# RESEARCH STATEMENT
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## 1. Overview of my Research Program

A fundamental question in geometry is to obtain concrete construction of manifolds. For instance, consider the classical Poincare polygon theorem that provides a recipe for building surfaces by gluing edges of a regular polygon. But such purely geometric methods become intractable as dimension goes up. A more systematic approach to this problem is the study of discrete subgroups of Lie groups. For instance, the above problem of constructing a surface - of genus at least two - can be recast as a problem of constructing a lattice $\Gamma$ in $\mathrm{PSL}_2(\mathbb{R})$. Any such co-compact lattice $\Gamma$ acts by Möbius transformations on the real hyperbolic plane $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma$ is a closed surface. More generally, if we construct discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that are 'thinner' than lattices, then we get can get nice infinite volume surfaces, e.g. convex co-compact or geometrically finite surfaces.



Closed surface            Convex co-compact surface            Geometrically finite surface
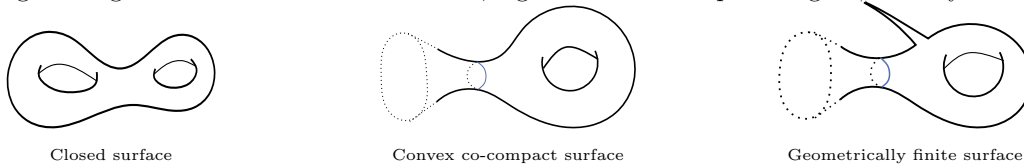
One can ask the same question for Lie groups that are more complicated than $\mathrm{PSL}_2(\mathbb{R})$, e.g. the so-called higher rank Lie groups like $\mathrm{SL}_d(\mathbb{R})$ when $d \geq 3$, $\mathrm{SO}(p,q)$, etc. Lattices in such groups are very restricted - they all come from arithmetic constructions and cannot be deformed, as proven by Margulis [Mar91]. But what about more general discrete subgroups? This is central question that I pursue in my research – the study of discrete subgroups of higher rank Lie groups, beyond lattices. One of the most exciting aspects of my research program is that it lies at the interface of geometry, topology, algebra, and analysis. I frequently use ideas and tools from geometry, topology, linear algebraic groups, Lie group theory, representation theory, dynamical systems, ergodic theory, and coarse geometry. I find it very rewarding to learn from a wide variety of areas of mathematics and see beautiful ideas emerging from their interplay.

Depending on their interest, the reader can choose to read only the first 3 pages of my research statement or opt all 9 pages. In the first 3 pages, I will do a quick exposition of the two key thrust areas of my research program, explain my results, and their context.

### Research Direction I: Convex Projective Structures and Groups Beyond Gromov Hyperbolicity

I adopt the perspective of studying discrete subgroups via their actions on spaces with rich geometry. The geometric spaces that I consider are *properly convex domains* in $\mathbb{P}(\mathbb{R}^d)$. I will explain this below. But first, the reader might think the following: if $\Gamma < \mathrm{SL}_d(\mathbb{R})$ is a discrete subgroup, why don't we consider the action on the associated Riemannian symmetric spaces $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$? This is indeed a great question because when $d$ is 2, the symmetric space is $\mathbb{H}^2$ and Möbius actions on $\mathbb{H}^2$ is exactly the tool that we use for studying subgroups of $\mathrm{SL}_2(\mathbb{R})$. However, the situation changes dramatically in higher rank. In independent works, Kleiner-Leeb[KL06] and Quint [Qui05] showed that when $d \geq 3$, the only discrete subgroups with reasonable actions (i.e. convex co-compact) on the symmetric space are uniform lattices. Hence, we are forced to look for new geometric spaces to study discrete subgroups in higher rank.

A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an open subset whose closure is a compact convex domain in an affine chart. A motivating example here is the projective Beltrami-Klein model of hyperbolic plane $\mathbb{H}^2$. It is an open disk $\mathbb{B}$ in an affine plane in $\mathbb{P}(\mathbb{R}^3)$ with a metric $d_{\mathbb{B}}$ whose geodesics are projective lines and the isometry group is $\mathrm{SO}(2,1)$. Any such domain $\Omega$ carries an analogous natural distance function $\mathrm{d}_\Omega$ – *the Hilbert metric on $\Omega$*– defined using cross-ratios. The subgroup of $\mathrm{PGL}_d(\mathbb{R})$ that preserves $\Omega$ – denoted by $\mathrm{Aut}(\Omega)$ – acts by isometries of $\mathrm{d}_\Omega$. The core of my research program is to study the Hilbert metric geometry of quotients $\Omega/\Gamma$ where $\Gamma < \mathrm{Aut}(\Omega)$ is a discrete group. Benoist had shown that in some special cases, the Hilbert metric on $\Omega/\Gamma$ behaves like negatively curved Riemannian metrics [Ben08, Qui10]. But typically, the metric $\mathrm{d}_\Omega$ is not CAT(0) (a popular generalization of non-positive curvature).

• [Isl19] *Rank One Hilbert Geometries*: I focus on developing tools to study generic Hilbert geometries. I introduce a notion of rank one automorphisms and develop the theory of rank one Hilbert geometries, analogous to rank one CAT(0) spaces. The main result is that a non-elementary rank one group in Hilbert geometry is an

acylindrically hyperbolic group. Acylindrically hyperbolic groups [Osi16] are a much-studied generalization of Gromov hyperbolic groups that include mapping class groups, $\mathrm{Out}(F_n)$, etc. *See Section 2.*

I will now discuss a second reason why I find convex projective manifolds (i.e. quotients of the form $\Omega/\Gamma$) exciting. Classical Teichmüller theory is the study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. Anosov representations are a dynamical approach to develop a "Teichmüller theory for higher rank Lie groups"[Lab06, BIW14, Wie18, KLP17, GGKW17]. It has received much success in studying many new classes of discrete subgroups in higher rank, e.g. Hitchin, maximal, and positive representations. However, until recently, there was a lack of geometric perspective on Anosov representations. Recently, convex projective geometry – more precisely, convex co-compact groups – was used to fill this gap and give a geometric interpretation of Anosov representations [DGK17, Zim17]. We call a group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ *convex co-compact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$ – the convex hull of its limit set in $\Omega$. [DGK17] and [Zim17] proved that for a large family of Gromov hyperbolic groups $\Gamma < \mathrm{PGL}_d(\mathbb{R})$, being convex co-compact is equivalent to the inclusion representation $\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ being Anosov. This inspired a natural question: what about non-Gromov hyperbolic groups [Kas18]? Note that the current definition of Anosov representations is limited to only Gromov hyperbolic groups. So one interpretation of this question is a quest to generalize Anosov representations, i.e. understand linear groups beyond Gromov hyperbolicity.

