RESEARCH STATEMENT

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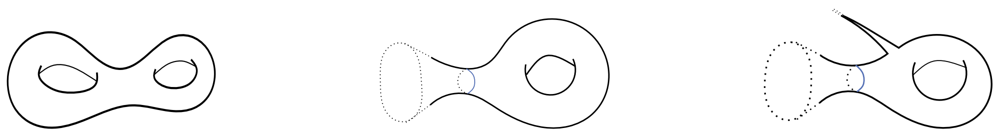
1. Overview of my Research Program

A fundamental question in geometry is to obtain concrete construction of manifolds. For instance, consider the classical Poincare polygon theorem, which provides a recipe for building surfaces by gluing the edges of a regular polygon. However such purely geometric methods become intractable as the dimension goes up. A more systematic approach to this problem is the study of discrete subgroups of Lie groups. For instance, the above problem of constructing a surface – of genus at least two – can be recast as a problem of constructing a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$. Any such uniform (i.e. co-compact) lattice Γ acts by Möbius transformations on the real hyperbolic plane $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma$ is a closed surface. More generally, if we construct discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that are 'thinner' than lattices, then we get nice infinite volume surfaces, e.g. convex co-compact or geometrically finite surfaces.



Closed surface of genus 2 — Convex co-compact surface — Geometrically finite surface

One can ask the same question for Lie groups that are more complicated than $\mathrm{PSL}_2(\mathbb{R})$, e.g. the so-called higher rank Lie groups like $\mathrm{SL}_d(\mathbb{R})$ when $d \geq 3$, $\mathrm{SO}(p,q)$, etc. Lattices in such groups are very restricted - they all come from arithmetic constructions and cannot be deformed, as proven by Margulis [Mar75, Mar91]. But what about more general discrete subgroups? This is the central question that I pursue in my research program – the study of discrete subgroups in higher rank Lie groups, beyond lattices. One of the most exciting aspects of my research program is that it lies at the interface of geometry, topology, algebra, and analysis. I use ideas and tools from geometry, linear algebraic groups, topology, Lie group theory, representation theory, dynamical systems, ergodic theory, and coarse geometry. I find it rewarding to learn from a wide variety of areas of mathematics and see beautiful ideas emerge from their interplay.

Research Direction I: Convex Projective Structures and Groups beyond Gromov Hyperbolicity

I adopt the perspective of studying discrete subgroups via their actions on the real projective space, more precisely, on *properly convex domains* in $\mathbb{P}(\mathbb{R}^d)$. I will explain this below. But first, the reader might think the following: if $\Gamma < \mathrm{SL}_d(\mathbb{R})$ is a discrete subgroup, why do we not consider its action on the associated Riemannian symmetric spaces $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$? This is a great question because when $d = 2$, this symmetric space is $\mathbb{H}^2$ and Möbius actions on $\mathbb{H}^2$ is indeed the classical tool that we use for studying subgroups of $\mathrm{SL}_2(\mathbb{R})$. However, the situation changes dramatically in higher rank. Kleiner-Leeb [KL06] and Quint [Qui05] independently showed that when $d \geq 3$, the only discrete subgroups with reasonable actions (i.e. convex co-compact) on the symmetric space are uniform lattices. Hence, we are forced to look for new geometric spaces to study discrete subgroups in higher rank.

A motivating example of a properly convex domain is an open disk $\mathbb{B}$ in an affine plane in $\mathbb{P}(\mathbb{R}^3)$ – the projective Beltrami-Klein model of the hyperbolic plane $\mathbb{H}^2$. It has projective lines as geodesics, and the isometry group is $\mathrm{SO}(2,1)$. Inspired by this, we say that an open subset $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a *properly convex domain* if its closure $\overline{\Omega}$ is a compact convex domain in an (affine) chart of $\mathbb{P}(\mathbb{R}^d)$. Any such domain Ω carries a natural distance function d_Ω – *the Hilbert metric on* Ω – defined using cross-ratios. The subgroup of $\mathrm{PGL}_d(\mathbb{R})$ that preserves Ω – denoted by $\mathrm{Aut}(\Omega)$ – acts by isometries of d_Ω. Benoist showed that in some special cases, the Hilbert metric behaves like a negatively curved Riemannian metric [Ben08, Qui10]. But typically, the metric d_Ω is not even $\mathrm{CAT}(0)$ (a popular generalization of non-positive curvature) [KS58]. An important aspect of my research is to study the metric geometry of $(\Omega, \mathrm{d}_\Omega)$ by analogy with non-positive curvature. I achieve this in the following papers:

