of the shear center (e.g. because the vertical coordinate is clear from symmetry), then you only need to apply a load S_y and thus you only need to know I_{xx}. Similarly, if you only need to compute the vertical coordinate, then you only need to apply a load S_x and thus you only need to know I_{yy}. This only holds if the section is symmetric.
3. For the horizontal coordinate ξ_s, only apply a vertical load S_y:
   (a) Find an easy point to evaluate moments due to internal shear flows around.
   (b) For all components that contribute to the moment due to internal shear flows, compute the shear flow distribution for that component, using

$$q_{ij} = \frac{S_y I_{xy}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\,ds - \frac{S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\,ds + q_i$$

   (c) For each component, evaluate the force generated by this shear flow, by applying

$$F_{ij} = \int_0^l q_{ij}\,ds$$

   where l is the length of the component.
   (d) Multiply each force F_{ij} by the distance to the point that's being evaluated and sum this to get the moment due to internal shear flows. Take note whether each contribution is clockwise or counterclockwise and take appropriate measures to take this into account.
   (e) Set $S_y \xi_s$ equal to this moment and find ξ_s by dividing both sides of the equation by S_y.
4. For the vertical coordinate η_s, only apply a horizontal load S_x and follow the exact same procedure as before; however, use

$$q_{ij} = -\frac{S_x I_{xx}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\,ds + \frac{S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\,ds + q_i$$

   and set $S_x \eta_s$ equal to the moment due to internal shear flows and find η_s by dividing both sides of the equation by S_x.

Example 3

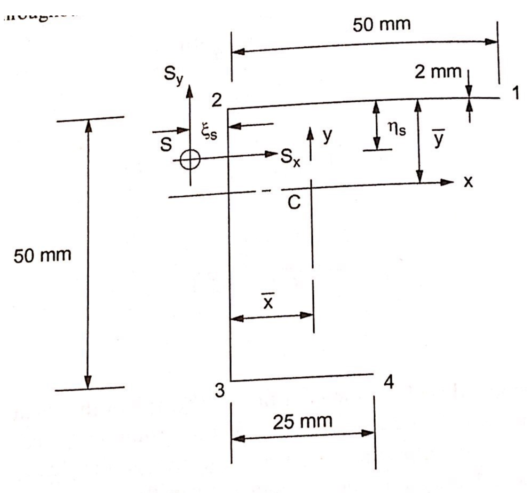


Figure 17.5: Thin-walled beam section of example.

Look at figure 17.5. Calculate the position of the shear center. The thickness of the section is 2 mm and is constant throughout.

The nasty thing about this question is that there are almost no nice things about this question. First, the coordinate system is established at the centroid, but to find the centroid, we locate a coordinate system at point 2. Then:

$$\begin{aligned}
\bar{x} &= \frac{\sum x_i A_i}{\sum A_i} = \frac{25\cdot(50\cdot 2)+0\cdot(50\cdot 2)+12.5\cdot(25\cdot 2)}{50\cdot 2+50\cdot 2+25\cdot 2} = 12.5\,\text{mm} \\
\bar{y} &= \frac{\sum y_i A_i}{\sum A_i} = \frac{0\cdot(50\cdot 2)+25\cdot(50\cdot 2)+50\cdot(25\cdot 2)}{50\cdot 2+50\cdot 2+25\cdot 2} = 20\,\text{mm}
\end{aligned}$$

The section properties are then

$$\begin{aligned}
I_{xx} &= \frac{2\cdot 50^3}{12}+(50\cdot 2)\cdot(0-20)^2+(50\cdot 2)\cdot(25-20)^2+(25\cdot 2)\cdot(50-20)^2 \\
&= 108\,333\,\text{mm}^4 \\
I_{yy} &= \frac{2\cdot 50^3}{12}+\frac{2\cdot 25^3}{12}+(50\cdot 2)\cdot(25-12.5)^2+(50\cdot 2)\cdot(0-12.5)^2+(25\cdot 2)\cdot(12.5-12.5)^2 \\
&= 54\,689\,\text{mm}^4 \\
I_{xy} &= (2\cdot 50)\cdot(25-12.5)(0-20)+(2\cdot 50)\cdot(0-12.5)(25-20)+(2\cdot 25)\cdot(12.5-12.5)(50-20) \\
&= 31\,250\,\text{mm}^4
\end{aligned}$$

Now, first focussing on ξ_s. An easy point to evaluate moments around is point 2. Only the flange 34 causes a moment, so we only need to evaluate

$$q_{34} = \frac{S_y I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s tx\,ds - \frac{S_y I_{yy}}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s ty\,ds + q_3$$

Let's think about what we'd do here: we'd evaluate q_{34} by starting at point 3 and then going to 4 (meaning s points in that direction as well). However, this means that we'd have to know q_3, and thus also q_2 which means we'd have to compute the shear flow distributions for the first two flanges as well. Naturally, we don't wanna do this, so what's actually much easier is to evaluate q_{43} instead, for which we know that $q_4 = 0$ as it's a free edge. However, now s starts at 4 and points in the direction of 3 (so we need to keep this in mind when evaluating whether it causes a positive or negative moment). So, we can also write

$$q_{43} = \frac{S_y I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s tx\,ds - \frac{S_y I_{yy}}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s ty\,ds$$

Plugging in the values of I_{xx}, I_{xy}, I_{yy} and t leads to

$$q_{43} = 1.26\cdot 10^{-5} S_y\int_0^s x\,ds - 2.21\cdot 10^{-5} S_y\int_0^s y\,ds$$

We have $x = 12.5 - s$ and $y = -30\,\text{mm}$. Please note the expression for x: we start evaluating at 4, for which the x-coordinate w.r.t. the centroid is given by $+12.5\,\text{mm}$. We then evaluate s in negative x-direction, hence the minus sign for s. Plugging this in yields

$$\begin{aligned}
q_{43} &= 1.26\cdot 10^{-5} S_y\int_0^s (12.5-s)\,ds - 2.21\cdot 10^{-5} S_y\int_0^s -30ds = S_y\cdot 10^{-5}\int_0^s (82.05-1.26s)\,ds \\
&= S_y\cdot 10^{-5}\left(82.05s-0.63s^2\right)
\end{aligned}$$

and thus the force created by this member is equal to

$$F_{43} = \int_0^{25} q_{43} ds = \int_0^{25} S_y \cdot 10^{-5} \left(82.05s - 0.63s^2\right) ds = S_y \cdot 10^{-5} \cdot \left[41.03s^2 - 0.21s^3\right]_0^{25} = 0.224S_y$$

pointing towards the left and thus the moment generated by it equals $0.224 \cdot 50 = 11.18S_y$ (clockwise), and thus we have

$$S_y \xi_s = 11.18S_y$$

and thus $\xi_s = 11.2$ mm. In the exact same way, but now using

$$q_{43} = -\frac{S_x I_{xx}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\, ds + \frac{S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\, ds = S_x \cdot 10^{-5} \left(2.19s^2 - 92.55s\right)$$

and thus

$$F_{43} = \int_0^{25} q_{43} ds = \int_0^{25} S_x \cdot 10^{-5} \left(2.19s^2 - 92.6s\right) ds = S_x \cdot 10^{-5} \cdot \left[0.73s^3 - 46.28s^2\right]_0^{25} = 0.0175S_x$$

again pointing to the left. The moment generated thus equals $0.0175S_y \cdot 50 = 8.77S_y$ (counterclockwise), and thus we have

$$S_x \eta_s = 8.77S_y$$

or $\eta_s = 8.77$ mm. For one reason, the book gets 7.2 mm as result, but I have no idea how.

Now, do yourself a favour and don't make any of the book exercises related to open section shear. They're honestly a horrible waste of time: the math is just treacherous; they'll never ask a question on an exam related to shapes such as figures P.17.1-P.17.3 (seriously, where would you ever use those kind of shapes in aircraft?), and you're just making your own Christmas break miserable. Furthermore, it's not as if you'll be likely to come across these type of problems in your life (not just the exam, but thereafter as well), for a very simple reason: in all of these calculations, we assume that the shear force acts through the shear center, but how likely is it exactly that a shear force acts exactly through that one specific point? Very unlikely.

If you want to practice more with shear for open sections, just do the following examples which I've taken directly from older exams (and after that, look for yourself for more exam questions).

Example 4

Determine the location of the shear center for the beam shown in figure 17.6 as a function of θ. The beam thickness is 1.75 mm.

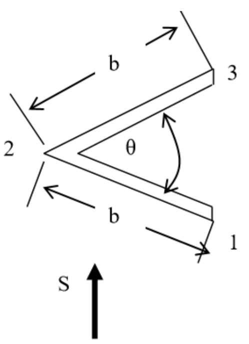


Figure 17.6: Example 4.

If a shear force acts through point 2, then if we evaluate moments around 2, then neither the shear force causes a moment, nor do the shear flows (as they go through point 2). Hence, point 2 is the shear center, and is that independent of θ.
And yes I've took this without simplifying anything from an old exam (so this was literally all you had to say), and it was worth a full 20 points (out of 100) on the exam.

Please note that this example shows an example of cross-sections where the shear center is easy to find; the same short-cut can be done for the beams shown in figure 17.7.

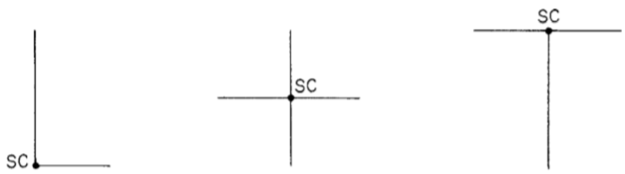


Figure 17.7: Shear center position for the type of open section beam shown.

Example 5

Determine the shear flow at point O for the cross-section shown in figure 17.8. The vertical shear force S is acting through point O. Thickness t everywhere.

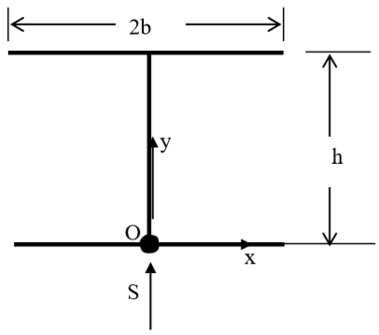


Figure 17.8: Example 5.

Again, not difficult at all. First we move our coordinate system to be halfway through the middle web, as that's the centroid of the beam and we need to use that for all of our computations. First, as we only have S_y and we have symmetry, the shear flow is given by

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s t y \, ds$$

Now, there's a small issue that may be confusing if you've never encountered it before: at point O, the shear flows of the lower left flange and lower right flange come together. Therefore, what we can do, is simply evaluate the shear flow of the lower left flange at point O, and then multiply this by 2 to take into account the shear flow of the lower right flange (due to symmetry, it's simply double this value). As long as you take into account that you don't have shear flow only coming from one side, you're fine. Now, computing I_{xx}:

$$I_{xx} = \frac{th^3}{12} + 2 \cdot (2b \cdot t) \cdot \left(\frac{h}{2}\right)^2 = \frac{h^3 t}{12} + bth^2$$

and thus we must compute, considering the lower left flange (where $y = -\frac{h}{2}$)

$$q_{12} = -\frac{S_y}{\frac{h^3 t}{12} + bth^2} \int_0^{s_1} t \frac{-h}{2} ds_1 = \frac{S_y}{\frac{h^2}{6} + 2bh} \left[s_1\right]_0^{s_1} = \frac{S_y}{\frac{h^2}{6} + 2bh} s_1$$

and thus, for $s_1 = b$, we have that at point O, the shear flow only due to the lower left flange equals

$$q_{12} = \frac{S_y}{\frac{h^2}{6} + 2bh} b$$

and thus that the total shear flow at O equals

$$q_{12} = \frac{S_y b}{\frac{h^2}{12} + bh}$$

17.3 *Shear of closed section beams*

For open sections, we had that exercises *always* had the shear force going through the shear center, as open sections could not efficiently carry torsional moments created when it didn't. However, closed section beams can carry torsional moments (you already had an entire quiz dedicated to it), so questions here will also take this into account, unfortunately. What complicates matters further is the fact that for an open section, you had an easy spot to start all of your integrations: you knew that at the cut, the shear flow would be zero, so you could start from there. If we were to apply the exact same procedure we did for the open sections, and just randomly somewhere along the cross-section start integrating, we'd say that the shear flow at the starting point is zero, even though in reality, it will have some value, say q_0. When we evaluate the shear flow at each point along the beam, the shear flow will everywhere be off by a constant number q_0 as well.

You may see where we're going here: what we'll do is that we'll make a cut somewhere along the beam, then evaluate the "base" shear stress q_b using the method of the previous section, which we know in advance to generate wrong results, and add to that the constant stress q_0 to make it correct.

So, what we're basically doing is depicted in figure 17.9: we replace the shear loads of the left figure by shear loads S_x and S_y acting through the shear center of the resulting open section (which we already know how to deal with from previous section) together with a torque T, as shown in the right figure.

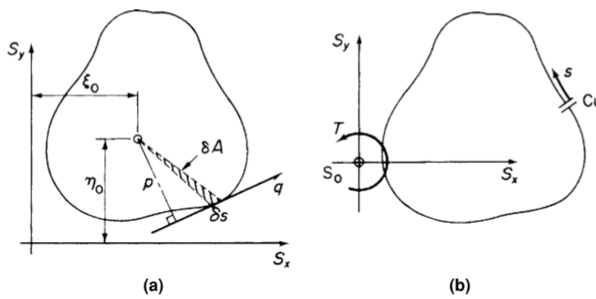


Figure 17.9: (a) Determination of $q_{s,0}$; (b) Equivalent loading of an "open" section beam.

Let's do an example to understand a bit better what you have to do:

Example 1

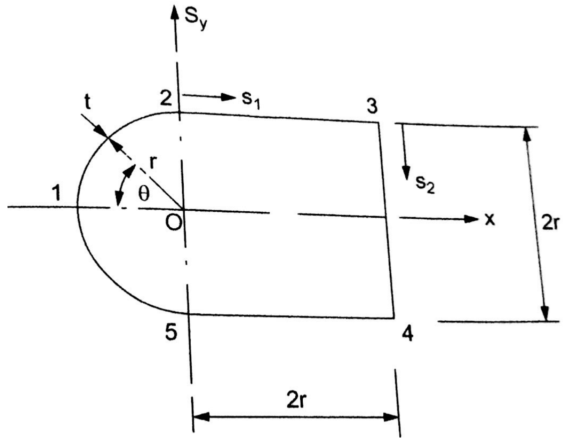


Figure 17.10: Example 1.

Determine the shear flow distribution in the walls of the thin-walled closed section beam shown in figure 17.10; the wall thickness, t, is constant throughout.

Since the x-axis is an axis of symmetry, $I_{xy} = 0$, and with $S_x = 0$, we simply have

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty\, ds + q_{s,0}$$

First, computing I_{xx}:

$$I_{xx} = \frac{\pi t r^3}{2} + \frac{t \cdot (2r)^3}{12} + 2 \cdot (2rt) \cdot r^2 = 6.24tr^3$$

Now, it's helpful to make your cut at an axis of symmetry to reduce computation time, so let's cut our section at 1 and then calculate the base shear flow. Please note: for all following computations, the positive direction for q will be clockwise, as we will evaluate along the cross-section in clockwise direction. If you would go from 1 to 5 to 4 etc., you'd be evaluating in counterclockwise direction, and positive values of q would indicate that it indeed points in counterclockwise direction locally, etc.

$$q_{b,12} = -\frac{S_y}{I_{xx}} \int_0^s ty\, ds$$

Now, let's find a relation between y and s. We have that $y = r\sin\theta$, and similarly $s = r\theta$, or $ds = rd\theta$, and thus we have to integrate (between 0 and θ)

$$q_{b,12} = -\frac{S_y}{6.24r^3} \int_0^\theta tr\sin\theta r\, d\theta = 0.16\frac{S_y}{r}\left[\cos\theta\right]_0^\theta = 0.16\frac{S_y}{r}\left(\cos\theta - 1\right)$$

At 2, when $\theta = \frac{\pi}{2}$, this becomes $q_{b,2} = -0.16\frac{S_y}{r}$. In wall 23, where $y = r$, we then get

$$\begin{aligned} q_{b,23} &= -0.16\frac{S_y}{tr^3}\int_0^{s_1} ty\, ds - 0.16\frac{S_y}{r} = -0.16\frac{S_y}{tr^3}\int_0^{s_1} tr\, ds - 0.16\frac{S_y}{r} = -0.16\frac{S_y}{r^2}\left[s_1\right]_0^{s_1} - 0.16\frac{S_y}{r} \\ &= -0.16\frac{S_y s_1}{r^2} - 0.16\frac{S_y}{r} = -0.16\frac{S_y\left(s_1 + r\right)}{r^2} \end{aligned}$$

and thus at 3, where $s_1 = 2r$, we get

$$q_{b,3} = -0.16\frac{S_y(2r+r)}{r^2} = -0.48\frac{S_y}{r}$$

For web 34, we get (with $y = r - s_2$) that

$$\begin{aligned} q_{b,34} &= -0.16\frac{S_y}{tr^3}\int_0^{s_2} ty\,ds - 0.48\frac{S_y}{r} = -0.16\frac{S_y}{r^3}\int_0^{s_2}\left(r - s_2\right)ds_2 - 0.48\frac{S_y}{r} \\ &= -0.16\frac{S_y}{r^3}\left[rs_2 - \frac{s_2^2}{2}\right]_0^{s_2} - 0.48\frac{S_y}{r} = -0.16\frac{S_y}{r^3}\left(rs_2 - 0.5s_2^2 + 3r^2\right) \end{aligned}$$

Now, note that the distribution will be symmetric with respect to the x-axis so we don't have to go any further (you may if you want, but it's not necessary). Now, we know that this shear flow distribution is completely wrong; at every point, it should be increased by a value $q_{s,0}$ (which may be negative, of course). What we can do is evaluate the moments around O: S_y does not create moment around this point (as it acts through O), so the internal shear flows shouldn't either. We have that the moment caused by the internal base shear flows equals

$$M_b = 2\left[\int_0^{\pi/2} q_{b,12}r^2 d\theta + \int_0^{2r} q_{b,23}r\,ds_1 + \int_0^{r} q_{b,34}2r\,ds_2\right]$$

where the 2 at the front is due to the symmetry which means that we can simply compute it for the top half and then multiply by 2. Furthermore, the term appearing between $q_{b,ij}$ and $d\theta$ or ds is simply the distance from the element to point O, as this is the moment arm. For flange 23, this distance is obviously r, for 34, it's obviously $2r$ and for 12 the distance is r, but as we normally integrate along ds, but we now use $ds = rd\theta$, we get an r^2 there. Unfortunately, this is an absolutely awful expression to evaluate, but we have to do it anyway. Doing it term by term:

$$\begin{aligned} \int_0^{\pi/2} q_{b,12}r^2 d\theta &= \int_0^{\pi/2} 0.16\frac{S_y}{r}(\cos\theta - 1)r^2 d\theta = 0.16S_y r\left[\sin\theta - \theta\right]_0^{\pi/2} = -0.0913S_y r \\ \int_0^{2r} q_{b,23}r\,ds_1 &= \int_0^{2r} -0.16\frac{S_y\left(s_1 + r\right)}{r^2}r\,ds_1 = -0.16\frac{S_y}{r}\left[\frac{s_1^2}{2} + rs_1\right]_0^{2r} = -0.64S_y r \\ \int_0^{r} q_{b,34}2r\,ds_2 &= \int_0^{r} -0.16\frac{S_y}{r^3}\left(rs_2 - 0.5s_2^2 + 3r^2\right)2r\,ds_2 = -0.32\frac{S_y}{r^2}\left[\frac{rs_2^2}{2} - \frac{s_2^2}{6} + 3r^2 s_2\right]_0^{r} \\ &= -1.067\frac{S_y}{r} \end{aligned}$$

Which means that the total moment becomes $M_b = 2\cdot\left(-0.0913S_y r - 0.64S_y r - 1.067S_y r\right) = -3.60S_y r$. As we've evaluated the shear flow in clockwise direction (each time, θ, s_1 and s_2 point in clockwise direction), this means that this moment points in counterclockwise direction as it's negative. Now, we must have that the "correcting" shear flow $q_{s,0}$ produces torque that's equal to this torque. As it's constant, we can simply use the fact that $T = 2Aq_{s,0}$, and with an enclosed area of $4r^2 + \frac{\pi r^2}{2} \approx 5.57r^2$, this means that this shear flow must create clockwise moment equal to $3.60S_y r$, and thus that

$$\begin{aligned} T &= 2Aq_{s,0} \\ q_{s,0} &= \frac{3.60S_y r}{5.57r^2} = 0.32\frac{S_y}{r} \end{aligned}$$

Thus, adding this to the previously found base shear flows we end up at

$$
\begin{aligned}
q_{12} &= 0.16\frac{S_y}{r}(\cos\theta + 1) \\
q_{23} &= 0.16\frac{S_y\left(r - s_1\right)}{r^2} \\
q_{34} &= 0.16\frac{S_y}{r^3}\left(0.5s_2^2 - rs_2 - r^2\right)
\end{aligned}
$$

This distribution is shown in figure 17.11. Note how the bottom shear flow can simply be deduced from line symmetry, and this is also why we could simply multiply the moment created by the top half by 2 to include the bottom half. If you really want to deduce the formulas for it, I think it's easier to just carry on with the integration rather than trying to get new formulas out of the old ones as that can get confusing.

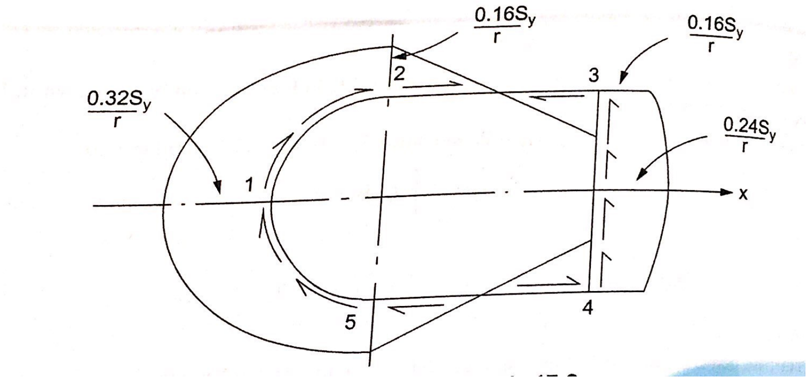


Figure 17.11: Shear flow distribution of example 1.

It may be slightly confusing how exactly you can deduce the shear flow distribution from line symmetry, especially considering which way the shear flow points if you apply line symmetry (because why doesn't it point in the same direction in the previous example for the horizontal bars, but for the vertical bar, it keeps pointing in the same direction). However, the rules for it are straightforward:

- If you use a horizontal axis of symmetry, then the horizontal direction of the shear flow is reversed but the vertical direction of the shear flow is preserved.
- If you use a vertical axis of symmetry, then the horizontal direction of the shear flow is preserved but the vertical direction of the shear flow is reserved.

FINDING THE SHEAR FLOW DISTRIBUTION DUE TO SHEAR FOR A CLOSED SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}.
3. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}$ and $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}$ as much as possible. If the expressions for I_{xx}, I_{yy} or I_{xy} are very complex, consider not plugging the values in just yet but let the fraction just stand there.
4. Make a cut somewhere, preferable at an axis of symmetry (if there's one). Then evaluate the base shear stress in exactly the same manner you did for an open section.
5. Pick a convenient point through which the applied shear force acts. Compute the moment M_b generated by this shear flow distribution.
6. Set
$$0 = M_b + 2Aq_{s,0} \tag{17.2}$$
and straightforwardly calculate $q_{s,0}$.
7. Add this $q_{s,0}$ to all shear flow distributions.

Unfortunately, the three exercises in the book are not really comparable to exam questions, so I wouldn't recommend doing any other than perhaps question 17.16.

17.3.1 Shear center

Computing the shear center is once again pretty much the same procedure as before. However, unfortunately, there's one complication. What we've done so far is that we knew the position of S_y or S_x, then computed the moments around it to compute $q_{s,0}$; in other words, we already knew what the location was for S_y and S_x. However, now we don't (because that's exactly what we're looking for) so we also can't compute $q_{s,0}$, meaning we end up with one equation with two unknowns. That's when the deflection comes in handy: the shear center is defined as the point such that a shear load acting through it does not cause twist, and thus

$$\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{Gt} ds = 0$$

From this, we can deduce that

$$\begin{aligned} 0 &= \oint \frac{1}{Gt}\left(q_b + q_{s,0}\right) ds \\ q_{s,0} &= \frac{\oint \left(q_b/Gt\right) ds}{\oint ds/Gt} \end{aligned}$$

If Gt is constant (which usually will be the case because otherwise it's just ridiculously much work), then we simply have

$$q_{s,0} = -\frac{\oint q_b ds}{\oint ds}$$

Remember that you can only use this property when you consider a load passing through the shear center: don't be stupid and don't use it e.g. for example 1, because there the load didn't pass through the shear center and thus there was twist, meaning that the entire derivation is invalid. Now, onto an absolutely horrible example.

Example 2

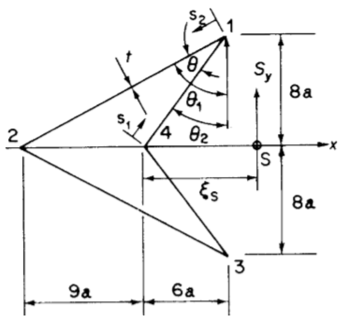


Figure 17.12: Example 2.

A thin-walled closed section beam has the singly symmetrical cross-section shown in figure 17.12. Each wall of the section is flat and has the same thickness t and shear modulus G. Calculate the distance of the shear center from point 4.

The shear center clearly lies on the horizontal axis of symmetry, so that it is necessary only to apply a shear load S_y through S and determine ξ_s. So, please note, even though the following computations are already horrendous, they could have been far more complicated if we had no line of symmetry.
First, as we only need to apply a load S_y, we only need to know I_{xx}. The centroid is obviously located at the line of symmetry, making the Steiner terms relatively easy. Furthermore, we can only calculate the stuff for the top half and then just double it to include the bottom half. Note that the length for wall 12 equals $\sqrt{(15a)^2+(8a)^2} = 17a$ and for wall 14 $\sqrt{(6a)^2+(8a)^2} = 10a$. Using the formulas from page 22 of this summary, we have that for wall 12 (around the line of symmetry):

$$I_{xx} = \frac{(17a)^3\, t \sin^2\beta}{12} + (17at)\cdot\left(\frac{8a}{2}\right)^2$$

Note that on page 22, we said that β was the angle between the bar and the positive x-axis; in this example, that clearly equals $\arcsin\frac{8a}{17a} = \arcsin\frac{8}{17}$. Thus, we get that

$$I_{xx} = \frac{4913a^3t}{12}\cdot\left(\frac{8}{17}\right)^2 + 272a^3t = \frac{1088}{3}a^3t$$

Similarly, for wall 14:

$$I_{xx} = \frac{(10a)^3\,t\sin^2\beta}{12} + (10at)\cdot\left(\frac{8a}{2}\right)^2 = \frac{1000a^3t}{12}\cdot\left(\frac{8}{10}\right)^2 = \frac{640}{3}a^3t$$

Meaning that the total second moment of area will be

$$I_{xx} = 2\cdot\left(\frac{1088}{3}a^3t + \frac{640}{3}a^3t\right) = 1152a^3t$$

Now, if you thought well this must be the only annoying thing, you're into several more stages of disappointment. Whereas for the open section we could suffice by picking a convenient point around which we could evaluate the moments, you here need to compute the shear flow distribution for all of the walls, so that we can compute $q_{s,0}$ afterwards (though we can afterwards compute the moments around a more convenient point that we can pick ourselves). So, let's start at wall 41. There, we have

$$q_{b,41} = -\frac{S_y}{1152a^3t}\int_0^{s_1} ty\,ds_1$$

with y being the projection of s_1 onto the vertical, i.e. $y = \frac{8}{10}s_1$, thus yielding

$$q_{b,41} = -\frac{S_y}{1152a^3}\int_0^{s_1}\frac{8}{10}s_1\,ds_1 = -\frac{S_y}{1152a^3}\left[\frac{4}{10}s_1^2\right]_0^{s_1} = -\frac{S_y}{1152a^3}\cdot\frac{2}{5}s_1^2$$

and thus $q_{b,1} = -\frac{S_y}{1152a^3}\cdot\frac{2}{5}\cdot(10a)^2 = -\frac{S_y}{1152a^3}\cdot 40a^2$. Note that I could have simplified this, but for further calculations, it's actually easier to let it be like this. For wall 12, we get (with $y = 8a - \frac{8}{17}s_2$)

$$\begin{aligned}
q_{b,12} &= -\frac{S_y}{1152a^3t}\int_0^{s_2} t\left(8a - \frac{8}{17}s_2\right)ds_2 - \frac{S_y}{1152a^3}\cdot 40a^2 = -\frac{S_y}{1152a^3}\left(\int_0^{s_2}\left(8a - \frac{8}{17}s_2\right)ds_2 + 40a^2\right)\\
&= -\frac{S_y}{1152a^3}\left(\left[8as_2 - \frac{4}{17}s_2^2\right]_0^{s_2} + 40a^2\right) = -\frac{S_y}{1152a^3}\left(8as_2 - \frac{4}{17}s_2^2 + 40a^2\right)
\end{aligned}$$

Again, due to symmetry, this is enough as we'll just double it everytime we have to use it. We use

$$q_{s,0} = -\frac{\oint q_b\,ds}{\oint ds}$$

to find the $q_{s,0}$ if the load passes through the shear center. $\oint ds$ is easy, that's just $2\cdot(10a + 17a) = 54a$. However, the other integral is more of a pain in the ass. Again, we can just compute it for the upper half and multiply this by two to include the bottom half. We have for wall 41:

$$\int_0^{10a} q_{b,41}\,ds = \int_0^{10a} -\frac{S_y}{1152a^3}\cdot\frac{2}{5}s_1^2\,ds_1 = \left[-\frac{S_y}{1152a^3}\cdot\frac{2}{15}s_1^3\right]_0^{10a} = -\frac{25S_y}{216}$$

and similarly, for wall 12:

$$\begin{aligned}
\int_0^{17a} q_{b,12}\,ds &= \int_0^{17a} -\frac{S_y}{1152a^3}\left(8as_2 - \frac{4}{17}s_2^2 + 40a^2\right)ds_2\\
&= \left[-\frac{S_y}{1152a^3}\left(4as_2^2 - \frac{4}{51}s_2^3 + 40a^2s_1\right)\right]_0^{17a} = -\frac{34S_y}{27}
\end{aligned}$$

and thus

$$\oint q_b ds = 2 \cdot \left(-\frac{25S_y}{216} - \frac{34S_y}{27}\right) = -\frac{11S_y}{4}$$

and

$$q_{s,0} = -\frac{\frac{-11S_y}{4}}{54a} = \frac{11S_y}{216a}$$

Now, finally, we can get on with computing ξ_s. Let's compute the moments around 4, so that we only have to deal with the shear flow in wall 12 (which we double to take into account for wall 23). We have that the moment arm of this shear flow is $\sin\theta \cdot 10a$. Now, finding θ is rather painful. We have $\sin\theta = \sin\left(\theta_1 - \theta_2\right) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$. Furthermore, $\sin\theta_1 = \frac{15}{17}$, $\cos\theta_2 = \frac{8}{10}$, $\cos\theta_1 = \frac{8}{17}$ and $\sin\theta_2 = \frac{6}{10}$, and thus $\sin\theta = \frac{15}{17}\cdot\frac{8}{10} - \frac{8}{17}\cdot\frac{6}{10} = \frac{36}{85}$. Thus, the moment arm equals $\frac{72a}{17}$. The force created in wall 12 equals

$$\begin{aligned} F_{12} &= \int_0^{17a} q_{12} ds = \int_0^{17a} \left[-\frac{S_y}{1152a^3}\left(8as_2 - \frac{4}{17}s_2^2 + 40a^2\right) + \frac{11S_y}{216a}\right] ds_2 \\ &= \left[-\frac{S_y}{1152a^3}\left(4as_2^2 - \frac{4}{51}s_2^3 + 40a^2 s_2\right) + \frac{11S_y}{216a}s_2\right]_0^{17a} = -\frac{85S_y}{216} \end{aligned}$$

and thus the moment created by *both walls 12 and 23* equals $2\cdot\frac{72a}{17}\cdot\frac{-85S_y}{216} = -\frac{10S_y a}{3}$. We must have

$$S_y \xi_s = -\frac{10S_y a}{3}$$

and thus $\xi_s = -\frac{10a}{3}$.

Finding the shear center for an closed section

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if you only need compute the horizontal coordinate of the shear center (e.g. because the vertical coordinate is clear from symmetry), then you only need to apply a load S_y and thus you only need to know I_{xx}. Similarly, if you only need to compute the vertical coordinate, then you only need to apply a load S_x and thus you only need to know I_{yy}. This only holds if the section is symmetric.
3. For the horizontal coordinate ξ_s, only apply a vertical load S_y:
   (a) Calculate the basic shear flow $q_{b,ij}$ distribution in all walls, using

   $$q_{ij} = \frac{S_y I_{xy}}{I_{xx} I yy - I_{xy}^2}\int_0^s tx\,ds - \frac{S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2}\int_0^s ty\,ds + q_i$$

   (b) Compute the redundant shear flow $q_{s,0}$ by applying

   $$q_{s,0} = -\frac{\oint q_b/Gt\,ds}{\oint ds/Gt}$$

   (c) Add this redundant shear flow to all the basic shear flows you've found so far.
   (d) Find an easy point to evaluate moments due to internal shear flows around.
   (e) For each component, evaluate the force generated by this shear flow, by applying

   $$F_{ij} = \int_0^l q_{ij} ds$$

where l is the length of the component.

(f) Multiply each force F_{ij} by the distance p_0 to the point that's being evaluated and sum this to get the moment due to internal shear flows. Take note whether each contribution is clockwise or counterclockwise and take appropriate measures to take this into account.

(g) Set $S_y\xi_s$ equal to this moment and find ξ_s by dividing both sides of the equation by S_y.

4. For the vertical coordinate η_s, only apply a horizontal load S_x and follow the exact same procedure as before; however, use

$$q_{ij} = -\frac{S_x I_{xx}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\, ds + \frac{S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\, ds + q_i$$

and set $S_x\eta_s$ equal to the moment due to internal shear flows and find η_s by dividing both sides of the equation by S_x.

Obviously, this previous example wasn't fun at all, and honestly, I don't recommend doing any of the exercises in the book as again you're just wasting your time (because I really don't think they're gonna ask us to do such a calculation on the quiz/real exam). The only example I can at least slightly recommend (still not fully recommend) is question 17.16, but that's literally about it. You can try the examples below the next subsection, but bear in mind that they're really hard so I recommend to first continue with the remainder of the summary and only do them once you've finished studying.

17.3.2 *Symmetry properties*

Last year, I didn't really recognize in what ways symmetry can be utilized when doing questions related to shear. However, there are four properties that are helpful (note: all of this holds for both open and closed sections). Please note that this is probably a bit overwhelming at first (especially the third and fourth one), so just remember: these only provide short-cuts. If you aren't feeling confident, it's absolutely fine to not take the short-cuts and just compute everything manually. However, later on you'll be more versed with calculating shear flow distributions, and then it's nice to have an overview of which short-cuts exactly exist, so that's why I included them here anyway:

1. The shear center will always lay on an axis of line symmetry. If there are two axes of line symmetry, then the shear center will be at the intersection of the two.
2. If there is point symmetry about a point, then this point will be the shear center[1].
3. If the shear force acts through the shear center in y-direction, and there is an axis of symmetry in y-direction, then the shear flow will be sort of 'anti-symmetric' with respect to this line (anti-symmetry isn't the correct word, but I don't know how else to call it): on opposite sides of the line of symmetry, the shear flow will be equal in magnitude but opposite in direction. See figure **??**: in the closed-section on the left, then we see that on opposite sides of the line of symmetry, the shear flows are equal in magnitude. However, on the left-side, they point in clockwise direction whereas on the right they point in counterclockwise direction. Similarly, for the open-section on the right, the magnitudes of the shear flows are the same on opposite sides of the line of symmetry, however, although they still point the same direction in y-direction, the horizontal x-component has been reversed in direction. The same proposition holds if the shear force acts through the shear center in x-direction, and there is an axis of symmetry in x-direction.
   - There is an important theorem that follows from this: if the shear force acts through the shear center in y-direction, and there is an axis of symmetry in y-direction, then if the material crosses the line of symmetry in a perpendicular fashion, the shear flow will be 0 at the intersection. See figure **??**

[1]For some reason, Dr. Rans keeps mentioning 'planes of antisymmetry' (where antisymmetry is the same as point symmetry btw). However, please do tell me if I'm wrong, but as far as my understanding of symmetry goes, there's no such thing as 'planes of antisymmetry'. They simply don't exist. It's called point symmetry for a reason. At most, you could say that there's an axis of anti-symmetry going through the point around which there is point symmetry that goes in and out of the page, but that's the most I could make of it. Planes of antisymmetry don't exist. I tried asking Dr. Rans via email and in the lecture but I didn't get a really convincing response tbh so I'm rather confident that he's just plain mistaken in thinking that planes of anti-symmetry are a thing (it's like referring to the corners of a circle (circles don't have corners after all). Point symmetry doesn't have planes of antisymmetry). But please do correct me if I'm wrong because I've been struggling with this for weeks now.

once more: in the closed-section on the left, the shear flow is zero at points A and C, as the material crosses the axis of symmetry in a perpendicular fashion. However, at point B, the material does not cross the axis of symmetry in a perpendicular fashion, so the shear flow there is not equal to zero.

4. If the shear force acts through the shear center in y-direction, and there is an axis of symmetry in x-direction, then the shear flow will be line-symmetric with resepct to this axis of symmetry: on opposite sides of the line of symmetry, the shear flow will be equal both in magnitude and in direction. Again, if you don't get what I mean, see figure 17.14: in the closed-section on the left, where the x-axis is an axis of symmetry, that the shear flow on opposite sides of this axis are equal in magnitude, and on both sides they point in counterclockwise and clockwise direction at the same locations (so the direction is preserved). In the open-section on the right, where the y-axis is an axis of symmetry and a horizontal shear force has been applied, the shear flows are equal in magnitude and direction.

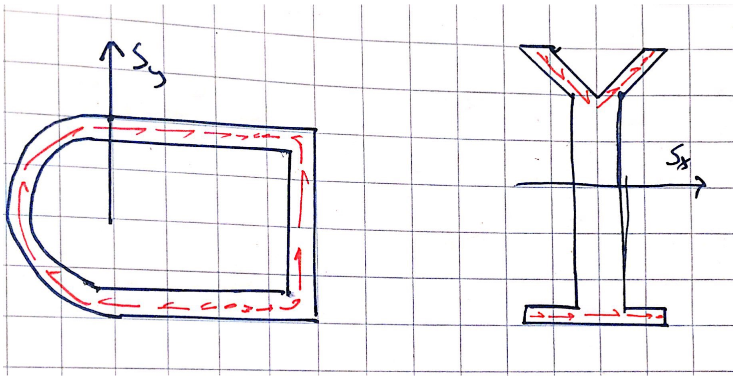


Figure 17.13: Two beams.

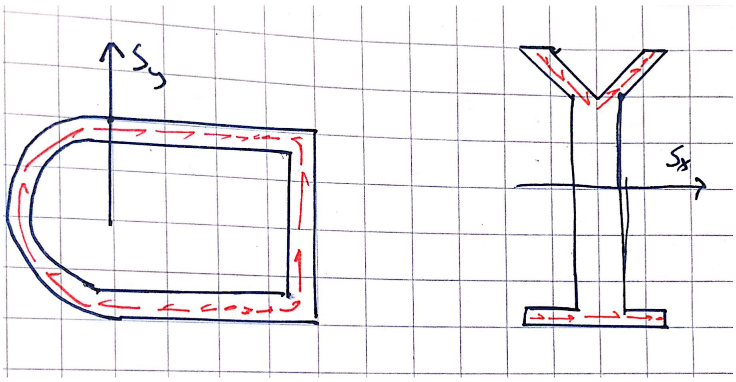


Figure 17.14: Two more beams.

Why are these properties useful? Well, the use of properties 1 and 2 is obvious. Properties 3 and 4 help in deciding on a good location to make a cut for the shear flow in a closed section:

- Property 3 means that for sections where there is a vertical shear force through the shear center and a vertical axis of symmetry (or horizontal/horizontal), you can identify places where the shear flow will be zero a priori. This means that if you make a cut on this place, you don't have to add the redundant shear flow later on, as you already know that the shear flow is 0 on this spot (so the redundant shear flow would be 0, so what's the point in computing it?). If that doesn't make sense: remember the reason for the existence of redundant shear flow. We had redundant shear flow because the location where we make a cut does not actually have zero shear flow, and the redundant shear flow 'corrects' it to the 'true' value. However, if we already know that the 'true' value will be zero as well, we don't need to compute the redundant shear flow any more as there will be no redundant shear flow.
- Property 4 means that for sections where there is a vertical shear force (not necessarily through the shear center) and a horizontal axis of symmetry (or horizontal/vertical), it suffices to make a cut at the horizontal axis of symmetry, then compute the shear flows for the upper part of the cross-section. To compute the redundant shear flow, you can then take moments around any point along the horizontal axis and compute

the moment caused by the base shear flow by computing the moment caused by the shear flow in the upper part of the cross-section and simply multiplying this by 2.

Now, Dr. Rans was nice enough to include a tip on where to make the cut, namely that you should always cut a corner of the cross-section. Unfortunately, this tip is pretty worthless, as there are better locations to make a cut, depending on your cross-section. In reality, this should be the preferred order on where you should make the cut in a closed-section:

WHERE TO MAKE THE CUT IN A CLOSED SECTION

1. If the shear force acts in vertical direction *through the shear center* and there is a vertical axis of symmetry, then make the cut at a location where the beam crosses the vertical axis of symmetry in a perpendicular fashion. The shear flow distribution will only have to be computed for one half of the cross-section, as the shear flow distribution on the other side will be anti-symmetric so you can sketch it easily if you sketch the shear flow distribution for the side you did compute. You do not have to compute the redundant shear flow. If this is not possible, try the next option.
2. If the shear force acts in vertical direction (not necessarily through the shear center) and there is a horizontal axis of symmetry, then make the cut at a location where the beam crosses the horizontal axis of symmetry. The shear flow distribution will only have to be computed for one half of the cross-section, as the shear flow distribution is line-symmetric so you can sketch it easily if you sketch the shear flow distribution for the side you did compute. You do have to compute the redundant shear flow, but to compute the moment caused by the base shear flow you simply only compute the contribution by one side of the cross-section and multiply this by two. If this is not possible, do the final resort.
3. Make the cut at a corner of the cross-section (Calvin's original advice).

In case you think, but hmm Calvin and the TAs probably thought well about the advice, maybe the two first options are so specific that it almost never happens? Well no, cause Calvin showed four examples on where to cut and for two of them they were incorrect (not optimal I mean), so that's not really a good success rate in my view. Let's analyse the four examples he gave, as shown in figures 17.15 and 17.16. The triangle in figure 17.15 is cut in the wrong place: it would make much more sense to cut it in the top corner: the shear force acts in vertical direction through the shear center and there is a vertical axis of symmetry, so we should make a cut at a location where the beam crosses the vertical axis of symmetry in a perpendicular fashion[2]. You also don't need to compute moments any more then. If the shear force would have been horizontal, then you'd still have cut at the top corner (or again, at the middle of the bottom part), by virtue of point 2 of above problem-solving guide.

The left cross-section in figure 17.16 is correct at a correct position, as there are no axes of symmetry at all. The cross-section in the middle of figure 17.16 is also cut correctly, but the right one is wrong: you should cut somewhere on the horizontal axis of symmetry so that you avoid having to compute the redundant shear flow.

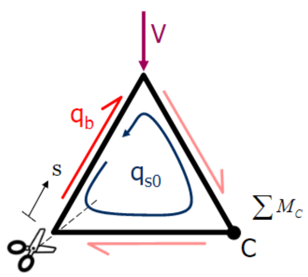


Figure 17.15: A beam.

[2]Okay may not seem like it, but trust me this is correct. Otherwise you can also cut in the middle of the lower side, doesn't lead to a longer computation.