My research focuses exactly on this question. I undertook a joint research program with Andrew Zimmer to develop a theory of convex co-compact groups which are relatively hyperbolic. We have written five joint paper in the area that I now summarize (*see Section 3 for details*). For the following, $\Gamma$ is a convex co-compact group and $\mathcal{P} := \{P_1, \ldots, P_m\}$ is a set of proper subgroups of $\Gamma$.

- [IZ23]<u>*Abelian peripherals*</u>(*published in Geom. & Top.*): Here we assume that each $P_i \cong \mathbb{Z}^{k_i}$ with $k_i \geq 2$. We introduce the notion of *properly convex domains with isolated simplices* and prove: $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P}$ if and only if $\mathcal{C}_\Omega(\Gamma)$ has isolated simplices. This is analogous to CAT(0) spaces with isolated flats [HK05].

- [IZ22]<u>*General peripherals:*</u> If the peripheral subgroups $P_i$ are arbitrary (not necessarily abelian like above), then we prove that relative hyperbolicity of $\Gamma$ is equivalent to the existence of an isolated family of convex submanifolds (not necessarily simplices). We also prove that the Bowditch boundary of $\Gamma$ is a quotient of the 'ideal' boundary of $\mathcal{C}_\Omega(\Gamma)$.

- [IZ21b] <u>*Flat Torus Theorem*</u> (*published in J. London Math. Soc.*): We prove that any maximal abelian subgroup of a convex co-compact group (not necessarily rel. hyp.) acts co-compactly on a simplex in $\mathcal{C}_\Omega(\Gamma)$. These properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ are analogues of totally geodesic flats in CAT(0) geometry.

- [IZ20], [IZ21a] <u>*Classification and examples*</u>: We study families of convex co-compact groups relatively hyperbolic with $\mathbb{Z}^2$ peripherals. We prove in [IZ21a] that if the boundary of $\mathcal{C}_\Omega(\Gamma)$ doesn't have any 2-dimensional faces, then $\Gamma$ must be of this type. We prove in [IZ20] that if $Y$ is closed non-geometric 3-manifold and $\pi_1(Y)$ is a convex co-compact group, then $Y$ is composed of hyperbolic pieces glued along 2-tori.

**Research Direction II: Group Action on Boundary by Homeomorphisms and their Deformations**

In my first research direction above, I study a discrete subgroup $\Gamma$ in higher rank by looking at its action on special subsets of $\mathbb{P}(\mathbb{R}^d)$, namely properly convex domains. But what about the action of $\Gamma$ on the entire projective space $\mathbb{P}(\mathbb{R}^d)$? This action is quite complicated because it has parts where the action is minimal (e.g., on the limit set of $\Gamma$ where every orbit is dense) as well as parts where the action is properly discontinuous (e.g., on properly convex domains $\Omega$ preserved by $\Gamma$). Nonetheless the action on $\mathbb{P}(\mathbb{R}^d)$ is very interesting because $\mathbb{P}(\mathbb{R}^d)$ can be interpreted as a boundary of the symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$. The rich symmetric space geometry enriches the boundary with a lot of structure, e.g. the Tits building structure. Classically, probing groups via their actions on boundaries has been a very successful endeavor. In Research Direction II, my goal is to pursue this philosophy for discrete groups in higher rank.

As a first step, I will discuss my joint work [CINS23] in the case of uniform (i.e. co-compact) lattices in $\mathrm{SL}_d(\mathbb{R})$, with $d \geq 3$. It has been known since the work of Margulis that these higher rank lattices are impossible to deform inside the group $\mathrm{SL}_d(\mathbb{R})$. In [CINS23], we ask the question: what about deformations of $\Gamma$ in the sense of deforming its boundary action on $\mathbb{P}(\mathbb{R}^d)$? Here we can deform in the infinite dimensional group $\mathrm{Homeo}(\mathbb{P}(\mathbb{R}^d))$ instead of the Lie group $\mathrm{SL}_d(\mathbb{R})$. Can we see new phenomena emerge? In this case, there are also many other

boundaries besides $\mathbb{P}(\mathbb{R}^d)$ where we can ask the same question. The various flag spaces in $\mathbb{R}^d$ are all examples of boundaries as well as the visual boundary of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$.

• [CINS23] *Rigidity of lattice actions on boundaries* (*with C. Connell, T. Nguyen, and R. Spatzier; see Section 4*) I find this question fascinating because it is a duel between the algebraic and the dynamical worlds. On one hand, we have immense freedom to distort the dynamics on $\mathbb{P}(\mathbb{R}^d)$ using very 'non-linear' homeomorphisms. One the other hand, the algebraic relations in the lattice are fighting back to keep things linear. We prove that for small deformations, the algebraic forces win.

Let $\rho_0$ be the standard action of a uniform lattice $\Gamma$ on $\mathbb{P}(\mathbb{R}^d)$ by left multiplication. We prove that: for any action $\rho$ that is sufficiently $C^0$-close to $\rho_0$, there exists a continuous surjective map $\phi_\rho : \mathbb{P}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$ such that the adjoining diagram commutes. Our theorem holds for all flag space boundaries, hence in particular for the Furstenberg boundary of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$.

$$\begin{array}{ccc} \mathbb{P}(\mathbb{R}^d) & \xrightarrow{\rho} & \mathbb{P}(\mathbb{R}^d) \\ \downarrow{\varphi_\rho} & & \downarrow{\varphi_\rho} \\ \mathbb{P}(\mathbb{R}^d) & \xrightarrow{\rho_0} & \mathbb{P}(\mathbb{R}^d) \end{array}$$

Moreover, this is the best we can hope for in the category of $C^0$ actions. By blowing up points along an orbit in $\mathbb{P}(\mathbb{R}^d)$, we construct in [CINS23] examples of deformations $\rho$ close to $\rho_0$ for which $\varphi_\rho$ cannot be a homeomorphism.

