RESEARCH STATEMENT
Mitul Islam

1. Overview

My research lies at the interface of geometry, topology, discrete subgroups of Lie groups, geometric group theory, and dynamical systems. The unifying theme of my research is understanding:

- representations of a finitely-generated discrete group Λ into another group H, and
- the deformation spaces of such representations, i.e. the set $\mathrm{Hom}(\Lambda, H)$ of all group homomorphisms from Λ into H, possibly up to some equivalence.

The domain group Λ is usually $\pi_1(N)$, the fundamental group of a manifold or an orbifold N. The target group H could either be a Lie group (e.g. $\mathrm{SL}_d(\mathbb{R})$, $\mathrm{PGL}_d(\mathbb{R})$, $\mathrm{SO}(p,q)$) or it could be the homeomorphism group $\mathrm{Homeo}(F)$ of a compact manifold F (e.g. where F is the sphere S^d, projective space $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$, or a flag variety). In the former case, when H is a Lie group, the representations of Λ can be interpreted as actions of Λ preserving some geometric structure that originates from the Lie group H. In the latter case, when H is $\mathrm{Homeo}(F)$, the representations of Λ correspond to much more wild 'non-linear' actions of Λ on the compact manifold F.

By virtue of the former point of view, my research program is connected to a diverse set of areas like Teichmüller theory (classical and higher), linear algebraic groups, hyperbolic and convex projective geometry, moduli spaces of geometric structures, and Anosov representations. The latter point of view connects my research program to a different diverse set of areas, viz. classification of group actions on compact manifolds (i.e. Zimmer program and its C^0-analogues [Zim87, Lab98, Wei11]), hyperbolic and partially hyperbolic dynamical systems, geometric group theory, boundaries of groups, and coarse geometry. I will now give an overview of these two viewpoints that form the two key thrust areas of my research program.

Research Direction I: Convex Projective Structures and Groups Beyond Gromov Hyperbolicity

When the target group H is a Lie group, the space $\mathrm{Hom}(\Lambda, H)$ up to conjugation in H, i.e. $\mathrm{Hom}(\Lambda, H)/H$, is finite dimensional. Moreover, if $\Lambda = \pi_1(N)$ where N is a manifold or an orbifold, then $\mathrm{Hom}(\Lambda, H)/H$ can be interpreted as the moduli space of 'new' geometric structures on N. The geometric structures that I focus on in my research are *convex projective structures*, a generalization of hyperbolic structures. A d-dimensional manifold M has a convex projective structure if M is diffeomorphic to Ω/Γ, where $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain[1] and $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ preserves Ω.

Convex projective structures arise naturally when $H = \mathrm{PGL}_d(\mathbb{R})$ or $\mathrm{SL}_d(\mathbb{R})$, as we will see shortly. Consider the case where $\Lambda = \pi_1(\Sigma)$ where Σ is a closed surface of genus at least 2. When the target is $H = \mathrm{PSL}_2(\mathbb{R})$, it is well-known that the Teichmüller space − the space of all hyperbolic structures on Σ − is a a connected component of $\mathrm{Hom}(\Lambda, H)/H$. When the target is $H = \mathrm{PSL}_3(\mathbb{R})$, there is analogous connected component called the PSL_3-Hitchin component (i.e. $\mathrm{Hit}_3(\Sigma)$) that properly contains the Teichmüller space[Hit92, Wie18]. The space $\mathrm{Hit}_3(\Sigma)$ is precisely space of all convex projective structures on Σ [CG97].

I study convex projective structures on manifolds M of arbitrary dimension, and often M has infinite 'volume' (convex co-compact, geometrically finite, etc.). The central theme of my research program here is to characterize group theoretic properties of $\pi_1(M)$ entirely in terms of the convex projective geometry of M. To this end, I take the viewpoint of developing the theory of convex projective structures in analogy with Riemannian non-positive curvature and CAT(0) geometry. A particularly challenging aspect of this work is that the natural metric (the so-called Hilbert metric) associated to convex projective structures is generically never CAT(0).

I now summarize my selected papers in this area (see Sections 2 and 3 for details).

[Isl19] *Rank One Hilbert Geometries*: I introduce the notion of rank one isometries for a properly convex domain (or, Hilbert geometries) and develop the theory of rank one Hilbert geometries in analogy with rank one CAT(0) spaces. The main result is that a non-elementary rank one group in Hilbert geometry is an acylindrically hyperbolic group. Acylindrically hyperbolic groups [Osi16] are a much-studied generalization of Gromov hyperbolic groups that include mapping class groups, $\mathrm{Out}(F_n)$, etc. *See Section 2.1.*

[1] An open set $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain if $\overline{\Omega}$ is compact and convex in an affine chart.

Convex Co-compact Groups and Relative Hyperbolicity: (*see Section 3.2*) A group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is called *convex co-compact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$ - the convex hull of its limit set in Ω. In joint works [IZ23b, IZ22] with A. Zimmer, I completely characterize the relative hyperbolicity of Γ in terms of the projective geometry of $\mathcal{C}_\Omega(\Gamma)$, as we now explain. Suppose Γ is a convex co-compact group and $\mathcal{P} := \{P_1, \dots, P_m\}$ is a family of subgroups of Γ.

- [IZ23b] *Abelian peripherals:* (*published in Geom. & Top., with A. Zimmer*) We introduce the notion of *properly convex domains with isolated simplices*, analogous to CAT(0) spaces with isolated flats [HK05]. If each $P_i \cong \mathbb{Z}^{k_i}$ ($k_i \geq 2$), then we prove: Γ is relatively hyperbolic with respect to $\mathcal{P}$ if and only if $\mathcal{C}_\Omega(\Gamma)$ has isolated simplices.
- [IZ22] *General peripherals:* (*with A. Zimmer*) If the peripheral subgroups P_i are arbitrary (not necessarily abelian like above), then we prove that relative hyperbolicity of Γ is equivalent to the existence of an isolated family of convex submanifolds (not necessarily simplices).

[IZ21b] *Flat Torus Theorem in Projective Geometry*: (*published in J. London Math. Soc., with A. Zimmer*) We study properly embedded simplices, the convex projective analogues of totally geodesic flats in CAT(0). Our theorem is an analogue of the Flat Torus theorem from CAT(0) geometry. *See Section 3.1.*

Classification of Convex Co-compact Groups: (*see Section 3.3*) This is my work in the direction of examples of groups like above, i.e. relatively hyperbolic convex co-compact groups. Our goal is to understand the building blocks of convex co-compact groups.