- [Isl] I develop the theory of *rank one Hilbert geometries*, analogous to rank one $\mathrm{CAT}(0)$ spaces. I prove that a rank one group is an *acylindrically hyperbolic group*, i.e. rank groups carry a form of generalized Gromov hyperbolicity akin to mapping class groups and $\mathrm{Out}(F_n)$.

- [IW24] We study *Morse geodesic rays* (a coarse geometric notion of 'hyperbolic' directions) in a *properly convex domain*. We characterize these 'hyperbolic' directions in terms of singular value growth gaps (linear algebraic data) as well as the regularity of the boundary (projective geometric notion).

The second reason why I care so deeply about convex projective geometry is its close connections with Teichmüller theory. A *convex projective manifold* is a quotient of the form Ω/Γ where Γ is a torsion-free discrete subgroup of $\mathrm{Aut}(\Omega)$. This clearly generalizes the notion of hyperbolic structures on manifolds and hints at an intimate connection with classical Teichmüller theory, the study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. In recent years, *Anosov representations* [Lab06, KLP17, GGKW17] have received much attention as a dynamical approach towards developing a "Teichmüller theory for higher rank Lie groups" [BIW14, Wie18]. Anosov representations correspond to rich families of discrete subgroups: they are stable under small deformations, and often fill out entire connected components in the character variety, e.g. Hitchin representations [Wie18].

Until recently, however, there was a lack of a geometric perspective on Anosov representations [Kas18]. Convex projective geometry – more precisely, convex co-compact groups – has recently been used to fill this gap [DGK17, Zim21]. We call a group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ *convex co-compact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$ – the convex hull of its limit set. [DGK17] and [Zim21] show that for a Gromov hyperbolic group Γ, convex co-compactness is closely related to the inclusion representation $\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ being Anosov. This inspires a natural question: what about non-Gromov hyperbolic groups? This question is a quest to understand discrete linear groups beyond the Gromov hyperbolic ones (i.e. the ones coming from Anosov representations).

I have addressed precisely this question in my joint research program with Andrew Zimmer. In the course of our five joint papers (summarized below), we develop a theory of convex co-compact groups that are relatively hyperbolic. Suppose Γ is a convex co-compact group that is relatively hyperbolic with respect to peripheral subgroups $\mathcal{P} := \{P_1, \dots, P_m\}$.

- In [IZ23], we solve the case of *abelian peripherals*, i.e. $P_i \cong \mathbb{Z}^{k_i}$ with $k_i \geq 2$. We introduce the notion of properly convex domains with isolated simplices, analogous to CAT(0) spaces with isolated flats [HK05]. We prove: Γ is relatively hyperbolic with respect to $\mathcal{P}$ if and only if $\mathcal{C}_\Omega(\Gamma)$ has isolated simplices.

- In [IZ22], we solve the case of *general peripheral subgroups*.

- In [IZ24b], we classify *3-manifold groups* that are convex co-compact while in [IZ24a], we classify convex co-compact groups with at most *1 dimensional boundary faces.* Both of these families are convex co-compact groups relatively hyperbolic with $\mathbb{Z}^2$ peripherals.

- In [IZ21], we prove the *flat torus theorem* and relate maximal abelian subgroups of a convex co-compact group (not necessarily rel. hyp.) to simplices in $\mathcal{C}_\Omega(\Gamma)$.

