RESEARCH STATEMENT

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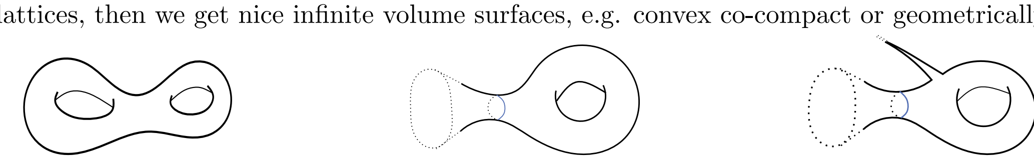
1. Overview of my Research Program

A fundamental question in geometry is to obtain concrete construction of manifolds. For instance, consider the classical Poincare polygon theorem that provides a recipe for building surfaces by gluing edges of a regular polygon. But such purely geometric methods become intractable as dimension goes up. A more systematic approach to this problem is the study of discrete subgroups of Lie groups. For instance, the above problem of constructing a surface - of genus at least two - can be recast as a problem of constructing a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$. Any such uniform (i.e. co-compact) lattice Γ acts by Möbius transformations on the real hyperbolic plane $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma$ is a closed surface. More generally, if we construct discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that are 'thinner' than lattices, then we get nice infinite volume surfaces, e.g. convex co-compact or geometrically finite surfaces.



Closed surface of genus 2 | Convex co-compact surface | Geometrically finite surface

One can ask the same question for Lie groups that are more complicated than $\mathrm{PSL}_2(\mathbb{R})$, e.g. the so-called higher rank Lie groups like $\mathrm{SL}_d(\mathbb{R})$ when $d \geq 3$, $\mathrm{SO}(p,q)$, etc. Lattices in such groups are very restricted - they all come from arithmetic constructions and cannot be deformed, as proven by Margulis [Mar75, Mar91]. But what about more general discrete subgroups? This is central question that I pursue in my research – the study of discrete subgroups of higher rank Lie groups, beyond lattices. One of the most exciting aspects of my research program is that it lies at the interface of geometry, topology, algebra, and analysis. I frequently use ideas and tools from geometry, linear algebraic groups, topology, Lie group theory, representation theory, dynamical systems, ergodic theory, and coarse geometry. I find it very rewarding to learn from a wide variety of areas of mathematics and see beautiful ideas emerge from their interplay.

Depending on their interest and time, the reader can choose to read only the first 3 pages of my research statement or opt for all 9 pages. In the first 3 pages, I will do a quick exposition of the two key thrust areas of my research program, explain my results, and their context.

Research Direction I: Convex Projective Structures and Groups Beyond Gromov Hyperbolicity

I adopt the perspective of studying discrete subgroups via their actions on spaces with rich geometry. The spaces that I consider are *properly convex domains* in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. I will explain this below. But first, the reader might think the following: if $\Gamma < \mathrm{SL}_d(\mathbb{R})$ is a discrete subgroup, why don't we consider its action on the associated Riemannian symmetric spaces $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$? This is a great question because when $d = 2$, this symmetric space is $\mathbb{H}^2$ and Möbius actions on $\mathbb{H}^2$ is the classical tool that we use for studying subgroups of $\mathrm{SL}_2(\mathbb{R})$. However, the situation changes dramatically in higher rank. Kleiner-Leeb [KL06] and Quint [Qui05] independently showed that when $d \geq 3$, the only discrete subgroups with reasonable actions (i.e. convex co-compact) on the symmetric space are uniform lattices. Hence, we are forced to look for new geometric spaces to study discrete subgroups in higher rank.

A *properly convex domain* $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is an open subset whose closure is a compact convex domain in an (affine) chart of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. A motivating example here is the projective Beltrami-Klein model of the hyperbolic plane $\mathbb{H}^2$. It is an open disk $\mathbb{B}$ in an affine plane in $\mathbb{P}(\mathbb{R}^3)$ carrying a metric $d_{\mathbb{B}}$ whose geodesics are projective lines and the isometry group is $\mathrm{SO}(2,1)$. Any properly convex domain Ω carries an analogous natural distance function d_Ω – *the Hilbert metric on* Ω– defined using cross-ratios. The subgroup of $\mathrm{PGL}_d(\mathbb{R})$ that preserves Ω – denoted by $\mathrm{Aut}(\Omega)$ – acts by isometries of d_Ω. The core of my research program is to study the Hilbert metric geometry of quotients Ω/Γ where $\Gamma < \mathrm{Aut}(\Omega)$ is a discrete group. Benoist had shown that in some special cases, the Hilbert metric on Ω/Γ behaves like negatively curved Riemannian metrics [Ben08, Qui10]. But typically, the metric d_Ω is not even CAT(0) (a popular generalization of non-positive curvature) [KS58].

• [Isl19] *Rank One Hilbert Geometries*: I focus on developing tools to study generic Hilbert geometries. I introduce a notion of rank one automorphisms and develop the theory of rank one Hilbert geometries, analogous to rank one CAT(0) spaces. The main result is that a non-elementary rank one group in Hilbert geometry is an acylindrically hyperbolic group. Acylindrically hyperbolic groups [Osi16] are a much-studied generalization of Gromov hyperbolic groups that include mapping class groups, $\mathrm{Out}(F_n)$, etc. *See Section 2.*

I will now discuss a second reason why I find *convex projective manifolds* (i.e. quotients of the form Ω/Γ) exciting. Classical Teichmüller theory is the study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. *Anosov representations* [Lab06, KLP17, GGKW17] are a dynamical approach to develop a "Teichmüller theory for higher rank Lie groups" [BIW14, Wie18]. This approach has been very successful in studying many new classes of discrete subgroups in higher rank, e.g. Hitchin, maximal, and positive representations [Wie18]. However, until recently, there was a lack of geometric perspective on Anosov representations [Kas18]. Recently, convex projective geometry – more precisely, convex co-compact groups – has been used to fill this gap and give a geometric interpretation of Anosov representations [DGK17, Zim21]. We call a group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ *convex co-compact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$ – the convex hull of its limit set. [DGK17] and [Zim21] prove that for a large family of Gromov hyperbolic groups $\Gamma < \mathrm{PGL}_d(\mathbb{R})$, being convex co-compact is equivalent to the inclusion representation $\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ being Anosov. This inspires a natural question: what about non-Gromov hyperbolic groups? Note that the current definition of Anosov representations is limited to only Gromov hyperbolic groups. So one interpretation of this question is a quest to generalize Anosov representations, i.e. understand linear groups beyond Gromov hyperbolicity.

