# CO-DIMENSION ONE SUBMANIFOLDS IN HIGHER RANK GEOMETRY

MITUL ISLAM

**Introduction.** The partner group under my leadership will conduct research on *discrete subgroups of higher rank Lie groups.* Such subgroups arise as holonomies of locally symmetric manifolds and are of much interdisciplinary interest – in geometry, physics, number theory, and algebraic geometry, to name a few. A key reason why the geometry of higher rank Lie groups (e.g. $\mathrm{SL}_n(\mathbb{R})$ where $n \geq 3$) is interesting is that it transcends rank one geometry (e.g. $\mathrm{SL}_2(\mathbb{R})$ and 2d hyperbolic geometry) and has room for a lot of flexibility, thus culminating in interesting deformation theories (e.g. the *higher rank Teichmüller theories*).

In our project titled "*Co-dimension one submanifolds in higher rank geometry*", the thrust of our research will be to initiate the study of discrete groups in higher rank via studying their 'slices'. The word 'slice' is our informal alias for the technical term *'codimension 1 totally geodesic submanifold'*(later abbreviated as TGS). The philosophy behind the project draws inspiration from the technology of 'Computerized Axial Tomography' (CAT) that is widely used in medical imaging. CAT reveals the inner structure of a 3d object by first scanning its 2d slices and then aggregating the information from slices. The mathematical principle underlying CAT is the theory of Radon transform in 3d space. So, philosophically, our project is an attempt to develop analogues of this circle of mathematical ideas in higher rank geometry and to apply them in the investigation of discrete subgroups.

The family of discrete subgroups that we will investigate in this project are called *Anosov subgroups.* These are higher rank generalizations of fundamental groups of hyperbolic surfaces. Interest in Anosov groups stem from the fact that they often admit nice deformations and have associated moduli spaces, akin to the moduli spaces of hyperbolic surfaces. Discovered by Wienhard (the MPI director who is endorsing my application for the partner group), Guichard, and Labourie, the theory of Anosov groups has blossomed into a rich area with many connections. For example, their moduli spaces have connections to geometric structures on manifolds; analysis (via Higgs bundles); dynamics (via Hamiltonian flows); combinatorics (via total positivity); algebra (via cluster algebraic coordinates); and algebraic geometry (via character varieties).

**Motivation.** Our project of studying Anosov groups in higher rank via its 'slices' is motivated by the success of the same idea in rank one. The study of totally geodesic submanifolds ('slices') in hyperbolic manifolds (rank one) dates back to Cartan and has been a very successful endeavour. We will say that $N$ is a *totally geodesic submanifold (TGS)* of $M$ if $N$ has co-dimension 1 in $M$ and the intrinsic geometry of $N$ coincides with the ambient one. More precisely, $N$ is a TGS if $N \hookrightarrow M$ is a closed, immersed submanifold of dimension $(\dim M - 1)$ and any geodesic in $N$ is also a geodesic in $M$ (here, geodesic is a length minimizing path for the relevant metric). The existence of a single TGS in $M$ introduces a lot of flexibility - one can "bend" $M$ about the TGS and produce many new Anosov groups that are distinct from the initial one given by $M$. On the other hand, having infinitely many TGS-s in $M$ leads to rigidity – it was recently shown that such an $M$ must be an arithmetic hyperbolic manifold, i.e. constructed via number theoretic methods.

The TGS-s are often a rich family that remembers much of the geometry. A theory of Radon transforms (alluded to above) of the hyperbolic space uses TGS-s as the analogue of 2d 'slices' in 3d space. Kahn-Markovic settled the famous surface subgroup conjecture for closed hyperbolic 3-manifolds by showing that they contain plenty of 2d subsurfaces that are 'almost TGS'. These 'almost TGS' subsurfaces have connections with cubulations and have been instrumental in settling the long-standing virtual Haken and virtual fibering conjectures by Agol and Wise.