Research Direction II: Group Action on Boundaries and their Deformations

In Research Direction I, I study discrete subgroups of $\mathrm{SL}_d(\mathbb{R})$ by looking at their action on properly convex subsets of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. But what about their action on the entire projective space $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$? This is the central question that drives my Research Direction II. The action on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is significant because $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ can be interpreted as a boundary of the symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$ and group actions on boundaries has a rich tradition, e.g. [Mos73, Sul85, Tuk95].

The question of deforming boundary actions is classical and was already investigated by Sullivan [Sul85] in the case of rank one Lie groups. Consider the example of a uniform lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$ which has a natural action ρ_0 on $\partial \mathbb{H}^2 \cong \mathbb{S}^1$. Sullivan asked: can we deform ρ_0 in $\mathrm{Diff}(\mathbb{S}^1)$? That is, he wanted to consider an action ρ for which both ρ and its derivative $d\rho$ are sufficiently close to ρ_0 and $d\rho_0$ respectively. In this C^1-deformation case, Sullivan proved that ρ_0 is *rigid.*, i.e. any such ρ is in fact *conjugate* to ρ_0. This means that there is a homeomorphism ϕ of $\partial \mathbb{H}^2$ such that $\phi \circ \rho_0 = \rho \circ \phi$. So from a dynamical viewpoint, the actions ρ and ρ_0 are indistinguishable.

In my joint paper [CINS23], we ask the same question as Sullivan, but for higher rank lattices. To consider a concrete example, let $\Gamma < \mathrm{SL}_d(\mathbb{R})$ be a uniform lattice where $d \geq 3$. There is a natural algebraic action $\rho_0 : \Gamma \to \mathrm{Homeo}(\mathbb{P}(\mathbb{R}^{\mathrm{d}}))$ and we ask, is ρ_0 rigid? However, unlike Sullivan, we pose our question in the more challenging world of C^0-deformations (i.e. any action ρ close to ρ_0). We consider the C^0-case because the C^1-case in higher rank had already been settled independently by [KS97] and [Kan96]. Similar to Sullivan, the answer is that any C^1-close ρ is conjugate to ρ_0.

Passing to the C^0-case gives rise to unique challenges. Since we can no longer control the derivative of the action, neither Sullivan's methods [Sul85] nor the dynamical methods [KS97] apply. In fact, the C^0-case had proven to be a formidable challenge even for rank one Lie groups – remaining open until the work [BM22] in 2019. The ingenuity of [BM22] lies in finding a perfect marriage between the use of dynamical foliations and coarse geometry to tame the C^0-deformations. Although the tools of [BM22] does not apply in higher rank, our work [CINS23] is inspired by their approach.

- [CINS23] (*joint with C.Connell, T.Nguyen, and R.Spatzier*)
  We prove that: for any ρ that is sufficiently C^0-close to ρ_0 (i.e. ρ and ρ_0 both map generators of the lattice close-by), there exists a continuous surjective map $\varphi_\rho : \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \to \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that the adjoining diagram commutes. Such a map φ_ρ is called a *semi-conjugacy* and we say that ρ_0 is '*semi-conujgacy rigid*'.

$$\begin{tikzcd} \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \arrow[r, "\rho"] \arrow[d, "\varphi_\rho"] & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \arrow[d, "\varphi_\rho"] \\ \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \arrow[r, "\rho_0"] & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \end{tikzcd}$$

Our theorem in [CINS23] also holds for all other flag space boundaries, in particular for the Furstenberg boundary. In [CINS23], we further show that 'semi-conjugacy rigidity' is the best that one can hope for in the C^0-deformation case. We construct examples of deformations ρ close to ρ_0 for which φ_ρ cannot be a homeomorphism. Finally, in [CINS23], we make another surprising discovery: there is a complete loss of rigidity if we replace the flag spaces with the visual boundary ∂X of $X = \mathrm{SL}_d(\mathbb{R})/\,\mathrm{SO}(d)$. The lattice Γ has a natural action ρ_{v} on ∂X. We construct deformations arbitrarily close to ρ_{v}, but not semi-conjugate to ρ_{v}.

