RELATIVELY HYPERBOLIC CONVEX CO-COMPACT GROUPS (EXPOSITORY NOTE ON MY JOINT WORK [IZ23])

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Abstract. This note provides a simplified account of the paper [IZ23] in the context of convex co-compact groups. The paper [IZ23] works in the very general context of naive convex co-compact groups which introduces many technical complications, making our paper quite long. In this note, we restrict ourselves to only convex co-compact groups and, as a result, obtain a more concise presentation. We write this note hoping to convey the main ideas underlying [IZ23] in a more streamlined way. This simplified treatment already appears in my thesis [Isl21] and this note is not intended for publication.

Contents

1. Introduction

The notion of convex co-compact subgroups of $\mathrm{PGL}_d(\mathbb{R})$ generalize convex co-compact Kleinian groups from the rank one Lie group $\mathrm{SO}(d,1)$ ($d \geq 2$) to higher rank Lie groups like $\mathrm{PGL}_d(\mathbb{R})$ for $d \geq 3$. We now define convex co-compact groups.

Definition 1.1 ([DGK17]). A discrete subgroup $\Gamma \leq \mathrm{PGL}_d(\mathbb{R})$ is called convex co-compact if:

(1) there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that $\Gamma \leq \mathrm{Aut}(\Omega)$, and
(2) the set $\mathcal{C}_\Omega(\Gamma) \subset \Omega$ is non-empty and Γ acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$, where $\mathcal{C}_\Omega(\Gamma)$ is the convex hull in Ω of the full orbital limit set $\mathcal{L}^{\mathrm{orb}}_\Omega(\Gamma) := \bigcup_{x \in \Omega} \left(\overline{\Gamma x} \setminus \Gamma x\right)$.

Since the Ω in the definition of convex co-compact groups is not canonical, we will remove this ambiguity by explicitly mentioning the properly convex domain wherever necessary, i.e. we will say that "$\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group" instead of "Γ is a convex co-compact group".

A recent result of Danciger-Guéritaud-Kassel, independently Zimmer, establishes a connection between the Hilbert geometry of the properly convex domain $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ and Anosov representations. More precisely, they prove the following.

Theorem 1.2 ([DGK17, Zim21]). *Suppose Ω is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ is a Gromov hyperbolic space,
(2) Γ is a Gromov hyperbolic group, and
(3) the inclusion $\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ is a projective Anosov representation.

Anosov representations are a class of representations of Gromov hyperbolic groups into real semi-simple Lie groups that generalizes classical Teichmüller theory, i.e. the study of discrete faithful representations of hyperbolic surface groups into $\mathrm{PSL}_2(\mathbb{R})$. They are discrete faithful representations that have good dynamical and geometric properties. Introduced by Labourie [Lab06] and studied subsequently by many authors, this area has received much attention lately, see for instance [GGKW17, KLP17, BIW14, Poz19].

The above Theorem 1.2 can be interpreted as a way associating convex projective structures to projective Anosov representations. This opens up a more geometric way of thinking about Anosov representations. In their paper, Danciger-Guéritaud-Kassel asked the following natural question [DGK17, Appendix A, Question A.2].

Question 1. *What geometric conditions on $\mathcal{C}_\Omega(\Gamma)$ will correspond to Γ relatively hyperbolic with respect to virtually Abelian subgroups of rank at least two?*

By virtue of Theorem 1.2, this question can be interpreted as seeking a generalization of projective Anosov representations to relatively hyperbolic groups. Note that the current definition of Anosov representations work only for Gromov hyperbolic groups and generalizing it beyond Gromov hyperbolicity is an area of active research, see for instance [Kas18, Gui19]. The notion of relative Anosov representations due to [KL18] and [Zhu22] provide an approach. But we note that those approaches do not provide an answer to the above question. Indeed, the peripheral subgroups in the work of [KL18, Zhu22] consist of unipotent elements while it is easy to verify that convex co-compact subgroups cannot contain unipotent elements other than the identity.

There are many interesting examples of convex co-compact groups coming from Coxeter groups and 3-manifold theory that satisfy the conditions in Question 1, see for instance Figure 1 or the papers [Ben06, BDL18, DGK17].

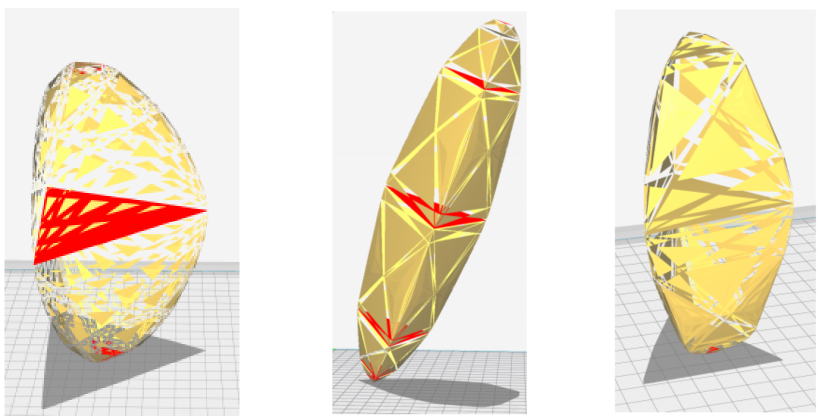

FIGURE 1. Examples of three-dimensional properly convex domains Ω that admit co-compact action by Γ where the groups Γ are relatively hyperbolic with respect to subgroups virtually isomorphic to $\mathbb{Z}^2$ [RSS+19]

We answer Question 1 in joint work with A. Zimmer in [IZ23] by introducing the notion of properly convex domains with strongly isolated simplices. We will now introduce the definition here. We will say that a properly embedded simplex is maximal if it is not properly contained in any other properly embedded simplex. Note that this notion of maximality does not mean that we are looking only at simplices of the maximal possible dimension.

Definition 1.3 ([IZ23, Definition 1.15])**.** Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $\mathcal{S}_\Gamma$ is the collection of all maximal properly embedded simplices in $\mathcal{C}$ of dimension at least two. We will say that $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has *strongly isolated simplices* provided: for any $r \geq 0$, there exists $D(r) \geq 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)\right) \leq D(r).$$

We answer Question 1 by proving a theorem connecting relative hyperbolicity (with respect to virtually Abelian subgroups of rank at least two) of a convex co-compact group and properly convex domains with strongly isolated simplices. The precise statement is as follows.

Theorem 1.4 ([IZ23, Theorem 1.7])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group, and $\mathcal{S}_\Gamma$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two. Then the following are equivalent:*

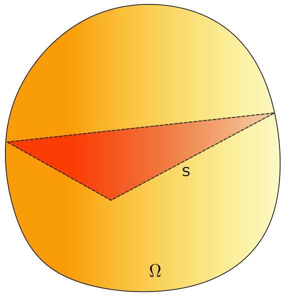

FIGURE 2. A maximal properly embedded 2-simplex

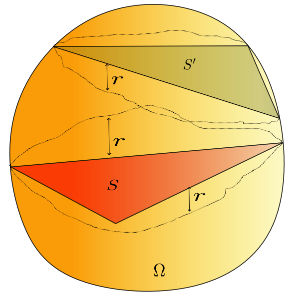

FIGURE 3. Strongly isolated simplices

(1) $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ *has strongly isolated simplices,*
(2) $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ *is a relatively hyperbolic space with respect to* $\mathcal{S}_\Gamma$*,*
(3) Γ *is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least two.*

Theorem 1.4 can be viewed as a real projective analogue of a CAT(0) result. In [HK05], Hruska-Kleiner study CAT(0) spaces with isolated flats and proves an analogoues result in that setting. In this analogy, maximal properly embedded simplices correspond to maximal totally geodesic flats in CAT(0) spaces (see [IZ21, Ben04]).

We also establish some finer geometric properties of $\mathcal{C}_\Omega(\Gamma)$ when $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has strongly isolated simplices. We will need some terminology to state the theorem precisely. The ideal boundary of $\mathcal{C}_\Omega(\Gamma)$ is defined as $\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma) := \overline{\mathcal{C}_\Omega(\Gamma)} \cap \partial\Omega$, i.e. it is the part of the boundary of $\mathcal{C}_\Omega(\Gamma)$ that is at "infinity" in $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$. If $C \subset \Omega$ is a convex subset and $x \in \overline{C}$, then

$$F_C(x) = \{x\} \cup \{y \in \overline{C} : \exists \text{ an open line segment in } \overline{C} \text{ containing both } x \text{ and } y\}.$$

Theorem 1.5 ([IZ23, Theorem 1.8])**.** *Suppose* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a convex co-compact group and* $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ *has strongly isolated simplices. Then*

(1) Γ *has finitely many orbits in* $\mathcal{S}_\Gamma$*.*
(2) *If* $S \in \mathcal{S}_\Gamma$*, then* $\mathrm{Stab}_\Gamma(S)$ *acts co-compactly on* S *and contains a finite index subgroup isomorphic to* $\mathbb{Z}^k$ *where* $k = \dim S$*.*
(3) *If* $A \leq \Gamma$ *is an infinite Abelian subgroup of rank at least two, then there exists a unique* $S \in \mathcal{S}_\Gamma$ *with* $A \leq \mathrm{Stab}_\Gamma(S)$*.*
(4) *If* $S \in \mathcal{S}_\Gamma$ *and* $x \in \partial S$*, then* $F_\Omega(x) = F_{\mathcal{C}_\Omega(\Gamma)}(x) = F_S(x)$*.*
(5) *If* $S_1, S_2 \in \mathcal{S}_\Gamma$ *are distinct, then* $\#(S_1 \cap S_2) \leq 1$ *and* $\partial S_1 \cap \partial S_2 = \emptyset$*.*
(6) *If* $\ell \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ *is a non-trivial line segment, then there exists* $S \in \mathcal{S}_\Gamma$ *with* $\ell \subset \partial S$*.*
(7) *If* $x, y, z \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ *form a half triangle in* $\mathcal{C}_\Omega(\Gamma)$ (i.e. $[x, y] \cup [y, z] \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ and $(x, z) \subset \mathcal{C}_\Omega(\Gamma)$), *then there exists* $S \in \mathcal{S}_\Gamma$ *such that* $x, y, z \in \partial S$*.*
(8) *If* $x \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ *is not a* C^1*-smooth point of* $\partial\Omega$ *(i.e.* Ω *does not have a unique supporting hyperplane at* x*), then there exists* $S \in \mathcal{S}_\Gamma$ *with* $x \in \partial S$*.*

2. Preliminaries

2.1. Properly Convex Domains and Hilbert Geometry.

A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is:

(1) *properly convex* if there exists an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a bounded convex subset.
(2) a *properly convex domain* if C is properly convex and open in $\mathbb{P}(\mathbb{R}^d)$.

Given a properly convex set $C \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset \overline{C}$, we define its *convex hull* as

$$\mathrm{ConvHull}_C(X) := \cap\big\{Y : Y \text{ is a closed convex subset such that } X \subset Y \subset \overline{C}\big\}.$$

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$ there is no canonical line segment with endpoints x and y, but we will use the following convention: if $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \overline{C}$, then (when the context is clear) we will let $[x, y]$ denote the closed line segment joining x to y which is contained in $\overline{C}$. In this case, we will also let $(x,y) = [x,y] \setminus \{x,y\}$, $[x,y) = [x,y] \setminus \{y\}$, and $(x,y] = [x,y] \setminus \{x\}$.

Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $x, y \in \Omega$ are distinct, let $[a,b] := \mathbb{P}(\mathrm{Span}\{x,y\}) \cap \overline{\Omega}$ where a and b labelled such that $x \in [a,y]$ (i.e. the points are ordered a, x, y, b along $[a,b]$). Then the *the Hilbert distance* between x and y is defined to be

$$\mathrm{d}_\Omega(x,y) := \frac{1}{2}\log[a,x,y,b]$$

where

$$[a,x,y,b] := \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}$$

is the cross ratio (here $|\cdot|$ is some (any) norm in some (any) affine chart which contains a,x,y,b). Then $(\Omega, \mathrm{d}_\Omega)$ is a complete geodesic metric space and $\mathrm{Aut}(\Omega)$ acts properly and by isometries on Ω (see for instance [BK53, Section 28]). Further, the projective line segment $[x,y]$ is a geodesic for the Hilbert distance.

Convexity is preserved under taking r-neighbourhoods in the Hilbert metric of closed convex sets. If $C \subset \Omega$ is a closed convex set and $r > 0$ then

$$\mathcal{N}_\Omega(C;r) := \{x \in \Omega : \mathrm{d}_\Omega(x,C) < r\}$$

is a convex set [CLT15, Corollary 1.10]. The corresponding closed neighbourhood of C, i.e. $\overline{\mathcal{N}_\Omega(C;r)}$, is also convex.

For this subsection, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Consider the equivalence relation $\sim_\Omega$ on $\overline{\Omega}$ is given by: $x \sim_\Omega y$ if and only if there exists an open projective line segment in $\overline{\Omega}$ containing x and y. The equivalence class of $x \in \overline{\Omega}$ is denoted by $F_\Omega(x)$ [CM14, Section 3.3]. The following results are simple consequences of convexity, see for instance [IZ21, Isl19].

Proposition 2.1.

(1) $F_\Omega(x)$ is open in its span.
(2) $F_\Omega(x) = \Omega$ whenever $x \in \Omega$ and $F_\Omega(x) \subset \partial\Omega$ whenever $x \in \partial\Omega$.
(3) $y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$.
(4) Suppose $x, y \in \overline{\Omega}$, $p \in F_\Omega(x)$, $q \in F_\Omega(y)$, and $z \in (x,y)$. Then

$$(p,q) \subset F_\Omega(z).$$

In particular, $(p,q) \subset \Omega$ if and only if $(x,y) \subset \Omega$.
(5) If $y \in \partial F_\Omega(x)$, then $F_\Omega(y) \subset \partial F_\Omega(x)$,

Proposition 2.1 shows that $F_\Omega(x)$ is a relatively open convex subset of $\partial\Omega$ for all $x \in \partial\Omega$. Thus $F_\Omega(x)$ can be equipped with a Hilbert metric $\mathrm{d}_{F_\Omega(x)}$ for any $x \in \partial\Omega$. We will now state some estimates that relate the Hilbert metric in the interior of Ω with the Hilbert metric on the faces $F_\Omega(x)$. These results are elementary and can be found in many places, for instance [IZ21].