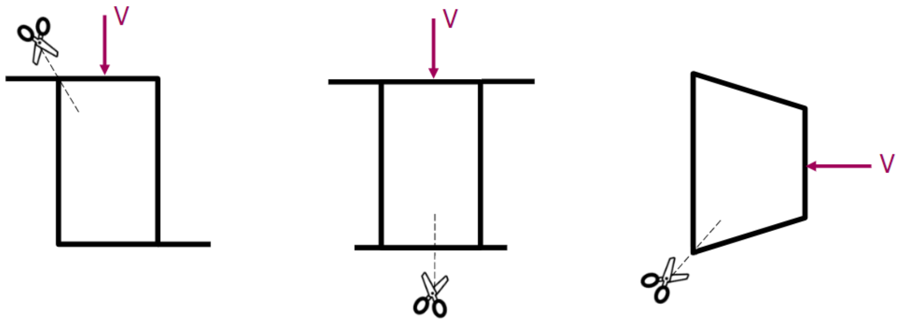


Figure 17.16: Three more beams.

Example 3

The rectangular beam cross-section shown in figure 17.17 is under a vertical shear load S. It fails in shear on the left hand vertical wall when S is applied. (Note that the right vertical wall does not fail). At failure the rate of twist is 0.1 rad/m. The material has $G = 79.5\,\text{GPa}$, $\tau_y = 630\,\text{MPa}$. Wall thickness $t = 1\,\text{mm}$, wall dimension $a = 10\,\text{cm}$. Determine the magnitude of S and where exactly it is applied (the horizontal distance of the line of action of S from one of the walls or the mid-point of the beam). Thin-walled assumptions hold. Neglect normal stresses. Maintain the numbering scheme shown.

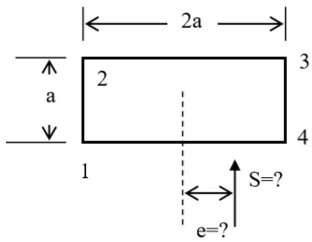


Figure 17.17: Example 3.

We immediately see the most striking difference between the exercises in the book and the actual exam questions. The actual exam questions use much, much easier cross-sections, but they ask for stuff that's far less straightforward than the book exercises. However, you can still come far by just applying the problem solving guides.

Let's first analyse what they actually ask from us. The left wall fails, and not the right wall. This means that there is clearly a torque, because this will increase the shear flow in one web, but will decrease it in the other. The shear force itself will cause a shear flow pointing upwards in the vertical walls, so we know that the torque must have been clockwise, because then it causes the shear flow in the left wall to point even more upwards and in the right wall to point somewhat less upwards. This means that, as the shear center is clearly in the middle of the beam due to double symmetry, the shear force actually acts towards the left of the dashed line. That aside, the question asks for two answers, but it also gives two requirements (namely that it fails at a certain point, so S and e must be such that the shear stress at that point is equal to 630 MPa; and that the rate of twist is 0.1 rad/m), meaning we expect to end up at two equations with two unknowns.

Now, let's decompose this problem into two parts: first calculate the shear flow due to the shear force (so as if it acts through the shear center) and then add to that the shear flow due to the torque it induces. This approach is simply what we've done at all times. When we've done this, we'll just see what we can do to solve the actual problem.

For the first part, make a cut as shown in figure 17.18.

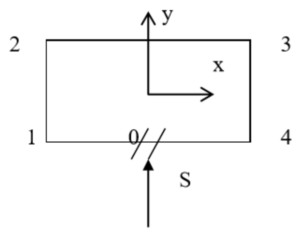


Figure 17.18: Example 3: cut.

The shear flow will be highest at the midpoint between 1 and 2 (it'll be equally high between 3 and 4, but we've already established that we only need to consider the left wall), so we don't have to go any further than that, fortunately. First, we have that

$$I_{xx} = 2 \cdot \frac{a^3 t}{12} + 2 \cdot (2at) \cdot \left(\frac{a}{2}\right)^2 = \frac{a^3 t}{6} + a^3 t = \frac{7a^3 t}{6}$$

Now, we simply have

$$q_{b,s} = -\frac{S}{I_{xx}} \int_0^s ty\,ds = -\frac{S}{\frac{7a^3 t}{6}} \int_0^s ty\,ds = -\frac{6S}{7a^3} \int_0^s y\,ds$$

For wall 01, we thus get (with $y = -\frac{a}{2}$)

$$q_{b,01} = -\frac{6S}{7a^3} \int_0^{s_1} -\frac{a}{2} ds_1 = \frac{3S}{7a^2} s_1$$

and thus $q_{b,1}$ (for which $s_1 = a$) equals $\frac{3S}{7a}$. Then, onto wall 12, we get (with $y = -\frac{a}{2} + s_2$)

$$\begin{aligned} q_{b,12} &= -\frac{6S}{7a^3} \int_0^{s_2} \left(-\frac{a}{2} + s_2\right) ds_2 + \frac{3S}{7a} = -\frac{6S}{7a^3} \left[-\frac{as_2}{2} + \frac{s_2^2}{2}\right]_0^{s_2} + \frac{3S}{7a} \\ &= \frac{6S}{7a^3} \left[\frac{as_2}{2} - \frac{s_2^2}{2} + \frac{a^2}{2}\right] \end{aligned}$$

and thus we have at the midpoint (at which $s_2 = \frac{a}{2}$) that the base shear flow equals

$$q_b = \frac{6S}{7a^3} \left(\frac{a \cdot \frac{a}{2}}{2} - \frac{\left(\frac{a}{2}\right)^2}{2} + \frac{a^2}{2} \right) = \frac{6S}{7a^3} \cdot \frac{5a^2}{8} = \frac{15S}{28a}$$

To this, we must add the extra shear flow caused by the torque, which has a value $S \cdot e$ (for simplicity, in figure 17.17, let's assume e points to the left of the dashed line, as we know S acts to the left of it), and we know that

$$T = Se = 2Aq_0$$

with A the enclosed area equalling $2a^2$ and thus

$$q_0 = \frac{Se}{2 \cdot 2a^2} = \frac{Se}{4a^2}$$

and thus the shear flow at the critical point equals

$$q = \frac{15S}{28a} + \frac{Se}{4a^2}$$

and the shear stress equals

$$\tau = \frac{q}{t} = \frac{\frac{15S}{28a} + \frac{Se}{4a^2}}{t}$$

That's all nice and all, and this leads to our first equation: this shear stress should be set equal to 630 MPa and then we have one equation with two unknowns. Now, invoking the rate of twist into this. For the first part, we placed the shear force through the shear center, so the shear flow induced by this *does not create a twist*. Only the shear flow due to the torque Se causes a twist:

$$\frac{d\theta}{dz} = \frac{1}{2A} \oint \frac{q}{Gt} ds$$

We already calculated q to equal $\frac{Se}{4a^2}$; and with G and t constant, and $A = 2a^2$, this leads to

$$\frac{d\theta}{dz} = \frac{1}{4a^2Gt} \cdot \frac{Se}{4a^2} \cdot 6a = \frac{3Se}{8a^3tG}$$

So, two equations, two unknowns, let's solve this. Unfortunately, as the system isn't linear, getting your graphical calculator out isn't the best idea unless you own a TI nspire 84, so we must solve this by hand, but it's relatively straightforward. From the second one, we have

$$Se = \frac{8a^3tG}{3} \frac{d\theta}{dz}$$

Plugging this into the first one yields

$$\tau = \frac{\frac{15S}{28a} + \frac{\frac{8a^3tG}{3}\frac{d\theta}{dz}}{4a^2}}{t}$$

Doing distances in mm and forces in N (so that we must use MPa and rad/mm):

$$630 = \frac{\frac{15S}{28 \cdot 100} + \frac{\frac{8 \cdot 100^3 \cdot 1 \cdot 79.5 \cdot 10^3}{3} \cdot 0.1 \cdot 10^{-3}}{4 \cdot 100^2}}{1} = \frac{3S}{560} - 530$$

From which it easily follows that $S = 18\,666.67$ N. Then, we easily have from $Se = \frac{8a^3tG}{3}\frac{d\theta}{dz}$ that

$$e = \frac{8a^3tG}{3S}\frac{d\theta}{dz} = \frac{8 \cdot 100^3 \cdot 1 \cdot 79.5 \cdot 1000}{3 \cdot 18666.67} \cdot 0.1 \cdot 10^{-3} = 1136\,\text{mm}$$

Example 4

During level flight, the wing of an aircraft is measured (using sensors for example) to have the following shape (see also figure 17.19):

$$w(x) = A_2x^2 + A_3x^3 + A_4x^4$$

where x is the distance from the wing root, w is the vertical deflection, and the span of each wing is 15 m. The wing cross section is constant and thin-walled with shape as shown in figure 17.20.

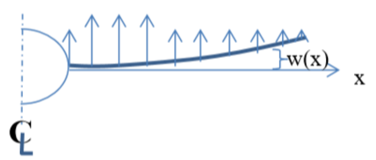


Figure 17.19: Example 4.

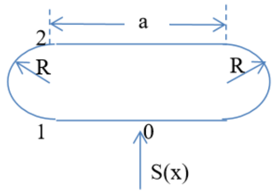


Figure 17.20: Example 4.

Assume that the net shear force $S(x)$ at any spanwise location x goes through the mid-point of the horizontal portion of the cross-section, that is S crosses at $a/2$. Pay attention to the fact that S is a function of x. You are also given that the moment of inertia of a thin-walled circular cross section is $I = \pi R^3 t$. If the bending stiffness of the wing is constant and equal to EI, then, using the numbering scheme, 0,1,2, shown above and cutting at 0:

(a) At $x = x_1$, determine a as a function of location x_1, and any of E, R, t, A_2, A_3, A_4, and τ_y the shear yield strength of the material, such that the cross-section does not fail. Do NOT check the cross-section for normal stresses (to save time) check ONLY for failure under the shear load $S(x_1)$.

(a) You are now given (and only for this part (b) that $E = 69\,\text{GPa}$, $R = 0.3\,\text{m}$, $t = 1\,\text{mm}$, $\tau_y = 220\,\text{MPa}$, $A2 = -0.0005$, $A_3 = -0.001$, $A_4 = -0.00001$ (A_2, A_3, A_4 have appropriate units so w comes out in m). Determine the value of a at $x_1 = 7.0\,\text{m}$.

This question seems rather weird, but by staying calm it's easily solvable. First, let's just derive the formula for the shear force $S(x)$, because we'll probably need it. Remember that $S = \frac{dM}{dx}$ and $M = -EI\frac{d^2w}{dx^2}$ and thus $S = -EI\frac{d^3w}{dx^3}$:

$$S(x) = -EI\left(6A_3 + 24A_4x\right)$$

So we get that out of the way. As we only have a vertical shear force, the basic shear flow is given by

$$q_{b,s} = -\frac{S}{I_{xx}}\int_0^s ty\,ds =$$

Let's now compute I_{xx}. That's simply equal to:

$$I_{xx} = \pi R^3 t + 2\cdot(at)\cdot R^2 = \pi R^3 t + 2R^2 at$$

Now, it's important to realize that due to perfect symmetry, the shear flow at 0 must be 0. Therefore, making a cut at 0 means that we do not have to take into account the redundant shear flow. So, starting at 0, where we have $y = -R$, we get

$$q_{01} = -\frac{S}{I_{xx}}\int_0^{s_1} ty\,ds = \frac{S}{\pi R^2 + 2Ra}\int_0^{s_1} ds = \frac{S}{\pi R^2 + 2Ra}s_1$$

and thus $q_1 = \frac{S}{\pi R^2+2Ra}\frac{a}{2}$. Now, onto the circular segment. As seen before, for circles, it's more convenient to use cylindrical coordinates, as shown in figure 17.21, from which we deduce that $y = -R\cos\theta$, and with $ds = Rd\theta$, this means that

$$\begin{aligned} q_{12} &= -\frac{S}{I_{xx}}\int_0^\theta ty\,ds + \frac{S}{\pi R^2 + 2Ra}\frac{a}{2} = \frac{S}{\pi R^2 + 2Ra}\int_0^\theta R\cos\theta\,d\theta + \frac{S}{\pi R^2 + 2Ra}\frac{a}{2} \\ &= \frac{S}{\pi R^2 + 2Ra}\left([R\sin\theta]_0^\theta + \frac{a}{2}\right) = \frac{S}{\pi R^2 + 2Ra}\left(R\sin\theta + \frac{a}{2}\right) \end{aligned}$$

This expression is maximised when $\theta = \frac{\pi}{2}$ (as expected), so the maximum shear flow is

$$q = \frac{S}{\pi R^2 + 2Ra}\left(R + \frac{a}{2}\right)$$

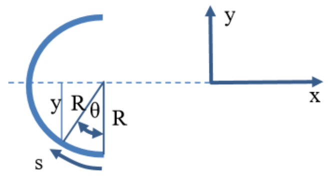


Figure 17.21: Example 4.

The shear stress of this must be equal to τ_y:

$$\begin{aligned}
\frac{\frac{S}{\pi R^2+2Ra}\left(R+\frac{a}{2}\right)}{t} &= \frac{\frac{-E\left(\pi R^3 t+2R^2 at\right)\cdot\left(6A_3+24A_4x\right)}{\pi R^2+2Ra}\left(R+\frac{a}{2}\right)}{t} = \tau_y \\
\tau_y &= \frac{-ERt\left(6A_3+24A_4x\right)\left(R+\frac{a}{2}\right)}{t} = -ER\left(6A_3+24A_x\right)\left(R+\frac{a}{2}\right) \\
\frac{a}{2}ER\left(6A_3+24A_4x\right) &= -\tau_y - ER\left(6A_3+24A_4x\right)R \\
a &= -\frac{2\tau_y}{ER\left(6A_3+24A_4x\right)} - 2R
\end{aligned}$$

So in the end it's mostly just a matter of rewriting stuff. Now, please note that you could have made your life perhaps slightly easier by noting that for the shear flow $q = \frac{S}{I_{xx}}\int_0^s ty\,ds$ and plugging in the function for S we'd have gotten

$$q = \frac{-EI\left(6A_3+24A_4x\right)}{I_{xx}}\int_0^s ty\,ds = -E\left(6A_3+24A_4x\right)\int_0^s ty\,ds$$

but I only realized this at the very end, and you may not have realized this before too. Just as long as you are able to rewrite equations, you're fine anyway. First step is always to look what parts cancel out, and then isolate the symbol you want to find the explicit expression for.

For b, it's literally just plugging in the numbers:

$$a = -\frac{2\cdot 220\cdot 10^6}{69\cdot 10^9\cdot 0.3\cdot(6\cdot -0.001+24\cdot -0.00001\cdot 7)} - 2\cdot 0.3 = 2.168\,\mathrm{m}$$

Example 5

The tailcone of the helicopter shown in figure 17.22 has length $L = 2.5$ m and is clamped to the transition where the tailcone radius is $R_1 = 0.5$ m. At the aft end, where the tailone attaches to the vertical pylon, the radius is R_2. There are no booms and the tailcone is assumed thin-walled with constant thickness $t = 2$ mm. The tail rotor is located at a distance $H = 0.6$ m from the center of the aft section of the tailcone and generates a thrust force F perpendicular to direction of forward motion (and out of the plane of this page). See also figure 17.23.

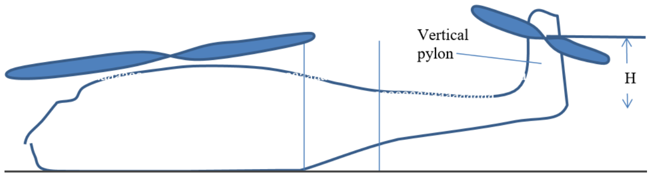


Figure 17.22: A helicopter.

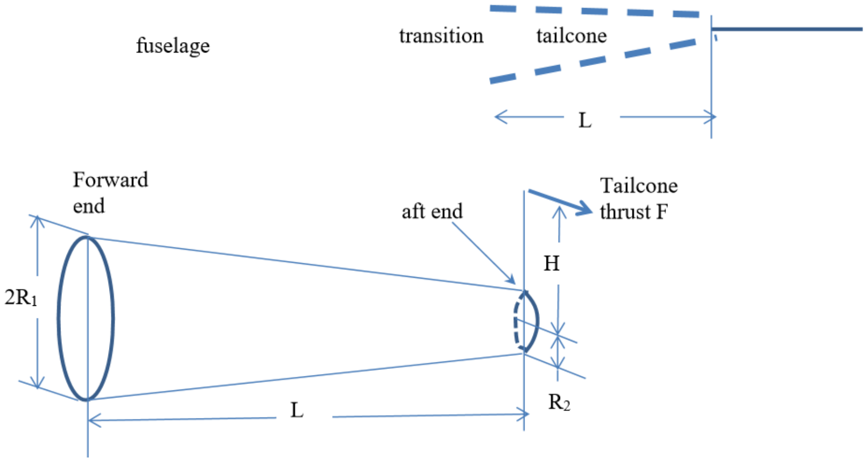


Figure 17.23: Tailcone isolated: clamped at left end and loaded off-center by thrust F.

During tests, the tailcone broke simultaneously at two places: At the forward and at the aft end. You are given: Tailcone material $E = 69$ GPa, $G = 26.5$ GPa, yield stress (tension and compression)=400 MPa. The moment of inertia for a thin-walled circular cross-section is $\pi R^3 t$. Finally, to save time, if normal and shear stresses are present at some location, do NOT use the von Mises stress (which would be the correct thing to do) but find the max normal stress location and the max shear stress location and compare them to their corresponding yield values. This would be a good approximation. Determine the thrust force F when the two simultaneous failures occurred and the radius R_2 at the aft end.

After such a long question, let's just analyse what exactly is asked of us: apparently, the aft and forward end fail simultaneously. How can that? What probably happened (though we still have to show some sort of mathematical proof for it) is that the forward end failed due to bending related stuff, and the aft end failed due to shear related stuff. Let's first focus on the effect of shear throughout the structure.
Let's first focus on the aft end. We can replace the load by a load passing through the shear center and a torque equal to Fh, as shown in figure 17.24.

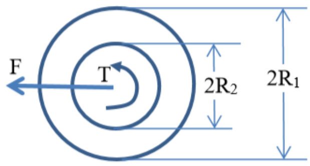


Figure 17.24: View of tailcone from aft looking forward.

Then we have two shear components: one due to the shear force through the shear center and one due to the torque. For the one due to the shear force through the shear center, we realize that if we make the cut at the right horizontal end as shown in figure 17.24, due to symmetry, the shear flow at the cut must be zero. Therefore, the basic shear flow will be the total shear flow and no constant shear flow needs to be added upon closing the cut. *However, you still need to add the shear flow due to the torque F creates.* You merely don't need to add another constant shear flow to correct the shear flow due to the shear force through the shear center. The basic shear flow is given by

$$q_b = -\frac{F}{I_{xx}} \int_0^s t y \, ds$$

where $I_{xx} = \pi R_2^3 t$. Furthermore, let's use $y = -R_2 \cos\theta$ and $ds = R_2 d\theta$:

$$q_b = -\frac{F}{\pi R_2^3 t} \int_0^\theta t \cdot -R_2^2 \cos\theta d\theta = \frac{F}{\pi R_2} \int_0^\theta \cos\theta d\theta = \frac{F}{\pi R_2} [\sin\theta]_0^\theta = \frac{F}{\pi R_2} \sin\theta$$

which is maximum when $\theta = \frac{\pi}{2}$ (as expected), and thus

$$q_{b,max} = \frac{F}{\pi R_2}$$

Now, adding the shear flow due to the torque to this:

$$q_{max} = \frac{F}{\pi R_2} + \frac{T}{2\pi R_2^2} = \frac{F}{\pi R_2} + \frac{FH}{2\pi R_2^2} = \frac{F}{\pi R_2}\left(1 + \frac{H}{2R_2}\right)$$

and thus the maximum shear stress equals

$$\tau_{max} = \frac{F}{t\pi R_2}\left(1 + \frac{H}{2R_2}\right)$$

Using the von Mises stress, we have that the yield shear stress equals:

$$\begin{aligned} Y &= \sqrt{\frac{1}{2}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2\right] + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{xz}^2} = \sqrt{3\tau_{xz}^2} \\ \tau_{xy} &= \frac{Y}{\sqrt{3}} \end{aligned}$$

where $Y = 400\,\text{MPa}$. So, this is one equation with two unknowns, F and R_2, we have here. We must find another one. For the forward end, we have that the maximum shear stress equals

$$\tau_{max} = \frac{F}{t\pi R_1}\left(1 + \frac{H}{2R_1}\right)$$

but as $R_1 > R_2$, the forward end will not have failed due to shear. The failure in the front end is caused by normal stress:

$$\sigma_z = \frac{My}{I}$$

where $M = FL$, $y = R_1$ and $I = \pi R_1^3 t$ and thus

$$\sigma_z = \frac{FLR_1}{\pi R_1^3 t} = \frac{FL}{\pi R_1^2 t} = \sigma_y$$

And thus we have two equations with two unknowns. Unfortunately, it's not linear, so we can't get our graphical calculator to do everything, but we can easily find F from the second equation:

$$F = \frac{\sigma_y \pi R_1^2 t}{L} = \frac{400 \cdot 10^6 \cdot \pi \cdot 0.5^2 \cdot 0.002}{2.5} = 251\,327\,\text{N}$$

and then plugging in all of these values into the first equation:

$$\begin{aligned}
\tau_{max} &= \frac{F}{t\pi R_2}\left(1+\frac{H}{2R_2}\right) \\
\frac{400\cdot 10^6}{\sqrt{3}} &= \frac{251327}{0.002\cdot\pi\cdot R_2}\left(1+\frac{0.6}{2\cdot R_2}\right) = \frac{40\cdot 10^6}{R_2}+\frac{12\cdot 10^6}{R_2^2} \\
230.94 &= \frac{40}{R_2}+\frac{12}{R_2^2} \\
230.94R_2^2 - 40R_2 - 12 &= 0 \\
R_2 &= \frac{40\pm\sqrt{(-40)^2 - 4\cdot 230.94\cdot -12}}{2\cdot 230.94}
\end{aligned}$$

and thus $R_2 = 0.330\,\mathrm{m}$ (the other solution can be discarded as it'd give a negative value for R_2). Again, we see that these exam questions require some rather extensive calculations. However, the trick for these questions is to first analyse what's exactly asked for you, so that you know how many equations you need to come up with. Furthermore, using common sense is also helpful: it's quite logical that the forward end suffered from bending stresses, whereas the aft end will have failed due to shear stress. Naturally, we'd expect to get one equation from the bending stress at the forward end, and one due to the shear stress at the aft end, so that's simply what you have to look for. When you've found those, you can simply solve the system of equations.

17.4 *A note regarding the concept of internal stresses*

Now, normally speaking, I'm not the biggest fan of conceptual things. However, if there is one concept I want you to take away from this course is the following. Remember how we computed the redundant/correction shear flow: so far, we've seen the case that we only had a single shear force acting on a cross-section. This allowed us to pick a point through which the shear force acted; we then computed the moment caused by the base shear flows M_b and then straightforwardly computed $q_{s,0}$ by setting $0 = M_b + 2Aq_{s,0}$. However, although this method is absolutely correct and it's the method you should use, it kinda makes one thing less obvious, namely: what would happen if you went mad and did not choose a point through which the shear force acts, but some other point? Well actually, the formula $0 = M_b + 2Aq_{s,0}$ should be interpreted as follows:

$$\sum M_{\text{external forces}} = \sum M_{\text{internal shear flows}} = M_b + 2Aq_{s,0}$$

In other words, the moment caused by the external forces about *any* point should *equal* the moment caused by the internal shear flows (both of them are evaluated in the same direction). So, it's actually not much of a deal if you pick a point other than a point through which the shear force acts to compute $q_{s,0}$. You simply now have to compute the moment caused by the external forces about that point, and set that equal to $M_b + 2Aq_{s,0}$. So, if there's another point that may be easier to compute moments around, you can do so as long as you remember above equation.

Please, please remember this concept that moments caused by external forces should equal the moment caused by internal shear flows: it also holds for what we did during torsion. In fact, we have even more such equations: maybe you didn't realize it before, but we should also have

$$\sum F_{\text{external force}_x} = \sum F_{\text{internal shear forces}_x}$$
$$\sum F_{\text{external force}_y} = \sum F_{\text{internal shear forces}_y}$$

In other words: the shear force generated in x-direction (found by integrating stuff) should equal the external shear force in x-direction, and similarly it holds for y! Indeed, if you consider example 1 of section 17.3, where a vertical shear force is applied, you realize that when looking at the corresponding sketch of the shear-flow distribution, that there is clearly an upward shear force in the structure (which turns out to have the same magnitude as the applied external shear load), but that in horizontal direction, the shear flows sort of cancel out.

Similarly, if we have pure torsion, then if you analyse the internal shear flow distribution, you'll notice that on net, if you take all the shear flows into account by integration, there won't be a resultant force in x nor y-direction! After all, there isn't an external force applied in x or y-direction, so there shouldn't be an internal force either.

In case you didn't think about it yet: realize that this also holds for bending! There, the bending moment caused by the bending stresses will equal the applied bending moment, and won't cause an additional vertical or horizontal force[3].

In my view, this is the most fundamental concept of this course that is the foundation of everything we do and thus it is an important concept to remember (especially because it's such an easy to remember concept, "forces/moments by external forces equals forces/moments by internal shear flows/stresses"), and although it may seem obvious, I'd like to point out that the TAs themselves last year where unaware of this. I already mentioned it, but during last year's final exam a catastrophic error was made by the TAs: there was a question about shear for a single-cell, non-idealized structure. It was a rectangular cross-section with different thicknesses for different walls; on the left vertical web a vertical shear force of 5000 N was applied and on the right web a vertical shear force of 8000 N was applied. This meant that it was impossible to pick a point through which both of the external shear forces acted. So, for example, what you could have done was to compute moments around the left-bottom corner of the beam, and then compute the moment caused by the vertical shear force on the right web (the distance was 500 mm, so the moment caused by the external shear force equalled $0.5 \cdot 8000 = 4000$ Nm). You then set this equal to $M_b + 2Aq_{s,0}$ and computed $q_{s,0}$. Really not that difficult to the normal situation where you only have one shear force.

Unfortunately, this was already too difficult for the TAs, who thought okay well I don't know what to do with the 2nd shear force so fuck logic I'll just take moments around the left-bottom corner and then take $0 = M_b + 2Aq_{s,0}$, completely ignoring the presence of the 8000 N. Even when my friends and I explained it to the TA for 2 hours that that was wrong and that it was literally written in the book that you had to take the moment caused by the external shear force into account, he just wouldn't understand this elementary concept. Although my friends and me ended up emailing the responsible professor who admitted we were right so that we did get compensated in the end, it has essentially meant that almost everyone who did this problem genuinely correctly got subtracted 1.3 points on their final exam grade, which is insane imo (last year the final exam counted for 70%, so this is about a full point on their final grade for the course. Honestly there gotta be so many people who failed because of it, all because the TAs were so horrifically incompetent). I'm not exactly sure what my point was with this paragraph other than it's a good idea to be critical of the official solutions for the exams[4]

However, in short, we have the following relations that form the fundamentals of this course:

For all types of loading, we should have

$$\sum F_{\text{external force}_x} = \sum F_{\text{internal shear forces}_x} \tag{17.3}$$
$$\sum F_{\text{external force}_y} = \sum F_{\text{internal shear forces}_y} \tag{17.4}$$
$$\sum M_{\text{external forces}} = \sum M_{\text{internal shear flows}} \tag{17.5}$$

in other words: the internal forces should *equal* the external forces (and not counteract them).

FUNDAMENTAL RELATIONS BETWEEN EXTERNAL AND INTERNAL FORCES

23.4 *Shear of multi-cell sections*

Not surprisingly, we can have shear of multi-cell sections as well.

Just like the case for torsion of multi-cell sections, you ultimately want to end up with N equations with N unknowns. How do we get there? First of all, we again will look at each cell individually. Fortunately, for each

[3] Remember that bending stresses act into and out of the paper.

[4] Also because miraculously the TA responsible for the error on the exam managed to become a TA again, and because there have been quite a number of errors so far in the lectures that haven't been noticed by the TAs.

cell, the base shear flow is relatively easily computed by

$$q_b = -\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s tx\,ds - \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\,ds$$

Just to be clear: I_{xx}, I_{xy} and I_{yy} are all with regards to the entire multicell section, and x and y are with respect to the centroid of the entire cross-section. At each cell, you make a cut somewhere, and for each cell, you evaluate along the perimeter as you've done for closed sections, using above formula. Not much has changed, basically.

Now, finding the redundant shear flow $q_{s,0}$ is a pain in the ass. For the closed section, we did this by evaluating moment around a point through which the shear force acts, so that the moment equation would become

$$0 = \oint q_b p_0 ds + 2Aq_{s,0}$$

where p_0 was the moment arm for a specific shear flow. Unfortunately, you can only do this once for a multicell; you quite obviously get

$$0 = \sum_{R=1}^{N} \oint_R q_b p_0 ds + \sum_{R=1}^{N} 2A_r q_{s,0,R}$$

where R denotes the cell number and $q_{s,0,R}$ is the residual shear flow for the Rth cell. Clearly, this is only one equation with N unknowns (namely $q_{s,0,1}$, $q_{s,0,2}$, ..., $q_{s,0,N}$. So, we need another bunch of equations. Again, we use compatibility for this. We have that for each cell

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \oint q_R \frac{ds}{t}$$

where q_R is the shear flow distribution in the Rth cell. This includes the base shear flow and the redundant shear flow. Now, when you have a "shared" wall in your cell, then you need to include the influence of the cell on the other side on that wall, complicating matters further. However, we now have another N equations with $N+1$ unknowns, meaning that we now have $N+1$ equations with $N+1$ unknowns, so we should be able to solve these questions. Now, the advantage is, these types of question, unless there are huge short cuts, take such an incredible amount of time, that I consider it very unlikely they'll ever ask an even slightly complicated question about it. Consider for example the example below: it's a relatively straightforward geometry, but I've already written more than 2.5 page and I couldn't be bothered to do the rest of the exercises. So, try to follow it to get an idea of what to do, but don't do another exercise like this afterwards because honestly it'll just bring back memories of wasting your time on a dynamics assignment.

Example 1

Consider the multicell structure with dimensions shown in figure 23.25.

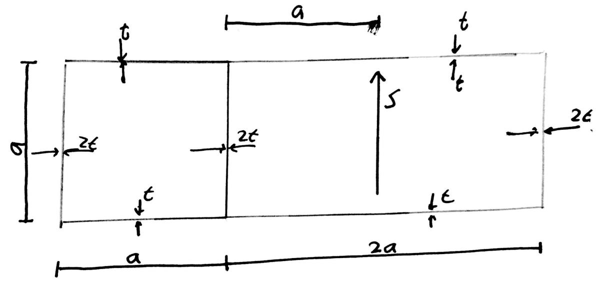


Figure 23.25: Multicell structure.

Calculate the shear flow in the middle web. As a note, I had to come up with this question myself because there were literally no suitable questions to be found anywhere. So, if you think I made a mistake somewhere, please let me know.

Like I said, we'll be considering each cell individually. For clarity, look at figure 23.26, which includes a numbering system and an overview of what shear flows we have to calculate. Note that q_{b_1} and q_{b_2} vary along both cells.

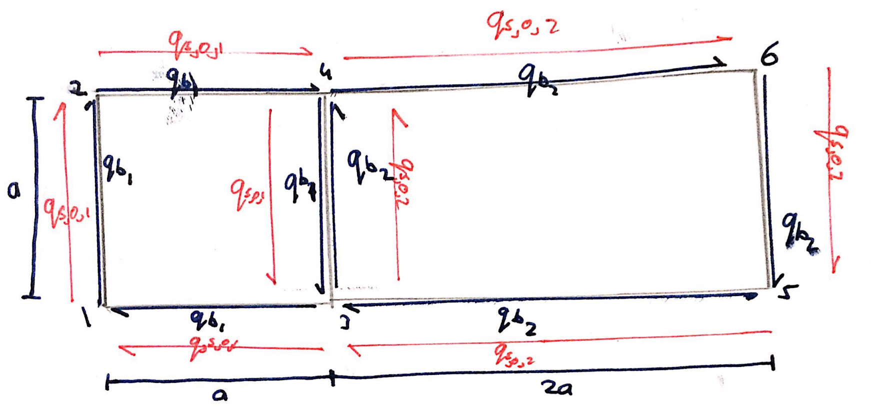


Figure 23.26: Numbering system.

First, remember that the basic shear flow is given by

$$q_b = -\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t x \, ds - \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t y \, ds$$

However, with $S_x = 0$ and $I_{xy} = 0$ due to symmetry, this reduces to

$$q_b = -\frac{S}{I_{xx}} \int_0^s t y \, ds$$

Computing I_{xx} first (the centroid obviously lies in the middle of the section):

$$I_{xx} = 3 \cdot \frac{2t \cdot a^3}{12} + 2 \cdot (3a \cdot t) \cdot \left(\frac{a}{2}\right)^2 = 2a^3 t$$

Then, let's first calculate the basic shear flow distribution for the left cell, going in clockwise direction. Then, for each wall, we have that $y_{12} = -\frac{a}{2} + s_1$, $y_{24} = \frac{a}{2}$, $y_{43} = \frac{a}{2} - s_3$, $y_{31} = -\frac{a}{2}$ and thus, doing the computations as you know how:

$$
\begin{aligned}
q_{b,12} &= -\frac{S}{2a^3t}\int\limits_0^{s_1} 2ty\,ds_1 = -\frac{S}{a^3}\int\limits_0^{s_1}\left(-\frac{a}{2}+s_1\right)\,ds_1 = -\frac{S}{a^3}\left[\frac{s_1^2}{2}-\frac{as_1}{2}\right]_0^{s_1} \\
&= -\frac{S}{a^3}\left(\frac{s_1^2}{2}-\frac{as_1}{2}\right) \\
q_{b,2}\big|_{s_1=a} &= -\frac{S}{a^3}\left(\frac{a^2}{2}-\frac{a^2}{2}\right) = 0 \\
q_{b,24} &= -\frac{S}{2a^3t}\int\limits_0^{s_2} ty\,ds_2 = -\frac{S}{2a^3}\int\limits_0^{s_2}\frac{a}{2}ds_2 = -\frac{S}{2a^3}\frac{as_2}{2} \\
q_{b,4}\big|_{s_2=a} &= -\frac{S}{2a^3}\cdot\frac{a^2}{2} \\
q_{b,43} &= -\frac{S}{2a^3t}\int\limits_0^{s_3} 2ty\,ds_3 - \frac{S}{2a^3}\cdot\frac{a^2}{2} = -\frac{S}{2a^3}\int\limits_0^{s_3} 2\left(\frac{a}{2}-s_3\right)ds_3 - \frac{S}{2a^3}\cdot\frac{a^2}{2} \\
&= -\frac{S}{2a^3}\left[as_3 - s_3^2\right]_0^{s_3} - \frac{S}{2a^3}\cdot\frac{a^2}{2} = \frac{S}{2a^3}\left(s_3^2 - as_3 - \frac{a^2}{2}\right) \\
q_{b,3}\big|_{s_3=a} &= \frac{S}{2a^3}\left(a^2 - a^2 - \frac{a^2}{2}\right) = -\frac{S}{2a^3}\cdot\frac{a^2}{2} \\
q_{b,31} &= -\frac{S}{2a^3t}\int\limits_0^{s_4} ty\,ds_4 - \frac{S}{2a^3}\cdot\frac{a^2}{2} = -\frac{S}{2a^3}\int\limits_0^{s_4} -\frac{a}{2}ds_4 - \frac{S}{2a^3}\cdot\frac{a^2}{2} \\
&= \frac{S}{2a^3}\left(\left[\frac{as_4}{2}\right]_0^{s_4} - \frac{a^2}{2}\right) = \frac{S}{2a^3}\left(\frac{as_4}{2}-\frac{a^2}{2}\right)
\end{aligned}
$$

Doing exactly the same for the right cell gives:

$$
\begin{aligned}
q_{b,53} &= -\frac{S}{2a^3t}\int\limits_0^{s_5} ty\,ds_5 = -\frac{S}{2a^3}\int\limits_0^{s_5} -\frac{a}{2}ds_5 = \frac{S}{2a^3}\cdot\frac{as_5}{2} \\
q_{b,3}\big|_{s_5=2a} &= \frac{S}{2a^3}\cdot a^2 \\
q_{b,34} &= -\frac{S}{2a^3t}\int\limits_0^{s_6} 2ty\,ds_6 + \frac{S}{2a^3}\cdot a^2 = -\frac{S}{2a^3}\int\limits_0^{s_6} 2\left(-\frac{a}{2}+s_6\right)ds_6 + \frac{S}{2a^3}\cdot a^2 \\
&= \frac{S}{2a^3}\left(\left[as_6 - s_6^2\right]_0^{s_6} + a^2\right) = \frac{S}{2a^3}\left(as_6 - s_6^2 + a^2\right) \\
q_{b,4}\big|_{s_6=a} &= \frac{S}{2a^3}\left(a^2 - a^2 + a^2\right) = \frac{S}{2a^3}\cdot a^2 \\
q_{b,46} &= -\frac{S}{2a^3t}\int\limits_0^{s_7} ty\,ds_7 + \frac{S}{2a^3}\cdot a^2 = -\frac{S}{2a^3}\int\limits_0^{s_7}\frac{a}{2}\,ds_7 + \frac{S}{2a^3}\cdot a^2 = \frac{S}{2a^3}\left(-\frac{as_7}{2}+a^2\right) \\
q_{b,6}\big|_{s_7=2a} &= \frac{S}{2a^3}\left(-\frac{a\cdot 2a}{2}+a^2\right) = 0 \\
q_{b,65} &= -\frac{S}{2a^3t}\int\limits_0^{s_8} 2ty\,ds_8 = -\frac{S}{2a^3}\int_0^{s_8} 2\left(\frac{a}{2}-s_8\right)ds_8 = -\frac{S}{2a^3}\left[\frac{as_8}{2}-\frac{s_8^2}{2}\right]_0^{s_8} \\
&= \frac{S}{2a^3}\left(\frac{s_8^2}{2}-\frac{as_8}{2}\right)
\end{aligned}
$$

Absolutely wonderful. Now, let's take a convenient point to compute the moments around. Let's take the point on the bottom flange where S crosses the flanges, so that we can ignore the contributions by the bottom walls. We then have that the moment caused by the left cell around this point is given by (clockwise positive)

$$\begin{aligned} M_{b,1} &= 2a\cdot\int_0^a -\frac{S}{a^3}\left(\frac{s_1^2}{2}-\frac{as_1}{2}\right)ds_1 + a\cdot\int_0^a -\frac{S}{2a^3}\frac{as_2}{2}ds_2 - a\int_0^a \frac{S}{2a^3}\left(s_3^2 - as_3 - \frac{a^2}{2}\right) \\ &= \frac{S}{2a^3}\left(4a\left[-\frac{s_1^3}{6}+\frac{as_1^2}{4}\right]_0^a - a\cdot\left[\frac{as_2^2}{4}\right]_0^a - a\cdot\left[\frac{s_3^3}{3}-\frac{as_3^2}{2}-\frac{a^2s_3}{2}\right]_0^a\right) \\ &= \frac{3Sa}{8} \end{aligned}$$

and similarly, the moment generated by the right cell:

$$\begin{aligned} M_{b,2} &= a\cdot\int_0^a \frac{S}{2a^3}\left(as_6 - s_6^2 + a^2\right)ds_6 + a\cdot\int_0^{2a}\frac{S}{2a^3}\left(-\frac{as_7}{2}+a^2\right)ds_7 + a\cdot\int_0^a\frac{S}{2a^3}\left(\frac{s_8^2}{2}-\frac{as_8}{2}\right) \\ &= \frac{S}{2a^3}\left(a\left[\frac{as_6^2}{2}-\frac{s_6^3}{3}+a^2s_6\right]_0^a + a\left[-\frac{as_7^2}{4}+a^2s_7\right]_0^{2a} + a\left[\frac{s_8^3}{6}-\frac{as_8^2}{4}\right]_0^a\right) = \frac{25Sa}{24} \end{aligned}$$

And thus we must have

$$0 = \frac{3Sa}{8} + \frac{25Sa}{24} + 2a^2 q_{s,0,1} + 4a^2 q_{s,0,2}$$

which will be our first equation with two unknowns, $q_{s,0,1}$ and $q_{s,0,2}$. Now, onto the other ones. We must first find the rate of twist for the left cell. For this, it is important to write down what exactly will be the shear flow in each wall, taking into account the shear flow caused by the other cell:

$$\begin{aligned} q_{12} &= -\frac{S}{a^3}\left(\frac{s_1^2}{2}-\frac{as_1}{2}\right) + q_{s,0,1} \\ q_{24} &= -\frac{S}{2a^3}\frac{as_2}{2} + q_{s,0,1} \\ q_{43} &= \frac{S}{2a^3}\left(s_3^2 - as_3 - \frac{a^2}{2}\right) + q_{s,0,1} - \frac{S}{2a^3}\left(as_6 - s_6^2 + a^2\right) - q_{s,0,2} \\ q_{31} &= \frac{S}{2a^3}\left(\frac{as_4}{2}-\frac{a^2}{2}\right) + q_{s,0,1} \end{aligned}$$

where you can relate s_6 to s_3 via $s_6 = a - s_3$. Now, I'm not going to work out this entire integral (nor the integral for the other $\frac{d\theta}{dz}$ because I think it's very clear that they'll never ask such a question on an exam (or the quiz), simply because it takes so ludicrously long to do it.

So, now that we've seen that it's very unlikely that they'll ask for such a problem, let's do some exam questions:

Example 2

Consider the 5-cell beam cross section shown in figure 23.27. Each cell is square of side a and the wall thickness is t. Thin-walled assumptions apply. A vertical shear force is applied through the shear center of the beam. Maintain the numbering scheme shown and cut where is shown.

(a) Determine the basic shear flow (that is: do NOT include the constant shear flow obtained after you close the cut; ONLY the one with the cut present) for segments 5/6 and 9/10.

(b) If now the beam had 9 identical cells numbered following the same pattern, use your results in part (a) to determine the basic shear flows in segments 9/10 and 13/14.

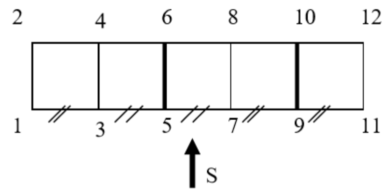


Figure 23.27: Part (a).

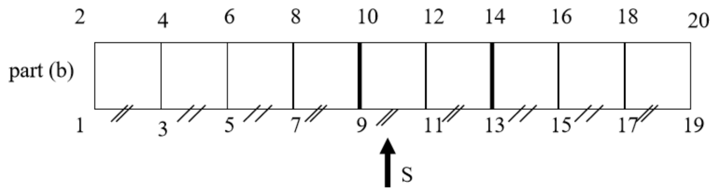


Figure 23.28: Part (b).

So, we only have to determine basic shear flows, and not deal with the residual shear flows. Furthermore, let's analyse

$$q_b = -\frac{S_y}{I_{xx}} \int_0^s t y \, ds$$

a bit. From this formula, it is apparent that the basic shear flow in each cell will be exactly the same: q_b does not specify a x-coordinate, after all, so it doesn't matter which cell you look at, the evaluation of the integral will be exactly the same. So, we know that the shear flow in all vertical walls will the same (except for 1/2 and 11/12), so we only have to calculate it for 5/6. Furthermore, note that wall 5/6 will experience shear flow due to 3/5 and 7/5, but again, due to symmetry, we can just calculate the influence of one and then double it to get the influence of the other.

Calling the cut in wall 5/7 point 0, we have

$$q_{05} = -\frac{S}{I} \int_0^s t y \, ds$$

with $y = -\frac{a}{2}$ and thus

$$\begin{aligned} q_{05} &= \frac{St}{I} \int_0^{s_1} \frac{a}{2} \, ds_1 = \frac{St}{I} \frac{as_1}{2} \\ q_5|_{s_1 = \frac{a}{2}} &= \frac{St}{I} \frac{a^2}{4} \end{aligned}$$

but we need to double this value to take into account wall 3/5:

$$q_{05} = \frac{Sta^2}{2I}$$

Onto the vertical wall:

$$q_{56} = -\frac{S}{I} \int_0^{s_2} t y \, ds_2$$

with $y = -\frac{a}{2} + s_2$ and thus

$$\begin{aligned} q_{56} &= -\frac{St}{I}\int_0^{s_2}\left(-\frac{a}{2}+s_2\right)ds_2 + \frac{Sta^2}{2I} = \frac{St}{I}\left[\frac{as_2}{2}-\frac{s_2^2}{2}\right]_0^{s_2} + \frac{Sta^2}{2I} \\ &= \frac{St}{I}\left(\frac{as_2}{2}-\frac{s_2^2}{2}+\frac{a^2}{2}\right) \end{aligned}$$

Finally, finding an expression for I:

$$I_{xx} = 6\cdot\frac{ta^3}{12} + 2\cdot(5at)\cdot\left(\frac{a}{2}\right)^2 = 3a^3t$$

and thus

$$q_{56} = \frac{St}{3a^3t}\left(\frac{as_2}{2}-\frac{s_2^2}{2}+\frac{a^2}{2}\right)$$

which is also the shear flow distribution in wall 9/10.