My research focuses exactly on this question. I have undertaken a joint research program with Andrew Zimmer to develop a theory of convex co-compact groups which are relatively hyperbolic. We have written five joint paper in the area that I now summarize (*see Section 3 for details*). For the following, Γ is a convex co-compact group and $\mathcal{P} := \{P_1, \ldots, P_m\}$ is a set of proper subgroups of Γ.

• [IZ23]<u>*Abelian peripherals*</u>(*published in Geom. & Topol.*): Here we assume that each $P_i \cong \mathbb{Z}^{k_i}$ with $k_i \geq 2$. We introduce the notion of properly convex domains with isolated simplices and prove: Γ is relatively hyperbolic with respect to $\mathcal{P}$ if and only if $\mathcal{C}_\Omega(\Gamma)$ has isolated simplices. This is analogous to CAT(0) spaces with isolated flats [HK05].

• [IZ22]<u>*General peripherals:*</u> For arbitrary peripheral subgroups P_i (not necessarily abelian), we prove that relative hyperbolicity of Γ is equivalent to the existence of an isolated family of convex submanifolds (not necessarily simplices). We also prove that the Bowditch boundary of Γ is a quotient of the 'ideal' boundary of $\mathcal{C}_\Omega(\Gamma)$.

• [IZ21b] <u>*Flat Torus Theorem*</u> (*published in J. London Math. Soc.*): We prove that any maximal abelian subgroup of a convex co-compact group (not necessarily rel. hyp.) acts co-compactly on a simplex in $\mathcal{C}_\Omega(\Gamma)$. These properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ are analogues of totally geodesic flats in CAT(0) geometry.

• [IZ20], [IZ21a] <u>*Classification and examples*:</u> We study families of convex co-compact groups relatively hyperbolic with $\mathbb{Z}^2$ peripherals. We prove in [IZ21a] that if the boundary faces of $\mathcal{C}_\Omega(\Gamma)$ are at most 1-dimensional, then Γ must be of this type. We prove in [IZ20] that if Y is closed non-geometric 3-manifold and $\pi_1(Y)$ is a convex co-compact group, then Y is composed of hyperbolic pieces glued along 2-tori (i.e. $\pi_1(Y)$ is of this type).

Research Direction II: Group Action on Boundary by Homeomorphisms and their Deformations

In my first research direction above, I study a discrete subgroup $\Gamma < \mathrm{SL}_d(\mathbb{R})$ in higher rank by looking at its action on special subsets of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$, namely properly convex domains. But what about the action of Γ on the entire projective space $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$? This action is quite complicated because it has parts where the action is minimal (e.g., on the limit set of Γ where every orbit is dense) as well as parts where the action is properly discontinuous (e.g., on properly convex domains Ω preserved by Γ). Nonetheless the action on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is interesting and worthy of study because $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ can be interpreted as a boundary of the symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$. Probing groups via their actions on boundaries has a long classical tradition, e.g. [Mos73, Sul85, Tuk95] among many others. In Research Direction II, my goal is to pursue this philosophy for discrete groups in higher rank.

I will discuss my first joint work [CINS23] in this direction via the example of uniform lattices in $\mathrm{SL}_d(\mathbb{R})$ when $d \geq 3$. It has been known since the work of Margulis that these higher rank lattices are impossible to deform inside the group $\mathrm{SL}_d(\mathbb{R})$ [Mar91]. In [CINS23], we ask the question: what about deformations of Γ in the sense of deforming its boundary action on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$? Here we can deform in the infinite dimensional group $\mathrm{Homeo}(\mathbb{P}(\mathbb{R}^{\mathrm{d}}))$ instead of the Lie group $\mathrm{SL}_d(\mathbb{R})$. Can we see new phenomena emerge? In this case, there are also many other boundaries besides $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ where we can ask the same question. The various flag spaces in $\mathbb{R}^d$ are all examples of boundaries as well as the visual boundary of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$.

• [CINS23] Rigidity of lattice actions on boundaries (*with C. Connell, T. Nguyen, and R. Spatzier; see Section 4*) I find this question fascinating because it is a duel between the algebraic and the dynamical worlds. On one hand, we have immense freedom to distort the dynamics on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ using very 'non-linear' homeomorphisms. One the other hand, the algebraic relations in the lattice are fighting back to keep things linear. We prove that for small deformations, the algebraic forces win.

Let ρ_0 be the standard action of a uniform lattice Γ on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ by left multiplication. We prove that: for any ρ that is sufficiently C^0-close to ρ_0 (i.e. ρ and ρ_0 both map generators of the lattice close-by), there exists a continuous surjective map $\phi_\rho : \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \to \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that the adjoining diagram commutes. Our theorem also holds for all other flag space boundaries, hence in particular for the Furstenberg boundary of $\mathrm{SL}_d(\mathbb{R})/\,\mathrm{SO}(d)$ (i.e. space of full flags).

$$\begin{array}{ccc} \mathbb{P}(\mathbb{R}^{\mathrm{d}}) & \xrightarrow{\rho} & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \\ \downarrow{\scriptstyle \varphi_\rho} & & \downarrow{\scriptstyle \varphi_\rho} \\ \mathbb{P}(\mathbb{R}^{\mathrm{d}}) & \xrightarrow{\rho_0} & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \end{array}$$

Moreover, this is the best kind of rigidity that one can hope for in the category of C^0-actions. By blowing up an orbit, we construct examples of deformations ρ close to ρ_0 for which φ_ρ cannot be a homeomorphism [CINS23].