**Project Goals.** *1. Develop a suitable notion of TGS for Anosov groups in higher rank.*
In higher rank, the Riemannian symmetric space – which is the natural analogue of the hyperbolic space – usually does not have any TGS (since a TGS has codimension 1). My goal is to address this issue by drawing on my expertise with projective structures. Often, an Anosov subgroup $\Gamma$ of $\mathrm{SL}_d(\mathbb{R})$ preserves a *convex projective structure* , i.e. an open convex domain $\Omega_\Gamma$ in the real projective space $\mathbb{P}(\mathbb{R}^d)$. Then $\Gamma$ preserves the *Hilbert metric* on $\Omega_\Gamma$, a Finsler metric whose geodesics are projective lines. The structure of TGS for this metric is particularly simple: they are intersections of projective hyperplanes with $\Omega_\Gamma$. I have a lot of expertise in studying 'zero cruvature' TGS-s in such domains $\Omega_\Gamma$. Using those ideas, I want to develop a theory for 'negatively curved' TGS-s in $\Omega_\Gamma$. This is a necessary step since all TGS-s for an Anosov group are always 'negatively curved'.

With this notion of TGS discussed above, we will pursue the two following directions:
*2. Rigidity.* We ask whether an abundance of TGS-s makes the geometry of Anosov groups rigid. More precisely, *if the closed manifold $\Omega_\Gamma/\Gamma$ contains infinitely many TGS-s, does it imply that $\Omega_\Gamma$ is the hyperbolic space and $\Gamma$ is an arithmetic hyperbolic lattice?* Recently, Filip-Fisher-Lowe answered this question in the affirmative in a different setting (variable negative curvature). I plan to attack this question by adapting their methods to convex projective geometry.
*3. Periods of the group and Crofton formula.* We seek to quantify how well the TGS-s remember the Anosov group when $\Omega_\Gamma/\Gamma$ is a closed manifold. Precisely, we ask: *is there a $\Gamma$-invariant measure $\mu$ on the space of TGS-s of $\Omega_\Gamma$ whose integrals record the periods of $\Gamma$, i.e. the lengths of all closed geodesics in $\Omega_\Gamma/\Gamma$?* For $\mathrm{SL}_3(\mathbb{R})$-Hitchin representations, Labourie gave a positive answer to this by finding a geodesic current defined using cross-ratios.
It is also natural to strengthen the question and ask for a *Crofton formula*, akin to hyperbolic spaces: *is there a measure $\mu$ on TGS-s of $\Omega_\Gamma$ whose integrals recover the Hilbert metric?*

For surface subgroups, I will investigate some other competing notion of TGS:
*4. Remembering positive representations.* *Positive representations* of a surface group in $\mathrm{SO}(2,n)$ are a special class of Anosov groups. For such groups, Lankers – a PhD student of Weinhard at MPI – has introduced a notion of codimension 1 submanifolds ('slices') in the Einstein universe of isotropic 2-planes. They want to use these 'slices' to 'bend' positive representations. But from my point of view, their 'slices' seem rich enough to remember the periods of the representation. With Lankers, I have begun discussing: *do these 'slices' remember the positive representation?*
*5. Building new surface subgroups.* Consider a closed complex hyperbolic manifold $M$ with $\dim_{\mathbb{R}} M = 4$. The 2-dimensional totally geodesic submanifolds of $M$ come in two varieties – either they are real hyperbolic (curvature -1) or complex hyperbolic (curvature -4). We ask: *are there any surface subgroups in $M$ that mix between these two different kinds of TGS-s?* We expect a positive answer, which has interesting consequences for the Euler characteristics of the resulting surfaces. Our idea is to build pairs of pants that are 'almost TGS' and then carry out a careful matching, like in Kahn-Markovic's work on hyperbolic 3-manifolds.

**Collaboration and value addition.** As a young faculty member at TIFR, I am determined to build my research group in the area of discrete subgroups of Lie groups. Setting up this partner group would be an invaluable contribution towards this goal. During the 5-year period, we will regularly host joint (with MPI-MIS) workshops and research meetings to train young researchers in this area. By doing so, we will bring in fresh ideas, discover new interdisciplinary connections, and contribute immensely towards the advancement of the field. As I explained in Question 3 above, during my stay at MPI-MIS, I started collaborating with Lankers on TGS-s in higher rank. This funding will instigate more such collaborations between TIFR and MPI-MIS and support the old ones. We plan to have short-term and long-term visits by members of the research groups to facilitate such an exchange of ideas. We will run independent working groups at each end to keep research activities alive between these events. They will periodically exchange ideas via web conferencing. Finally we intend to organize a few conferences to report our findings to the broader mathematics research community.