For b, the only thing that changes is I:

$$I_{xx} = 10\cdot\frac{ta^3}{12} + 2\cdot(9at)\cdot\left(\frac{a}{2}\right)^2 = \frac{16}{3}a^3t$$

so that

$$q_{9/10} = q_{13/14} = \frac{3St}{16a^3t}\left(\frac{as_2}{2}-\frac{s_2^2}{2}+\frac{a^2}{2}\right)$$

Quite clearly, this example is much, much easier than the previous one.

Example 3

A rectangular cross-section is under two shear forces as shown. Determine the shear flows everywhere. Maintain numbering scheme and shear flow orientations shown.

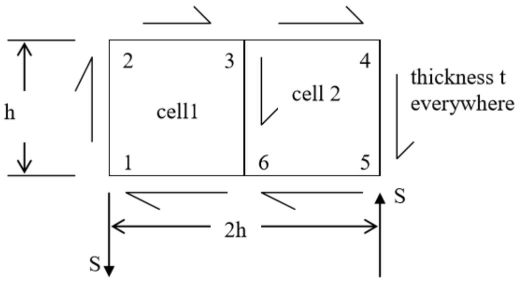


Figure 23.29: Example 3.

Due to symmetry, the shear center is at the mid-point of the middle web. Therefore, the shear forces actually only create a torque around this point, but there's no "real" shear force acting on it. By symmetry, we must therefore have

$$q_1 = q_2 = q$$

and additionally, we must have

$$T = 2Sh = 2A_1q_1 + 2A_2q_2 = 4h^2q$$

and consequently $q = \frac{S}{2h}$. This is the shear flow in all of the external walls; in wall 36, the shear flows cancel out. That's literally all there is to this question.

23.4.1 Shear center

Again, we can, theoretically, calculate a shear center. However, this is so ridiculously complicated that you shouldn't bother spending time on it. If there's a question about, it's glaringly obvious where it's located.

Example 4

In the following thin-walled hollow beam the applied shear force S is found to cause zero rate of twist. Determine the location of the shear center.

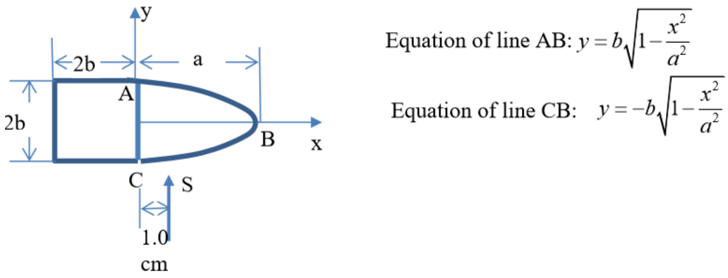


Figure 23.30: Example 5.

Very simple: the shear center is due to symmetry located at the x-axis. Furthermore, as the shear force does not create twist, it must be passing through the shear center, meaning that the shear center is located on the x-axis, 1.0 cm to the right of AC.

19 Combined open- and closed sections

Consider a section that is both open and closed at the same time, as shown in figure 19.1.

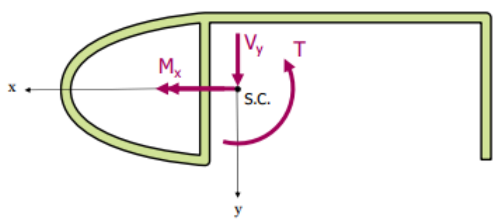


Figure 19.1: Combined open and closed section.

How would this influence our calculations for bending, torsion, and shear, respectively? Let's find out.

19.1 Bending

For bending, literally nothing changes. Just use

$$\sigma_z = \frac{\left(M_x I_{yy} - M_y I_{xy}\right) y + \left(M_y I_{xx} - M_x I_{xy}\right) x}{I_{xx} I_{yy} - I_{xy}^2}$$

19.2 Shear

For shear, literally nothing changes if you understood correctly what we were doing before. Let's do an example to show how you should approach it:

Example 1

Determine the shear flow distribution in the beam section shown in figure 19.2, when it is subjected to a shear load in its vertical plane of symmetry. The thickness of the walls of the section is 2 mm throughout.

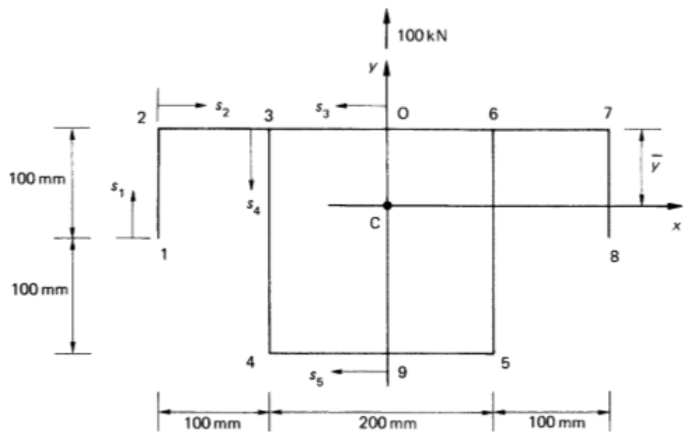


Figure 19.2: Beam section of example 1.

The centroid obviously lays on the y-axis. If you measure y from the bottom wall 495, you get

$$\bar{y} = \frac{\sum \tilde{x}_i A_i}{\sum A_i} = \frac{2 \cdot 100 \cdot 2 \cdot \frac{200-100}{2} + 400 \cdot 2 \cdot 200 + 2 \cdot 200 \cdot 2\frac{200-0}{2} + 200 \cdot 2 \cdot 0}{2 \cdot 100 \cdot 2 + 400 \cdot 2 + 2 \cdot 200 \cdot 2 + 200 \cdot 2} = 125\,\text{mm}$$

where the first term in the sums is the contribution by walls 12 and 78; the second term is the contribution of wall 23067; the third term is the contributions of walls 34 and 56; the fourth term is the contribution by wall 495.

We then have for I_{xx}:

$$\begin{aligned} I_{xx} &= 2 \cdot \left(\frac{2 \cdot 100^3}{12} + 2 \cdot 100 \cdot \left(\frac{200-100}{2} - 125\right)^2\right) + 400 \cdot 2 \cdot (200-125)^2 \\ &\quad +2 \cdot \left(\frac{2 \cdot 200^3}{12} + 2 \cdot 200 \cdot \left(\frac{200-0}{2} - 125\right)^2\right) + 200 \cdot 2 \cdot (0-125)^2 = 14.5 \times 10^6\,\text{mm}^4 \end{aligned}$$

Since the section is symmetric, $I_{xy} = 0$ and with $S_x = 0$, we simply can use

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s t y ds$$

Now, how do we start our analysis? Note that the shear force goes through the shear center. This means that the shear flow distribution in the two open sections can easily be calculated. First, let's analyse the open section 123; clearly, the shear flow at 1 will be zero. Thus, we get

$$q_{12} = -\frac{S_y}{I_{xx}} \int_0^s t y ds = -\frac{100 \cdot 10^3}{14.5 \cdot 10^6} \int_0^{s_1} 2 \cdot \left(-25 + s_1\right) ds_1 = -69.0 \cdot 10^{-4} \cdot \left(-50 s_1 + s_1^2\right)$$

so that $q_2 = -34.5\,\text{N/mm}$. In wall 23, we then get

$$q_{23} = -\frac{S_y}{I_{xx}} \int_0^s t y ds - 34.5 = -\frac{100 \cdot 10^3}{14.5 \cdot 10^6} \int_0^{s_2} 2 \cdot 75 ds_2 = -1.04 s_2 - 34.5$$

so that $q_3 = -138.5\,\text{N/mm}$. Please note: this is the shear flow *just before* point 3, when you are approaching point 3 from wall 23 (we'll later see why this is important). The shear flow distribution in 876 follows directly due to symmetry.

Then, for the closed section: we have a vertical shear force applied through the shear center in a section with a vertical axis of symmetry. Thus, we should make a cut at the midpoint of point 9 or point 0, as we know that the shear flow will be zero there, meaning that we won't have to compute the redundant shear flow. So, let's make the cut at point 0; then

$$q_{03} = -\frac{S_y}{I_{xx}} \int_0^s t y ds = -\frac{100 \cdot 10^3}{14.5 \cdot 10^6} \int_0^{s_3} 2 \cdot 75 ds_3 = -1.04 s_3$$

and thus $q_3 = -104\,\text{N/mm}$ in the point just before point 3, when you are approaching point 3 from wall 03. Why is this different from the shear flow found before? Well, you have to apply Kirchoff's law: the sum of the incoming shear flow should equal the sum of the outcoming shear flow. Note that in figure 19.2, s_2 and s_3 point towards point 3, whereas s_4 points away from point 3. Thus, we must have

$$q_{23,3} + q_{03,3} = q_{34,3}$$

where $q_{23,3}$ is the shear flow just before point 3 in wall 23, etc. Thus, $q_{34,3} = -138.5 - 104 = -242.5\,\text{N/mm}$. So, in wall 34 we get

$$q_{34} = -\frac{S_y}{I_{xx}} \int_0^s t y ds = -\frac{100 \cdot 10^3}{14.5 \cdot 10^6} \int_0^{s_4} 2\left(75 - s_4\right) ds_4 - 242.5 = -1.04 s_4 + 69.0 \cdot 10^{-5} s_4^s - 242.5$$

Then, for wall 49, note that we've drawn s_5 as if it originates from 9! The reason for doing so is that we know that the shear flow is zero at point 9, so it's easier to just evaluate

$$q_{94} = -\frac{S_y}{I_{xx}} \int_0^s tyds = -\frac{100 \cdot 10^3}{14.5 \cdot 10^6} \int_0^{s_5} 2(-125)\, ds_5 = 1.73 s_5$$

All in all, this leads to the shear flow distribution as shown in figure 19.3 (just use symmetry to sketch it for the other half of the beam).

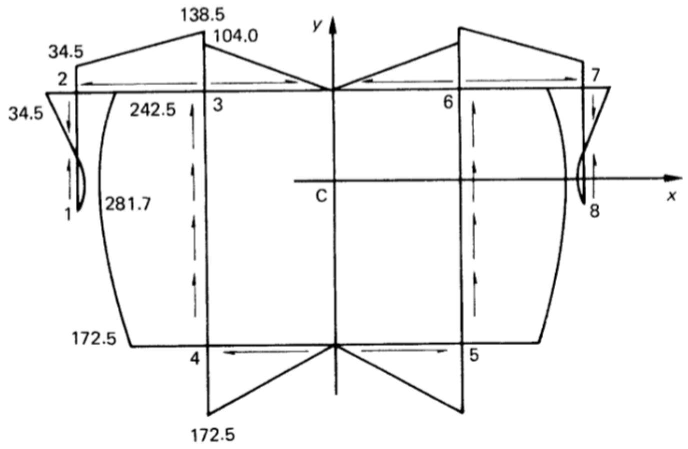


Figure 19.3: Shear flow distribution of example 1.

Not really that hard, imo. You only have to apply Kirchoff's law for the junctions.

FINDING THE SHEAR FLOW DISTRIBUTION DUE TO SHEAR FOR A CLOSED SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}.
3. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ and $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ as much as possible. If the expressions for I_{xx}, I_{yy} or I_{xy} are very complex, consider not plugging the values in just yet but let the fraction just stand there.
4. Make a cut somewhere in the closed section, preferable at an axis of symmetry (if there's one). Then evaluate the base shear stress in exactly the same manner you did for an open section. Note that at junctions, Kirchoff's law should always be satisfied (incoming shear flow should equal outgoing shear flow).
5. Pick a convenient point through which the applied shear force acts. Compute the moment M_b generated by this shear flow distribution.
6. Set
$$0 = M_b + 2Aq_{s,0} \tag{19.1}$$
and straightforwardly calculate $q_{s,0}$.
7. Add this $q_{s,0}$ to all shear flow distributions in only the closed section.

Bear in mind that the redundant shear flow $q_{s,0}$ should only be applied to the closed section of the beam (even though in the calculation of M_b, you also take the moment caused by the open section into account). For the shear center, not much changes either.

FINDING THE SHEAR CENTER FOR AN CLOSED SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if you only need compute the horizontal coordinate of the shear center (e.g. because the vertical coordinate is clear from symmetry), then you only need to apply a

load S_y and thus you only need to know I_{xx}. Similarly, if you only need to compute the vertical coordinate, then you only need to apply a load S_x and thus you only need to know I_{yy}. This only holds if the section is symmetric.

3. For the horizontal coordinate ξ_s, only apply a vertical load S_y:
   (a) Calculate the basic shear flow $q_{b,ij}$ distribution in all walls, using

$$q_{ij} = \frac{S_y I_{xy}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\, ds - \frac{S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\, ds + q_i$$

   (b) Compute the redundant shear flow $q_{s,0}$ by applying

$$q_{s,0} = -\frac{\oint q_b / Gt\, ds}{\oint ds / Gt}$$

   (c) Add this redundant shear flow to all the basic shear flows in the closed section you've found so far.
   (d) Find an easy point to evaluate moments due to internal shear flows around.
   (e) For each component, evaluate the force generated by this shear flow, by applying

$$F_{ij} = \int_0^l q_{ij} ds$$

   where l is the length of the component.
   (f) Multiply each force F_{ij} by the distance p_0 to the point that's being evaluated and sum this to get the moment due to internal shear flows. Take note whether each contribution is clockwise or counterclockwise and take appropriate measures to take this into account.
   (g) Set $S_y \xi_s$ equal to this moment and find ξ_s by dividing both sides of the equation by S_y.
4. For the vertical coordinate η_s, only apply a horizontal load S_x and follow the exact same procedure as before; however, use

$$q_{ij} = -\frac{S_x I_{xx}}{I_{xx} I yy - I_{xy}^2} \int_0^s tx\, ds + \frac{S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s ty\, ds + q_i$$

   and set $S_x \eta_s$ equal to the moment due to internal shear flows and find η_s by dividing both sides of the equation by S_x.

Again, bear Kirchoff's law in mind and note that $q_{s,0}$ only applies to the closed section of the beam (even though in the calculation of M_b, you also take the moment caused by the open section into account).

Note: problem 19.2 has a slightly weird cross-section: it's two closed sections connected via an open section. However, note that the shear force acts through the shear center; therefore, the shear flow in the middle of beam 23 is 0 due to symmetry. Therefore, you can simply cut the section in half and evaluate the effect of a single shear force of 50 kN.

19.3 Torsion

Again, for some reason or another, Dr. Rans likes to complicate stuff and needlessly introduce compatibility equations. In my opinion, the easiest way to do it as follows:

Example 1

Find the angle of twist per unit length in the wing whose cross section is shown in figure 19.4 when it is subjected to a torque of 10 kNm. Find also the maximum shear stress in the section.

$G = 25\,000\,\text{N/mm}^2$. Wall 12 (outer) has length 900 mm. The nose cell area is 20 000 mm^2.

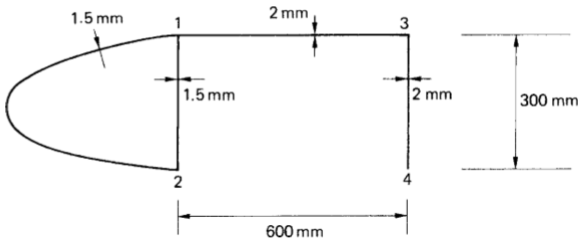


Figure 19.4: Wing section of example 1.

Remember that for a closed section, the torsional constant is given by

$$J_c = \frac{4A^2}{\oint \frac{ds}{t}} = \frac{4 \cdot 20000^2}{\frac{900+300}{1.5}} = 2 \times 10^6\,\text{mm}^4$$

and for an open section, it's given by

$$J_o = \sum \frac{st^3}{3} = \frac{900 \cdot 2^3}{2} = 2400\,\text{mm}^4$$

so the combined torsional constant is

$$J = J_c + J_o = 2 \cdot 10^7 + 2400 = 2\,002\,400\,\text{mm}^4$$

So, the rate of twist is

$$\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{10 \cdot 10^6}{25000 \cdot 2002400} \approx 2 \times 10^{-4}\,\text{rad/mm}$$

You could have neglected the torsional stiffness of the open section and you'd arrived at approximately the same result, but this calculation wasn't much work any way so I'd prefer to be inclusive. To compute the shear stresses, you take different approaches for the closed section and open section. For the closed section, you simply compute the shear flow using

$$\begin{aligned} \frac{d\theta}{dz} &= \frac{T}{GJ} = \frac{2Aq}{GJ_c} \\ q &= \frac{GJ_c}{2A}\frac{d\theta}{dz} = \frac{25000 \cdot 2 \cdot 10^6}{2 \cdot 20000} \cdot 2 \cdot 10^{-4} = 249.7\,\text{N/mm} \end{aligned}$$

The maximum shear stress in the closed section is thus $\tau_{\max} = q/t_{\min} = 249.7/1.5 = 166.47\,\text{N/mm}^2$. In the open section, we combine

$$\begin{aligned} \tau &= \frac{Tt}{J} \\ \frac{d\theta}{dz} &= \frac{T}{GJ} \end{aligned}$$

by rewriting the first to $\tau \cdot J/t$ so that we get

$$\begin{aligned} \frac{d\theta}{dz} &= \frac{\tau J_o}{GJ_o t} = \frac{\tau}{Gt} \\ \tau &= \frac{d\theta}{dz} \cdot Gt = 2 \cdot 10^{-4} \cdot 25000 \cdot 2 = 10\,9\,99 N/mm^2 \end{aligned}$$

You cannot use $\tau = Tt/J_o$ directly as here T is the torsion that is taken up by the open section (which is only part of the applied 10 kNm.

So, in short, what you have to do:

FINDING THE DISTRIBUTION OF TWIST AND MAXIMUM SHEAR STRESS DUE TO TORSION: COMBINED OPEN AND CLOSED CELL

1. Identify the magnitude of the maximum torque acting on the beam.
2. Find the torsional constant J_c of the closed section by application of

$$J_c = \frac{4A^2}{\oint \frac{ds}{t}} \tag{19.2}$$

3. Find the torsional constant J_o of the open section by application of

$$J_o = \sum \frac{st^3}{3} \tag{19.3}$$

4. Find the torsional constant of the section as a whole by simply summing the two torsional constants, i.e. $J = J_c + J_o$.
5. Compute the rate of twist by

$$\frac{d\theta}{dz} = \frac{T}{GJ} \tag{19.4}$$

6. For the closed section, find the shear flow by

$$q_c = \frac{GJ_c}{2A}\frac{d\theta}{dz} \tag{19.5}$$

   Find the maximum shear stress by $\tau_{\max} = q_c/t_{\min}$.
7. For the open section, find the maximum shear stress by

$$\tau = \frac{d\theta}{dz}Gt \tag{19.6}$$

20 Structural idealization, taper and cut-outs

20.1 The concept of structural idealization

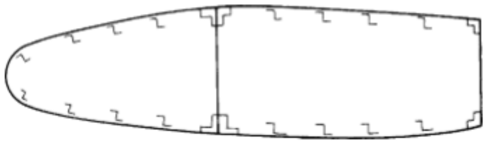


Figure 20.1: Typical wing section.

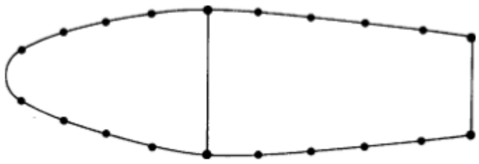


Figure 20.2: Idealization of a wing section.

Look at figure 20.1. We can make our lives much easier by assuming the stringers and spar flanges by concentrations of area, known as **booms**, over which the direct stress (bending stress) is constant and which are located at the mid-line of the skin, as shown in figure 20.2. This makes calculating the moment of inertia much easier, for example. We can go even further, however: we can lump the area of the adjacent skin panels to the booms, so that the booms carry all of the direct stresses, as shown in figure 20.3.

(a) Actual.

(b) Idealized.

Figure 20.3: Idealization of a panel.

Why is the latter such a helpful idealization? Remember that we had

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

If we assume the skins do not carry any bending stresses, then in the skins, $\sigma_z = 0$, everywhere along the skin. Thus, $\frac{\partial q}{\partial s} = 0$, and thus we know the shear flow in the skin must be constant (not necessarily equal to 0, but it's constant, and that's very nice indeed). Now, how exactly do we lump the area of the adjacent skin panels to the booms? There are multiple ways to do this: first, you can just split skin panels in half and just lump each half to the closest boom. This is rather simplistic and can give very inaccurate results if there is not a sufficient amount of booms.

Another way to do it is by comparing figures 20.3a and 20.3b. We see that as the skin panel loses its thickness in figure 20.3b, the bending stress distribution is lost. This sucks. But, we can try to keep the extremes σ_1 and σ_2 at the booms; these extremes are actually the most important thing when designing your nice beam of course.

How can we do this? Remember that $F = \sigma A$, and thus taking moments around the right-hand edge of each panel leads to[1]

$$\sigma_2 t_D \frac{b^2}{2} + \frac{1}{2}\left(\sigma_1 - \sigma_2\right) t_d b \frac{2}{3} b \quad = \quad \sigma_1 B_1 b$$

From this, and doing the exact same computation around the left edge, we can deduce that

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$$B_1 = \frac{t_D b}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right) \qquad (20.1)$$

$$B_2 = \frac{t_D b}{6}\left(2 + \frac{\sigma_1}{\sigma_2}\right) \qquad (20.2)$$

Now of course, you're wondering, how do I determine σ_1 and σ_2, without having to solve all of the stuff of bending theory for the non-idealized panel? However, for relatively simple geometry (as long as there is one axis of symmetry) this is easier than it seems: we only need to know the ratio σ_2/σ_1:

Example 1

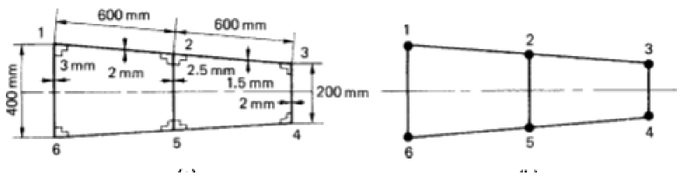


Figure 20.4: Idealization of a wing section.

Part of a wing section is in the form of the two-cell box shown in figure 20.4, in which the vertical spars are connected to the wing skin through angle sections all having constant cross-sectional area of 300 mm^2. Idealize the structure into an arrangement of direct stress carrying booms and shear stress only carrying panels suitable for resisting bending moments in a vertical plane. Position the booms at the spar/skin junctions.

From symmetry, we easily deduce that $B_1 = B_6$, $B_2 = B_5$ and $B_3 = B_4$. For B_1, we can write

$$B_1 = 300 + \frac{3.0 \cdot 400}{6}\left(2 + \frac{\sigma_6}{\sigma_1}\right) + \frac{2.0 \cdot 600}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right)$$

Now, what are σ_6/σ_1 and σ_2/σ_1? From symmetry, we can deduce that $\sigma_6 = -\sigma_1$: the bending stress will be equal in magnitude but opposite of sign due to the distance to the neutral axis. This means that $\sigma_6/\sigma_1 = -1$. We can deduce σ_2/σ_1 in an equally straightforward manner by looking at the neutral axis. The distance from boom 1 to the neutral axis is 200 mm, the distance from boom 2 to the neutral axis is 150 mm; thus, $\sigma_2 = 150\sigma_1/200$, and $\sigma_2/\sigma_1 = 150/200$. This leads to

$$B_1 = B_6 = 300 + \frac{3.0 \cdot 400}{6}(2-1) + \frac{2.0 \cdot 600}{6}\left(2 + \frac{150}{200}\right) = 1050\,\text{mm}^2$$

[1] Don't get where the first equation comes from? In figure 20.3a, the stress distribution is basically split in two: a rectangular part with height σ_2, and a rectangular triangular part, making up the remaining part. The rectangular part causes a force $\sigma_2 \cdot t_d \cdot b$, and this force is located halfway through the beam, at $b/2$, so that the moment it creates equals $\sigma_2 t_d b^2/2$. For the rectangular triangular part, the height equals $\sigma_1 - \sigma_2$, so that the force created equals $\left(\sigma_1 - \sigma_2\right) \cdot t_d \cdot b/2$. The centroid is located at $2b/3$, so that the moment it creates equals $\left(\sigma_1 - \sigma_2\right) \cdot t_d \cdot b/2 \cdot 2b/3$. The right hand side of the equation should be pretty self-explanatory.

In exactly the same way, B_2 and B_3 can be deduced:

$$\begin{aligned}
B_2 = B_5 &= 2\cdot 300 + \frac{2.0\cdot 600}{6}\left(2+\frac{\sigma_1}{\sigma_2}\right) + \frac{2.5\cdot 300}{6}\left(2+\frac{\sigma_5}{\sigma_2}\right) + \frac{1.5\cdot 600}{6}\left(2+\frac{\sigma_3}{\sigma_2}\right)\\
&= 2\cdot 300 + \frac{2.0\cdot 600}{6}\left(2+\frac{200}{150}\right) + \frac{2.5\cdot 300}{6}(2-1) + \frac{1.5\cdot 600}{6}\left(2+\frac{100}{150}\right) = 1791.7\,\mathrm{mm}^2\\
B_3 = B_4 &= 300 + \frac{1.5\cdot 600}{6}\left(2+\frac{\sigma_2}{\sigma_3}\right) + \frac{2.0\cdot 200}{6}\left(2+\frac{\sigma_4}{\sigma_3}\right)\\
&= 300 + \frac{1.5\cdot 600}{6}\left(2+\frac{150}{100}\right) + \frac{2.0\cdot 200}{6}(2-1) = 891.7\,\mathrm{mm}^2
\end{aligned}$$

IDEALIZING A STRUCTURE: BOOM AREAS

1. Sum the areas of the stringers and spar flanges located at the point where you put the boom.
2. For each adjacent panel, calculate

$$B_i = \frac{t_D b}{6}\left(2+\frac{\sigma_j}{\sigma_i}\right)$$

where j denotes the jth adjacent boom. Compute the stress ratio by inspection of the distances to the neutral axis.

20.2 Effects of structural idealization: single cell

The concept of structural idealization should be quite clear by now. Let's see how it affects (almost) all of our calculations we have discussed so far (we are only not discussing the effect on buckling because if your entire wing column buckles you have done an terrible job anyway (I mean, skin panels buckle, stringers buckle, but having the entire wing buckle buckle is an extraordinary achievement)).

20.2.1 Bending of open and closed section beams

Calculations simply become so much easier for bending:

Example 1

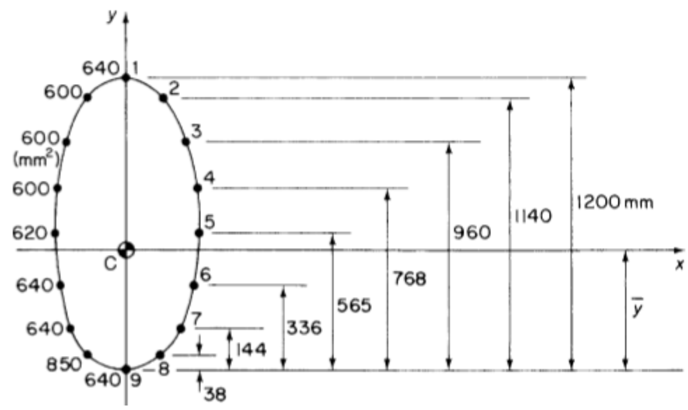


Figure 20.5: Idealized fuselage section of example 1.

The fuselage section shown in figure 20.5 is subjected to a bending moment of 100 kNm appliead in the vertical plane of symmetry. If the section has been completely idealized into a combination of direct stress carrying booms and shear stress only carrying panels, determine the direct stress in each boom.

We simply can still use

$$\sigma_z = \frac{M_x}{I_{xx}} y$$

for this. First, finding the centroid:

$$\bar{y} = \frac{640 \cdot 0 + 2 \cdot 850 \cdot 38 + 2 \cdot 640 \cdot 144 + 2 \cdot 640 \cdot 336 + 2 \cdot 620 \cdot 565 + 2 \cdot 600 \cdot 768 + 2 \cdot 600 \cdot 960}{640 + 2 \cdot 850 + 2 \cdot 640 + 2 \cdot 640 + 2 \cdot 620 + 2 \cdot 600 + 2 \cdot 600 + 2 \cdot 600 + 640} + \frac{2 \cdot 600 \cdot 1140 + 640 \cdot 1200}{640 + 2 \cdot 850 + 2 \cdot 640 + 2 \cdot 640 + 2 \cdot 620 + 2 \cdot 600 + 2 \cdot 600 + 2 \cdot 600 + 640} = 540\,\text{mm}$$

Then, computing the second moment of inertia is simply a case of calculating the Steiner term for each boom; e.g. for the lowest boom, we have

$$\Delta I_{xx} = 640 \cdot (0 - 540)^2 = 187 \times 10^6\,\text{mm}^4$$

Doing this for all booms leads to the results tabulated in column (4) of table 20.1.

Table 20.1: Example 1.

(1) Boom	(2) y (mm)	(3) B (mm^2)	(4) $\Delta I_{xx} = By^2$ (mm^2)	(5) σ_z (N/mm^2)
1	+660	640	$278 \cdot 10^6$	35.6
2	+600	600	$216 \cdot 10^6$	32.3
3	+420	600	$106 \cdot 10^6$	22.6
4	+228	600	$31 \cdot 10^6$	12.3
5	+25	620	$0.4 \cdot 10^6$	1.3
6	-204	640	$27 \cdot 10^6$	-11.0
7	-396	640	$100 \cdot 10^6$	-21.4
8	-502	850	$214 \cdot 10^6$	-27.0
9	-540	640	$187 \cdot 10^6$	-29.0

The bending stress can then be straightforwardly calculated from

$$\sigma_z = \frac{M_x}{I_{xx}}$$

and the results are tabulated in column (5) of table 20.1.

FINDING THE DIRECT STRESS DISTRIBUTION FOR BENDING OF AN IDEALIZED BEAM AS A FUNCTION OF z

1. Establish a coordinate system, preferably as described in subsection 16.2.1.
2. Analyse M_x by looking at w_y and S_y and finding a function for M_x as function of z. Note that it may be that this function consists of multiple parts, for example if a force only acts halfway through the beam, and not at $z = 0$.
3. Do the same for M_y by looking at w_x and S_x.
4. Calculate the centroid of the cross-section. You only have to consider the booms.
5. Calculate the second moments of area, I_{xx}, I_{xy} and I_{yy}, of the cross-section. You only have to consider the Steiner terms of the booms.
6. Find the stress at any boom by applying

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2} x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2} y \tag{20.3}$$

20.2.2 *Shear of open section beams*

Remember that we had

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right) \int_0^s t_D x\, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right) \int_0^s t_D y\, ds$$

where t_D is the direct stress carrying thickness of the skin; $t_D = t$ if the skin is fully effective in carrying direct stress or $t_D = 0$ if the skin is assumed to only carry shear stresses. This formula does not yet account for the booms that are present. We can easily modify the equation to include it, fortunately.

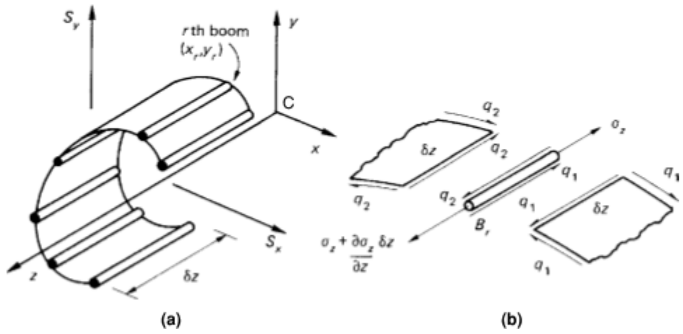


Figure 20.6: (a) Elemental length of shear loaded open section with booms; (b) equilibrium of boom element.

From figure 20.6, we see that

$$\begin{aligned}
\left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\delta z\right) B_r - \sigma_z B_r + q_2 \delta z - q_1 \delta z &= 0 \\
q_2 - q_1 &= -\frac{\partial \sigma_z}{\partial z} \\
&= -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r x_r - \left(\frac{S_x I_{xx} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r y_r
\end{aligned}$$

The derivation of this is exactly the same as the derivation for the first formula, which involved writing down the long formula for σ_z from bending theory, then substituting $\frac{\partial M_y}{\partial z} = -S_x$ etc. This formula means that each time you encounter a boom, the shear flow jumps by a certain amount. The final formula for shear flow can thus be written as

FORMULA

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x \, ds + \sum_{r=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y \, ds + \sum_{r=1}^{n} B_r y_r\right) \tag{20.4}$$

In this formula, if a distance s around the profile of the section has been travelled, then n booms have been passed. Let's just do an example to clear it a bit up because the sum makes things look more difficult than they are:

Example 2

Calculate the shear flow distribution shown in figure 20.7 produced by a vertical shear load of 4.8 kN acting through its shear center. Assume that the walls of the section are effective in resisting only shear stresses, while the booms, each of area 300 mm^2, carry all the direct stresses.

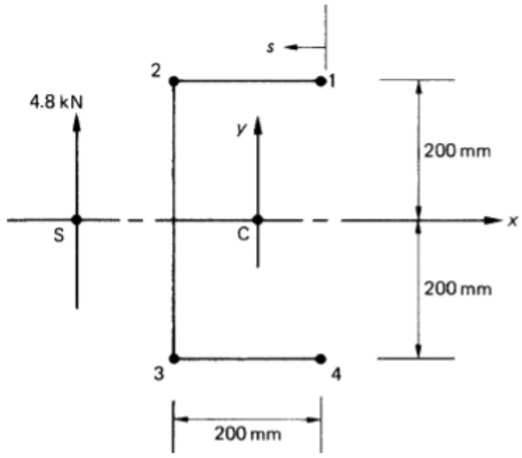


Figure 20.7: Idealized channel section of example 2.

The effective direct stress carrying thickness t_D of the walls of the section is zero. This means that

$$q_s = -\frac{S_x}{I_{xx}} \sum_{r=1}^{n} B_r y_r$$

First, finding I_{xx} (obviously, $\bar{y}$ is located at the line of symmetry):

$$I_{xx} = 300 \cdot 200^2 + 300 \cdot 200^2 + 300 \cdot (-200)^2 + 300 \cdot (-200)^2 = 48 \times 10^6 \text{ mm}^4$$

So we have

$$q_s = -\frac{4.8 \cdot 10^3}{48 \cdot 10^6} \sum_{r=1}^{n} B_r y_r = -10^{-4} \sum_{r=1}^{n} B_r y_r$$

Starting at boom 1: we have that to the right of boom 1, the shear flow is obviously zero. To the left of it, we have

$$q_{12} = -10^{-4} \cdot 300 \cdot 200 = -6 \text{ N/mm}$$

For the vertical web, we get

$$q_{23} = -10^{-4} \cdot 300 \cdot 200 - 6 = -12 \text{ N/mm}$$

We see that just like before, we have to add the constant of integration, equal to the shear flow just before boom 2. Finally:

$$q_{34} = -10^{-4} \cdot 300 \cdot -200 - 12 = -6 \text{ N/mm}$$

If we'd evaluate beyond boom 4, we indeed see that the shear flow there is

$$-10^{-4} \cdot 300 \cdot -200 - 6 = 0$$

as it should. The shear flow distribution is shown in figure 20.8.

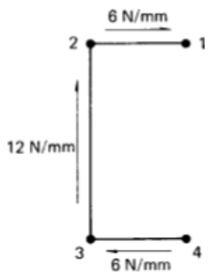


Figure 20.8: Shear flow in channel section of example 2.

We clearly see that the shear flow is constant in between booms, which makes calculations much easier.

FINDING THE SHEAR FLOW DISTRIBUTION DUE TO SHEAR FOR AN IDEALIZED OPEN SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}.
3. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ and $\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ as much as possible. If the expressions for I_{xx}, I_{yy} or I_{xy} are very complex, consider not plugging the values in just yet but let the fraction just stand there.
4. Start at an open edge. Perform the integration for the first geometric component, between points 1 and 2, where point 1 is the open edge and point 2 a discontinuity (typically a junction) as required by

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x\, ds + \sum_{r=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y\, ds + \sum_{r=1}^{n} B_r y_r\right)$$

   by first finding the correct expression for x and y (note that these are *always* with respect to the centroid; you cannot make your life easier by picking a convenient coordinate system), relating them to s_1 if applicable. Note that the plugging in of integration limits is incredibly straightforward and that you should get a medal if you manage to fuck that up. Furthermore, for idealized section, if not always, they'll make the walls have zero thickness, meaning you're only left with the booms. Calculate the shear flow at point 2.
5. For the second component, between points 2 and 3, perform the integration as required by

$$\begin{aligned} q_s &= -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x\, ds + \sum_{r=1}^{n} B_r x_r\right) \\ &\quad -\left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y\, ds + \sum_{r=1}^{n} B_r y_r\right) + q_2 \end{aligned}$$

   by first finding the correct expression for x and y, relating them to s_2 if applicable. Again, most of the time it'll simply reduce to only summing the booms.
6. Do the same for the other components. As a final note; if q_{ij} is positive, then it points in the same direction as s_{ij}; if it's negative, then it points in the opposite direction.

20.2.3 *Shear loading of closed section beams*

Again, the only difference is that you now have to apply

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x\, ds + \sum_{r=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y\, ds + \sum_{r=1}^{n} B_r y_r\right)$$

Other than that, you can use exactly the same problem solving guide as you've used previously.

Example 3

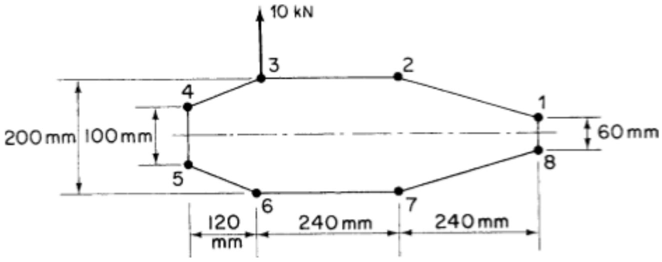


Figure 20.9: Closed section of beam of example 3.

The thin-walled single cell beam shown in figure 20.9 has been idealized into a combination of direct stress carrying booms and shear stress only carrying walls. If the section supports a vertical load of 10 kN acting in a vertical plane through booms 3 and 6,calculate the distribution of shear flows around the section. The boom areas are $B_1 = B_8 = 200\,\text{mm}^2$, $B_2 = B_7 = 250\,\text{mm}^2$, $B_3 = B_6 = 400\,\text{mm}^2$ and $B_4 = B_5 = 100\,\text{mm}^2$.

The centroid is obviously located at the line of symmetry. Furthermore, t_D equals zero, and with $S_x = 0$, this means that the base shear flow (to which we shall add the correcting shear flow at the end) equals

$$q_b = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r$$

The second moment of area around the x-axis equals

$$2 \cdot \left(200 \cdot 30^2 + 250 \cdot 100^2 + 400 \cdot 100^2 + 100 \cdot 50^2\right) = 13.86 \times 10^6\,\text{mm}^4$$

leading to

$$q_b = -\frac{10 \cdot 10^3}{13.86 \cdot 10^6} \sum_{r=1}^{n} B_r y_r = -7.22 \cdot 10^{-4} \sum_{r=1}^{n} B_r y_r$$

Let's make a cut in wall 23 (does not matter where in wall 23). Going in counterclockwise direction, we then have

$$\begin{aligned}
q_{23} &= 0 \\
q_{34} &= -7.22 \cdot 10^{-4} \cdot (400 \cdot 100) = -28.9\,\text{N/mm} \\
q_{45} &= -28.9 - 7.22 \cdot 10^{-4} \cdot (100 \cdot 50) = -32.5\,\text{N/mm} \\
q_{56} &= -32.5 - 7.22 \cdot 10^{-4} \cdot (100 \cdot -50) = -28.9\,\text{N/mm} \\
q_{67} &= -28.9 - 7.22 \cdot 10^{04} \cdot (400 \cdot -100) = 0 \\
q_{78} &= -7.22 \cdot 10^{-4} \cdot (250 \cdot -100) = 18.1\,\text{N/mm} \\
q_{81} &= 18.1 - 7.22 \cdot 10^{-4} \cdot (200 \cdot -30) = 22.4\,\text{N/mm} \\
q_{12} &= 22.4 - 7.22 \cdot 10^{-4} \cdot (200 \cdot 30) = 18.1\,\text{N/mm}
\end{aligned}$$

Now, remember how we calculated $q_{s,0}$ for closed sections: we evaluated moments around a convenient point and then set the moment generated by the shear flows equal to the applied torque. A convenient point to compute moments around is the intersection of the line of symmetry with the line on which the shear force acts (though 3 and 6 are also arguably convenient).
There is something about the constant shear flow that makes these calculations much less tedious then before (where we had to integrate to find the force generated by each wall, etc.). For this, I will shortly interrupt this example to discuss some theorem.

Suppose we have some arbitrary shaped wall 12, as shown in figure 20.10.

You can mathematically proof that the vertical shear force developed in this wall, if q_{12} is constant, is equal

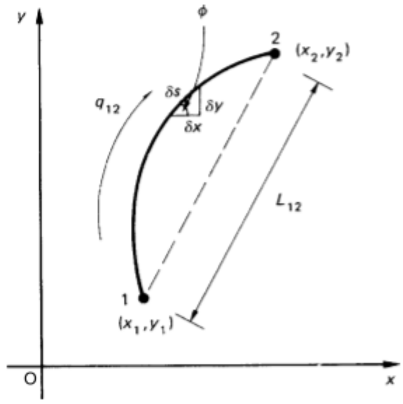


Figure 20.10: Curved web with constant shear flow.

to

FORMULA

$$S_y = q_{12}\left(y_2 - y_1\right) \tag{20.5}$$

Similarly, the horizontal shear force developed in the wall is equal to

FORMULA

$$S_x = q_{12}\left(x_2 - x_1\right) \tag{20.6}$$

Where we place the resultant shear force is irrelevant; as long as it is on the line connecting 1 and 2. For consistency's sake, I'll always place it at the first boom (so for shear flow q_{ij}, I'll place it at boom i). The fact that we now have a very elegant resultant shear force means that we can easily compute the moment it creates. For the signs, you can either do it manually for each shear flow, checking whether they point in the clockwise or counterclockwise direction, or you can apply the following sign convention: looking at figure 20.11, where all arrows point in positive direction, and evaluating all moments and shear flows in counterclockwise direction, the moment created about point O equals

FORMULA

$$M_o = -q_{1,2}\left(x_2 - x_1\right)\cdot\eta + q_{1,2}\left(y_2 - y_1\right)\cdot\xi = -S_x\eta + S_y\xi \tag{20.7}$$

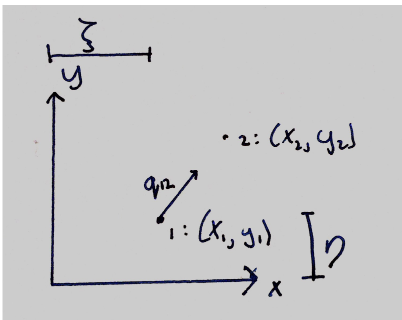


Figure 20.11: Sign convention for moment due to shear flows.

Follow this sign convention like a madman and you're safe. If you decide for whatever reason to evaluate the shear flows in clockwise direction, then a positive value of M_o also means that M_o is clockwise direction.

Example 3: continued

Take the shear flow in wall q_{34}, which we'll replace by a single shear force at boom 3. For wall 34, we have $x_2 = -120\,\text{mm}$, $x_1 = 0$, $\eta = 100\,\text{mm}$ and $\xi = 0$ (thus we do not care for y_1 and y_2). Doing this for every wall yields (I assume writing it out this way is clear enough):

$$
\begin{aligned}
M_o = & -q_{3,4}(-120-0)\cdot 100 + q_{3,4}\left(y_2-y_1\right)\cdot 0 \\
& -q_{4,5}(-120--120)\cdot 50 + q_{4,5}(-50-50)\cdot -120 \\
& -q_{5,6}(0--120)\cdot -50 + q_{5,6}(-100--50)\cdot -120 \\
& -q_{6,7}(240-0)\cdot -100 + q_{6,7}(-100--100)\cdot 0 \\
& -q_{7,8}(480-240)\cdot -100 + q_{7,8}(-30--100)\cdot 240 \\
& -q_{8,1}(480-480)\cdot -30 + q_{8,1}(30--30)\cdot 480 \\
& -q_{1,2}(240-480)\cdot 30 + q_{1,2}(100-30)\cdot 480 \\
& -q_{2,3}(0-240)\cdot 100 + q_{2,3}(100-100)\cdot 240 \\
& +2Aq_{s,0}
\end{aligned}
$$

From simple geometry, we have $A = 97\,200\,\text{mm}^2$ (again, A is the *enclosed* area. It is not the total boom area). From this, we can calculate easily that $q_{s,0} = -5.4\,\text{N/mm}$. The negative sign indicates that $q_{s,0}$ acts in clockwise sense. Add this shear flow to all the other shear flows, and the final shear flow distribution is shown in figure 20.12.