Proposition 2.2. *Suppose $\{x_n\}$ is a sequence in Ω and $x_n \to x \in \overline{\Omega}$. If $\{y_n\}$ is another sequence in Ω, $y_n \to y \in \overline{\Omega}$, and*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n) < +\infty,$$

then $y \in F_\Omega(x)$ and

$$\mathrm{d}_{F_\Omega(x)}(x,y) \leq \liminf_{n\to\infty} \mathrm{d}_\Omega(x_n,y_n).$$

Corollary 2.3. *Suppose $A, B \subset \Omega$ be non-empty subsets such that $A \subset \mathcal{N}_\Omega(B;r)$ for some $r > 0$. If $a \in \overline{A}$, then there exists $b \in \overline{B}$ such that $a \in F_\Omega(b)$ and $\mathrm{d}_{F_\Omega(b)}(a,b) \leq r$.*

Corollary 2.4 ([DGK17, Corollary 3.5])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain, $y \in \partial\Omega$, and $\{y_m\}$ and $\{z_m\}$ are two sequences in Ω. If $y_m \to y$ and $\mathrm{d}_\Omega(y_m, z_m) \to 0$, then $z_m \to y$.*

Lemma 2.5 ([Cra09, Lemma 8.3])**.** *Suppose that $\sigma_1, \sigma_2 : [0,T] \to \Omega$ are two unit speed projective line geodesics, then for $0 \leq t \leq T$,*

$$\mathrm{d}_\Omega(\sigma_1(t), \sigma_2(t)) \leq \mathrm{d}_\Omega(\sigma_1(0), \sigma_2(0)) + \mathrm{d}_\Omega(\sigma_1(T), \sigma_2(T)).$$

We introduce two metric geometric tools: closest-point projection and center of mass.

Definition 2.6. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $A \subset \Omega$ is a non-empty closed convex set. If $p \in \Omega$, the *closest-point projection of p on A* is

$$\pi_A(p) := A \cap \overline{\mathcal{B}_\Omega(p, \mathrm{d}_\Omega(p, A))}.$$

If $p \in \Omega$, $\pi_A(p)$ is a compact convex set. Moreover, if $g \in \mathrm{Aut}(\Omega)$, then $g \circ \pi_A = \pi_{gA} \circ g$.

There is a notion of "center of mass" for compact subsets of a properly convex domain. Let $\mathcal{K}_d$ denote the set of all pairs (Ω, K) where $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $K \subset \Omega$ is a compact subset.

Proposition 2.7 ([IZ21]). *There exists a function*

$$(\Omega, K) \in \mathcal{K}_d \longmapsto \mathrm{CoM}_\Omega(K) \in \mathbb{P}(\mathbb{R}^d)$$

such that:

(1) $\mathrm{CoM}_\Omega(K) \in \mathrm{ConvHull}_\Omega(K)$,
(2) $\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_\Omega(\mathrm{ConvHull}_\Omega(K))$, *and*
(3) if $g \in \mathrm{PGL}_d(\mathbb{R})$, *then* $g\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_{g\Omega}(gK)$,

for every $(\Omega, K) \in \mathcal{K}_d$.

There are several ways of proving the existence of such a "center of mass". Proposition 2.7 appears in [IZ21] and their argument is inspired by Frankel [Fra89]. An alternative approach to this construction appears in [Mar14, Lemma 4.2].

2.2. **Dynamics of Automorphisms in Hilbert Geometry.** If $g \in \mathrm{GL}_d(\mathbb{R})$, let $\lambda_1(g),\ \lambda_2(g),\ \ldots,\ \lambda_d(g)$ denote the absolute values of eigenvalues of g (over $\mathbb{C}$), indexed such that

$$\lambda_1(g) \geq \lambda_2(g) \geq \ldots \geq \lambda_d(g).$$

In particular, we will use the notation $\lambda_{\max}(g) := \lambda_1(g)$ and $\lambda_{\min}(g) := \lambda_d(g)$. If $h \in \mathrm{PGL}_d(\mathbb{R})$, we define

$$\frac{\lambda_i}{\lambda_j}(h) := \frac{\lambda_i(\widetilde{h})}{\lambda_j(\widetilde{h})}$$

where $\widetilde{h} \in \mathrm{GL}_d(\mathbb{R})$ is some (hence any) lift of h.

Proposition 2.8 ([CLT15]). *Suppose Ω is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$. Then the translation length of g, defined as*

$$\tau_\Omega(g) := \inf_{x \in \Omega} \mathrm{d}_\Omega(x, gx),$$

is given by

$$\tau_\Omega(g) = \log\left(\frac{\lambda_1}{\lambda_d}(g)\right).$$

The next two results relate the faces of a convex domain with the behavior of automorphisms.

Proposition 2.9 ([IZ21]). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, and $g_n \in \mathrm{Aut}(\Omega)$ is a sequence such that*

(1) $g_n(p_0) \to x \in \partial\Omega$,
(2) $g_n^{-1}(p_0) \to y \in \partial\Omega$, *and*
(3) g_n *converges to* T *in* $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$.

Then $\mathrm{image}(T) \subset \mathrm{Span}\, F_\Omega(x)$, $\mathbb{P}(\ker T) \cap \Omega = \emptyset$, *and* $y \in \mathbb{P}(\ker T)$.

Given a group $G \leq \mathrm{PGL}_d(\mathbb{R})$ define $\overline{G}^{\mathrm{End}}$ to be the closure of the set

$$\{g \in \mathrm{GL}_d(\mathbb{R}) : [g] \in G\}$$

in $\mathrm{End}(\mathbb{R}^d)$.

Proposition 2.10 ([IZ21]). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\mathcal{C} \subset \Omega$ is a non-empty closed convex subset, and $G \leq \mathrm{Stab}_\Omega(\mathcal{C})$ acts co-compactly on $\mathcal{C}$. If $x \in \partial_{\mathrm{i}}\,\mathcal{C}$, then there exists $T \in \overline{G}^{\mathrm{End}}$ such that*

(1) $\mathbb{P}(\ker T) \cap \Omega = \emptyset$,

(2) $T(\Omega) = F_\Omega(x)$, *and*
(3) $T(\mathcal{C}) = F_\Omega(x) \cap \partial_{\mathrm{i}} \mathcal{C}$.

2.3. **(Local) Hausdorff topology.** Let ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}$ denote the *Hausdorff distance* on subsets of Ω induced by d_Ω, that is: for subsets $A, B \subset \Omega$ define

$$ {\mathrm{d}_\Omega}^{\mathrm{Hauss}}(A,B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \mathrm{d}_\Omega(a,b), \sup_{b \in B} \inf_{a \in A} \mathrm{d}_\Omega(a,b) \right\}. $$

Proposition 2.11. *Assume* $p_1, p_2, q_1, q_2 \in \overline{\Omega}$, $F_\Omega(p_1) = F_\Omega(p_2)$, *and* $F_\Omega(q_1) = F_\Omega(q_2)$. *If* $(p_1, q_1) \cap \Omega \neq \emptyset$, *then* $(p_2, q_2) \subset \Omega$ *and*

$$ \mathrm{d}_\Omega^{\mathrm{Hauss}} \Big((p_1, q_1), (p_2, q_2) \Big) \leq \max \{ \mathrm{d}_{F_\Omega(p_1)}(p_1, p_2), \mathrm{d}_{F_\Omega(q_1)}(q_1, q_2) \}. $$

The local Hausdorff topology is a natural topology on the set of all closed subsets of Ω induced by the Hausdorff distance ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}$. For a closed subset $C_0 \subset \Omega$, $r_0, \varepsilon_0 > 0$, and $x_0 \in \Omega$, define $U(C_0, r_0, \varepsilon_0, x_0)$ to be the set of all closed subsets C of Ω such that

$$ {\mathrm{d}_\Omega}^{\mathrm{Hauss}}(\mathcal{B}_\Omega(x_0, r_0) \cap C, \mathcal{B}_\Omega(x_0, r_0) \cap C_0) < \varepsilon_0. $$

The local Hausdorff topology is the topology generated by $U(\cdot, \cdot, \cdot, \cdot)$ on the set of closed subsets of Ω.

2.4. **Projective Simplices.** For $0 \leq k \leq d$, consider the following subsets of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$:

$$ S_k := \left\{ [x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \ldots, x_k + 1 > 0 \right\}. $$

A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a $k-$dimensional *simplex* if there exists $g \in \mathrm{PGL}_d(\mathbb{R})$ such that $S = gS_k$. In this case, the k points

$$ g[1 : 0 : \cdots : 0], g[0 : 1 : 0 : \cdots : 0], \ldots, g[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \partial S $$

are the vertices of S. We now discuss some basic properties of projective simplices (see [Nus88, Proposition 1.7], [dlH93], or [IZ23, Section 5]). Choosing suitable projective coordinates, we write a $(d-1)$ dimensional projective simplex as

$$ S = \left\{ [x_1 : \cdots : x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \ldots, x_d > 0 \right\}. $$

The Hilbert metric on S can be explicitly computed as:

$$ \mathrm{d}_S \Big([x_1 : \cdots : x_d], [y_1 : \cdots : y_d] \Big) = \max_{1 \leq i,j \leq d} \frac{1}{2} \left| \log \frac{x_i y_j}{y_i x_j} \right|. $$

Let $G \leq \mathrm{GL}_d(\mathbb{R})$ denote the group generated by the group of diagonal matrices with positive entries and the group of permutation matrices. Then $\mathrm{Aut}(S) = \{[g] \in \mathrm{PGL}_d(\mathbb{R}) : g \in G\}$.

Proposition 2.12. *If* $S \subset \mathbb{P}(\mathbb{R}^d)$ *is a simplex, then* (S, H_S) *is quasi-isometric to real Euclidean space of dimension* $\dim S$.

We will frequently use the following observation about the faces of properly embedded simplices.

Observation 2.13. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain and* $S \subset \Omega$ *is a properly embedded simplex. If* $x \in \partial S$, *then*

(1) $F_S(x)$ *is properly embedded in* $F_\Omega(x)$.
(2) $F_S(x) = \overline{S} \cap F_\Omega(x)$.

Proposition 2.14. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain. The set of all properly embedded simplices in* Ω *of dimension at least two is a closed set in the local Hausdorff topology.*

2.5. **Linear Projection on Simplices.** In this section, we construct certain linear projection maps associated to a properly embedded simplex in a properly convex domain. This notion was introduced in [IZ23] and all results in this section appear in [IZ23].

Definition 2.15. Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with $\dim S = (q-1) \geq 1$. A set of co-dimension one linear subspaces $\mathcal{H} := \{H_1, \dots, H_q\}$ is *S-supporting* when:

(1) Each $\mathbb{P}(H_j)$ is a supporting hyperplane of Ω,
(2) If $F_1, \dots, F_q \subset \partial S$ are the boundary faces of maximal dimension, then (up to relabelling) $F_j \subset \mathbb{P}(H_j)$ for all $1 \leq j \leq q$.

Proposition 2.16 ([IZ23])**.** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Then*

$$\operatorname{Span} S \oplus (\cap_{H \in \mathcal{H}} H) = \mathbb{R}^d \quad \textit{and} \quad \Omega \cap \mathbb{P}\left(\cap_{H \in \mathcal{H}} H\right) = \emptyset.$$

Using Proposition 2.16, we define the following linear projection.

Definition 2.17. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Define $L_{S,\mathcal{H}} \in \operatorname{End}(\mathbb{R}^d)$ to be the linear projection

$$\operatorname{Span} S \oplus (\cap_{H \in \mathcal{H}} H) \longrightarrow \operatorname{Span} S$$

We call $L_{S,\mathcal{H}}$ the *linear projection of Ω onto S relative to $\mathcal{H}$.*

We now derive some basic properties of these projection maps. We use the notation

$$F_\Omega(X) = \cup_{x \in X} F_\Omega(x)$$

where $\Omega \subset \mathbb{P}(\mathrm{\mathbb{R}^d})$ and $X \subset \overline{\Omega}$.

Proposition 2.18 ([IZ23])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Then*

(1) $L_{S,\mathcal{H}}(\Omega) = S$.
(2) If $x \in \partial\Omega \cap \mathbb{P}\left(\cap_{H \in \mathcal{H}} H\right)$ and $y \in \partial S$, then $[x, y] \subset \partial\Omega$.
(3) $\mathbb{P}\left(\cap_{H \in \mathcal{H}} H\right) \cap F_\Omega(\partial S) = \emptyset$.

For a general properly embedded simplex, there could be many different sets of supporting hyperplanes, but the next result shows that the corresponding linear projections form a compact set.

Definition 2.19. Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Define

$$\mathcal{L}_S := \{L_{S,\mathcal{H}} : \ \mathcal{H} \text{ is a set of } S\text{-supporting hyperplanes}\} \subset \operatorname{End}(\mathbb{R}^d).$$

Proposition 2.20 ([IZ23])**.** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Then $\mathcal{L}_S$ is a compact subset of* $\operatorname{End}(\mathbb{R}^d)$.

2.6. **Convex Co-compact Groups and Flat Torus Theorem.**

Definition 2.21. A discrete subgroup $\Gamma \leq \mathrm{PGL}_d(\mathbb{R})$ is called convex co-compact if:

(1) there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathrm{\mathbb{R}^d})$ such that $\Gamma \leq \operatorname{Aut}(\Omega)$, and
(2) the set $\mathcal{C}_\Omega(\Gamma) \subset \Omega$ is non-empty and Γ acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$, where $\mathcal{C}_\Omega(\Gamma)$ is the convex hull in Ω of the full orbital limit set $\mathcal{L}^{\mathrm{orb}}_\Omega(\Gamma) := \bigcup_{x \in \Omega} \left(\overline{\Gamma x} \setminus \Gamma x\right)$.

Remark 2.22. *If Γ acts co-compactly on a properly convex domain Ω, then $\mathcal{C}_\Omega(\Gamma) = \Omega$ and Γ is a convex co-compact group.*

If Γ is convex co-compact, the boundary of $\mathcal{C}_\Omega(\Gamma)$ splits into the *ideal boundary* $\partial_{\mathrm{i}}\,\mathcal{C}_\Omega(\Gamma) := \partial\Omega \cap \overline{\mathcal{C}_\Omega(\Gamma)}$ and the *non-ideal boundary* $\partial_{\mathrm{n}}\,\mathcal{C}_\Omega(\Gamma) := \Omega \cap \overline{\mathcal{C}_\Omega(\Gamma)}$. We recall some results from [DGK17] regarding properties of convex co-compact groups.

Theorem 2.23 ([DGK17])**.** *Suppose $\Gamma \leq \operatorname{Aut}(\Omega)$ is a convex co-compact group. Then:*

(1) $\mathcal{C}_\Omega(\Gamma)$ is the minimal non-empty Γ-invariant closed convex subset of Ω,

(2) $\mathcal{L}^{\mathrm{orb}}_{\Omega}(\Gamma) = \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$,
(3) *if* $x \in \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$, *then* $F_{\mathcal{C}_{\Omega}(\Gamma)}(x) = F_{\Omega}(x)$.