Broader context. My research program in Direction II fits into the broader context of studying the actions of higher rank lattices on compact manifolds. One incarnation of this quest is the *Zimmer program* and its analogues [Zim87, Wei11]. Robert Zimmer started the program in the wake of G. Margulis's superrigidity and R. Zimmer's cocycle superrigidity theorems [Mar91, Zim87]. The driving philosophy is that differentiable actions of higher rank lattices on 'low dimensional' manifolds should be essentially trivial or classifiable. The program has seen tremendous activity and success in recent years [Fis20]. While the Zimmer program concerns measure preserving differentiable actions, questions have often been asked about the possibility of a C^0-analogue and for actions that do not preserve measure, e.g. lattices acting on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ [Wei11]. To the best of our knowledge, our paper [CINS23] is the first one that tackles such C^0-rigidity problems in higher rank.

2. Ongoing Work and Future Directions

2.1. **Totally geodesic submanifolds (TGS) in higher rank.** A submanifold N of a Riemannian manifold M is called totally geodesic if, for any pair of points in N, the geodesic joining them (in the metric on M) lies entirely in N. Totally geodesic submanifolds (TGS) are objects of classical interest in geometry and dynamics, with tradition dating back to Riemann [Rie54] and Cartan [Car28].

The simplest example of a TGS is of course the one-dimensional ones, i.e. geodesics, and they are found in aplenty. Moreover, if M is a compact manifold with infinite fundamental group, then we can find infinitely many closed geodesics. However, TGS-s become a lot more rare when we look for them with the dimension of the TGS strictly greater than 1. In fact, guided by some recent results that we will explain below, the emerging consensus is that plenty of TGS-s of dimension greater than 1 can only arise under stringent restrictions on the manifold like homogeneity and arithmeticity. To simplify our current discussion, we will only focus on codimension 1 TGS-s, i.e. $\dim(TGS) = \dim(M) - 1$.

Context. Consider a closed hyperbolic 3-manifold M, i.e. $\pi_1(M)$ is a uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$. Then, each closed totally geodesic submanifold $N \subset M$ corresponds to a Fuchsian subgroup in $\pi_1(M)$, i.e. the subgroup $\pi_1(N)$ can be conjugated to a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$. Motivated by questions of Reid and McMullen, there has been much progress in understanding such closed TGS-s. There are now examples of M that contain exactly k closed totally geodesic N for any non-zero k [FLMS21, LP22]. However, the existence of infinitely many closed TGS $\{N_1, N_2, \dots\}$ in M forces extra-ordinary rigidity conditions of a number-theoretic nature. More precisely, [BFMS21] and [MM22] show that in this case, M must be an arithmetic hyperbolic 3-manifold. In recent work, [FFL24] has generalized this result to the case where M carries an analytic metric of variable negative curvature.

In another direction, Kahn-Markovic [KM12] had shown that any closed hyperbolic 3-manifold M contains immersed surfaces N that are nearly-totally geodesic. In geometric group theory language, $\pi_1(N)$ is a quasi-convex subgroup of $\pi_1(M)$. The existence of plenty of quasi-convex subgroups is connected to cubulability of the group $\pi_1(M)$ [BW12]. Note that a group is cubulable if it acts nicely on a CAT(0) cube complex. Cubulability has a variety of applications: from Agol's proof of virtual Haken and fibering theorem [Ago14] to providing a strong negation of Property (T) [CCJ$^+$01, CMV04].

Research Projects. The above results indicate that in rank one Lie groups, e.g. $\mathrm{PSL}_2(\mathbb{C})$, the study of TGS-s unlock a wealth of information. My research goal is to initiate a similar study of TGS-s in the context of discrete subgroups of higher rank Lie groups. Moving to higher rank produces a variety of unique challenges. For instance, the Riemannian symmetric space now – unlike $\mathbb{H}^3$ – is non-positively curved with plenty of flats. Moreover, the TGS-s of a higher rank symmetric space are highly constrained by Lie theory.