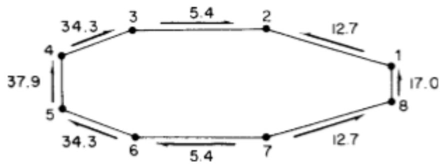


Figure 20.12: Shear flow distribution N/mm in walls of the beam section of example 3.

FINDING THE SHEAR FLOW DISTRIBUTION DUE TO SHEAR FOR AN IDEALIZED CLOSED SECTION

1. Establish a coordinate system at the centroid of the cross-section.
2. Make a sketch showing the assumed direction of the shear flow in each wall.
3. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric, then $I_{xy} = 0$. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}.
4. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ and $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$ as much as possible. If the expressions for I_{xx}, I_{yy} or I_{xy} are very complex, consider not plugging the values in just yet but let the fraction just stand there.
5. Make a cut somewhere, preferable at an axis of symmetry (if there's one). Then evaluate the base shear stress in exactly the same manner you did for an open section.
6. Pick a convenient point through which the applied shear force acts. Compute the moment M_b generated by this shear flow distribution. Do this by replacing the shear flow in each wall by a single shear force q_{ij}, placed at boom i, which creates a moment
$$M_{o,i} = -q_{ij}\cdot\left(x_j - x_i\right)\xi_i + q_{ij}\left(y_j - y_i\right)\eta_i$$
Alternatively, make quick sketch to make sure the signs are correct.
7. Set
$$0 = \sum_{i=1}^{n} M_{o,i} + 2Aq_{s,0} \tag{20.8}$$
and straightforwardly calculate $q_{s,0}$.
8. Add this $q_{s,0}$ to all shear flow distributions.

20.2.4 Torsion

There's literally nothing that changes for the torsion case, as the shear flow is constant under pure torsion. Literally nothing changes.

FINDING THE MAXIMUM SHEAR STRESS DUE TO TORSION AND DISTRIBUTION OF TWIST: IDEALIZED SINGLE CELL

1. Identity the magnitude of the maximum torque acting on the beam.
2. Find the value of the shear flow q by $q = \frac{T}{2A}$; remember that this is constant throughout the entire cross-section.
3. Calculate the maximum shear stress by dividing the shear flow by the minimum wall thickness of the section, i.e. $\tau_{\text{max}} = \frac{q}{t_{\text{min}}}$.
4. Find the formula for $\frac{d\theta}{dz}$ by plugging in $\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt}$. Please note that T is *not* the maximum torque, but the formula expressing T as a function of z. Evaluate $\oint \frac{ds}{Gt}$ carefully, by taking the sum of the lengths of each part of the cross-section divided by Gt.
5. Integrate $\frac{d\theta}{dz}$ to find a function for θ as function of z. This will lead to one integration constant, which can easily be determined by applying a boundary condition.

20.2.5 *Alternative method for the calculation of shear flow distribution*

We can write

$$q_2 - q_1 = -\frac{\partial \sigma_z}{\partial z} B_r = \frac{\partial P_r}{\partial z}$$

More practically, suppose it is known that the boom load varies linearly with z (the case when over a length of the beam the shear forces are constant), then

$$q_2 - q_1 = -\Delta P_r$$

in which ΔP_r is the change in boom load over unit length of the rth boom. ΔP_r may be calculated by first determining the change in bending moment between two sections of a beam a unit distance apart then calculating the corresponding change in boom stress, the multiplying this change by the boom area B_r.

Example 2: revisited

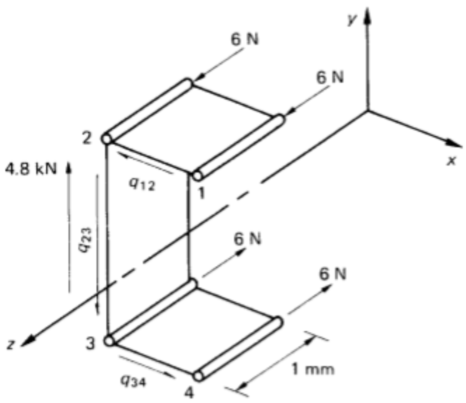


Figure 20.13: Alternative solution to example 2.

This is the same example as example 2 we did before. Let's do it in a different way this time. With a shear load of 4.8 kN, the change in bending moment on the beam over 1 mm is $4.8 \cdot 1 = 4.8$ kNmm. Thus, we have

$$\Delta P_r = \frac{\Delta M_x y}{I_{xx}} B_r = \frac{4.8 \cdot 10^3 \cdot 200}{48 \cdot 10^6} \cdot 300 = 6\,\text{N}$$

for the top two booms and a -6 N change for the lower two booms. Note that P is the tensile load in a boom increasing with increasing z. If the tensile load increases with decreasing z, then ΔP_r is negative. The positive values for the top beams confirm this: clearly, for decreasing z, the tensile load in the top booms decrease (they become more compressed), so for increasing z, the tensile load increases, so it should be positive. We thus have

$$
\begin{aligned}
q_{12} - q_{01} & \quad q_{12} = -6\,\mathrm{N/mm} \\
q_{23} - q_{12} &= q_{23} + 6 = -6 \\
q_{23} &= -12\,\mathrm{N/mm} \\
q_{34} - q_{23} &= q_{34} + 12 = 6 \\
q_{34} &= -6\,\mathrm{N/mm}
\end{aligned}
$$

Although this method is called alternative, you *do* need to know and master this method.

20.3 *Effects of structural idealization: multicell*

There is very little theory in this section: it's mostly about modifying the problem solving guides of the initial multicell sections we've seen so far (in the chapters about torsion and shear) to have them include structural idealization. Furthermore, all examples are taken from the book; however, I have modified most of them slightly: the book likes to use really not nice numbers, taking different values for t and G for fucking every single wall. This makes calculating stuff quite a mess. Furthermore, looking at the practice exam and the previous quizzes we've had, it seems as Dr. Abdalla prefers "nicer" numbers. Therefore, I've modified the problems to have them have nicer numbers without harming the actual educational value of them.

20.3.1 *Bending*

For bending, nothing changed anyway for a multi-cell section (even without structural idealization):

FINDING THE DIRECT STRESS DISTRIBUTION FOR BENDING OF AN IDEALIZED MULTICELL BEAM AS A FUNCTION OF z

1. Establish a coordinate system, preferably as described in subsection 16.2.1.
2. Analyse M_x by looking at w_y and S_y and finding a function for M_x as function of z. Note that it may be that this function consists of multiple parts, for example if a force only acts halfway through the beam, and not at $z = 0$.
3. Do the same for M_y by looking at w_x and S_x.
4. Calculate the centroid of the cross-section. You only have to consider the booms.
5. Calculate the second moments of area, I_{xx}, I_{xy} and I_{yy}, of the cross-section. You only have to consider the Steiner terms of the booms.
6. Find the stress at any boom by applying

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \tag{20.9}$$

Example 1

The central cell of a wing has the idealized section shown in figure 20.14.

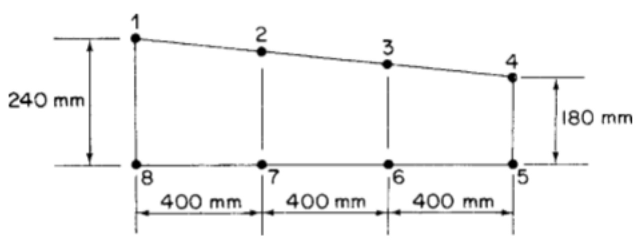


Figure 20.14: Example 1.

If the lift and drag loads on the wing produce bending moments of -120 000 Nm and -30 000 Nm, respectively, at the section shown, calculate the direct stresses in the booms. Neglect axial constraint effects and assume that the lift and drag vectors are in vertical and horizontal planes. The boom areas are

$$B_1 = B_4 = B_5 = B_8 = 1000\,\text{mm}^2$$
$$B_2 = B_3 = B_6 = B_7 = 600\,\text{mm}^2$$

For this question, draw the coordinate system as shown in figure 20.15.

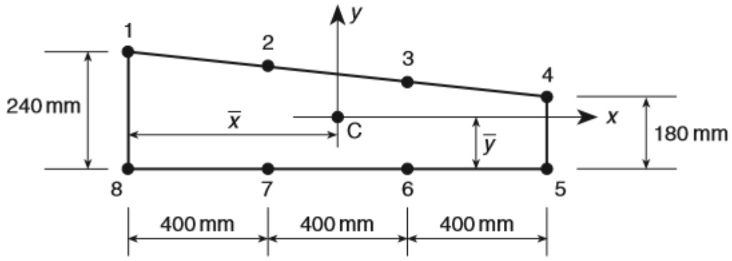


Figure 20.15: Example 1.

We then have, from inspection, $\bar{x} = 600\,\text{mm}$. For $\bar{y}$, we have

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{240 \cdot 1000 + 220 \cdot 600 + 200 \cdot 600 + 180 \cdot 1000}{4 \cdot 1000 + 4 \cdot 600} = 105\,\text{mm}$$

which, truthfully, could also be deduced from inspection. We then have the second moments of area:

$$\begin{aligned}
I_{xx} &= 2 \cdot 1000 \cdot 105^2 + 2 \cdot 600 \cdot 105^2 + 1000 \cdot 135^2 + 600 \cdot 115^2 + 600 \cdot 95^2 + 1000 \cdot 75^2 \\
&= 72.5 \times 10^6\,\text{mm}^4 \\
I_{yy} &= 4 \cdot 1000 \cdot 600^2 + 4 \cdot 600 \cdot 200^2 = 1536 \times 10^6\,\text{mm}^4 \\
I_{xy} &= 1000 \cdot (-600 \cdot -105 - 600 \cdot 135 + 600 \cdot -105 + 600 \cdot 75) \\
&\quad +600 \cdot (-200 \cdot -105 - 200 \cdot 115 + 200 \cdot -105 + 200 \cdot 95) = -38.4 \times 10^6\,\text{mm}^4
\end{aligned}$$

Now, faithfully apply

$$\sigma_z = \frac{I_{xx} M_y - I_{xy} M_x}{I_{xx} - I_{xy}^2} x + \frac{I_{yy} M_x - I_{xy} M_y}{I_{xx} I_{yy} - I_{xy}^2} y$$

with $M_x = -120\,000\,\text{Nm}$ and $M_y = -30\,000\,\text{Nm}$ for each boom and end up at the results shown in table 20.2.

Table 20.2: Results of example 1.

Boom	1	2	3	4	5	6	7	8
x (mm)	-600	-200	200	600	600	200	-200	-600
y (mm)	135	115	95	75	-105	-105	-105	-105
σ_z (N/mm^2)	-190.7	- 181.7	-172.8	-163.8	140.0	164.8	189.6	214.4

20.3.2 *Torsion*

Again, literally nothing has changed from the non-idealized multicell problem.

FINDING THE SHEAR FLOW FOR EACH WALL FOR MULTICELL PROBLEMS

1. Note that the total torque over the entire cross-section is equal to the sum of the torques over each individual cell, leading to
$$T = 2A_1 q_1 + 2A_2 q_2 + ... + 2A_n q_n$$

2. Note that the angle of twist must be the same for each cell, i.e.

$$\frac{d\theta}{dz} = \frac{1}{2A_r}\oint \frac{q\,ds}{Gt}$$

Carefully perform this integration: for each cell, sketch what the shear flow is along each wall by taking into account the influence of surrounding cells. For each wall, multiply this shear flow with the length and divide it by the thickness and shear modulus, and then take the sum of this.
3. This will yield n equations with $n+1$ unknowns (the extra unknown being $\frac{d\theta}{dz}$). Combine this with the equation resulting from the torsion distribution, and you have a system of $n+1$ equations and $n+1$ unknowns.
4. Solve this system. Be smart and bring your graphical calculator so that you can use the matrix solver there. Do note that sometimes you have to manipulate the system slightly to make sure your graphical calculator can solve it.

Example 2

The idealized cross section of a two-cell thin-walled box is shown in figure 20.16. If the wing box is subjected to a clockwise torque of 10 000 Nm, calculate the maximum shear stress in the wing box. Use the values given in table 20.3. All walls have the same shear modulus.

Table 20.3: Values.

Wall	Length (mm)	Thickness (mm)	Boom/cell	Area (mm^2)
16	250	2	1, 6	1500
25	400	2.5	2, 5	2000
34	200	1	3, 4	500
12, 56	650	1	Cell I	225000
23,45	800	0.5	Cell II	250000

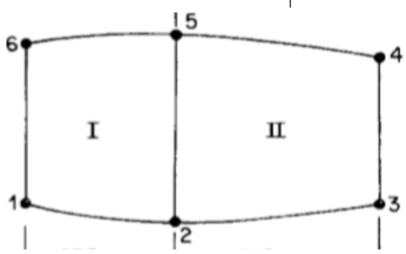


Figure 20.16: Example 2.

Then, we first have that

$$\begin{aligned} T &= 2A_1q_1 + 2A_2q_2 \\ 10000000 &= 2\cdot 225000q_1 + 2\cdot 250000q_2 \end{aligned}$$

which is our first equation. For the other two, we have that the angle of twist must be the same for each cell. We have

$$\frac{d\theta}{dz} = \frac{1}{2A_r}\oint \frac{q\,ds}{Gt}$$

but with G constant, this can rather be written as

$$G\frac{d\theta}{dz} = \frac{1}{2A_r}\oint \frac{q\,ds}{t}$$

For cell I, this leads to (starting at boom 1, going in clockwise direction)

$$\begin{aligned} G\frac{d\theta}{dz} &= \frac{1}{2\cdot 225000}\cdot\left[\frac{q_1\cdot 250}{2}+\frac{q_1\cdot 650}{1}+\frac{\left(q_1-q_2\right)\cdot 400}{2.5}+\frac{q_1\cdot 650}{1}\right] \\ &= 3.522\cdot 10^{-3}q_1 - 0.3556\cdot 10^{-3}q_2 \end{aligned}$$

For cell II, this leads to (starting at boom 3, going in clockwise direction)

$$\begin{aligned} G\frac{d\theta}{dz} &= \frac{1}{2\cdot 250000}\cdot\left[\frac{q_2\cdot 800}{0.5}+\frac{\left(q_2-q_1\right)\cdot 400}{2.5}+\frac{q_2\cdot 800}{0.5}+\frac{q_2\cdot 200}{1}\right] \\ &= -0.320\cdot 10^{-3}q_1 + 7.12\cdot 10^{-3}q_2 \end{aligned}$$

Solving this system of equations with your graphical calculator, by multiplying the equation for the sum of the torques with 10^{-6}:

$$\begin{bmatrix} 0.450 & 0.5 & 0 & 0.01 \\ 3.522\cdot 10^{-3} & -0.3556\cdot 10^{-3} & -1 & 0 \\ -0.320\cdot 10^{-3} & 7.12\cdot 10^{-3} & -1 & 0 \end{bmatrix}$$

leading to $q_1 = 14.144\,\mathrm{N/mm}$ and $q_2 = 7.269\,\mathrm{N/mm}$. May seem quite small, but we apply a relatively small torque (only 10 kNm) on a relatively large structure. From inspection, we have that the maximum shear stress occurs in walls 23 and 54, where it is equal to

$$\tau_{\max} = \frac{q}{t} = \frac{7.269}{0.5} = 14.54\,\mathrm{N/mm^2}$$

20.3.3 *Shear: shear flow distribution*

For shear, shit is a bit more complicated, as I said during the non-idealized discussion of multicells that you shouldn't bother with shear for multicells as it is just too much work. However, for idealized structures, it's actually more reasonable (it still takes a lot of time, but it's within the acceptable limits of what they could reasonable ask from you on the exam). First, for the shear flow distribution:

FINDING THE SHEAR FLOW DISTRIBUTION FOR AN IDEALIZED MULTI-CELL SECTION

1. Establish a coordinate system at the centroid of the cross-section. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}. Work out $\frac{S_xI_{xx}-S_yI_{xy}}{I_{xx}I_{yy}-I_{xy}^2}$ and $\frac{S_yI_{yy}-S_xI_{xy}}{I_{xx}I_{yy}-I_{xy}^2}$ as much as possible.
2. Make a sketch showing the assumed direction of the shear flow in each wall.
3. For each cell, make a cut somewhere. As we're talking about wing boxes if not always, preferably make them in the lower flanges for each cell.
4. For each wall, calculate the base shear flow distribution by use of

$$\begin{aligned} q_{ij} &= -\left(\frac{S_xI_{xx}-S_yI_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\left(\int_0^s tx\,ds+\sum_{r=1}^n B_rx_r\right) \\ &\quad -\left(\frac{S_yI_{yy}-S_xI_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\left(\int_0^s ty\,ds+\sum_{r=1}^n B_ry_r\right)+q_i \end{aligned}$$

   Make use of symmetry to speed up your calculations: on opposite sides of the line of symmetry, the shear flow will have the same magnitude *and* sign. Note that the wall where you make the cut will have zero shear flow in it.
5. Calculate the moment generated by each wall around a convenient point due to its base shear flow,

typically a point through which the shear force acts, and sum to that the torques generated by the redundant shear flows, and set this equal to the moment generated around this point by the shear force. If a point is picked through which the shear force acts, this means that

$$0 = \sum M_i + \sum_{R=1}^{N} 2A_R q_{0,R}$$

where A_R is the enclosed area of a cell and $q_{0,R}$ the redundant shear flow in that cell. This is the first of $N+1$ equations necessary to calculate each $q_{0,R}$.

6. For each cell, calculate

$$\frac{d\theta}{dz} = \frac{1}{2A_R} \oint \frac{q\,ds}{Gt}$$

by carefully integrating. Note that q combines all of the influences of the other cells as well. This will result in N equations with $N+1$ unknowns, allowing the system to be solved by use of your graphical calculator.

Please note: for the base shear flow distribution, you should *not* evaluate the base shear flow for each cell individually; when you make the cuts, it'll simply become a single open section. If you would do it for each cell individually (by evaluating in (counter)clockwise direction), you'd end up inadvertently calculating some walls twice. Furthermore, evaluating the moments generated by the shear flows is most easily done by just making a sketch of the shear flows.

Example 3

The idealized cross section of a two-cell thin-walled box is shown in figure 20.17. If the wing box supports a load of 50 000 N acting along the web 25, calculate the maximum shear stress in the wing box. Use the values given in table 20.4. All walls have the same shear modulus.

Table 20.4: Values. Because I only realized my mistake once I was finished, also assume the horizontal distances between webs 16 and 25 are 650 mm, and between between webs 25 and 34 800 mm (yes that's obviously incorrect, as the actual distances will be smaller, but I can't be bothered to change it.

Wall	Length (mm)	Thickness (mm)	Boom/cell	Area (mm^2)
16	250	2	1, 6	1500
25	400	2.5	2, 5	2000
34	200	1	3, 4	500
12, 56	650	1	Cell I	225000
23,45	800	0.5	Cell II	250000

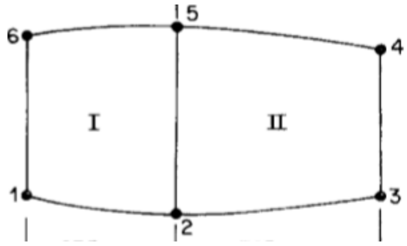


Figure 20.17: Example 3.

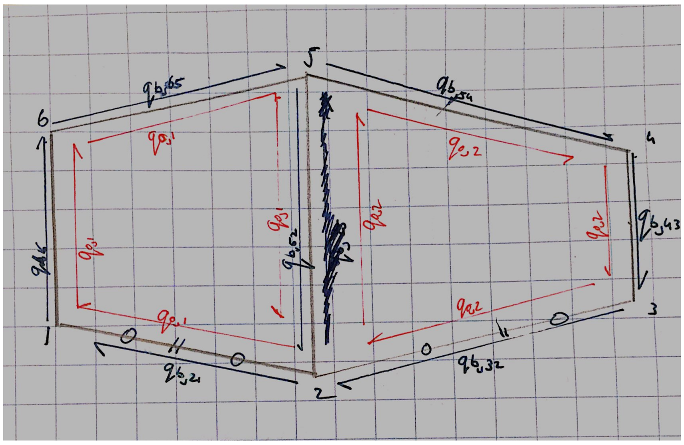


Figure 20.18: Example 3.

Make the cuts as shown in figure 20.18. We then automatically have $q_{b,21} = q_{b,32} = 0$, and from symmetry, $q_{b,65} = q_{b,54} = 0$. Now, again, as we have line symmetry around the x-axis, and we only apply a vertical shear force, we simply have (as we also have an idealized cross-section):

$$q_{ij} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_i$$

To compute I_{xx}:

$$I_{xx} = 2 \cdot 1500 \cdot 125^2 + 2 \cdot 2000 \cdot 200^2 + 2 \cdot 500 \cdot 100^2 = 216.875 \times 10^6 \,\text{mm}^4$$

and thus

$$\frac{S_y}{I_{xx}} = \frac{50000}{216.875 \cdot 10^6} = 230.5 \times 10^{-6} \,\text{N/mm}^4$$

Finally, we can relatively simply compute $q_{b,16}$, $q_{b,52}$ and $q_{b,43}$ as we know that we don't have to add any constant shear flows to them due to the presence of "previous" walls:

$$\begin{aligned}
q_{b,16} &= -230.5 \cdot 10^{-6} \cdot 1500 \cdot -125 = 43.22 \,\text{N/mm} \\
q_{b,52} &= -230.5 \cdot 10^{-6} \cdot 2000 \cdot 200 = -92.2 \,\text{N/mm} \\
q_{b,43} &= -230.5 \cdot 10^{-6} \cdot 500 \cdot 100 = -11.53 \,\text{N/mm}
\end{aligned}$$

Now, let's take moments around boom 2, so that we can disregard the shear flow in wall 25 (and also the moment generated by the shear force). We then have, by looking at the sketch, that the moment in clockwise direction equals:

$$\begin{aligned}
0 &= q_{b,16} \cdot 250 \cdot 650 + q_{b,43} \cdot 200 \cdot 800 + 2 \cdot 225000 \cdot q_{0,1} + 2 \cdot 250000 \cdot q_{0,2} \\
&= 43.22 \cdot 250 \cdot 650 - 11.53 \cdot 200 \cdot 800 + 450000 q_{0,1} + 500000 q_{0,2} \\
&= 5179250 + 450000 q_{0,1} + 500000 q_{0,2}
\end{aligned}$$

We then have to compute $d\theta/dz$ for each cell. We have

$$\frac{d\theta}{dz} = \frac{1}{2A_R} \oint \frac{q\,ds}{Gt}$$

but as G is constant for each cell, we have

$$G\frac{d\theta}{dz} = \frac{1}{2A_R} \oint \frac{q\,ds}{t}$$

For cell I, we then have (starting at boom 1, going in clockwise direction):

$$\begin{aligned}G\frac{d\theta}{dz} &= \frac{1}{2\cdot 225000}\cdot\left[\frac{\left(q_{b,16}+q_{0,1}\right)\cdot 250}{2}+\frac{q_{0,1}\cdot 650}{1}+\frac{\left(q_{b,52}+q_{0,1}-q_{0,2}\right)\cdot 400}{2.5}+\frac{q_{0,1}\cdot 650}{1}\right]\\ &= -0.02078+3.522\cdot 10^{-3}q_{0,1}-356\cdot 10^{-6}q_{0,2}\end{aligned}$$

For cell II, we then have (starting at boom 3, going in clockwise direction):

$$\begin{aligned}G\frac{d\theta}{dz} &= \frac{1}{2\cdot 250000}\cdot\left[\frac{q_{0,2}\cdot 800}{0.5}+\frac{\left(q_{0,2}-q_{b,52}-q_{0,1}\right)\cdot 400}{2.5}+\frac{q_{0,2}\cdot 800}{0.5}+\frac{\left(q_{b,43}+q_{0,2}\right)\cdot 200}{1}\right]\\ &= 0.024892-320\cdot 10^{-6}q_{0,1}+2.32\cdot 10^{-3}q_{0,2}\end{aligned}$$

Please note: I made an error here; it should be $7.12\cdot 10^{-3}q_{0,2}$. However, subsequent calculations use $2.32\cdot 10^{-3}$. This all can be written as the following system of equations:

$$\begin{aligned}450000q_{0,1}+500000q_{0,2} &= -5179250\\ 3.522\cdot 10^{-3}q_{0,1}-356\cdot 10^{-6}q_{0,2}-G\frac{d\theta}{dz} &= 0.0278\\ -320\cdot 10^{-6}q_{0,1}+2.32\cdot 10^{-3}q_{0,2}-G\frac{d\theta}{dz} &= -0.024892\end{aligned}$$

Divide the first row by 1000 and multiply the other two rows by one million to get

$$\begin{bmatrix}450 & 500 & 0 & -5179.25\\ 3522 & -356 & -10^6 & 27800\\ -320 & 2320 & -10^6 & -24892\end{bmatrix}$$

Row reducing gives $q_{0,1}=3.99\,\text{N/mm}$ and $q_{0,2}=-13.95\,\text{N/mm}$. From inspection, we then have that the maximum shear stress is achieved in web 25, where the shear flow in downwards direction equals $q_{b,52}+q_{0,1}-q_{0,2}=-92.2+3.99--13.95=-74.26\,\text{N/mm}$, leading to a maximum shear stress of

$$\tau=\frac{q}{t}=\frac{74.26}{2.5}=29.704\,\text{N/mm}^2$$

20.3.4 Shear: shear center

Again, I previously stated that you shouldn't bother with multicell shear, but with structural idealization, it actually becomes doable.

FINDING THE SHEAR CENTER FOR AN IDEALIZED MULTI-CELL SECTION

1. Establish a coordinate system at the centroid of the cross-section. Calculate the section properties I_{xx}, I_{yy} and I_{xy}. Note that if the section is symmetric. Furthermore, if there is only S_y, you only need to compute I_{xx}; and if there's only S_x, you only need to compute I_{yy}. Work out $\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}$ and $\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}$ as much as possible.
2. Make a sketch showing the assumed direction of the shear flow in each wall.
3. For the horizontal coordinate ξ_s, only apply a vertical load S_y:
   - For each closed cell, make a cut somewhere. As we're talking about wing boxes if not always, preferably make them in the lower flanges for each cell.
   - For each wall, calculate the base shear flow distribution by use of

     $$\begin{aligned} q_{ij} &= \left(\frac{S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s tx\,ds + \sum_{r=1}^n B_r x_r\right) \\ &\quad -\left(\frac{S_y I_{yy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s ty\,ds + \sum_{r=1}^n B_r y_r\right) + q_i \end{aligned}$$

     Make use of symmetry to speed up your calculations: on opposite sides of the line of symmetry, the shear flow will have the same magnitude *and* sign. Note that the wall where you make the cut will have zero shear flow in it.
   - Compute the redundant shear flow of each cell by setting

     $$\frac{d\theta}{dz} = \frac{1}{2A_R}\oint \frac{q\,ds}{Gt}$$

     equal to zero, i.e.

     $$\oint \frac{q\,ds}{Gt} = 0$$

     by carefully integrating, making use of your initial sketch showing the assumed direction of the shear flows.
   - If you have N cells, this will give you N equations with N unknowns. Solve this system, using your graphical calculator.
   - Set the moment created by all of the shear flows equal to the moment generated by the vertical shear force around a convenient point.
4. Do the same for the horizontal coordinate of the shear center, but now use

   $$\begin{aligned} q_{ij} &= -\left(\frac{S_x I_{xx}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s tx\,ds + \sum_{r=1}^n B_r x_r\right) \\ &\quad +\left(\frac{S_x I_{yy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s ty\,ds + \sum_{r=1}^n B_r y_r\right) + q_i \end{aligned}$$

Example 4

A singly symmetric wing section consists of two closed cells and one open cell (see figure 20.19. The webs 25, 34 and the walls 12, 56 are straight, while all others are curved. Note that there is no web 16. All walls of the section are assumed to be effective in carrying shear stresses only, direct stresses being carried by booms 1 to 6. Calculate the distance ξ_s of the shear center S aft of the web 34. The shear modulus G is the same for all walls. Use the data given in table 20.5 and 20.6.

Table 20.5: Values, part I. 34^o indicates the outer wall connecting 34 (the curved one), 34^i the veritcal wall.

Wall	Length (mm)	Thickness (mm)	Boom/cell	Area (mm^2)
12, 56	500	0.5	1, 6	750
23, 45	800	1	2, 5	1250
34^o	1000	0.5	3, 4	2000
34^i	300	2	Cell I	100000
25	300	1.5	Cell II	250000

Table 20.6: Values, part II.

Between booms	Horizontal distance (mm)	Between booms	Vertical distance (mm)
1, 2	450	1, 6	200
2, 3	750	2, 5	300
3, LE	600	3, 4	300

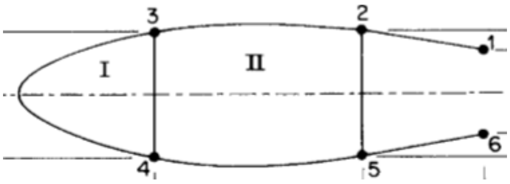


Figure 20.19: Example 4.

We only need I_{xx} to do this exercise. The line of symmetry is obviously in the middle. We thus have

$$I_{xx} = 2 \cdot 2000 \cdot 150^2 + 2 \cdot 1250 \cdot 150^2 + 2 \cdot 750 \cdot 100^2 = 161.25 \times 10^6 \text{ mm}^4$$

so that

$$\frac{S_y}{I_{xx}} = \frac{S_y}{161.25 \cdot 10^6} = 6.202 \cdot 10^{-9} S_y$$

A sketch showing the assumed directions of the shear flows is shown in figure 20.20.

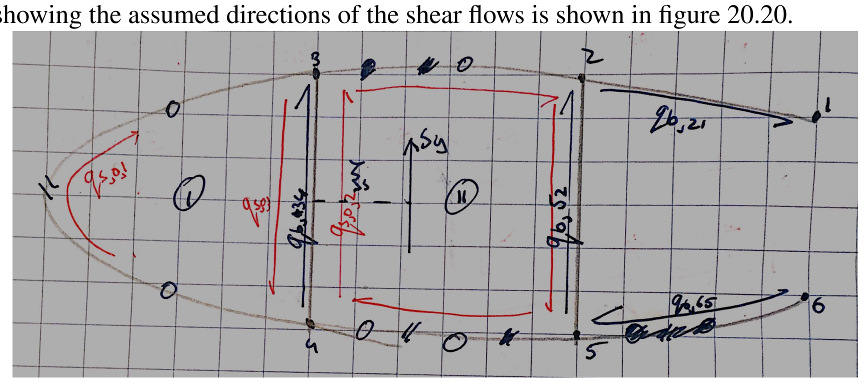


Figure 20.20: Sketch of shear flows.

Now, first computing the base shear flows. We make cuts in walls 34^o and 45, making the shear flow zero in those walls, as well in wall 23, due to line symmetry. We do not have to cut in wall 56 or 21 as that "cell" is already open. Now, computing the shear flows in walls 56, 52, 21 and 34 ($q_{56} = q_{21}$ due to line symmetry):

$$\begin{aligned} q_{b,34} &= -6.202 \cdot 10^{-9} S_y \cdot 2000 \cdot -150 = 1.8605 \cdot 10^{-3} S_y \\ q_{b,65} = q_{b,21} &= -6.202 \cdot 10^{-9} S_y \cdot 750 \cdot -100 = 465.1 \cdot 10^{-6} S_y \\ q_{b,52} &= -6.202 \cdot 10^{-9} S_y \cdot 1250 \cdot -150 + q_{b,65} = 1.6279 \cdot 10^{-3} S_y \end{aligned}$$

Then, for each cell, we compute the redundant shear flow by setting

$$\oint \frac{q\,ds}{Gt} = 0$$

but as G is constant, this can be further reduced to

$$\oint \frac{q\,ds}{t} = 0$$

For the left cell, we then have (clockwise direction, starting from boom 4):

$$0 = \frac{q_{s,0,1} \cdot 1000}{0.5} + \frac{\left(q_{s,0,1} - q_{b,34} - q_{s,0,4}\right) \cdot 300}{2} = 2150 q_{s,0,1} - 150 q_{s,0,2} - 279.075 \cdot 10^{-3} S_y$$

For the right cell, we get (clockwise direction, starting from boom 5):

$$\begin{aligned} 0 &= \frac{q_{s,0,2} \cdot 800}{1} + \frac{\left(q_{s,0,2} + q_{b,34} - q_{s,0,1}\right) \cdot 300}{2} + \frac{q_{s,0,2} \cdot 800}{1} + \frac{\left(q_{s,0,2} - q_{b,52}\right) \cdot 300}{1.5} \\ &= -150 q_{s,0,1} + 1950 q_{0,2} - 46.505 \cdot 10^{-3} S_y \end{aligned}$$

Which means we have two equations with two unknowns, meaning we can solve for $q_{s,0,1}$ and $q_{s,0,2}$ by writing the system of equations in matrix format[a]

$$\begin{bmatrix} 2150 & -150 & 0.79075 \\ -150 & 1950 & 0.046505 \end{bmatrix}$$

which leads to $q_{s,0,1} = 371 \cdot 10^{-6} S_y$ and $q_{s,0,2} = 52 \cdot 10^{-6} S_y$. Then, take moments around boom 5, so that we do not have to deal with $q_{b,52}$ and $q_{b,65}$. Note that $q_{b,21}$ is curved; therefore, we need to apply some theorem of before and replace the shear flow in 21 with a single horizontal force and single vertical force, acting on boom 2 (so that the vertical force goes through boom 5, meaning we do not have to bother with that component). The horizontal force of it simply equals $q_{b,21} \cdot 450 = 465.1 \cdot 10^{-6} S_y \cdot 450 = 209.295 \cdot 10^{-3} S_y$, which has an arm of 300 mm w.r.t. boom 5. Thus, the total moment in clockwise direction equals

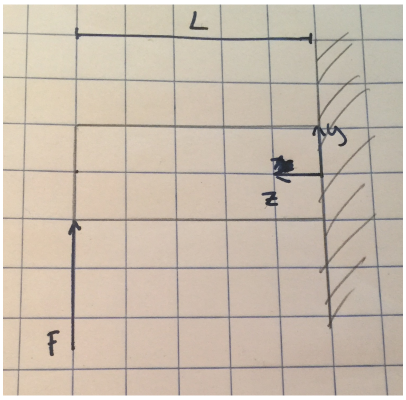
$$\begin{aligned} S_y \left(750 - \xi_s\right) &= q_{b,21} \cdot 450 \cdot 300 + q_{b,34} \cdot 300 \cdot 750 + 2 \cdot 100000 q_{s,0,1} + 2 \cdot 250000 q_{s,0,2} \\ &= 62.79 S_y + 418.6 S_y + 74.2 S_y + 26 S_y \\ 750 S_y - \xi_s S_y &= 581.59 S_y \\ \xi_s &= 168.41\,\text{mm} \end{aligned}$$

[a] Please note: I made a typo here: the topright entry should obviously be 0.279075. However, I continued calculating with 0.79075 there, and I can't be bothered to fix it.

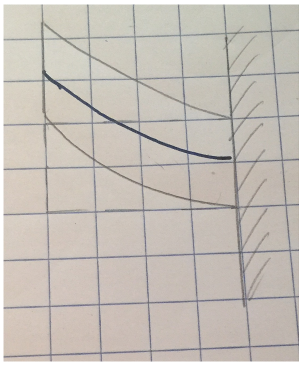
8 Columns

8.1 Euler buckling of columns

Normally, when you compress a column, it'll start to buckle a lot earlier than you'd expect based on the compressive stress. This occurs for example due to material imperfections: the applied compressive force is slightly offset from the centroid, causing a moment which causes it to buckle out of plane. The load at which buckling P_{cr} occurs is called the **critical buckling load**.



(a) Bending moment.



(b) Change in slope.

Figure 8.1: Bending moment and change in slope.

Now, you can derive that

FORMULA

$$M = -EIv'' \tag{8.1}$$

Why does the minus sign makes sense? Suppose we have the cantilever beam shown in figure 8.1a. Using the method I described in the previous chapter, we realize that (by making a cut somewhere and evaluating the part of the beam in positive z-direction)

$$M_x = F \cdot -(L - z) = -F(L - z)$$

However, looking at figure 8.1b, we realize that the slope of the slope v'' caused by this moment is actually positive, and hence we need to add another minus sign to make $M = -EIv''$ hold.

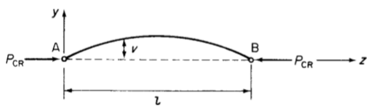


Figure 8.2: Determination of load for a pin-ended column.

How can we apply this equation? Take the simplest case possible, shown in figure 8.2. We have that

$$EIv'' = -M = -P_{cr}v$$

Note that $M = P_{cr}v$. This may have been something you would not have been able to directly find using the method I previously described (as it's now suddenly in different directions), but this is something that you'll come across so many times this chapter that it's better to just remember that $M = Pv$ for the compressive force.

Note that we now have a homogeneous second order linear differential equation, something you are able to solve with relative ease:

$$\begin{aligned} EIv'' + P_{cr}v &= 0 \\ v'' + \frac{P_{cr}}{EI}v &= 0 \end{aligned}$$

Now, let $\mu^2 = \frac{P_{cr}}{EI}$, then we have

$$\begin{aligned} v'' + \mu^2 v &= 0 \\ r^2 + \mu^2 &= 0 \\ r &= \pm \mu i \\ v(z) &= A\cos\mu z + B\sin\mu z \end{aligned}$$

Where A and B need to be determined, based on the boundary conditions. What are the boundary conditions here? Obviously, $v(0) = v(l) = 0$. Hence, we get that

$$v(0) = A\cos\mu 0 + B\sin\mu 0 = A = 0$$

And hence $A = 0$. For $z = L$ we get

$$v(l) = B\sin\mu l = 0$$

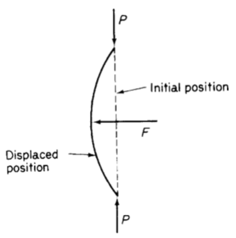
Which has nontrivial solutions (that is, solutions for $B \neq 0$) for $\mu = \frac{n\pi}{l}$. Hence, we have

$$\mu^2 = \left(\frac{n\pi}{l}\right)^2 = \frac{P_{cr}}{EI}$$

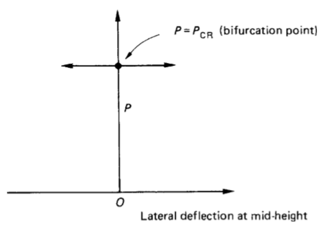
FORMULA

$$P_{cr} = \frac{n^2\pi^2 EI}{l^2} \tag{8.2}$$

with $n = 1, 2, 3, ...$[1]. Hence, the lowest critical buckling load is $P_{cr} = \frac{\pi^2 EI}{l^2}$. After this load, the column becomes unstable. What do we mean by this? Normally, under a pure compressive stress, the column will only compress. When the load is released, it will go back to its original state. This is also true when you apply a small lateral load; if you release the load, the beam will bend back to the original state. However, once you pass the critical buckling load, if you *then* apply a lateral load, then you'll permanently deform the beam as shown in figure 8.3a. Keep in mind that as long as you don't apply this lateral load, the beam will simply keep compressing; it's unstable in the sense that as soon as you touch it with a very small lateral load, it'll buckle. We thus have essentially three deflection paths, shown in figure 8.3b: once you reach the critical buckling load, you can either deflect to the left or to the right, or continue the load, with the beam undisturbed but unstable. The point at which this branching occurs is called a **bifurcation point**; further bifurcation points occur at higher values of P_{cr} (so $4\pi^2 EI/l^2$, $9\pi^2 EI/l^2$, etc.).



(a) Determination of load for a perfect column.



(b) Behaviour of a perfect pin-ended column.

Figure 8.3: Two nice drawings.

[1]The lecturer also stated that $n = -1, -2, ...$ but those solutions are *not* valid: $\mu l = n\pi$, where μ must be positive considering $\mu = \sqrt{\frac{P_{cr}}{EI}}$ which is always positive, and l and π are both positive as well. Hence, n cannot be negative.

Furthermore, higher values of n are associated with the shapes shown in figure 8.4. These occur when there are fixed points in the beam that cannot be displaced. Finally, please note that the buckling mode is *not* per definition a simple sine wave of the form $\sin\frac{n\pi z}{L}$. What we have done so far has all been done for simply supported beams. For these beams, the boundary conditions are $v(0) = v(L) = 0$. However, suppose we now have a cantilever beam, that is, one end fixed and the other free. Then we have $v(0) = 0$ and $v'(0) = 0$, but not $v(L) = 0$. If you'd have used the solution $v = A\cos\lambda z + B\sin\lambda z$, then you'd quickly find that both A and B would need to be zero, so you'd only have found the trivial solution. In these cases, different solutions ought to be tried, and different shapes of v would be the outcome. Generally speaking, this is fucked up and since the academic year of 2017-2018, you don't need to worry about doing this yourself. However, for a handful of very simple cases, the solution for the buckling load can be found relatively straightforwardly. We can write the buckling load as

$$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$

where l_e is the **effective length** of the column. This is the length of a pin-ended column that has the same critical load as that of a column of length l but with different end conditions. The ratio l_e/l can be simply taken from table 8.1.

Table 8.1: Column length solutions

Ends	l_e/l	Boundary conditions
Both pinned	1.0	$v(0) = v(l) = 0$
Both fixed	0.5	$v(0) = v(l) = 0$, $v'(0) = v'(l) = 0$
One fixed, the other free	2.0	$v(0) = 0$, $v'(0) = 0$
One fixed, one pinned	0.6998	$v(0) = v(l) = 0$, $v'(0) = 0$

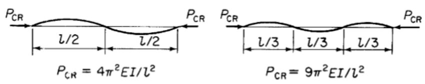


Figure 8.4: Buckling loads for different buckling modes of a pin-ended column.

Now, you may wonder, welps this is easy, what can they do to make it harder? The difficulty with this chapter lays in what M is:

SOLVING BUCKLING PROBLEMS: UNBENT BEAMS

1. Draw the free-body diagram of the beam, using the coordinate system that the book uses.
2. Determine the moment M, which can be determined by applying the method I've shown in chapter 8, and then adding the term Pv to it.
3. Set up the differential equation
$$EIv'' + M = 0$$
by substituting for M and then rewriting, such that you get a second order, linear, (non)homogeneous differential equation. Make sure that you properly indicate for which values of z this differential equation applies.
4. Write down *all* of the boundary conditions. Note that as it is a second order differential equation, you strictly only need two boundary conditions[a]. However, as I'll explain in the next step, you need another boundary condition, so you need three in total.
5. Solve the differential equation. The unknowns will be the two integration constants and P. If not always, the first boundary condition is incredibly easy to solve and determines the value of one integration constant. Then, plug in the two other boundary conditions, which leads to a system of two equations with two unknowns, where P and one constant of integration are the unknowns.
6. Solve this by first finding an expression for the other integration constant, picking the easier of the two equations. Then plug this is into the other equation and solve for P. This will usually lead to an implicit expression for P, i.e. you will write down $P = ...$, where P is still appearing after the equal sign. Note that the constant of integration must be gone in this equation.

[a]Had it been a fourth order differential equation, you'd need four boundary conditions.

Now, let's do some examples.

Example 1

A uniform column of length L and flexural stiffness EI is simply supported at its ends and has an additional elastic support at midspan. This support is such that if a lateral displacement v_c occurs at this point, a restoring force kv_c is generated at this point, as shown in figure 8.5. Derive an equation giving the buckling load of the column. If the buckling load is $4\pi^2 EI/L^2$, find the value of k. Also, if the elastic support is infinitely stiff, show that the buckling load is given by equation $\tan \lambda L/2 = \lambda L/2$, where $\lambda = \sqrt{P/EI}$.

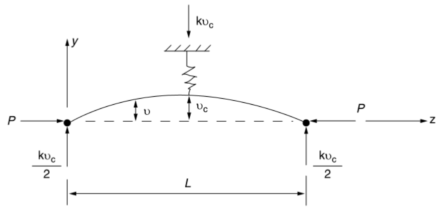


Figure 8.5: Column of first example.