In joint work with A. Zimmer, we prove a key technical result concerning the Abelian subgroups of a convex co-compact group [IZ21]. It is a convex projective analog of the well-known Flat Torus theorem in CAT(0) geometry [BH99].

Theorem 2.24 (Convex Projective Flat Torus Theorem, [IZ21, Theorem 1.6])**.** *Suppose that* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a convex co-compact group. If* $A \leq \Gamma$ *is a maximal Abelian subgroup of* Γ*, then there exists a properly embedded simplex* $S \subset \mathcal{C}_{\Omega}(\Gamma)$ *such that*

(1) S *is* A*-invariant,*
(2) A *acts co-compactly on* S*, and*
(3) A *fixes each vertex of* S*.*

Moreover, A *has a finite index subgroup isomorphic to* $\mathbb{Z}^{\dim(S)}$*.*

3. Properly Convex Domains with Strongly Isolated Simplices

In this chapter, we will introduce a special class of properly convex domains called "properly convex domains with strongly isolated simplices". This definition is motivated by Hruska-Kleiner's work on CAT(0) spaces with isolated flats [HK05]. We will work with convex co-compact groups; recall the definition from Section 2.6.

If Ω' is a properly convex domain and $S \subset \Omega'$ is a properly embedded simplex of dimension at least two, then S is called maximal provided S is not properly contained in any other properly embedded simplex in Ω'. If $X \subset \Omega$, let $\mathrm{diam}_{\Omega}(X) := \sup_{x_1, x_2 \in X} \mathrm{d}_{\Omega}(x_1, x_2)$.

Definition 3.1 ([IZ23, Definition 1.15])**.** Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $\mathcal{S}_{\Gamma}$ is the collection of all maximal properly embedded simplices in $\mathcal{C}_{\Omega}(\Gamma)$ of dimension at least two.

(1) We will say that $\mathcal{S} \subset \mathcal{S}_{\Gamma}$ is *strongly isolated* provided: for any $r \geq 0$, there exists $D(r) \geq 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then
$$\mathrm{diam}_{\Omega}\left(\mathcal{N}_{\Omega}(S_1; r) \cap \mathcal{N}_{\Omega}(S_2; r)\right) \leq D(r).$$
(2) We will say that $(\mathcal{C}_{\Omega}(\Gamma), \mathrm{d}_{\Omega})$ has *strongly isolated simplices* if $\mathcal{S}_{\Gamma}$ is strongly isolated.

Observation 3.2. *If* $\mathcal{S} \subset \mathcal{S}_{\Gamma}$ *is strongly isolated, then* $\mathcal{S}$ *is closed and discrete in the local Hausdorff topology induced by* d_{Ω}*.*

Proof. See the proof of Proposition 3.3 part (1). □

Now we need to understand the geometric consequences of the property - "strongly isolated simplices".

3.1. Geometric Properties: Proof of Theorem 1.5.

In this section, we will prove the Theorem 1.5. It establishes some key geometric properties of $(\mathcal{C}_{\Omega}(\Gamma), \mathrm{d}_{\Omega})$ with strongly isolated simplices where $\Gamma \leq \mathrm{Aut}(\Omega)$ is convex co-compact. We will use this theorem in the next chapter for proving Theorem 1.4. We restate Theorem 1.5 before beginning the proof.

Theorem 1.5 ([IZ23, Theorem 1.8]) *Suppose* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a convex co-compact group and* $(\mathcal{C}_{\Omega}(\Gamma), \mathrm{d}_{\Omega})$ *has strongly isolated simplices. Then*

(1) Γ *has finitely many orbits in* $\mathcal{S}_{\Gamma}$*.*
(2) If $S \in \mathcal{S}_{\Gamma}$*, then* $\mathrm{Stab}_{\Gamma}(S)$ *acts co-compactly on* S *and contains a finite index subgroup isomorphic to* $\mathbb{Z}^k$ *where* $k = \dim S$*.*
(3) If $A \leq \Gamma$ *is an infinite Abelian subgroup of rank at least two, then there exists a unique* $S \in \mathcal{S}_{\Gamma}$ *with* $A \leq \mathrm{Stab}_{\Gamma}(S)$*.*
(4) If $S \in \mathcal{S}_{\Gamma}$ *and* $x \in \partial S$*, then* $F_{\Omega}(x) = F_{\mathcal{C}_{\Omega}(\Gamma)}(x) = F_S(x)$*.*
(5) If $S_1, S_2 \in \mathcal{S}_{\Gamma}$ *are distinct, then* $\#(S_1 \cap S_2) \leq 1$ *and* $\partial S_1 \cap \partial S_2 = \emptyset$*.*
(6) If $\ell \subset \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$ *is a non-trivial line segment, then there exists* $S \in \mathcal{S}_{\Gamma}$ *with* $\ell \subset \partial S$*.*
(7) If $x, y, z \in \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$ *form a half triangle in* $\mathcal{C}_{\Omega}(\Gamma)$ (i.e. $[x, y] \cup [y, z] \subset \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$ and $(x, z) \subset \mathcal{C}_{\Omega}(\Gamma)$)*, then there exists* $S \in \mathcal{S}_{\Gamma}$ *such that* $x, y, z \in \partial S$*.*
(8) If $x \in \partial_{\mathrm{i}} \mathcal{C}_{\Omega}(\Gamma)$ *is not a* C^1*-smooth point of* $\partial\Omega$*, then there exists* $S \in \mathcal{S}_{\Gamma}$ *with* $x \in \partial S$*.*

For the rest of this chapter, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a convex co-compact subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$. Let $\mathcal{S}_\Gamma$ denote the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two. For ease of notation, we set $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$. The proof of Theorem 1.5 is split into the next few sections in the following order:

- ☐ parts (1) – (3) of is proven in Section 3.2,
- ☐ part (5) is proven in Section 3.3,
- ☐ part (4) is proven in Section 3.4
- ☐ parts (6) and (7) are proven in Section 3.5, and
- ☐ part (8) is proven in Section 3.6.

3.2. **Maximal Simplices are Periodic.** In this section we show that if $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has strongly isolated simplices, then each simplex $S \in \mathcal{S}_\Gamma$ is periodic, i.e. $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S.

Proposition 3.3 ([IZ23, Proposition 8.1])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. Then the collection $\mathcal{S}_\Gamma$ satisfies the following properties:*

(1) $\mathcal{S}_\Gamma$ is closed and discrete in the local Hausdorff topology.
(2) $\mathcal{S}_\Gamma$ is a locally finite collection, that is, for any compact set $K \subset \Omega$ the set $\{S \in \mathcal{S}_\Gamma : S \cap K \neq \emptyset\}$ is finite.
(3) Γ has finitely many orbits in $\mathcal{S}$.
(4) If $S \in \mathcal{S}_\Gamma$, then $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.
(5) If $A \leq \Gamma$ is an infinite Abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_\Gamma$ with $A \leq \mathrm{Stab}_\Gamma(S)$.

We spend the rest of this section proving this proposition. The proofs are almost analogous to results in the CAT(0) setting, see Wise [Wis96, Proposition 4.0.4], Hruksa [Hru05, Theorem 3.7], or Hruska-Kleiner [HK05, Section 3.1].

(1) Suppose S_n is a sequence in $\mathcal{S}_\Gamma$ that converges to S in the local Hausdorff topology. By Proposition 2.14, S is a properly embedded simplex in Ω of dimension at least two. It is enough to show that $S_n = S$ for n large enough.

Fix $\varepsilon > 0$. Since $\mathcal{S}_\Gamma$ is strongly isolated, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Gamma$ are distinct, then

$$\mathrm{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon)\right) \leq D(\varepsilon).$$

Let $r_\varepsilon := D(\varepsilon) + 1$ and fix $x \in S$. Since $S_n \to S$, there exists $N_0 \in \mathbb{N}$ for all $n \geq N_0$,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(S_n \cap \mathcal{B}_\Omega(x, r_\varepsilon), S \cap \mathcal{B}_\Omega(x, r_\varepsilon)) < \varepsilon.$$

Observe that there exists $x_1, x_2 \in S$ such that $(x_1, x_2) \subset S \cap \mathcal{B}_\Omega(x, r_\varepsilon)$ and $\mathrm{d}_\Omega(x_1, x_2) = r_\varepsilon$. Thus, for any $m \neq n \geq N_0$,

$$(x_1, x_2) \subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_m; \varepsilon).$$

Thus

$$\mathrm{diam}_\Omega \left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_m; \varepsilon)\right) \geq r_\varepsilon > D(\varepsilon)$$

implying that $S_m = S_n$ for all $m, n \geq N_0$. Thus $S_n = S$ for all n large enough.

(2) Follows from part (1).

(3) Follows from part (2).

(4) Fix $S \in \mathcal{S}_\Gamma$ and a compact set $K \subset \Omega$. Let

$$X := \{g \in \Gamma : S \cap gK \neq \emptyset\}.$$

Then $S = \cup_{g \in X} S \cap gK$. Since $(g^{-1}S) \cap K \neq \emptyset$ when $g \in X$, Part (2) implies that the set

$$\{g^{-1}S : g \in X\}$$

is finite. Since $g^{-1}S = h^{-1}S$ if and only if $gh^{-1} \in \mathrm{Stab}_\Gamma(S)$ if and only if $\mathrm{Stab}_\Gamma(S)g = \mathrm{Stab}_\Gamma(S)h$, there exists $g_1, \ldots, g_m \in X$ such that

$$\bigcup_{g \in X} \mathrm{Stab}_\Gamma(S)g = \bigcup_{j=1}^{m} \mathrm{Stab}_\Gamma(S)g_j.$$

Then the set $\widehat{K} := \cup_{j=1}^m S \cap g_j K$ is compact and

$$\mathrm{Stab}_\Gamma(S) \cdot \widehat{K} = \cup_{g \in X} S \cap gK = S.$$

So $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S.

It is now easy to show that $\mathrm{Stab}_\Gamma(S)$ contains a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

(5) This is a straightforward application of the convex projective Flat Torus Theorem. Suppose $\widehat{A} \geq A$ is a maximal abelian subgroup containing A. By Theorem 2.24, there exists $\widehat{S} \in \mathcal{S}_\Gamma$ such that $\widehat{A}$ acts co-compactly on $\widehat{S}$. Thus $A \leq \mathrm{Stab}_\Gamma(\widehat{S})$. If A preserves another simplex, then it violates the strong isolation property because A is infinite. Thus $\widehat{S}$ is the unique properly embedded simplex preserved by A.

3.3. **Intersections of Simplices.** This result follows easily from the strong isolation property.

Proposition 3.4 ([IZ23, Section 12])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $S_1, S_2 \in \mathcal{S}_\Gamma$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.*

Suppose $x \neq y \in S_1 \cap S_2$. Let $(x_1, y_1) \subset S_1$ be the maximal projective line segment in S_1 containing $[x, y]$. By convexity, $(x_1, y_1) \subset S_1 \cap S_2$. However $\mathrm{diam}_\Omega((x_1, y_1)) = \infty$ which implies $S_1 = S_2$ since $\mathcal{S}_\Gamma$ is a strongly isolated. This is a contradiction.

For the second part, suppose $y \in \partial S_1 \cap \partial S_2$. Let $p_1 \in S_1$ and $p_2 \in S_2$. By Proposition 2.11

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}([p_1, y), [p_2, y)) \leq R := \mathrm{d}_\Omega(p_1, p_2).$$

Then,

$$[p_1, y) \subset S_1 \cap \mathcal{N}_\Omega(S_2; R).$$

As $\mathcal{S}_\Gamma$ is strongly isolated, if S_1 and S_2 are distinct, then there exists $D(R) \geq 0$ such that

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S_1; R) \cap \mathcal{N}_\Omega(S_2; R)) \leq D(R) < \infty.$$

But $\mathrm{diam}_\Omega(p_1, y) = \infty$. Thus $S_1 = S_2$, a contradiction.

3.4. **Boundary Faces of Simplices.** In this subsection, we will prove the following result about boundary faces of simplices.

Proposition 3.5 ([IZ23])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $S \in \mathcal{S}_\Gamma$ and $x \in \partial S$, then $F_\Omega(x) = F_{\mathcal{C}}(x) = F_S(x)$.*

Fix $S \in \mathcal{S}_\Gamma$ and $x \in \partial S$. By Theorem 2.23 part (3), $F_\Omega(x) = F_{\mathcal{C}}(x)$. So it is enough to show that $F_S(x) = F_{\mathcal{C}}(x)$. Also observe that if $\dim(F_{\mathcal{C}}(x)) = 0$, then $F_{\mathcal{C}}(x) = F_S(x) = \{x\}$ and the result is immediate. So, without loss of generality, we can assume that $\dim(F_{\mathcal{C}}(x)) \geq 1$.

In order to prove this theorem, it is enough to show that $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$. Indeed,

$$F_{\mathcal{C}}(x) = \text{rel-int}(\mathrm{ConvHull}_\Omega(\partial F_{\mathcal{C}}(x)))$$

and

$$F_S(x) = \text{rel-int}(\mathrm{ConvHull}_\Omega(\partial F_S(x))).$$

So, $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$ implies that $F_{\mathcal{C}}(x) \subset F_S(x)$. On the other hand, $F_S(x) \subset F_{\mathcal{C}}(x)$ since $S \subset \mathcal{C}$. This shows that proving $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$ is enough to prove the theorem.

In order to prove $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$, we will require the following general result about convex co-compact groups. Note that the following lemma does not require the assumption that $\mathcal{S}_\Gamma$ is strongly isolated.

Lemma 3.6. *Suppose $w \in \partial_{\mathrm{i}} \mathcal{C}$ with $\dim(F_{\mathcal{C}}(w)) \geq 1$ and $w' \in \partial_{\mathrm{i}} F_{\mathcal{C}}(w)$. For any $r, \varepsilon > 0$ and $p \in \mathcal{C}$, there exists $N \geq 0$ such that: if $y \in (w, w')$ with $\mathrm{d}_{F_\Omega(w)}(w, y) > N$, then there exists $p_y \in [p, y)$ such that whenever $q \in [p_y, y)$,*

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(q, r) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

for some $S_q \in \mathcal{S}_\Gamma$.