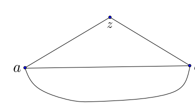
In [CINS23], we also make a surprising discovery special to higher rank. We discover a complete loss of rigidity if we replace the flag spaces by the visual boundary ∂X of $X = \mathrm{SL}_d(\mathbb{R})/\,\mathrm{SO}(d)$. A lattice Γ has a natural action ρ_{v} on ∂X. We construct deformations ρ of ρ_{v}, arbitrarily C^0-close to ρ_{v}, but not semi-conjugate to ρ_{v} (i.e. do not admit such continuous surjective maps φ_ρ on ∂X, like above). *See Section 4.*

Broader context. My research program in this direction can also be interpreted in the broader context of studying actions of higher rank lattices on compact manifolds. One incarnation of this quest is the *Zimmer program* and its analogues [Zim87, Wei11]. Zimmer started the program in the wake of Margulis's superrigidity and Zimmer's cocycle superrigidity theorems [Mar91, Zim87]. The driving philosophy is that differentiable actions of higher rank lattices on 'low dimensional' manifolds should be essentially trivial or classifiable. The program has seen tremendous activity and success in recent years [Fis20]. While Zimmer program concerns measure preserving differentiable actions, questions have often been asked about the possibility of a C^0-analogue and for actions that do not preserve measure (e.g. lattices acting on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$) [Wei11]. To the best of our knowledge, our paper [CINS23] is the first one that tackles such C^0-rigidity problems in higher rank.

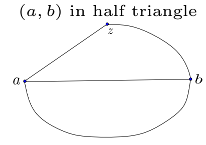
2. Rank One Properly Convex Domains: Beyond Hyperbolicity

2.1. **Rank One Hilbert Geometries** [Isl19]**.** In my research on convex projective structures, I focus on developing a theory for properly convex domains that is analogous to the theory of non-positively curved Riemannian manifolds, or more generally, CAT(0) metric spaces. A particularly challenging aspect of this work is that the natural metric - the Hilbert metric - is typically non-Riemannian (only Finsler) and not CAT(0). But this also makes things exciting because I get to use an eclectic mix of ideas from coarse geometry, geometric group theory, linear groups, and projective geometry.

In Riemannian non-positive curvature, a geodesic direction is hyperbolic – or rank one – if it witnesses an abundance of negative curvature. The precise formulation is that the space of parallel Jacobi fields along the geodesic is one dimensional. This definition doesn't work for typical properly convex domains because the projective geodesic flow is not regular enough to talk about Jacobi fields. In my paper [Isl19], I bypass this issue and introduce a notion of rank one for properly convex domains. I call a projective line geodesic $(a, b) \subset \Omega$ *rank one* if it is not contained in any half triangle, see the adjoining figure. An element $\gamma \in \mathrm{Aut}(\Omega)$ is called a *rank one automorphism* if γ acts by translating along a rank one geodesic in Ω.



(a, b) in half triangle



(a, b) is rank one

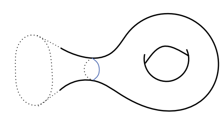
My key insight in [Isl19] is that such rank one automorphisms have a special coarse geometric 'contraction property' that I will now explain. Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a rank one automorphism that translates along a rank one geodesic $\ell := (\gamma^+, \gamma^-)$. Then the closest point projection π_ℓ (in d_Ω) of Ω onto ℓ has the following special property: any metric ball disjoint from ℓ, projects under π_ℓ to a set whose diameter is uniformly bounded. The existence of such 'contracting projections' has major geometric group theoretic conseqeunces.

Theorem 2.1 ([Isl19])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ contains a rank one automorphism and is non-elementary (i.e. does not contain a finite index cyclic subgroup). Then Γ is an acylindrically hyperbolic group.*

This theorem adds the rank one groups in projective geometry to a large list of groups with interesting properties: Gromov hyperbolic groups, relatively hyperbolic groups, most mapping class groups, rank one CAT(0) groups, $\mathrm{Out}(F_n)$, etc. I can then exploit the well-developd theory of acylindrically hyperbolic groups and obtain several interesting applications, e.g. counting of conjugacy classes (roughly speaking, closed geodesics), infinite-dimensionality of the space of quasi-morphisms, etc. I will discuss the counting result. The critical exponent $\delta(\Gamma)$ of a discrete group $\Gamma < \mathrm{Aut}(\Omega)$ is, $\delta(\Gamma) := \limsup_{n\to\infty} \frac{1}{n} \cdot \log \# \{g \in \Gamma : \mathrm{d}_\Omega(x, gx) \leq n\}$ for some $x \in \Omega$. We write $f(t) \sim g(t)$ if there is a constant $C > 0$ such that $(1/C)g(t) \leq f(t) \leq Cg(t)$ as $t \to \infty$.

Proposition 2.2 ([Isl19]). *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is non-elementary, contains a rank one automorphism, and Ω/Γ is compact. Let $\mathrm{Per(t)}$ count the number of conjugacy classes in Γ of rank one automorphisms whose translation length in Ω is at most t. Then $\frac{e^{t\delta(\Gamma)}}{t} \sim \mathrm{Per}(t)$.*

3. Convex Co-compact Groups and Relative Hyperbolicity


$\mathbb{H}^2/\Gamma$

A prototypical example of a convex co-compact group in $\mathrm{SO}(2,1) < \mathrm{SL}_3(\mathbb{R})$ is a free group $\Gamma = \langle g, h\rangle$, where g, h are hyperbolic isometries in 'ping-pong' configuration (i.e. the fixed points of g, h are disjoint and both g, h translate points in $\mathbb{H}^2$ a lot). In this case, $\mathbb{H}^2/\Gamma$ is an infinite volume surface but it has a compact 'convex core'. The limit set of Γ is a Cantor set in $\partial\,\mathbb{H}^2$ and the convex hull $\mathcal{C}_{\mathbb{H}^2}(\Gamma)$ of this limit set has a co-compact Γ action. Similarly, a discrete group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is called *convex co-compact* provided Γ preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and the *convex core* – $\mathcal{C}_\Omega(\Gamma)/\Gamma$ – is compact. Here, $\mathcal{C}_\Omega(\Gamma)$ is the convex hull in Ω of the limit set. We will write - $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group* - to convey that $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ acts convex co-compactly on Ω.