Please note that this example was discussed in class and in the book, both using different methods to solve the differential equation. However, both are unnecessarily complicated in my honest opinion and not fully in compliance with the problem solving guide, so I'll just provide you with my own solution. Now, the moment is given by

$$M = Pv - \frac{kv_c}{2}z \qquad \text{for } 0 \le z \le \frac{L}{2}$$

Where P is the load at which is starts to buckle (so P_{cr}). Again, just remember that Pv is positive. $\frac{kv_c}{2}z$ is negative as you look in the negative z-direction. This yields us

$$\begin{aligned} EIv'' &= -M = -Pv + \frac{kv_c}{2}z \\ v'' + \frac{P}{EI}v &= \frac{kv_c}{2EI}z \end{aligned}$$

Note that this is a nonhomogeneous second order linear differential equation. Note that the right hand side of the equation is simply a first order term, hence the particular solution for that is straightforwardly $\frac{kv_c}{2P}z$ (if you plug that in for v, you see that $v'' = 0$ and $\frac{P}{EI}\frac{kv_c}{2P}z$ gives $\frac{kv_c}{2EI}z$, so the equation holds. Hence:

$$v(z) = A\cos\lambda z + B\sin\lambda z + \frac{kv_c}{2P}z$$

Note that A, B and P are the unknowns[a]. We thus need three boundary conditions: $v(0) = 0$, $v(L/2) = v_c$ and $v'(L/2) = 0$. Please note that $v(L) = 0$ is *not* a valid boundary condition here, as this solution was only derived for $0 \le z \le L/2$. From the first one, it's fairly straightforward to see that $A = 0$. We then apply the other two boundary conditions. First, finding $v'(z)$:

$$v'(z) = \lambda B\cos\lambda z + \frac{kv_c}{2P}$$

And hence:

$$\begin{aligned} v\left(\frac{L}{2}\right) &= B\sin\lambda\frac{L}{2} + \frac{kv_c}{2P}\frac{L}{2} = v_c \\ v'\left(\frac{L}{2}\right) &= \lambda B\cos\lambda\frac{L}{2} + \frac{kv_c}{2P} = 0 \end{aligned}$$

Now, again, following the problem solving guide: we must find an expression for B, using the easiest equation. Though it's a subjective matter of course, the second equation is arguably easier as it only consists of two terms rather than three. Hence, we can rewrite that one to get:

$$B = \frac{\frac{-kv_c}{2P}}{\lambda \cos \lambda \frac{L}{2}} = \frac{-kv_c}{2P\lambda \cos \lambda \frac{L}{2}}$$

Which we then substitute back into the first one, and then we rewrite that one to get an implicit expression for P:

$$\begin{aligned}
\frac{-kv_c}{2P\lambda \cos \lambda \frac{L}{2}} \sin \lambda \frac{L}{2} + \frac{kv_c}{2P}\frac{L}{2} &= v_c \\
-\frac{k}{2P}\frac{\tan \lambda \frac{L}{2}}{\lambda} + \frac{k}{2P}\frac{L}{2} &= 1 \\
\frac{k}{2P}\left(\frac{L}{2} - \frac{\tan \lambda \frac{L}{2}}{\lambda}\right) &= 1 \\
\frac{2P}{k} &= \frac{L}{2} - \frac{\tan \lambda}{\frac{L}{2}} \\
P &= \frac{k}{2}\left(\frac{L}{2} - \frac{\tan \lambda}{\lambda}\right) = \frac{kL}{4}\left(1 - \frac{\tan(\lambda L/2)}{\lambda L/2}\right)
\end{aligned}$$

We also already know that it buckles at $4\pi^2 EI/L^2$, then we have:

$$\lambda^2 = \frac{P}{EI} = \frac{\frac{4\pi^2 EI}{L^2}}{EI} = \frac{4\pi^2}{L^2}$$

or $\lambda = \frac{2\pi}{L}$, and hence

$$\frac{\lambda L}{2} = \frac{\frac{2\pi}{L}L}{2} = \pi$$

and hence

$$P = \frac{kL}{4}\left(1 - \frac{\tan(\lambda L/2)}{\lambda L/2}\right) = \frac{kL}{4}$$

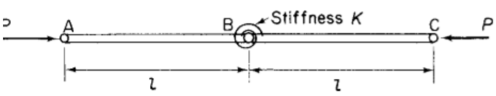
and hence $k = \frac{4P}{L}$. Furthermore, if instead of that it buckled at $4\pi^2 EI/L^2$, we'd have that $k \to \infty$, then the midpoint of figure of the beam becomes almost fixed. We know, however, that P_{cr} must still have a finite value, hence, to get a finite value for P, we must have that $\tan \frac{\lambda L}{2} \approx \frac{\lambda L}{2}$.[b]

[a] Note that λ also 'contains' P.
[b] To be honest, I'm not sure whether this reasoning is perfectly correct, but the book doesn't give any other reason at all.

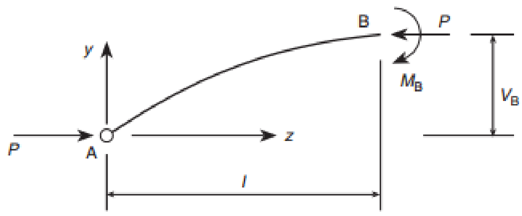
Example 2

The system shown in figure 8.6a consists of two bars AB and BC, each of bending stiffness EI, elastically hinged together at B by a spring of stiffness K (i.e., bending moment applied by spring = $K \times v'$ at B). Regarding A and C as simple pin joints, obtain an equation for the first buckling load of the system. What are the lowest buckling loads when $K \to \infty$, and what when $EI \to \infty$? Note that B is free to move vertically.

First, draw the free body diagram of figure 8.6b.



(a) System of second example.



(b) Free body diagram of second example.

Figure 8.6: Example 2.

Why is there no vertical reaction force (or horizontal force for that matter) at A, to counteract the moment M_B? Note how M_B is created: it's solely created by the compression caused by P; it's a reaction moment in itself. Hence, we simply have $M = Pv$, and hence we must solve

$$EIv'' + Pv = 0 \qquad \text{for } z \le l$$

with boundary conditions $v(0) = 0$, $M_B = K \cdot v'(l)$ and $P \cdot v(l) = M_B$. Solving is straightforward in the beginning:

$$\begin{aligned} v'' + \frac{P}{EI}v = v'' + \mu^2 v &= 0 \\ v(z) &= A\cos\mu z + B\sin\mu z \\ v(0) &= A\cos\mu 0 + B\sin\mu 0 = A = 0 \\ v(z) &= B\sin\mu z \end{aligned}$$

where we still have to determine B and P, though P is the only one the question asks for. We can use the two other boundary conditions for this: $v(l) = \frac{M_B}{P}$ and $v'(l) = \frac{M_B}{K}$. We then have

$$v(l) = \frac{M_B}{P} = B\sin\mu l \tag{8.3}$$

$$v'(l) = \frac{M_B}{K} = \mu B\cos\mu l \tag{8.4}$$

From the first row, we can write

$$B = \frac{M_B}{P\sin\mu l}$$

and substitution of this into the second one leads to

$$\begin{aligned} \frac{M_B}{K} &= \mu\frac{M_B}{P\sin\mu l}\cos\mu l = \frac{\mu M_B}{P\tan\mu l} \\ P &= \frac{\mu K}{\tan\mu l} \end{aligned}$$

Again, note that $\mu = \sqrt{\frac{P}{EI}}$ and hence this is not an explicit expression for P. However, note that they only ask for an *equation* of the buckling load of the system. This means that you only have to provide one equation with one unknown (the buckling load), but you do not necessarily have to solve this equation. Of course, you may do so if you like, but you're seriously wasting your time and you'll probably end up wanting to kill yourself so it's not that advisable.

Now, we had two other questions, what happens when $K \to \infty$ and what happens when $EI \to \infty$? When $K \to \infty$, we see that the only way to keep P at a finite value is to make $\tan\mu l \to \infty$, which means that $\mu l \to \frac{\pi}{2}$, and hence:

$$\begin{aligned} \sqrt{\frac{P}{EI}}l &\to \frac{\pi}{2} \\ P &\to \frac{\pi^2 EI}{4l^2} \end{aligned}$$

which is the Euler buckling load of a pin-ended column of length $2l$.[a] What happens when $EI \to \infty$? Then $\mu = \sqrt{\frac{P}{EI}} \to 0$, and hence $\tan \mu l \to \mu l$, and hence

$$P = \frac{\mu K}{\tan \mu l} \to \frac{\mu K}{\mu l} = \frac{K}{l}$$

[a]This is, in fact, the Euler buckling load you'd get if you'd removed the spring, as the length of the beam is $2l$ as well. This is expected: if the spring is infinitely stiff, then v' would be zero halfway the beam as well. This is the same boundary condition that a simply-supported beam has when the bending mode is assumed to be symmetric.

Example 3

A uniform column of length l and bending stiffness EI is built-in at one end and free at the other and has been designed so that its lowest flexural buckling load is P, as shown in figure 8.7.

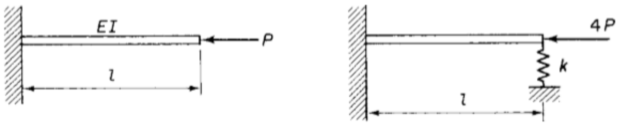


Figure 8.7: Example 3.

Subsequently, it has to carry an increased load, and for this, it is provided with a lateral spring at the free end. Determine the necessary spring stiffness k so that the buckling load becomes $4P$.

So, we're no longer interested in P_{cr} apparently (as P_{cr} is now $4P$, where P was the 'old' critical buckling load), but in k. How does this change anything? It changes very little, actually. The only difference is that now, when solving the differential equation, your third unknown (in addition to the two constants of integration) is k, and not P. Again, first we draw the free-body diagram, shown in figure 8.8.

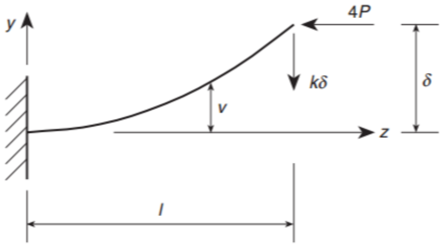


Figure 8.8: Free body diagram of example 3.

We then have that[a]

$$M = 4P \cdot (v - \delta) + k\delta (L - z)$$

Hence, we must solve

$$EIv'' + 4P(v - \delta) + k\delta (L - z) = 0$$

Which has boundary conditions $v(0) = 0$, $v'(0) = 0$ and $v(l) = \delta$. Rearranging yields

$$\begin{aligned} EIv'' + 4Pv &= 4P\delta - k\delta L + k\delta z \\ v'' + \frac{4P}{EI}v &= \delta \frac{1}{EI}\left[4P + k(z - l)\right] \end{aligned}$$

Now, let $\mu^2 = \frac{4P}{EI}$. The homogeneous solution then becomes

$$v = A \cos \mu z + B \sin \mu z$$

Furthermore, realize again that the particular solution to this problem is again very easy: we can just plug in

$$v_p = \frac{\delta}{4P}\left[4P + k(z - l)\right]$$

and you can verify that this indeed leads to the particular solution, as $v'' = 0$, obviously. The $\frac{\delta}{4P}$ was perhaps the hardest to come up with; but it's just a matter of how you can get those factors in front of the brackets to be the same. In short:

$$\begin{aligned} v(z) &= A\cos\mu z + B\sin\mu z + \frac{\delta}{4P}\left[4P + k(z-l)\right] \\ v'(z) &= -A\sin\mu + B\cos\mu z + \frac{k\delta}{4P} \end{aligned}$$

Unfortunately, the first boundary condition does not lead to $A = 0$:

$$v(0) = A\cos\mu 0 + B\sin\mu 0 + \frac{\delta}{4P}\left[4P + k(0-l)\right] = A + \frac{\delta}{4P}\left[4P - kl\right] = 0$$

However, this is still rather simply $A = \frac{\delta(kl-4P)}{4P}$. Note that this is a constant; in subsequent calculations, you do not have to substitute it already, but you can wait for a more convenient time, as to avoid some writing. The second and third boundary condition are simply a matter of working stuff out:

$$\begin{aligned} v'(0) &= -A\sin\mu 0 + \mu B\cos\mu 0 + \frac{k\delta}{4P} = \mu B + \frac{k\delta}{4P} = 0 \\ v(l) &= A\cos\mu l + B\sin\mu l + \frac{\delta}{4P}\left[4P + k(l-l)\right] = A\cos\mu z + B\sin\mu l + \delta = \delta \end{aligned}$$

The first equation is arguably the easiest this time, and it leads to $B = \frac{-k\delta}{4P\mu}$. Then, plugging both A and B into the second equation:

$$\begin{aligned} A\cos\mu l + B\sin\mu l &= 0 \\ \frac{\delta(kl-4P)}{4P}\cos\mu l - \frac{k\delta}{4P\mu}\sin\mu l &= 0 \\ (\delta kl - 4P)\cos\mu l &= \frac{k\delta}{\mu}\sin\mu l \\ kl - 4P &= \frac{k}{\mu}\tan\mu l \\ kl - \frac{k}{\mu}\tan\mu l &= 4P \\ k\left(l - \frac{\tan\mu l}{\mu}\right) &= 4P \\ k &= \frac{4P}{l - \frac{\tan\mu l}{\mu}} = \frac{4P\mu}{\mu l - \tan\mu l} \end{aligned}$$

[a]For $k\delta z$, remember that we evaluate in the negative z-direction. Hence, $-M$ is given by $-k\delta z$, as $k\delta$ acts in negative direction. Furthermore, P causes a positive moment along as it acts 'below' the location of the cross-section. Hence, the distance is given by $v - \delta$: when $\delta = 0$, it causes a positive moment, and when δ is very large in comparison to v, it will create a negative moment, etc.

Example 4

A simply-supported thin-walled beam (with no imperfections) with the cross section shown and length 1.2 m is under a compressive load of 181 601 kN. The Young's modulus is 69 GPa. (a) What is the failure index (= applied load/allowable load) for buckling? (b) What is the minimum number of hinges and their location one should use for the *beam to just buckle* under the given load?

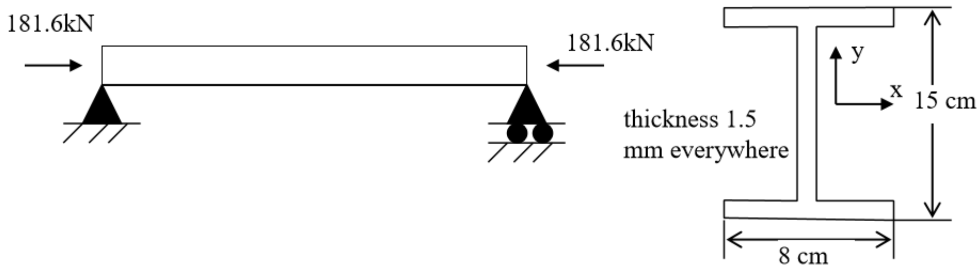


Figure 8.9: Example 4.

Before we do anything, there's an important note to make. The beam buckles about the axis with the lowest second moment of area. As it's not visible by inspection what axis that will be, we need to calculate both I_{xx} and I_{yy}, unfortunately:

$$\begin{aligned} I_{xx} &= \frac{1.5 \cdot 150^3}{12} + 2 \cdot 1.5 \cdot 80 \cdot 75^2 = 1.772 \times 10^6\,\text{mm}^4 = 1.772 \times 10^{-6}\,\text{mm}^4 \\ I_{yy} &= 2 \cdot \frac{1.5 \cdot 80^3}{12} = 1.28 \times 10^5\,\text{mm}^4 = 1.28 \times 10^{-7}\,\text{m}^4 \end{aligned}$$

And hence it'll buckle around I_{yy}, and $I = 1.28 \times 10^{-7}\,\text{m}^4$. Now, note how this problem is a bit different from the problems you've encountered so far. For question (a), you may again derive the buckling load for a simply supported beam. However, this is utterly useless as it's already given on the formula sheet to be

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \cdot 69 \cdot 10^9 \cdot 1.28 \cdot 10^{-7}}{1.2^2} = 60\,533\,\text{N}$$

So, no need to solve any differential equation this time. Furthermore, the failure index hence equals

$$\frac{181601}{60533} = 3$$

Question (b) is even more different from what you've seen so far, but is not impossible, at all. Note how the failure index indicates that the beam buckles at one third the applied load. A hinge acts as an additional simple support, so if you'd place one exactly in the middle, you'd get basically two beams, with half of the length each, hence you'd increase the buckling load by a factor 4, which is sufficient. By placing it at a different location, you can increase the length of one beam a bit, making the buckling load for that beam be exactly 181 601 kN, even though the other beam will have an even higher buckling load. See also figure 8.10.

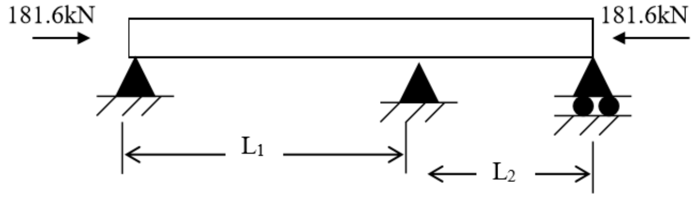


Figure 8.10: Example 4.

If $L_1 > L_2$, we have that

$$\begin{aligned} 181601 &= \frac{\pi^2 \cdot 69 \cdot 10^9 \cdot 1.28 \cdot 10^{-7}}{L_1^2} \\ L_1 &= 0.693\,\text{m} \end{aligned}$$

8.3 *Effect of initial imperfections*

This section wasn't covered in class, but it does appear in older exams, so I think you ought to know it anyway. Its content is pretty short anyway. Suppose we have a beam that is initially bent, where v_0 is given by $v_0(z)$, as shown in figure 8.11.

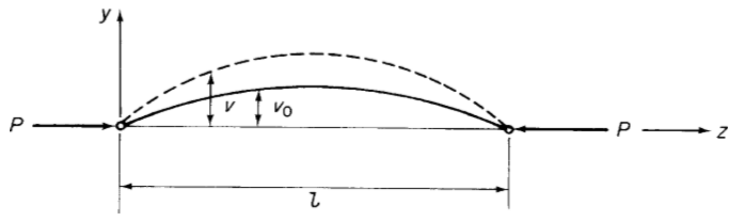


Figure 8.11: Initially bent column.

Then, rather than solving

$$EI\frac{d^2v}{dz^2} + M = 0$$

you need to solve

FORMULA

$$EI\frac{d^2v}{dz^2} + M = EI\frac{d^2v_0}{dz^2} \tag{8.5}$$

Please note that if a beam is initially bend, it will start bending further immediately; hence, the concept of the critical buckling load sort of disappears. Instead, we'll be interested in other stuff, such as bending stress.

SOLVING BUCKLING PROBLEMS: BENT BEAMS: CALCULATING THE NORMAL STRESS

1. Draw the free-body diagram of the beam, using the coordinate system that the book uses.
2. Determine the moment M, which can be determined by applying the method I've shown in chapter 8, and then adding the term Pv to it. If not always, there will be no other forces and moments acting on an already bent beam, so $M = Pv$ most of the time.
3. Set up the differential equation
$$EIv'' + M = EIv_0''$$
by substituting for M and then rewriting, such that you get a second order, linear, nonhomogeneous differential equation. Make sure that you properly indicate for which values of z this differential equation applies.
4. Write down *all* of the boundary conditions. As P is no longer the critical buckling load, but simply a load, you only need two boundary conditions.
5. Solve the differential equation. The unknowns will be the two integration constants. Plug in the two boundary conditions, which leads to a system of two equations with two unknowns.
6. Solve this by first finding an expression for one integration constant, picking the easier of the two equations. Then plug this is into the other equation and solve for the other integration constant.
7. Calculate the bending moment at the desired location.
8. If asked, calculate the normal stress via (when it bends around the x-axis):
$$\sigma = \frac{P}{A} + \frac{M_x y}{I_{xx}}$$
or (when it bends around the y-axis:
$$\sigma = \frac{P}{A} + \frac{M_y x}{I_{yy}}$$
Note that in both cases, you need to plug in the negative value for P, as P is compressive, hence the axial stress due to the axial load is negative as well.

Example 1

A uniform, pin-ended column of length l and bending stiffness EI has an initial curvature such that the lateral displacement between the column and the straight line joining its ends is given by

$$v_0 = a\frac{4z}{l^2}(l-z)$$

See also figure 8.12. Show that the maximum bending moment due to a compressive end load P is given by[a]

$$M_{\max} = -\frac{8aP}{(\lambda L^2)}\left(\sec\frac{\lambda l}{2} - 1\right)$$

where

$$\lambda^2 = \frac{P}{EI}$$

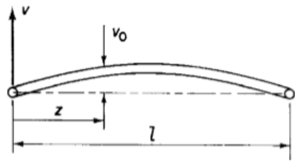


Figure 8.12: Example 1.

[a] $\sec x = \frac{1}{\cos x}$

So, in this example, we simply have $M = Pv$, so we must solve

$$EI\frac{d^2v}{dz^2} + Pv = \frac{d^2v_0}{dz^2}$$

with $v(0) = 0$ and $\frac{dv}{dz}(l/2) = 0$. Note that these are only two boundary conditions. However, please note that we'll also simply only *have* two unknowns, namely the two integration constants. P is simply the load you apply, not the critical buckling load[b]. Furthermore, I don't really like how this question is posed; you can also pick $v(l) = 0$ as second boundary condition (so getting rid of $\frac{dv}{dz}(l/2) = 0$). Both lead to exactly the same answer, but differently written and rewriting from one to the other is rather cumbersome. For now, I'll use these two boundaries, as that'll give you the solution as stated in the book; had you chosen the other boundary, your answer would have been just as correct. First, we have

$$\begin{aligned}
v_0 &= a\frac{4z}{l^2}(l-z) \\
\frac{dv_0}{dz} &= a\frac{4}{l^2}(l-z) - a\frac{4z}{l^2} \\
\frac{d^2v_0}{dz^2} &= -a\frac{4}{l^2} - a\frac{4}{l^2} = -\frac{8a}{l^2}
\end{aligned}$$

and hence

$$\begin{aligned}
EI\frac{d^2v}{dz^2} + Pv &= -EI\frac{8a}{l^2} \\
\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{d^2v}{dz^2} + \lambda^2 v &= -\frac{8a}{l^2}
\end{aligned}$$

First, the homogeneous solution is simply $v_h(z) = A\cos\lambda z + B\sin\lambda z$. The particular solution is also simply $v_p(z) = \frac{-8a}{(\lambda l)^2}$. Hence, we have

$$v(z) = A\cos\lambda z + B\sin\lambda z - \frac{8a}{(\lambda l)^2} = 0$$

For $z = 0$, we have

$$v(0) = A\cos\lambda 0 + B\sin\lambda 0 - \frac{8a}{(\lambda l)^2} = A - \frac{8a}{l^2} = 0$$

And hence $A = \frac{8a}{(\lambda l)^2}$. Then, for $\frac{dv}{dz}(l/2) = 0$:

$$\begin{aligned}\frac{dv}{dz}(z) &= -\frac{8\lambda a}{(\lambda l)^2}\sin\lambda z + \lambda B\cos\lambda z \\ \frac{dv}{dz}(l/2) &= -\frac{8\lambda a}{(\lambda l)^2}\sin\frac{\lambda l}{2} + \lambda B\cos\frac{\lambda l}{2} = 0\end{aligned}$$

And hence $B = \frac{8a}{(\lambda l)^2}\tan\frac{\lambda l}{2}$. We thus have
for $z = l$ we end up at

$$v(l) = \frac{8a}{(\lambda l)^2}\cos\lambda l + \frac{8a}{(\lambda l)^2}\tan\frac{\lambda l}{2}\sin\lambda l - \frac{8a}{(\lambda l)^2}$$

The moment will be maximum when $v(z)$ is maximum as $M = Pv$. By virtue of logic, we realize that this is maximum when $z = \frac{l}{2}$, and hence we get[c]

$$M_{\text{max}} = -Pv_{\text{max}} = -\frac{8aP}{(\lambda l)^2}\left(\cos\frac{\lambda l}{2} + \tan\frac{\lambda l}{2}\sin\frac{\lambda l}{2} - 1\right)$$

This can be rewritten: remember that $\tan = \frac{\sin}{\cos}$ and $\sin^2 = 1 - \cos^2$:

$$\begin{aligned}M_{\text{max}} &= -\frac{8aP}{(\lambda l)^2}\left(\cos\frac{\lambda l}{2} + \frac{\sin^2\frac{\lambda l}{2}}{\cos\frac{\lambda l}{2}} - 1\right) = -\frac{8aP}{(\lambda l)^2}\left(\cos\frac{\lambda l}{2} + \frac{1-\cos^2\frac{\lambda l}{2}}{\cos\frac{\lambda l}{2}} - 1\right) \\ &= -\frac{8aP}{(\lambda l)^2}\left(\frac{1}{\cos\frac{\lambda l}{2}} - 1\right) = -\frac{8aP}{(\lambda l)^2}\left(\sec\frac{\lambda l}{2} - 1\right)\end{aligned}$$

[b] As I mentioned, the concept of the critical buckling load kind of vanishes for bent beams. Instead, a bent beam starts bending further immediately, so we're more interested in what happens with the bending stress for certain values of P.

[c] Please note that the solutions put a minus sign here. I honest to God have no idea why they put it here, as it makes no sense to me: a compressive load is assumed positive when solving the differential equation, so P is positive. v will also be positive. We also have a positive bending moment, as it causes tension in the positive y-direction (upwards). If you see why they add it here, please let me know so that I can update this.

Example 2

The pin-jointed column shown in figure 8.13 carries a compressive load P applied eccentrically at a distance e from the axis of the column. Determine the maximum bending moment in the column.

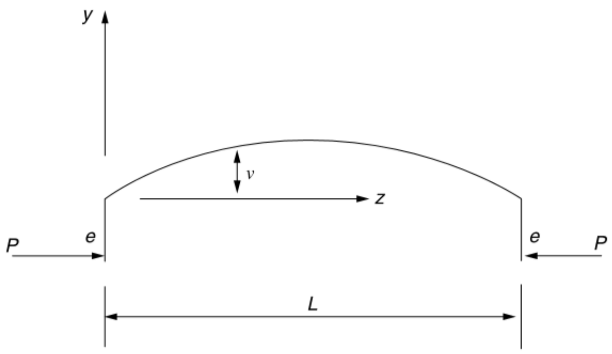


Figure 8.13: Example 2.

First, note that $\frac{d^2 v_0}{dz^2}$ is obviously zero, as the beam is initially completely straight. However, the bending moment is now given by $P(e+v)$, hence we must solve

$$\begin{aligned} EI\frac{d^2v}{dz^2} + P(e+v) &= 0 \\ \frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{d^2v}{dz^2} + \mu^2 v &= -\frac{eP}{EI} = -\mu^2 e \end{aligned}$$

For which the homogeneous solution once again simply equals $v_h(z) = A\cos\mu z + B\sin\mu z$ and the particular solution simply $v_p(z) = -e$:

$$v(z) = A\cos\mu z + B\sin\mu z - e$$

We have boundary conditions $v(0) = 0$ and $\frac{dv}{dz}(l/2) = 0$.[a] This leads to

$$\begin{aligned} v(0) &= A\cos\mu 0 + B\sin\mu 0 - e = A - e = 0 \\ \frac{dv}{dz}(z) &= -\mu A\sin\mu z + \mu B\cos\mu z \\ \frac{dv}{dz}(l/2) &= -\mu A\sin\mu l/2 + \mu B\cos\mu 1/2 = 0 \end{aligned}$$

From the first one, we obviously have $A = e$, leading to:

$$\begin{aligned} -\mu e\sin\mu l/2 + \mu B\cos\mu 1/2 &= 0 \\ B\cos\mu l/2 &= e\sin\mu l/2 \\ B &= e\tan\frac{\mu l}{2} \end{aligned}$$

Hence

$$v(z) = e\cos\mu z + e\tan\frac{\mu l}{2}\sin\mu z - e = e\left(\cos\mu z + \tan\frac{\mu l}{2}\sin\mu z - 1\right)$$

Note that you can rewrite this in various ways; both the official formula sheet and the book have different expressions, though they are essentially exactly the same. In further exercises, you can probably best use the one that's given on the formula sheet.

[a] Again, please note that you may also have used $v(l) = 0$. If you do everything correctly, you end up at exactly the correct answer, though it may be a slightly different expression.

Example 3

A thin-walled simply-supported beam of length 1.3 m with the cross-section as shown is loaded in compression at point A with a force F of 6 kN.The beam Young's modulus is $E = 210$ GPa. Determine the maximum compressive stress and its location. (Watch out for the units!) See also figure 8.14.

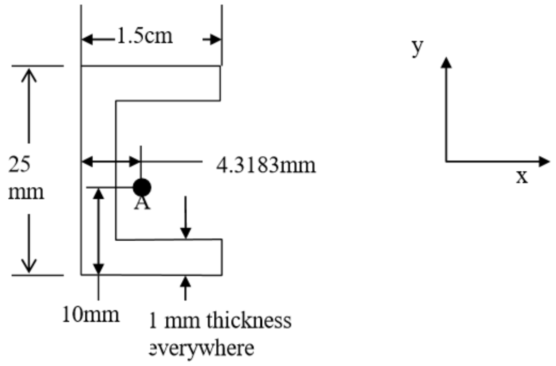


Figure 8.14: Example 3.

Now, you may be wondering, what does this have to do with buckling? Note that the beam extends 1.3 m into the page. The force does not seem to be applied at the centroid of the beam, hence, you essentially get a force with an eccentricity, as shown in figure 8.15.

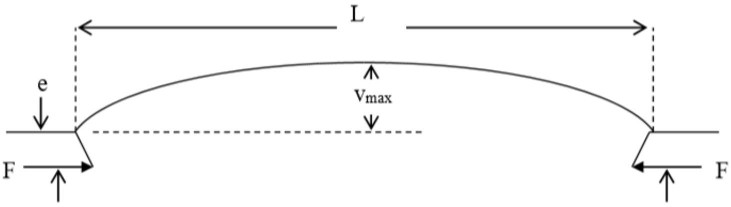


Figure 8.15: Example 3.

Let's calculate the location of the centroid first. From symmetry, $\bar{y} = 12.5$ mm.

$$\bar{x} = \frac{7.5 \cdot 15 \cdot 1 + 0.5 \cdot 1 \cdot 25 + 7.5 \cdot 15 \cdot 10}{15 \cdot 1 + 25 \cdot 1 + 15 \cdot 1} = 4.3182\,\text{mm}$$

This is close enough to the 4.3183 mm, so we'll only consider bending around the x-axis. We have

$$I_{xx} = \frac{1 \cdot 25^3}{12} + 2 \cdot 15 \cdot 1 \cdot 12.5^2 = 5989.6\,\text{mm}^4$$

We have that the moment generated by F equals $M = F(e + v)$, where $e = 2.5$ mm as the load is applied 2.5 mm below the x-axis of the centroid, and v is given by[a]

$$\begin{aligned}
v &= e\left[-1 + \cos\sqrt{\frac{P}{EI}}z + \frac{1 - \cos\sqrt{\frac{P}{EI}}L}{\sin\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}z\right] \\
&= 2.5\left[-1 + \cos\sqrt{\frac{6000}{210 \cdot 10^9 \cdot 5989.6 \cdot 10^{-12}}}z\right. \\
&\quad \left. + \frac{1 - \cos\sqrt{\frac{6000}{210\cdot10^9\cdot5989.6\cdot10^{-12}}}1.3}{\sin\sqrt{\frac{6000}{210\cdot10^9\cdot5989.6\cdot10^{-12}}}1.3}\sin\sqrt{\frac{6000}{210 \cdot 10^9 \cdot 5989.6 \cdot 10^{-12}}}z\right] \\
&= 2.5\left[-1 + \cos 2.18z + \frac{1.955}{0.298}\sin 2.18z\right]
\end{aligned}$$

This is maximum when $z = \frac{L}{2} = 0.65$ m. Hence,

$$v_{\text{max}} = 2.5\left[-1 + \cos 2.18 \cdot 0.65 + 6.57\sin 2.18 \cdot 0.65\right] = 14.1\,\text{mm}$$

Hence, the maximum bending moment is given by

$$M_{\text{max}} = P \cdot \left(e + v_{\text{max}}\right) = 6000 \cdot \left(2.5 \cdot 10^{-3} + 14.1 \cdot 10^{-3}\right) = 99.6\,\text{Nm} = 99.6 \times 10^3\,\text{Nmm}$$

The maximum compressive stress then occurs at the bottom flange, where the stress will be

$$\sigma = \frac{F}{A} + \frac{M_x}{I_{xx}}y = \frac{-6000}{15 \cdot 1 + 25 \cdot 1 + 15 \cdot 1} + \frac{99.6 \cdot 10^3}{5989.6} \cdot -12.5 = -316.95\,\text{Nm}$$

Hence, the maximum compressive stress occurs along the entire bottom flange, where the stress equals 316.95 Nm.

[a] Please note the units! I *must* be converted to m^4, or E must be in MPa, and then z should be in mm as well. I'm using the former as converting I to m^4 is less work.

8.4 Stability of beams under transverse and axial loads

For beams under transverse loads, such as the one shown below, nothing much changes, really (compared to the previous section).

Example 1

Consider the pin-ended beam carrying a uniformly distributed load of intensity w per unit length and an axial load P as shown in figure 8.16. Compute the maximum deflection and the maximum bending moment.

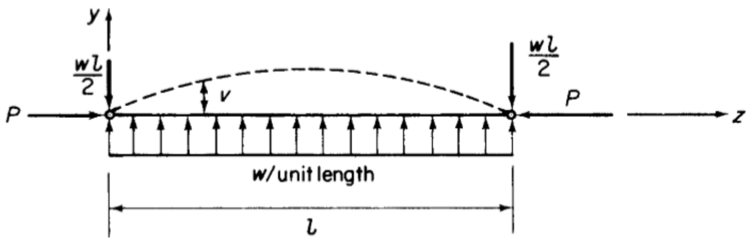


Figure 8.16: Example 1.

First, see the sketch of figure 8.17.

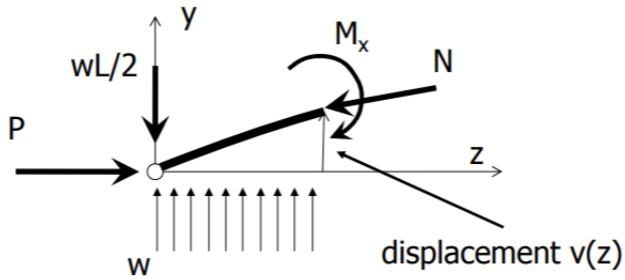


Figure 8.17: Example 1.

Then, for M_x, we simply have

$$M = P \cdot v + \frac{wLz}{2} - \frac{wz^2}{2}$$

The initial beam is not bent, so we simply have

$$\begin{aligned} EIv'' + M &= 0 \\ EIv'' + Pv + \frac{wLz}{2} - \frac{wz^2}{2} &= 0 \end{aligned}$$

with boundary conditions $v(0) = 0$ and $v(L) = 0$. We rewrite the differential equation to

$$v'' + \frac{P}{EI}v = \frac{wz^2}{2EI} - \frac{wLz}{2EI}$$

Let $\lambda^2 = P/(EI)$, then the homogeneous solution is simply

$$v(z) = A\cos(\lambda z) + B\sin(\lambda z)$$

For the particular solution, we use ~~Sturm-Louiville theorem~~ just the easy way (method of undetermined coefficients), namely assuming a solution of the form

$$\begin{aligned} v(z) &= \alpha z^2 + \beta z + \gamma \\ v'(z) &= 2\alpha z + \beta \\ v''(z) &= 2\alpha \end{aligned}$$

Plugging this into the ODE leads to

$$2\alpha + \frac{P}{EI}\left(\alpha z^2 + \beta z + \gamma\right) = \frac{wz^2}{2EI} - \frac{wLz}{2EI}$$

From it follows that we have the system of equations

$$\begin{aligned} 2\alpha + \frac{P}{EI}\gamma &= 0 \\ \frac{P}{EI}\beta &= -\frac{wL}{2EI} \\ \frac{P}{EI}\alpha &= \frac{w}{2EI} \end{aligned}$$

You don't need your graphical calculator for this, as this simply leads to

$$\begin{aligned} \alpha &= \frac{w}{2P} \\ \beta &= -\frac{wL}{2P} \\ \gamma &= -2\alpha \cdot \frac{EI}{P} = -2 \cdot \frac{w}{2P} \cdot \frac{EI}{P} = -\frac{w}{\lambda^2 P} \end{aligned}$$

Thus, our general solution will be

$$v(z) = A\cos(\lambda z) + B\sin(\lambda z) + \frac{w}{2P}z^2 - \frac{wL}{2P}z - \frac{w}{\lambda^2 P} = A\cos(\lambda z) + B\sin(\lambda z) + \frac{w}{2P}\left(z^2 - Lz - \frac{2}{\lambda^2}\right)$$

Then, determining A and B can be done from the boundary conditions $v(0) = 0$ and $v(l) = 0$. First, $v(0)$ leads to

$$\begin{aligned} v(0) = A\cos(\lambda \cdot 0) + B\sin(\lambda \cdot 0) + \frac{w}{2P}\left(0^2 - L \cdot 0 - \frac{2}{\lambda^2}\right) &= 0 \\ A - \frac{w}{\lambda^2 P} &= 0 \\ A &= \frac{w}{\lambda^2 P} \end{aligned}$$

From $v(L) = 0$, it follows that

$$\begin{aligned} v(L) = \frac{w}{\lambda^2 P}\cos(\lambda \cdot L) + B\sin(\lambda \cdot L) + \frac{w}{2P}\left(L^2 - L \cdot L - \frac{2}{\lambda^2}\right) &= 0 \\ \frac{w}{\lambda^2 P}\cos(\lambda \cdot L) + B\sin(\lambda \cdot L) - \frac{w}{\lambda^2 P} &= 0 \\ B\sin(\lambda \cdot L) &= \frac{w}{\lambda^2 P}(1 - \cos(\lambda \cdot L)) \\ B &= \frac{w}{\lambda^2 P} \cdot \frac{1 - \cos(\lambda \cdot L)}{\sin(\lambda \cdot L)} \end{aligned}$$

So, in the end we have

$$v(z) = \frac{w}{\lambda^2 P}\left(\cos(\lambda z) + \frac{1 - \cos(\lambda L)}{\sin(\lambda L)}\sin(\lambda z)\right) + \frac{w}{2P}\left(z^2 - Lz - \frac{2}{\lambda^2}\right)$$

Evidently, maximum deflection will occur at $z = L/2$, where the deflection will be

$$v_{\max} = \frac{w}{\lambda^2 P}\left(\cos\left(\lambda \cdot \frac{L}{2}\right) + \frac{1 - \cos(\lambda L)}{\sin(\lambda L)}\sin\left(\lambda \cdot \frac{L}{2}\right)\right) + \frac{w}{2P}\left(\left(\frac{L}{2}\right)^2 - L \cdot \frac{L}{2} - \frac{2}{\lambda^2}\right)$$

Now, $\sin(\lambda L) = 2\sin(\lambda L/2)\cos(\lambda L/2)$, meaning we can write

$$\frac{1 - \cos(\lambda L)}{\sin(\lambda L)}\sin\left(\lambda \cdot \frac{L}{2}\right) = \frac{1 - \cos(\lambda L)}{2\sin\left(\frac{\lambda L}{2}\right)\cos\left(\frac{\lambda L}{2}\right)}\sin\left(\lambda \cdot \frac{L}{2}\right) = \frac{1 - \cos(\lambda L)}{2\cos\left(\frac{\lambda L}{2}\right)}$$

Furthermore, $\cos(\lambda L) = 2\cos^2(\lambda L/2) - 1$, so we get

$$\begin{aligned}
\cos\left(\lambda \cdot \frac{L}{2}\right) + \frac{1 - 2\cos^2(\lambda L/2) + 1}{2\cos\left(\frac{\lambda L}{2}\right)} &= \cos\left(\lambda \cdot \frac{L}{2}\right) + \frac{2\sin^2(\lambda L/2)}{2\cos\left(\frac{\lambda L}{2}\right)} \\
&= \cos(\lambda L/2) + \frac{\sin^2(\lambda L/2)}{\cos(\lambda L/2)} = \frac{\cos^2(\lambda L/2) + \sin^2(\lambda L/2)}{\cos(\lambda L/2)} \\
&= \frac{1}{\cos(\lambda L/2)}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
v_{\max} &= \frac{w}{\lambda^2 P}\left(\cos\left(\lambda \cdot \frac{L}{2}\right) + \frac{1 - \cos(\lambda L)}{\sin(\lambda L)}\sin\left(\lambda \cdot \frac{L}{2}\right)\right) + \frac{w}{2P}\left(\left(\frac{L}{2}\right)^2 - L \cdot \frac{L}{2} - \frac{2}{\lambda^2}\right) \\
&= \frac{w}{\lambda^2 P} \cdot \frac{1}{\cos(\lambda L/2)} - \frac{w}{2P} \cdot \frac{L^2}{4} - \frac{w}{\lambda^2 P} = \frac{w}{\lambda^2 P}\left(\frac{1}{\cos(\lambda L/2)} - 1\right) - \frac{wL^2}{8P}
\end{aligned}$$

Note that this is different from what appears in the slides (the $-wL^2/8P$ is missing there), but I'm really sure I'm correct. The corresponding maximum bending moment is

$$M_{\max} = -Pv - \frac{wLz}{2} + \frac{wz^2}{2} = -Pv - \frac{wL \cdot L/2}{2} + \frac{w \cdot (L/2)^2}{2} = -Pv - \frac{wL^2}{8}$$

Note that this is the negative of what I listed at the beginning of this solution. The reason for this is that the bending moment at the beginning of this solution was the *internal* bending moment. This means that the actual applied external bending moment has the opposite sign, hence the minus signs are all switched around. With the known maximum value of v, this becomes

$$M_{\max} = -P \cdot \left[\frac{w}{\lambda^2 P}\left(\frac{1}{\cos(\lambda L/2)} - 1\right) - \frac{wL^2}{8P}\right] - \frac{wL^2}{8} = \frac{w}{\lambda^2}\left(1 - \frac{1}{\cos(\lambda L/2)}\right)$$

If we replace λ with $\sqrt{P/(EI)}$ and substitute $P_{cr} = \pi^2 EI/L^2$, you can show that this may also be written as

$$M_{\max} = \frac{wL^2}{\pi^2}\frac{P_{cr}}{P}\left(1 - \frac{1}{\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)}\right)$$

So, if P approaches P_{cr}, then the cosines goes to 0, meaning $1/\cos$ blows up and the bending moment (and deflection) becomes infinite. How can this? The reason is that we assumed in this derivation that the deflection is small,since otherwise v'' is not a close approximation for the curvature (apparently). So, if P approaches P_{cr}, then the derivation becomes invalid as the derivation would lead to very large deflections, invalidating the initial assumption on which the derivation was based. So, in real life, if $P \to P_{cr}$ the bending moment and deflection won't become infinite, although they'll become sufficiently large so that the above derivation does not hold any more, and you'd need to use other methods.

8.4.1 *A final remark regarding this chapter*

This example was not too difficult I think, just the mathematics were a bit ugly but other than that there was no magic involved. Furthermore, in case you have been wondering the following, why is it that in the previous sections we wanted to know stuff like maximum bending moment and maximum deflection etc., but in the beginning of this chapter we wanted to know the critical buckling load? In particular, in the last example P_{cr} was suddenly introduced but we never calculated it, so what's the point of putting it in there?

This can be answered as follows: the critical buckling load is a beam property that depends on the geometry of the beam and the boundary conditions of the beam. It does not depend on the applied loads! The buckling load is found by *assuming* a certain buckling shape; based on three boundary conditions are then needed to determine two integration constants and the critical buckling load. Except for very simple cases, this is often very treacherous and we often use energy methods for this; these are discussed in section 8.5 but are not part of the exam any more (unfortunately, because section 8.5 was relatively straightforward in that it was always literally the same thing you had to do).

The critical buckling load is determined based on sort of 'ideal' circumstances: merely a compressive force P, the beam is initially unbent etc. If we apply those kind of things (as done in sections 8.3 and 8.4), then more often than not we are interested in other things[2]. These things are not *directly* related to buckling: in the examples of section 8.3 and 8.4 the main part of the question was focussed on computing maximum bending moment in the beam, and we don't really do anything with the critical buckling load. So why did we discuss them in this chapter and not in chapter 16 (the chapter dedicated to bending)? There is a fundamental difference between the bending of these chapters: in chapter 16, we assumed that the moment is caused by shear forces; however, in this chapter, the moment is caused by a compressive force. This is obviously a distinct difference. Furthermore, it turns out that the calculation for the moment caused by a compressive force is based on the same mathematics as the calculations for the critical buckling load (namely solving of the 2nd order ODE equation $EIv'' + M = 0$), which is why it is fitting to put bending caused by compression in this chapter.

And thus, in case it was not clear: why we were able to introduce P_{cr} in the example of section 8.4 is simply because we assume we can look up the value of P_{cr} in literature; in fact since the example was about a beam of length L that was simply supported on both sides we know that $P_{cr} = \pi^2 EI/L^2$, as calculated in the very beginning of this chapter (so that's the literature we can look it up from). Had the beam had different boundary conditions, then most likely P_{cr} would have differed from this value $\pi^2 EI/L^2$, but you get the point, hopefully.