Proof of Lemma 3.6. Suppose this fails. Then there exist $r, \varepsilon > 0$ and $p \in \mathcal{C}$ such that: if $n \geq 1$, there exist $y_n \in (w, w')$ with $\mathrm{d}_{F_\Omega(w)}(w, y_n) \geq n$ and $q_{n,m} \in [p, y_n)$ with $\lim_{m\to\infty} q_{n,m} = y_n$ such that

$$\mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right) \cap \mathcal{B}_\Omega(q_{n,m}, r) \not\subset \mathcal{N}_\Omega(S; \varepsilon) \tag{1}$$

for any properly embedded simplex $S \in \mathcal{S}_\Gamma$. By Proposition 2.2,

$$\liminf_{m\to\infty} \mathrm{d}_\Omega(q_{n,m}, [p, w] \cup [p, w']) \geq \mathrm{d}_{F_\Omega(w)}(y_n, w) \geq n.$$

Then for each n, we choose m_n large enough such that

$$\mathrm{d}_\Omega(q_{n,m_n}, [p, w] \cup [p, w']) \geq n. \tag{2}$$

Set $q'_n := q_{n,m_n}$.

Since Γ acts co-compactly on $\mathcal{C}$, we can pass to a subsequence and choose $\gamma_n \in \Gamma$ such that $\gamma_n q'_n \to q'_\infty \in \mathcal{C}$. Up to passing to another subsequence, we can assume that

$$\gamma_n w', \gamma_n w, \gamma_n p, \gamma_n y_n \to w'_0, w_0, p_0, y_\infty \in \overline{\mathcal{C}}.$$

By construction and by Equation (2),

$$[p_0, w'_0] \cup [w'_0, w_0] \cup [w_0, p_0] \subset \partial_{\mathrm{i}} \mathcal{C}.$$

But $(p_0, y_\infty) \subset \mathcal{C}$ since $q'_\infty \in (p_0, y_\infty) \cap \mathcal{C}$. Thus,

$$S := \text{rel-int}\left(\mathrm{ConvHull}\{w_0, w'_0, p_0\}\right)$$

is a properly embedded two dimensional simplex in $\mathcal{C}$. Note that

$$S_n := \text{rel-int}\left(\mathrm{ConvHull}\{\gamma_n w, \gamma_n w', \gamma_n p\}\right)$$

converges to S in the local Hausdorff topology. Thus, for n large enough,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}\left(\mathcal{B}_\Omega(q'_\infty, r) \cap S, \mathcal{B}_\Omega(q'_\infty, r) \cap S_n\right) < \varepsilon/2.$$

Since $\gamma_n q'_n \to q'_\infty$,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(\mathcal{B}_\Omega(q'_\infty, r), \mathcal{B}_\Omega(\gamma_n q'_n, r)) < \varepsilon/2$$

when n is large enough. Thus, for large enough n,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}\left(\mathcal{B}_\Omega(q'_\infty, r) \cap S, \mathcal{B}_\Omega(\gamma_n q'_n, r) \cap S_n\right) < \varepsilon.$$

Since $q'_\infty \in \mathcal{S}$, this implies that

$$\mathcal{B}_\Omega(q'_n, r) \cap \gamma_n^{-1} S_n \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \varepsilon).$$

Now observe that

$$\mathcal{B}_\Omega(q'_n, r) \cap \gamma_n^{-1} S_n = \mathcal{B}_\Omega(q'_n, r) \cap \mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right).$$

Thus, for n large enough,

$$\mathcal{B}_\Omega(q'_n, r) \cap \mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \varepsilon).$$

Let $\widehat{S}_n \in \mathcal{S}_\Gamma$ be a maximal properly embedded simplex such that $\gamma_n^{-1} S \subset \widehat{S}_n$. Thus,

$$\mathcal{B}_\Omega(q'_n, r) \cap \mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right) \subset \mathcal{N}_\Omega(\widehat{S}_n; \varepsilon)$$

This contradicts Equation (1) and concludes the proof of this lemma. □

Now we finish the proof of Proposition 3.5. We want to prove $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$. Recall that $\dim(F_{\mathcal{C}}(x)) \geq 1$ which implies that $\partial F_{\mathcal{C}}(x) \neq \emptyset$. Let $x' \in \partial F_{\mathcal{C}}(x)$. We will show that $x' \in \partial F_S(x)$.

Fix $\varepsilon > 0$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Gamma$ are distinct, then

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon)\right) \leq D(\varepsilon). \tag{3}$$

Fix $R_\varepsilon := D(\varepsilon) + 1$. Fix $p \in S$. Applying the above Lemma 3.6 with $R_\varepsilon, \varepsilon$ and p, we get $N \geq 0$ which satisfies the conclusions of the lemma. Choose $y \in (x, x')$ such that $\mathrm{d}_{F_\Omega(x)}(x, y) > N$. Then there exists $p_y \in [p, y)$ such that whenever $q \in [p_y, y)$, there exists $S_q \in \mathcal{S}_\Gamma$ such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q, r) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

Pick a sequence $q_n \in [p_y, y)$ with $q_n \to y$ such that $\mathrm{d}_\Omega(q_n, q_{n+1}) = R_\varepsilon$ for all $n \geq 1$. There exist properly embedded simplices S_n such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q_n, R_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon)$$

for all $n \geq 1$. Then, for $n \geq 1$,

$$\begin{aligned}(q_n, q_{n+1}) &\subset \mathcal{B}_\Omega(q_n, R_\varepsilon) \cap \mathcal{B}_\Omega(q_{n+1}, R_\varepsilon) \cap \mathbb{P}\left(\mathrm{Span}\{x, x', p\}\right) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon).\end{aligned}$$

Thus

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon)\right) \geq \mathrm{d}_\Omega(q_n, q_{n+1}) = R_\varepsilon > D_\varepsilon.$$

Then Equation (3) implies that $S_n = S_{n+1} = S'$ for all $n \geq 1$. Thus,

$$[p_y, y) \subset \mathcal{N}_\Omega(S'; \varepsilon). \tag{4}$$

Let $p_x \in S$ be such that $\mathrm{d}_\Omega(p_x, p_y) = \mathrm{d}_\Omega(p_y, S)$. Then Proposition 2.11 implies that

$${\mathrm{d}_\Omega}^{\mathrm{Hauss}}([p_x, x), [p_y, y)) \leq R_0 := \max\{\mathrm{d}_\Omega(p_x, p_y), \mathrm{d}_{F_\Omega(x)}(x, y)\}.$$

Then equation (4) implies that

$$[p_x, x) \subset S \cap \mathcal{N}_\Omega(S'; R_0 + \varepsilon),$$

that is, $\mathrm{diam}_\Omega(S \cap \mathcal{N}_\Omega(S'; R_0 + \varepsilon)) = \infty$. This violates equation (3) unless $S = S'$. Thus, by equation (4),

$$[p_y, y) \subset \mathcal{N}_\Omega(S; \varepsilon).$$

Then, by Corollary 2.3, there exists $a_y \in \partial S$ such that $y \in F_\Omega(a_y)$ and

$$\mathrm{d}_{F_\Omega(y)}(y, a_y) = \mathrm{d}_{F_\Omega(x)}(y, a_y) \leq \varepsilon.$$

Note that this is true for any $y \in (x, x')$ with $\mathrm{d}_{F_\Omega(x)}(x, y) > N$ (here N depends on ε, see Lemma 3.6). Thus, for $m \geq 1$, we can find a sequence $y_m \in (x, x')$ and $a_m \in \partial S$ with $y_m \to x'$ and $\mathrm{d}_{F_\Omega(x)}(y_m, a_m) < 1/m$. Then, by Corollary 2.4, $\lim_{m\to\infty} a_m = x'$. Thus, $x' \in \partial S \cap \partial F_\Omega(x) = \partial F_S(x)$. This finishes the proof.

3.5. **Lines and Half Triangles in the Boundary.**

Proposition 3.7 ([IZ23])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $\ell \subset \partial_{\mathrm{i}} \mathcal{C}$ is a non-trivial line segment, then there exists $S \in \mathcal{S}_\Gamma$ with $\ell \subset \partial S$.*

Proof. We can assume that ℓ is an open line segment with x' as one of its endpoints. Fix some $x \in \ell$, that is $\ell \subset F_{\mathcal{C}}(x)$. Then $x' \in \partial_{\mathrm{i}} F_{\mathcal{C}}(x)$. Now fix $\varepsilon > 0$ and $p \in \mathcal{C}$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Gamma$ are distinct, then

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon)\right) \leq D(\varepsilon). \tag{5}$$

Fix $r_\varepsilon = D(\varepsilon) + 1$. Applying Lemma 3.6 with $r_\varepsilon, \varepsilon$, and p, let $N \geq 0$ be such that it satisfies the conclusions of the lemma. Choose $y \in \ell$ with $\mathrm{d}_{F_\Omega(x)}(x, y) > N$. Then there exists $p_y \in [p, y)$ such that: if $q \in [p_y, y)$, there exists $S_q \in \mathcal{S}_\Gamma$ such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q, r_\varepsilon) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

Pick a sequence $q_n \in [p_y, y)$ with $q_n \to y$ such that $\mathrm{d}_\Omega(q_n, q_{n+1}) = r_\varepsilon$. Let $S_n \in \mathcal{S}_\Gamma$ be such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q_n, r_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon).$$

Then

$$\begin{aligned}(q_n, q_{n+1}) &\subset \mathcal{B}_\Omega(q_n, r_\varepsilon) \cap \mathcal{B}_\Omega(q_{n+1}, r_\varepsilon) \cap \mathbb{P}\left(\mathrm{Span}\{x, x', p\}\right) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon).\end{aligned}$$

Thus,

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon)\right) \geq r_\varepsilon = D(\varepsilon) + 1 > D(\varepsilon)$$

Then equation (5) implies that $S_n = S_{n+1} = S$ for all $n \geq 1$. Then $\{q_n : n \geq 1\} \subset \mathcal{N}_\Omega(S;\varepsilon)$. Then Corollary 2.3 implies that $y \in F_\Omega(c)$ for some $c \in \partial S$. As $c \in \partial S$, Proposition 3.5 implies that $F_\Omega(c) = F_S(c) \subset \partial S$. Since $y \in F_\Omega(x)$, this implies that

$$F_\Omega(x) = F_\Omega(y) = F_\Omega(c) \subset \partial S.$$

Finally, since $\ell \subset F_\Omega(x)$,

$$\ell \subset \partial S. \qquad \square$$

Proposition 3.8 ([IZ23])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $x, y, z \in \partial_{\mathrm{i}}\mathcal{C}$ form a half triangle in $\mathcal{C}$, then there exists $S \in \mathcal{S}_\Gamma$ such that $x, y, z \in \partial S$.*

Proof. By Proposition 3.7, there exist $S_1, S_2 \in \mathcal{S}_\Gamma$ such that $[x, y] \subset \partial S_1$ and $[y, z] \subset \partial S_2$. Thus $y \in \partial S_1 \cap \partial S_2$. Then Proposition 3.4 implies that $S_1 = S_2 = S$. Hence $x, y, z \in \partial S$. $\square$

3.6. **Corners in the Boundary.** A supporting hyperplane of Ω at $z \in \partial\Omega$ is a co-dimension one projective subspace $\mathbb{P}(H)$ such that $\mathbb{P}(H) \cap \Omega = \emptyset$ and $z \in \mathbb{P}(H) \cap \overline{\Omega}$. We will say that a point $z \in \partial_{\mathrm{i}}\mathcal{C}$ is not C^1-smooth if Ω does not have a unique supporting hyperplane at z. We will show that such a point is necessarily contained in the boundary of a properly embedded simplex.

Proposition 3.9 ([IZ23])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $z \in \partial_{\mathrm{i}}\mathcal{C}$ is not a C^1-smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_\Gamma$ with $z \in \partial S$.*

In order to prove this, we first establish the following lemma about general convex co-compact subgroups. Note that this lemma does not require that $\mathcal{S}_\Gamma$ is strongly isolated.

Lemma 3.10 ([IZ23])**.** *Suppose that $z \in \partial_{\mathrm{i}}\mathcal{C}$ is not a C^1-smooth point of $\partial\Omega$ and $q \in \mathcal{C}$. For any $r > 0$ and $\epsilon > 0$ there exists $N > 0$ such that: if $p \in [q, z)$ with $\mathrm{d}_\Omega(q, p) > N$, then there exists a properly embedded simplex $S_p \subset \mathcal{C}$ of dimension at least two such that*

$$\mathcal{B}_\Omega(p; r) \cap (z, q] \subset \mathcal{N}_\Omega(S_p; \epsilon). \tag{6}$$

Proof. Fix $r > 0$ and $\epsilon > 0$. Suppose for a contradiction that such a N does not exist. Then we can find $p_n \in (z, q]$ such that $\lim_{n\to\infty} p_n = z$ and

$$\mathcal{B}_\Omega(p, r) \cap (z; q] \not\subset \mathcal{N}_\Omega(S; \varepsilon)$$

for any properly embedded simplex S in $\mathcal{C}$ of dimension at least two.

We can find a 3-dimensional linear subspace V such that $(z, q] \subset \mathbb{P}(V)$ and $z \in \partial_{\mathrm{i}}\mathcal{C}$ is not a C^1-smooth boundary point of $\mathbb{P}(V) \cap \Omega$. By changing coordinates we can suppose that

$$\begin{aligned} \mathbb{P}(V) &= \{[w : x : y : 0 : \cdots : 0] : w, x, y \in \mathbb{R}\}, \\ \mathbb{P}(V) \cap \Omega &\subset \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R},\ y > |x|\}, \\ z &= [1 : 0 : 0 : \cdots : 0], \text{ and} \\ q &= [1 : 0 : 1 : 0 \cdots : 0]. \end{aligned}$$

We may also assume that $\mathbb{P}(V) \cap \Omega$ is bounded in the affine chart

$$\{[1 : x : y : 0 : \cdots : 0] : x, y \in \mathbb{R}\}.$$

Then

$$p_n = [1 : 0 : y_n : 0 : \cdots : 0]$$

where $0 < y_n < 1$ and y_n converges to 0. Let

$$L_n := \{[1 : x : y_n : 0 : \cdots : 0] : x \in \mathbb{R}\} \cap \Omega.$$

By passing to a subsequence we can suppose that $(y_n)_{n\geq 1}$ is a decreasing sequence and

$$\lim_{n\to\infty} \mathrm{d}_\Omega(p_n, L_{n-1}) = \infty. \tag{7}$$

Then

$$\lim_{n\to\infty} \frac{y_{n-1}}{y_n} = \infty.$$

Let $a_n, b_n \in \partial\Omega$ be the endpoints of $L_n = (a_n, b_n)$. We claim that

$$\lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, a_{n-1}) \Big) = \infty = \lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, b_{n-1}) \Big). \tag{8}$$

Consider $g_n \in \mathrm{PGL}(V)$ defined by

$$g_n([w : x : y : 0 : \cdots : 0]) = \left[w : \frac{1}{y_n} x : \frac{1}{y_n} y : \cdots : 0 \right].$$

Since $(y_n)_{n\geq 1}$ is a decreasing sequence converging to zero, $D_n := g_n(\mathbb{P}(V) \cap \Omega)$ is an increasing sequence of properly convex domains in $\mathbb{P}(V)$ and

$$D := \cup_{n\geq 1} D_n \subset \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R},\ y > |x|\}$$

is also a properly convex domain. Notice that d_{D_n} converges to d_D uniformly on compact subsets of D. Also, by construction, there exist $t \leq -1$ and $1 \leq s$ such that

$$D = \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R},\ y > \max\{sx, tx\}\}.$$

Then $a_n = [1 : t_n^{-1} y_n : y_n : 0 : \ldots : 0]$ where $t_n \to t$.