At this juncture, convex projective geometry proves to be a great boon. The Hilbert metric, although not CAT(0), has a wide variety of TGS-s that have a simple description. In particular, for a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, any codimension 1 projective hyperplane in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ that intersects Ω is a codimension 1 TGS of Ω in the d_Ω-metric. Thus, I adopt the viewpoint of studying discrete groups in higher rank via studying TGS-s in properly convex domains. Under this broad research umbrella of understanding TGS-s in higher rank, I will discuss two focused projects that I am currently invested in.

• **Haagerup property or Gromov's a-T-menability:** My goal here is to use TGS-s to establish the Haagerup property for a large class of discrete subgroups in higher rank. This question is due to Benoist[Ben12]. As a starting point, we consider discrete groups $\Gamma \leq \mathrm{SL}_d(\mathbb{R})$ that preserve and act co-compactly on strictly convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ for $d \geq 3$. In ongoing joint work with T. Nguyen [IN], we are proving that any such Γ is a-T-menable.

• **Rigidity and homogeneity:** In this project, my goal is to show that plenty of closed TGS-s force the Hilbert metric to become a homogeneous Riemannian metric, akin to the situation in [FFL24] alluded to above. However, in contrast to the negative curvature (i.e. rank one) setting of [FFL24], my work will be in the context of higher rank groups, thus introducing some unique challenges. To achieve my target theorem in this project, my strategy is to import tools from homogeneous dynamics into convex projective geometry. In particular, my idea is to use a Ratner-like theorem [Rat91, MS95] outside the purview of classical homogeneous dynamics.

To state my target theorem more precisely, consider $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$ that preserves a strictly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ and acts co-compactly on Ω. Then $M := \Omega/\Gamma$ is a closed strictly convex projective manifold of dimension d. Assume that M has infinitely many codimension 1 closed TGS-s $\{N_1, N_2, \dots\}$ where each N_i is 'hyperbolic', i.e. the lift $\widetilde{N_i}$ of N_i in the universal cover Ω is an open $(d-1)$ ball so that $\widetilde{N_i}$ is isometric to $\mathbb{H}^{d-1}$. My conjecture is that this will force Ω itself to be a projective d-ball, i.e. Ω is isometric to $\mathbb{H}^d$. In particular, the result of [BFMS21] would then imply that the group Γ that we started with is in fact an arithmetic (uniform) lattice inside some conjugate of $\mathrm{SO}(d,1)$.

2.2. **Geometric finiteness in projective geometry and representations beyond Anosov.** In hyperbolic geometry, *geometrically finite(GF) subgroups of* $\mathrm{SO}(d,1)$ encompass the broadest class of discrete subgroups that can be studied well using tools from geometry and dynamics. To get a sense of such groups, let us look at $\mathrm{SO}(2,1)$. A discrete subgroup $\Gamma \leq \mathrm{SO}(2,1)$ is GF if the surface $\mathbb{H}^2/\Gamma$ is a geometrically finite, i.e. it decomposes as the union of a compact surface with boundary and a union of cusps and funnels (see Figure). This topological decomposition has an equivalent, albeit dynamical, reinterpretation [Bow93]. The dynamical definition says, Γ is GF if it has convergence dynamics (a generalization of 'north-south' dynamics) on its limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^2$, and each point in Λ_Γ is either bounded parabolic (cusp point) or conical (endpoint of a geodesic lying in the compact part) [Yam04].

Context. In stark contrast to the rank one case above, the notion of geometrical finiteness in higher rank groups (e.g. $\mathrm{SL}_d(\mathbb{R})$ with $d \geq 3$) has remained elusive. In an ongoing joint research project [FIZ24] (with B. Fléchelles and F. Zhu; scheduled to be finished by the end of 2024), I am developing a theory of geometrically finite subgroups of $\mathrm{SL}_d(\mathbb{R})$ and fill this gap. A major challenge in this work is to control the behavior near cusp points. In fact, as Bowditch showed in [Bow95], cusp points already prove to be a delicate issue even in variable negative curvature. Our goal is to use convex projective geometry to tame the cusp regions and get a good theory of geometric finiteness in higher rank.

