PROSEMINAR/SEMINAR: MATRIX (LIE) GROUPS

Lecturer: Dr. Mitul Islam
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Meeting Time: Thursdays 16.00 - 18.00
Meeting Place: TBD
Language: The talks are supposed to be in English.

Abstract

This is a seminar targeted at mid-level as well as advanced undergraduate students. The goal of the course is to introduce some basic ideas in Lie theory through matrix groups. There will be a strong emphasis on looking at plenty of examples of matrix groups.

The first part of the course will be a fairly general introduction to matrix groups and their Lie algebras - both as algebraic objects and as tangent spaces at identity to the respective groups. In the second part, we will focus on compact matrix groups and discuss their structure theory. At the end, we will cover the classification theorem of compact semi-simple Lie groups and look at some concrete examples.

Recommended Pre-requisite

Linear Algebra, Analysis (Basic topology in $\mathbb{R}^n$ and the notion of derivatives), Basic abstract algebra (group theory).

Roughly speaking, if you feel comfortable with the first three chapters of Reference (1) below, this class is well-suited for you.

Key References

(1) **(A very simple introductory textbook)** Matrix Groups for undergraduates, Kristopher Tapp (AMS). (**only for talks 1-6**)
(2) **(Main Reference Text)** Matrix Groups: an Introduction to Lie Group Theory, Andrew Baker, Springer, 2002.

Rules

Grading. You must give a talk to receive a grade. The quality of your talk will determine your grade. Please register on Müsli for the class, if you haven't.

If for some reason, you are unable to give the talk that you had signed up for, you should let me know at least two weeks ahead of your scheduled talk. Otherwise, you risk attracting a grade penalty. This of course does not apply to an emergency situation (e.g. medical or family). In case of such emergencies, a decision will be made on a case-by-case basis.

Participation. While attendance and participation isn't mandatory, you will receive bonus grade points for active participation in the seminar.

Date: February 2023.

Seminar Presentation

Preparing for the talk. Ideally one should start preparing for their talk around 3-4 weeks ahead of time. Suppose your talk is on the day T.

- Between $T-20$ to $T-8$, you should have a meeting with me to discuss the material of your talk: give me a short overview of your talk, ask me questions about the material, etc.
- Between $T-7$ and $T-3$, you should either send me your talk notes or have a second meeting to give me a 10-minute version of your talk.

Day of the talk. Presentations using a board is preferred. But if you have other ideas like using a tablet or slides, please discuss this with me ahead of time.

For your talk, you are expected to prepare two things besides your presentation:

- a 1-page handout (could be typed, written on a tablet, legibly hand-written, etc.) that contains a short list of things you are covering. You can send me this before the talk and I can make copies for everyone.
- a problem hand-out with 1 or 2 problems based on your presentation.

The talk. You should plan to speak for 35-40 minutes. Then you ask the class to work on one problem from your problem hand-out and give them a few minutes to brainstorm. Then there is a 5-minute session when there is some discussion and then you explain the solution.

Class time. Per the usual convention, we will start 15 minutes after the hour (i.e. 16.15 if the official start time is 16.00) and we will try to end 15 minutes before the two-hour mark (i.e. 17.45 if the official end time is 18.00). But if the last talk is going over the expected time, please be patient and do not interrupt the speaker.

There will often be two talks on the same day. On those days, there will be a 5-minute break between the two talks.

Questions. You are strongly encouraged to ask questions - both at the end during discussions as well as during the talk. You must not deter yourself from asking questions because you are worried that you will interrupt the talk.

Schedule

Date	Talk
April 20	Talks 1 and 2
April 27	Talk 3
May 4	Talk 6
May 11	Talks 4 and 7
May 18	–
May 25	Talks 9 and 5
June 1	–
June 8	–
June 15	Talk 8
June 22	–
June 29	Talks 10 and 11

LIST OF TALKS

1. Introduction to Matrix Groups and Examples

Emilian Arnold

We will introduce families of matrix groups which will serve as the main examples throughout the course.

Key examples: $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{SL}_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$; orthogonal groups $O(n), U(n), SO(n), SU(n)$.

Explain how to think of complex matrix groups as real matrix groups.

Have a discussion on the group of isometries of the Euclidean space $\mathbb{R}^n$. Explain how orthogonal groups arise as groups that preserve certain bilinear forms.

It is also important to show some low-dimensional examples. Write down $SO(2)$ and $SO(3)$ and explain that it encodes positions on a cricle and a 2-dimensional sphere. Then write down $SU(2)$. Prove (or at least give some ideas about) the following: $SU(2)$ is a double cover of $SO(3)$.

Very important: give examples to show that not all groups are matrix groups (Section 7.7 of Baker's book (Ref 2) gives such an example)!

If time permits, explain the group of quarternions and the matrix groups over them. Explain how they can be realized as matrix groups over $\mathbb{C}$ and hence $\mathbb{R}$.

2. Topology of Matrix Groups

Friedrich Homann

Discuss the topology on matrix groups - open sets, closed sets, limit points, homeomorphisms, connectedness, compactness.

