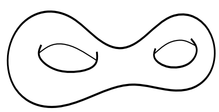
RESEARCH STATEMENT

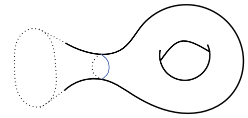
Mitul Islam

1. Overview of my Research Program

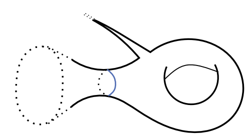
A fundamental question in geometry is to obtain concrete construction of manifolds. For instance, consider the classical Poincare polygon theorem, which provides a recipe for building surfaces by gluing the edges of a regular polygon. However such purely geometric methods become intractable as the dimension goes up. A more systematic approach to this problem is the study of discrete subgroups of Lie groups. For instance, the above problem of constructing a surface – of genus at least two – can be recast as a problem of constructing a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$. Any such uniform (i.e. co-compact) lattice Γ acts by Möbius transformations on the real hyperbolic plane $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma$ is a closed surface. More generally, if we construct discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that are 'thinner' than lattices, then we get nice infinite volume surfaces, e.g. convex co-compact or geometrically finite surfaces.



Closed surface of genus 2



Convex co-compact surface



Geometrically finite surface

One can ask the same question for Lie groups that are more complicated than $\mathrm{PSL}_2(\mathbb{R})$, e.g. the so-called higher rank Lie groups like $\mathrm{SL}_d(\mathbb{R})$ when $d \geq 3$, $\mathrm{SO}(p,q)$, etc. Lattices in such groups are very restricted - they all come from arithmetic constructions and cannot be deformed, as proven by Margulis [Mar75, Mar91]. But what about more general discrete subgroups? This is the central question that I pursue in my research program – the study of discrete subgroups in higher rank Lie groups, beyond lattices. One of the most exciting aspects of my research program is that it lies at the interface of geometry, topology, algebra, and analysis. I use ideas and tools from geometry, linear algebraic groups, topology, Lie group theory, representation theory, dynamical systems, ergodic theory, and coarse geometry. I find it rewarding to learn from a wide variety of areas of mathematics and see beautiful ideas emerge from their interplay.

Depending on their interest and time, the reader can choose to read only the first 3 pages (for a concise version) or opt for all 14 pages (complete with details of past, ongoing, and future work).

Research Direction I: Convex Projective Structures and Groups beyond Gromov Hyperbolicity

I adopt the perspective of studying discrete subgroups via their actions on the real projective space, more precisely, on *properly convex domains* in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. I will explain this below. But first, the reader might think the following: if $\Gamma < \mathrm{SL}_d(\mathbb{R})$ is a discrete subgroup, why do we not consider its action on the associated Riemannian symmetric spaces $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$? This is a great question because when $d = 2$, this symmetric space is $\mathbb{H}^2$ and Möbius actions on $\mathbb{H}^2$ is indeed the classical tool that we use for studying subgroups of $\mathrm{SL}_2(\mathbb{R})$. However, the situation changes dramatically in higher rank. Kleiner-Leeb [KL06] and Quint [Qui05] independently showed that when $d \geq 3$, the only discrete subgroups with reasonable actions (i.e. convex co-compact) on the symmetric space are uniform lattices. Hence, we are forced to look for new geometric spaces to study discrete subgroups in higher rank.

A motivating example of a properly convex domain is an open disk $\mathbb{B}$ in an affine plane in $\mathbb{P}(\mathbb{R}^3)$ – the projective Beltrami-Klein model of the hyperbolic plane $\mathbb{H}^2$. It has projective lines as geodesics, and the isometry group is $\mathrm{SO}(2,1)$. Inspired by this, we say that an open subset $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a *properly convex domain* if its closure $\overline{\Omega}$ is a compact convex domain in an (affine) chart of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Any such domain Ω carries a natural distance function d_Ω – *the Hilbert metric on* Ω – defined using cross-ratios. The subgroup of $\mathrm{PGL}_d(\mathbb{R})$ that preserves Ω – denoted by $\mathrm{Aut}(\Omega)$ – acts by isometries of d_Ω. Benoist showed that in some special cases, the Hilbert metric behaves like a negatively curved Riemannian metric [Ben08, Qui10]. But typically, the metric d_Ω is not even CAT(0) (a popular generalization of non-positive curvature) [KS58]. An important aspect of my research is to study the metric geometry of $(\Omega, \mathrm{d}_\Omega)$ by analogy with non-positive curvature.

- [Isl] *Rank One Hilbert Geometries*: I develop the theory of rank one Hilbert geometries, analogous to rank one CAT(0) spaces, and prove that a rank one group is an acylindrically hyperbolic group. Acylindrically hyperbolic groups are a much-studied generalization of Gromov hyperbolic groups that include mapping class groups, $\mathrm{Out}(F_n)$, etc. *See Section 2.*
- [IW24] *Morse geodesic rays in projective geometry*: These are the 'hyperbolic' directions – a coarse geometric notion – in a properly convex domain. We characterize these 'hyperbolic' directions in terms of singular value growth gaps (linear algebraic data) as well as the regularity of the boundary (projective geometric notion). *See Section 2.*

The second reason why I care so deeply about convex projective geometry is its close connections with Teichmüller theory. A *convex projective manifold* is a quotient of the form Ω/Γ where Γ is a torsion-free discrete subgroup of $\mathrm{Aut}(\Omega)$. This clearly generalizes the notion of hyperbolic structures on manifolds and hints at an intimate connection with classical Teichmüller theory, the study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. In recent years, *Anosov representations* [Lab06, KLP17, GGKW17] have received much attention as a dynamical approach towards developing a "Teichmüller theory for higher rank Lie groups" [BIW14, Wie18]. Anosov representations correspond to rich families of discrete subgroups: they are stable under small deformations, and often fill out entire connected components in the character variety, e.g. Hitchin representations [Wie18].