I started working on convex co-compact groups through my collaboration with Andrew Zimmer. Zimmer visited University of Michigan in 2018 when I was a graduate student there. I had just finished working on my first paper [Isl19] on rank one properly convex domains, so I had some expertise in studying non-Gromov hyperbolic groups acting on projective domains. Andrew talked about his result, that Gromov hyperbolic convex co-compact groups are essentially the same as Anosov subgroups of $\mathrm{PGL}_d(\mathbb{R})$ [Zim21]. I had a strong feeling that the geometric group theoretic tools that I had developed, could be useful in studying non-Gromov hyperbolic convex co-compact groups. I shared my enthusiasm with him and we started discussing the case of relatively hyperbolic groups as a first step in this direction. The discussion ended up taking the shape of a joint research program on relatively hyperbolic convex co-compact groups. We wrote five joint papers that I will now explain.

3.1. Projective Flat Torus Theorem [IZ21b].

In order to study a non-Gromov hyperbolic convex co-compact group Γ, it is vital to understand its $\mathbb{Z}^k$ subgroups where $k \geq 2$. For a non-positively curved Riemannian manifold M, the flat torus theorem provides this understanding: any $\mathbb{Z}^k$ subgroup in $\pi_1(M)$ is the fundamental group of an immersed k-dimensional torus in M. This is an amazing theorem as it provides a recipe for constructing a totally geodesic flat – a geometric object – from an abelian subgroup – a purely algebraic object. But prior to our work, this powerful theorem was missing in convex projective geometry. In convex projective geometry, the analogues of totally geodesic flats are properly embedded simplices. In a properly convex domain Ω, $S \subset \Omega$ is a *properly embedded simplex* if S is a simplex in some affine chart and the inclusion map $S \hookrightarrow \Omega$ is proper. Our projective flat torus theorem says that a maximal abelian subgroup in a convex co-compact group Γ is the fundamental group of an immersed torus in $\mathcal{C}_\Omega(\Gamma)/\Gamma$.

Theorem 3.1 ([IZ21b]). *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Gamma$ is a maximal abelian subgroup. Then there exists a properly embedded simplex $S_A \subset \mathcal{C}_\Omega(\Gamma)$ such that A preserves S_A and acts co-compactly on it.*

3.2. Relatively Hyperbolic Convex Co-compact Groups [IZ23, IZ22].

A group Γ is relatively hyperbolic with respect to its subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$ if the group Γ is 'hyperbolic' away from the conjugates of P_i and the only 'non-hyperbolic' regions, i.e. conjugates of P_i, interact in a very restricted manner. A classic example is a non-uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$, i.e. the fundamental group of a cusped hyperbolic 3-manifold. It is hyperbolic with respect to the cusp subgroups which are all isomorphic to $\mathbb{Z}^2$.

Abelian Peripherals. Our projective flat torus theorem enabled us to study relatively hyperbolic convex co-compact groups whose peripheral subgroups are isomorphic to $\mathbb{Z}^k$ with $k \geq 2$. By our theorem 3.1, each peripheral subgroup then acts co-compactly on a k-dimensional simplex in $\mathcal{C}_\Omega(\Gamma)$[1]. In [IZ23], we proved that relative hyperbolicity is precisely characterized by a geometric isolation property of these simplices. This isolation property (cf. 3.2) essentially says that the simplices do not intersect each other in $\mathcal{C}_\Omega(\Gamma)$. This non-intersection should be interpreted in a coarse sense, i.e. metric neighborhoods of simplices intersect in sets of uniformly bounded diameter. I state the definition for an arbitrary collection of subsets $\mathcal{X}$ in $\mathcal{C}_\Omega(\Gamma)$, not just simplices.

Definition 3.2. *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. A collection $\mathcal{X}$ of closed unbounded convex subsets of $\mathcal{C}_\Omega(\Gamma)$ is called* strongly isolated *if: for every $r > 0$ there exists $D_1(r) > 0$ such that for any distinct pair $X_1, X_2 \in \mathcal{X}$, $\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)\right) \leq D_1(r)$.*

Our main theorem in [IZ23] concerns strong isolation of $\mathcal{S}_{\max}$, the set of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least 2. A simplex is maximal if it is not contained in a simplex of bigger dimension.

Theorem 3.3 ([IZ23])**.** *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) Γ is a relatively hyperbolic group with respect to a family of subgroups isomorphic to $\mathbb{Z}^k$ with $k \geq 2$.

(2) $\mathcal{S}_{\max}$ is strongly isolated.

We were inspired by Hruska-Kleiner's work on CAT(0) spaces with isolated flats [HK05]. With very different methods, they had proven an analogous theorem in CAT(0) geometry. Although their CAT(0) tools did not work in our setting, their geometric ideas shaped our intuition. The challenging part in proving the above theorem was (2) implied (1). The strong isolation property only provides some metric geometric data in the form of an isolated family of subsets. Leveraging that to extract group theoretic properties clearly needed some fresh ideas.

During my work [Isl19] on contraction properties in projective geometry, I had learnt about a characterization of relative hyperbolicity in terms of contracting projections [Sis13]. My idea was to use the closest-point projection for the Hilbert metric d_Ω and this characterization of relative hyperbolicity. However, this closest-point projection had a shortcoming - it depended on the boundary structure of the domain Ω and seemed quite unwieldy. Meanwhile, Andrew had defined a linear projection map for a simplex – a linear map that projects $\mathcal{C}_\Omega(\Gamma)$ onto a simplex. A priori, this projection seemed unrelated to the Hilbert metric. Our key insight was to establish a coarse equivalence between these two projections. Then we used the metric characterization of relative hyperbolicity [Sis13] to establish Theorem 3.3.