- [IZ20] *3-manifold groups*: (*with A. Zimmer*) Consider closed 3-manifolds Y that admit a representation $\rho : \pi_1(Y) \to \mathrm{PGL}_d(\mathbb{R})$ with image a convex co-compact group. We prove that: if Y has non-trivial geometric decomposition, then Y is relatively hyperbolic (composed of hyperbolic pieces glued along 2-tori) and we construct a topological model for the Bowditch boundary of $\pi_1(Y)$.

- [IZ21a] *One dimensional faces*: (*with A. Zimmer*) We prove that if all boundary faces of $\mathcal{C}_\Omega(\Gamma)$ have dimension at most one, then the convex co-compact group Γ is relatively hyperbolic with respect to $\mathbb{Z}^2$ subgroups.

Research Direction II: Group Action on Boundary by Homeomorphisms and their Deformations

When the target group H is $\mathrm{Homeo}(F)$ (or $\mathrm{Diff}(F)$) for some compact manifold F, then the space $\mathrm{Hom}(\Lambda, H)$ is infinite dimensional. Such infinite dimensional spaces appear naturally while studying boundaries of groups and spaces, as we now explain. (*see below for a summary of my work in this direction, see Section 4 for details*)

For example, let Λ be a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$. Then Λ acts co-compactly on $\mathbb{H}^2$ and S^1 - the visual boundary of $\mathbb{H}^2$ - is a 'boundary' of the group Λ. In fact, Λ has a natural action ρ_0 on S^1 given by the Mobius action of $\mathrm{PSL}_2(\mathbb{R})$ on S^1. Sullivan showed that the action ρ_0 – arising from a hyperbolic structure on $\mathbb{H}^2/\Lambda$ – is a 'special' point in $\mathrm{Hom}(\Lambda, \mathrm{Diff}(S^1))$ [Sul85]. In particular, he proved that any $\rho \in \mathrm{Hom}(\Lambda, \mathrm{Diff}(S^1))$ that is sufficiently close to ρ_0 in C^1-topology, is conjugate to ρ_0, i.e. there exists $\phi_\rho \in \mathrm{Homeo}(S^1)$ such that $\phi_\rho \circ \rho = \rho_0 \circ \phi_\rho$. Sullivan's main tools come from hyperbolic dynamics and automatic coding of hyperbolic groups. *See Section 4.1.*

[CINS23] *Rigidity of lattice actions on boundaries*: (*with C. Connell, T. Nguyen, and R. Spatzier*) We ask the same question as Sullivan but for more complicated groups and in the category of actions by homeomorphisms. More precisely, we let Γ be uniform lattice in a semi-simple linear group G of higher rank, e.g. $\mathrm{SL}_3(\mathbb{R})$ (or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$). Such a group Γ has a natural action ρ_0 (by left multiplication) on the Furstenberg boundary G/P, e.g. the full flag variety in $\mathbb{R}^3$ (or $S^1 \times S^1$).

Sullivan's tools do not quite work here, because the group Γ isn't Gromov hyperbolic and we consider only C^0-deformations of ρ_0. Nonetheless, we can prove an analogous rigidity result, albeit with semi-conjugacies instead of conjugacies. More precisely, we prove that: for any action ρ that is sufficiently C^0-close to ρ_0, there exists a continuous surjective map $\phi_\rho : G/P \to G/P$ such that the adjoining diagram commutes. *See Section 4.2.*

$$\begin{tikzcd} G/P \arrow[r, "\rho"] \arrow[d, "\varphi_\rho"] & G/P \arrow[d, "\varphi_\rho"] \\ G/P \arrow[r, "\rho_0"] & G/P \end{tikzcd}$$

To the best of our knowledge, [CINS23] is the first paper that tackles the C^0 rigidity question in higher rank. Our methods are inspired by the rank one results in [BM22]. In [CINS23], we also make a surprising discovery special to higher rank. We discover a complete loss of rigidity if we replace the Furstenberg boundary by the visual boundary. Suppose X is the symmetric space associated to G, e.g. $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$ (or $\mathbb{H}^2 \times \mathbb{H}^2$), and $\rho_{\mathrm{vis}} : \Gamma \to \mathrm{Homeo}(\partial X)$ the action on the visual boundary of X. We construct a sequence (ρ_n) in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(\partial X))$ that converges to ρ_{vis} but none of the ρ_n are semi-conjugate to ρ_{vis}. *See Section 4.3.*

2. Rank One Properly Convex Domains: Beyond Hyperbolicity

Focus Area. In my research on convex projective structures, I focus on developing a theory for properly convex domains that is analogous to the theory of non-positively curved Riemannian manifolds, or more generally, CAT(0) metric spaces. However, the natural metric associated to convex projective structures (the so-called Hilbert metric) is generically not CAT(0). Hence a major focus of my work is to develop new tools for studying properly convex domains. To this end, I draw on ideas from geometric group theory, CAT(0) geometry, linear groups, dynamical systems, and metric geometry. I have written one paper in this direction - [Isl19].

Definition. A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an open subset such that $\overline{\Omega}$ is a compact convex domain in some affine chart. The closed subgroup $\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_d(\mathbb{R}) : g\Omega = \Omega\}$ of $\mathrm{PGL}_d(\mathbb{R})$ is called the automorphism group of the domain Ω. The natural metric structure on Ω is given by a canonical distance function d_Ω defined using projective cross-ratios. The distance d_Ω is called the Hilbert metric on Ω, $(\Omega, \mathrm{d}_\Omega)$ is called a Hilbert geometry, and $\mathrm{Aut}(\Omega)$ acts properly isometrically $(\Omega, \mathrm{d}_\Omega)$.

We say that a d-dimensional manifold M has a convex projective structure if M is homeomorphic to Ω/Γ where $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\Gamma < \mathrm{Aut}(\Omega)$ is a discrete group.

Context. Consider a manifold $M = \Omega/\Gamma$. We say that M is *strictly convex* if the topological boundary of Ω does not contain any non-trivial projective segment. Benoist had shown that compact strictly convex manifolds are reminiscent of compact negatively curved Riemannian manifolds. For instance, the geodesic flow on T^1M is Anosov and $\pi_1(M)$ is a Gromov hyperbolic group. This motivates a natural conjectural analogy between Riemannian non-positive curvature and convex projective geometry. However, a 2-simplex with its Hilbert metric has infinitely many geodesics between pairs of points and thus fails to be a CAT(0) metric space. This is the context of my research focus.

2.1. Rank One Hilbert Geometries [Isl19].

In Riemannian non-positive curvature, a compact manifold M is rank one if it has an abundance of 'hyperbolic' or 'negative curvature'-like properties. This is a notion that can be made precise in terms of parallel Jacobi fields orthogonal to closed geodesics in M. Thus this notion of rank one will not generalize to convex projective geometry, where the projective geodesic flow is generically only C^0.

