PROSEMINAR/SEMINAR: MATRIX (LIE) GROUPS

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Meeting Time: TBD
Language: The talks are supposed to be in English.
Pre-semester Meeting (Vorbesprechung):
- 17.02.2023 (Friday) 16.00-18.00 - Seminarraum 8 (Floor 4, Mathematisches Institut)

Abstract

This is a seminar targeted at mid-level as well as advanced undergraduate students. The goal of the course is to introduce some basic ideas in Lie theory through matrix groups. There will be a strong emphasis on looking at plenty of examples of matrix groups.

The first part of the course will be a fairly general introduction to matrix groups and their Lie algebras - both as algebraic objects and as tangent spaces at identity to the respective groups. In the second part, we will focus on compact matrix groups and discuss their structure theory. At the end, we will cover the classification theorem of compact semi-simple Lie groups and look at some concrete examples.

Recommended Pre-requisite

Linear Algebra, Analysis (Basic topology in $\mathbb{R}^n$ and the notion of derivatives), Basic abstract algebra (group theory).

Roughly speaking, if you feel comfortable with the first three chapters of Reference (1) below, this class is well-suited for you.

Key References

(1) Matrix Groups for undergraduates, Kristopher Tapp (AMS). (**for talks 1-6**)
(2) Matrix Groups: an Introduction to Lie Group Theory, Andrew Baker, Springer, 2002. (**for talks 1-11**)

Rules

Talk Timeline Suppose your talk is on day T. Between $T-14$ to $T-8$, you should have a meeting with me to discuss the material of your talk - give me a short overview of your talk, ask me questions if you have any, etc. Between $T-7$ and $T-3$, you should either send me your talk notes or have a second meeting to give me a short version of your talk.

On day T, you are expected to prepare two things besides your presentation:
- a handout (≤ 1 page) that is a list of things you are covering
- a problem handout with 1-2 problems. You will present the solution to one of them.

Date: February 2023.

Schedule

Date	Talk
April 20	Talk 1 and 2
April 27	Talk 3
May 4	Talk 6
May 11	Talk 4 and Talk 7
May 18	–
May 25	Talk 5
June 1	–
June 8	–
June 15	Talk 8
June 22	–
June 29	Talk 9, 10 and 11

LIST OF TALKS

1. Introduction to Matrix Groups and Examples

Jonas Biba and David Barth

We will introduce families of matrix groups which will serve as the main examples throughout the course.

Key examples: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$; orthogonal groups $O(n), U(n), SO(n), SU(n)$.

Explain how to think of complex matrix groups as real matrix groups.

Have a discussion on the group of isometries of the Euclidean space $\mathbb{R}^n$. Explain how orthogonal groups arise as groups that preserve certain bilinear forms.

It is also important to show some low-dimensional examples. Write down $SO(2)$ and $SO(3)$ and explain that it encodes positions on a cricle and a 2-dimensional sphere. Then write down $SU(2)$. Prove (or at least give some ideas about) the isomorphism of the groups $SO(3)$ and $SU(2)$.

Very important: give examples to show that not all groups are matrix groups!

If time permits, explain the group of quarternions and the matrix groups over them. Explain how they can be realized as matrix groups over $\mathbb{C}$ and hence $\mathbb{R}$.

2. Topology of Matrix Groups

Friedrich Homann

Discuss the topology on matrix groups - open sets, closed sets, limit points, homeomorphisms, connectedness, compactness.

The ideal way to structure this class is to first say that we will think of matrix groups as subsets of $\mathbb{R}^m$ by looking at the coordinate map. Then introduce all these concepts in $\mathbb{R}^m$. Then explain how to give a matrix group the subspace topology.

It is important that you give examples of open matrix groups, closed matrix groups, compact matrix groups, connected and disconnected matrix groups.

3. Lie Algebra

Robin Campbell and Dominik Svorad

In this class, we will define the Lie algebra of a matrix group at the tangent space of the group at identity.

For this, first define tangent vectors and tangent space for subsets of the Euclidean spaces. Then explain how this defines the Lie algebra.

Examples are crucial. Compute the Lie algebras in all of our key examples. For each of the key examples, compute the dimension of the Lie algebras.

Explain how the Lie algebras can be thought of as the space of left-invariant vector fields.

4. Matrix Exponentiation

Amelia Faber

Define exponential map for matrices. For this, you will first need to explain what it means for a series in a matrix group to converge.

Then show that matrix exponentiation is the solution of a differential equation that should remind one of $x'(t) = ax(t)$.

Explain the properties of this exponential map. Along the way, show that this lets you go from the Lie algebra back to the Lie group. Computation of examples is again crucial here.

Discuss the application of matrix exponentiation in solving a system of linear differential equations.

5. Matrix groups as manifolds

Simon Weiß

6. Lie Bracket

(1 person may sign up) + Emilian Arnold

Define Lie bracket as the commutator for matrix groups. Show the properties of the Lie bracket. Then show that we can also think of the Lie bracket as the derivative of the Adjoint.

Then explain that all the Lie algebras we had seen earlier have a Lie bracket (given by the commutator.)

Explain Lie algebra homomorphisms. Explain its relation to Lie group homomorphisms.

Finally, explain that Lie brackets can be defined more abstractly and give the abstract definition of a Lie algebra.

7. Lie groups and Homogeneous Spaces

Anton Tapking

Introduce the notion of matrix group actions on spaces and define homogeneous spaces.

As a first example, define real and complex projective spaces and realize them as homogeneous spaces.

As a second example, consider the full flag variety and realize it as a homogeneous space.

As the third example, define grassmanians and realize them as homogeneous spaces.

Compute tangent space to homogeneous spaces in these examples and show that they are subsets of Lie algebras but not necessarily Lie algebras.

8. Adjoint Representation

Philip Nazari and Alexander Marwitz

Define the adjoint representation of a Lie group. Show the relationship between the maps Ad and ad.

Compute the adjoint representation of $SO(3)$. More generally discuss the case of adjoint representation of compact matrix groups.

Use the tools developed so far to show the isomorphism between $SO(3)$ and $SU(2)$.

9. Maximal Tori

(2 people may sign up)

Introduce the maximal tori in a compact Lie group. Compute examples of maximal tori in the key examples of orthogonal groups. Explain what happens to maximal tori under conjugation and discuss the uniqueness of such tori.

Define the notion of rank. Also introduce the notion of Weyl group.

It is important that you properly compute the cases $SO(3), SU(2), U(2)$.

Compute the center of the group in the key examples of orthogonal groups. Show that

10. Structure of the Adjoint Representation

(2 people may sign up)

We work in the case of compact matrix groups only. Explain the root space decomposition of the adjoint representation of a compact, connected simple Lie group. It is important to do a concrete example. Possibly do $SO(3)$ and $SU(2)$.

Then explain how you can reduce the general case to the first case. This will require you to discuss the semi-simple factorization of the Lie algebra.

11. Classification of compact Lie groups: Roots Systems and Dynkin Diagrams

(3 people may sign up)

Explain what is an abstract root system.

Then quickly move to examples. It is important that you do the examples $A_1, A_2, B_2, A_1 \times A_1$ at least.

Then introduce the notion of Dynkin diagrams and draw them for the above ones.

Then show them the classification table to show much more exists. But we are not going to go into further details.

Then finally say how to go from the data of a root system back to Lie algebra.

Do a few concrete examples - maybe A_1, A_2, and B_2.

In the end, show the classification table of compact simple Lie groups. And say a few words about how this gives a complete list.