Research Project. The importance of our ongoing work [FIZ24] is that we develop geometric tools to study *rel-Anosov representations*, a broad generalization of Anosov representations. Our project [FIZ24] has two phases: first, we introduce a notion of geometric finiteness for subgroups of $\mathrm{SL}_d(\mathbb{R})$ (using convex projective geometry), and second, we identify a new family of discrete groups in $\mathrm{SL}_d(\mathbb{R})$ that encompass the geometrically finite ones. We christen this new family as *asymmetrically Anosov subgroups* and propose them as the right candidate for being *rel-Anosov*, owing to their geometric origin. We can already prove that *asymmetrically Anosov* subgroups are stable under certain small deformations (with some mild conditions on deformations of the 'cusps'). The broadest class of deformations that they can admit is the subject of our current investigation.

2.3. **Higher rank hyperbolicity in projective geometry.** My work [Isl] shows that rank one properly convex domains have a lot of 'hyperbolicity'. However, there are plenty of domains that are not rank one and hence, lack all 'hyperbolicity' in a strong sense. To digest this point, consider Pos_n, the space of positive definite n-by-n matrices with trace 1 and $n \geq 3$. This space Pos_n is the projective model of the symmetric space $X_n := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. As is well-known, X_n is very 'non-hyperbolic' for any $n \geq 3$.

However, it has recently been discovered that even when $n \geq 3$, the higher rank globally symmetric space X_n – equipped with its homogeneous Riemannian metric – exhibit a host properties that are now being popularly called 'higher rank hyperbolicity' [KL20]. An informal way to explain this notion is that, X_n satisfies a higher-rank Morse lemma above the dimension $(n-1)$, where $(n-1)$ is the maximal dimension of any flat in X_n. That is, any quasi-isometric embedding of $\mathbb{R}^{n-1}$ into X_n stays close to an actual flat (much like a quasi-geodesic in $\mathbb{H}^2$ staying close to a geodesic).

Inspired by this higher rank hyperbolicity phenomena, we asked: does Pos_n – with its Hilbert metric – also satisfy some notion of higher rank hyperbolicity? In spite the amazing results of [KL20] at our disposal, this is a complicated question. The Hilbert geometry of Pos_n is starkly different from the Riemannian geometry of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ when $n \geq 3$. In particular, much of the technology in [KL20] (and related work) uses some form of convexity of the distance function and the distance function d_Ω is far from being convex.

Research Project. In ongoing joint work with G. Raggo [IR], we have recently discovered a geometric approach to answering this question. In fact, we can address an even more general question: do 'higher rank' convex projective domains [Zim20] satisfy some notion of higher rank hyperbolicity? Our idea is to consider a higher dimensional "slim k-simplex" condition to charaterize higher rank hyperbolicity. The notion of slim simplices is inspired by results in [KL20, GL23] and is a higher dimensional generalization of Gromov's slim triangle condition, used in the theory of Gromov hyperbolic metric spaces [BH99]. The slim simplex condition proves to be extremely fruitful in convex projective geometry. To be more precise, in [IR], we consider a properly convex domain Ω such that Ω/Γ is compact and let m be the maximal dimension of any projective simplex (properly embedded) in Ω. Then, we prove that any projective $(m+1)$-simplex in Ω is D-thin for some constant D, independent of the simplex [IR].

Our immediate next goal is to use the slimness of $(m+1)$-simplices to define a *coarse median* for $(m+1)$-simplices. Using this as a motivation, we plan to introduce the notion *coarse k-median spaces* generalizing the notion of *coarse median spaces* that works for triples [Bow13, NWZ19, BL23]. Our work would imply that the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ has the structure of a coarse $(n-1)$-median space. This will answer a question raised in [BL23], where the authors prove a 2-median property for CAT(0) polygonal 2-complexes.

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