Until recently, however, there was a lack of a geometric perspective on Anosov representations [Kas18]. Convex projective geometry – more precisely, convex co-compact groups – has recently been used to fill this gap [DGK17, Zim21]. We call a group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ *convex co-compact* if it preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$ – the convex hull of its limit set. [DGK17] and [Zim21] show that for a Gromov hyperbolic group Γ, convex co-compactness is closely related to the inclusion representation $\Gamma \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ being Anosov. This inspires a natural question: what about non-Gromov hyperbolic groups? This question is a quest to understand discrete linear groups beyond the Gromov hyperbolic ones (i.e. the ones coming from Anosov representations).

I have addressed precisely this question in my joint research program with Andrew Zimmer. In the course of our five joint papers (summarized below), we develop a theory of convex co-compact groups that are relatively hyperbolic. Suppose Γ is a convex co-compact group that is relatively hyperbolic with respect to peripheral subgroups $\mathcal{P} := \{P_1, \dots, P_m\}$.

- In [IZ23], we solve the case of *abelian peripherals*, i.e. $P_i \cong \mathbb{Z}^{k_i}$ with $k_i \geq 2$. We introduce the notion of properly convex domains with isolated simplices, analogous to CAT(0) spaces with isolated flats [HK05]. We prove: Γ is relatively hyperbolic with respect to $\mathcal{P}$ if and only if $\mathcal{C}_\Omega(\Gamma)$ has isolated simplices.
- In [IZ22], we solve the case of *general peripheral subgroups*.
- In [IZ24b], we classify *3-manifold groups* that are convex co-compact while in [IZ24a], we classify convex co-compact groups with at most *1 dimensional boundary faces.* Both of these families are convex co-compact groups relatively hyperbolic with $\mathbb{Z}^2$ peripherals.
- In [IZ21], we prove the *flat torus theorem* and relate maximal abelian subgroups of a convex co-compact group (not necessarily rel. hyp.) to simplices in $\mathcal{C}_\Omega(\Gamma)$.

Research Direction II: Group Action on Boundaries and their Deformations

In Research Direction I, I study discrete subgroups of $\mathrm{SL}_d(\mathbb{R})$ by looking at their action on properly convex subsets of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. But what about their action on the entire projective space $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$? This is the central question that drives my Research Direction II. The action on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is significant because $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ can be interpreted as a boundary of the symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$ and group actions on boundaries has a rich tradition, e.g. [Mos73, Sul85, Tuk95].

The question of deforming boundary actions is classical and was already investigated by Sullivan [Sul85] in the case of rank one Lie groups. Consider the example of a uniform lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$ which has a natural action ρ_0 on $\partial\,\mathbb{H}^2 \cong \mathbb{S}^1$. Sullivan asked: can we deform ρ_0 in $\mathrm{Diff}(\mathbb{S}^1)$? That is, he wanted to consider an action ρ for which both ρ and its derivative $d\rho$ are sufficiently close to ρ_0 and $d\rho_0$ respectively. In this C^1-deformation case, Sullivan proved that ρ_0 is *rigid.*, i.e. any such ρ is in fact *conjugate* to ρ_0. This means that there is a homeomorphism ϕ of $\partial\,\mathbb{H}^2$ such that $\phi \circ \rho_0 = \rho \circ \phi$. So from a dynamical viewpoint, the actions ρ and ρ_0 are indistinguishable.

In my joint paper [CINS23], we ask the same question as Sullivan, but for higher rank lattices. To consider a concrete example, let $\Gamma < \mathrm{SL}_d(\mathbb{R})$ be a uniform lattice where $d \geq 3$. There is a natural algebraic action $\rho_0 : \Gamma \to \mathrm{Homeo}(\mathbb{P}(\mathbb{R}^{\mathrm{d}}))$ and we ask, is ρ_0 rigid? However, unlike Sullivan, we pose our question in the more challenging world of C^0-deformations (i.e. any action ρ close to ρ_0). We consider the C^0-case because the C^1-case in higher rank had already been settled independently by [KS97] and [Kan96]. Similar to Sullivan, the answer is that any C^1-close ρ is conjugate to ρ_0.

Passing to the C^0-case gives rise to unique challenges. Since we can no longer control the derivative of the action, neither Sullivan's methods [Sul85] nor the dynamical methods [KS97] apply. In fact, the C^0-case had proven to be a formidable challenge even for rank one Lie groups – remaining open until the work [BM22] in 2019. The ingenuity of [BM22] lies in finding a perfect marriage between the use of dynamical foliations and coarse geometry to tame the C^0-deformations. Although the tools of [BM22] does not apply in higher rank, our work [CINS23] is inspired by their approach.

- [CINS23] (*joint with C.Connell, T.Nguyen, and R.Spatzier*)
  We prove that: for any ρ that is sufficiently C^0-close to ρ_0 (i.e. ρ and ρ_0 both map generators of the lattice close-by), there exists a continuous surjective map $\varphi_\rho : \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \to \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that the adjoining diagram commutes. Such a map φ_ρ is called a *semi-conjugacy* and we say that ρ_0 is '*semi-conujgacy rigid*'. *See Section 4.*

$$\begin{array}{ccc} \mathbb{P}(\mathbb{R}^{\mathrm{d}}) & \xrightarrow{\rho} & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \\ \downarrow{\scriptstyle\varphi_\rho} & & \downarrow{\scriptstyle\varphi_\rho} \\ \mathbb{P}(\mathbb{R}^{\mathrm{d}}) & \xrightarrow{\rho_0} & \mathbb{P}(\mathbb{R}^{\mathrm{d}}) \end{array}$$