Now pick $v_n \in (z, a_{n-1})$ such that

$$\mathrm{d}_\Omega \Big(p_n, (z, a_{n-1}) \Big) = \mathrm{d}_\Omega(p_n, v_n).$$

Since

$$\lim_{n\to\infty} g_n a_{n-1} = \lim_{n\to\infty} \left[1 : t_{n-1}^{-1} \frac{y_{n-1}}{y_n} : \frac{y_{n-1}}{y_n} : 0 : \ldots : 0 \right] = [0 : t^{-1} : 1 : 0 : \cdots : 0]$$

any limit point of $g_n v_n$ is in

$$\{[0 : t^{-1} : 1 : 0 : \cdots : 0]\} \cup \{[1 : rt^{-1} : r : 0 : \cdots : 0] : r \geq 0\} \subset \partial D.$$

Then

$$\lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, a_{n-1}) \Big) = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, v_n) = \lim_{n\to\infty} \mathrm{d}_{D_n} \Big(g_n p_n, g_n v_n \Big) = \infty$$

since $g_n p_n \to [1 : 0 : 1 : 0 : \cdots : 0] \in D$.

For the same reasons,

$$\lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, b_{n-1}) \Big) = \infty.$$

This establishes Equation (8).

Next we can pass to a subsequence and find $\gamma_n \in \Gamma$ such that $\gamma_n p_n \to p_\infty \in \mathcal{C}$. Passing to a further subsequence we can suppose that $\gamma_n a_{n-1} \to a_\infty$, $\gamma_n b_{n-1} \to b_\infty$, $\gamma_n z \to z_\infty$, and $\gamma_n q \to q_\infty$.

Equation (7) implies that $[a_\infty, b_\infty] \subset \partial\Omega$ and Equation (8) implies that

$$[z_\infty, a_\infty] \cup [z_\infty, b_\infty] \subset \partial\Omega.$$

Thus $a_\infty, b_\infty, z_\infty$ are the vertices of a properly embedded simplex $S \subset \Omega$ which contains p_∞. Further, for n sufficiently large we have

$$\mathcal{B}_\Omega(\gamma_n p_n, r) \cap \gamma_n(z, q] \subset \mathcal{N}_\Omega(S; \epsilon)$$

and so

$$\mathcal{B}_\Omega(p_n, r) \cap (z, q] \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \epsilon).$$

To obtain a contradiction we have to show that $\gamma_n^{-1} S \subset \mathcal{C}$ for every n or equivalently that $S \subset \mathcal{C}$. By construction, $q_\infty \in \partial_{\mathrm{i}} \mathcal{C} \cap (a_\infty, b_\infty)$. Then Theorem 2.23 implies that

$$(a_\infty, b_\infty) \subset F_\Omega(q_\infty) = F_{\mathcal{C}}(q_\infty) \subset \partial_{\mathrm{i}} \mathcal{C}.$$

Thus $[a_\infty, b_\infty] \subset \partial_{\mathrm{i}} \mathcal{C}$. Since $z_\infty \in \partial_{\mathrm{i}} \mathcal{C}$ and S has vertices $a_\infty, b_\infty, z_\infty$ we then see that $S \subset \mathcal{C}$. □

We now finish the proof of Proposition 3.9. The strategy is similar to the proof of Proposition 3.7. Fix $\varepsilon > 0$ and $q \in \mathcal{C}$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D_\varepsilon \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Gamma$ are distinct, then

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_1;\varepsilon) \cap \mathcal{N}_\Omega(S_2;\varepsilon)\right) \leq D(\varepsilon). \tag{9}$$

Fix $r_\varepsilon := D(\varepsilon) + 1$. Applying the above Lemma 3.6 with $r_\varepsilon, \varepsilon$ and q, we get $N \geq 0$ which satisfies the conclusions of the lemma. Pick a sequence $z_n \in [q, z)$ with $z_n \to z$, $\mathrm{d}_\Omega(z_n, z_{n+1}) = r_\varepsilon$, and $\mathrm{d}_\Omega(q, z_n) \geq N$ for $n \geq 1$. Then, for each $n \geq 1$, there exist $S_n \in \mathcal{S}_\Gamma$ such that:

$$[q, z) \cap \mathcal{B}_\Omega(z_n, r_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon).$$

Then, if $n \geq 1$,

$$\begin{aligned}(z_n, z_{n+1}) &\subset \mathcal{B}_\Omega(z_n, r_\varepsilon) \cap \mathcal{B}_\Omega(z_{n+1}, r_\varepsilon) \cap [q, z) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon).\end{aligned}$$

Thus

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_n;\varepsilon) \cap \mathcal{N}_\Omega(S_{n+1};\varepsilon)\right) \geq \mathrm{d}_\Omega(z_n, z_{n+1}) = r_\varepsilon > D_\varepsilon.$$

Then Equation (9) implies that $S_n = S_{n+1} = S$ for all $n \geq 1$. Thus $\{z_n : n \in \mathbb{N}\} \subset \mathcal{N}_\Omega(S; \varepsilon)$. Corollary 2.3 then implies that there exists $c \in \partial S$ such that $z \in F_\Omega(c)$. As $c \in \partial S$, Proposition 3.5 implies that $F_\Omega(c) = F_S(c) \subset \partial S$. Thus, $z \in \partial S$.

4. Relative Hyperbolicity, Convex Co-compactness, and Strongly Isolated Simplices

This section is devoted to the proof of Theorem 1.4 which we now restate.

Theorem 1.4.([IZ23, Theorem 1.7]) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \leq \operatorname{Aut}(\Omega)$ is a convex co-compact group, and $\mathcal{S}_\Gamma$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two. Then the following are equivalent:*

(1) $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has strongly isolated simplices,
(2) $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Gamma$,
(3) Γ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least two.

The most difficult part of the proof is (1) implies (2). This is done in Section 4.2. For this proof, we rely on Sisto's characterization of relative hyperbolicity (cf. Theorem A.8). A key ingredient of this proof is the notion of closest-point projection onto properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ (cf. Definition 2.6) and its comparison with linear projections on simplices (cf. Definition 2.17). Results proven in Section 4.1 play a key role in Section 4.2.

The proof of (3) implies (1) is also quite involved since we have to prove that $\mathcal{C}_\Omega(\Gamma)$ is relatively hyperbolic with respect to the collection of **all** simplices in $\mathcal{S}_\Gamma$. This is done in Section 4.5. The rest of the parts of the proof of Theorem 1.4 is also in this section.

4.1. Closest-point Projections on Simplices.

For the rest of this section fix a convex co-compact group $\Gamma \leq \operatorname{Aut}(\Omega)$. Set $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$ and $\mathcal{S} := \mathcal{S}_\Gamma$. We will assume that $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has strongly isolated simplices for rest of this section.

Suppose S is a properly embedded simplex in $\mathcal{C}$. Since S is a closed convex subset, we can follow Definition 2.6 and define the closest-point projection onto S. We will denote it by π_S. On the other hand, if $\mathcal{H}$ is a set of S-supporting hyperplanes, then we have a notion of linear projection onto S which we will denote by $L_{S,\mathcal{H}}$, see Definition 2.17.

We will establish a coarse equivalence between the two projections. But first we need a continuity lemma for linear projections.

Lemma 4.1 ([IZ23, Lemma 13.4]). *If $S \in \mathcal{S}_\Gamma$, then the map*

$$(L, x) \in \mathcal{L}_S \times \overline{\mathcal{C}} \to L(x) \in \overline{S}$$

is continuous.

Proof. We first show that $\mathbb{P}(\ker L) \cap \overline{\mathcal{C}} = \emptyset$ for all $L \in \mathcal{L}_S$. Suppose for a contradiction that $L \in \mathcal{L}_S$ and

$$x \in \mathbb{P}(\ker L) \cap \overline{\mathcal{C}}.$$

Proposition 2.16 implies that $x \in \partial_{\rm i}\,\mathcal{C}$. Then Proposition 2.18 implies that $[y, x] \subset \partial_{\rm i}\,\mathcal{C}$ for every $y \in \partial S$. Next fix $y_1, y_2 \in \partial S$ such that $(y_1, y_2) \subset S$. Then y_1, x, y_2 form a half triangle. By Theorem 1.5, $y_1, x, y_2 \in \partial S$ for some $S \in \mathcal{S}_\Gamma$. Since $x \in \partial S \subset \mathrm{Span}(S)$, $x \notin \ker L$, a contradiction.

Thus $\mathbb{P}(\ker L) \cap \overline{\mathcal{C}} = \emptyset$ for all $L \in \mathcal{L}_S$.

Now suppose that $\lim_{n\to\infty}(L_n, x_n) = (L, x)$ in $\mathcal{L}_S \times \overline{\mathcal{C}}$. Let $\widetilde{x}_n, \widetilde{x}$ denote lifts of x_n, x respectively such that $\lim_{n\to\infty} \widetilde{x}_n = \widetilde{x}$. Then

$$L(\widetilde{x}) = \lim_{n\to\infty} L_n(\widetilde{x}_n) \in \mathbb{R}^d.$$

Since $\mathbb{P}(\ker L) \cap \overline{\mathcal{C}} = \emptyset$, we have $L(\widetilde{x}) \neq 0$. So

$$L(x) = [L(\widetilde{x})] = \lim_{n\to\infty} [L_n(\widetilde{x}_n)] = \lim_{n\to\infty} L_n(x_n). \qquad \square$$

Now the proof of equivalence.

Proposition 4.2 ([IZ23, Proposition 13.7])**.** *There exists $\delta_1 \geq 0$ such that: if $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $x \in \mathcal{C}$, then*

$$\max_{p \in \pi_S(x)} \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), p) \leq \delta_1.$$

Proof. Since $\mathcal{S}$ has finitely many Γ orbits (see Proposition 3.3), it is enough to prove the result for some fixed $S \in \mathcal{S}$.

Suppose the proposition is false. Then, for every $n \geq 0$, there exist $x_n \in \mathcal{C}$, a set of S-supporting hyperplanes $\mathcal{H}_n$, and $p_n \in \pi_S(x_n)$ such that

$$\mathrm{d}_\Omega(p_n, L_{S,\mathcal{H}_n}(x_n)) \geq n.$$

Let m_n be the midpoint of the projective line segment $[p_n, L_{S,\mathcal{H}_n}(x_n)]$ in the Hilbert distance. Since $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S (see Proposition 3.3), translating by elements of $\mathrm{Stab}_\Gamma(S)$ and passing to a subsequence, we can assume that $m := \lim_{n\to\infty} m_n$ exists in S. Passing to a further subsequence and using Proposition 2.20, we can assume that there exists $x, p, x' \in \partial_{\rm i}\,\mathcal{C}$ and $L_{S,\mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n\to\infty} x_n$, $p := \lim_{n\to\infty} p_n$, $x' := \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n)$, and $L_{S,\mathcal{H}} := \lim_{n\to\infty} L_{S,\mathcal{H}_n}$. By Lemma 4.1,

$$L_{S,\mathcal{H}}(x) = \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n) = x'.$$

We first show that $[x', x] \subset \partial_{\rm i}\,\mathcal{C}$. Observe that $L_{S,\mathcal{H}}(v) = x'$ for all $v \in [x', x]$ since $L_{S,\mathcal{H}}$ is linear and $L_{S,\mathcal{H}}(x') = x' = L_{S,\mathcal{H}}(x)$. But $L_{S,\mathcal{H}}(\Omega) = S$, implying $[x', x] \cap \Omega = \emptyset$. Hence,

$$[x', x] \subset \partial_{\rm i}\,\mathcal{C}.$$

Next we show that $[p, x] \subset \partial_{\rm i}\,\mathcal{C}$. Suppose not, then $(p, x) \subset \mathcal{C}$. Choose any $v \in (p, x) \cap \mathcal{C}$ and a sequence $v_n \in [p_n, x_n]$ such that $v = \lim_{n\to\infty} v_n$. Since $p \in \partial_{\rm i}\,\mathcal{C}$ and $v \in \mathcal{C}$,

$$\lim_{n\to\infty} \mathrm{d}_\Omega(v_n, p_n) = \infty.$$

Fix any $v_S \in S$. Then, choosing n large enough so that $\mathrm{d}_\Omega(v_n, p_n) \geq 2 + \mathrm{d}_\Omega(v, v_S)$ and $\mathrm{d}_\Omega(v, v_n) \leq 1$,

$$\begin{aligned} \mathrm{d}_\Omega(x_n, v_S) &\leq \mathrm{d}_\Omega(x_n, v_n) + \mathrm{d}_\Omega(v_n, v) + \mathrm{d}_\Omega(v, v_S) \\ &= \mathrm{d}_\Omega(x_n, p_n) - \mathrm{d}_\Omega(p_n, v_n) + \mathrm{d}_\Omega(v_n, v) + \mathrm{d}_\Omega(v, v_S) \\ &\leq \mathrm{d}_\Omega(x_n, p_n) - 1, \end{aligned}$$

which is a contradiction since $p_n \in \pi_S(x_n)$. Hence, $[p, x] \subset \partial_{\rm i}\,\mathcal{C}$.

Thus, $[p, x] \cup [x, x'] \subset \partial_{\rm i}\,\mathcal{C}$ and by construction, $m \in (p, x) \subset \mathcal{C}$. Thus the three points p, x, x' form half triangle in $\mathcal{C}$. Then Theorem 1.5 part (7) implies that $p, x, x' \in \partial S$ for some $S \in \mathcal{S}_\Gamma$. Then $x' = L_{S,\mathcal{H}}(x) = x$ which implies $[p, x] = [p, x'] \subset \partial_{\rm i}\,\mathcal{C}$. This is a contradiction since $(p, x) \subset \mathcal{C}$ by construction. $\square$

The next result proves δ-slimness of some special triangles built using linear projections.