In [CINS23], we also make a surprising discovery special to higher rank. We discover a complete loss of rigidity if we replace the flag spaces by the visual boundary of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$. A lattice $\Gamma$ has a natural visual action $\rho_v$ on the visual boundary $\partial X$. We construct a sequence $(\rho_n)$ of deformations of $\rho_v$ that converges to $\rho_v$ but none of the $\rho_n$ are semi-conjugate to $\rho_v$. *See Section 4.*

*Broader context.* My research program in this direction could also be interpreted in the broader context of studying actions of higher rank lattices on compact manifolds. One incarnation of this quest is the Zimmer program and its analogues [Zim87, Wei11]. Zimmer started the program in the wake of Margulis's superrigidity and Zimmer's cocycle superrigidity theorems [Mar91]. The driving philosophy was that differentiable actions of higher rank lattices on 'low dimensional' manifolds should be essentially trivial or classifiable. The program has seen tremendous activity and success in recent years[Fis20]. While Zimmer program concerns measure preserving differentiable actions, questions have often been asked about the possibility of a $C^0$-analogue and for actions that do not preserve measure (e.g. lattices acting on $\mathbb{P}(\mathbb{R}^d)$) [Wei11]. To the best of our knowledge, our paper [CINS23] is the first one that tackles the $C^0$-rigidity problem in higher rank.

## 2. Rank One Properly Convex Domains: Beyond Hyperbolicity

2.1. **Rank One Hilbert Geometries** [Isl19]. In my research on convex projective structures, I focus on developing a theory for properly convex domains that is analogous to the theory of non-positively curved Riemannian manifolds, or more generally, CAT(0) metric spaces. A particularly challenging aspect of this work is that the natural metric - the Hilbert metric - is generically non-Riemannian (it is only Finsler) and it is not CAT(0). The most exciting part of my work here is the eclectic mix of ideas from coarse geometry, geometric group theory, linear groups, and projective geometry.

In Riemannian non-positive curvature, a geodesic direction is hyperbolic – or rank one – if it witnesses an abundance of negative curvature. The precise formulation is that the space of parallel Jacobi fields along the geodesic is one dimensional. This definition doesn't work for typical properly convex domains because the projective geodesic flow is not regular enough to talk about Jacobi fields. In my paper [Isl19], I bypass this issue and introduce a notion of rank one for properly convex domains. I call a projective line geodesic $(a, b) \subset \Omega$ rank one if it is not contained in any half triangle, i.e. for $z$ in the boundary $\partial\Omega$, either $(a, z) \subset \Omega$ or $(z, b) \subset \Omega$. An element $\gamma \in \mathrm{Aut}(\Omega)$ is called a rank one automorphism if $\gamma$ acts by translating along a rank one geodesic in $\Omega$.



$(a, b)$ in half triangle



$(a, b)$ is rank one
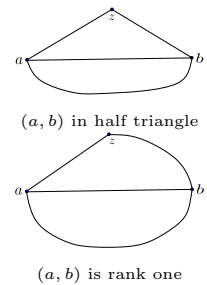
My key insight in [Isl19] was that such rank one automorphisms have a special coarse geometric 'contraction' property that I will now explain. Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a rank one automorphism that translates along a rank one geodesic $\ell := (\gamma^+, \gamma^-)$. Then the closest point projection $\pi_\ell$ (in $d_\Omega$) of $\Omega$ onto $\ell$ has the following special property: any metric ball disjoint from $\ell$ projects under $\pi_\ell$ to a set of uniformly bounded diameter. The existence of such 'contracting' projections has major consequences for the group theoretic properties of a rank one group.

**Theorem 2.1** ([Isl19])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ contains a rank one automorphism and is non-elementary (i.e. does not contain a finite index cyclic subgroup). Then $\Gamma$ is an acylindrically hyperbolic group.*

This theorem adds the rank one groups in projective geometry to a large list of groups with interesting properties: Gromov hyperbolic groups, relatively hyperbolic groups, most mapping class groups, rank one CAT(0) groups, $\mathrm{Out}(F_n)$, etc. This theorem lets me exploit the rich theory of acylindrically hyperbolic groups and obtain interesting applications, e.g. counting of conjugacy classes (roughly speaking, closed geodesics), infinite-dimensionality of the space of quasi-morphisms, etc. I will discuss the counting result. The critical exponent $\delta(\Gamma)$ of a discrete group $\Gamma < \mathrm{Aut}(\Omega)$ is, $\delta(\Gamma) := \limsup_{n\to\infty} \frac{1}{n} \cdot \log \# \{g \in \Gamma : \mathrm{d}_\Omega(x, gx) \leq n\}$ for some $x \in \Omega$.

**Proposition 2.2** ([Isl19])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is non-elementary rank one group such that $\Omega/\Gamma$ is compact. Let $\mathrm{Per}(\mathrm{t})$ count the number of conjugacy classes in $\Gamma$ of rank one automorphisms whose translation length is at most $t$. Then $\frac{e^{t\delta(\Gamma)}}{t} \sim \mathrm{Per}(t)$ [1].*

## 3. Convex Co-compact Groups and Relative Hyperbolicity

A prototypical example of a convex co-compact group in $\mathrm{SO}(2,1) \subset \mathrm{PSL}_3(\mathbb{R})$ is a free group $\Gamma = \langle g, h \rangle$ where $g, h$ are hyperbolic isometries in 'ping-pong' configuration (i.e. the fixed points of $g, h$ are disjoint and both $g, h$ translates points in $\mathbb{H}^2$ a lot). In this case, $\Gamma$ acts on the projective model $\mathbb{B}^2$ of the hyperbolic plane and $\mathbb{B}^2/\Gamma$ is an infinite volume manifold. But it has a compact 'convex core', i.e. the limit set of $\Gamma$ in $\mathbb{B}^2$ is a Cantor set and the convex hull of this limit set – $\mathcal{C}_{\mathbb{H}^2}(\Gamma)$ – has a co-compact $\Gamma$ action. In similar vein, an infinite discrete group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is called *convex co-compact* provided $\Gamma$ preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and the *convex core* – $\mathcal{C}_\Omega(\Gamma)/\Gamma$ – is compact. Here, $\mathcal{C}_\Omega(\Gamma)$ is the convex hull in $\Omega$ of the limit set. We will write – $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group*– to convey that $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ acts convex co-compactly on $\Omega$.