In my paper [Isl19], I introduce a notion of rank one for properly convex domains. The definition of rank one automorphisms for a Hilbert geometry is motivated by the properties of Riemannian rank one isometries.

Definition 2.1. ([Isl19]) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain. An element $\gamma \in \mathrm{Aut}(\Omega)$ is called a* rank one automorphism *if*

(1) γ has an axis in Ω, i.e. γ acts by a translation along a projective line segment $(a, b) \subset \Omega$,

(2) none of the axes of γ are contained in half triangles, i.e. if $(a, b) \subset \Omega$ is an axis of γ and $z \in \partial\Omega$, then either $(a, z) \subset \Omega$ or $(z, b) \subset \Omega$.

We call a discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$ a rank one group *if Γ contains a rank one automorphism.*

This notion, although projective geometric at first glance, is secretly a dynamical one. For instance, it ensures that a rank one automorphism $\gamma \in \mathrm{Aut}(\Omega)$ acts on $\partial\Omega$ with 'north-south' dynamics. This mimics the boundary action of rank one isometries.

A rank one group that is generated by a single rank one automorphism will have two global fixed points on the boundary and would be an uninteresting interesting. Thus we will focus on *non-elementary* rank one groups, i.e. a rank one group that does not contain a finite index cyclic subgroup. In [Isl19], I develop the theory of non-elementary rank one groups in analogy with non-elementary rank one CAT(0) groups [Bal95]. This analogy between rank one in projective geometry and CAT(0) geometry has to be understood in a coarse sense using geometric group theory. In particular, my main result in [Isl19] is that non-elementary rank one groups belong to the family of acylindrically hyperbolic groups. Previously, Sisto had proven that non-elementary rank one CAT(0) groups are acylindrically hyperbolic. Other prominent examples of acylindrically hyperbolic groups include Gromov hyperbolic groups, relatively hyperbolic groups, non-exceptional mapping class groups, and outer automorphisms of free groups on at least two generators.

Theorem 2.2. (I. [Isl19, Theorem 1.6]) *A rank one group $\Lambda \leq \mathrm{Aut}(\Omega)$ is either virtually $\mathbb{Z}$ (i.e. has a finite index subgroup isomorphic to $\mathbb{Z}$) or is an acylindrically hyperbolic group.*

This theorem lets us exploit the rich theory of acylindrically hyperbolic groups and obtain several interesting applications in [Isl19], e.g. counting of closed geodesics, infinite-dimensionality of the space of quasi-morphisms, etc. I will discuss the counting result for closed geodesics. For any discrete group $\Gamma < \mathrm{Aut}(\Omega)$, we define its critical exponent as $\delta_\Gamma := \limsup_{n\to\infty} \frac{1}{n} \cdot \log \# \{g \in \Gamma : \mathrm{d}_\Omega(x, gx) \leq n\}$ for some $x \in \Omega$.

Proposition 2.3. (I. [Isl19]) *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is non-elementary rank one group such that Ω/Γ is compact. Let $\mathrm{Per(t)}$ count the number of conjugacy classes in Γ of rank one automorphisms whose translation length is at most t. Then there exist constants $D, T > 0$ such that for all $t \geq T$,*

$$\frac{1}{D}\frac{e^{t\delta_\Gamma}}{t} \leq \mathrm{Per}(t) \leq D\frac{e^{t\delta_\Gamma}}{t}.$$

I will now briefly discuss the proof of Theorem 2.2. The main technical tool that I use in the proof of Theorem 2.2 is the notion of contracting elements from geometric group theory that we now roughly explain. Suppose g is acts on a proper metric space (X, d) by 'translating' along an axis ℓ_g. Consider the closest-point projection for $\pi_g : X \to \ell_g$ for the metric d. If any metric ball in X that is disjoint from ℓ_g, projects to a set of uniformly bounded diameter, then g is called a contracting element. It turns out that this contraction behavior lends g a strong hyperbolic or negative curvature-like behavior. In [Isl19], I prove that an automorphism $\gamma \in \mathrm{Aut}(\Omega)$ is rank one if and only if it is a contracting element for its action on $(\Omega, \mathrm{d}_\Omega)$. Sisto established an analogous characterization of CAT(0) rank one isometries [Sis18].

3. Convex Co-compact Groups and Relative Hyperbolicity

Focus Area. In my research program into convex co-compact groups, I seek to develop an understanding of convex co-compact groups beyond Gromov hyperbolic groups. I develop tools to study convex co-compact groups that are motivated from a diverse set of areas - geometric group theory, dynamics of group actions on boundaries, Riemannian non-positive curvature and CAT(0) geometry. In this direction, I have written five papers - [IZ21b, IZ23b, IZ20, IZ21a, IZ22],

Context. In independent works [DGK17] and [Zim17], Danciger-Guéritaud-Kassel and Zimmer showed that convex co-compactness can be interpreted as a geometric characterization of the notion of much-studied notion of Anosov representations. More precisely, they prove that a Gromov hyperbolic group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is convex co-compact if and only if$\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ is a P_1-Anosov representation that preserves a properly convex domain in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. In the light of this theorem, understanding non-Gromov hyperbolic convex co-compact groups in $\mathrm{PGL}_d(\mathbb{R})$ becomes a point of convergence of several research programs: studying discrete linear groups, extending the theory of Anosov representations beyond Gromov hyperbolic groups, and so forth. In fact, the authors pose a question in [DGK17] about understanding relatively hyperbolic convex co-compact groups [DGK17, Question A.2]. I have since answered the question in joint work with A. Zimmer [IZ23b, IZ22]. But the area of non-Gromov linear groups is witnessing constant, rapid and exciting new developments, for instance [KL18, Zhu19, ZZ22, Wei22, IZ23a] to mention a few.

Definition. A prototypical example of a convex co-compact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is a free group $\Gamma = \langle g, h\rangle$ where g, h are hyperbolic isometries in 'ping-pong' configuration (i.e. fixed point sets of g and h are disjoint and both g, h have sufficiently large translation length). In this case, $\mathbb{H}^2/\Gamma$ is an infinite volume manifold, but it has a compact 'convex core'. More precisely, the limit set of Γ in $\mathbb{H}^2$ is a Cantor set and the convex hull of this limit set (denoted by $\mathcal{C}_{\mathbb{H}^2}(\Gamma)$) has a co-compact Γ action. Thus the *convex core*, $\mathcal{C}_{\mathbb{H}^2}(\Gamma)/\Gamma$, is compact (in fact, a compact surface with boundary where the boundary is the closed geodesic representative of $ghg^{-1}h^{-1}$).