[2] Yes they could also ask to compute the critical buckling load but then you'd ignore all the stuff that they applied to the beam, so they could just as well leave those modifications out in the first place.

9 Buckling of reinforced structures

In aerospace structures, we often have skins (essentially thin plates) that are reinforced by stiffeners (essentially a set of beams, as discussed in the previous chapter). In this chapter, we'll first analyse skins and stiffeners (we'll see that the previous chapter did not discuss *everything* yet), and at the end we'll put them together.

9.1 Buckling of thin plates

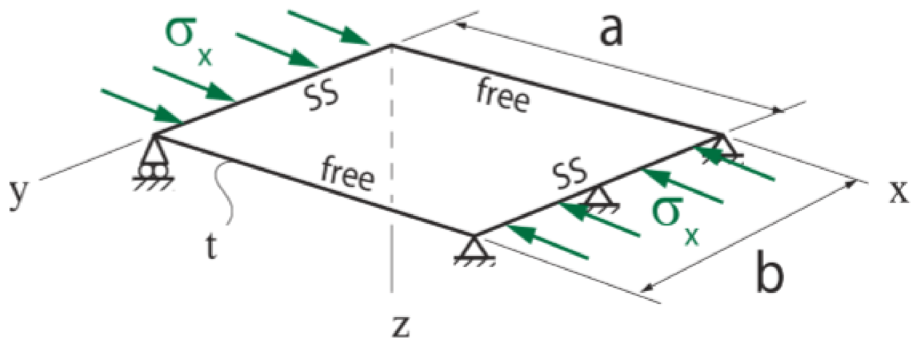


Figure 9.1: A simply supported plate.

Consider a simply supported plate, as shown in figure 9.1. It has length a, width b, and experiences a compressive stress σ_x. With thickness t, its smallest moment of inertia in the yz-plane is[1]

$$I = \frac{bt^3}{12}$$

Then, remember that for a simply supported *beam* we have

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI/L^2}{A} = \frac{\pi^2 EI}{AL^2}$$

So why don't we just try this for a plate? Why don't we just take $L = a$, $A = bt$, $I = bt^3/12$ and see what happens?

$$\sigma_{cr} = \frac{\pi^2 E \cdot \frac{bt^3}{12}}{btL^2} = \frac{\pi^2 Et^2}{12a^2}$$

What could possibly be wrong? Well, actually, we're making a mistake in E: recall Hooke's Law:

$$\epsilon_x = \frac{\sigma_x}{E} - \upsilon\frac{\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \upsilon\frac{\sigma_x}{E}$$

where υ is the Poisson ratio. Now, for a beam, we simply have $\sigma_y = 0$, so $\epsilon_x = \sigma_x/E$ as expected. However, for a plate, the thing that's true is that away from a free edge, $\epsilon_y \approx 0$, but $\sigma_y \neq 0$.[2] Indeed, from $\epsilon_y = 0$ we

[1]Remember that buckling happened around the cross-sectional axis with the smallest moment of inertia. In figure 9.1, the cross-section of the beam is in the yz-plane (as the length of the beam is in x-direction). Then, with y and z as drawn in the figure, $I_{yy} = bt^3/12$ and $I_{zz} = b^3t/12$. Thus, I_{yy} is clearly the smallest.

[2]If you're wondering, but why? The reason for this is simple: at free edges, we clearly must have $\sigma_y = 0$. For a beam (and a beam is essentially a plate shown in figure 9.1 with very small b), almost everything is close to a free edge, thus we can say that $\sigma_y = 0$ everywhere. However, if b is sufficiently large, then the parts in the middle are not close to a free edge any more, so for those parts in the middle not necessarily $\sigma_y = 0$. However, you can say that $\epsilon_y = 0$ there.

get

$$\begin{aligned} 0 &= \frac{\sigma_y}{E} - \upsilon\frac{\sigma_x}{E} \\ \sigma_y &= \upsilon\sigma_x \end{aligned}$$

Substituting this into the expression for ϵ yields

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\upsilon\sigma_x}{E} = \frac{\sigma_x}{E}\left(1-\upsilon^2\right) = \frac{\sigma_x}{E^*}$$

where $E^* = E/(1-\upsilon^2)$. Thus, for our simply supported plate (free on two edges), our critical buckling stress is

$$\sigma_{cr} = \frac{\pi^2 E^* t^2}{12a^2} = \frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{a}\right)^2$$

Now, this is the critical buckling stress for a plate that's simply supported on two edges and free on the other two. As you may remember from the previous chapter, the critical buckling stresses for beams that have different boundary conditions follow by multiplication by some constant that can be looked up from literature; for plates the same holds, here we have the constant C^* (we'll shortly see how exactly to determine this constant), so that in general, we have

CRITICAL BUCKLING STRESS OF A PLATE

The critical buckling stress of a plate is given by

$$\sigma_{cr} = C^*\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{a}\right)^2 \tag{9.1}$$

where C^* is a value that can be looked up from literature, based on the boundary conditions of the plate; a is the length of the plate. The buckling stress is more commonly written as

$$\sigma_{cr} = C\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2 \tag{9.2}$$

where C is a value that can be looked up from literature, based on the boundary conditions of the plate; b is the width of the plate.

Note that the equations are pretty much the same except that a and b have been switched; consequentially this also changes how C^* or C is determined (I mean, C^* is not equal to C most of the time). However, in literature, pretty much always the second equation is used which is why from now on we will stick to using that one. So remember that the b there is the *width* of the plate, not the length.

Now, as promised, how do we determine C? Well, we just look it up in figure 9.2, which should look familiar to you from your beautifully designed wing box project. On the x-axis, the aspect ratio a/b is plotted. On the vertical axis, the value of C is plotted. Then, which line should you pick? What do the CCSS, CCCC etc. mean? They mean the following:

- SSFF means it has two sides that are **s**imply supported, and two sides that are **f**ree (i.e. the example shown in figure 9.1).
- CCSS means it has two sides that are **c**lamped, and two sides that are **s**imply supported.
- SSSS means it has four sides that are **s**imply supported.

I hope you get the idea now. In case you're wondering, what exactly are free, simply supported and clamped edges?

- Simply supported means that something is supporting that side, but it's free to rotate around that side. Looking at figure 9.1, it's easy to envision that the plate will remain in place at the y-axis in the back; however, there will be a slope there. So, it's position is fixed, but it's free to rotate.
- Clamped means that something is supporting that side and preventing it to rotate around as well. Think back to your wing box: when we started destructing it, one of the sides was clamped: the support had a very firm grip on it, preventing it from even rotate at the root of the wing (the tip of the wing was lifted up and was merely simply supported, iirc). So both the position and rotation are fixed.

- Free means free: it's position isn't fixed and it's free to rotate as well. The free edges in figure 9.1 clearly satisfy this: they'll bent upwards (so it's position is not fixed) and there'll be most likely also be rotation around the x-axis.

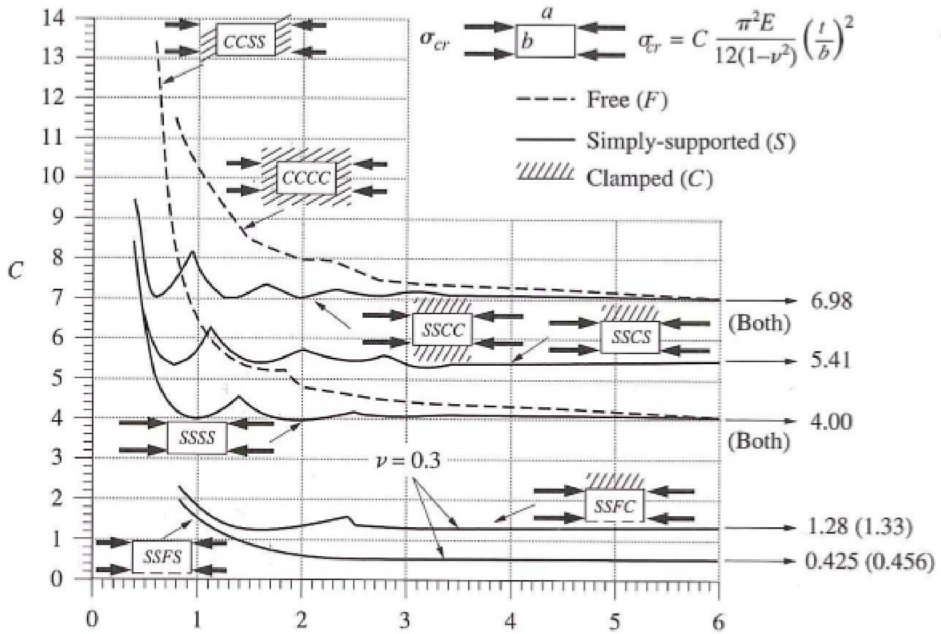


Figure 9.2: Coefficient C versus aspect ratio for several different boundary conditions. Only the bottommost curves depend on Poisson's ratio, and their asymptotes for $\upsilon = 0.25$ are shown in parentheses.

Now, the aspect ratio may be slightly confusing, so let me explain: it's just the length of the plate divided by the width of the plate. So, if the plate is very long in comparison to how wide it is, the aspect ratio is very large. If the plate is short, the aspect ratio is small. Short plates buckle slower than long plates (I hope you have this intuition by now), so it makes sense that C is larger as well. Now, you may wonder, why those peaks for small aspect ratios? The reason is actually pretty logical: when stuff buckles, it becomes shaped like sine-waves. These peaks then occur when these sine-waves fit particularly 'nicely' into the plate: the sine-waves have a sort of 'natural' wave-length, and if the aspect ratio is such that e.g. exactly two waves fit in the length of the plate and exactly one wave in the width of the beam, then it kinda makes sense that this is beneficial to the buckling performance, so C is increased. When you would then slightly change the aspect ratio, the waves don't fit 'nicely' any more, so then you suddenly get a significant decrease in the value of C. That's why those peaks are present. However, in this course, we'll mostly consider plates that are very long, i.e. the aspect ratio $a/b > 3$. For $a/b > 3$, the value of C is more or less constant, and the associated values have been denoted in the plot. Note that for SSFF (the boundary conditions of figure 9.1) $C = 0.92$; this is due to imperfections and due to the fact that the derivation was not 100% accurate (I guess).

Also, just because I'm sure there'd be idiots drawing this plot on their formula sheet: obviously you'd be given this graph on the exam if you need it (I assume at least).

9.2 *Buckling and crippling of stiffeners*

We now shift our focus to stiffeners, as shown in figure 9.3. Looking at that stiffener, you may very well wonder, but wait a sec, how would this buckle? Cause essentially it's just two plates put together, so what do I do? Do I do something with the analysis of thin plates or is it column buckling of chapter 8?

The matter of truth is, it's a bit of both. If the stiffener is very slender (i.e. very long in comparison to the dimensions of the cross-section), the stiffener will buckle everywhere simultaneously at some applied load (chapter 8 buckling). If it's not so slender, then the plates will each buckle individually first (section 9.1). If the first plate starts buckling, the stiffener does not immediately fail yet! Instead, what will happen is the following: suppose one plate of the stiffener shown in figure 9.3 buckles at 20 MPa; then if the applied load is increased,

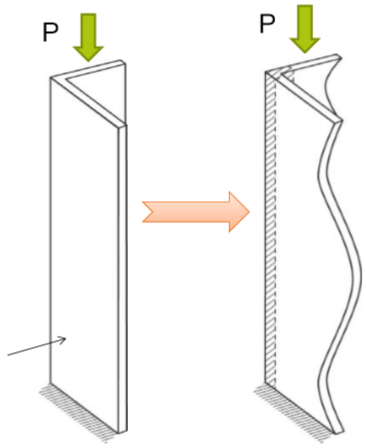


Figure 9.3: A stiffener under compression.

this plate will still take up 20 MPa of stress. However, the other plate will simply take up more stress, meaning the applied load can still be carried by the stiffener. Only once the stress in the other plate also exceeds some critical stress, this remaining plate will also buckle and once all plates (in this case only two) have buckled, the stiffener has failed. However, it could be that the remaining plate only buckles when the stress in this plate reaches a value of 40 MPa, meaning it could have been possible that the applied stress was increased to 30 MPa (so the failure stress is not 20 MPa but 30 MPa).[3]. Please note that we'll do an example shortly hereafter to make everything clearer so don't freak out if you don't 100% follow what I'm saying (same holds for how to determine at what slenderness ratio you need to take into account for column buckling as a whole, etc.).

Now, the buckling of individual plates, and the subsequent failure, is called **crippling**. So, in the rest of this chapter, when I write buckling, I mean Euler buckling[4]. Then, before we do an example, you need to remember the following differences between buckling and crippling:

- Buckling occurs at high slenderness ratios; crippling at low slenderness ratios (again, we'll later see exactly how this influences calculations. For now I want you to remember this qualitative difference).
- Column buckling is the buckling of a column/stiffener/beam as a whole. It's also known as global buckling for that reason: everything buckles simultaneously. It's the calculations you did in chapter 8. The critical buckling stress is denoted by σ_{cr}.
- Crippling is the sequential buckling of individual plates of a stiffener. Each plate cripples at some level of applied stress; it then stops taking up more stress (but it stays taking up as much as stress as when it started crippling). The remaining plates will still be able to take up load, so the stiffener has not failed yet[5]. As the applied load is increased, more and more of the plates will start to cripple; once the last plate has crippled there is no plate available any more to take up an increased load, and thus failure has occurred. The crippling stress is denoted by σ_{cc}. To be specific, the crippling stress at which plate i starts to cripple is $\sigma_{cc}^{(i)}$. The crippling stress at which the stiffener fails as a whole due to crippling is denoted by merely σ_{cc}.

Let's do two examples to clear this all up. However, first, don't ask me why, but for the crippling of plates of stiffeners, we no longer use

$$\sigma_{cc}^{(i)} = C \cdot \frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2$$

but we use some correction factors α and n, to arrive at

[3]If you get confused by these numbers: suppose both plates would have had a cross-sectional area of 20 mm^2. Then the force carried by plate that crippled first would be $F = \sigma \cdot A = 20 \cdot 20 = 400$ N. On the other hand, the second plate apparently crippled at 40 MPa; the load in this plate is then $F = \sigma \cdot A = 40 \cdot 20 = 800$ N. The total load carried by the stiffener is thus $400 + 800 = 1200$ N, and the total applied stress is thus $\sigma = F/A = 1200/(20+20) = 30$ MPa.

[4]Also known as column buckling, or the buckling described in chapter 8. But this last name isn't quite as catchy and if you ever write it in a report that a beam is suffering from the buckling described in chapter 8 of Aircraft Structural Analysis, I wouldn't really trust you to design anything.

[5]Failure happens when the structure is not able to take up any more load.

CRIPPLING STRESS

The crippling stress of flanges in a stiffener are given by

$$\frac{\sigma_{cc}^{(i)}}{\sigma_y} = \alpha \left[\frac{C}{\sigma_y} \frac{\pi^2 E}{12\left(1-v^2\right)} \left(\frac{t}{b}\right)^2 \right]^{1-n} \tag{9.3}$$

Note that σ_y is the yield stress of the material; note that the result is thus the ratio of the crippling stress of the ith flange divided by the yield stress of the material.

Let's finally do an example:

Example 1

Consider the stiffener shown in figure 9.4. Compute the crippling stress of the stiffener. Take the following values:

- $b_1 = b_3 = b_4 = 20\,\text{mm}$.
- $b_2 = 30\,\text{mm}$.
- $t_1 = t_2 = t_3 = t_4 = 1.5\,\text{mm}$.
- $E = 72\,\text{GPa}$.
- $v = 0.3$.
- $\sigma_y = 450\,\text{MPa}$.

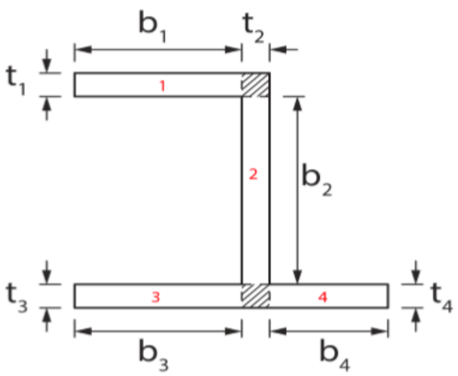


Figure 9.4: Example 1.

So, how do we do this? Well, hopefully you'll agree with me that elements 1, 3 and 4 are essentially the same: they all have the same thickness and width, and they are all connected on one side to a corner whilst they are free on the other end. Then, let us consider the formula

$$\frac{\sigma_{cc}^{(i)}}{\sigma_y} = \alpha \left[\frac{C}{\sigma_y} \frac{\pi^2 E}{12\left(1-v^2\right)} \left(\frac{t}{b}\right)^2 \right]^{1-n}$$

We actually know all of these values, except for C which is a bit weird. However, we can look it up from figure 9.2; we'll assume that $a/b > 3$ (this is an assumption that can be made unless otherwise stated). Probably it makes sense that we have to take a graph associated with at least one side being free (since we noted that all elements 1, 3 and 4 have one edge free in the drawing). However, what do we do with the other edges? First, the edge that's connected to a corner are simply supported edges: their position is fixed, but they are free to rotate, I hope you can agree with that. That leaves us with two edges to establish: *these are not the remaining two edges shown in figure 9.4.* Instead, these edges refer to the top and bottom edge shown in figure 9.3 (note that you need to extrude the cross-section shown in figure 9.4 to get the three-dimensional stiffener): they are the edges where the load is applied (i.e. the top and bottom edge of figure 9.3, the edges obtained by extruding the cross-section of figure 9.4). These edges are simply supported edges most of the time, unless otherwise indicated.

Thus, we have three simply supported edges and one free edge, so we must take the graph associated with SSFS, for which $C = 0.425$. In general, for these questions, we *always* have two simply supported edges (these originate from the 3D-shape of the stiffener). Additionally, we have an additional simply supported edge for each edge that is connected to a corner, and a free edge for each edge that is not connected to a corner. Now, plugging in the numbers leads to

$$\begin{aligned}\frac{\sigma_{cc}^{(1,3,4)}}{\sigma_y} &= \alpha\left[\frac{C}{\sigma_y}\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2\right]^{1-n}\\ &= 0.8\left[\frac{0.425}{450\cdot 10^6}\frac{\pi^2\cdot 72\cdot 10^9}{12\cdot\left(1-0.3^2\right)}\left(\frac{1.5}{20}\right)^2\right]^{1-0.6}\\ &= 0.523\end{aligned}$$

so $\sigma_{cc}^{(1,3,4)} = 235\,\text{MPa}$.

For element number 2, we see that it has two edges connected to corners (and not edges connected to no corner), so it is SSSS. Furthermore, note that $b = 30\,\text{mm}$ now. Thus, $C = 4.0$ and thus

$$\begin{aligned}\frac{\sigma_{cc}^{(2)}}{\sigma_y} &= \alpha\left[\frac{C}{\sigma_y}\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2\right]^{1-n}\\ &= 0.8\left[\frac{4}{450\cdot 10^6}\frac{\pi^2\cdot 72\cdot 10^9}{12\cdot\left(1-0.3^2\right)}\left(\frac{1.5}{30}\right)^2\right]^{1-0.6}\\ &= 0.927\end{aligned}$$

so $\sigma_{cc}^{(2)} = 417\,\text{MPa}$.

Now, what is thus the crippling stress of the stiffener as a whole? Note that when the stress in element 2 exceeds 417 MPa, the entire structure will fail, as all of the webs will have crippled by then. However, note that the *applied* stress will be lower than 417 MPa at the moment the stress in element 2 exceeds 417 MPa: after all, the applied stress is more or less the 'weighted' average of the stresses in the individual elements (weighted by area), so the crippling stress of the entire stiffener is computed as

$$\sigma_{cc} = \frac{\sum \sigma_{cc}^{(i)} A_i}{\sum A_i} = \frac{3\cdot 235\cdot 20\cdot 1.5 + 417\cdot 30\cdot 1.5}{3\cdot 20\cdot 1.5 + 30\cdot 1.5} = 296\,\text{MPa}$$

In case it's really not clear to you what I mean with applied stress and why it's the weighted average etc.: when none of the elements have crippled, then if you increase the applied stress by 1 MPa, the stresses in each of the flanges increases by exactly 1 MPa as well (this is due to compatibility); this means that when the applied stress reaches a value slightly below 235 MPa, the stress in each of the elements is 235 MPa as well. However, if the applied stress is increased slightly, elements 1, 3 and 4 will cripple, and so the stresses in those elements will remain at 235 MPa. To make up for this, the stress in element 2 will increase at a faster rate when the applied stress increased, after all, the underlying equation is

$$\begin{aligned}F_{\text{applied}} &= \sum F_i\\ \sigma_{\text{applied}}\cdot A_{\text{total}} &= \sum \sigma_i A_i\end{aligned}$$

where F_i and σ_i are the force respectively stress in element i. Thus, to increase the stress in element 2 to 417 MPa, the applied stress does not nearly have to be increased to 417 MPa itself. Instead, it turns out it only needs to be increased to 296 MPa. If the applied stress is increased beyond this, not even element 2 is going to take up any more load: thus, the stiffener has failed. This is why the crippling stress of the stiffener is 296 MPa.

So, let's see what we actually have to do because in hindsight, the steps are pretty easy.

FINDING THE CRIPPLING LOAD OF A STIFFENER

1. For each element, find the value of C from the graph (this will be provided to you on the exam in some form or shape), based on the boundary conditions.
2. Find the ratio $\sigma_{cc}^{(i)}/\sigma_y$ for each element by use of

$$\frac{\sigma_{cc}^{(i)}}{\sigma_y} = \alpha \left[\frac{C}{\sigma_y} \frac{\pi^2 E}{12\left(1-\upsilon^2\right)} \left(\frac{t}{b}\right)^2 \right]^{1-n} \tag{9.4}$$

3. If this ratio is smaller than 1, then the crippling stress of the ith element is equal to

$$\sigma_{cc}^{(i)} = \left(\frac{\sigma_{cc}^{(i)}}{\sigma_y}\right) \cdot \sigma_y$$

   If this ratio is equal to or larger than 1, then for this element the yield stress needs to be used in subsequent calculations.
4. Find the the crippling stress for the stiffener as a whole by use of

$$\sigma_{cc} = \frac{\sum \sigma_{cc}^{(i)} A_i}{\sum A_i}$$

   where A_i is the area of each element.
5. Find the crippling load for the stiffener as a whole by use of

$$P_{cc} = \sigma_{cc} A$$

   where A is the total cross-sectional area, including the corners.

Two things may appear strange from this guide: what happens when the ratio is equal to or larger than 1, and what am I on about with regards to the corners? Regarding the first, we'll get to this in the next example. Regarding the latter, please note that for the width of each element, we do not count the corners (the shaded areas in figure 9.4) *at all*. This is because corners are very stiff and thus do not count. We also do not use them for the calculation of $\sum A_i$ to find the crippling stress. However, we do take those areas into account to compute P_{cc}. This is mostly just a matter of being convention and being conservative.

Example 2

Consider the omega stiffener shown in figure 9.5. Compute the crippling stress. Take the following values:

- $\alpha = 0.8$.
- $n = 0.6$.
- $\upsilon = 0.334$.
- $E = 72\,\text{GPa}$.
- $\sigma_y = 503\,\text{MPa}$.
- Dimensions shown in the figure.

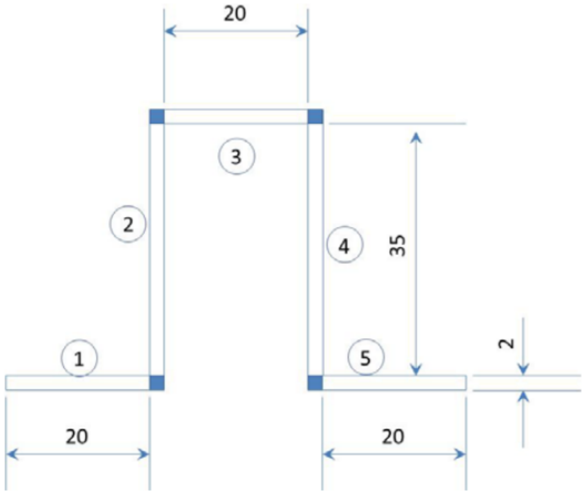


Figure 9.5: Example 2.

Note that element 1 is exactly the same as element 5, and element 2 is exactly the same as element 4. Element 1 and 5 have one free edge and three simply supported edges, thus $C = 0.425$. With $b = 20$ and $t = 2$, we get

$$\begin{aligned}\frac{\sigma_{cc}^{(1,5)}}{\sigma_y} &= \alpha\left[\frac{C}{\sigma_y}\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2\right]^{1-n} \\ &= 0.8\cdot\left[\frac{0.425}{503\cdot 10^6}\frac{\pi^2\cdot 72\cdot 10^9}{12\left(1-0.334^2\right)}\left(\frac{2}{20}\right)^2\right]^{1-0.6} = 0.6356\end{aligned}$$

Since this ratio is smaller than 1, it means that the crippling stress is lower than the yield stress, thus it will cripple before yielding. Therefore, the crippling stress will be $\sigma_{cc}^{(1,5)} = 0.6356\sigma_y = 0.6356\cdot 503 = 319.8\,\text{MPa}$.

For element 2 and 4, both edges are connected to corners, so we have to use the C for SSSS, which is 4.0. Thus, with $b = 35$ and $t = 2$, we get

$$\begin{aligned}\frac{\sigma_{cc}^{(1,5)}}{\sigma_y} &= \alpha\left[\frac{C}{\sigma_y}\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2\right]^{1-n} \\ &= 0.8\cdot\left[\frac{4}{503\cdot 10^6}\frac{\pi^2\cdot 72\cdot 10^9}{12\left(1-0.334^2\right)}\left(\frac{2}{35}\right)^2\right]^{1-0.6} = 0.996\end{aligned}$$

Again, it's smaller than 1, so $\sigma_{cc}^{(2,4)} = 0.996\sigma_y = 0.996\cdot 503 = 501.1\,\text{MPa}$.

For element 3, we have again that both edges are connected to corners, so $C = 4.0$. With $b = 20$, we get

$$\begin{aligned}\frac{\sigma_{cc}^{(3)}}{\sigma_y} &= \alpha\left[\frac{C}{\sigma_y}\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2\right]^{1-n} \\ &= 0.8\cdot\left[\frac{4}{503\cdot 10^6}\frac{\pi^2\cdot 72\cdot 10^9}{12\left(1-0.334^2\right)}\left(\frac{2}{20}\right)^2\right]^{1-0.6} = 1.55\end{aligned}$$

This is *larger* than 1: thus, this element would cripple after yielding. Thus, yielding is critical, and in subsequent calculations, we will assume that the crippling stress is $\sigma_{cc}^{(3)} = \sigma_y = 503\,\text{MPa}$.

The crippling stress of the stiffener is then

$$\sigma_{cc} = \frac{\sum \sigma_{cc}^{(i)} A_i}{\sum A_i} = \frac{2 \cdot 319.8 \cdot (20 \cdot 2) + 2 \cdot 501.1 \cdot (35 \cdot 2) + 503 \cdot (20 \cdot 2)}{2 \cdot 20 \cdot 2 + 2 \cdot 35 \cdot 2 + 20 \cdot 2} = 445.6\,\text{MPa}$$

The crippling load is then found by taking into account the area of the corners, i.e.

$$P_{cc} = \sigma_{cc} A = 445.6 \cdot (2 \cdot 20 \cdot 2 + 2 \cdot 35 \cdot 2 + 20 \cdot 2 + 4 \cdot 2 \cdot 2) = 122\,987.7\,\text{N} = 123\,\text{kN}$$

Again, don't ask me why we take the corners into account now, it's just to be conservative.

I think this is pretty clear now. The only question is, how do I know whether I need to consider Euler buckling? Cause I said it had to do with how slender the stiffener was. We'll discuss this issue now.

9.2.1 *Johnson-Euler column curves*

Let us consider the critical buckling stress of a column once more:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E I_{xx}}{A L_e^2} = \pi^2 E \cdot \frac{I_{xx}}{A L_e^2}$$

where L_e is the effective length of the column, which takes into account the boundary conditions. Now, if we introduce the radius of gyration,

RADIUS OF GYRATION

The radius of gyration ρ is defined as

$$\rho \equiv \sqrt{\frac{I_{xx}}{A}} \tag{9.5}$$

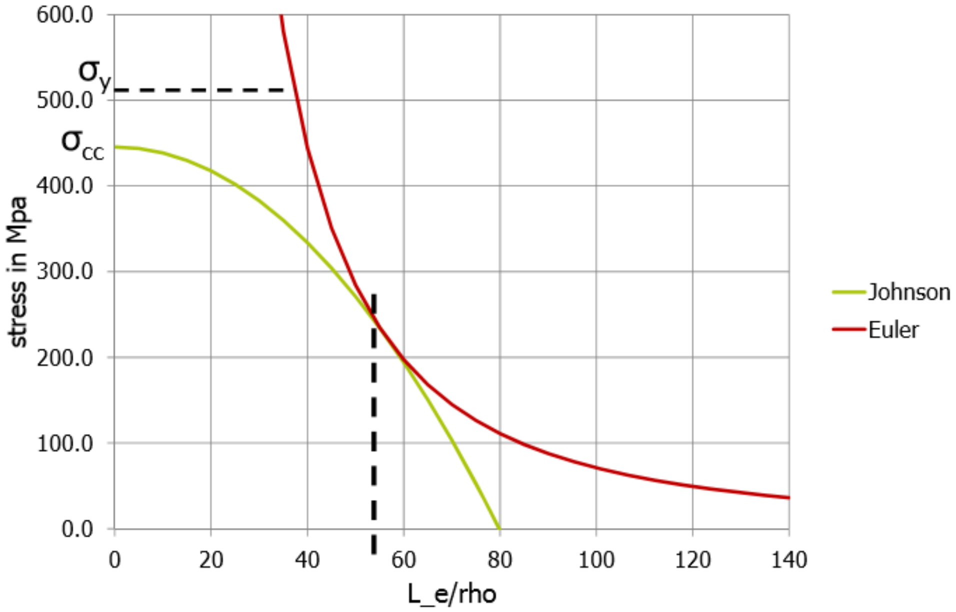
Essentially, if you have a cross-section with area A, and put an area A at a distance ρ from the neutral axis, then the Steiner term (so you ignore any dimensions of this area, as if it was a boom) corresponding to this new area is equal to the moment of inertia of the cross-section. You may remember the radius of gyration from dynamics: there, if a disk had a certain radius of gyration ρ, it meant that if you placed all the mask of the disk in a thin circular disk of radius ρ, it'd lead to the same mass moment of inertia as the original disk, allowing you to straightforwardly compute the mass moment of inertia straightforwardly. Similar idea happens here, although here we're talking about the area moment of inertia. Writing $I_{xx} = A\rho^2$ into the formula for buckling leads to

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E I_{xx}}{A L_e^2} = \pi^2 E \cdot \frac{A\rho^2}{A L_e^2} = \pi^2 E \left(\frac{\rho}{L_e}\right)^2 \tag{9.6}$$

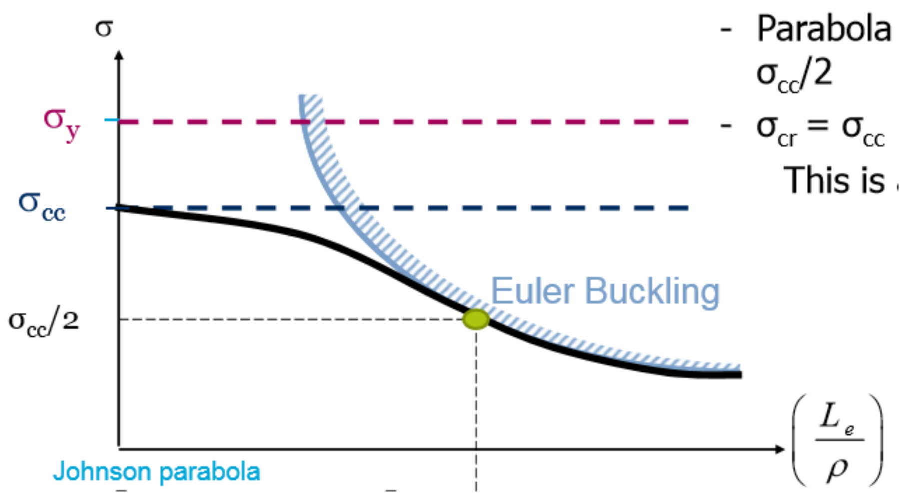
The nice thing about ρ is that it is a value that you can look up from literature for various cross-sections; essentially it is an indication of how 'wide' the cross-section is. Thus, the ratio L_e/ρ is essentially the slenderness (length over width) of a beam, and so we now have a nice expression for buckling stress as function of slenderness. We see that it behaves like $\sigma_{cr} \sim 1/x^2$ (with x being the slenderness ratio).

We can conclude that if the slenderness is very large (i.e. it's very long compared to how wide the cross-section is), the buckling stress decreases. On the other hand, in our previous two examples, we never actually took the slenderness into account (the only time it actually mattered was when the aspect ratio was smaller than 3, in which case C changed significantly). Thus, it makes sense to say that at small slenderness ratios, it's the cripple stress that's dominant. However, after a certain slenderness ratio, the Euler buckling stress takes over. In fact, we can graphically depict this as shown in figure 9.6a: in red, the Euler buckling stress as a function of L_e/ρ is shown, as dictated by above equation. In green, the crippling stress is shown as function of L_e/ρ. In figure 9.6b, it is then depicted what value you should pick as critical stress: before the graphs touch each other, you should take the crippling stress; afterwards, you should take the Euler buckling stress.

Now, you probably are confused by the green graph of the crippling stress. Why is it suddenly a function of L_e/ρ? And after the graphs touch, it'll still be lower than the Euler buckling stress, so why should we not still



(a) Johnson-Euler column curves.



(b) Johnson parabola.

Figure 9.6: Johsnon-Euler column curves.

take the crippling stress as being critical? The reason may be slightly anticlimatic, but basically it boils down to that most of what we're doing is empirically derived. This was the original reason why the weird α and n suddenly appeared in our equation: purely because what we've obtained in experimental results, we realized we had to modify the equations a bit to get results that are more in line with real life. This is why our obtained results did not depend at all on how long the stiffener was, even though in real life we know that more elongated plates are more prone to crippling. Thus, the green graph in figure 9.6a is also a graph that is partly based on experimental results: some guy (named Johnson) observed that the crippling stress depended on the slenderness ratio as shown, and magically he found a curve describing the green graph:

$$\sigma = \sigma_{cc}\left[1 - \frac{\sigma_{cc}\left(L_e/\rho\right)^2}{4\pi^2 E}\right] \tag{9.7}$$

where σ_{cc} is the crippling stress we computed before; this is more or less the 'base' crippling stress. This curve *always* touches the Euler buckling curve; it was designed to do so. After it has touched the red curve, the red curve does take over and the Johnson curve can be ignored, as shown in figure 9.6b. So, to keep it short: the green curve shown in figure 9.6b is mostly just an experimental curve.

Using above equation, we can compute at which slenderness ratio the failure mode switches from crippling to Euler buckling. After all, at this slenderness ratio, we must have that the green curve (given by equation (9.7)) equals the red curve (given by equation (9.6)), i.e.

$$\begin{aligned}
\pi^2 E \cdot \left(\frac{\rho}{L_e}\right)^2 &= \sigma_{cc}\left[1 - \frac{\sigma_{cc}\left(L_e/\rho\right)^2}{4\pi^2 E}\right] \\
\frac{\pi^2 E}{\sigma_{cc}} &= \left(\frac{L_e}{\rho}\right)^2 - \frac{\sigma_{cc}\left(L_e/\rho\right)^4}{4\pi^2 E} \\
\left(\frac{L_e}{\rho}\right)^4 - \frac{4\pi^2 E}{\sigma_{cc}}\left(\frac{L_e}{\rho}\right)^2 + \frac{4\pi^4 E^2}{\sigma_{cc}^2} &= 0 \\
\left[\left(L_e/\rho\right)^2 - \frac{2\pi E}{\sigma_{cc}}\right]\left[\left(L_e/\rho\right)^2 + \frac{2\pi E}{\sigma_{cc}}\right] &= 0
\end{aligned}$$

So the only positive root is the critical slenderness ratio:

CRITICAL SLENDERNESS RATIO

The critical slenderness ratio is given by

$$\frac{L_e}{\rho} = \sqrt{\frac{2\pi^2 E}{\sigma_{cc}}} \tag{9.8}$$

Obviously, you don't need to know this derivation by heart since you'll just write it on your formula sheet. Furthermore, just to repeat myself: the σ_{cc} in this equation is the σ_{cc} that was computed in the previous section (this is also the case for equation (9.7). The corresponding stress at which we switch from crippling failure to Euler buckling failure is $\sigma_{cc}/2$. The resultant curve shown in figure 9.6b is called the Johnson-Euler approximation.

9.2.2 *Bulb/lip reinforcements*

As it was mentioned in the slides, I'll also mention it. You can strengthen stiffeners by adding bulb/lip reinforcements as shown in figure 9.7. These work because they are not prone to buckling at all because their width b is very small; thus the $\sigma_{cc}^{(i)}$ of those elements is very large, delaying local instability.

9.3 *Buckling of stiffened panels*

So, what happens if we put stiffeners on top of panels? How does this change buckling? What should you really have done last year for the redesign of the top plate of the wing box?

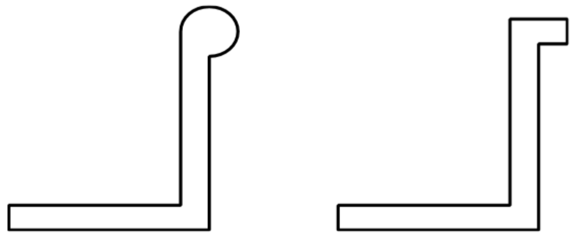


Figure 9.7: A bulb/lip reinforcement.

It's honestly best explained by virtue of an example. The basic outline will be as follows:

1. You calculate the skin buckling stress; this is Euler buckling since it's just a single plate (crippling per definition requires multiple plates because otherwise you couldn't have plates crippling at differing plates can you?).
2. You calculate the stiffener critical stress using the approach of section 9.2.
3. You calculate the effect the stiffener has on the skin.

So, the first two steps aren't too hard as they are just section 9.1 and 9.2 respectively; the third one we'll see in more detail in the next example (after which I'll show a better problem-solving guide, I promise). The important thing to remember is the following: buckling of individual elements is fine. It's very likely that the skin will buckle at some low applied stress; this is fine, no one dies yet, as the stiffeners will simply take up more stress. There may be local buckling in the stiffeners as well (i.e. some of the elements of the stiffener start to cripple), but again, this is fine. It's only problematic if *all* elements of the stiffener have failed. At this point, everything in your skin has buckled/crippled, and if the load is increased more, there's nothing available to soak up more stress. People die then if it's the skin of a wing, and we prefer to avoid that. However, just individual elements buckling isn't that bad, it becomes ugly but no one dies so yay we did something right.

Example 1

Consider the stiffened panel shown in figure 9.8. It has stiffener pitch $b = 200$ mm; the thickness of the skin is 1.5 mm. Instead of the Z-stringers shown, we will use the omega stringers discussed in the second example of section 9.2 to reduce computations. Compute the buckling stress of the stiffened panel (note: panel refers to the skin + stiffeners, the skin refers to merely the bare skin without any stiffeners on it).

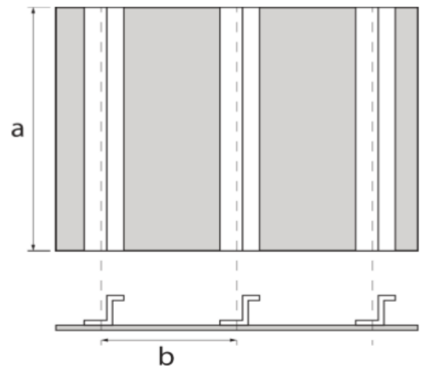


Figure 9.8: Example 1.

First, we will compute the buckling stress of the bare skin without stiffeners using the following expression:

$$\sigma_{cr} = C\frac{\pi^2 E}{12\left(1-v^2\right)}\left(\frac{t}{b}\right)^2$$

Now, for the skin, what should be our C and our b? Essentially, the stiffeners act as simple supports; we essentially get plates with width $b = 200$ mm (as this is the stringer pitch), and $C = 4$ as follows from that graph. Thus, plugging in the numbers, we get

$$\sigma_{cr} = 4 \cdot \frac{\pi^2 \cdot 72 \cdot 10^9}{12 \cdot \left(1 - 0.334^2\right)} \cdot \left(\frac{1.5}{200}\right)^2 = 15\,\text{MPa}$$

which is really low so if don't add any stiffeners and you fly in this plane you basically know you are going to die. This is why we add stiffeners. We calculated before that those omega stiffeners had a crippling stress of 445.6 MPa, so I won't repeat those calculations here. So, even though the plate can't handle a stress larger than 15 MPa, the stiffener is able to withstand stresses up to 445.6 MPa, meaning the applied stress can be significantly increased beyond 15 MPa before everyone dies. The applied stress that the panel can now withstand follows also from a weighted average of the two stresses, based on the areas of the elements, so e.g. a first order estimate would be

$$\sigma_{cc_{\text{panel}}} = \frac{\sum \sigma A_i}{\sum A_i} = \frac{445.6 \cdot 260 + 15 \cdot 200 \cdot 1.5}{260 + 200 \cdot 1.5} = 215\,\text{MPa}$$

i.e. the panel would be able to withstand an applied stress 215 MPa, up from 15 MPa without stiffeners! We definitely feel much safer in our plane now. However, it gets even better: this is an underestimate of the buckling strength of the panel. The stiffeners have an added benefit, namely that they make the skin close to the stiffeners has been reinforced significantly. This means it only starts buckling once the stiffener cripples (since it's riveted to the stiffener), and thus makes it able to withstand the same stress as the crippling stress of the stiffeners. Essentially what happens to the stress distribution is shown in figure 9.9.

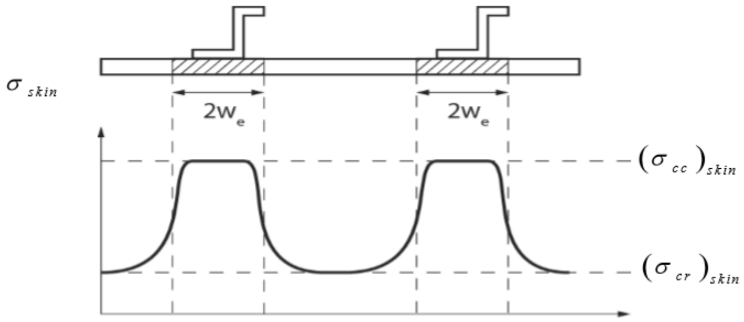


Figure 9.9: Stress distribution in skin due to presence of stiffeners.

In some small area, denoted by $2w_e$ (where w_e is called the effective sheet width), the skin is able to withstand the same stress as the crippling stress of the stiffeners. This significantly improves the buckling strength of the panel, as more area is able to withstand a high applied stress, thus increasing the weighted average. All we need to do is find that width $2w_e$.[a] Now, we know that the buckling stress of the plate will be the same as the crippling stress of the stiffener, in other words

$$\left(\sigma_{cr}\right)_{2w_e} = \left(\sigma_{cc}\right)_{\text{stiffener}}$$

We know the formula for the buckling stress in a plate, so that we can write

$$\begin{aligned} \left(\sigma_{cr}\right)_{2w_e} &= \left(\sigma_{cc}\right)_{\text{stiffener}} \\ C\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{2w_e}\right)^2 &= \left(\sigma_{cc}\right)_{\text{stiffener}} \end{aligned}$$

We can rewrite this to an explicit expression for $2w_e$:

EFFECTIVE SHEET WIDTH

The effective sheet width $2w_e$ is given by

$$2w_e = t\sqrt{\frac{C\pi^2}{12\left(1-\upsilon^2\right)}}\sqrt{\frac{E}{\left(\sigma_{cc}\right)_{\text{stiffener}}}} \tag{9.9}$$

Now, one issue is, what is the value of C we need to take here? The thing is, if $2w_e$ is small in comparison to the stiffener width, then you need to take SSCC, for which $C = 6.98$: the two edges of the effective sheet width will be very much clamped. If the effective sheet width is relatively large, then you need to take SSSS, i.e. $C = 4$, because then the ends of the effective sheet width are located quite a distance away from the stiffener and thus are merely simply supported by the rest of the skin instead. Note that the torsional rigidity also matters: if the stiffener is not torsionally rigid (which is the case if the stiffener width is small), then you should also take SSSS since the stiffeners do not prevent rotation. If the stiffener is torsionally rigid (which is the case when the stiffener width is large), you should take SSCC cause then they indeed prevent rotation. As a general rule of thumb, you can use figure 9.10, where the value of C has been plotted against b/t, where b is the stiffener pitch.