General Peripherals. In spite of the neat statement, we were not satisfied by the assumption on peripherals in Theorem 3.3. What if the convex co-compact group has a more complicated peripheral structure, say closed surface groups? In fact, a very recent work [BV23] has constructed such examples and confirmed that our question wasn't solely of intellectual interest. So, we were keen on a characterization of relative hyperbolicity without assumptions on peripherals. We achieved this in [IZ22]. Although it might sound like a simple extension of [IZ23], it is very much not so in reality! In particular, one of our key tools in [IZ23] – the linear projection maps – do not generalize for arbitrary peripheral subgroups. In fact, we had to discard our entire metric geometric viewpoint of [IZ23] and adopt the dynamical perspective on relative hyperbolicity [Yam04]. Before explaining the idea further, I will first state our theorem precisely. Given a convex co-compact group Γ, a *peripheral family for* Γ is a collection $\mathcal{X}$ of convex subsets of $\mathcal{C}_\Omega(\Gamma)$ that is:(a) Γ-invariant, (b) strongly isolated (cf. 3.2), and (c) *coarsely contains all simplices in* $\mathcal{C}_\Omega(\Gamma)$ (i.e. there is a constant D so that, any properly embedded simplex in $\mathcal{C}_\Omega(\Gamma)$ of dim ≥ 2 is inside the D-neighborhood of some $X \in \mathcal{X}$).

Theorem 3.4 ([IZ22])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) If Γ is relatively hyperbolic with respect $\mathcal{P} = \{P_1, \dots, P_m\}$, then $\mathcal{X} := \Gamma \cdot \{X_1, \dots, X_m\}$ is a peripheral family for Γ where each X_j is the convex hull of the limit set of P_j in Ω.

(2) If $\mathcal{X}$ is a peripheral family for Γ and $\mathcal{P} := \{P_1, \dots, P_m\}$ is a set of representatives of the Γ-conjugacy classes in $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$, then Γ is relatively hyperbolic with respect $\mathcal{P}$.

As in the proof of Theorem 3.3, the main challenge was (2) implied (1). Our key idea in [IZ22] was to construct a topological quotient of the boundary of $\mathcal{C}_\Omega(\Gamma)$ and study the dynamics of Γ on it. We worked with the ideal boundary of $\mathcal{C}_\Omega(\Gamma)$, i.e. $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma) := \overline{\mathcal{C}_\Omega(\Gamma)} \cap \partial\Omega$. Given a set of subgroups $\mathcal{P} := \{P_1, \dots, P_m\}$ of a convex

[1] This implies that projective convex co-compact groups are very different from non-uniform lattices in $\mathrm{PSL}_2(\mathbb{C})$.

co-compact group Γ, we defined the quotient space $[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ by collapsing the limit set of each $\gamma P_i \gamma^{-1}$ to a point. Here $\gamma \in \Gamma$ and $P_i \in \mathcal{P}$. When $\mathcal{P}$ is a peripheral family, $[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ is a nice space (compact, Hausdorff) with a particularly nice Γ-action (geometrically finite convergence group action). Then Yaman's results [Yam04] implied that Γ is relative hyperbolic with peripherals $\mathcal{P}$. We were inspired by related work of Weisman [Wei23].

An immediate consequence of this proof strategy was the construction of a topological model for the abstract Bowditch boundary $\partial(\Gamma, \mathcal{P})$ of the group Γ.

Corollary 3.5 ([IZ22, Wei23]). *If $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group relatively hyperbolic with respect to $\mathcal{P}$, then there is a Γ-equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$.*

3.3. **Classification and Examples.** Relatively hyperbolic convex co-compact groups have several examples from diverse areas of geometry, for instance from linear Coxeter reflection groups [Ben06, DGK$^+$21], cusp deformations of finite volume hyperbolic manifolds [BDL18], and bending constructions [BV23]. But does this mean that all convex co-compact groups are relatively hyperbolic (barring free products and amalgamations)? To investigate this natural question, Andrew and I looked at certain families of test cases.

Three manifold groups [IZ20]**.** Consider a closed irreducible 3-manifold Y and a convex co-compact group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ (d arbitrary) isomorphic to $\pi_1(Y)$. The most basic case is when Y supports one of the eight Thurston geometries, i.e. Y is a geometric 3-manifold. In this case, we proved that Γ must either be $\mathbb{Z}^3$, or a uniform lattice in $\mathrm{SO}(3,1)$, or the product of $\mathbb{Z}$ with a closed surface group. The key interesting case is when Y is non-geometric, i.e. obtained by gluing several geometric pieces along tori or Klein bottle (as given by the famous geometric decomposition theorem for 3-manifolds). Our main result was that $\pi_1(Y)$ being isomorphic to a convex co-compact group forces strong restrictions on the topology of Y. In particular, Y must be built out of gluing only hyperbolic pieces along tori or Klein bottle.

Theorem 3.6 ([IZ22]). *Suppose Y is a non-geometric 3-manifold as above and $\pi_1(Y)$ is isomorphic to a convex co-compact group in $\mathrm{PGL}_d(\mathbb{R})$. Then each component in the geometric decomposition of M supports a $\mathbb{H}^3$ structure and $\pi_1(Y)$ is a relatively hyperbolic group with $\mathbb{Z}^2$ peripherals.*

One Dimensional Faces [IZ21a]**.** Suppose Y is a non-geometric 3-manifold as in Theorem 3.6 above and $\Gamma \cong \pi_1(Y)$ acts convex co-compactly on Ω. Then our work [IZ20] imposes strong restrictions on the boundary faces[2] of the convex domain Ω. The restriction is that the boundary face $F_\Omega(x)$ of any $x \in \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$ is at most one dimensional. Hence, we had conjectured: this bound on the dimension of boundary faces is sufficient to force relative hyperbolicity. In [IZ21a], we verified this conjecture.