Motivated by this, Danciger-Guéritaud-Kassel introduces the notion of convex co-compact subgroups of $\mathrm{PGL}_d(\mathbb{R})$ in [DGK17]. Suppose $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is an infinite discrete group that preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. The *full orbital limit set* of Γ consists of all accumulation points of all Γ orbits in Ω, that is $\mathcal{L}_\Omega(\Gamma) = \cup_{x\in\Omega} \left(\overline{\Gamma \cdot x} \cap \partial\Omega\right)$. The convex hull of the full orbital limit set is $\mathcal{C}_\Omega(\Gamma) := \mathrm{ConvHull}_\Omega\left(\mathcal{L}_\Omega(\Gamma)\right) \cap \Omega$.

Definition 3.1. *A group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is called convex co-compact if there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ that Γ preserves (i.e. $\Gamma < \mathrm{Aut}(\Omega)$), $\mathcal{C}_\Omega(\Gamma)$ is non-empty, and $\mathcal{C}_\Omega(\Gamma)/\Gamma$ is compact.*

This is a rich class of groups with several interesting sources of examples - for instance, projective structures on Gromov-Thurston manifolds, projective reflection groups, deformation and gluing of cusped hyperbolic 3-manifolds, etc. In many of these examples, the convex co-compact groups are not Gromov hyperbolic. This is in sharp contrast to the case of convex co-compact Kleinian groups (i.e. convex co-compact groups in $\mathrm{SO}(d,1)$ as opposed to $\mathrm{PGL}_{d+1}(\mathbb{R})$). A major advantage of convex co-compactness is its stablity under small deformations. More precisely, if $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is convex co-compact, then $\iota_\Gamma : \Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ is an interior point of $\mathrm{Hom}(\Gamma, \mathrm{PGL}_d(\mathbb{R}))/\mathrm{PGL}_d(\mathbb{R})$.

In the next three subsections, I will discuss the results of my research program in convex co-compact groups.

3.1. **Flat Torus Theorem for Convex Co-compact Groups** [IZ21b]**.** In order to study a non-Gromov hyperbolic convex co-compact group Γ, it is vital to understand the free abelian subgroups $\mathbb{Z}^k$ of Γ whose free rank k is at least 2. For non-positively curved Riemannian manifold M (more generally, CAT(0) spaces), the flat torus theorem provides this understanding: any free abelian subgroup $\mathbb{Z}^k$ in $\pi_1(M)$ is the fundamental group of an immersed k-dimensional torus in M. Since the Hilbert metric d_Ω on a k-simplex is not CAT(0) for any $k \geq 2$, the flat torus theorem doesn't apply. Nonetheless, we prove a convex projective analog of the flat torus theorem in [IZ21b], jointly with A. Zimmer.

Properly embedded simplices are convex projective analogues of totally geodesic flats in CAT(0) [Ben06, IZ21b]. Suppose Ω is a properly convex domain. Then $S \subset \Omega$ is a *properly embedded simplex* (or *PES* in short) if S is a simplex in some affine chart and the inclusion map $S \hookrightarrow \Omega$ is proper. A PES is called maximal if it is not properly contained in any other PES. We prove the following *'Flat Simplices Theorem'* characterizing free abelian subgroups of a convex co-compact group. Although we work in the convex co-compact case, this fundamental technical result is new even for co-compact groups in convex projective geometry.

Theorem 3.2. (I.-Zimmer [IZ21b, Theorem 1.6]) *If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Gamma$ is a maximal abelian subgroup, then:*

(1) there exists a PES $S_A \subset \mathcal{C}_\Omega(\Gamma)$ such that $A \cdot S_A = S_A$ and A fixes each vertex of S_A,

(2) A acts co-compactly on the PES S_A, and

(3) A contains a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S_A)}$.

3.2. **Relatively Hyperbolic Convex Co-compact Groups** [IZ23b, IZ22]**.** In my two papers [IZ23b] and [IZ22] written jointly with A. Zimmer, we provide a complete characterization of relatively hyperbolicity in terms of the projective geometry of the convex co-compact group (or $\mathcal{C}_\Omega(\Gamma)$, to be precise). This answers the question posed in [DGK17] alluded to above. Although our methods are very different (owing to the fact that the Hilbert metric d_Ω isn't a CAT(0) metric), our work is strongly influenced by Hruska and Kleiner's work on *CAT(0) spaces with isolated flats* [HK05].

Notation. *For the rest of this section, $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ will always denote a convex co-compact group, $\Gamma < \mathrm{Aut}(\Omega)$ where $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain, and Γ is relatively hyperbolic with respect to peripheral subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$.*

Our key contribution is introducing a geometric isolation property using the Hilbert metric that captures relative hyperbolicity. This isolation property is the notion of a *strongly isolated family* that we now define. Intuitively, the definition says that metric neighborhoods of distinct elements in $\mathcal{X}$ can only have uniformly bounded intersection.

Definition 3.3 ([IZ23b, IZ22])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. A collection $\mathcal{X}$ of closed unbounded convex subsets of $\mathcal{C}_\Omega(\Gamma)$ is calle* strongly isolated *if: for every $r > 0$ there exists $D_1(r) > 0$ such that for any distinct pair $X_1, X_2 \in \mathcal{X}$,*

$$\mathrm{diam}_\Omega \left(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)\right) \leq D_1(r).$$

We will now explain that existence of a large enough strongly isolated family characterizes relative hyperbolicity.

Case I (Abelian Peripherals). We first consider the case where each peripheral subgroup P_i is isomorphic to $\mathbb{Z}^{k_i}$ where $k_i \geq 2$. By Theorem 3.2, each P_i then corresponds to a properly embedded simplex of dimension at least 2. Let $\mathcal{S}_{\max}$ be the set of all maximal PES in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two.

We prove that in this setting, Γ is relatively hyperbolic if and only if $\mathcal{S}_{\max}$ is a strongly isolated family.

Theorem 3.4. (I.-Zimmer [IZ23b, Theorem 1.7]) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) Γ is a relatively hyperbolic group with respect to a family of virtually free abelian subgroups of rank at least two.

(2) $\mathcal{S}_{\max}$ is a strongly isolated family.

The proof of Theorem 3.4 is metric geometric. We introduce a new technical tool in [IZ23b] that we call *linear projections on simplices*. A key result in our paper is that in the setting of Theorem 3.4, a linear projection on a simplex S is coarsely equivalent to the closest-point projection on S for the Hilbert metric d_Ω. This coarse equivalence and geometric group theory results in [Sis13] are instrumental in proving (2) implies (1). The proof of (1) implies (2) essentially comes from using the tools in [DS05] and the convex projective flat torus theorem Theorem 3.2.