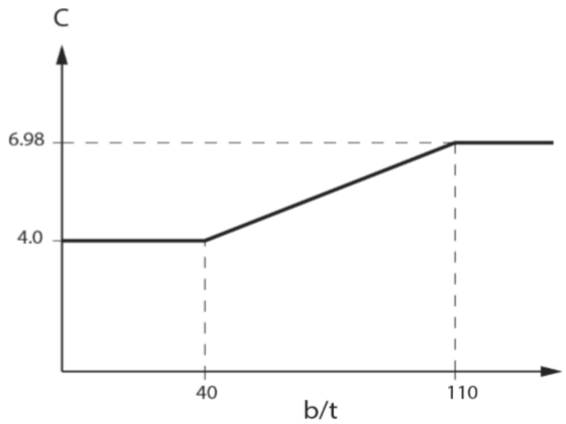


Figure 9.10: Rule of thumb.

[a] If you're thinking, why is it $2w_e$ and not simply w_e? No one knows.

Going back to this example: the omega stiffeners form a closed section together with the skin, making it torsionally very stiff. Thus, we should use SSCC, and thus $C = 6.98$. This leads to

$$\begin{aligned}
2w_e &= t\sqrt{\frac{C\pi^2}{12\left(1-v^2\right)}}\sqrt{\frac{E}{\left(\sigma_{cc}\right)_{\text{stiffener}}}} \\
&= 1.5\sqrt{\frac{6.98\cdot\pi^2}{12\left(1-0.334^2\right)}}\sqrt{\frac{72\cdot 10^9}{445.6\cdot 10^6}} = 48.4\,\text{mm}
\end{aligned}$$

which is indeed smaller than the stiffener width (which is 64 mm, based on the geometry of the omega stiffener), thus our assumption for C was valid. This also means that the stiffener width that is not supported by the stiffener is merely $200 - 48.4 = 151.6$ mm instead of 200 mm. If you want, you can redo the calculations for the buckling stress in this plate with reduced width b; this yields $\sigma_{cr} = 26.1$ MPa. The weighted average of the stresses now becomes

$$\sigma_{cc_{\text{panel}}} = \frac{\sum_{cc}^{(i)} A_i}{\sum A_i} = \frac{445.6\cdot(260+48.4\cdot 1.5)+26.1\cdot 151.6\cdot 1.5}{260+48.4\cdot 1.5+151.6\cdot 1.5} = 275\,\text{MPa}$$

So, only when the applied stress exceeds 275 MPa our wing breaks and we die! That's a significant improvement over the 15 MPa buckling stress of the original skin, to be precise, a factor 18.3 higher, even though the weight has only gone up by a factor 1.92. So, the specific strength has increased by a factor 9.5, which is enormous. Note that if you did not recompute the buckling stress for the skin based on the smaller width that is not supported by the stiffener, your result would only be 1.2% lower than what we got now. So, this iteration is quite often omitted.

FINDING THE BUCKLING STRESS OF A PANEL

1. Find the buckling stress of the bare skin using

$$\sigma_{cr} = C\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b}\right)^2$$

with b the stiffener pitch, and take $C = 4$ as we will assume it is simply supported on all sides.
2. Find the crippling stress of the stiffener, using the method of section 9.2.
3. Find the effective sheet width $2w_e$ using

$$2w_e = t\sqrt{\frac{C\pi^2}{12\left(1-\upsilon^2\right)}}\sqrt{\frac{E}{\left(\sigma_{cc}\right)_{\text{stiffener}}}}$$

For C, take 6.98 if the stiffener pitch is large (use the rule of thumb shown in figure 9.10; if the stiffener is torsionally stiff or if the crippling stress of the stiffener is close to the yield stress. Take $C = 4$ else. Most important is that you justify your choice.
4. If wanted, recompute the buckling stress of the skin, using

$$\sigma_{cr} = C\frac{\pi^2 E}{12\left(1-\upsilon^2\right)}\left(\frac{t}{b-2w_e}\right)^2$$

5. Compute the buckling stress of the panel using

$$\sigma_{cc_{\text{panel}}} = \frac{\left(A_{\text{stiffener}} + 2w_e t_{\text{skin}}\right)\left(\sigma_{cc}\right)_{\text{stiffener}} + \left(b-2w_e\right)t_{\text{skin}}\sigma_{cr}}{A_{\text{stiffener}} + bt_{\text{skin}}}$$

where b is the stiffener pitch.

Now, it is important to consider, what design variables are important in our design of a panel, and how do they influence things? There are a lot of things you can think about:

- Thickness of the skin: the thicker the skin, the harder it is to buckle. However, increasing the thickness obviously significantly increases the weight of the skin. Furthermore, since the skin will not carry that much load any way, it is questionable whether this is an efficient way of increasing the buckling stress of the panel.
- Material choice: stiffer materials (higher E-modulus) are harder to buckle, although they typically are also heavier. However, having a higher E-modulus is certainly beneficial as shown by the equations in this chapter. Furthermore, for various materials, the Poison ratio υ may also differ, also influencing results.
- Stiffener pitch: the farther away the stiffeners, the less of a reinforcement they provide, which obviously is detrimental for performance.
- Stiffener shape: if the stiffener is torsionally stiff (e.g. by creating a closed section in combination with the skin), then this reinforces the skin better by creating a clamped support for the skin in the vicinity of the stiffener, meaning that the effective sheet width becomes larger, thus increasing the stress-bearing capabilities of the panel.
- Stiffener shape: the stiffener should have a shape such that the crippling stress is high. This can be achieved by having many corners in the stiffener, or by adding small limps and bulbs to the stiffener, as these will delay crippling.
- Cross-sectional area of stiffener: the higher the cross-sectional area of the stiffener, the more load it can soak up, increasing the critical stress of the panel as that is the weighted average of the critical stress of each element in the panel (weighted by area). However, a higher cross-sectional area inevitably leads to a higher weight, obviously.
- Radius of gyration of the stiffener: the higher the radius of gyration of the stiffener, the less slender it is for a fixed length. Less slender stiffeners are less prone to crippling/Euler buckling as follows from the Johnson-Euler approximation, making it better able to withstand compressive stresses.

These are just some of the things you need to take into account. The best way to think of stuff that you need to take into account is to just look at all the formulas you used and see what things mattered then: these are obviously things that you should take into account when designing your plate.

Two final remarks regarding these chapters (in general about chapters 8, 9, 4 and 5, actually): note that we *never* ever use the thin-walled assumption. We just don't. Similarly, we never idealize cross-sections. Just in case one of you people is stupid enough to think so.

4 Energy methods part I: work

So yeah these two chapters are basically totally new to me. I did kinda have it last year, but this year everything is explained differently (and the stuff with trusses is totally new).

First, you have to see these two chapters kinda like one big chapter, but for the sake of the structure I've split in two because otherwise the sections would get too long. The first chapter is about work, whereas the second is about energy. They are very similar, but the equations are slightly different and before you spend hours thinking about why some equations are divided by 2 and others not (even if they look almost exactly the same), it's because the equations in this chapter are based on the concept of work, and the equations in the next are based on the concept of energy.

Now, sometimes in life you find yourself stuck in a job interview for dynamics T.A. and suddenly you are asked to explain what energy conceptually means, what would you say? Turns out that the correct answer is that "energy is the ability to perform work". This is how these two are related. In this chapter, we will analyse work.

4.1 Introduction to work

Remember that work is defined as

$$W = \int \mathbf{F} \cdot d\mathbf{s}$$

When they are aligned, this is simply

$$W = \pm \int F\,ds$$

where you get a minus sign if F acts in opposite direction of the displacement s.

Now, how can we apply this? Remember that if a body is in equilibrium, then the internal forces act in opposite direction of the external forces (the ones that are applied to a structure). In other words, $\mathbf{F}_i = -\mathbf{F}_e$, or $\mathbf{F}_i + \mathbf{F}_e = 0$, where $\mathbf{F}_i$ is the vector sum of the internal forces and $\mathbf{F}_e$ the vector sum of the external forces. Then, it should not come as a surprise that if a body is in equilibrium, then also

EQUILIBRIUM OF WORK

If a deformable body is in equilibrium, then

$$W_{\text{tot}} = W_i + W_e = 0 \tag{4.1}$$

Let us see an example of how this can be applied for a very elementary case:

Example 1

Consider a simply supported beam as shown in figure 4.1, to which a load W is applied. Compute the support reactions.

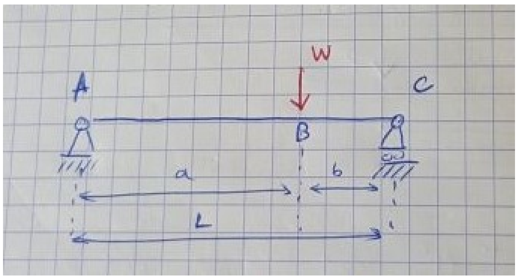


Figure 4.1: A beautiful sketch of a beam.

Of course, any first year student (well maybe not all of them considering that some of them seem to think that gravity acts upwards) is able to solve this by just considering the equilibrium equations. However, we're gonna do it the hard way using virtual work. Obviously, the horizontal reaction force at A will be zero so we won't even consider that one.

Since we have two unknown reaction forces, we must introduce *two* virtual displacements that are independent of each other (I'll explain afterwards what this exactly means). We can use the displacements shown in figure 4.2, where Δ_v and θ_v are proposed as virtual displacements: these displacements are *virtual*, as we know that in real life, these displacements will not occur.

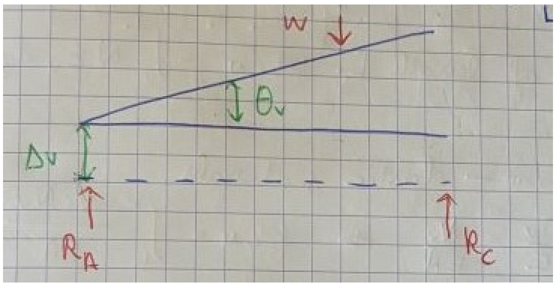


Figure 4.2: Proposed virtual displacements.

The work done by R_a is then $R_a\Delta_v$. The work done by W will be $-W_a\left(\Delta v + \theta_v \cdot a\right)$ (the minus sign is because W points in opposite direction of the displacement; furthermore, the vertical displacement due to the rotation will be $\theta_v \cdot a$ (see the geometry of figure 4.1). The work done by R_c will be $R_c\left(\Delta v + \theta_v \cdot (a+b)\right)$. Thus, the total external work is

$$W_{\text{tot}} = W_i + W_e = R_a\Delta v - W\left(\Delta v + \theta_v \cdot a\right) + R_c\left(\Delta v + \theta_v \cdot (a+b)\right) = 0$$

This can also be written as

$$\Delta v\left(R_a - W + R_c\right) + \theta_v\left(R_c\,(a+b) - W \cdot a\right) = 0$$

This needs to be satisfied for *all* Δv and θ_v. Thus, we cannot say $\Delta_v = 0$, $\theta_v = 0$ problem solved. Instead, we must solve the system of equations

$$\begin{aligned} R_a - W + R_c &= 0 \\ R_c\,(a+b) - W \cdot A &= 0 \end{aligned}$$

which has unknowns R_a and R_c. Solving the second equation yields $R_c = Wa/(a+b)$. Solving the first leads to

$$R_a = W - R_c = W - \frac{Wb}{a+b} = W\left(1 - \frac{a}{a+b}\right) = W\frac{b}{a+b}$$

Thus we have now found our solution.

Now, what did I mean with two independent displacements? Well, suppose you'd be interested in a third support reaction, then you need a third displacement. However, you could not, for example, take the displacement $\Delta_{v,B}$ (virtual displacement at B), as this is already uniquely determined by the combination of Δ_v and θ_v (namely, $\Delta_{v,B} = \Delta_v + \theta_v \cdot L$). Instead, you should come up with a displacement that is 'independent' of the displacements that you already defined. For example, a horizontal displacement would be suitable as it would not be already determined by Δ_v and θ_v.

To be clear: this kinda stuff is really easy as they ask more complicated problems than this even in statics. Nonetheless I'll provide the problem solving guide to make it clear what the general approach is.

DETERMINING REACTION FORCES BY USE OF PRINCIPLE OF VIRTUAL WORK

1. For each reaction force that one is interested in, come up with a virtual displacement such that only the desired reaction forces are performing work.
2. Set up the equation for the total external work. Note that $W_i = 0$.
3. Set the total work equal to 0.
4. Collect terms corresponding to each virtual displacement.
5. Note that the term corresponding to each virtual displacement needs to be 0, and solve the resultant system of equations to find your reaction forces.

So, if you are only interested in one reaction force, you only need *one* virtual displacement, and this displacement should be such that only this reaction force is performing work. If this is not possible (i.e. it will always happen that another reaction force also performs work), you can add more virtual displacements until you've reached the same number of unknown reaction forces and independent virtual forces.

4.2 *Virtual forces and internal work*

Virtual work can also be used to find displacements of a structure that's subject to some forces; this will be analysed in the next section. However, we first need some theory before we commence our analysis of that. The basic idea behind it is the following: remember again that

$$W_{\text{total}} = W_i + W_e = 0$$

Now, consider a structure that's subject to some external forces. This structure will be in equilibrium (we're not doing Dynamics here after all). *This means we don't have to calculate the internal and external work done due to the external forces that are originally applied.* After all, we know $W_i + W_e = 0$ as it is in equilibrium, and believe me or not, but that's sufficient information.

Instead, we'll assume that the structure with the external forces applied is the 'original' situation, and then we'll apply a virtual force at the position where we want to know the deflection. W_e is then simply $P\delta$ where P is the virtual force and δ the deflection. Based on the original loading, we can then compute W_i, and based on $W_i + W_e = 0$ we can then compute δ, as we'll see in the next section.

Now, how do we compute the internal work? Consider first simply the internal work due to an axial load. Suppose we have an actual axial load in a member N_A; if a virtual strain ϵ_v is then imposed upon the structure, the virtual work done by N_A is simply

$$w_{i,N} = \int_L N_A \epsilon_v dx$$

If we assume that the material is linearly elastic (it obeys Hooke's law), we can write

$$\epsilon_v = \frac{\sigma_v}{E} = \frac{N_v}{EA}$$

where N_v is the magnitude of the virtual normal force that's imposed. Thus, we get

$$w_{i,N} = \int_L \frac{N_A N_v}{EA} dx$$

If there are multiple members in a structure (e.g. a truss), we simply get

$$w_{i,N} = \sum \int_L \frac{N_A N_v}{EA} dx$$

where you sum over all of the members. I must admit that this derivation is a bit short through the turn but I doubt they'll ask for a derivation on the exam. The only thing important that was stressed during the lecture was that the principle of virtual work initially does not assume linearly elastic material, and that $w_{i,N} = \int_L N_A \epsilon_v dx$ holds for *all* kinds of materials, since it does not impose any assumption at all. It is only after we assume $\epsilon_v = N_v/(EA)$ that we restrict our analysis to linearly elastic materials. However, in theory, virtual work can be applied to non-linearly elastic materials as well, although you won't be required to perform any calculations with it. This is in contrast with our bff Castigliano's theorem: his theorem is *always* limited to only linearly elastic materials.

You can find the virtual work done by shear forces, bending moments and torsion in very similar ways. This leads to the following results:

INTERNAL VIRTUAL WORK

The internal virtual work due to a virtual normal force N_v, virtual shear force S_v, virtual bending moment M_v, virtual torque T_v and virtual rotation θ_v are respectively

$$w_{i,N} = \sum \int_L \frac{N_A N_v}{EA} dz \tag{4.2}$$

$$w_{i,S} = \sum \beta \int_L \frac{S_A S_v}{GA} dz \tag{4.3}$$

$$w_{i,M} = \sum \int_L \frac{M_A M_v}{EI} dz \tag{4.4}$$

$$w_{i,T} = \sum \int_L \frac{T_A T_v}{GJ} dz \tag{4.5}$$

$$w_{i,h} = \sum M_A \theta_v \tag{4.6}$$

Okay yeah I also don't really get the last equation but I've never seen it used so I don't really care about it. Furthermore, you may think, what the hell is that β for the shear force one? It's a form factor that'd be given to you; it takes into account that the shear stress is not uniform throughout the cross-section.

The four 'main' equations are all very similar I think. Only the denominator really changes based on the type of loading but even makes a fair bit of sense imo. Furthermore, note that the external work is simply

EXTERNAL VIRTUAL WORK

The external virtual work due to a virtual normal force N_v, virtual shear force S_v, virtual bending moment M_v, virtual torque T_v and distributed load $w(z)$ is given by

$$w_e = \sum \left[P\delta_z + S_x\delta_x + S_y\delta_y + M_x\theta_y + M_y\theta_x + T\phi + \int_L w_x(z)\,\delta_x dz + \int_L w_y(z)\,\delta_y dz \right] \tag{4.7}$$

Time for some applications now.

4.3 Applications of the principle of virtual work

Example 1

Determine the vertical deflection of the free end of the cantilever beam shown in figure 4.3.

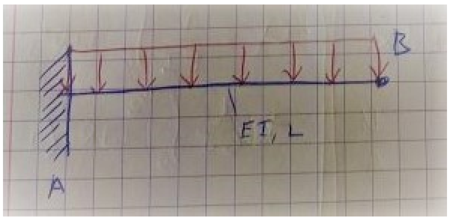


Figure 4.3: Example 1.

The approach is not that difficult. We have to apply a virtual force at the end of the beam. This force may have any nonzero value, so let us just take 1 N, i.e. a unit load. This leads to the sketch shown in figure 4.4.

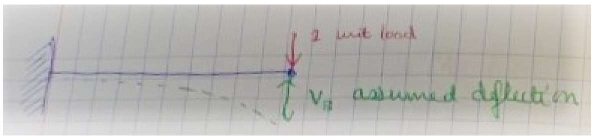


Figure 4.4: Example 1.

The application of this virtual load leads to a bending moment, so we shall use

$$W_{i,M} = \int_0^L \frac{M_A M_v}{EI} dz$$

where z is measured from the wall in the direction of B. Using our problem-solving guide of chapter 16 (let us assume positive direction is downwards here), we then have

$$\begin{aligned} M_A(z) = w \cdot (L-z) \cdot -\frac{L-z}{2} &= -w\frac{(L-z)^2}{2} \\ M_v(z) = 1 \cdot -(L-z) &= -(L-z) \end{aligned}$$

Thus,

$$\begin{aligned} W_{i,M} &= \int_0^L \frac{-w\frac{(L-z)^2}{2} \cdot -(L-z)}{EI} dz = \int_0^L \frac{w(L-z)^3}{2EI} dz \\ &= -\frac{w}{2EI}\left[\frac{1}{4}(L-z)^4\right]_0^L = \frac{wL^4}{8EI} \end{aligned}$$

Since $W_e = 1 \cdot v_B$, we have $v_B = wL^4/8EI$.

The hardest thing is determining the moments which is always hard simply due to the minus signs etc. Still, if you follow the problem-solving guide of chapter 16 rigorously to the letter, you'll be fine. Furthermore, you may have wondered, but is there not a shear force as well? Should we not take that into account? Yes, theoretically speaking you should have. However, note that $S = dM/dz$: this means that the deflection due to shear would only be dependent on L^2 rather than L^4 (as we differentiated with respect to distance twice (once for both S_A and S_v)). Thus, assuming L is large, it can be neglected, and this assumption usually holds up.

APPLYING VIRTUAL WORK TO FIND DISPLACEMENTS OF BEAMS

1. Apply a force/couple in the direction in which you want to find the displacement. Let the magnitude of this force/couple be
2. Find the bending moment and torque distribution due to the actual loading and due to the virtual loading. Note that discontinuities in either functions are possible; in this case you have to set up functions each covering different parts of the beam.
3. Find the internal work by the virtual force by computing

$$w_i = \int_L \frac{M_A M_v}{EI} dz + \int_L \frac{T_A T_v}{GJ} dz$$

over the entire length of the beam; this may require multiple integrations for the different parts of the beam if there are bending moment discontinuities.
4. Solve $w_e = w_i$ for the desired deflection.

If you forgot how to set up the bending moment function, remember this problem solving guide:

FINDING THE INTERNAL MOMENTS

1. Make a cut somewhere in the beam.
2. Decide which side is easier to evaluate (probably the one with the fewest reaction forces).
3. Use sign convention of figure 16.6, that is:
4. If you want to evaluate the side in the negative z-direction:
   - Then $-M_x$ (note the minus sign!) is given by multiplying all forces in y-direction with the distances and summing them. Minus signs are added as follows:
     - If the force acts in negative direction, a minus sign should be added[a].
   - $-M_y$ (note the minus sign!) is found in exactly the same way.
5. If you want to evaluate the side in the positive z-direction:
   - Then M_x is given by multiplying all forces in y-direction with the distances and summing them. Minus signs are added as follows:
     - If the force acts in negative direction, a minus sign should be added.
     - As you have to travel *from* the point where the force is applied *towards* the point where you evaluate the moments in negative z-direction, you have to add another minus sign in front of each term designating the distance. If there is no distance in play (e.g. when there's a couple working on your beam), then you do not add a minus sign.

     Two minus signs obviously cancel out.
   - M_y is found in exactly the same way.

[a]Though don't be a stupid kid: if you already had to include a minus sign in the expression of the force due to the fact that it acts in negative direction, then you don't have to add another minus sign now obviously.

Please note that above problem-solving guide holds for simple beams. For trusses consisting of multiple members, you need to apply the guides of next chapter. Let me show a few examples to clear it a bit up.

Example 2

Consider the curved beam shown in figure 4.5. On the left, a front-view of the beam has been shown; on the right, a side-view has been shown. A load P is applied in the direction shown.

- Compute the vertical deflection at B.
- Compute the slope of the beam at B.

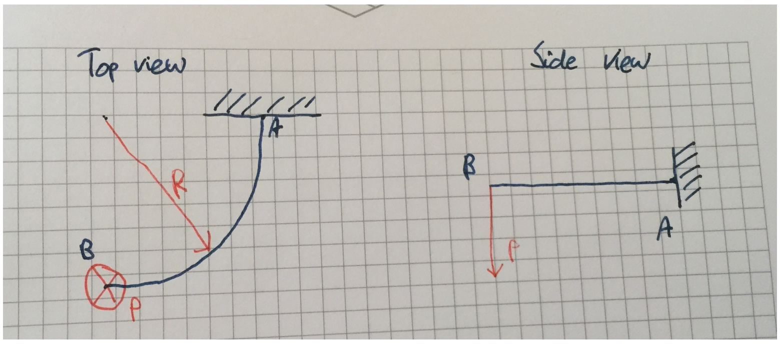


Figure 4.5: Example 1.

This question was literally in last year's resit. It was made so badly that they decided that of the 25 points it was worth (100 points needed for a 10), 20 points would count for bonus (i.e. you'd only need 80 points in total to get a 10). So, let's see if you can understand it.

First, the second biggest difficulty I had with this question (I did the resit) is *which direction is the fucking vertical direction if they provide two views*? Honestly, vertical direction is different in the left picture compared to the right. The largest difficulty I'll mention after this example has been completed.

Now, from the way the force P points into the paper at B in the top view, it is apparent that a torque will be created around A. From the way the force P points in the side view, it is also apparent that a bending moment will be created.

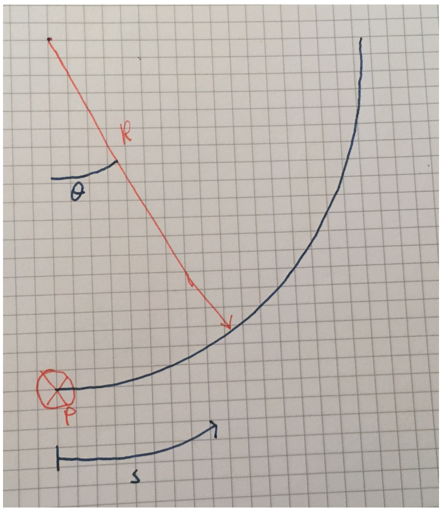


Figure 4.6: Example 1.

This torque and bending moment can be computed from figure 4.6. If you look well enough, you see that the torque created by P is equal to

$$T_A(\theta) = P \cdot R \sin\theta$$

measure θ as shown in figure 4.6. Don't see why? We start measuring the internal torque $T(\theta)$ from point B, and then go along the beam towards the root. In the top view, the torque is created by the horizontal distance between the application of P and point A. In other words, the internal torque depends on the horizontal distance between B and the point you're evaluating. Thus, the internal torque is P times the horizontal distance for the corresponding value of θ, which is $R\sin(\theta)$.

For the bending moment, we need to take the vertical distance between B and the point we're evaluating at. This distance will be $R(1-\cos(\theta))$. Thus, we have

$$M_A(\theta) = P \cdot R(1-\cos\theta)$$

Now, which direction is vertical? They meant vertical as vertical in the side view, i.e. in the same direction as P. Thus, we apply a virtual load with magnitude 1 in the same direction as P, meaning that we have

$$\begin{aligned} T_v(\theta) &= R\sin\theta \\ M_v(\theta) &= R(1-\cos\theta) \end{aligned}$$

Then, we compute

$$w_i = \int_L \frac{M_A M_v}{EI} ds + \int_L \frac{T_A T_v}{GJ} ds$$

Note that we have found all the moments and torques as functions of θ; thus, let's use $ds = Rd\theta$ to integrate:

$$w_i = \int_L \frac{M_A M_v}{EI} Rd\theta + \int_L \frac{T_A T_v}{GJ} Rd\theta$$

Let's first perform the first integral:

$$\begin{aligned} \int_L \frac{M_A M_v}{EI} Rd\theta &= \frac{1}{EI}\int_L PR(1-\cos\theta)\cdot R(1-\cos\theta)\,Rd\theta \\ &= \frac{PR^3}{EI}\int_L (1-\cos\theta)^2\,d\theta = \frac{PR^3}{EI}\int_0^{\pi/2}\left[1-2\cos\theta+\cos^2\theta\right]d\theta \\ &= \frac{PR^3}{EI}\int_0^{\pi/2}\left[1-2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right]d\theta = \frac{PR^3}{EI}\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \\ &= \frac{PR^3}{EI}\left[\frac{3}{4}\pi-2\right] \end{aligned}$$

where we used $\cos^2(x) = 1/2\cos(2x)+1/2$. Then, for the second integral, we get

$$\begin{aligned} \int_L \frac{T_A T_v}{GJ} Rd\theta &= \frac{1}{GJ}\int_L PR\sin\theta\cdot R\sin\theta\cdot Rd\theta \\ &= \frac{PR^3}{GJ}\int_0^{\pi/2}\sin^2\theta d\theta = \frac{PR^3}{GJ}\frac{\pi}{4} \end{aligned}$$

where I made use of a standard integral. You can work it out yourself by use of $\sin^2\theta = 1/2-\cos(\theta)/2$.

In other words, we have

$$w_i = \frac{PR^3}{GJ}\frac{\pi}{4} + \frac{PR^3}{EI}\left[\frac{3}{4}\pi-2\right]$$

Furthermore, $w_e = 1\delta_B$, so

$$\begin{aligned} w_e &= w_i \\ \delta_B &= \frac{PR^3}{GJ}\frac{\pi}{4} + \frac{PR^3}{EI}\left[\frac{3}{4}\pi-2\right] \end{aligned}$$

and there you go.

For the slope at B, you impose a unit moment at B, meaning that

$$\begin{aligned} M_v &= 1 \\ T_v &= 0 \end{aligned}$$

So, we merely have to compute

$$\begin{aligned} w_i &= \int_L \frac{M_A M_v}{EI} R d\theta = \frac{1}{EI} \int_0^{\pi/2} PR(1-\cos\theta) R d\theta = \frac{PR^2}{EI} \int_0^{\pi/2} (1-\cos\theta] d\theta \\ &= \frac{PR^2}{EI} [\theta - \sin\theta]_0^{\pi/2} = \frac{PR^2}{EI} \left(\frac{\pi}{2} - 1\right) \end{aligned}$$

Now, $w_e = 1\theta_B$, so

$$\begin{aligned} w_e &= w_i \\ \theta_B &= \frac{PR^2}{EI} \left(\frac{\pi}{2} - 1\right) \end{aligned}$$

In hindsight, it's not that difficult, once you actually see what's going on (which is unfortunately very hard, I'll admit). Btw, the thing that made this question so incredibly hard for us in 2016-2017 is not because we're incompetent bunch of idiots (honestly, you people win that competition), but because we were not taught this material beforehand. We did not learn about virtual work at all. We only learned about Castigliano (though more extensively than you guys are studying it this year). So how could they expect us to solve it? The entire resit was a joke tbh btw, but that aside[1].

Anyway, let's do another example, this time slightly easier to understand what's going on.

Example 3

The tubular steel post shown in figure 4.7 supports a load of 250 N at the free end C. The outside diameter of the tube is 100 mm, and the wall thickness is 3 mm. Neglecting the weight of the tube find the horizontal deflection at C. The modulus of elasticity is 206 000 MPa.

[1] Literally every question had issues. Question 1 was about a beam in bending, but they did not tell us the length of the beam which was rather vital information. Question 2 was this question that was so difficult they had to make it count as bonus points as otherwise the passing rate would have been 7% (I'm serious, this would have been the passing rate). Question 3 did not contain an actual question. They did give us a hint to help us, but they did not actually include the question itself in the exam. Questions 4 and 5 were a total mess as well: as you know by now, Dr. Abdalla left the university after the final exam. Under him, the final exam contained five questions, each worth 25 points, but you only needed to get 100 points to receive a 10. However, you were allowed to attempt all 5 problems, and you could get a 12.5 in the end as a result. This was really, really helpful imo. Now, the lecturers who took over the course for the resit assured us that the format of the exam would be *exactly* as the written exam given by Abdalla. That gives the impression that the format would be *exactly* the same as before, with no changes. However, on the resit itself, they wrote on the front page that you were only allowed to do either question 4 or 5; if you did both, only question 4 would count (which is in stark contrast to Abdalla's exam, where you had complete freedom in the questions you made). Apparently, a lot of people don't bother to read front pages on exams, so they decided to do both questions 4 and 5, because well they were really similar so it was easy to score points since if you knew how to do one, you knew how to do the other as well. So, a lot of people lost a lot of points because they could not read. They started complaining, and initially the lecturers were just like lol you're incompotent we're not gonna compensate if you did both question 4 and 5 and therefore did not have time to do the other ones. Weeks later it turns out that they did compensate the people who did both question 4 and 5 completely and that they decided to give them full marks if they did both questions. The people who *did* read correctly and knew not to do question 4 and 5 both were now angry, as they would have liked to do both question 4 and 5 as well since those were the easiest questions of the exam. Thus, they got compensated in the end as well. In conclusion, in the end you could get a 15.625 as final grade for an exam for which there were no bonus points intended. Absolutely beautiful drama show that was. Then in the Summer there was an additional resit for which the passing rate was 5%.

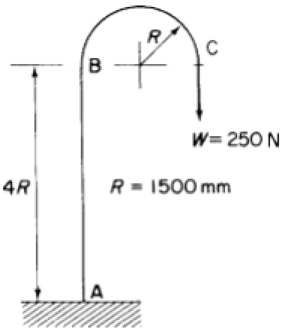


Figure 4.7: Example 3.

We want to find the horizontal displacement at C, thus we impose a virtual load of magnitude 1 in horizontal direction at C (let's assume it points outward). Now, consider figure 4.8 for our symbol definitions.

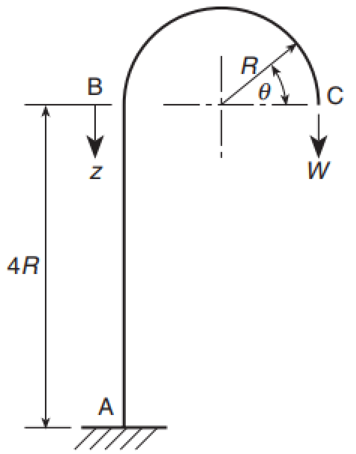


Figure 4.8: Example 3.

Let's first find the moment distribution due to W. In CB we have (clockwise positive)

$$M_A = W\left(R - R\cos\theta\right)$$

and in BA we simply have $M_A = W \cdot 2R$ (clockwise positive). Then let's find the moment due to virtual load. In CB, we have (clockwise positive)

$$M_v = -1 \cdot R\sin\theta$$

and in BA we simply have $M_v = 1 \cdot z$ (clockwise positive). Thus, performing the integration for

$$w_i = \int_L \frac{M_A M_v}{EI} dz = \int_{CB} \frac{M_A M_v}{EI} dz + \int_{BA} \frac{M_A M_v}{EI} dz$$

For the integration over CB it is easier to use cylindrical coordinates, i.e. $dz = Rd\theta$, so that we get for the first integral:

$$\begin{aligned}\int_{CB} \frac{M_A M_v}{EI} dz &= \frac{1}{EI}\int_0^\pi W\left(R - R\cos\theta\right)\cdot -R\sin\theta \cdot Rd\theta = \frac{-WR^3}{EI}\int_0^\pi \left(1-\cos\theta\right)\sin\theta d\theta \\ &= \frac{-WR^3}{EI}\int_0^\pi \left[\sin\theta - \cos\theta\sin\theta\right] d\theta = \frac{WR^3}{EI}\left[-\cos\theta - \sin^2\theta\right]_0^\pi = \frac{-2WR^3}{EI}\end{aligned}$$

For the second integral, we simply get

$$\int_{BA} \frac{M_A M_v}{EI} dz = \frac{1}{EI} \int_0^{4R} 2WR \cdot z dz = \frac{WR}{EI} \left[z^2\right]_0^{4R} = \frac{16WR^3}{EI}$$

Thus, we have

$$\begin{aligned} w_i &= w_e \\ 1\delta_C &= -\frac{2WR^3}{EI} + \frac{16WR^3}{EI} \\ \delta_C &= \frac{14WR^3}{EI} \end{aligned}$$

Here, $W = 250\,\mathrm{N}$, $R = 1500\,\mathrm{mm}$, $E = 206\,000\,\mathrm{MPa}$, $I = \pi/64\left(100^4 - (100 - 2 \cdot 3)^4\right) = 1\,076\,246\,\mathrm{mm}^4$. Plugging this in yields $\delta_C = 53.3\,\mathrm{mm}$.

5 Energy methods part II: energy

Okay yeah the problem is that in the lecture, this chapter was only discussed with regards to trusses. However, I asked Ir. Melkert and he said that you need to know more than what was discussed in the lectures, as you also need to be able to apply it to stuff that's not a truss. This kinda sucks as it adds a lot of material to study and more importantly I'm sacrificing even more of my own time as it's starting to look like I had to write close to 60-70 pages of summaries and solution manuals this Christmas break even though I'm not doing the exam myself.

5.1 Strain energy and complementary energy

Just as a warning, this section is just one big derivation of which you barely have to know anything, except for the red box and the final paragraph in the end. So don't bother studying it too much.

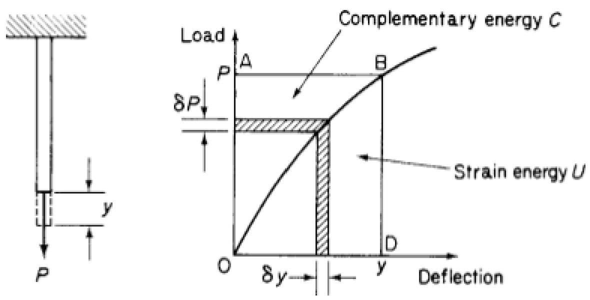


Figure 5.1: Load-deflectio curve for a nonlinearly elastic member.

Consider figure 5.1: if we subject a member to simple tension, we get the load-deflection curve shown on the right. Note that we do not assume linearly elastic just yet.

The complementary and strain energy are then defined as shown in the right of figure 5.1; mathematically these are defined as

COMPLEMENTARY AND STRAIN ENERGY

The complementary energy is defined as

$$C = \int_0^P y\,dP \tag{5.1}$$

The strain energy is defined as

$$U = \int_0^y P\,dy \tag{5.2}$$

Now, suppose that the curve in the right of figure 5.1 is represented by

$$P = by^n \tag{5.3}$$

where b and n are constants. Then,

$$dP = nby^{n-1}dy \tag{5.4}$$

by virtue of implicit differentiation. Now, note that this can also be rewritten as

$$dP = \frac{nby^n}{y}dy = \frac{nP}{y}dy \tag{5.5}$$

since $by^n = P$ (equation (5.3)) but we also have $y = (P/b)^{1/n}$ from equation (5.3), meaning that

$$dP = \frac{nP}{(P/b)^{1/n}}dy \tag{5.6}$$

$$Pdy = \frac{1}{n}\left(\frac{P}{b}\right)^{1/n} dP \tag{5.7}$$

Substitute equation (5.7) into equation (5.1) to get

$$U = \int_0^P dy = \frac{1}{n}\int_0^P \left(\frac{P}{b}\right)^{1/n} dP$$

Furthermore, substitute equation (5.4) into equation (5.2) to get

$$C = \int_0^P ydP = n\int_0^y by^n dy$$

This means that we have

$$\frac{dU}{dy} = P, \qquad \frac{dU}{dP} = \frac{1}{n}\left(\frac{P}{b}\right)^{1/n} = \frac{1}{n}y$$

$$\frac{dC}{dP} = y, \qquad \frac{dC}{dy} = bny^n = nP$$

In fact, when $n = 1$, i.e. when it's linearly elastic, then

$$\frac{dU}{dy} = \frac{dC}{dy} = P$$

$$\frac{dU}{dP} = \frac{dC}{dP} = y$$

which are Castigliano's first and second theorem respectively. He wrote it in his master's thesis and then died at an early age. These are incredibly power relations of which we'll see the use later on. The entire point of this section was to show one important assumption: Castigliano's theorems only hold when the material is linearly elastic. If it's not, then you *have* to use the og equations (equations (5.1) and (5.2)). I genuinely do not know whether you are expected to be able to do calculations for non-linearly elastic materials; this is normally the time for me to go through all of the old exams and see whether it's been asked before, but obviously that doesn't work this year. So, if someone knows for sure (e.g. because they sent an email), I'd greatly appreciate to know the answer.

5.2 The principle of the stationary value of the total complementary energy

Consider an elastic system supporting forces $P_1, P_2, ..., P_n$ producing real displacements $\Delta_1, \Delta_2, ..., \Delta_n$. If we then impose virtual forces $\delta P_1, \delta P_2, ..., \delta P_n$ on the system acting through the real displacements, then the total virtual work done by the system is (this is basically chapter 4)

TOTAL VIRTUAL WORK DONE BY A SYSTEM

$$-\int_{\text{vol}} ydP + \sum_{r=1}^{n} \Delta_r \delta P_r \tag{5.8}$$

The minus sign comes from the fact that the internal forces point in opposite direction of the real displacements. From the principle of virtual work, above equation must equal 0.

Now, consider equation (5.1) again. If you look very closely and really squish your eyes, you can compare it with equation (5.1). Then you see that if we define the internal and external complementary energy as

INTERNAL AND EXTERNAL COMPLEMENTARY ENERGY

The internal complementary energy is defined as

$$C_i = \int_{\text{vol}} \int_0^P y \, dP \tag{5.9}$$

The external complementary energy is defined as

$$C_e = -\sum_{r=1}^{n} \Delta_r P_r \tag{5.10}$$

then

PRINCIPLE OF THE STATIONARY VALUE OF THE TOTAL COMPLEMENTARY ENERGY

$$\delta \left(C_i + C_e \right) = 0 \tag{5.11}$$

after all, if you put a differential δ in front of equations (5.9) and (5.10), then add them together[1], you get equation (5.8) again.

If you're like, where does the additional integral come from for C_i (it wasn't there in equation (5.1) after all)? Why is there $\int_{\text{vol}}$ in front? The reason is that the book is inconsistent. In above equation, we consider $\int_0^P y \, dP$ to be the internal complementary energy of an infinitesimal amount of material[2]. Thus, to find the complementary energy of the structure, you need to integrate over the whole body as well:

$$C_i = \int_{\text{vol}} \delta C_i dV \int_{\text{vol}} \int_0^P y \, dP \, dV$$

but they leave the dV out for some reason[3]. However, the book is a bit inconsistent about when to include the volume integral and when not so I'll indicate every time we include it and when not. In general, if a beam is simply axially loaded, it's allowed to simply use equation (5.1) for the internal complementary energy as everything is pretty much constant. However, if the beam is under bending, then you can't as the stress is not constant throughout the material, thus you need to use the equation (5.9).

The summation $C_i + C_e$ is referred to as total complementary energy and is denoted by simply C:

TOTAL COMPLEMENTARY ENERGY

The total complementary energy is given by

$$C = C_i + C_e = C_i = \int_{\text{vol}} \int_0^P y \, dP + C_e = -\sum_{r=1}^{n} \Delta_r P_r \tag{5.12}$$

I'll repeat again: we have $\delta C = \delta \left(C_i + C_e \right)$. This is a very important principle and is called the principle of stationary value of the total complementary energy. It means that no matter what you differentiate C with respect to, it should always be zero. We'll apply it various shapes and forms over the next sections. Please note: *this is not Castigliano's theorem yet*. It's called the principle of stationary value of the total complimentary energy.

[1] Note that the integral then disappears.
[2] Whereas in equation (5.1) we took the complementary energy of the whole body.
[3] To be honest, this entire book is more often a bit loose with mathematical rules, so I think this is just another of those instances.

5.3 Finding deflections

In this section, we'll analyse how to find deflections in trusses first. Afterwards, we'll try to find deflections in simple beams (i.e. stuff that's not a truss).

5.3.1 Stuff that's a truss

If we have a truss like we often did in statics, then all members are purely axially loaded. Then equation (5.12) simplifies a bit

$$C = \sum_{i=1}^{k} \int_0^{F_1} \lambda_i dF_i - \sum_{r=1}^{n} \Delta_r P_r$$

where λ_i is the deflection of the ith member and F_i the force in the ith member. We can simply use equation (5.1) as it merely axially loaded. Now, the deflection of the ith member is simply

$$\lambda_i = \frac{F_i L_i}{E_i A_i}$$

thus in fact we'll use

TOTAL COMPLEMENTARY ENERGY OF A TRUSS

The total complementary energy of a truss consisting of k members is given by

$$C = \sum_{i=1}^{k} \int_0^{F_1} \frac{F_i L_i}{E_i A_i} dF_i - \sum_{r=1}^{n} \Delta_r P_r \tag{5.13}$$

Now, we can use the principle of stationary value of the total complementary energy: no matter with what we differentiate C to, the result should be zero. Thus, suppose we want to find Δ_j, i.e. the deflection caused by load P_j, we can simply differentiate with respect to P_j, to obtain

TOTAL COMPLEMENTARY ENERGY OF A TRUSS

The total complementary energy of a truss consisting of k members is given by

$$\frac{\partial C}{\partial P_j} = \sum_{i=1}^{k} \frac{F_i L_i}{E_i A_i} \frac{\partial F_i}{\partial P_j} - \Delta_j = 0 \tag{5.14}$$

as the partial derivative of the sum will simply be merely Δ_j. This equation can be solved for Δ_j straightforwardly if the left term is known.

It's best explained with an example, to be honest, so let's do one now.

Example 1

Calculate the vertical deflection of the point B and the horizontal movement of D in the pin-jointed framework shown in the top-left part of figure 5.2. All members are linearly elastic and have cross-sectional areas of 1800 mm^2. E for the material of the members is 200 000 MPa.

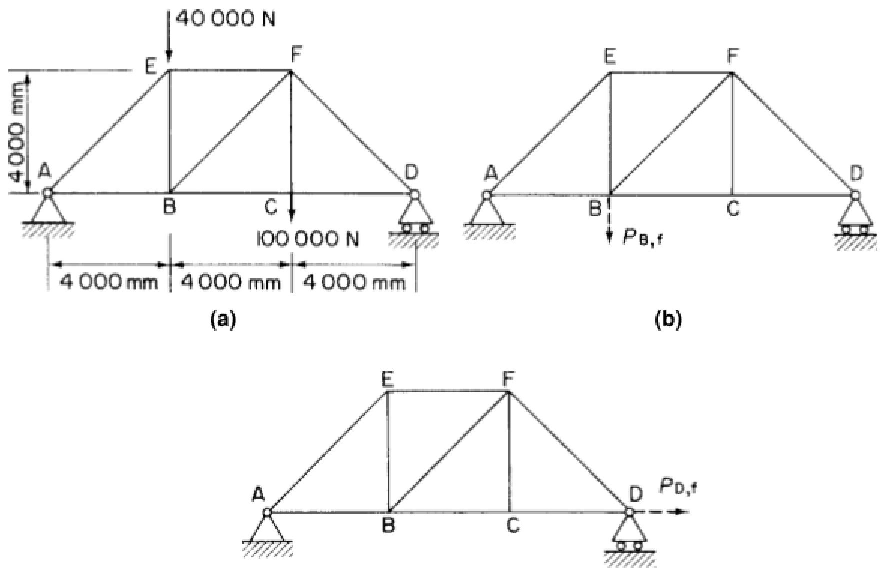


Figure 5.2: Example 1.