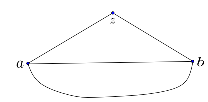
Our theorem in [CINS23] also holds for all other flag space boundaries, in particular for the Furstenberg boundary. In [CINS23], we further show that 'semi-conjugacy rigidity' is the best that one can hope for in the C^0-deformation case. We construct examples of deformations ρ close to ρ_0 for which φ_ρ cannot be a homeomorphism. Finally, in [CINS23], we make another surprising discovery: there is a complete loss of rigidity if we replace the flag spaces with the visual boundary ∂X of $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$. The lattice Γ has a natural action ρ_{v} on ∂X. We construct deformations arbitrarily close to ρ_{v}, but not semi-conjugate to ρ_{v}. *See Section 4.*

Broader context. My research program in Direction II fits into the broader context of studying the actions of higher rank lattices on compact manifolds. One incarnation of this quest is the *Zimmer program* and its analogues [Zim87, Wei11]. Robert Zimmer started the program in the wake of G. Margulis's superrigidity and R. Zimmer's cocycle superrigidity theorems [Mar91, Zim87]. The driving philosophy is that differentiable actions of higher rank lattices on 'low dimensional' manifolds should be essentially trivial or classifiable. The program has seen tremendous activity and success in recent years [Fis20]. While the Zimmer program concerns measure preserving differentiable actions, questions have often been asked about the possibility of a C^0-analogue and for actions that do not preserve measure, e.g. lattices acting on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ [Wei11]. To the best of our knowledge, our paper [CINS23] is the first one that tackles such C^0-rigidity problems in higher rank.

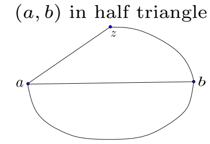
2. Rank One Properly Convex Domains: Beyond Hyperbolicity

In my research on convex projective structures, I focus on developing a theory for properly convex domains that is analogous to the theory of non-positively curved Riemannian manifolds, or more generally, CAT(0) metric spaces. A particularly challenging aspect of this work is that the natural metric - the Hilbert metric d_Ω - is typically non-Riemannian (only Finsler) and not CAT(0). A second challenge is the lack of a good notion of 'hyperbolic' directions for the geodesic flow. Recall that in the classical case of Riemannian non-positive curvature, 'hyperbolicity' stems from the vanishing of parallel Jacobi fields along a geodesic. For a typical properly convex domain, the projective geodesic flow is not even C^1. Hence, Jacobi fields or the classical notion of 'hyperbolicity' is meaningless. A key contribution of my work in [Isl] and [IW24] is providing a good interpretation of 'hyperbolicity' in absence of Jacobi fields and convexity of the distance function. I interpret 'hyperbolicity' from the viewpoint of coarse geometry and geometric group theory.

In [Isl], I call a projective line geodesic $(a, b) \subset \Omega$ *rank one* if it is not contained in any half triangle, see the adjoining figure. If an element $\gamma \in \mathrm{Aut}(\Omega)$ acts by a translation along a rank one geodesic $\ell := (\gamma^+, \gamma^-) \subset \Omega$, then I call γ a *rank one automorphism*. My key insight in [Isl] is that a such rank one automorphism has a special coarse geometric 'contraction property'. Roughly speaking, this contraction property says: any metric ball disjoint from ℓ, projects to ℓ (under the closest-point projection for d_Ω) as a set of uniformly bounded diameter. The existence of such 'contracting projections' has the following geometric group theoretic consequence.



(a, b) in half triangle



(a, b) is rank one

Theorem 2.1 ([Isl])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ contains a rank one automorphism and is non-elementary (i.e. does not contain a finite index cyclic subgroup). Then Γ is an acylindrically hyperbolic group.*

Acylindrically hyperbolic groups [Osi16] are a broad class of groups with a lot of 'hyperbolicity' and includes Gromov hyperbolic groups, relatively hyperbolic groups, most mapping class groups, rank one CAT(0) groups, $\mathrm{Out}(F_n)$, etc. As an immediate application, we get an asymptotic counting result for conjugacy classes (roughly speaking, closed geodesics). The asymptotics are reminiscent of the formula obtained by Margulis in negative curvature. Here $\delta(\Gamma)$ is the critical exponent, i.e. the exponential growth rate of Γ-orbits in Ω.

Proposition 2.2 ([Isl])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is non-elementary, contains a rank one automorphism, and Ω/Γ is compact. Let $\mathrm{Per(t)}$ count the number of conjugacy classes in Γ of rank one automorphisms whose translation length in Ω is at most t. Then $\frac{e^{t\delta(\Gamma)}}{t} \sim \mathrm{Per}(t)$*[1].

In my paper [IW24] with Weisman, we look at the more general setup of 'hyperbolic' geodesics that are not necessarily the 'axis' of some element in $\mathrm{Aut}(\Omega)$. In this case, we interpret 'hyperbolicity' in the coarse geometric sense of *Morse geodesic rays*. Intuitively, a geodesic ray σ is *M-Morse* if any (K, C)-quasigeodesic with endpoints on σ, is within a distance $M(K, C)$ of σ. For example, in $\mathbb{R}^2 * \mathbb{R}$, a Morse ray is one that keeps switching between the two free factors, while an example of a non-Morse ray is one that stays in some copy of $\mathbb{R}^2$ forever. Our main result is that this coarse geometric notion of Morse-ness can be captured through the purely linear algebraic data of singular values. Let $\mu_i(g)$ denote the logarithm of the i-th singular value of g and let $\mu_{i,j}(g) := \mu_i(g) - \mu_j(g)$.