Proposition 4.3 ([IZ23, Proposition 13.9]). *There exists $\delta_2 \geq 0$ such that: if $x \in \mathcal{C}$, $S \in \mathcal{S}$, $z \in S$, and $\mathcal{H}$ is a set of S-supporting hyperplanes, then the geodesic triangle*

$$\big[x, z\big] \cup \big[z, L_{S,\mathcal{H}}(x)\big] \cup \big[L_{S,\mathcal{H}}(x), x\big]$$

is δ_2-thin.

Proof. Since $\mathcal{S}$ has finitely many Γ orbits (see Proposition 3.3), it is enough to prove the result for some fixed $S \in \mathcal{S}$. By Lemma **??**, it is enough to show that there exists $\delta_2 \geq 0$ such that

$$[L_{S,\mathcal{H}}(x), z] \subset \mathcal{N}_{\delta_2/2}([z, x] \cup [x, L_{S,\mathcal{H}}(x)])$$

for all $x \in \mathcal{C}$, $z \in S$, and $\mathcal{H}$ a set of S-supporting hyperplanes.

Suppose such a δ_2 does not exist. Then, for every $n \geq 0$, there exist $z_n \in S$, a set of S-supporting hyperplanes $\mathcal{H}_n$, $p_n := L_{S,\mathcal{H}_n}(x_n)$, and $u_n \in [z_n, p_n]$ such that

$$\mathrm{d}_\Omega(u_n, [z_n, x_n] \cup [x_n, p_n]) \geq n.$$

Since $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S, translating by elements of $\mathrm{Stab}_\Gamma(S)$ and passing to a subsequence, we can assume that $u := \lim_{n\to\infty} u_n$ exists and $u \in S$. Passing to a further subsequence and using Proposition 2.20, we can assume there exist $x, z, p \in \overline{\mathcal{C}}$ and $L_{S,\mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n\to\infty} x_n$, $z := \lim_{n\to\infty} z_n$, $p := \lim_{n\to\infty} p_n$, and $L_{S,\mathcal{H}} := \lim_{n\to\infty} L_{S,\mathcal{H}_n}$. Since

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{d}_\Omega(u, [x_n, z_n] \cup [x_n, p_n]) \\
&\qquad \geq \lim_{n\to\infty} \Big(\mathrm{d}_\Omega(u_n, [x_n, z_n] \cup [x_n, p_n]) - \mathrm{d}_\Omega(u, u_n) \Big) = \infty,
\end{aligned}$$

we have

$$[x, z] \cup [x, p] \subset \partial_{\mathrm{i}}\, \mathcal{C}.$$

By construction, $u \in (p, z) \subset \mathcal{C}$. Thus, p, x, z form a half triangle in $\mathcal{C}$.

By Theorem 1.5 part (7), $p, x, z \in \partial S$ for some $S \in \mathcal{S}_\Gamma$. Lemma 4.1 then implies that

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n) = L_{S,\mathcal{H}}(x) = x.$$

Thus $[p, z] = [p, x] \subset \partial_{\mathrm{i}}\, \mathcal{C}$ which is a contradiction since $(p, z) \subset \mathcal{C}$ by construction. □

Let δ_1 and δ_2 be the constants as in Propositions 4.2 and 4.3.

Proposition 4.4 ([IZ23, Proposition 13.10]). *Set $\delta_3 := \delta_1 + 3\delta_2$. If $x \in \mathcal{C}$, $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $z \in S$, then $\mathrm{d}_\Omega\big(L_{S,\mathcal{H}}(x), [x, z]\big) \leq \delta_3$.*

Proof. By Proposition 4.3, the geodesic triangle

$$[x, z] \cup [z, L_{S,\mathcal{H}}(x)] \cup [L_{S,\mathcal{H}}(x), x]$$

is δ_2-thin. Thus, there exist $y \in [L_{S,\mathcal{H}}(x), z]$, $y_1 \in [x, L_{S,\mathcal{H}}(x)]$, and $y_2 \in [x, z]$ such that $\mathrm{d}_\Omega(y, y_1) \leq \delta_2$ and $\mathrm{d}_\Omega(y, y_2) \leq \delta_2$.

We claim that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y_1) \leq \delta_1 + \delta_2$. Choose any $p \in \pi_S(x)$. Since $[L_{S,\mathcal{H}}(x), z] \subset S$,

$$\mathrm{d}_\Omega(x, p) = \mathrm{d}_\Omega(x, S) \leq \mathrm{d}_\Omega(x, y).$$

Then, using Proposition 4.2,

$$\mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) \leq \mathrm{d}_\Omega(x, p) + \mathrm{d}_\Omega(p, L_{S,\mathcal{H}}(x)) \leq \mathrm{d}_\Omega(x, y) + \delta_1.$$

Then,

$$\begin{aligned}
\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y_1) &= \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x) - \mathrm{d}_\Omega(y_1, x) \\
&\leq \mathrm{d}_\Omega(x, y) + \delta_1 - \mathrm{d}_\Omega(y_1, x) \\
&\leq \mathrm{d}_\Omega(y, y_1) + \delta_1 \leq \delta_2 + \delta_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), [x, z]) &\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y_2) \\
&\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y_1) + \mathrm{d}_\Omega(y_1, y) + \mathrm{d}_\Omega(y, y_2) \\
&\leq \delta_1 + 3\delta_2 = \delta_3.
\end{aligned}$$

□

Our next goal is to prove if the distance between the linear projections of two points onto a simplex $S \in \mathcal{S}$ is large, then the geodesic between the two points spends a significant amount of time in a tubular neighborhood of S. This is accomplished in Corollary 4.6 using the next result.

Proposition 4.5 ([IZ23, Proposition 13.11]). *There exists a constant $\delta_4 \geq 0$ such that: if $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, $x, y \in \mathcal{C}$, and $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_4$, then*

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), [x,y]) \leq \delta_4 \quad \textit{and} \quad \mathrm{d}_\Omega(L_{S,\mathcal{H}}(y), [x,y]) \leq \delta_4.$$

Proof. Observe that the linear projections are Γ-equivariant, that is,

$$L_{gS, g\mathcal{H}} \circ g = g \circ L_{S,\mathcal{H}}$$

for any $g \in \Gamma$, $S \in \mathcal{S}$, and $\mathcal{H}$ a set of S-supporting hyperplanes. Moreover, by Proposition 3.3 there are only finitely many Γ-orbits in $\mathcal{S}$. Thus, it is enough to prove this proposition for a fixed $S \in \mathcal{S}$.

Suppose the proposition is false. Then, for every $n \geq 0$, there exist $x_n, y_n \in \mathcal{C}$ and a set of S-supporting hyperplanes $\mathcal{H}_n$ with

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}_n}(x_n), L_{S,\mathcal{H}_n}(y_n)) \geq n$$

and

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}_n}(x_n), [x_n, y_n]) \geq n.$$

Let $a_n := L_{S,\mathcal{H}_n}(x_n)$ and $b_n := L_{S,\mathcal{H}_n}(y_n)$. Then pick $c_n \in [a_n, b_n]$ such that

$$\mathrm{d}_\Omega(c_n, a_n) = n/2. \tag{10}$$

Then,

$$\mathrm{d}_\Omega(c_n, b_n) \geq \mathrm{d}_\Omega(a_n, b_n) - \mathrm{d}_\Omega(c_n, a_n) \geq n/2 \tag{11}$$

and

$$\mathrm{d}_\Omega\left(c_n, [x_n, y_n]\right) \geq \mathrm{d}_\Omega\left(a_n, [x_n, y_n]\right) - \mathrm{d}_\Omega(c_n, a_n) \geq n/2. \tag{12}$$

Since $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S (see Proposition 3.3), translating by elements of $\mathrm{Stab}_\Gamma(S)$ and passing to a subsequence, we may assume that $c := \lim_{n\to\infty} c_n$ exists and $c \in S$. After taking a further subsequence, we can assume that the following limits exist in $\overline{\mathcal{C}}$: $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$, $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$.

We now observe that $a, b, x, y \in \partial_{\mathrm{i}} \mathcal{C}$. Equation (10) and (11) imply that $a, b \in \partial_{\mathrm{i}} \mathcal{C}$. Equation (12) implies that $[x, y] \subset \partial_{\mathrm{i}} \mathcal{C}$.

We claim that $x \in F_\Omega(a)$ and $y \in F_\Omega(b)$. Since $c_n \in S$, by Proposition 4.4, there exists $a_n' \in [x_n, c_n]$ such that $\mathrm{d}_\Omega(a_n, a_n') \leq \delta_3$. Up to passing to a subsequence, we can assume that $a' := \lim_{n\to\infty} a_n'$ exists in $\overline{\mathcal{C}}$. Observe that $a' \in \partial_{\mathrm{i}} \mathcal{C}$ since

$$\lim_{n\to\infty} \mathrm{d}_\Omega(a_n', c) \geq \lim_{n\to\infty} \Big(\mathrm{d}_\Omega(a_n, c_n) - \mathrm{d}_\Omega(c_n, c) - \mathrm{d}_\Omega(a_n, a_n')\Big) = \infty.$$

Since $a_n' \in [x_n, c_n]$,

$$a' \in \partial_{\mathrm{i}} \mathcal{C} \cap [x, c] = \{x\}.$$

Thus, $\lim_{n\to\infty} a_n' = x$. Since $\lim_{n\to\infty} a_n = a$ and $\mathrm{d}_\Omega(a_n, a_n') \leq \delta_3$, Proposition 2.2 implies that $x \in F_\Omega(a)$. Similar reasoning shows that $y \in F_\Omega(b)$.

Since $[x, y] \subset \partial_{\mathrm{i}} \mathcal{C}$, Proposition 2.1 part (4) implies that $[a, b] \subset \partial_{\mathrm{i}} \mathcal{C}$. This is a contradiction since $c \in (a, b) \cap \mathcal{C} \neq \emptyset$. □

Corollary 4.6 ([IZ23, Corollary 13.12]). *If $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, $R > 0$, $x, y \in \mathcal{C}$, and $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq R + 2\delta_4$, then:*

(1) there exists $[x_0, y_0] \subset [x, y]$ such that $[x_0, y_0] \subset \mathcal{N}_\Omega(S; \delta_4)$,
(2) $\big[L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)\big] \subset \mathcal{N}_\Omega\big([x, y]; \delta_4\big)$, and,
(3) $\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S; \delta_4) \cap [x, y]\Big) \geq R$.

Proof. Since $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) > \delta_4$, Proposition 4.5 implies that there exists $x_0, y_0 \in [x,y]$ such that

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x_0) \leq \delta_4 \quad \text{and} \quad \mathrm{d}_\Omega(L_{S,\mathcal{H}}(y), y_0) \leq \delta_4.$$

By Proposition 2.11,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}\Big([x_0, y_0], [L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)] \Big) \leq \delta_4$$

and, by convexity, $[L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)] \subset S$. This proves parts (1) and (2). To prove part (3), observe that

$$\begin{aligned} \mathrm{d}_\Omega(x_0, y_0) &\geq \mathrm{d}_\Omega\left(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)\right) - \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x_0) - \mathrm{d}_\Omega(L_{S,\mathcal{H}}(y), y_0) \\ &\geq R. \end{aligned}$$

Then, $\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap [x,y]\Big) \geq \mathrm{d}_\Omega(x_0, y_0) \geq R$. □

4.2. **Strongly Isolated Simplices implies Relative Hyperbolicity.** For the rest of this section fix a convex co-compact group $\Gamma \leq \mathrm{Aut}(\Omega)$ for which $(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ has strongly isolated simplices. Set $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$ and $\mathcal{S} := \mathcal{S}_\Gamma$.

We will prove that (1) $\implies$ (2) in Theorem 1.4, that is, $(\mathcal{C}, \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Gamma$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, the results of Section 4.1 hold. For each $S \in \mathcal{S}$, fix a set $\mathcal{H}_S$ of S-supporting hyperplanes. Consider the family of projection maps

$$\Pi_{\mathcal{S}} := \{L_{S,\mathcal{H}} : S \in \mathcal{S}, \mathcal{H} = \mathcal{H}_S\}$$

and the geodesic path system

$$\mathcal{G} := \{[x,y] : x, y \in \mathcal{C}\}$$

on $\mathcal{C}$. By Theorem A.8, it is enough to verify that $\Pi_{\mathcal{S}}$ is an almost projection system and that $\mathcal{S}$ is asymptotically transverse-free relative to $\mathcal{G}$. We complete this in the next two subsections (cf. 4.3 and 4.4).

4.3. **$\Pi_{\mathcal{S}}$ is an Almost Projection System.** Let δ_3 be the constant in Proposition 4.4.

Lemma 4.7 ([IZ23, Lemma 13.13])**.** *If $S \in \mathcal{S}$, $\mathcal{H}$ a set of S-supporting hyperplanes, $x \in \mathcal{C}$, and $z \in S$, then*

$$\mathrm{d}_\Omega(x,z) \geq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), z) - 2\delta_3.$$

Proof. By Proposition 4.4, there exists $q \in [x,z]$ such that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), q) \leq \delta_3$. Then,

$$\mathrm{d}_\Omega(x,z) = \mathrm{d}_\Omega(x,q) + \mathrm{d}_\Omega(q,z) \geq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), z) - 2\delta_3.$$ □

Lemma 4.8 ([IZ23, Lemma 13.14])**.** *There exists a constant $\delta_5 \geq 0$ such that: if $S \neq S' \in \mathcal{S}$ and $\mathcal{H}$ is a set of S-supporting hyperplanes, then*

$$\mathrm{diam}_\Omega(L_{S,\mathcal{H}}(S')) \leq \delta_5.$$

Proof. Since $\mathcal{S}$ is strongly isolated, for every $r > 0$ there exists $D(r) > 0$ such that

$$\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2, r)\Big) \leq D(r) \tag{13}$$

for all $S_1, S_2 \in \mathcal{S}$ distinct.

Let δ_4 be the constant in Proposition 4.5. Set $\delta_5 := D(\delta_4) + 2\delta_4 + 1$. Fix $x, y \in S'$ and suppose for a contradiction that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_5$. Then, by Corollary 4.6,

$$\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap S'\Big) \geq \mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap [x,y]\Big) \geq D(\delta_4) + 1.$$

which contradicts Equation (13). □

Let δ_1 and δ_4 be the constants in Proposition 4.2 and 4.5 respectively.