I started working on convex co-compact groups through my collaboration with Andrew Zimmer. Zimmer visited University of Michigan in 2018 when I was a graduate student there. I had just finished working on my first paper [Isl19] on rank one properly convex domains, so I had some expertise in studying non-Gromov hyperbolic groups acting on projective domains. Andrew talked about his result that Gromov hyperbolic convex co-compact groups are essentially the same as Anosov subgroups of $\mathrm{PGL}_d(\mathbb{R})$ [Zim17]. I had a strong feeling that the geometric group theoretic tools that I had developed, could be useful in studying non-Gromov hyperbolic convex co-compact groups. I shared my enthusiasm with him and we started discussing the case of relatively hyperbolic groups as a first step in this direction. The discussion ended up taking the shape of a joint research program on relatively hyperbolic convex co-compact groups. We wrote five joint papers that I will now explain.

3.1. **Projective Flat Torus Theorem** [IZ21b]**.** In order to study a non-Gromov hyperbolic convex co-compact group $\Gamma$, it is vital to understand the subgroups $\mathbb{Z}^k$ of $\Gamma$ where $k \geq 2$. For a non-positively curved Riemannian manifold $M$, the flat torus theorem provides this understanding: any free abelian subgroup $\mathbb{Z}^k$ in $\pi_1(M)$ is the fundamental group of an immersed $k$-dimensional torus in $M$. This is an amazing theorem as it provides a recipe for constructing a totally geodesic flat – a geometric object – from an abelian subgroup – a purely algebraic object. But prior to our work, this powerful theorem was missing in convex projective geometry. In convex projective geometry, the analogues of totally geodesic flats are properly embedded simplices. In a properly convex domain $\Omega$, $S \subset \Omega$ is a *properly embedded simplex* if $S$ is a simplex in some affine chart and the inclusion map $S \hookrightarrow \Omega$ is proper. Our projective fla torus theorem says that a maximal abelian subgroup in a convex co-compact group $\Gamma$ is the fundamental group of an immersed torus in $\mathcal{C}_\Omega(\Gamma)/\Gamma$.

**Theorem 3.1** ([IZ21b])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Gamma$ is a maximal abelian subgroup. Then there exists a properly embedded simplex $S_A \subset \mathcal{C}_\Omega(\Gamma)$ such that $A$ preserves $S_A$ and acts co-compactly on it.*

3.2. **Relatively Hyperbolic Convex Co-compact Groups** [IZ23, IZ22]**.**

**Abelian Peripherals.** With the projective flat torus theorem at our disposal, we could embark on the project to study relatively hyperbolic convex co-compact groups whose peripheral subgroups are isomorphic to $\mathbb{Z}^k$ with $k \geq 2$. By our flat torus theorem, each peripheral subgroup then corresponds to $k$-dimensional simplex in $\mathcal{C}_\Omega(\Gamma)$.

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[1]We write $f(t) \sim g(t)$ if $(1/C)g(t) \leq f(t) \leq Cg(t)$ as $t \to \infty$ for some constant $C$.

In [IZ23], we proved that relative hyperbolicity is precisely characterized by a geometric isolation property of these simplices. This isolation property (cf. 3.2) essentially says that the simplices do not intersect each other in $\mathcal{C}_\Omega(\Gamma)$. This non-intersection should be interpreted in a coarse sense, i.e. metric neighborhoods of simplices intersect in sets of uniformly bounded diameter. We state the definition for an arbitrary collection of subsets $\mathcal{X}$ in $\mathcal{C}_\Omega(\Gamma)$, not just simplices.

**Definition 3.2.** *Suppose* $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group. A collection $\mathcal{X}$ of closed unbounded convex subsets of $\mathcal{C}_\Omega(\Gamma)$ is called* strongly isolated *if: for every $r > 0$ there exists $D_1(r) > 0$ such that for any distinct pair $X_1, X_2 \in \mathcal{X}$, $\mathrm{diam}_\Omega \left( \mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r) \right) \leq D_1(r)$.*

Our main theorem in [IZ23] concerns strong isolation of $\mathcal{S}_{\max}$, the set of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least 2. A simplex is maximal if it is not contained in a simplex of bigger dimension.

**Theorem 3.3** ([IZ23]). *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

*(1) $\Gamma$ is a relatively hyperbolic group with respect to a family of subgroups isomorphic to $\mathbb{Z}^k$ with $k \geq 2$.*

*(2) $\mathcal{S}_{\max}$ is strongly isolated.*

We were inspired by Hruska-Kleiner's work on CAT(0) spaces with isolated flats [HK05]. With very different methods, they had proven an analogous theorem in CAT(0) geometry. Although their CAT(0) tools did not work in our setting, their geometric ideas shaped our intuition.

The challenging part in proving the above theorem is (2) implies (1). The strong isolation property only provides some metric geometric data in the form of an isolated family of subsets. Leveraging that to extract group theoretic properties clearly needed some fresh ideas. During my work [Isl19] on contraction properties in projective geometry, I had learnt about a characterization of relative hyperbolicity in terms of contracting projections [Sis13]. My idea was to use the closest-point projection for the Hilbert metric $\mathrm{d}_\Omega$ and this characterization of relative hyperbolicity. However, this closest-point projection had a shortcoming - it depended on the boundary structure of the domain $\Omega$ and seemed quite unwieldy. Meanwhile, Andrew had defined a linear projection map for a simplex – a linear map that projects $\mathcal{C}_\Omega(\Gamma)$ onto a simplex. A priori, this projection seemed unrelated to the Hilbert metric. Our key insight was to establish a coarse equivalence between these two projections. Then we used the metric characterization of relative hyperbolicity to establish Theorem 3.3.

