Talk 1: Introduction to Matrix Groups and Examples of Them

1 Matrix Groups

Definition 1.1. A subgroup G $\leqslant GL_n(\mathbb{K})$ which is also a closed Subspace is called a Matrix Group or a $\mathbb{K}$ -matrix Group [Ba, Prop. 1.30]

Not all Groups of Matrices are Matrix Groups!

Example 1.2. SL_n is a Matrix Group

Definition 1.3. for a Vector x $\in \mathbb{K}^n$ the lenght is defined as |x|=$\sqrt{(x_1)^2 + ... + (x_n)^2}$

Proposition 1.4. *The following Statements are equivalent: A is a linear Isometry, $Ax * Ay = x * y$, $A^T * A = I_n$ [Ba, Prop. 1.38]*

Lemma 1.5. $Isom_n(\mathbb{R}) = O(n) \ltimes Trans_n(\mathbb{R})$ $= \{AT : A \in O(n), T \in Trans_n\}$ *[Ba, Prop. 1.39]*

Lemma 1.6. *SU(2) is a double cover of So(3)*

Lemma 1.7. *The Group $Heis_3$ is not linear*

2 Overview of Groups

Overview 2.1. .
$GL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : det(A) \neq 0\}$
$SL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : det(A) = 1\}$
$UT_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K})$:A is upper triangular$\}$
$SUT_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K})$:A is unipotent$\}$
$O(n) = \{A \in GL_n(\mathbb{R}) : A^T A = I_n\}$
$SO(n) = \{A \in GL_n(\mathbb{R}) : A^T A = I_n, det(A) = 1\}$
U(n)=$\{A \in GL_n(\mathbb{C}) : A^* A = I_n\}$
SU(n)=$\{A \in GL_n(\mathbb{C}) : A^* A = I_n, det(A) = 1\}$
$Trans_n(\mathbb{K}) = \left\{ \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{K}^n \right\}$
$\mathrm{Aff}_n(\mathbb{K}) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{K}^n, A \in GL_n(\mathbb{K}) \right\}$
$\mathrm{Isom}_n(\mathbb{K}) = \left\{ \text{f: } \mathbb{K}^n \to \mathbb{K}^n \text{ :f is an isometry} \right\}$

Example 2.2.
$\mathrm{SO}(2) = \left\{ \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} : \theta \in [0,2\pi) \right\}$

SO(3) can be imagined as all the proper rotations of a Sphere

Exercise 2.3. Prove for that any Eigenvalue λ of a Matrix A $\in$ U(n) $|\lambda|$ =1

References

[Ba] Andrew Baker: *Matrix Groups.*

Topology (of Matrix Groups)

by Friedrich Homann

Talk 2

- **Topology**
  - What is topology?
  - **a topology**
    - axioms
      - Let X be a set and let $\tau \subseteq P(x)$. Then τ is called a topology if
      - i. Both the empty set and X are elements of τ.
      - ii. Any union of elements of τ is an element of τ.
      - iii. Any intersection of finitely many elements of τ is an element of τ.
    - examples
      - chaotic/trivial/indiscrete topology
      - discrete topology
      - standard topology
  - **topological space**
    - Def.: the pair of a set and a topology on that set
  - **Open sets**
    - Analysis 1 notion: similar to open interval
    - Def.: U is an <u>open set</u> if and only if it is an element of the topology.
    - Therefore $GL_n(\mathbb{R}, \mathbb{C}) \subseteq M_n(\mathbb{R}, \mathbb{C})$ can be open subsets.
  - **Closed sets**
    - Def.: A set P is <u>closed</u> if and only if the complement is open.
  - **Continuity**
    - Def.: Let (M, τ_M) and (N, τ_N) be topological spaces. Then a map $f: M \to N$ is <u>continuous</u> if $\forall\ V \in \tau_N: preim_f(V) \in \tau_M$.
    - theorem: the composition of continuous maps is continuous.
  - example:
    - Let S $= \{1,2,3,4\}$ be a set. $\tau = \{\{\emptyset\}, \{1\}, \{1,2,3,4\}\}$ is a topology on S.
    - $\{1\}$ is an open set
    - $\{2,3,4\}$ is a closed set
    - Let $\tau' = \{\{\emptyset\}, S\}$ be a different topology on S. Let $f: (S, \tau) \to (S, \tau')$ be the identity map. Then f is continuous, but not its inverse, since the preimage of $\{1\}$ is not an open set in τ'.
  - **Compactness**
    - simplified notion from Analysis 1 lectures: closed & bounded
    - **Open cover**
      - Def.: If U is a family of open subsets u, then U is an <u>open cover</u> of E if $E \subseteq \bigcup\{u | u \in U\}$
      - example:
        - $U = \{B((M, N), 1); M, N \in \mathbb{Z}\}$
    - **Subcover**
      - Def.: V is a <u>subcover</u> of U if V is a subset of U that also covers E.
    - Def.: E is <u>compact</u> if every open cover U has a finite subcover V.
    - Compactness is preserved by continuous functions.
    - Heine-Borel theorem states: for any set S in $\mathbb{R}^n$, S is closed and compact $\Leftrightarrow$ S is compact i.e. every open cover has a finite subcover.
    - examples:
      - $[a, b]$
      - ~~open balls in $\mathbb{R}^n$~~ closed balls of finite radius
      - $O(n)$ and $SO(n)$
    - non-examples:
      - $\mathbb{R}$ ($U = \{(-n, n) | n \in \mathbb{N}\}$, finitely many elements do not suffice)
      - $(0,1)$ ($U = \left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\}$, again, finitely many elements do not suffice)