Lemma 4.9 ([IZ23, Lemma 13.15])**.** *If $x \in \mathcal{C}$, $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $R := \mathrm{d}_\Omega(x, S)$, then*

$$\mathrm{diam}_\Omega\left(L_{S,\mathcal{H}}\Big(\mathcal{B}_\Omega(x,R) \cap \mathcal{C}\Big)\right) \leq 8(\delta_4 + \delta_1).$$

Proof. Fix $y \in \mathcal{B}_\Omega(x,R) \cap \mathcal{C}$. We claim that

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \leq 4(\delta_4+\delta_1).$$

It is enough to consider the case when $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_4$. Then by Proposition 4.5, there exists $x' \in [x,y]$ such that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') \leq \delta_4$. By Proposition 4.2,

$$\mathrm{d}_\Omega(x,y) \leq R = \mathrm{d}_\Omega(x, \pi_S(x)) \leq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \delta_1.$$

Then,

$$\begin{aligned}
\mathrm{d}_\Omega(x',y) = \mathrm{d}_\Omega(x,y) - \mathrm{d}_\Omega(x,x') &\leq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) - \mathrm{d}_\Omega(x,x') + \delta_1 \\
&\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \delta_1 \\
&\leq \delta_4 + \delta_1.
\end{aligned}$$

Thus,

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y) \leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \mathrm{d}_\Omega(x',y) \leq 2\delta_4 + \delta_1.$$

Since $L_{S,\mathcal{H}}(x) \in S$, using Proposition 4.2 again,

$$\begin{aligned}
\mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(y)) &\leq \mathrm{d}_\Omega(y, \pi_S(y)) + \delta_1 \leq \mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(x)) + \delta_1 \\
&\leq 2(\delta_4+\delta_1).
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) &\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \mathrm{d}_\Omega(x',y) + \mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(y)) \\
&\leq 4(\delta_4+\delta_1).
\end{aligned}$$

□

4.4. $\mathcal{S}$ **is Asymptotically Transverse-free relative to** $\mathcal{G}$**.** This section is essentially the proof of [IZ23, Theorem 13.16]. Let δ_4 be the constant in Proposition 4.5. We will show that exists $\Gamma > 0$ such that for each $\Delta \geq 1$ and $\kappa \geq 2\delta_4$ the following holds: if $\mathcal{T} \subset \mathcal{C}$ is a geodesic triangle whose sides are in $\mathcal{G}$ and is $\mathcal{S}$-almost-transverse with constants κ and Δ, then $\mathcal{T}$ is $(\Gamma\Delta)$-thin.

Suppose such a $\Gamma > 0$ does not exist. Then, for every $n \geq 1$, there exist $\kappa_n \geq 2\delta_4$, $\Delta_n \geq 1$, and a $\mathcal{S}$-almost-transverse triangle $\mathcal{T}_n \subset \mathcal{C}$ with constants κ_n and Δ_n such that $\mathcal{T}_n$ is not $(n\Delta_n)$-thin. Let a_n, b_n, and c_n be the vertices of $\mathcal{T}_n$, labeled in a such a way that there exists $u_n \in [a_n,b_n] \subset \mathcal{T}_n$ with

$$\mathrm{d}_\Omega\Big(u_n, [a_n,c_n] \cup [c_n,b_n]\Big) > n\Delta_n \geq n. \tag{14}$$

Then the geodesic triangles $\mathcal{T}_n$ are also $\mathcal{S}$-almost-transverse with constants $2\delta_4$ and Δ_n since $\kappa_n \geq 2\delta_4$.

Since Γ acts co-compactly on $\mathcal{C}$, translating by elements of Γ and passing to a subsequence, we can assume that $u := \lim_{n\to\infty} u_n$ exists and $u \in \mathcal{C}$. By passing to a further subsequence, we can assume that $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$, and $c := \lim_{n\to\infty} c_n$ exist in $\overline{\mathcal{C}}$. By Equation (14),

$$[a,c] \cup [c,b] \subset \partial_{\mathrm{i}}\, \mathcal{C}$$

whereas, by construction, $u \in (a,b) \subset \mathcal{C}$. Thus, the points a,b,c form a half triangle. Then, by Theorem 1.5 part (7), there exists $S \in \mathcal{S}_\Gamma$ such that $a,b,c \in \partial S$.

Fix a set of S-supporting hyperplanes $\mathcal{H}$. Let $a_n' := L_{S,\mathcal{H}}(a_n)$, $b_n' := L_{S,\mathcal{H}}(b_n)$, and $c_n' := L_{S,\mathcal{H}}(c_n)$. Up to passing to a subsequence, we can assume that the limits $a' := \lim_{n\to\infty} a_n'$, $b' := \lim_{n\to\infty} b_n'$ and $c' := \lim_{n\to\infty} c_n'$ exist. By Lemma 4.1,

$$a' = \lim_{n\to\infty} L_{S,\mathcal{H}}(a_n) = L_{S,\mathcal{H}}(a) = a.$$

Similarly, $b' = b$ and $c' = c$.

Since a,b,c form a half triangle, the faces $F_\Omega(a')$, $F_\Omega(b')$, and $F_\Omega(c')$, are pairwise disjoint. Then, by Proposition 2.2,

$$\lim_{n\to\infty} \mathrm{d}_\Omega(a_n', b_n') = \infty.$$

Thus, for n large enough, Corollary 4.6 part (2) and part (3) implies

$$[a_n', b_n'] \subset \mathcal{N}_\Omega([a_n,b_n]; \delta_4) \tag{15}$$

and

$$\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap [a_n,b_n]\Big) \geq \mathrm{d}_\Omega(a_n',b_n') - 2\delta_4. \tag{16}$$

Since $\mathcal{T}_n$ is $\mathcal{S}$-almost-transverse with constants $2\delta_4$ and Δ_n, by Equation (16),

$$\mathrm{d}_\Omega(a_n', b_n') \leq \Delta_n + 2\delta_4. \tag{17}$$

Similarly, for n large enough,

$$[b_n', c_n'] \subset \mathcal{N}_\Omega([b_n, c_n]; \delta_4) \quad \text{and} \quad \mathrm{d}_\Omega(b_n', c_n') \leq \Delta_n + 2\delta_4 \tag{18}$$

$$[c_n', a_n'] \subset \mathcal{N}_\Omega([c_n, a_n]; \delta_4) \quad \text{and} \quad \mathrm{d}_\Omega(c_n', a_n') \leq \Delta_n + 2\delta_4. \tag{19}$$

Let m_n^{ab}, m_n^{bc}, and m_n^{ca} be the Hilbert distance midpoints of $[a_n', b_n']$, $[b_n', c_n']$, and $[c_n', a_n']$ respectively. By Equations (15), (18), and (19), there exists w_n^{ab}, w_n^{bc}, and w_n^{ca} in $[a_n, b_n]$, $[b_n, c_n]$, and $[c_n, a_n]$ respectively such that:

$$\mathrm{d}_\Omega(w_n^{ab}, m_n^{ab}) \leq \delta_4, \ \mathrm{d}_\Omega(w_n^{bc}, m_n^{bc}) \leq \delta_4, \quad \text{and} \quad \mathrm{d}_\Omega(w_n^{ca}, m_n^{ca}) \leq \delta_4.$$

Then,

$$\begin{aligned}
\mathrm{d}_\Omega(w_n^{ab}, w_n^{bc}) &\leq \mathrm{d}_\Omega(w_n^{ab}, m_n^{ab}) + \mathrm{d}_\Omega(m_n^{ab}, m_n^{bc}) + \mathrm{d}_\Omega(m_n^{bc}, w_n^{bc}) \\
&\leq \delta_4 + \mathrm{d}_\Omega(m_n^{ab}, b_n') + \mathrm{d}_\Omega(b_n', m_n^{bc}) + \delta_4 \\
&= 2\delta_4 + \frac{\mathrm{d}_\Omega(a_n', b_n') + \mathrm{d}_\Omega(b_n', c_n')}{2} \\
&\leq 4\delta_4 + \Delta_n \quad \text{(by Equations (17) and (18))}
\end{aligned}$$

Similarly,

$$\mathrm{d}_\Omega(w_n^{bc}, w_n^{ca}) \leq \Delta_n + 4\delta_4 \quad \text{and} \quad \mathrm{d}_\Omega(w_n^{ca}, w_n^{ab}) \leq \Delta_n + 4\delta_4. \tag{20}$$

Then, for n large enough, the triangles $\mathcal{T}_n$ are $(\Delta_n + 4\delta_4)$-thin, since

$$\begin{aligned}
&{\mathrm{d}_\Omega}^{\mathrm{Hauss}}\Big([a_n, w_n^{ab}], [a_n, w_n^{ca}]\Big) \leq \Delta_n + 4\delta_4, \\
&{\mathrm{d}_\Omega}^{\mathrm{Hauss}}\Big([b_n, w_n^{bc}], [b_n, w_n^{ab}]\Big) \leq \Delta_n + 4\delta_4, \ \text{and} \\
&{\mathrm{d}_\Omega}^{\mathrm{Hauss}}\Big([c_n, w_n^{ca}], [c_n, w_n^{bc}]\Big) \leq \Delta_n + 4\delta_4.
\end{aligned}$$

Since $\Delta_n \geq 1$, we have $\Delta_n + 4\delta_4 \leq (1 + 4\delta_4)\Delta_n$. Thus, for n large enough, $\mathcal{T}_n$ is $(\Gamma\Delta_n)$-thin for $\Gamma := 1 + 4\delta_4$, which contradicts the assumption that $\mathcal{T}_n$ is not $(n\Delta_n)$-thin.

4.5. **Proof of Theorem 1.4.** For the rest of the section suppose that $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Set $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$ and let $\mathcal{S}_\Gamma$ be the family of all maximal properly embedded simplices in $\mathcal{C}$ of dimension at least two.

(1) implies (2). See Section 4.2.

(2) implies (1). This follows from Theorem A.5 part (1).

(1) and (2) implies (3). Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}_\Gamma$. Equivalence of (1) and (2) implies that $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. Then, by Theorem 1.5 part (1), there exists $m \in \mathbb{N}$ such that

$$\mathcal{S}_\Gamma = \sqcup_{i=1}^m \Gamma \cdot S_i. \tag{21}$$

Theorem 1.5 part (2) implies that for each $i \in \{1, \dots, m\}$, there exists an Abelian subgroup $A_i \leq \Gamma$ of rank at least two such that A_i acts co-compactly on S_i. We will show that Γ is a relatively hyperbolic group with respect to the subgroups $\{A_1, \dots, A_m\}$.

Fix $p \in \mathcal{C}$. Since Γ acts co-compactly on $\mathcal{C}$, Theorem **??** implies that the orbit map $F : (\Gamma, \mathrm{d}_S) \to (\mathcal{C}, \mathrm{d}_\Omega)$ defined by $F(g) = g \cdot p$ is a quasi-isometry . Here d_S is a word metric on Γ obtained by fixing a finite generating set S of Γ.

Since A_i acts co-compactly on S_i for $1 \leq i \leq m$, there exists $R > 0$ such that

$$\sup_{g \in \Gamma} \sup_{1 \leq i \leq m} {\mathrm{d}_\Omega}^{\mathrm{Hauss}}(gA_i \cdot p, gS_i) \leq R.$$

Then equation (21) implies that up to modifying the map F by a bounded quantity determined by R, we can assume that

$$F(\{gA_i : g \in \Gamma, 1 \leq i \leq m\}) = \mathcal{S}_\Gamma\,.$$

Then by Proposition A.4, (Γ, d_S) is a relatively hyperbolic space with respect to the collection of left cosets $\{gA_i : g \in \Gamma, 1 \leq i \leq m\}$. This completes the proof.

(3) implies (2). Suppose that Γ is a relatively hyperbolic group with respect to a collection of subgroups $\{H_1, \dots, H_k\}$ each of which is a virtually Abelian group of rank at least two. For each $1 \leq j \leq k$, let $A_j \leq H_j$ be a finite index Abelian subgroup with rank at least two. Then, by definition, Γ is a relatively hyperbolic group with respect to $\{A_1, \dots, A_k\}$.

Fix some $x_0 \in \Omega$ and consider the orbit map $F : (\Gamma, \mathrm{d}_\Gamma) \to (\mathcal{C}, \mathrm{d}_\Omega)$ defined by $F(g) = g \cdot x_0$. By Proposition **??**, F is a quasi-isometry. Let $G : \mathcal{C} \to \Gamma$ be a quasi-inverse. Fix a word metric d_Γ on Γ. We will use the following notation: if $U \subset \Gamma$ and $r > 0$, let

$$\mathcal{N}_\Gamma(U; r) := \{g \in \Gamma : \mathrm{d}_\Gamma(g, U) < r\}$$

and

$$\mathrm{diam}_\Gamma(U) = \sup\{\mathrm{d}_\Gamma(g_1, g_2) : g_1, g_2 \in U\}.$$

For each $1 \leq j \leq k$, let $\widehat{A}_j$ be a maximal Abelian subgroup of Γ that contains A_j. By Theorem 2.24, there exists a properly embedded simplex $S_j \subset \mathcal{C}$ such that $\widehat{A}_j \leq \mathrm{Stab}_\Gamma(S_j)$, $\widehat{A}_j$ acts co-compactly on S_j, and $\widehat{A}_j$ has a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S_j)}$. Since A_j (and hence $\widehat{A}_j$) has rank at least two, this implies that $\dim S_j \geq 2$.

Claim 4.10. $(\mathcal{C}, \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to

$$\mathcal{S}_0 := \{gS_j : g \in \Gamma, 1 \leq j \leq k\}.$$

Proof of Claim. We claim that $A_j \leq \widehat{A}_j$ has finite index and hence A_j also acts co-compactly on S_j. By Observation 2.12, the metric space (S_j, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S_j}$. So, by the fundamental lemma of geometric group theory [BH99, Chapter I, Proposition 8.19], $(\widehat{A}_j, \mathrm{d}_\Gamma)$ is also quasi-isometric to $\mathbb{R}^{\dim S_j}$. Since $\dim S_j \geq 2$, Theorem A.5 part (2) implies that there exists $r_1 > 0$, $g_j \in \Gamma$, and $1 \leq i_j \leq k$ such that

$$\widehat{A}_j \subset \mathcal{N}_\Gamma(g_j A_{i_j}; r_1).$$

Then

$$\mathrm{diam}_\Gamma \left(\mathcal{N}_\Gamma(g_j A_{i_j}; r_1) \cap \mathcal{N}_\Gamma(A_j; r_1)\right) \geq \mathrm{diam}_\Gamma (A_j) = \infty.$$

So Theorem A.5 part (1) implies that $g_j A_{i_j} = A_j$. Then,

$$\widehat{A}_j \subset \mathcal{N}_\Gamma(A_j; r_1)$$

and hence $A_j \leq \widehat{A}_j$ has finite index.