Case II (General Peripherals). In [IZ23b], we drop the assumptions on the peripheral subgroups. In the case of abelian peripheral, the maximal PES – the analogues of totally geodesic flats – were the canonical candidates for non-hyperbolic behavior. The trade-off for dropping the abelian assumption is that we no longer have any such canonical choice for 'non-hyperbolic' regions. Rather, the choice will be influenced by the structure of the peripheral subgroups $\mathcal{P}$. In order to capture this, we require the notion of a *peripheral family* of subsets of $\mathcal{C}_\Omega(\Gamma)$.

Definition 3.5 ([IZ22])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. A collection $\mathcal{X}$ of closed unbounded convex subsets of $\mathcal{C}_\Omega(\Gamma)$ is callled a* peripheral family for Γ *if:*

(1) $\mathcal{X}$ is a strongly isolated family and $\mathcal{X}$ is Γ-invariant *(i.e. $\Gamma \cdot \mathcal{X} = \mathcal{X}$).*

(2) $\mathcal{X}$ coarsely contains the properly embedded simplices of $\mathcal{C}$*, i.e. there exists $D_2 > 0$ such that: if $S \subset \mathcal{C}_\Omega(\Gamma)$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset \mathcal{N}_\Omega(X, D_2)$.*

We prove that the relatively hyperbolicity of Γ is equivalent to the existence of a peripheral family for Γ.

Theorem 3.6. (I.-Zimmer [IZ22]) *Suppose that $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) If Γ is relatively hyperbolic with respect $\mathcal{P} = \{P_1, \ldots, P_m\}$, then

$$\mathcal{X} := \Gamma \cdot \{X_1, \ldots, X_m\}$$

is a peripheral family for Γ where each X_j is the closed convex hull of the limit set of P_j in Ω.

(2) If $\mathcal{X}$ is a peripheral family for Γ and $\mathcal{P} := \{P_1, \ldots, P_m\}$ is a set of representatives of the Γ-conjugacy classes in $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$, then Γ is relatively hyperbolic with respect $\mathcal{P}$.

Although Theorem 3.6 is similar to Theorem 3.4 in appearence, the proofs are completely different. In particular, the proof of Theorem 3.6 is dynamical and relies on Yaman's characterization of relative hyperbolicity. In fact, this change of viewpoint in [IZ22] was absolutely crucial since our main tool in Theorem 3.4 - linear projections - do not generalize to arbitrary peripheral subgroups. In the proof, we construct an appropriate quotient of $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ where the group acts by a geometrically finite convergence group action. Interestingly, the same proof also implies that the quotient of $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ that we consider is a model for the Bowditch boundary. We discuss this below.

Topological model for the Bowditch boundary. If Γ is a convex co-compact group relatively hyperbolic with respect to $\mathcal{P} = \{P_1, \ldots, P_m\}$, then our work provides a topological model for the abstract Bowditch boundary $\partial(\Gamma, \mathcal{P})$. We construct this topological model as quotient of the ideal boundary $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ of $\mathcal{C}_\Omega(\Gamma)$. Recall that $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma) = \overline{\mathcal{C}_\Omega(\Gamma)} \cap \partial\Omega$. Consider the equivalence relation $\sim_{\mathcal{P}}$ on $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$: $x \sim_{\mathcal{P}} y$ if and only if x and y belong to the limit set $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ of the group $\gamma P_j \gamma^{-1}$ where $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$. We define

$$[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}} := \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma) \,/ \sim_{\mathcal{P}},$$

i.e. the quotient of $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ obtained by collapsing the limit set of each $\gamma P_j \gamma^{-1}$. A priori, this doesn't even need to be a Hausdorff space. However, for relatively hyperbolic convex co-compact groups, we show that $[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ is a topological model for the Bowditch boundary of Γ.

Theorem 3.7. (I.-Zimmer [IZ22]) *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group and that Γ is relatively hyperbolic with respect to $\mathcal{P} = \{P_1, \ldots, P_m\}$. Then there is a Γ-equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$.*

3.3. **Classification of Convex Co-compact Groups and Examples** [IZ20, IZ21a]**.** Most of the examples of convex co-compact groups in the literature are relatively hyperbolic, unless they come out of a product or a free-product construction. Hence, it begs the question whether the basic building blocks of convex co-compact groups are necessarily relatively hyperbolic. I would prefer to label this as the *classification problem for convex co-compact groups*. I pursue this question in my two papers [IZ20] and [IZ21a] that I will now discuss.

Three Manifold Groups [IZ20]**.**

Many examples of convex co-compact groups come from 3-manifold groups [Ben06, CLM16, BDL18]. Moreover, all these examples are relatively hyperbolic with abelian peripherals. Thus, in [IZ20], we investigate the following.

Question: *Suppose M is a closed, irreducible, orientable 3-manifold and $\rho : \pi_1(M) \to \mathrm{PGL}_d(\mathbb{R})$ is a faithful representation whose image is a convex co-compact group in $\mathrm{PGL}_d(\mathbb{R})$. Can we determine the geometry of the 3-manifold M?*

This question can also be interpreted as the quest for a Higher Teichmüller theory [BIW14, Wie18] for 3-manifolds, i.e. studying discrete faithful linear representations of closed 3-manifold groups into $\mathrm{PGL}_d(\mathbb{R})$. We provide the following answer to the question.

Theorem 3.8. (I.-Zimmer [IZ20, Theorem 1.3]) *Suppose M is as in the* **Question** *above. Then:*

(1) either M is geometric and supports a $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{R} \times \mathbb{H}^2$ structure, or

(2) M is non-geometric and each component in the geometric decomposition of M supports a $\mathbb{H}^3$ structure.

Moreover, when M is non-geometric, $\pi_1(M)$ is a relatively hyperbolic group with respect to a family of subgroups isomorphic to $\mathbb{Z}^2$.

In [Ben06], Benoist had obtained similar results for the case $d = 3$ and co-compact groups. But he used very different methods. While Benoist relied on actions on $\mathbb{R}$-trees for proving his result, our main tools are more algebraic. Our main tool is a characterization result for centralizers in convex co-compact groups.

One Dimensional Boundary Faces [IZ21a]**.**

We prove a sufficient condition for a convex co-compact group to be relatively hyperbolic with $\mathbb{Z}^2$ peripherals. The condition is a geometric one. Roughly speaking, it demands that every point in the ideal boundary $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ is contained in a face of dimension at most 1. We now explain a more precise formulation.

Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. We say that boundary of $\mathcal{C}_\Omega(\Gamma)$ has dimension at most 1 if for any $x \in \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$, $\dim \mathbb{P}(\mathrm{Sp}\, F_\Omega(x)) \leq 1$. Note that since $F_\Omega(x) \subset \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$, this says that the boundary face of Ω that contains x is at most one dimensional.

Theorem 3.9. [IZ21a] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ is convex co-compact, and $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$. Then the following are equivalent:*

(1) boundary of $\mathcal{C}_\Omega(\Gamma)$ has dimension at most 1

(2) Γ is a relatively hyperbolic group with respect to $\mathcal{P} = P_1, \ldots, P_m$ where each P_i is isomorphic to $\mathbb{Z}^2$.

4. Local Rigidity of Boundary Actions

Focus Area. My research program here is focused on groups acting on their boundaries by homeomorphisms and the rigidity/flexibility of such actions. I use tools from differential geometry, theory of foliations, (partially) hyperbolic dynamical systems, Lie theory, geometric group theory, and coarse geometry to study such boundary actions. In contrast to my research program discussed in the previous two sections, these boundary actions are not amenable to tools from linear groups and geometric structures. I have written one paper in this direction - [CINS23].

Context. My research program as stated above encompasses a large variety of problems since the words group and boundary could be interpreted in several ways. For instance, one could consider Gromov hyperbolic groups acting on their Gromov boundaries. Alternatively, one could consider groups $\pi_1(M)$ where M is a compact non-positively curved Riemannian manifold and consider the action of $\pi_1(M)$ on the visual boundary $\partial\widetilde{M}$ of the universal cover $\widetilde{M}$. A third example could be lattices in semi-simple Lie groups, like $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$, acting on its Furstenberg boundary. The first case, that of Gromov hyperbolic groups, has a long tradition and has been investigated by many authors since Sullivan [Sul85, Ghy93, Man18, BM22, MM21, MMW22]. My first paper [CINS23] in this area analyzes the third example - lattice actions on Furstenberg and visual boundaries. With my collaborators, I am actively working on better understanding the second class of examples.

My research program in this direction could also be interpreted in the broader context of studying actions of higher rank lattices on compact manifolds. One incarnation of this quest is the Zimmer program and its analogues [Zim87, Lab98, Wei11]. Zimmer started the program in the wake of Margulis's superrigidity and Zimmer's cocycle superrigidity theorems [Mar91, Fis20]. The driving philosophy was that differentiable actions of lattices on 'low dimensional' manifolds should be essentially trivial or classifiable. The program has seen tremendous activity and success in recent years [BFH22]. While Zimmer program concerns measure preserving differentiable actions, questions have often been asked about the possibility of a C^0-analogue and for actions that do not preserve measure (e.g. lattices acting on the Furstenberg boundary) [Wei11]. To the best of our knowledge, our paper [CINS23] is the first one that tackles this problem.

Definitions.

Actions on manifolds. Suppose Λ is a finitely generated discrete group with a finite symmetric generating set S_Λ. Suppose F is a compact manifold (equipped with a distance function d_F compatible with the manifold topology). We will use the notation

$$\mathcal{X}(\Lambda, F) := \mathrm{Hom}(\Lambda, \mathrm{Homeo}(F)).$$

We equip $\mathcal{X}(\Lambda, F)$ with the so-called C^0-topology that we now explain. Two points $\rho, \rho_0 \in \mathcal{X}(\Lambda, F)$ are C^0-close (i.e. close in C^0-topology) if the continuous maps $\rho_0(s)$ and $\rho_0(s)$ are close for each $s \in S_\Lambda$.

The right notion of rigidity in this C^0-category is C^0-*local semi-rigidity*. We will say that $\rho_0 \in \mathcal{X}(\Lambda, F)$ is C^0-*locally semi-rigid* if for any $\rho \in \mathcal{X}(\Lambda, F)$ sufficiently C^0-close to ρ_0, there exists $\varphi_\rho : F \to F$ such that the diagram commutes for all $\gamma \in \Lambda$.
The map φ_ρ is called a (ρ, ρ_0) *semi-conjugacy* and we say that ρ_0 is a *topological factor* of ρ.

$$\begin{tikzcd} F \arrow[r, "\rho(\gamma)"] \arrow[d, "\varphi_\rho"] & F \arrow[d, "\varphi_\rho"] \\ F \arrow[r, "\rho_0(\gamma)"] & F \end{tikzcd}$$

If instead, we consider $\mathrm{Hom}(\Lambda, \mathrm{Diff}(F))$, then the right topology to consider is the C^1-topology. Two actions $\rho, \rho_0 \in \mathrm{Hom}(\Lambda, \mathrm{Diff}(F))$ are C^1-close if the maps $\rho(s)$ and $\rho_0(s)$ are close and their derivatives $d\rho(s)$ and $d\rho_0(s)$ are also close, for each $s \in S_\Lambda$.

Lattice actions on boundaries. Suppose Γ is a uniform lattice in a semi-simple linear Lie group G without compact factors, for instance $G = \mathrm{SO}(d, 1), \mathrm{SL}_d(\mathbb{R})$, or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Then Γ acts properly discontinuously on the associated symmetric space $X = G/K$, where $K < G$ is the maximal compact subgroup. There are two kinds of boundary actions that we will consider:

(1) Let $P < G$ be the minimal parabolic (e.g. the group of upper triangular matrices in $\mathrm{SL}_d(\mathbb{R})$). Then G/P is the so-called Furstenberg boundary of the symmetric space X. More, generally we can consider any other parabolic subgroup $Q < G$ and consider the boundary G/Q.

(2) The symmetric space X carries a complete non-positively curved G-invariant metric (unique up to scaling in each factor). The visual boundary ∂X of X is another boundary and $\rho_{\mathrm{vis}} : \Gamma \to \mathrm{Homeo}(\partial X)$ is the natural action of Γ on it.

4.1. **Rigidity in Rank One.** We first discuss the case of rank one group $G = \mathrm{SO}(d+1,1)$ where $d \geq 1$. In this case, a uniform lattice $\Gamma < G$ is the fundamental group a hyperbolic orbifold $\mathbb{H}^{d+1}/\Gamma$ and both the Furstenberg and the visual boundaries are $\partial\,\mathbb{H}^{d+1} \cong S^d$, a d-dimensional sphere. The boundary actions ρ_0 and ρ_{vis} are both induced by the conformal action of $\mathrm{SO}(d+1,1)$ on S^d.