- **Connectedness**
    - Def.: not disconnected
    - **Disconnectedness**
        - Def.: E is <u>disconnected</u> if there are nonempty, open and disjoint subsets of E such that the union is E.
        - analogy/example:
            - Puzzle
        - property
            - If $f: C \to f(C)$ is continuous and C is connected, then $f(C)$ is connected.
                - Therefore: a continuous function $f: (0,1) \to (0; 0{,}5) \cup (1{,}5; 2)$ is impossible. *proof is left as an exercise/problem*
    - example:
        - $\mathbb{R}$
        - $GL_n(\mathbb{R})$ isn't connected because it has two disjoint components. The matrices with positive and the matrices with negative determinants.
        - $GL_n(\mathbb{C})$ is
- **Homeomorphisms**
    - *not* homomorphisms
    - a.k.a. the donut = coffee mug part of topology
    - preserve the topological structure
    - Def.: $f: M \to N$ is a <u>homeomorphism</u> if f is bijective and continuous "in both directions".
    - examples:
        - $f: [0,1] \to [0,2]$ (f could be $x \mapsto 2x$)
        - $(0,1)$ and $\mathbb{R}$ are homeomorphic
            - *exercise:*
                a) *First find a homeomorphism between* $[0,100]$ *and* $[0,1]$.
                b) *Then, find a homeomorphism between* $(0,1)$ *and* $\mathbb{R}$.
- **Metric**
    - creates the notion of distance
    - has to meet certain properties:
        i. $\mathrm{d}(x,y) \geq 0$ and $\mathrm{d}(x,y) = 0 \Leftrightarrow x = y$
        ii. $\mathrm{d}(x,y) = \mathrm{d}(y,x)$
        iii. $\mathrm{d}(x,y) + \mathrm{d}(y,z) \geq \mathrm{d}(x,z)$
    - A pair of a set S and a metric d (S,d) is called a metric space.
    - Closely related notion of norm.
    - examples:
        - Euclidean metric $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ in $\mathbb{R}^2$
        - taxicab metric ($|x_1 - x_2| + |y_1 - y_2|$ in $\mathbb{R}^2$
        - L^P- metrics $(|x_1 - x_2|^P + |y_1 - y_2|^P)^{\frac{1}{P}}$ in $\mathbb{R}^2$
    - possible norm on $M_n(\mathbb{R})$: $\|A\| = \{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$
    - can be used to define a metric on $M_n(\mathbb{R})$ ($\mathrm{d}(A,B) = \|A - B\|$)
- **Subspace topology**
    - Induction of topologies on subsets
    - Def.: Let (M,τ) be a topological space, $N \subset M$, then $\tau|_N := \{u \cap N | u \in \tau\}$.
        - *proof that* $\tau|_N$ *is a topology is left as an exercise.*
    - properties:
        - If N is an open set in M, then v is open in N if and only if it is open in M.
    - examples:
        - We can equip $S_1 \subset \mathbb{R}^2$ with $\tau_{std.}|_{s_1}$.