Then, using the fact that A_j acts co-compactly on S_j, there exists $r_2 > 0$ such that

$$F(gA_j) \subset \mathcal{N}_\Omega(gS_j; r_2)$$

and

$$G(gS_j) \subset \mathcal{N}_\Gamma(gA_j; r_2)$$

for all $g \in \Gamma$ and $1 \leq j \leq k$. Then, by Theorem A.5 part (3), $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}_0$. □

In order to finish the proof, we will now show that $\mathcal{S}_0 = \mathcal{S}_\Gamma$.

We first show that $\mathcal{S}_0 \subset \mathcal{S}_\Gamma$. Suppose $S \in \mathcal{S}_0$ is properly contained in a maximal properly embedded simplex S'. Then, by Theorem A.5 part (2), there exists $S''' \in \mathcal{S}_0$ such that

$$S \subset S' \subset \mathcal{N}_\Omega(S'''; M).$$

This implies that $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S'''; M) \cap \mathcal{N}_\Omega(S; M)) = \infty$. Then Theorem A.5 part (1) implies that $S''' = S$, i.e. $S' \subset \mathcal{N}_\Omega(S; M)$. Hence, $\dim(S') \leq \dim(S)$ which is a contradiction since S' properly contains S.

For proving $\mathcal{S}_\Gamma \subset \mathcal{S}_0$, we first need the following claim.

Claim 4.11. If $S \in \mathcal{S}_0$ and $x \in \partial S$, then $F_\Omega(x) = F_\mathcal{C}(x) = F_S(x)$.

Proof of Claim. Fix $S \in \mathcal{S}_0$. Recall that since Γ is a convex co-compact subgroup, $F_\Omega(x') = F_{\mathcal{C}}(x')$ for any $x' \in \partial_{\mathrm{i}} \mathcal{C}$. Then, as in the proof of Proposition 3.5, if $x \in \partial S$, then the claim fails only when

$$\partial F_S(x) \subsetneq \partial F_{\mathcal{C}}(x).$$

We will prove that $\partial F_S(x) = \partial F_{\mathcal{C}}(x)$ by induction on $\dim(F_S(x))$.

Base case: $\dim(F_S(x)) = 0$.

Then x is a vertex of S. Suppose the claim fails, i.e. $\partial F_S(x) \subsetneq \partial F_{\mathcal{C}}(x)$. Let $w_0 \in \partial F_{\mathcal{C}}(x) \setminus \overline{F_S(x)}$. Then $(w_0, x) \cap \overline{S} = \emptyset$. Otherwise, if there was $x'' \in (w_0, x) \cap \overline{S}$, then Observation 2.13 would imply that

$$F_S(x'') = \overline{S} \cap F_\Omega(x'') = \overline{S} \cap F_\Omega(x) = F_S(x) = \{x\},$$

that is $x'' = x$, a contradiction.

Then, we fix $w \in (w_0, x) \subset F_\Omega(x)$ such that $M + 1 \leq \mathrm{d}_{F_\Omega(x)}(w, x)$. Set $R_w := \mathrm{d}_{F_\Omega(x)}(w, x)$. Let us label the vertices of S as $v_1, \dots, v_m$ where $m = \dim(S) + 1$ and $v_1 = x$. By Proposition **??**, $S' := \mathrm{ConvHull}_\Omega(w, v_2, \dots, v_m)$ is a properly embedded simplex in $\mathcal{C}$ such that $\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(S', S) \leq R_w$. Observe that

$$S' \not\subset \mathcal{N}_\Omega(S; M).$$

Indeed, if $S' \subset \mathcal{N}_\Omega(S; M)$, then applying Corollary 2.3 to $w \in \partial S'$, we get $s \in \partial S$ such that $w \in F_\Omega(s)$ and $\mathrm{d}_{F_\Omega(s)}(w, s) \leq M$. Since $w \in F_\Omega(x)$, this implies that $x \in F_\Omega(s)$. Since $x, s \in \partial S$ and x is a vertex of S, $s = x$. Thus $\mathrm{d}_{F_\Omega(x)}(w, x) \leq M$, a contradiction.

Then, by Theorem A.5 part (2), there exists $S_1 \neq S \in \mathcal{S}_0$ such that $S' \subset \mathcal{N}_\Omega(S_1; M)$. Then we have

$$S' \subset \mathcal{N}_\Omega(S; R_w) \cap \mathcal{N}(S_1; M)$$

which implies that $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S; R_w) \cap \mathcal{N}(S_1; 2M)) = \infty$ for distinct $S, S_1 \in \mathcal{S}_0$. This contradicts Theorem A.5 part (1), as $\mathcal{C}$ is a relatively hyperbolic space with respect to $\mathcal{S}_0$. This completes the proof in the base case.

Induction step: Suppose the proposition is true when $\dim(F_S(x)) = k$ for some $k \geq 0$.

Now suppose $\dim(F_S(x)) = k + 1$. Let $y \in \partial F_{\mathcal{C}}(x)$. We will show that $y \in \partial F_S(x)$. For $n \geq 1$, choose $y_n \in (y, x)$ such that $\mathrm{d}_{F_{\mathcal{C}}(x)}(y_n, x) = n$.

Let us label the vertices of S as $v_1, \dots, v_m$ where $m = \dim(S) + 1$ and $v_1 = x$. By Proposition **??**,

$$S_n := \mathrm{ConvHull}_\Omega(y_n, v_2, \dots, v_m)$$

is a properly embedded simplex in $\mathcal{C}$ of dimension at least two and $\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(S, S_n) \leq n$. Then, by Theorem A.5 part (2), there exist $T_n \in \mathcal{S}_0$ such that

$$S_n \subset \mathcal{N}_\Omega(T_n; M)$$

for each $n \geq 1$. Then

$$S \subset \mathcal{N}_\Omega(T_n; M + n).$$

Since $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(T_n; M+n) \cap \mathcal{N}_\Omega(S; M+n)) = \infty$, Theorem A.5 part (1) implies that $S = T_n$ for all $n \geq 1$. Since $S_n \subset \mathcal{N}_\Omega(S; M)$ and $y_n \in \partial S_n$, Corollary 2.3 implies that for each $n \geq 1$, there exists $z_n \in \partial S \cap F_\Omega(y_n)$ such that $\mathrm{d}_{F_\Omega(y_n)}(y_n, z_n) \leq M$. Since $y_n \in F_\Omega(x)$, $z_n \in \partial S \cap F_\Omega(x)$ and $\mathrm{d}_{F_\Omega(x)}(y_n, z_n) \leq M$. Up to passing to a subsequence, we can assume that $z_n \to z \in \overline{S}$. By Proposition 2.2,

$$y \in F_{F_\Omega(x)}(z) = F_\Omega(z).$$

Observe that $z \in \partial F_S(x) = \partial S \cap \partial F_\Omega(x)$ since

$$\begin{aligned} \mathrm{d}_{F_\Omega(x)}(x, z) = \lim_{n \to \infty} \mathrm{d}_{F_\Omega(x)}(x, z_n) &\geq \lim_{n \to \infty} \mathrm{d}_{F_\Omega(x)}(x, y_n) - \mathrm{d}_{F_\Omega(x)}(y_n, z_n) \\ &\geq \lim_{n \to \infty} (n - M) = \infty. \end{aligned}$$

Since $z \in \partial F_S(x)$, $F_S(z) \subset \partial F_S(x)$. Then

$$\dim(F_S(z)) \leq \dim(\partial F_S(x)) = \dim(F_S(x)) - 1 = k.$$

The induction hypothesis then implies that $F_\Omega(z) = F_S(z)$. Thus,

$$y \in F_S(z) \subset \partial F_S(x).$$

Hence $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$ which finishes the proof of this claim. $\square$

Now fix any $S \in \mathcal{S}_\Gamma$. By Theorem A.5 part (2), there exists M such that $S \subset \mathcal{N}_\Omega(S_0; M)$ for some $S_0 \in \mathcal{S}_0$. If $q \in \partial S$, then Corollary 2.3 implies that there exists $q_0 \in \partial S_0$ such that $q \in F_\Omega(q_0)$. Since $S_0 \in \mathcal{S}_0$, the above claim implies that $F_\Omega(q_0) = F_{S_0}(q_0) \subset \partial S_0$. Thus $q \in \partial S_0$. This implies that $\partial S \subset \partial S_0$, that is, $S \subset S_0$. Since S is a maximal properly embedded simplex, $S = S_0$ and $S \in \mathcal{S}_0$. This finishes the proof that $\mathcal{S}_0 = \mathcal{S}_\Gamma$. Then, by Claim 4.10, $\mathcal{C}$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Gamma$.

Appendix A. Relatively Hyperbolic Groups

Relatively hyperbolic groups generalize fundamental groups of finite volume non-compact hyperbolic manifolds. There are several equivalent definitions due to Bowditch, Farb, and Druţu-Sapir to name a few. In this section, we will follow the approach taken by Druţu-Sapir in [DS05]. For this, we recall the notion of asymptotic cones and asymptotically tree-graded metric spaces.

Definition A.1. Suppose ω is a non-principal ultrafilter, (X, d) is a metric space, (x_n) is a sequence of points in X, and (Γ_n) is a sequence of positive numbers with $\lim_\omega \Gamma_n = \infty$. The *asymptotic cone of X with respect to (x_n) and (Γ_n)*, denoted by $C_\omega(X, x_n, \Gamma_n)$, is the ultralimit $\lim_\omega(X, \Gamma_n^{-1}\,\mathrm{d}, x_n)$.

For more background on asymptotic cones, see [Dru02].

Definition A.2 ([DS05, Definition 2.1])**.** Let (X, d) be a complete geodesic metric space and let $\mathcal{S}$ be a collection of closed geodesic subsets (called *pieces*).

1. We say that (X, d) is *tree-graded with respect to $\mathcal{S}$* if:
   - (a) every two different pieces have at most one common point.
   - (b) every simple geodesic triangle (a simple loop composed of three geodesics) in X is contained in one piece.
2. We say that (X, d) is *asymptotically tree-graded with respect to $\mathcal{S}$* if all its asymptotic cones, with respect to a fixed non-principal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of $\mathcal{S}$.

Using asymptotically tree-graded metric spaces, we now introduce the definition of relatively hyperbolic spaces and groups respectively.

Definition A.3 ([DS05])**.**

1. A complete geodesic metric space (X, d) is *relatively hyperbolic with respect to a collection of subsets $\mathcal{S}$* if (X, d) is asymptotically tree-graded with respect to $\mathcal{S}$.
2. A finitely generated group (G, S) is *relatively hyperbolic with respect to a family of subgroups* $\{H_1, \dots, H_k\}$ if (G, d_S) is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G, i = 1, \dots, k\}$.

Relative hyperbolicity is well-behaved under quasi-isometries.

Proposition A.4. *Suppose (X, d_X) is a relatively hyperbolic space with respect to $\mathcal{S}_X$ and $f : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is a quasi-isometry. Then, (Y, d_Y) is a relatively hyperbolic space with respect to $\mathcal{S}_Y := f(\mathcal{S}_X)$.*

We will require the following results about relatively hyperbolic spaces.

Theorem A.5. *Suppose (X, d) is a relatively hyperbolic space with respect to $\mathcal{S}$.*

(1) [DS05, Theorem 4.1] *For any $r > 0$ there exists $Q(r) > 0$ such that: if $S_1, S_2 \in \mathcal{S}$ are distinct, then*
$$\operatorname{diam}_X \left(\mathcal{N}_X(S_1; r) \cap \mathcal{N}_X(S_2; r)\right) \leq Q(r).$$

(2) [DS05, Corollary 5.8] *If $A \geq 1$, $B \geq 0$, and $f : \mathbb{R}^k \to X$ is an (A, B)-quasi-isometric embedding, then there exists $M = M(A, B)$ such that: if $k \geq 2$, then there exists some $S \in \mathcal{S}$ such that*
$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S; M).$$

(3) [DS05, Theorem 5.1] *If (Y, d_Y) are complete geodesic metric spaces and $f : X \to Y$ is a quasi-isometry, then (X, d_X) is relatively hyperbolic with respect to $\mathcal{S}$ if and only if (Y, d_Y) is relatively hyperbolic with respect to $f(\mathcal{S})$.*

We end this section by stating a characterization of relative hyperbolicity due to Sisto [Sis13]. In order to state his characterization, we introduce two notions: "almost-projection system" and "asymptotically transverse-free with respect to a geodesic path system".

Definition A.6 ([Sis13])**.** Let (X, d) be a complete geodesic metric space and $\mathcal{S}$ a collection of subsets of X. A family of maps $\Pi_{\mathcal{S}} = \{\pi_S : X \to S\}_{S \in \mathcal{S}}$ is an *almost-projection system for* $\mathcal{S}$ if there exists $C > 0$ such that for all $S \in \mathcal{S}$:

(1) if $x \in X$ and $p \in S$, then $\mathrm{d}(x, p) \geq \mathrm{d}(x, \pi_S(x)) + \mathrm{d}(\pi_S(x), p) - C$,
(2) $\mathrm{diam}_X\, \pi_S(S') \leq C$ for all $S, S' \in \mathcal{S}$ distinct, and
(3) if $x \in X$ and $\mathrm{d}(x, S) = R$, then $\mathrm{diam}_X\, \pi_S(\mathcal{B}_X(x, R)) \leq C$.

Definition A.7 ([Sis13])**.** Let (X, d) be a complete geodesic metric space and $\mathcal{S}$ a collection of subsets of X.

(1) A geodesic triangle $\mathcal{T}$ in X is $\mathcal{S}$*-almost-transverse with constants* κ *and* Δ if
$$\mathrm{diam}_X(\mathcal{N}_X(S; \kappa) \cap \gamma) \leq \Delta$$
for every $S \in \mathcal{S}$ and edge γ of $\mathcal{T}$.
(2) The collection $\mathcal{S}$ is *asymptotically transverse-free relative to a geodesic path system* $\mathcal{G}$ if there exists Γ, σ such that for each $\Delta \geq 1, \kappa \geq \sigma$ the following holds: if $\mathcal{T}$ is a geodesic triangle in X whose sides are in $\mathcal{G}$ and is $\mathcal{S}$-almost-transverse with constants κ and Δ, then $\mathcal{T}$ is $(\Gamma\Delta)$-thin.

We finally state Sisto's characterization of relative hyperbolicity.

Theorem A.8 ([Sis13, Theorem 2.14])**.** *Let* (X, d) *be a complete geodesic metric space and* $\mathcal{S}$ *a collection of subsets of* X*. Then the following are equivalent:*

(1) X *is relatively hyperbolic with respect to* $\mathcal{S}$*,*
(2) $\mathcal{S}$ *is asymptotically transverse-free relative to a geodesic path system and there exists an almost-projection system for* $\mathcal{S}$*,*

In [Sis13], the theorem is stated for path systems instead of geodesic path systems. But his methods also imply this result, see [IZ23, Appendix] for details.

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