Theorem 2.3 ([IW24])**.** *Suppose $\sigma : [0, \infty) \to \Omega$ is a geodesic ray that is shadowed by a sequence $\{\gamma_n\}$ in $\mathrm{Aut}(\Omega)$, i.e. $(\sup_{n \in \mathbb{N}} \mathrm{d}_\Omega(c(n), \gamma_n x_0)) < \infty$ for any $x_0 \in \Omega$. If σ is M-Morse, then there exist $1 < \alpha(M)$ and $\beta(M) < \infty$ such that*

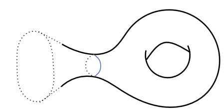
$$\liminf_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,d-1}(\gamma_n)} \geq \alpha(M) \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)} \leq \beta(M).$$

[1]We write $f(t) \sim g(t)$ if there is a constant $C > 0$ such that $(1/C)g(t) \leq f(t) \leq Cg(t)$ as $t \to \infty$.

The numbers $\alpha(M)$ and $\beta(M)$ have an interesting and purely geometric interpretation. Roughly speaking, it says that by choosing appropriate affine charts near $c(\infty)$, we can write the boundary $\partial\Omega$ as the graph of a convex function $f(x)$ that is sandwiched between $x^{\beta(M)}$ and $x^{\alpha(M)}$.

3. Convex Co-compact Groups and Relative Hyperbolicity

A prototypical example of a convex co-compact group in $\mathrm{SO}(2,1) < \mathrm{SL}_3(\mathbb{R})$ is a free group $\Gamma = \langle g, h \rangle$, where g, h are hyperbolic isometries in 'ping-pong' configuration (i.e. the fixed points of g, h are disjoint and both g, h translate points in $\mathbb{H}^2$ a lot). In this case, $\mathbb{H}^2/\Gamma$ is an infinite volume surface but it has a compact 'convex core'. The limit set of Γ is a Cantor set in $\partial\mathbb{H}^2$ and the convex hull $\mathcal{C}_{\mathbb{H}^2}(\Gamma)$ of this limit set has a co-compact Γ action.



$\mathbb{H}^2/\Gamma$

Similarly, a discrete group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ is called *convex co-compact* provided Γ preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and the *convex core* – $\mathcal{C}_\Omega(\Gamma)/\Gamma$ – is compact. Here, $\mathcal{C}_\Omega(\Gamma)$ is the convex hull in Ω of the limit set. We will write - $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group* - to convey that $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ acts convex co-compactly on Ω.

I started working on convex co-compact groups through my collaboration with Andrew Zimmer. Zimmer visited the University of Michigan in 2018 when I was a graduate student there. I had just finished working on my first paper [Isl] on rank one properly convex domains, so I had some expertise in studying non-Gromov hyperbolic groups acting on projective domains. Andrew talked about his result, that Gromov hyperbolic convex co-compact groups are essentially the same as Anosov subgroups of $\mathrm{PGL}_d(\mathbb{R})$ [Zim21]. I had a strong feeling that the geometric group theoretic tools that I had developed, could be useful in studying non-Gromov hyperbolic convex co-compact groups. I shared my enthusiasm with him and we started discussing the case of relatively hyperbolic groups as a first step in this direction. The discussion ended up taking the shape of a joint research program on relatively hyperbolic convex co-compact groups. We wrote five joint papers that I will now explain.

3.1. Projective Flat Torus Theorem [IZ21].

In order to study a non-Gromov hyperbolic convex co-compact group Γ, it is vital to understand its $\mathbb{Z}^k$ subgroups where $k \geq 2$. For a non-positively curved Riemannian manifold M, the flat torus theorem provides this understanding: any $\mathbb{Z}^k$ subgroup in $\pi_1(M)$ is the fundamental group of an immersed k-dimensional torus in M. This is an amazing theorem as it provides a recipe for constructing a totally geodesic flat – a geometric object – from an abelian subgroup – a purely algebraic object. But prior to our work, this powerful theorem was missing in convex projective geometry. In convex projective geometry, the analogues of totally geodesic flats are properly embedded simplices. In a properly convex domain Ω, $S \subset \Omega$ is a *properly embedded simplex* if S is a simplex in some affine chart and the inclusion map $S \hookrightarrow \Omega$ is proper. Our projective flat torus theorem says that a maximal abelian subgroup in a convex co-compact group Γ is the fundamental group of an immersed torus in $\mathcal{C}_\Omega(\Gamma)/\Gamma$.

Theorem 3.1 ([IZ21])**.** *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Gamma$ is a maximal abelian subgroup. Then there exists a properly embedded simplex $S_A \subset \mathcal{C}_\Omega(\Gamma)$ such that A preserves S_A and acts co-compactly on it.*

3.2. Relatively Hyperbolic Convex Co-compact Groups [IZ23, IZ22].

A group Γ is relatively hyperbolic with respect to its subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$ if the group Γ is 'hyperbolic' away from the conjugates of P_i and the only 'non-hyperbolic' regions, i.e. conjugates of P_i, interact in a very restricted manner. A classic example is a non-uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$, i.e. the fundamental group of a cusped hyperbolic 3-manifold. It is hyperbolic with respect to the cusp subgroups which are all isomorphic to $\mathbb{Z}^2$.

Abelian Peripherals. Our projective flat torus theorem enabled us to study relatively hyperbolic convex co-compact groups whose peripheral subgroups are isomorphic to $\mathbb{Z}^k$ with $k \geq 2$. By our theorem 3.1, each peripheral subgroup then acts co-compactly on a k-dimensional simplex in $\mathcal{C}_\Omega(\Gamma)$[2]. In [IZ23], we proved that relative hyperbolicity is precisely characterized by a geometric isolation property of these simplices. This isolation property (cf. 3.2) essentially says that the simplices do not intersect each other in $\mathcal{C}_\Omega(\Gamma)$. This non-intersection should be interpreted in a coarse sense, i.e. metric neighborhoods of simplices intersect in sets of uniformly bounded diameter. I state the definition for an arbitrary collection of subsets $\mathcal{X}$ in $\mathcal{C}_\Omega(\Gamma)$, not just simplices.