Theorem 3.7 ([IZ21a]). *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then Γ is a relatively hyperbolic group with $\mathbb{Z}^2$-peripherals if and only if $\dim F_\Omega(x) \leq 1$ for each $x \in \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$.*

4. Local Rigidity of Boundary Actions

As I mentioned in Section 1, my interest in boundaries originates from a desire to study discrete subgroups in higher rank via their boundary actions. As a first step, I was curious about the case of lattices in higher rank groups, e.g. $\mathrm{SL}_d(\mathbb{R})$ with $d \geq 3$. I was still a Ph.D. student the University of Michigan and I started discussing with Ralf Spatzier, my advisor. In the 90s, Katok and Spatzier had addressed the same question, but for C^1-deformations. In [KS97], they had proven a rigidity result – deformations that are C^1 close (i.e. the deformation modifies the generators of the lattice slightly without changing the derivatives too much) are in fact conjugate. That is, from a dynamicist's viewpoint, C^1 close deformations are the same as the original. The question about C^1-deformations goes back to Sullivan [Sul85] who had solved the problem in the complementary case of rank one groups, e.g. uniform lattices in $\mathrm{SL}_2(\mathbb{R})$.

But I was interested in the case of C^0-deformations, sans information about derivatives. Hence a dynamical approach would not be helpful. Around this time, Bowden-Mann's paper [BM22] on C^0-deformations appeared. Although they were working with negatively curved manifolds – very far away from higher rank lattices – their approach was inspiring. Ralf had already been discussing with Chris Connell and Thang Nguyen other questions in the same spirit, i.e. rigidity properties in symmetric spaces. So we joined forces and started working together on the project.

[2]The boundary face of $x \in \partial\Omega$ – denoted $F_\Omega(x)$ – is the union of all open projective lines in $\partial\Omega$ that contain x.

I will explain our work [CINS23] in the case of $G = \mathrm{SL}_3(\mathbb{R})$, but our results hold for more general higher rank Lie groups, see Theorem 4.1. In this case, a lattice $\Gamma < G$ acts on the rank two *Riemannian symmetric space* $X := \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. There are several notions of boundaries of X. For instance, the full flag space of $\mathbb{R}^3$ is called the *Furstenberg boundary* of X. More generally, the partial flag spaces are called the *generalized Furstenberg boundaries* of X. We will always denote by ρ_0 the action of a lattice $\Gamma < G$ on such boundaries. The *visual boundary* of X, denoted by ∂X, gives a different notion of boundary. It is the visual boundary of X with respect to a complete non-positively curved G-invariant Riemannian metric on X (given by the Killing form). We will always denote the action of Γ on ∂X by ρ_{v}.

Before proceeding with a discussion about deformations of actions, it is imperative that I explain the topology on the space of actions. Given a lattice $\Gamma < G$, fix a finite generating set S of Γ. Suppose F is a boundary of X equipped with a metric d_F. We will say that two actions ρ and θ, i.e. homomorphisms from Γ to $\mathrm{Homeo}(F)$, are C^0-close if $\rho(s)$ and $\theta(s)$ are close (as continuous maps on F) for each generator $s \in S$.

We say that θ is a *topological factor* of ρ if there is a continuous surjective map φ on F such that the adjoining diagram commutes for all $\gamma \in \Gamma$. The map φ is called a *semi-conjugacy*. Note that this is weaker than a conjugacy – two actions ρ, θ are *conjugate* if φ is a homeomorphism.

$$\begin{tikzcd} F \arrow[r, "\rho(\gamma)"] \arrow[d, "\varphi"] & F \arrow[d, "\varphi"] \\ F \arrow[r, "\theta(\gamma)"] & F \end{tikzcd}$$

I now give an example to illustrate that semi-cojugacy, and not conjugacy, gives the right notion of rigidity in the context of C^0-actions. I will consider $\mathbb{Z}$ actions on S^1. Consider $g \in \mathrm{PSL}_2(\mathbb{R})$ that acts on S^1 with north-south dynamics, i.e. g has an attracting and a repelling fixed point $g^{\pm}$ on S^1. Now consider $h \in \mathrm{Homeo}(S^1)$ that acts by fixing tiny intervals $\mathcal{I}^{\pm}$ around $g^{\pm}$ pointwise and the same north-south dynamics as g outside of $\mathcal{I}^{\pm}$. Then h is C^0-close to g but cannot be conjugate to it as h has uncountably many fixed points. However there is a semi-cojugacy. The map that collapses $\mathcal{I}^{\pm}$ to $g^{\pm}$ intertwines the actions by g and h.

Rigidity in generalized Furstenberg boundaries. In [CINS23], we investigate the question: what happens if we deform the boundary actions ρ_0 and ρ_{v} of a higher rank lattice Γ in the category of C^0-actions? For the actions ρ_0 on the generalized Furstenberg boundaries, we prove that any C^0-close deformation ρ has ρ_0 as a topological factor. Intuitively, this means that ρ differs from ρ_0 only in the sense that finitely many Γ-orbits could have 'blown up' during the deformation. But otherwise, the action hasn't changed. In fact, we also show in [CINS23] that this semi-conjugacy is the best that we can hope in this category of C^0-actions. I will explain that below, but first I want to discuss our rigidity result.