Sullivan[Sul85] had proven that any action $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(S^d))$ that is sufficiently C^1-close to ρ_0 (equivalently, ρ_{vis}) is conjugate to ρ_0 (i.e. there exists $\varphi_\rho \in \mathrm{Homeo}(S^d)$ such that $\varphi \circ \rho = \rho_0 \circ \varphi$). But the case of C^0-deformations, i.e. when we consider ρ C^0-close to ρ_0 remained open until the recent work [BM22]. In [BM22], Bowden-Mann prove that for any ρ sufficiently C^0-close to ρ_0 in $\mathcal{X}(\Gamma, S^d)$, ρ_0 is a topological factor of ρ. It is important to note that this is the best possible answer that one can hope for in the C^0 category of actions by homeomorphisms. In fact, [BM22] perform a 'Denjoy-like' construction and produce actions $\rho \in \mathcal{X}(\Gamma, S^d)$ that are arbitrarily C^0-close to ρ_0, but none of them are conjugate to ρ_0. One would be remiss to not mention the work of Matsumoto who had solved the problem in C^0-case but only for uniform lattices in $\mathrm{SO}(2,1)$. But Matsumoto's methods were specific to surfaces and does not immediately generalize to higher dimensions.

We now briefly explain the key idea in [BM22]. As is perhaps already clear from the timegap between [Sul85] and [BM22], it is well-known that C^0-deformations are notoriously hard to tame. In [Sul85], Sullivan's main tools were C^1-structural stability of hyperbolic dynamical systems and the coding of geodesics in a hyperbolic group using a finite automaton. Unfortunately, the hyperbolicity property of a dynamical system can be destroyed with C^0-deformations. A clever trick in [BM22] is to utilize the hyperbolicity of the underlying space $\mathbb{H}^d$ instead of the dynamical system. More precisely, Bowden-Mann uses the fact that a uniform lattice in $\mathrm{SO}(d+1,1)$ acts on S^d by a uniform convergence action and that $\mathbb{H}^d$ satisfies a Morse lemma (i.e. quasi-geodesics in $\mathbb{H}^d$ uniformly fellow travel a geodesic in $\mathbb{H}^d$).

4.2. **Rigidity in Higher rank for G/Q boundaries.** In the paper [CINS23], we tackle the C^0 rigidity question for uniform lattices in higher rank Lie groups, e.g. $G = \mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. In this case, the Furstenberg boundary and the visual boundary ∂X are distinct and inequivalent. Here we consider the case of Furstenberg (more generally, G/Q) boundary. Our main result is a local rigidity result of the same flavor as in [BM22]. We prove that if $\rho \in \mathcal{X}(\Gamma, G/Q)$ is sufficiently C^0-close to ρ_0, then ρ has ρ_0 as a topological factor. Moreover, the semi-conjugacy map φ_ρ gets uniformly close to id.

Theorem 4.1. *Let G be a connected linear semi-simple Lie group without compact factors, Γ be a uniform lattice in G, Q be a parabolic subgroup of G and $\rho_0 : \Gamma \to \mathrm{Homeo}(G/Q)$ is the boundary action (i.e. action by left multiplication). Then there is a neighborhood $\mathcal{U}_0$ of ρ_0 (in $\mathcal{X}(\Gamma, G/Q)$ in C^0-topology) and a neighborhood V_0 of* id *(in the space of continuous maps on G/Q) such that: for every $\rho \in \mathcal{U}_0$, there exists a unique (ρ, ρ_0)-semiconjugacy $\phi_\rho \in V_0$. Moreover, ϕ_ρ converges to* id *uniformly as ρ converges to ρ_0.*

Sketch of Proof. We now briefly sketch the key idea in the proof of Theorem 4.1 in [CINS23] in the case where $Q = P$, the minimal parabolic. Although we take inspiration from [BM22], note that we do not have much hyperbolicity in this higher rank setting. Our main trick is to work with the Weyl chamber flow which is a partially hyperbolic dynamical system. In the first step, we convert the data of an action ρ into the data of a 'foliation' of a flat bundle over X/Γ. In the second step, we manipulate this 'foliation' to obtain quasi-flats in the symmetric space X/Γ. Now the powerful theorems about quasi-flats in higher rank symmetric space enables us to produce the semi-conjugacy map.

We now discuss an application of our Theorem 4.1 in the local C^1-rigidity problem. As a corollary to our theorem, we recover the result that ρ_0 is locally C^1-rigid. This well-known result is originally due to Katok-Spatzier [KS97], and independently Kanai [Kan96]. Recently [KKL22] proved this for deformations in Lipschitz topology.

Corollary 4.2. [Kan96, KS97, KKL22] *Assume the same setup as in the previous theorem. Then ρ_0 is C^1-locally rigid, i.e. for any $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(G/Q))$ that is sufficiently C^1-close to ρ_0, there exists $\varphi_\rho \in \mathrm{Homeo}(G/Q)$ such that $\varphi \circ \rho = \rho_0 \circ \varphi$.*

It is worthwhile to remark that the conclusion of our Theorem 4.1 is the best possible in the C^0-category. We adapt the 'Denjoy-like' construction to show that 'semi-conjugacy' cannot be replaced by 'conjugacy'. We essentially follow the same strategy as in [BM22] of 'blowing up' a dense Γ orbit. While the [BM22] construction applies only to topological spheres, our construction works for any compact manifold F.

Proposition 4.3. *Consider any C^1-action $\rho_0 \in \mathcal{X}(\Lambda, F)$ of a countable group Λ on a compact smooth manifold F. If $\rho_0(\Lambda)$ has a dense orbit in F, then there exists an action $\rho \in \mathcal{X}(\Lambda, F)$ that is not C^0 conjugate to ρ_0. Moreover, ρ can be chosen to be arbitrarily C^0 close to ρ_0.*

4.3. Non-rigidity of the Visual Actions.

Theorem 4.4. (Connell-I.-Nguyen-Spatzier [CINS23]) *Suppose X is a simply connected higher rank symmetric space of non-compact type and G is the connected component of identity in $\mathrm{Isom}(X)$. Let $\Gamma < G$ be a lattice (not necessarily uniform) and $\rho_{\mathrm{vis}} \in \mathcal{X}(\Gamma, \partial X)$ be its natural action on the visual boundary ∂X. Then, there exists $\rho \in \mathcal{X}(\Gamma, \partial X)$, arbitrarily C^0-close to ρ_{vis}, such that ρ_0 isn't a topological factor of ρ.*

We now discuss the construction used in proving the above theorem. Note that the construction will produce a deformation of the full G-action on ∂X, not just the Γ action. First recall that ∂X has the structure of a spherical building. In particular, ∂X can be constructed by gluing together spherical simplies $\{gW_{\mathrm{mod}} : g \in G\}$ where W_{mod} is the model closed Weyl chamber. Thus it suffices to construct the action ρ on each simplex.