Definition 3.2. *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. A collection $\mathcal{X}$ of closed unbounded convex subsets of $\mathcal{C}_\Omega(\Gamma)$ is called* strongly isolated *if: for every $r > 0$ there exists $D_1(r) > 0$ such that for any $X_1 \neq X_2 \in \mathcal{X}$, $\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)\right) \leq D_1(r)$.*

Our main theorem in [IZ23] concerns strong isolation of $\mathcal{S}_{\max}$, the set of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least 2. A simplex is maximal if it is not contained in a simplex of bigger dimension.

Theorem 3.3 ([IZ23])**.** *Suppose $\Gamma < \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) Γ is a relatively hyperbolic group with respect to a family of subgroups isomorphic to $\mathbb{Z}^k$ with $k \geq 2$.

(2) $\mathcal{S}_{\max}$ is strongly isolated.

We were inspired by Hruska-Kleiner's work on CAT(0) spaces with isolated flats [HK05]. With very different methods, they had proven an analogous theorem in CAT(0) geometry. Although their CAT(0) tools did not work in our setting, their geometric ideas shaped our intuition. The challenging part in proving the above theorem was (2) implied (1). The strong isolation property only provides some metric geometric data in the form of an isolated family of subsets. Leveraging that to extract group theoretic properties clearly needed some fresh ideas.

During my work [Isl] on contraction properties in projective geometry, I had learnt about a characterization of relative hyperbolicity in terms of contracting projections [Sis13]. My idea was to use the closest-point projection for the Hilbert metric d_Ω and this characterization of relative hyperbolicity. However, this closest-point projection had a shortcoming - it depended on the boundary structure of the domain Ω and seemed quite unwieldy. Meanwhile, Andrew had defined a linear projection map for a simplex – a linear map that projects $\mathcal{C}_\Omega(\Gamma)$ onto a simplex. A priori, this projection seemed unrelated to the Hilbert metric. Our key insight was to establish a coarse equivalence between these two projections. Then we used the metric characterization of relative hyperbolicity [Sis13] to establish Theorem 3.3.

General Peripherals. In spite of the neat statement, we were not satisfied by the assumption on peripherals in Theorem 3.3. What if the convex co-compact group has a more complicated peripheral structure, say closed surface groups? In fact, a very recent work [BV23] has constructed such examples and confirmed that our question wasn't solely of intellectual interest. So, we were keen on a characterization of relative hyperbolicity without assumptions on peripherals. We achieved this in [IZ22]. Although it might sound like a simple extension of [IZ23], it is very much not so in reality! In particular, one of our key tools in [IZ23] – the linear projection maps – does not generalize for arbitrary peripheral subgroups. In fact, we had to discard our entire metric geometric viewpoint of [IZ23] and adopt the dynamical perspective on relative hyperbolicity [Yam04]. Before explaining the idea further, I will first state our theorem precisely. Given a convex co-compact group Γ, a *peripheral family for* Γ is a collection $\mathcal{X}$ of convex subsets of $\mathcal{C}_\Omega(\Gamma)$ that is:(a) Γ-invariant, (b)

[2] This implies that projective convex co-compact groups are very different from non-uniform lattices in $\mathrm{PSL}_2(\mathbb{C})$.

strongly isolated (cf. 3.2), and (c) *coarsely contains all simplices in* $\mathcal{C}_\Omega(\Gamma)$ (i.e. there is a constant D so that, any properly embedded simplex in $\mathcal{C}_\Omega(\Gamma)$ of dim ≥ 2 is inside the D-neighborhood of some $X \in \mathcal{X}$).

Theorem 3.4 ([IZ22])**.** *Suppose* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a convex co-compact group. Then the following are equivalent:*

(1) If Γ *is relatively hyperbolic with respect* $\mathcal{P} = \{P_1, \ldots, P_m\}$*, then* $\mathcal{X} := \Gamma \cdot \{X_1, \ldots, X_m\}$ *is a peripheral family for* Γ *where each* X_j *is the convex hull of the limit set of* P_j *in* Ω*.*

(2) If $\mathcal{X}$ *is a peripheral family for* Γ *and* $\mathcal{P} := \{P_1, \ldots, P_m\}$ *is a set of representatives of the* Γ*-conjugacy classes in* $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$*, then* Γ *is relatively hyperbolic with respect* $\mathcal{P}$*.*

As in the proof of Theorem 3.3, the main challenge was (2) implied (1). Our key idea in [IZ22] was to construct a topological quotient of the boundary of $\mathcal{C}_\Omega(\Gamma)$ and study the dynamics of Γ on it. We worked with the ideal boundary of $\mathcal{C}_\Omega(\Gamma)$, i.e. $\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma) := \overline{\mathcal{C}_\Omega(\Gamma)} \cap \partial\Omega$. Given a set of subgroups $\mathcal{P} := \{P_1, \ldots, P_m\}$ of a convex co-compact group Γ, we defined the quotient space $[\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}$ by collapsing the limit set of each $\gamma P_i \gamma^{-1}$ to a point. Here $\gamma \in \Gamma$ and $P_i \in \mathcal{P}$. When $\mathcal{P}$ is a peripheral family, $[\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}$ is a nice space (compact, Hausdorff) with a particularly nice Γ-action (geometrically finite convergence group action). Then Yaman's results [Yam04] implied that Γ is relatively hyperbolic with peripherals $\mathcal{P}$. We were inspired by the related work of Weisman [Wei23a]. As an immediate consequence of this proof strategy, we get a topological model for the abstract Bowditch boundary $\partial(\Gamma, \mathcal{P})$ of the group Γ.