The ideal way to structure this class is to first say that we will think of matrix groups as subsets of $\mathbb{R}^m$ by looking at the coordinate map. Then introduce all these concepts in $\mathbb{R}^m$. Then explain how to give a matrix group the subspace topology.

It is important that you give examples of open matrix groups, closed matrix groups, compact matrix groups, connected and disconnected matrix groups.

3. Lie Algebra

Robin Campbell and Dominik Svorad

In this class, we will define the Lie algebra of a matrix group at the tangent space of the group at identity.

For this, first define tangent vectors and tangent space for subsets of the Euclidean spaces. Then explain how this defines the Lie algebra.

Examples are crucial. Compute the Lie algebras in all of our key examples. For each of the key examples, compute the dimension of the Lie algebras.

Explain how the Lie algebras can be thought of as the space of left-invariant vector fields.

4. Matrix Exponentiation

Amelia Faber

Define exponential map for matrices. For this, you will first need to explain what it means for a series in a matrix group to converge.

Then show that matrix exponentiation is the solution of a differential equation that should remind one of $x'(t) = ax(t)$.

Explain the properties of this exponential map. Along the way, show that this lets you go from the Lie algebra back to the Lie group. Computation of examples is again crucial here.

Discuss the application of matrix exponentiation in solving a system of linear differential equations.

5. Matrix groups as manifolds

Simon Weiß

Define the notion of a smooth manifold (using charts, atlases, and transition maps). Then explain the notion of tangent vectors (as derivatives of curves) and tangent space at a point. Give some good and concrete examples here - say sphere $\mathbb{S}^2$, torus, and cylinder.

Then explain the notion of derivative for a map between two manifolds (before doing this, recall the notion of derivative for functions between $\mathbb{R}^m$ and $\mathbb{R}^n$). Give an example of the derivative of a function, say $f : \mathbb{S}^2 \to \mathbb{R}$ given by $f(x, y, z) = z$.

Then explain what a smooth map is.

Explain that matrix groups have the structure of smooth manifolds. Remind the audience of the earlier definition of Lie algebras that they are tangent spaces at identity.

Use this abstract notion of manifolds, to define Lie groups. Conclude the discussion by explaining that:

(a) all matrix groups are Lie groups (so we have already seen many examples of Lie groups!)
(b) but not conversely (remind the audience of the example from Talk 1 of a group that isn't a matrix group. It should be clear now that the example was a Lie group. The example came from Section 7.7 of Baker's book.)

6. Lie Bracket

Jonas Biba and David Barth

Define Lie bracket as the commutator for matrix groups. Show the properties of the Lie bracket. Then show that we can also think of the Lie bracket as the derivative of the Adjoint.

Then explain that all the Lie algebras we had seen earlier have a Lie bracket (given by the commutator.)

Explain Lie algebra homomorphisms. Explain its relation to Lie group homomorphisms.

Finally, explain that Lie brackets can be defined more abstractly and give the abstract definition of a Lie algebra.

7. Lie groups and Homogeneous Spaces

Anton Tapking

Introduce the notion of matrix group actions on spaces and define homogeneous spaces.

As a first example, define real and complex projective spaces and realize them as homogeneous spaces.

As a second example, consider the full flag variety and realize it as a homogeneous space.

As the third example, define grassmanians and realize them as homogeneous spaces.

Compute tangent space to homogeneous spaces in these examples and show that they are subsets of Lie algebras but not necessarily Lie algebras.

8. Adjoint Representation

Philip Nazari and Alexander Marwitz

Define the adjoint representation of a Lie group. Show the relationship between the maps Ad and ad.

Compute the adjoint representation of $SO(3)$. More generally discuss the case of adjoint representation of compact matrix groups.

Use the tools developed so far to show the isomorphism between $SO(3)$ and $SU(2)$.

9. Maximal Tori

Julia Piazolo

Introduce the maximal tori in a compact Lie group. Compute examples of maximal tori in the key examples of orthogonal groups. Explain what happens to maximal tori under conjugation and discuss the uniqueness of such tori.

Define the notion of rank. Also introduce the notion of Weyl group.

It is important that you properly compute the cases $SO(3), SU(2), U(2)$.

Compute the center of the group in the key examples of orthogonal groups. Show that

10. Structure of the Adjoint Representation

Mitul Islam

We work in the case of compact matrix groups only. Explain the root space decomposition of the adjoint representation of a compact, connected simple Lie group. It is important to do a concrete example. Possibly do $SO(3)$ and $SU(2)$.

Then explain how you can reduce the general case to the first case. This will require you to discuss the semi-simple factorization of the Lie algebra.

11. Classification of compact Lie groups: Roots Systems and Dynkin Diagrams

Mitul Islam

Explain what is an abstract root system.

Then quickly move to examples. It is important that you do the examples $A_1, A_2, B_2, A_1 \times A_1$ at least.

Then introduce the notion of Dynkin diagrams and draw them for the above ones.

Then show them the classification table to show much more exists. But we are not going to go into further details.

Then finally say how to go from the data of a root system back to Lie algebra.

Do a few concrete examples - maybe A_1, A_2, and B_2.

In the end, show the classification table of compact simple Lie groups. And say a few words about how this gives a complete list.