In general, we have that the total complimentary energy is given by

$$C = \sum_{i=1}^{k} \int_0^{F_1} \frac{F_i L_i}{E_i A_i} dF_i - \sum_{r=1}^{n} \Delta_r P_r$$

To find the requested deflections, we impose a vertical fictitious load $P_{B,f}$ at B as shown in part (b) of figure 5.2; we also impose a horizontal fictitious load $P_{D,f}$ at D as shown in part (c) of figure 5.2. To find Δ_B we then differentiate C with respect to $P_{B,f}$, since we know that the total complimentary energy is stationary:

$$\frac{\partial C}{\partial P_{B,f}} = \sum_{i=1}^{k} \frac{F_i L_i}{E_i A_i} \frac{\partial F_i}{\partial P_{B,f}} - \Delta_B = 0$$

Thus, the deflection at B is simply

$$\Delta_B = \sum_{i=1}^{k} \frac{F_i L_i}{E_i A_i} \frac{\partial F_i}{\partial P_{B,f}}$$

L_i, E_i and A_i are all given to us. Furthermore, E_i and A_i are constant for all members, thus we in fact have

$$\Delta_B = \frac{1}{EA} \sum_{i=1}^{k} F_i L_i \frac{\partial F_i}{\partial P_{B,f}} \tag{5.15}$$

In a similar fashion,

$$\Delta_D = \frac{1}{EA} \sum_{i=1}^{k} F_i L_i \frac{\partial F_i}{\partial P_{D,f}} \tag{5.16}$$

Now we must compute F_i for the loading, including both the original loading and the fictitious loading. I won't include the entire calculation here since it requires a lot of sketches and calculations and I don't really have time for that tbh. However, I will include a problem solving guide about finding the forces in the members after this guide and for the practice exam I'll also provide a full worked-out solution in the solution manual so if you need refreshment I'd like to refer you to that.

Anyway, we get the results shown in the first three columns of table 5.1.

Table 5.1: Results.

Member	L (mm)	F (N)	$\partial F/\partial P_{B,f}$	$\partial F/\partial P_{D,f}$	$\frac{FL\partial F}{\partial P_{B,f}}$	$\frac{FL\partial F}{\partial P_{D,f}}$
AE	$4000\sqrt{2}$	$-60000\sqrt{2}-2\sqrt{2}P_{B,f}/3$	$-2\sqrt{2}/3$	0	$320\sqrt{2}$	0
EF	4000	$-60000-2P_{B,f}/3$	-2/3	0	160	0
FD	$4000\sqrt{2}$	$-80000\sqrt{2}-\sqrt{2}P_{B,f}/3$	$-\sqrt{2}/3$	0	$640\sqrt{2}/3$	0
DC	4000	$80000+P_{B,f}/3+P_{D,f}$	$1/3$	1	320/3	320
CB	4000	$80000+P_{B,f}/3+P_{D,f}$	$1/3$	1	320/3	320
BA	4000	$60000+2P_{B,f}/3+P_{D,f}$	$2/3$	1	480/3	240
EB	4000	$20000+2P_{B,f}/3$	$2/3$	0	160/3	0
FB	$4000\sqrt{2}$	$-20000\sqrt{2}+\sqrt{2}P_{B,f}/3$	$\sqrt{2}/3$	0	$-160\sqrt{2}/3$	0
FC	4000	100000	0	0	0	0

Let's first focus on Δ_B. For the $\partial F_i/\partial P_{B,f}$ we will differentiate all the loading with respect to $P_{B,f}$; this should not be too hard and the results are listed in the fourth column of table 5.1. Then, for each member, you multiply this derivative with the length and the internal load; for the internal load you have to take $P_{B,f}$ and $P_{D,f}$ equal to 0 as they are fictitious loadings! Thus they do not have a value and need to be set equal to 0. This yields the values in column 6 (note that they need to be multiplied by a million but I could not fit that into the table). Δ_B is then simply the summation of these values divided by EA, i.e.

$$\Delta_B = \frac{1}{EA}\sum_{i=1}^{k} F_i L_i \frac{\partial F_i}{\partial P_{B,f}} = \frac{1268\cdot 10^6}{1800\cdot 200000} = 3.52\,\text{mm}$$

where 1268×10^{106} N is the sum of the terms in the sixth column.
For Δ_D you do almost exactly the same. Now you differentiate with respect to $P_{D,f}$, multiply with L_i and F_i by setting the fictitious loads equal to 0, sum the terms and divide by EA to get

$$\Delta_D = \frac{1}{EA}\sum_{i=1}^{l} F_i L_1 \frac{\partial F_i}{\partial P_{D,f}} = \frac{1}{1800\cdot 200000}\cdot 880\cdot 10^6 = 2.44\,\text{mm}$$

I hope you agree with me that it's very easy: yes it's a total bitch to compute the loading, but if that's everything, you could ask a bunch of first years to do it for you and they could do it as well. Then it's just a bunch of bookkeeping.

You may wonder, if $P_{B,f}$ and $P_{D,f}$ are both zero, when can we actually say they are 0? Couldn't we have said so from the beginning? No, you can only say it once you have differentiated with respect to the loads. Furthermore, I know that the lecture slides and book does it slightly differently (they compute the internal loadings separately for the three loading cases depicted in figure 5.2. However, in my opinion, this has a few disadvantages: first and most obvious, they have to compute it three times, I only once. I call that a win already. Secondly, and more importantly perhaps depending on the question: my method is more general and is capable of more. Consider the following: suppose I'd want to what force I'd need to apply at D such that D does not move horizontally. The method the slides and the book use would be flabbergasted and not be able to answer that question for you, but my method is: this time, the load $P_{D,f}$ is *not* fictitious (as we are asking what the value of it should be such that there is no deflection), but we do know $\Delta_D = 0$. So, what you can do is do the multiplications $FL\partial F/\partial P_{D,f}$ by still setting $P_{B,f} = 0$ but not setting $P_{D,f} = 0$; this will give you one equation with one unknown that you can solve for the desired $P_{D,f}$.

Note that above problem is also more common than it may seem at first: asking for the load at D such that D does not displace is equivalent to asking what would be the horizontal reaction force at D had the connection to the wall constrained horizontal movement as well (i.e. it'd be an equivalent support as the support at A). You wouldn't have been able to determine this horizontal reaction force using static equilibrium, as the structure becomes statically indeterminate: there are four reaction forces but only three equilibrium equations. We'll get into this in more detail in the next section (called statically indeterminate systems), but I already wanted to give you a flavour of it so that you understood why I used a more general approach.

FINDING THE LOADING IN A TRUSS DUE TO EXTERNAL LOADINGS

1. Find the reaction forces due to the external loading.
2. Start at one of the supports, preferably one that does not have more than 2 members connected to it. Assume that each member connected to it is in tension, i.e. the force in each member points along the direction of the member, and points away from the joint.
3. Set up the two equilibrium equations corresponding to the link ($\sum F_x = 0$ and $\sum F_y = 0$).
4. Compute the forces in the members. If a force is negative, it means that the member is in compression.
5. Continue to an adjacent joint, preferably one that is connected to at most two members of which the loading is yet unknown. Set up the equilibrium equations, and find the forces of the two unknown members. Again, be consistent and assume that all forces act in tension. If a previously found force was negative, do not forget to include the minus sign again.
6. Repeat step 5 until finished.

So, in above example I'd first find the support reactions, then analyse joint A, then joint E[4], then I can go to joint B[5], then to C, then to F and I don't need to bother with D since I already found the forces of CD and FD by then.

One thing I'd like to point out: it's beneficial to just be super consistent with your approach: just *always*, literally *always*, assume that the loads are tensile forces, even if you already computed that it is in compression. You merely plug in the negative value in the equilibrium equations of the next joints, but that's it. Once people start drawing it in the 'correct' direction, they start fucking up because they get confused whether they should insert a minus sign etc.

FINDING DISPLACEMENTS IN A TRUSS USING STATIONARY VALUE OF TOTAL COMPLEMENTARY ENERGY

1. Note that the total complementary energy is given by
$$C = \sum_{i=1}^{k} \int_0^{F_i} \frac{F_i L_i}{A_i E_i} dF_i - \sum_{r=1}^{n} \Delta P_r$$
2. Write that to find the displacement Δ in a certain direction at a certain point, you'll apply a fictitious force P at this point in the same direction as the displacement you want to find.
3. Write that the complementary energy is stationary, i.e. its derivative with respect to an arbitrary variable should be 0. This means that
$$\frac{\partial C}{\partial P} = \sum_{i=1}^{k} \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P} - \Delta = 0$$
and thus
$$\Delta = \sum_{i=1}^{k} \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P}$$
4. Find the internal loading F_i in each of the members due to the loading of the external loads and the fictitious loads.
5. Differentiate this loading with respect to P.
6. For each member, note that $P = 0$ when it appears in F_i, and compute
$$\frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P}$$

[4] Since joint B has three members that connect to a joint I did not analyse before, it has three members carrying unknown forces. Since I'll only get two equilibrium equations, this sucks cause it results in a system I cannot solve. So I prefer to go to E first.

[5] Now the force in EB is known, so joint B has only two unknown forces attached to it.

7. Sum these quantities to find

$$\Delta = \sum_{i=1}^{k} \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P}$$

5.3.2 *Stuff that's not a truss*

Although not mentioned in the slides, you can also use complementary energy for stuff that's not a truss (also, you are supposed to know this, unfortunately).

We pretty much always get bending related problems to this. So, let's focus on that. For bending problems, the stress is *not* very straightforward and thus we need to use equation (5.9) for the internal complementary energy rather than equation (5.1). In fact, rather than

$$C = \int_{\text{vol}} \int_0^P y dP$$

we must use its bending brother

INTERNAL COMPLIMENTARY ENERGY OF A BEAM UNDER BENDING

$$C_i = \int_L \int_0^M d\theta dM \tag{5.17}$$

which seems super weird and it truly is weird. However, if we assume a linear load-displacement relationship, then

$$d\theta = \frac{M}{EI} dz$$

and thus this simply becomes

$$C_i = \int_0^L \int_0^M \frac{M}{EI} dM dz$$

Now consider for example the cantilever beam, shown in figure 5.3. The total complementary energy will be

$$C = \int_0^L \int_0^M \frac{M}{EI} dM dz - P\Delta_v$$

Differentiating with respect to P yields

$$\frac{\partial C}{\partial P} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dz - \Delta_v = 0$$

and this is a pretty useful formula. After all, for figure 5.3 we have

$$M = Pz$$

so that $\partial M / \partial P = z$, so we get

$$\Delta v = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dz = \int_0^L \frac{Pz}{EI} \cdot z dz = \int_0^L \frac{Pz^2}{EI} dz = \frac{PL^3}{3EI}$$

and that's it basically.

This went really fast probably, so let's do two examples to show it a bit more carefully.

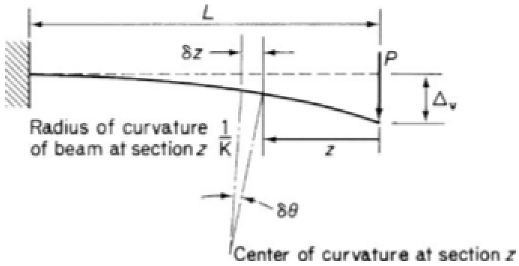


Figure 5.3: Beam deflection by the method of complementary energy.

Example 2

Consider the cantilever beam shown in figure 5.4, subject to a distributed load. Find the displacement at the tip, Δ_T.

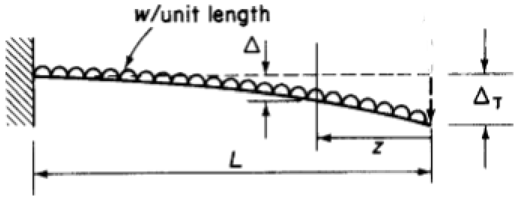


Figure 5.4: Example 2.

We start our analysis by applying a fictitious load at the tip, P_f, pointing downward. The total complementary energy will then be

$$C = \int\limits_L \int\limits_0^M d\theta dM - \Delta_T P_f - \int\limits_0^L \Delta w dz$$

Differentiating with respect to P_f, and remembering that the total complementary energy is stationary, we get

$$\frac{\partial C}{\partial P_f} = \int\limits_L d\theta \frac{dM}{dP_f} - \Delta_T = 0$$

Assuming linearly elastic material, $d\theta = M/(EI)dz$, so

$$\Delta_T = \int\limits_L \frac{M}{EI}\frac{dM}{dP_f}dz$$

Now,

$$\begin{aligned} M &= P_f z + \frac{wz^2}{2} \\ \frac{\partial M}{\partial P_f} &= z \end{aligned}$$

However, $P_f = 0$ so $M = wz^2/2$. Then,

$$\Delta_T = \int\limits_0^L \frac{M}{EI}\frac{dM}{dP_f}dz = \int\limits_0^L \frac{wz^2}{2EI}zdz = \int\limits_0^L \frac{wz^3}{2EI}dz = \frac{wL^4}{EI}$$

Again, note that you can assume that any fictitious loads are 0 only after you differentiated! Let's do another one since you're having so much fun right now.

Example 3

Calculate the vertical displacements of the quarter and the midspan points B and C of the simply supported beam of length L and the flexural rigidity EI loaded, as shown in figure 5.5.

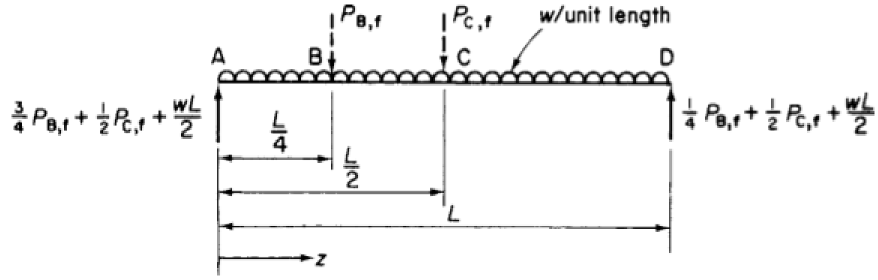


Figure 5.5: Example 3.

The fictitious loads have already been applied and the corresponding reaction forces have also been depicted in figure 5.5. Once more, the total complimentary energy of the system including the fictitious loads is

$$C = \int_L \int_0^M d\theta dM - P_{B,f}\Delta_B - P_{C,f}\Delta_C - \int_0^L \Delta w dz$$

so that

$$\begin{aligned}
\frac{\partial C}{\partial P_{B,f}} &= \int_L d\theta \frac{\partial M}{\partial P_{B,f}} - \Delta_B = 0 \\
\frac{\partial C}{\partial P_{C,f}} &= \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_C = 0
\end{aligned}$$

Assuming linearly elastic material, we have

$$\begin{aligned}
\frac{\partial C}{\partial P_{B,f}} &= \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_{B,f}} dz - \Delta_B = 0 \\
\frac{\partial C}{\partial P_{C,f}} &= \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_{C,f}} dz - \Delta_C = 0
\end{aligned}$$

Let's first focus on Δ_B. Note that we have a discontinuous moment function due to the presence of the fictitious loads at B and C. From A to B we have

$$\begin{aligned}
M &= \left(\frac{3}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right) z - \frac{wz^2}{2} \\
\frac{\partial M}{\partial P_{B,f}} &= \frac{3}{4} z
\end{aligned}$$

Between B and C we have

$$\begin{aligned}
M &= \left(\frac{3}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right) z - \frac{wz^2}{2} - P_{B,f}\left(z - \frac{L}{4}\right) \\
\frac{\partial M}{\partial P_{B,f}} &= \frac{1}{4}(L - z)
\end{aligned}$$

Between C and D we get

$$\begin{aligned} M &= \left(\frac{1}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right)(L-z) - \frac{w}{2}(L-z)^2 \\ \frac{\partial M}{\partial P_{B,f}} &= \frac{1}{4}(L-z) \end{aligned}$$

Thus, with $P_{B,f} = P_{C,f} = 0$, we get

$$\begin{aligned} \Delta_B &= \frac{1}{EI}\left[\int_0^{L/4}\left(\frac{wLz}{2} - \frac{wz^2}{2}\right)\frac{3}{4}z\,dz + \int_{L/4}^{L/2}\left(\frac{wLz}{2} - \frac{wz^2}{2}\right)\frac{1}{4}(L-z)\,dz + \int_{L/2}^{L}\left(\frac{wLz}{2} - \frac{wz^2}{2}\right)\frac{1}{4}(L-z)\,dz\right] \\ &= \frac{119wL^4}{24576EI} \end{aligned}$$

Yes, absolutely awful integral, but it's just polynomials so it's not actually that bad, just a lot of work. Similarly, you can obtain that $\Delta_C = 5wL^4/(384EI)$ but icba to write that all out.

FINDING DISPLACEMENTS USING TOTAL COMPLEMENTARY ENERGY

1. If there is no force yet applied in the direction of the displacement you're interested in, apply a fictitious load (or couple, if you are interested in the slope) at this position.
2. Write down that

$$C = \int_L \int_0^M d\theta\, dM - \sum \Delta_r P_r$$

3. Note that the total complementary energy is stationary, and thus that you differentiate with respect to P (the force aligned with the deflection Δ you are interested in), this becomes

$$\frac{\partial C}{\partial P} = \int_L d\theta \frac{\partial M}{\partial P} - \Delta = 0$$

4. Note that if the material is linearly elastic, that $d\theta = M/(EI)dz$, i.e. that

$$\Delta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dz$$

5. Find a function for the bending moment along the beam; if there are discontinuities in the bending moment, you will find multiple functions each covering a different part of the beam.
6. Find the partial derivatives of the bending moments with respect to the dummy force/couple.
7. Set P (or c) equal to zero.
8. Integrate

$$\int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dz$$

over the entire length of the beam; this may require multiple integrations for the different parts of the beam if there are bending moment discontinuities.
9. Solve the equation for the desired deflection.

Note that this method can be applied to all examples you did in chapter 4 as well! It's just another method to do the same thing (and you may have noticed the equations being very similar). However, I'm not sure whether they'll instruct you to specifically use virtual work or to use total complementary energy.

In case you want to see it applied for a couple, see the example below.

Example 4

Find the rotation at the tip of the cantilevered beam of figure 5.2.

We now simply place a fictitious couple at the tip: we then have

$$C = \int_0^L \frac{M^2}{2EI} dz - c\theta - P\Delta$$

Again, differentiating with respect c gives

$$\frac{\partial C}{\partial c} = \int_0^L \frac{M^2}{2EI} dz - \theta = 0$$

where the moment function is simply given by (for $0 \leq z \leq L$):

$$M(z) = P(L-z) + c$$

and thus

$$\frac{\partial M}{\partial c} = 1$$

and thus

$$M\frac{\partial M}{\partial c} = P(L-z) + c$$

Again, we realize now that $c = 0$, and thus we only have to integrate

$$\theta = \int_0^L \frac{P(L-z)}{EI} dz = \frac{P}{EI}\int_0^L (L-z)\, dz = \frac{P}{EI}\left[Lz - \frac{z^2}{2}\right]_0^L = \frac{PL^2}{2EI}$$

5.4 *Castigliano's theorem*

Now is the time for the biggest anti-climax of the course, namely Castigliano's theorem. Castigliano is a very fine dude who came up with two theorems. Basically, it comes down to:

CASTIGLIANO'S THEOREMS

Castigliano's first theorem is

$$\frac{\partial C_i}{\partial \Delta_i} = P_i \tag{5.18}$$

In other words, the partial derivative of the internal complimentary energy with respect to the displacement of the jth applied load P_j is equal to P_j.

Castigliano's second theorem is

$$\frac{\partial C_i}{\partial P_i} = \Delta_i \tag{5.19}$$

In other words, the partial derivative of the internal complimentary energy with respect to the jth applied load P_j is equal to the displacement of this load.

The first theorem we won't use that much (I don't think we use it at all tbh), but the second one we've already applied frequently: when we wanted to know the deflection in a certain direction, we differentiated the total complementary energy with respect to a load that's in the same direction, and that gave us the deflection (I hope this doesn't sound totally alien to you). Thus, essentially it's literally exactly the same, it seems. So what's different?

Literally only the way you write stuff down is different. What we did before was explicitly write down the total complementary energy, so also the external complementary energy, to arrive at

$$\frac{\partial C_i}{\partial P_j} = \Delta_j$$

However, when you need to apply Castigliano's theorem, you compute C_i itself and then differentiate with respect to the load of interest; you never explicitly mention the external complementary energy. Yes I agree this is being very anal about it because it's literally the exact same thing (if you use the principle of stationary total complementary energy, you just derive Castigliano's theorem on the spot), but unfortunately I'm not the one grading your exams so I can't change anything about it. However, I'd think that it needs to be explicitly clear from your answer that you understand the difference between the concept of stationary total complementary energy and Castigliano's theorem.

Let's do another example about trusses to show the difference in how your answer should be written down, but before that, I need to show you the internal complementary energy for simple loading cases.

If a bar is loaded in tension, then we have

$$C_i = \int_0^F \lambda dF$$

where λ is the deflection of the beam (you've seen this before). We have $\lambda = FL/(EA)$, thus we get

$$C_i = \int_0^F \frac{FL}{EA} dF = \frac{F^2 L}{2EA}$$

For a beam under bending, note that we had

$$C_i = \int_L \int_0^M d\theta dM = \int_0^L \int_0^M \frac{M}{EI} dz dM = \int_0^L \frac{M^2}{2EI} dz$$

since $\int_0^M M dM = M^2/2$. If the beam is under a shear load and the beam is not very long (cause then shear *was* important[6]), then $M = Sz$. Thus,

$$C_i = \int_0^L \frac{S^2 z^2}{2EI} dz = \frac{S^2 L^3}{6EI}$$

In short,

INTERNAL COMPLIMENTARY ENERGY FOR SIMPLE LOADING CASES

The internal complimentary energy for axially loaded beams, beams under bending moments and beams under shear are given by

$$C_i = \frac{F^2 L}{2EA} \tag{5.20}$$

$$C_i = \int_0^L \frac{M^2}{2EI} dz \tag{5.21}$$

$$C_i = \frac{S^2 L^3}{6EI} \tag{5.22}$$

[6] If beams are long, then the deflection at the tip is primarily caused by bending and shear can be neglected. If beams are not slender, then shear does play a dominant role.

Note, for axial and shear loaded beams, it is assumed that the forces are constant. If they're not, you have to perform the integration similar to the one for bending moments.

Let's now do another truss example to see how different it is in notation.

Example 1

Consider the truss with applied loading shown in figure 5.6. Compute:

a. The deflection at point D.
b. The force Q at B to have no horizontal deflection.
c. The deflection at D when force Q is applied.

The E-modulus and cross-sectional area are the same for each member.

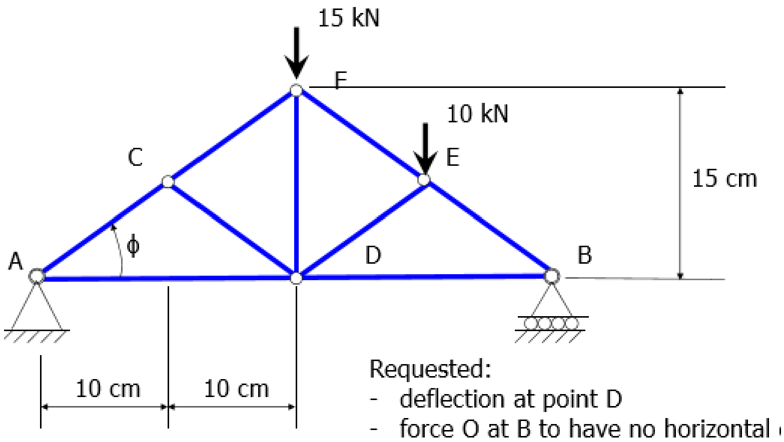


Figure 5.6: Example 1.

The basic strategy is largely the same as it was when we used the principle of stationary value of total complementary energy. We first write that the internal complementary energy for a truss is given by

$$C_i = \sum_{i=1}^{k} \frac{F_i^2 L_i}{2E_i A_i}$$

To find the displacement in vertical direction at D, we will apply a fictitious load P at D. We will also apply a fictitious load Q at B in horizontal direction (pointing outward, to the right). However, rather than differentiating now already, we first find the internal loading F_i now. The results are shown in figure 5.7 as I'm too lazy to write the calculations all out. However, after finding the reaction forces, you start your truss analysis at A, then go to C, then go to F, then to either E or D and afterwards to either D or E (depending on which you did before).

Member	Force
AC	$-\frac{10}{\sin\phi}-\frac{P}{2\sin\phi}$
AD	$\frac{10}{\tan\phi}+\frac{P}{2\tan\phi}+Q$
CD	0
CF	$-\frac{10}{\sin\phi}-\frac{P}{2\sin\phi}$
BD	$\frac{15}{\tan\phi}+\frac{P}{2\tan\phi}+Q$
DE	$-\frac{5}{\sin\phi}$
DF	$P+5$
EF	$-\frac{10}{\sin\phi}-\frac{P}{2\sin\phi}$
BE	$-\frac{15}{\sin\phi}-\frac{P}{2\sin\phi}$

Figure 5.7: Results.

Please note that it's advisable to multiply everything yourself by 1000 to get the force in N rather than kN (personally I find it a bit confusing how units carry over when I'm using kN so I prefer to just write a few more digits myself). In any case, we get

$$\begin{aligned}
C_i &= \sum_{i=1}^{k}\frac{F_i^2L_i}{2E_iA_i}=\frac{1}{2EA}\sum_{i=1}^{k}F_i^2L_i=\frac{1}{2EA}\left[\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)^2L_{ac}\right.\\
&+\left(\frac{10000}{\tan\phi}+\frac{P}{2\tan\phi}+Q\right)^2L_{AD}+0+\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)^2L_{CF}\\
&+\left(\frac{15000}{\tan\phi}+\frac{P}{2\tan\phi}+Q\right)^2L_{BD}+\left(\frac{-5000}{\sin\phi}\right)^2L_{DE}+(P+5000)^2L_{DF}\\
&\left.+\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)^2L_{EF}+\left(-\frac{15000}{\sin\phi}-\frac{P}{2\sin\phi}\right)^2L_{BE}\right]
\end{aligned}$$

which is really annoying to write down but just remember kids in Africa have it worse than you.

We only start differentiating now. By Castigliano's second theorem, the deflection Δ_D is equal to the derivative of C_i with respect to the load P; thus we get

$$\begin{aligned}
\Delta_D &= \frac{1}{2EA}\left[-2\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)\frac{L_{AC}}{2\sin\phi}+2\left(\frac{10000}{\tan\phi}+\frac{P}{2\tan\phi}+Q\right)\frac{L_{AD}}{2\tan\phi}\right.\\
&-2\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)\frac{L_{CF}}{2\sin\phi}+2\left(\frac{15000}{\tan\phi}+\frac{P}{2\tan\phi}+Q\right)\frac{L_{BD}}{2\tan\phi}\\
&\left.+2(P+5000)L_{DF}-2\left(-\frac{10000}{\sin\phi}-\frac{P}{2\sin\phi}\right)\frac{L_{EF}}{2\sin\phi}-2\left(-\frac{15000}{\sin\phi}-\frac{P}{2\sin\phi}\right)\frac{L_{BE}}{2\sin\phi}\right]\\
&= \frac{1}{2EA}\left[\left(\frac{10000}{\sin\phi}+\frac{P}{2\sin\phi}\right)\frac{L_{AC}+L_{CF}+L_{EF}}{\sin\phi}+\frac{10000}{\tan^2\phi}L_{AD}+\frac{150000}{\tan^2\phi}L_{BD}\right.\\
&\left.+\left(\frac{P}{2\tan\phi}+Q\right)\frac{L_{BD}+L_{AD}}{\tan\phi}+2(P+5000)L_{DF}+\left(\frac{15000}{\sin\phi}+\frac{P}{2\sin\phi}\right)\frac{L_{BE}}{\sin\phi}\right]
\end{aligned}$$

However, P and Q are fictitious loads so they're both zero, resulting in

$$\Delta_D=\frac{1}{2EA}\left[\left(\frac{10000}{\sin\phi}\right)\frac{L_AC+L_{CF}+L_{EF}}{\sin\phi}+\frac{10000}{\tan^2\phi}L_{AD}+\frac{15000}{\tan^2\phi}L_{BD}+10000L_{DF}+\frac{15000}{\sin^2\phi}L_{BE}\right]$$

Yes absolutely awful to write out but nothing you can do about it.

For b), to restrain the motion at B, we differentiate with respect to Q to find the deflection at B:

$$\Delta_B = \frac{\partial C_i}{\partial Q} = \frac{1}{2EA}\left[2\left(\frac{10000}{\tan\phi} + \frac{P}{2\tan\phi} + Q\right)L_{AD} + 2\left(\frac{15000}{\tan\phi} + \frac{P}{2\tan\phi} + Q\right)L_{BD}\right]$$

Now, if we restrain the motion at B, then $Q \neq 0$, but $\Delta_B = 0$! However, still $P = 0$ so we need to solve

$$\Delta_B = \frac{1}{2EA}\left[\left(\frac{20000}{\tan\phi} + 2Q\right)L_{AD} + \left(\frac{30000}{\tan\phi} + 2Q\right)L_{BD}\right] = 0$$

which has the solution

$$Q = -\frac{1}{\tan\phi}\frac{10000L_{AD} + 15000L_{BD}}{L_{AD} + L_{BD}}$$

For c), we again use our expression for Δ_D, but now we don't have $Q = 0$ but the above expression for Q. Thus, we get

$$\begin{aligned}\Delta_D &= \frac{1}{2EA}\left[\left(\frac{10000}{\sin\phi}\right)\frac{L_AC + L_{CF} + L_{EF}}{\sin\phi} + \frac{10000}{\tan^2\phi}L_{AD} + \frac{15000}{\tan^2\phi}L_{BD}\right. \\ &\quad \left. + \left(-\frac{1}{\tan\phi}\cdot\frac{10000L_{AD} + 15000L_{BD}}{L_{AD} + L_{BD}}\right) + 10000L_{DF} + \frac{15000}{\sin^2\phi}L_{BE}\right]\end{aligned}$$

It's not really hard imo, just a lot of bookkeeping and not making stupid mistakes.

FINDING DISPLACEMENTS USING CASTIGLIANO'S THEOREM

1. If there is no force yet applied in the direction of the displacement you're interested in, apply a fictitious load (or couple, if you are interested in the slope) at this position.
2. Write down that the internal complementary energy is given by

$$C_i = \sum_{i=1}^{k}\frac{F_i^2 L_i}{2E_i A_i}$$

3. Compute the internal loading in the structure due to the presence of both the actual loading and the fictitious loading.
4. Compute C_i using the equation above.
5. State that Castigliano's theorem states that the deflection in the direction of force P is equal to the partial derivative of the internal complementary energy with respect to this load, i.e.

$$\Delta = \frac{\partial C}{\partial P}$$

6. Find Δ by stating that the fictitious forces are 0.

There's one thing I'd like to point out: you can arguably differentiate before. You could say after bullet number 3; the deflection in the direction of P is found by differentiating C_i with respect to P, so that we get

$$\Delta = \frac{\partial C_i}{\partial P} = \sum_{i=1}^{k}\frac{F_i L_i}{E_i A_i}\frac{\partial F_i}{\partial P}$$

and then the solution procedure becomes exactly the same as for the stationary value of total complementary energy.

5.5 Statically indeterminate structures

Again, not something that was specifically discussed in the lectures but you are required to know it. Statically indeterminate structures are structures for which it is not possible to determine all reaction forces as there are more reaction forces than equilibrium equations. We'll see it applied to both trusses and non-trusses.

5.5.1 Stuff that's a truss

We actually already saw it applied in the previous example: consider part b) again. Here, we found the force Q such that B did not move: this is essentially finding the horizontal reaction force if the support at B would have restrained horizontal movement as well. You'd not have found this reaction found by equilibrium equations: there'd be four reaction forces (2 at A and 2 at B) and only three equilibrium equations, but this way you were able to. In other words, the problem solving guide only needs to be slightly amended:

FINDING REACTION FORCES USING CASTIGLIANO'S THEOREM

1. *Assume* that there is a displacement in the direction of the reaction force you want to know.
2. Write down that the internal complementary energy is given by

$$C_i = \sum_{i=1}^{k} \frac{F_i^2 L_i}{2E_i A_i}$$

3. Compute the internal loading in the structure due to the presence of the loading.
4. Compute C_i using the equation above.
5. State that Castigliano's theorem states that the deflection in the direction of reaction force R is equal to the partial derivative of the internal complementary energy with respect to this load, i.e.

$$\Delta = \frac{\partial C_i}{\partial R}$$

6. Note that this displacement must be 0, and solve for R.

Again, note that I think it should be fine to move step 5 to be in front of step 3. The example in the next subsection will show how that changes how you write stuff down.

5.5.2 Stuff that's not a truss

We can also find the reaction force for stuff that's not a truss. For that, let me take an example of last year's practice exam.

Example 1

The pinned-pinned portal frame shown in figure 5.8 is symmetrically loaded by a distributed load w over the horizontal member. The bending stiffness EI is the same everywhere. Find the bending moment distribution.

[Hint: use the horizontal reaction as redundant, and calculate the complementary energy only due to bending.]

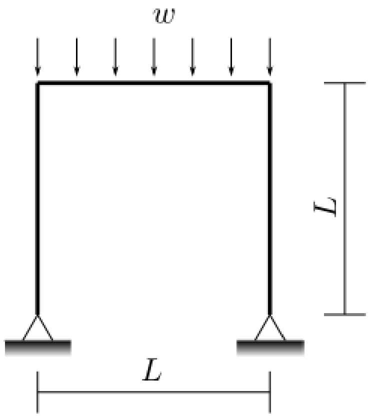


Figure 5.8: Example 1.

First of all, let's look at the reaction forces and deflections as shown in figure 5.9 (the vertical reaction forces don't really require explanation hopefully).

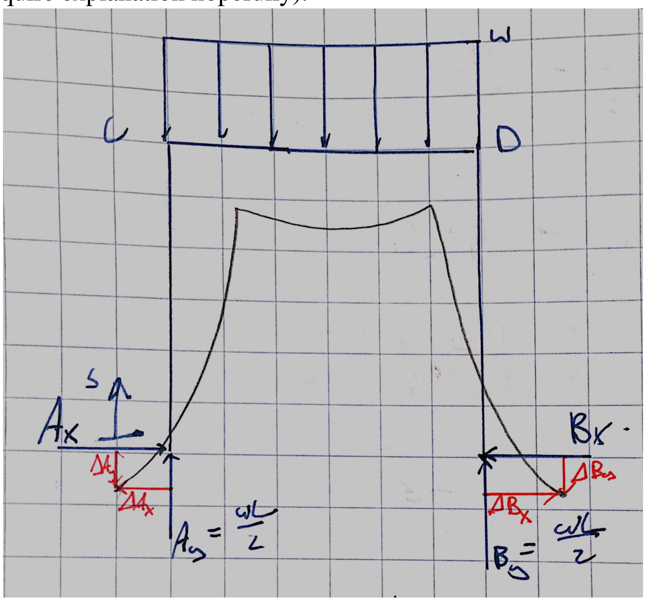


Figure 5.9: Reaction forces and deflections.

We simply have

$$C_i = \int_L \frac{M^2}{2EI} ds$$

By Castigliano's second theorem, the horizontal deflection at A is found by differentiating with respect to the horizontal reaction force at A, i.e A_x:

$$\Delta_{A_x} = \frac{\partial C_i}{\partial A_x} = \int_L \frac{M}{EI} \frac{\partial M}{\partial A_x} ds$$

However, as A is a reaction force, in reality we have $\Delta A_x = 0$ as well, and as EI is constant everywhere, we only have to compute

$$\int_L M \frac{\partial M}{\partial A_x} ds = 0$$

Now, we have to find a function for M everywhere along the entire structure. For this, we divide the structure in three parts:

$$\int_L M \frac{\partial M}{\partial A_x} ds = \int_0^L M \frac{\partial M}{\partial A_x} ds_1 + \int_0^L M \frac{\partial M}{\partial A_x} ds_2 + \int_0^L M \frac{\partial M}{\partial A_x} ds_3$$

where the first part refers to beam AC, the second to beam CD and the third for beam DB. For beam AC, we have the sketch shown in figure 5.10.

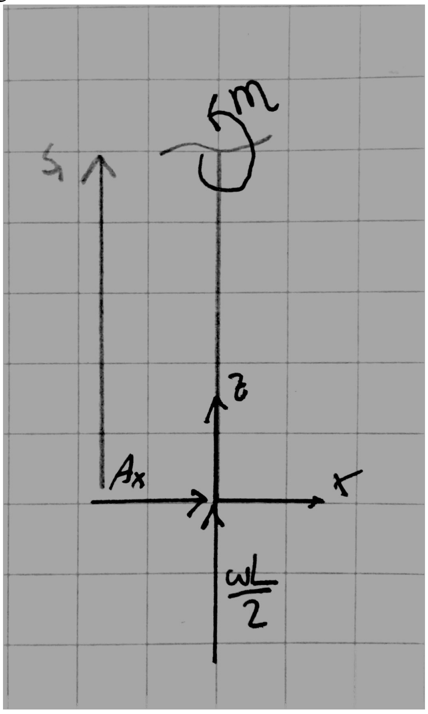


Figure 5.10: Beam AC.

Drawing the internal moment in counterclockwise direction, so that when it is positive it causes tension for positive values of x (which is the sign convention we've always used); then we have (taking moments in counterclockwise direction around the cut):

$$\begin{aligned} M + A_x s_1 &= 0 \\ M &= -A_x s_1 \\ \frac{\partial M}{\partial A_x} &= -s_1 \\ M \frac{\partial M}{\partial A_x} &= A_x s_1^2 \end{aligned}$$

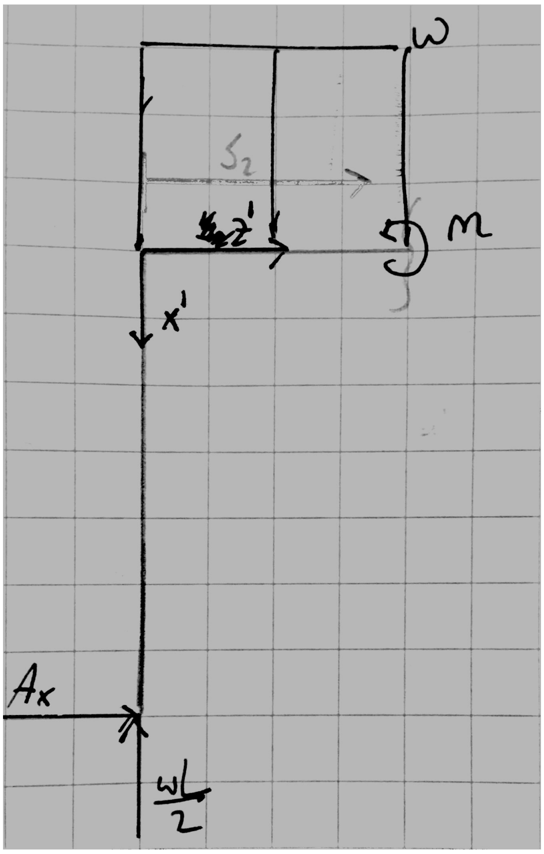


Figure 5.11: Beam CD.

For beam CD, we take a look at figure 5.11, where we see that (again, counterclockwise moment around the cut):

$$
\begin{aligned}
M + A_x L + w s_2 \cdot \frac{s_2}{2} - \frac{wL}{2} s_2 &= 0 \\
M &= -A_x L - w\frac{s_2^2}{2} + wL\frac{s_2}{2} \\
\frac{\partial M}{\partial A_x} &= -L \\
M\frac{\partial M}{\partial A_x} &= A_x L^2 + wL\frac{s_2^2}{2} - wL^2\frac{s_2}{2}
\end{aligned}
$$

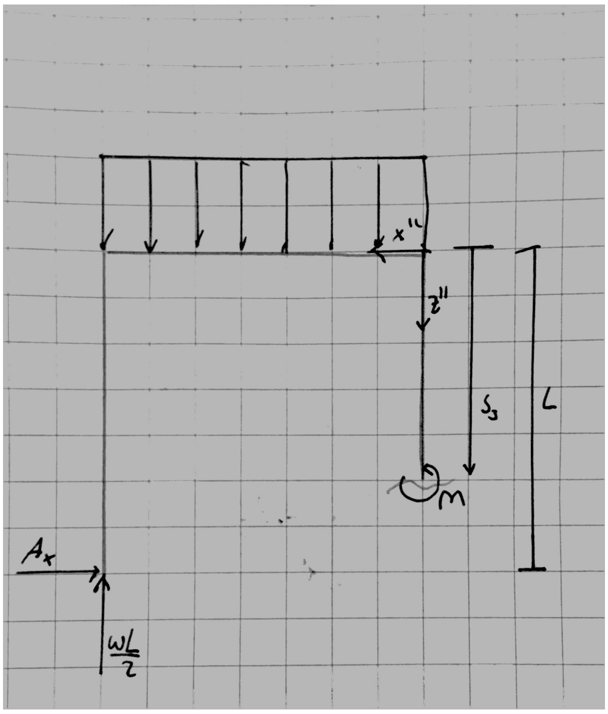


Figure 5.12: Beam DB.

For beam DB, we take a look at figure 5.12, where we see that (again, counterclockwise moment around the cut):

$$\begin{aligned}
M + A_x\left(L - s_3\right) + \frac{wL^2}{2} - \frac{wL^2}{2} &= 0 \\
M &= -A_x\left(L - s_3\right) = A_x\left(s_3 - L\right) \\
\frac{\partial M}{\partial A_x} &= s_3 - L \\
M\frac{\partial M}{\partial A_x} &= A_x s_3^2 - 2A_x L s_3 + A_x L^2
\end{aligned}$$

Then performing every integration:

$$\begin{aligned}
0 &= \int_0^L A_x s_1^2 ds_1 + \int_0^L \left(A_x L^2 + wL\frac{s_2^2}{2} - wL^2\frac{s_2}{2}\right) ds_2 + \int \left(A_x s_3^2 - 2A_x L s_3 + A_x L^2\right) ds_3 \\
&= \frac{A_x L^3}{3} + A_x L^3 + \frac{wL^4}{6} - \frac{wL^4}{4} + \frac{A_x L^3}{3} - A_x L^3 + A_x L^3 = \frac{5A_x L^3}{3} - \frac{wL^4}{12} = 0 \\
A_x &= \frac{wL}{20}
\end{aligned}$$

Thus, in beam AC, the bending moment is given by

$$M = -A_x s_1 = -\frac{wL}{20}s_1$$

In beam CD, the bending moment is given by

$$M = -A_x L - w\frac{s_2^2}{2} + wL\frac{s_2}{2} = -\frac{wL^2 0}{2} - w\frac{s_2^2}{2} + wL\frac{s_2}{2} = -\frac{w}{2}\left(\frac{L^2}{10} - Ls_2 + s_2^2\right)$$

In beam DB, we find

$$M = A_x\left(s_3 - L\right) = \frac{wL}{20}\left(s_3 - L\right) = \frac{wL}{20}s_3 - \frac{wL^2}{20}$$

CASTIGLIANO'S THEOREM: SOLVING A STATICALLY INDETERMINATE SYSTEM

1. Draw *all* of the reaction forces acting on the system. Note that in a statically indeterminate system, you cannot simply put one of the reaction forces equal to zero to solve the static equilibriume equations.
2. Set up the equation
$$C_i = \int_0^L \frac{M^2}{2EI} dz$$
3. State that Castigliano's theorem states that the deflection in the direction of reaction force R is equal to the partial derivative of the internal complementary energy with respect to this load, i.e.
$$\Delta = \frac{\partial C_i}{\partial R}$$
4. Note that this must be 0 as the displacement caused by a reaction force will be 0.
5. Find a function for the bending moment along the beam; if there are discontinuities in the bending moment, you will find multiple functions each covering a different part of the beam.
6. Find the partial derivatives of the bending moments with respect to the desired reaction force.
7. Integrate
$$\int_0^L \frac{M}{EI} \frac{\partial M}{\partial f} dz$$
over the entire length of the beam; this may require multiple integrations for the different parts of the beam if there are bending moment discontinuities.
8. Solve the equation for the desired reaction force.