Corollary 3.5 ([IZ22, Wei23a])**.** *If* $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group relatively hyperbolic with respect to* $\mathcal{P}$*, then there is a* Γ*-equivariant homeomorphism between the Bowditch boundary* $\partial(\Gamma, \mathcal{P})$ *and* $[\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}$*.*

3.3. **Classification and Examples.** Relatively hyperbolic convex co-compact groups have several examples from diverse areas of geometry, for instance from linear Coxeter reflection groups [Ben06, DGK+21], cusp deformations of finite volume hyperbolic manifolds [BDL18], and bending constructions [BV23]. But does this mean that all convex co-compact groups are relatively hyperbolic (barring free products and amalgamations)? To investigate this natural question, Andrew and I looked at certain families of test cases.

Three manifold groups [IZ24b]**.** Consider a closed irreducible 3-manifold Y and a convex co-compact group $\Gamma < \mathrm{PGL}_d(\mathbb{R})$ (d arbitrary) isomorphic to $\pi_1(Y)$. The most basic case is when Y supports one of the eight Thurston geometries, i.e. Y is a geometric 3-manifold. In this case, we proved that Γ must either be $\mathbb{Z}^3$, or a uniform lattice in $\mathrm{SO}(3,1)$, or the product of $\mathbb{Z}$ with a closed surface group. The key interesting case is when Y is non-geometric, i.e. obtained by gluing several geometric pieces along tori or Klein bottle (as given by the famous geometric decomposition theorem for 3-manifolds). Our main result was that $\pi_1(Y)$ being isomorphic to a convex co-compact group forces strong restrictions on the topology of Y. In particular, Y must be built out of gluing only hyperbolic pieces along the tori or Klein bottle.

Theorem 3.6 ([IZ24b])**.** *Suppose* Y *is a closed non-geometric 3-manifold and* $\pi_1(Y)$ *is isomorphic to a convex co-compact group in* $\mathrm{PGL}_d(\mathbb{R})$*. Then each component in the geometric decomposition of* Y *supports a* $\mathbb{H}^3$*-geometric structure and* $\pi_1(Y)$ *is a relatively hyperbolic group with* $\mathbb{Z}^2$ *peripherals.*

One Dimensional Faces [IZ24a]**.** Suppose Y is a non-geometric 3-manifold as in Theorem 3.6 above and $\Gamma \cong \pi_1(Y)$ acts convex co-compactly on Ω. Then our work [IZ24b] imposes strong restrictions on the boundary faces[3] of the convex domain Ω. The restriction is that the boundary face $F_\Omega(x)$ of any $x \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ is at most one-dimensional. Hence, we conjectured: this bound on

[3]The boundary face of $x \in \partial\Omega$ – denoted $F_\Omega(x)$ – is the union of all open projective lines in $\partial\Omega$ that contain x.

the dimension of boundary faces is sufficient to force relative hyperbolicity. In [IZ24a], we verified this conjecture.

Theorem 3.7 ([IZ24a])**.** *Suppose* $\Gamma < \mathrm{Aut}(\Omega)$ *is a convex co-compact group. Then* Γ *is a relatively hyperbolic group with* $\mathbb{Z}^2$ *peripherals if and only if* $\dim F_\Omega(x) \leq 1$ *for each* $x \in \partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Gamma)$.

4. Local Rigidity of Boundary Actions

As I outlined in Section 1, I got interested in studying discrete subgroups using boundary actions toward the end of my Ph.D. As a first step, I was curious about the case of lattices in higher rank groups, e.g. $\mathrm{SL}_d(\mathbb{R})$ with $d \geq 3$. I was still a Ph.D. student at the University of Michigan and I started discussing with Ralf Spatzier, my Ph.D. advisor. In the 90s, Katok and Spatzier had asked the same question, but for C^1-deformations (i.e. the deformation modifies the generators of the lattice slightly without changing the derivatives too much). In [KS97] (independently [Kan96]), they had proven a rigidity result – deformations that are C^1-close are in fact conjugate. In other words, C^1-small deformations are the same as the original action. The question about C^1-deformations, in fact, goes back to Sullivan [Sul85]. He had proven an analogous rigidity result but in the complementary case of rank one groups, e.g. uniform lattices in $\mathrm{SL}_2(\mathbb{R})$. Very recently, some global rigidity results have been proven in this C^1-case for higher rank lattice actions [BHW24].

In the C^1-case, the main tools come from smooth dynamics, see e.g. [KS97]. Thus, such an approach would not work for C^0-deformations, the case that I was interested in. We drew inspiration from [BM22] that had just appeared. Bowden-Mann [BM22] worked with C^0-deformations, although only for negatively curved manifolds – very far away from our case of higher rank lattices. Ralf Spatzier had already been discussing with Chris Connell and Thang Nguyen other questions in the same spirit, i.e. rigidity properties in symmetric spaces. So we joined forces and started working together on the project.

I will explain our work [CINS23] in the case of $G = \mathrm{SL}_3(\mathbb{R})$, but our results hold for more general higher rank Lie groups, see Theorem 4.1. In this case, a lattice $\Gamma < G$ acts on the rank two *Riemannian symmetric space* $X := \mathrm{SL}_3(\mathbb{R})/\,\mathrm{SO}(3)$. There are several notions of boundaries of X. For instance, the full flag space of $\mathbb{R}^3$ is called the *Furstenberg boundary* of X. More generally, the partial flag spaces are called the *generalized Furstenberg boundaries* of X. We will always denote by ρ_0 the action of a lattice $\Gamma < G$ on such boundaries. The *visual boundary* of X, denoted by ∂X, gives a different notion of boundary. It is the visual boundary of X with respect to a complete non-positively curved G-invariant Riemannian metric on X (given by the Killing form). We will always denote the action of Γ on ∂X by ρ_{v}.