Theorem 4.1 ([CINS23])**.** *Let G be a connected linear semi-simple Lie group without compact factors, Γ be a uniform lattice in G, Q be a parabolic subgroup of G and ρ_0 be the standard boundary action of Γ on G/Q. Then for any action ρ that is sufficiently C^0-close to ρ_0, ρ is a topological factor of ρ_0. Moreover, the semi-conjugacy φ_ρ converges to* id *uniformly as ρ converges to ρ_0.*

Key ideas underlying Theorem 4.1. The first idea (or step) involves 'taming' the wild deformations ρ of ρ_0. This step is differential geometric in nature where we construct some leafwise C^1 'foliations'. These foliations are the 'tame' objects that encode the 'wild' actions ρ of the lattice. To make this a little more precise, consider $E := M \backslash G/\Gamma$ where M consists of diagonal matrices with diagonal entries ± 1. The space E is a a fiber bundle over X/Γ with fibers G/P, where P consists of upper triangular matrices. For each action ρ of Γ on G/P close to ρ_0, we construct a 'foliation' $\mathcal{H}_\rho$ of E. When $\rho = \rho_0$, the foliation $\mathcal{H}_{\rho_0}$ is very special. It is the center-stable foliation of an algebraic flow on E called the Weyl chamber flow (a higher rank analogue of the geodesic flow on the unit tangent bundle of a closed hyperbolic surface). When ρ is a deformation of ρ_0, we interpret the 'foliation' $\mathcal{H}_\rho$ as a perturbation of the dynamical foliation $\mathcal{H}_{\rho_0}$. This $\mathcal{H}_\rho$ is a more tame object than ρ, since it is related to an algebraic flow! The construction of this 'foliation' is inspired by [BM22]. However, their setup is quite different – they construct foliations on the unit tangent bundle of a negatively curved manifold and relate it to the geodesic flow. A particularly interesting part of our construction is the use of barycenters (i.e. center of mass of measures) for constructing these 'foliations'. I use the word 'foliation' as it conveys the idea well; the actual construction is involved and only yields leaf-wise immersions.

The second idea (or step) is to use coarse geometry and 'straighten' the above foliations $\mathcal{H}_\rho$ of E. This straightening essentially pops out a continuous semi-conjugacy between the actions ρ and ρ_0. Coarse geometry enters the picture via theorems about quasi-isometric embeddings of flats into higher rank symmetric spaces. I find this step amazing because it uses coarse geometry to prove a completely dynamical result. This is our key insight - using coarse geometry to study problems in the C^0-category. The coarse geometry of higher rank symmetric spaces (e.g. higher rank Morse lemma, quasi-flats theorem, etc.) is an area of my core expertise and is one of the key reasons behind my interest in this problem. We again draw inspiration from [BM22] which uses convergence group actions and the Morse lemma for Gromov hyperbolic groups. Higher rank lattices are very far from being Gromov hyperbolic, but the coarse geometry viewpoint proves to be fruitful nonetheless.

Can we get better than topological factors, i.e. conjugacies? In [CINS23], we show that the answer is 'no'. Via a 'Denjoy-like' construction, we produce actions ρ that are C^0-close to ρ_0 but are not conjugate to ρ_0. Intuitively, the idea is to blow up a Γ orbit. While such blow up constructions are standard in dynamics, our main contribution is finding a blow up method that does not alter the topology of the underlying manifold.

Since the group Γ acts minimally on G/Q, we fix a dense orbit Γx on G/Q. To blow up this orbit, we replace each orbit point γx by a tiny ball B_γ. The new action ρ coincides with ρ_0 outside these balls B_γ. Inside the ball B_γ, ρ is defined to act by the derivative of ρ_0. This derivative action makes sense because tiny balls on the manifold and the tangent space are essentially the same. As mentioned before, we then need to ensure that the blown-up manifold is still homeomorphic G/Q. Here, our main tools are topological results on the homeomorphism type of spaces obtained by quotienting Sierpinski spaces. In fact, our result holds for any compact manifold F - not just G/Q. [BM22] also has a similar construction, but their result is limited to topological spheres.

Proposition 4.2 ([CINS23])**.** *Suppose ρ_0 is a C^1-action of a countable group Λ on a compact smooth manifold F with a dense orbit. Then there exist actions ρ arbitrarily C^0-close to ρ_0 but not C^0 conjugate to ρ_0.*

Non-rigidity in the visual boundary. In [CINS23], we also discover that this dynamical rigidity vanishes in the visual boundary. Consider a lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$ and its action ρ_{v} on the visual boundary of $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. We construct arbitrarily small deformations of ρ_{v} that do not have ρ_{v} as a topological factor. Our key idea is to exploit the spherical building structure on ∂X and construct deformations that destroy the G-orbit structure on ∂X.

The spherical building structure on ∂X tell us that ∂X is constructed by gluing together circles (i.e. visual boundaries of flats in X) and each circle is obtained by gluing closed intervals (i.e. visual boundaries of sectors in a flat). Owing to this combinatorial structure, we can construct a deformation ρ of ρ_{v} by specifying it on a single closed interval. The action ρ on a closed interval is simple - it fixes the centre of the interval and dilates around it. The challenge is to coherently control the dilation amount depending on which element of Γ is acting. Our idea is to do this using a cocycle on the Furstenberg boundary of X. For this, we use the well-known Radon-Nikodym derivative cocycle for the Lebesgue measure class. In fact, by taking powers of the cocycle, we produce a one parameter family of deformations ρ that converges to ρ_{v}.

Theorem 4.3 ([CINS23])**.** *Suppose X is a simply connected higher rank symmetric space of non-compact type, G is the identity component in $\mathrm{Isom}(X)$, $\Gamma < G$ is a lattice, and ρ_{v} is the natural action of Γ on ∂X. Then, there exist actions ρ, arbitrarily C^0-close to ρ_{v}, such that ρ_{v} isn't a topological factor of ρ.*

5. Future Directions

I will now outline some of my future research goals along with some concrete problems. Parts of these problems would make interesting research projects for doctoral students. There are also computer visualization problems and toy cases that would make fantastic undergraduate research projects.

5.1. Boundary Structures for Convex Co-compact Groups.

I want to pursue the philosophy of studying discrete groups via their boundaries further. My focus will be on convex co-compact groups and their boundary structures. I now outline two research directions.