**General Peripherals.** In spite of the neat statement, we were not satisfied by the assumption on peripherals in Theorem 3.3. What if the convex co-compact group has a more complicated peripheral structure - say closed surface groups? In fact, a very recent work [BV23] has constructed such examples and confirmed that our question wasn't solely of intellectual interest. So, we were keen on a characterization of relative hyperbolicity without assumptions on peripherals. We achieved this in [IZ22]. Although it might sound like a simple generalization of [IZ23], in reality it is very much not so! In particular, one of our key tools – the linear projection maps – do not generalize for arbitrary peripheral subgroups. In fact, we had to discard our entire metric geometric viewpoint of [IZ23]. Instead, we adapted the dynamical perspective on relative hyperbolicity [CITE]. Before explaining the idea further, we first state our theorem precisely. Given a convex co-compact group $\Gamma$, a *peripheral family for* $\Gamma$ is a collection $\mathcal{X}$ of convex subsets of $\mathcal{C}_\Omega(\Gamma)$ that is:(a) $\Gamma$-invariant, (b) strongly isolated (cf. 3.2), and (c) coarsely contains all simplices in $\mathcal{C}_\Omega(\Gamma)$[2].

**Theorem 3.4** ([IZ22]). *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

*(1) If $\Gamma$ is relatively hyperbolic with respect $\mathcal{P} = \{P_1, \ldots, P_m\}$, then $\mathcal{X} := \Gamma \cdot \{X_1, \ldots, X_m\}$ is a peripheral family for $\Gamma$ where each $X_j$ is the convex hull of the limit set of $P_j$ in $\Omega$.*

*(2) If $\mathcal{X}$ is a peripheral family for $\Gamma$ and $\mathcal{P} := \{P_1, \ldots, P_m\}$ is a set of representatives of the $\Gamma$-conjugacy classes in $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$, then $\Gamma$ is relatively hyperbolic with respect $\mathcal{P}$.*

As in the proof of Theorem 3.3, the main challenge was (2) implies (1). Our key idea in [IZ22] was to construct a topological quotient of the boundary of $\mathcal{C}_\Omega(\Gamma)$ and study the dynamics of $\Gamma$ on it. We worked with the ideal boundary of $\mathcal{C}_\Omega(\Gamma)$, i.e. $\partial_i \mathcal{C}_\Omega(\Gamma) := \overline{\mathcal{C}_\Omega(\Gamma)} \cap \partial\Omega$. Given a set of subgroups $\mathcal{P} := \{P_1, \ldots, P_m\}$ of a convex co-compact group $\Gamma$, we defined the quotient space $[\partial_i \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}$ by collapsing the limit set of each $\gamma P_i \gamma^{-1}$ to a

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[2]There is a fixed $D$ so that, any properly embedded simplex in $\mathcal{C}_\Omega(\Gamma)$ of dim $\geq 2$ is inside the $D$-neighborhood of some $X \in \mathcal{X}$.

point. Here $\gamma \in \Gamma$ and $P_i \in \mathcal{P}$. When $\mathcal{P}$ is a peripheral family, $[\partial_{\mathsf{i}}\,\mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ is a nice space (compact, Hausdorff) with a particularly nice $\Gamma$-action (geometrically finite convergence group action). Then Yaman's results implied that $\Gamma$ is relative hyperbolic with peripherals $\mathcal{P}$.

An immediate consequence of this proof strategy was the construction of a topological model for the abstract Bowditch boundary $\partial(\Gamma, \mathcal{P})$ of the group $\Gamma$.

**Corollary 3.5** ([IZ22])**.** *If $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group relatively hyperbolic with respect to $\mathcal{P}$, then there is a $\Gamma$-equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_{\mathsf{i}}\,\mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$.*

3.3. **Classification and Examples.** Relatively hyperbolic convex co-compact groups have several examples from diverse areas of geometry, for instance from linear Coxeter reflection group constructions a la Vinberg [Ben06, DGK$^+$21], cusp deformations of finite volume hyperbolic manifolds [BDL18], and bending constructions [BV23]. But does this mean that all convex co-compact groups are relatively hyperbolic (barring free products and amalgamations)? To investigate this natural question, Andrew and I looked at certain families of test cases – 3-manifold groups and examples inspired by them.

**Three manifold groups** [IZ20]**.** We consider a closed irreducible 3-manifold $Y$ and a convex co-compact group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ ($d$ arbitrary) isomorphic to $\pi_1(Y)$. The most basic case is when $Y$ supports one of the eight Thurston geometries, i.e. $Y$ is a geometric 3-manifold. In this case, we prove that $\Gamma$ must be $\mathbb{Z}^3$, a uniform lattice in $\mathrm{SO}(3,1)$, or the product of $\mathbb{Z}$ with a closed surface group. The key interesting case is when $Y$ is non-geometric, i.e. obtained by gluing several geometric pieces along tori or Klein bottle (as given by the famous geometric decomposition theorem for 3-manifolds). Our main result is that $\pi_1(Y)$ being isomorphic to a convex co-compact group forces a very strong restriction on the topology of $Y$. In particular, $Y$ must be built out of gluing only hyperbolic pieces along tori or Klein bottle.

**Theorem 3.6** ([IZ22])**.** *Suppose $Y$ is a non-geometric 3-manifold as above and $\pi_1(Y)$ is isomorphic to a convex co-compact group in $\mathrm{PGL}_d(\mathbb{R})$. Then each component in the geometric decomposition of $M$ supports a $\mathbb{H}^3$ structure and $\pi_1(M)$ is a relatively hyperbolic group with $\mathbb{Z}^2$ peripherals.*

**One Dimensional Faces** [IZ21a]**.** Suppose $Y$ is a non-geometric 3-manifold as in Theorem 3.6 above and $\Gamma \cong \pi_1(Y)$ acts convex co-compactly on $\Omega$. Then our work [IZ20] imposes a strong restriction on the boundary faces[3] of the convex domain $\Omega$. The restriction is that the boundary face $F_\Omega(x)$ of any $x \in \partial_{\mathsf{i}}\,\mathcal{C}_\Omega(\Gamma)$ is at most one dimensional. Hence, we had the following conjecture: this bound on the dimension of boundary faces is sufficient to force relative hyperbolicity. In [IZ21a], we verify this conjecture.