Before proceeding with a discussion about deformations of actions, it is imperative that I explain the topology of the space of actions. Given a lattice $\Gamma < G$, fix a finite generating set S of Γ. Suppose F is a boundary of X equipped with a metric d_F. We will say that two actions ρ and θ, i.e. homomorphisms from Γ to $\mathrm{Homeo}(F)$, are C^0-close if $\rho(s)$ and $\theta(s)$ are close (as continuous maps on F) for each generator $s \in S$.

We say that θ is a *topological factor* of ρ if there is a continuous surjective map φ on F such that the adjoining diagram commutes for all $\gamma \in \Gamma$. The map φ is called a *semi-conjugacy*. Note that this is weaker than a conjugacy. Two actions ρ and θ are *conjugate* if φ is a homeomorphism.

$$\begin{tikzcd} F \arrow[r, "\rho(\gamma)"] \arrow[d, "\varphi"] & F \arrow[d, "\varphi"] \\ F \arrow[r, "\theta(\gamma)"] & F \end{tikzcd}$$

I now give an example to illustrate that semi-conjugacy, and not conjugacy, gives the right notion of rigidity in the context of C^0-actions. I will consider $\mathbb{Z}$ actions on S^1. Consider $g \in \mathrm{PSL}_2(\mathbb{R})$ that acts on S^1 with north-south dynamics, i.e. g has an attracting and a repelling fixed point $g^\pm$ on S^1. Now consider $h \in \mathrm{Homeo}(S^1)$ that acts by fixing tiny intervals $\mathcal{I}^\pm$ around $g^\pm$ pointwise and the same north-south dynamics as g outside of $\mathcal{I}^\pm$. Then h is C^0-close to g but cannot be conjugated to it as h has uncountably many fixed points. However, there is a semi-conjugacy. The map that collapses $\mathcal{I}^{\pm}$ to $g^{\pm}$ intertwines the actions by g and h.

Rigidity in generalized Furstenberg boundaries. In [CINS23], we investigate the question: what happens if we deform the boundary actions ρ_0 and ρ_{v} of a higher rank lattice Γ in the category of C^0-actions? For the actions ρ_0 on the generalized Furstenberg boundaries, we prove that any C^0-close deformation ρ has ρ_0 as a topological factor. Intuitively, this means that ρ differs from ρ_0 only in the sense that finitely many Γ-orbits could have 'blown up' during the deformation. But otherwise, the action hasn't changed.

Theorem 4.1 ([CINS23])**.** *Let G be a connected linear semi-simple Lie group without compact factors, Γ be a uniform lattice in G, Q be a parabolic subgroup of G, and ρ_0 be the standard boundary action of Γ on G/Q. Then for any action ρ that is sufficiently C^0-close to ρ_0, ρ_0 is a topological factor of ρ. Moreover, the semi-conjugacy φ_ρ converges to* id *uniformly as ρ converges to ρ_0.*

Can we get better than topological factors and semi-conjugacies? In [CINS23], we show that the answer is 'no'. Via a 'Denjoy-like' construction, we produce actions ρ that are C^0-close to ρ_0 but are not conjugate to ρ_0. Intuitively, the idea is to blow up a Γ orbit. While such blow-up constructions are standard in dynamics, our main contribution is finding a blow-up method that does not alter the topology of the underlying manifold.

Proposition 4.2 ([CINS23])**.** *Suppose θ_0 is a C^1-action of a countable group Λ on a compact smooth manifold F with a dense orbit. Then there exist actions θ arbitrarily C^0-close to θ_0 but not conjugate to θ_0.*

Key ideas underlying Theorem 4.1. I will now discuss some ideas that go into proving Theorem 4.1. The first idea (or step) involves 'taming' the wild deformations ρ of ρ_0. This step is differential geometric in nature where we construct some leaf-wise C^1 'foliations'. These foliations are the 'tame' objects that encode the 'wild' actions ρ of the lattice. To make this a little more precise, consider $E := M\backslash G/\Gamma$ where M consists of diagonal matrices with diagonal entries ± 1. The space E is a fiber bundle over X/Γ with fibers G/P, where P consists of upper triangular matrices. For each action ρ of Γ on G/P close to ρ_0, we construct a 'foliation' $\mathcal{H}_\rho$ of E. When $\rho = \rho_0$, the foliation $\mathcal{H}_{\rho_0}$ is very special. It is the center-stable foliation of an algebraic flow on E called the Weyl chamber flow (a higher rank analogue of the geodesic flow on the unit tangent bundle of a closed hyperbolic surface). When ρ is a deformation of ρ_0, we interpret the 'foliation' $\mathcal{H}_\rho$ as a perturbation of the dynamical foliation $\mathcal{H}_{\rho_0}$. This $\mathcal{H}_\rho$ is a more tame object than ρ, since it is related to an algebraic flow! The construction of this 'foliation' is inspired by [BM22]. However, their setup is quite different – they construct foliations on the unit tangent bundle of a negatively curved manifold and relate it to the geodesic flow. A particularly interesting part of our construction is the use of barycenters (i.e. center of mass of measures) for constructing these 'foliations'. I use the word 'foliation' as it conveys the idea well; the actual construction is involved and only yields leaf-wise immersions.