• _Pseudo-Riemannian convex co-compact groups._ The group $\mathrm{SO}(p, q+1)$ preserves a quadratic form of signature $(p, q+1)$ and is the isometry group of the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q} \subset \mathbb{P}(\mathbb{R}^{p+q+1})$. Intuitively, the pseudo-Riemannian convex co-compact groups are discrete subgroups of $\mathrm{SO}(p, q+1)$ that preserve a properly convex domain $\Omega \subset \mathbb{H}^{p,q}$ and acts convex co-compactly on Ω. Such groups preserve a particularly nice structure at their bonudary [GM18], reminiscent of the conformal structure at the boundary of the hyperbolic space. But in $\mathrm{SO}(p, q+1)$, since the structure comes from a pseudo-Riemannian metric, it is only partly conformal. Hence I will call it a 'pseudo-conformal' structure. This structure has not been investigated much for convex co-compact groups. I plan to undertake a systematic program to study it.

I now explain a concrete question in this direction. In [GM18], the authors explore the following question: if a convex co-compact group $\Gamma < \mathrm{SO}(p, q+1)$ has a $(p-1)$-dimensional sphere as its limit set, does Γ necessarily preserve a totally geodesic copy of $\mathbb{H}^p$ inside $\mathbb{H}^{p,q}$? They used dynamical tools to answer it for $\mathrm{SO}(2,2)$ but it is open in general. I want to attack this question using a quasiconformal geometry perspective. In particular, Bonk-Kleiner's work [BK02] discusses – in the quasiconformal setting – how to fill in a $(p-1)$-dimensional boundary sphere with a totally geodesic copy of $\mathbb{H}^p$. Can this be adapted to the 'pseudo-conformal' setting?

• *Divisible convex domains.* What about the boundaries of more general projective domains in $\mathbb{P}(\mathbb{R}^d)$? Morally, the question that I want to ask here is whether a properly convex domain can be re-built from the trace that it leaves at its boundary. A motivating example is the result that a quasi-Möbius map between circles extends to a quasi-isometry between two copies of $\mathbb{H}^2$.

For a more precise formulation, consider two rank one groups $\Gamma_i < \mathrm{Aut}(\Omega_i)$ with Ω_i/Γ_i compact and $i = 1, 2$. In this case, there is a notion of Morse boundary $\partial_M \Omega_i$, a subset of the projective boundary $\partial\Omega_i$ that records all the rank one/hyperbolic directions. Can we find a good notion of 'quasiconformality' on this Morse boundary? I want to leverage this 'quasiconformal' structure and prove that: an equivariant 'quasiconformal' map between $\partial_M \Omega_1$ and $\partial_M \Omega_2$ extends to a quasi-isometry between Ω_1 and Ω_2.

My strategy to attack this question is inspired by the contracting boundary of CAT(0) spaces. The paper [CCM19] develops a notion of 'quasiconformality' on such CAT(0) boundaries and proves a quasiisometry extension result. In our case, the key insight necessary is the candidate for 'quasiconformality'. A promising candidate originates from my upcoming work with Weismann [IW23]. In [IW23], we relate a Morse boundary point to the 'regularity' of the boundary $\partial\Omega$ at that point. To quantify regularity, we define a function determined by the singular values along a sequence in the rank one group Γ. This function also seems to control distortion of balls and would be a natural candidate for 'quasiconformality'.

5.2. **Constructions of discrete linear groups.** We currently have only a limited number of methods for constructing discrete subgroups of linear groups. I want to address this dearth of examples. I will focus on Coxeter groups, i.e. groups of the form $W = \langle s_1, \ldots, s_m | s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1$ where $m_{ij} \in \{2, \infty\}$ for $i \neq j\rangle$. My goal is to develop new constructions of discrete linear representations of Coxeter groups that admit nice deformation spaces.Vinberg's work [Vin71] provides a criteria for constructing discrete linear representations of W where each s_i acts by a linear reflection. The resulting discrete groups preserve a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$. But what about other possible linear representations of W? For instance, what if s_i acts an involution on $\mathbb{R}^d$ but not a reflection (i.e. the fixed hyperplane of s_i has codimension greater than 1)? What if the group W does not preserve a convex domain? My goal is to study representations that address these natural questions.

• $\underline{\mathrm{Sp}_4(\mathbb{R})}$. In the symplectic group, I let each s_i act by a symplectic reflection, i.e. s_i acts by Id on a 2-dimensional subspace $W_i \subset \mathbb{R}^4$ and by $(-\mathrm{Id})$ on its symplectic orthogonal. In ongoing work with Beatrice Pozzetti, we have been studying the case of W_∞, 3-generated Coxeter group with each $m_{ij} = \infty$, i.e. (∞, ∞, ∞)-triangle group. As W_∞ contains an index 2 free subgroup, we use higher rank ping-pong dynamics and prove a discreteness criteria. Moreover, we have nice deformations indexed by a triple of fixed symplectic subspaces (one for each s_i). But there are plenty of questions to ask - what happens when there is torsion, i.e. $m_{ij} < \infty$? More generators? Can we get a complete picture for all triangle groups? Are there nice parametrizations of the moduli spaces?

• $\underline{\mathrm{SL}_d(\mathbb{R})}$. Here we let s_i act with a fixed subspace of codimension ≥ 2. For the W_∞ group as above, even the $\mathrm{SL}_3(\mathbb{R})$ case is interesting as it is complementary to Vinberg's case [Vin71]. Jointly with Ludovic Marquis, we have partial results for discreteness in $\mathrm{SL}_3(\mathbb{R})$. We explicitly construct a fundamental domain in an affine chart in $\mathbb{P}(\mathbb{R}^3)$. Again, there are many open questions - what about the other $\mathrm{SL}_d(\mathbb{R})$? More generators? Torsion elements? Even the case $\mathrm{SL}_4(\mathbb{R})$ is unclear, as our geometric ideas from the $\mathrm{SL}_3(\mathbb{R})$ case do not generalize. This area seems ripe for undergraduate research projects. Computer visulaizations of these group actions on $\mathbb{P}(\mathbb{R}^4)$ would provide valuable geometric intuition that we desperately need. A sample problem could be: consider a concrete example, e.g. W_∞, in $\mathrm{SO}(2,1)$ whose tiling in $\mathbb{H}^2$ we can generate. Then observe what happens to this tiling as we deform the group in $\mathrm{SL}_4(\mathbb{R})$.

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