**Theorem 3.7** ([IZ21a])**.** *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then $\Gamma$ is a relatively hyperbolic group with $\mathbb{Z}^2$-peripherals if and only if $\dim F_\Omega(x) \leq 1$ for each $x \in \partial_{\mathsf{i}}\,\mathcal{C}_\Omega(\Gamma)$.*

## 4. Local Rigidity of Boundary Actions

As I mentioned in Section 1, my interest in boundaries originate from a desire to study discrete subgroups in higher rank via their actions on boundaries. As a first step, I was curious about the case of lattices in higher rank groups, e.g. $\mathrm{SL}_d(\mathbb{R})$ with $d \geq 3$. I was still at the University of Michigan and I started discussing this with Ralf Spatzier, my Ph.D. advisor. In the 90s, Katok and Spatzier had addressed the same question, but for $C^1$-deformations. In [KS97], they had proven a rigidity result – deformations that are $C^1$ close (i.e. the deformation modifies the generators of the lattice slightly without changing the derivatives too much) are in fact conjugate. That is, from a dynamicist's viewpoint, $C^1$ close deformations are the same as the original. But I was interested in the case of $C^0$-deformations sans any information about derivatives. Hence a dynamical approach would not be helpful. Around this time, Bowden-Mann's paper [BM22] on $C^0$-deformations appeared. Although they were working with negatively curved manifolds – very far away from higher rank lattices – their approach was inspiring. Ralf had already been discussing with Chris Connell and Thang Nguyen other questions in the same spirit, i.e. rigidity properties in symmetric spaces. So we joined forces and started working together on the project.

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[3]For $x \in \partial\Omega$, the boundary face of $x$ – denoted by $F_\Omega(x)$– is the union of all open projective lines in $\partial\Omega$ that contain $x$. It is an open convex subset of $\partial\Omega$ containing $x$.

I will explain our work [CINS23] in the case of $G = \mathrm{SL}_3(\mathbb{R})$, but our results hold for more general higher rank Lie groups, see Theorem 4.1. In this case, a lattice $\Gamma < G$ acts on the rank two Riemannian symmetric space $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. There are several notions of boundaries of $X$. For instance, the full flag space of $\mathbb{R}^3$ is called the Furstenberg boundary of $X$. More generally, the other partial flag spaces are called the generalized Furstenberg boundaries of $X$. We will always denote by $\rho_0$ the action of a lattice $\Gamma < G$ on such boundaries. The visual boundary of $X$, denoted by $\partial X$, gives a very different and geometric boundary. The space $X$ has a complete non-positively curved $G$-invariant metricand $\partial X$ is the visual boundary of $X$ with respect to this metric. We will always denote the action of $\Gamma$ on $\partial X$ by $\rho_{\mathrm{v}}$.

Before proceeding with a discussion about deformations of actions, it is imperative that I explain the topology on the space of actions. Given a lattice $\Gamma < G$, we fix a finite generating set $S$ of $\Gamma$. We will denote by $F$ a boundary of $X$, that is equipped with a metric $d_F$. We will say that two actions $\rho$ and $\rho_1$, i.e. homomorphisms from $\Gamma$ to $\mathrm{Homeo}(F)$, are $C^0$-close if $\rho(s)$ and $\rho_1(s)$ are close (as continuous maps on $F$) for each generator $s \in S$.

We will say that $\rho_1$ is a *topological factor* of $\rho$ if there is a continuous surjective map $\varphi : F \to F$ such that the adjoining diagram commutes for all $\gamma \in \Gamma$. The map $\varphi$ is called a *semi-conjugacy*. Note that this is weaker than a conjugacy – two actions $\rho, \rho_1$ are conjugate if $\varphi$ is a homeomorphism. The next example will show that for deformations of actions in this $C^0$-category, semi-conjugacy is the right notion of rigidity.

$$\begin{array}{ccc} F & \xrightarrow{\rho(\gamma)} & F \\ \downarrow{\varphi_\rho} & & \downarrow{\varphi_\rho} \\ F & \xrightarrow{\rho_1(\gamma)} & F \end{array}$$

For an example, consider $g \in \mathrm{PSL}_2(\mathbb{R})$ that acts on $S^1$ with north-south dynamics, i.e. it has an attracting and a repelling fixed point on $g^\pm \in S^1$. We can consider $h \in \mathrm{Homeo}(S^1)$ that acts by fixing tiny intervals around $g^\pm$ pointwise and the same north-south dynamics as $g$ outside of those. Then $h$ is $C^0$-close to $g$ but cannot be conjugate to it as $h$ has uncountably many fixed points. However $g$ is topological factor of $h$ via the semi-conjugacy that collapses the intervals around $g^\pm$ to the point $g^\pm$.

**Rigidity in generalized Furstenberg boundaries.** In the paper [CINS23], we investigate the question: what happens if we deform the boundary actions $\rho_0$ and $\rho_{\mathrm{v}}$ of a higher rank lattice $\Gamma$ in the $C^0$-topology? For the actions $\rho_0$ on the generalized Furstenberg boundaries, we prove that any $C^0$-close deformation $\rho$ has $\rho_0$ as a topological factor. Intuitively, this means that $\rho$ differs from $\rho_0$ only in the sense that finitely many $\Gamma$-orbits could have 'blown up' during the deformation. But other than that, the action hasn't changed. In fact, we also show in [CINS23] that this semi-conjugacy is the best that we can hope in this category of $C^0$-actions. I will explain that below.

**Theorem 4.1** ([CINS23]). *Let $G$ be a connected linear semi-simple Lie group without compact factors, $\Gamma$ be a uniform lattice in $G$, $Q$ be a parabolic subgroup of $G$ and $\rho_0$ be the standard boundary action of $\Gamma$ on $G/Q$. Then for any action $\rho$ that is sufficiently $C^0$-close to $\rho_0$, $\rho$ is a topological factor of $\rho_0$. Moreover, the semi-conjugacy $\varphi_\rho$ converges to* id *uniformly as $\rho$ converges to $\rho_0$.*