We fix the following ingredients: (a) fix the center of mass c_{mod} of W_{mod}, and (b) fix the Radon-Nikodym derivative cocycle $\beta : G \times G/P \to \mathbb{R}$, coming fixing a Haar measure μ on the Furstenberg boundary G/P. Then, for $g \in G$, we define the action $\rho(g)$ by the following: for any $h \in G$, let $\rho(g) : hW_{\mathrm{mod}} \to ghW_{\mathrm{mod}}$ take $h \cdot c_{\mathrm{mod}}$ to $gh \cdot c_{\mathrm{mod}}$ and then act by a linear dilation of magnitude $\beta(g, hP)$ around hc_{mod}. Since the action ρ doesn't alter the boundary of any closed Weyl chamber, it easily extends to an action of G. In [CINS23], we use the action of 'regular' elements on ∂X to prove that ρ_{vis} cannot be a factor of ρ. Finally, to construct actions arbitrarily close to ρ_{vis}, it suffices to replace the cocycle β by β^α where $\alpha \in (0, 1)$ is sufficiently close to 0.

5. Future Directions

Geometric Finiteness and Relative Anosov Representations. Geometrically finite groups in $\mathrm{SO}(d, 1)$ are a rich class of groups that extend the family of convex co-compact Kleinian groups in $\mathrm{SO}(d, 1)$. For example, a non-uniform lattice $\Gamma < \mathrm{SO}(d, 1)$ is geometrically finite, but not convex co-compact.

I am interested in developing a theory of *geometrically finite subgroups of* $\mathrm{PGL}_d(\mathbb{R})$. Following the idea of [DGK17], I plan to use convex projective geometry to define geometric finiteness. In work in progress with F. Zhu [IZ23a], we now have a definition of *projectively geometrically finite (PGF)* groups in $\mathrm{PGL}_d(\mathbb{R})$. We say that a discrete group $\Gamma < \mathrm{Aut}(\Omega)$ is PGF provided:

(1) Γ acts on the limit set $\mathcal{L}_\Omega(\Gamma)$ by a geometrically finite convergence group action,

(2) every point in the limit set $\mathcal{L}_\Omega(\Gamma)$ is extreme and C^1 (i.e. has a unique supporting hyperplane for Ω), and

(3) $\mathcal{L}_\Omega(\Gamma)$ is visible (i.e. for any $x \neq y \in \mathcal{L}_\Omega(\Gamma)$, the projective segment $(x, y) \subset \Omega$).

In [IZ23a], we develop the theory of PGF groups and prove that they are closely related to relatively Anosov subgroups, as defined and studied in [KL18, Zhu19, ZZ22]. Moreover our notion of PGF is also closely related to the notion of transverse subgroups recently introduced in [CZZ22]. I must also compare our notion of PGF to the existing notion of *strong geometric finiteness* in projective geometry that was introduced in [CM14]. The notion in [CM14] is very restrictive as it forces all cusp subgroups to be conjugate to a cusp subgroup in some $\mathrm{SO}(d, 1)$. But our definition of PGF doesn't enforce any such special conditions on cusp subgroups.

In the next step of our project, we are working on developing a more dynamical notion of geometric finiteness and constructing new examples of geometrically finite groups.

Quantitative characterization of geometric group theoretic properties. Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is an infinite discrete group. As Γ is a linear group, there are various numerical quantities attached to individual elements of Γ, e.g. singular values, eigenvalues, their ratios, etc. In the setting where Γ is Gromov hyperbolic, such numerical data has already been useful in analyzing the properties of Γ, for instance using exponential growth of singular values to characterize Anosov representation [GGKW17].

My current research goal is to develop an analogous quantitative understanding of other geometric group theoretic properties of Γ in terms of these numerical quantities. To this end, I have a joint project with T. Weisman that

I will now outline[IW23]. Consider a group $\Gamma < \mathrm{Aut}(\Omega)$ such that Ω/Γ is compact, but Γ is not necessarily Gromov hyperbolic. Consider a sequence (γ_n) in Γ that 'tracks' along a projective geodesic ray in Ω. Now suppose that the geodesic ray is *Morse* in (the sense of geometric group theory), i.e. the ray witnesses an abundance of negative curvature-like behavior in $(\Omega, \mathrm{d}_\Omega)$ as it goes towards infinity. It is natural to expect that some singular values of the sequence (γ_n) should detect this Morse property. In [IW23], we already have partial results affirming this expectation. In particular, we can show that the singular value ratio $\frac{\log \sigma_1 - \log \sigma_2}{\log \sigma_1 - \log \sigma_d}(\gamma_n)$ indeed detects Morse-ness. The next point of investigation is to relate Morse-ness to the 'regularity' of the boundary of Ω near the endpoint of a Morse geodesic ray.

Boundary actions of non-positively curved Riemannian manifolds. Suppose M is a compact rank one non-positively curved Riemannian manifold and let $\rho_0 : \pi_1(M) \to \mathrm{Homeo}(\partial\widetilde{M})$ be the action of $\pi_1(M)$ on the visual boundary $\partial\widetilde{M}$ of the universal cover $\widetilde{M}$. I have the following broad and ambitious research question:

Question: Investigate the C^0-local rigidity of the aforementioned action ρ_0.

Note that I have posed the question only for rank one manifolds. Indeed, rank rigidity theorem [BS87, Bal95] would imply that the higher rank case is already covered by our Theorem 4.1. Since the question is quite broad, a good place to start is by looking at two families of examples: (a) Manifolds with isolated flats, and (b) Three dimensional graph manifolds.

I am currently investigating the case of three dimensional graph manifolds jointly with C.Connell, T.Nguyen, and R. Spatzier. In this case, our strategy is to use the Tits boundary which has been made quite explicit by the work of Croke-Kleiner [CK00]. In fact, the Tits geometry of the boundary provides some invariants that could be used to distinguish semi-conjugacy classes of boundary actions. But the combinatorial data involved is rather complex. Hence we have only been able to study some specific examples. Our investigation so far indicates that local C^0 semi-rigidity might not hold for all rank one graph manifolds. But on the other hand, graph manifolds are extreme examples in a certain sense. We next plan to investigate manifolds with isolated flats. We expect them to be a more tame class of examples.

In a similar direction, there are several questions asked by Bowden-Mann in [BM22].

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