**Key ideas underlying Theorem 4.1.** The first idea (or step) involves 'taming' the wild deformations $\rho$ of $\rho_0$. This step is differential geometric in nature where we construct some leaf-wise $C^1$ 'foliations'. These foliations are the 'tame' objects that encode the 'wild' actions $\rho$ of the lattice. To make this a little more precise, consider the space $E := M\backslash G/\Gamma$[4], which is a fiber bundle over $X/\Gamma$ with fibers $G/P$. For each action $\rho$ of $\Gamma$ on $G/P$ close to $\rho_0$, we construct a 'foliation' $\mathcal{H}_\rho$ of $E$. When $\rho = \rho_0$, the foliation $\mathcal{H}_{\rho_0}$ is very special. In that case, it is the center-stable foliation of an algebraic flow on $E$ called the Weyl chamber flow (a higher rank analogue of the geodesic flow in unit tangent bundle of a closed hyperbolic surface). When $\rho$ is a deformation of $\rho_0$, we interpret the foliation $\mathcal{H}_\rho$ as the perturbation of the dynamical foliation $\mathcal{H}_{\rho_0}$. This $\mathcal{H}_\rho$ is a more tame object than $\rho$, since it is related to an algebraic flow! The construction of this foliation is inspired by [BM22]. However, their setup is quite different – they construct foliations on the unit tangent bundle of a negatively curved manifold and relate it to the geodesic flow. A particularly interesting part of our construction is the use of barycenters (i.e. center of mass of measures) for constructing these 'foliations'. I use the word 'foliation' as it conveys our idea better; the actual construction is involved and only yields leaf-wise immersions.

The second idea (or step) is to use coarse geometry and 'straighten' the above foliations $\mathcal{H}_\rho$ on $E$. This straightening essentially pops out a continuous semi-conjugacy between the actions $\rho$ and $\rho_0$. Coarse geometry enters the picture via theorems about quasi-isometric embeddings of flats into higher rank symmetric spaces. I find this step amazing because it uses coarse geometry to prove completely dynamical result. This was our key

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[4] Here $M$ consists of diagonal matrices with diagonal entries $\pm 1$ and $P$ consists of upper triangular matrices.

insight - using coarse geometry for studying problems in the $C^0$-category. The coarse geometry of higher rank symmetric spaces (e.g. higher rank Morse lemma, quasi-flats theorem, etc.) is an area of my core expertise and was one of the key reasons that I got interested in this problem. We drew inspiration from [BM22] which uses convergence group actions and the Morse lemma for Gromov hyperbolic groups. Higher rank lattices are very far from being Gromov hyperbolic, but coarse geometry viewpoint proves to be fruitful nonetheless.

**Can we get better than topological factors, i.e. conjugacies?** In [CINS23], we construct actions $\rho$ that are $C^0$-close to $\rho_0$ but are not conjugate to $\rho_0$. The key idea is to perform a 'Denjoy-like' construction, i.e. blowing up a $\Gamma$ orbit). While such blow up constructions are standard in dynamics, our main contribution is finding a blow-up method that does not alter the topology of the underlying manifold.

Since the group $\Gamma$ acts minimally on $G/Q$, we fixed a dense orbit $\Gamma \cdot x$ on $G/Q$. To blow up this orbit, we replaced each orbit point $\gamma \cdot x$ by a tiny ball $B_\gamma$ of dimension $\dim(G/Q)$. The new action $\rho$ coincided with $\rho_0$ outside these balls $B_\gamma$. Inside the ball $B_\gamma$, $\rho$ was defined to act by the derivative of $\rho_0$. This derivative action makes sense because tiny balls on the manifold and the tangent space are essentially the same. As mentioned before, we then needed to ensure that the blown-up manifold is still homeomorphic $G/Q$. Here, our main tools were topological results on the homeomorphism type of spaces obtained by quotienting Sierpinski spaces. In fact, our result holds for any compact manifold $F$ - not just $G/Q$. Bowden-Mann also had a similar construction, but their method was limited to topological spheres [BM22].

**Proposition 4.2** ([CINS23])**.** *Suppose $\rho_0$ is a $C^1$-action of a countable group $\Lambda$ on a compact smooth manifold $F$ with a dense orbit. Then there exist actions $\rho$ arbitrarily $C^0$-close to $\rho_0$ but not $C^0$ conjugate to $\rho_0$.*

**Non-rigidity in the visual boundary.** In [CINS23], we also discover a complete failure of dynamical rigidity in the visual boundary for higher rank lattices. Consider a lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$ and its action $\rho_v$ on the visual boundary of $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. We construct arbitrarily small deformations of $\rho_v$ that do not have have $\rho_v$ as a topological factor. Our key idea is to exploit the spherical building structure on $\partial X$ and construct deformations that destroy the $G$-orbit structure on $\partial X$.

The spherical building structure on $\partial X$ tell us that $\partial X$ is constructed by gluing together circles (i.e. visual boundaries of flats in $X$) and each circle is obtained by gluing closed intervals (i.e. visual boundaries of sectors in a flat). Owing to this combinatorial structure, we can construct a deformation $\rho$ of $\rho_v$ by specifying it on a single closed interval. The action $\rho$ on a closed interval is simple - it fixes the centre of the interval and dilates around it. The challenge is to coherently control the dilation amount depending on which element of $\Gamma$ is acting. Our idea is to do this using a cocycle on the Furstenberg boundary of $X$. For this, we use the well-known Radon-Nikodym derivative cocycle for the Lebesgue measure class. In fact, by taking powers of the cocycle, we produce a one parameter family of deformations $\rho$ that converges to $\rho_v$.

**Theorem 4.3** ([CINS23])**.** *Suppose $X$ is a simply connected higher rank symmetric space of non-compact type, $G$ is the identity component in $\mathrm{Isom}(X)$, $\Gamma < G$ is a lattice, and $\rho_\mathrm{v}$ is the natural action of $\Gamma$ on $\partial X$. Then, there exist actions $\rho$, arbitrarily $C^0$-close to $\rho_\mathrm{v}$, such that $\rho_v$ isn't a topological factor of $\rho$.*

## 5. FUTURE DIRECTIONS

5.1. **Boundary Structures for Convex Co-compact Groups.** I would like to pursue the philosophy of studying discrete groups via their boundaries further. My current goal is to focus on convex co-compact groups and the boundary structures they preserve. I will outline several research questions here, parts of which would make interesting research projects for doctoral students.

• *Pseudo-Riemannian convex co-compact groups.* Intuitively, the pseudo-Riemannian convex co-compact groups are discrete subgroups of $\mathrm{SO}(p, q + 1)$ that preserve a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{p+q+1})$ and acts convex co-compactly on $\Omega$. Such groups also preserve a particularly nice structure at their bonudary, reminiscent of the conformal structure at the boundary $\partial \mathbb{H}^d$ of the hyperbolic space [GM18]. But in $\mathrm{SO}(p, q+1)$, since the structure comes from a pseudo-Riemannian metric, it is only partly conformal. Hence I will call it as 'pseudo-conformal' structure. This structure has not investigated much for convex co-compact groups and I plan to undertake a program to study it.

I will now explain a concrete question that I want to pursue in this direction. In [GM18], the authors wanted to explore the following question: if a convex co-compact group $\Gamma < \mathrm{SO}(p, q + 1)$ has a $p$-dimensional sphere as its

limit set, does $\Gamma$ necessarily preserve a totally geodesic copy of $\mathbb{H}^p$ inside $\mathbb{H}^{p,q}$? They used dynamical tools to answer it for $SO(2,2)$ but it is open in general. I want to attack this question using a quasiconformal geometry perspective. In particular, Bonk-Kleiner's work [BK02] discusses – in the quasiconformal setting – how to fill in a $p$-dimensional sphere in the boundary with a totally geodesic copy of $\mathbb{H}^{p+1}$. Can their methods be adapted to the "pseudo-conformal" setting? This is the question that I would like to investigate. If this works, it will immediately solve the above rigidity question.

• *Divisible convex domains.* Morally, the question that I want to ask here is whether a geometric space can be re-built from the trace that it leaves at its boundary. A motivating example here is the result that a Möbius map between two circles extends to a quasi-isometry between two copies of $\mathbb{H}^2$. Is there an analogous result for convex projective domains?

For a precise formulation of the question, consider two rank one groups $\Gamma_i < \mathrm{Aut}(\Omega_i)$ for $i = 1, 2$ where $\Omega_i/\Gamma_i$ is compact. In this case, there is a notion of Morse boundary $\partial_M \Omega_i$, a subset of the projective boundary $\partial \Omega_i$ that records all the rank one or hyperbolic directions. Can we find a good notion of 'quasiconformality' on the Morse boundary? I want to leverage this 'quasiconformal' structure and prove that: an equivariant 'quasiconformal' map between $\partial_M \Omega_1$ and $\partial_M \Omega_2$ extends to a quasi-isometry between $\Omega_1$ and $\Omega_2$.

My strategy to attack this question is to adapt ideas from the contracting boundary of CAT(0) spaces. In[CCM19], they develop a notion of quasiconformality on contracting boundary of CAT(0) spaces and prove an analogous extension result. So a key insight necessary in our problem is the candidate for quasiconformality. I have a promising candidate in my mind that originates from my upcoming work with Weismann [IW23]. In this paper, we relate characterize Morse boundary points using "regularity" of the boundary $\partial \Omega$ at that point. The quantity we define is determined by the singular values of a sequence in the rank one group $\Gamma$. The availability of these above ideas makes this project particularly well-suited for a doctoral student.

## 5.2. Constructions of discrete linear groups.
There are only a very limited number of methods of constructing discrete subgroups of linear groups. My goal is to address this dearth of examples – develop new constructions of discrete linear groups that admit many deformations and yield families of examples. My strategy is to consider linear representations of Coxeter groups, i.e. groups of the form $W = \langle s_1, \ldots, s_m | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1$ where $m_{ij} \in \{2, \infty\}$ for $i \neq j \rangle$. Vinberg's work [Vin71] provides a criteria for constructing discrete linear representations of $W$ where each $s_i$ acts by a linear reflection. The resulting discrete groups preserve a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$. But what about other possible linear representations of $W$? For instance, what if $s_i$ acts an involution on $\mathbb{R}^d$ but not a reflection (i.e. the fixed hyperplane of $s_i$ has codimension greater than 1)? What if the group $W$ does not preserve a convex domain? My goal is to study representations that address these natural questions.

• $\underline{\mathrm{Sp}_4(\mathbb{R})}$. Here we let each $s_i$ act by a symplectic reflection, i.e. $s_i$ acts by Id on a 2-dimensional subspace $W_i$ and by $(-\mathrm{Id})$ on its symplectic orthogonal. Then $W_i$ must be a symplectic (i.e. not Lagrangian) subspace. In ongoing work with Beatrice Pozzetti, we have been studying the case of 3-generated $W_\infty$ with each $m_{ij} = \infty$, i.e. $(\infty, \infty, \infty)$-triangle group. As this contains an index 2 free subgroup, we use higher rank ping-pong dynamics to prove a discreteness criteria for some $\mathrm{Sp}(4, \mathbb{R})$ representations of $W_\infty$. Moreover, these representations admit nice deformations indexed by a triple of fixed symplectic subspaces (one for each $s_i$). But there are plenty of questions to ask - what happens for more generators? Or when there is torsion, i.e. $m_{ij} < \infty$? Can we get a complete picture for all triangle groups? Is there a nice parametrization of the moduli space?

• $\underline{\mathrm{SL}_d(\mathbb{R})}$. For the $W_\infty$ group as above, even the $\mathrm{SL}_3(\mathbb{R})$ case is interesting as it is complementary to Vinberg's case [Vin71]. IJointly with Ludovic Marquis, we have partial results for discreteness in $\mathrm{SL}_3(\mathbb{R})$. We explicitly construct a fundamental domain in an affine chart in $\mathbb{P}(\mathbb{R}^3)$. Again, there are many open questions - what about the other $\mathrm{SL}_d(\mathbb{R})$? More generators? Torsion elements? Even the case $\mathrm{SL}_4(\mathbb{R})$ is unclear, as our geometric ideas from the $\mathrm{SL}_3(\mathbb{R})$ case does not generalize well.

This area seems ripe for undergraduate research projects, or even the stepping stones for a graduate research projects. Computer visualizations of these Coxeter group actions in $\mathbb{P}(\mathbb{R}^4)$ would provide valuable geometric intuition in solving the problem. A sample problem could be: consider a concrete example in $SO(2, 1)$ whose tiling in $\mathbb{H}^2$ we can generate. Then observe what happens to this tiling as we deform the group in $\mathrm{SL}_4(\mathbb{R})$.

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