The second idea (or step) is to use coarse geometry and 'straighten' the above foliations $\mathcal{H}_\rho$ of E. This straightening essentially pops out a continuous semi-conjugacy between the actions ρ and ρ_0. Coarse geometry enters the picture via theorems about quasi-isometric embeddings of flats into higher rank symmetric spaces. I find this step amazing because it uses coarse geometry to prove a completely dynamical result. This is our key insight - using coarse geometry to study problems in the C^0-category. The coarse geometry of higher rank symmetric spaces (e.g. higher rank Morse lemma, quasi-flats theorem, etc.) is an area of my core expertise and is one of the key reasons behind my interest in this problem. We again draw inspiration from [BM22] which uses convergence group actions and the Morse lemma for Gromov hyperbolic groups. Higher rank lattices are very far from being Gromov hyperbolic, but the coarse geometry viewpoint proves to be fruitful nonetheless.

Non-rigidity in the visual boundary. To wrap up the discussion, I want to discuss a non-rigid situation that we discover in [CINS23]. We discover that this dynamical rigidity vanishes in the visual boundary. Consider a lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$ and its action ρ_{v} on the visual boundary of $X = \mathrm{SL}_3(\mathbb{R})/\operatorname{SO}(3)$. We construct arbitrarily small deformations of ρ_{v} that do not have ρ_{v} as a topological factor. Our key idea is to exploit the spherical building structure on ∂X and construct deformations that destroy the G-orbit structure on ∂X.

The spherical building structure on ∂X tells us that ∂X is constructed by gluing together circles (i.e. visual boundaries of flats in X) and each circle is obtained by gluing closed intervals (i.e. visual boundaries of sectors in a flat). Owing to this combinatorial structure, we can construct a deformation ρ of ρ_{v} by specifying it on a single closed interval. The action ρ on a closed interval is simple - it fixes the center of the interval and dilates around it. The challenge is to coherently control the dilation amount depending on which element of Γ is acting. Our idea is to do this using a cocycle on the Furstenberg boundary of X. For this, we use the well-known Radon-Nikodym derivative cocycle for the Lebesgue measure class. In fact, by taking powers of the cocycle, we produce a one-parameter family of deformations ρ that converges to ρ_{v}.

Theorem 4.3 ([CINS23])**.** *Suppose X is a simply connected higher rank symmetric space of non-compact type, G is the identity component in $\operatorname{Isom}(X)$, $\Gamma < G$ is a lattice, and ρ_{v} is the natural action of Γ on ∂X. Then, there exist actions ρ, arbitrarily C^0-close to ρ_{v}, such that ρ_{v} is not a topological factor of ρ.*

5. Ongoing Work and Future Directions

5.1. **Totally geodesic submanifolds (TGS) in higher rank.** A submanifold N of a Riemannian manifold M is called totally geodesic if, for any pair of points in N, the geodesic joining them (in the metric on M) lies entirely in N. Totally geodesic submanifolds (TGS) are objects of classical interest in geometry and dynamics, with tradition dating back to Riemann [Rie54] and Cartan [Car28].

The simplest example of a TGS is of course the one-dimensional ones, i.e. geodesics, and they are found in aplenty. Moreover, if M is a compact manifold with infinite fundamental group, then we can find infinitely many closed geodesics. However, TGS-s become a lot more rare when we look for them with the dimension of the TGS strictly greater than 1. In fact, guided by some recent results that we will explain below, the emerging consensus is that plenty of TGS-s of dimension greater than 1 can only arise under stringent restrictions on the manifold like homogeneity and arithmeticity. To simplify our current discussion, we will only focus on codimension 1 TGS-s, i.e. $\dim(TGS) = \dim(M) - 1$.

Context. Consider a closed hyperbolic 3-manifold M, i.e. $\pi_1(M)$ is a uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$. Then, each closed totally geodesic submanifold $N \subset M$ corresponds to a Fuchsian subgroup in $\pi_1(M)$, i.e. the subgroup $\pi_1(N)$ can be conjugated to a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$. Motivated by questions of Reid and McMullen, there has been much progress in understanding such closed TGS-s. There are now examples of M that contain exactly k closed totally geodesic N for any non-zero k [FLMS21, LP22]. However, the existence of infinitely many closed TGS $\{N_1, N_2, \dots\}$ in M forces extra-ordinary rigidity conditions of a number-theoretic nature. More precisely, [BFMS21] and [MM22] show that in this case, M must be an arithmetic hyperbolic 3-manifold. In recent work, [FFL24] has generalized this result to the case where M carries an analytic metric of variable negative curvature.

In another direction, Kahn-Markovic [KM12] had shown that any closed hyperbolic 3-manifold M contains immersed surfaces N that are nearly-totally geodesic. In geometric group theory language, $\pi_1(N)$ is a quasi-convex subgroup of $\pi_1(M)$. The existence of plenty of quasi-convex subgroups is connected to cubulability of the group $\pi_1(M)$ [BW12]. Note that a group is cubulable if it acts nicely on a CAT(0) cube complex. Cubulability has a variety of applications: from Agol's proof of virtual Haken and fibering theorem [Ago14] to providing a strong negation of Property (T) [CCJ+01, CMV04].

Research Projects. The above results indicate that in rank one Lie groups, e.g. $\mathrm{PSL}_2(\mathbb{C})$, the study of TGS-s unlock a wealth of information. My research goal is to initiate a similar study of TGS-s in the context of discrete subgroups of higher rank Lie groups. Moving to higher rank produces a variety of unique challenges. For instance, the Riemannian symmetric space now – unlike $\mathbb{H}^3$ – is non-positively curved with plenty of flats. Moreover, the TGS-s of a higher rank symmetric space are highly constrained by Lie theory.

At this juncture, convex projective geometry proves to be a great boon. The Hilbert metric, although not CAT(0), has a wide variety of TGS-s that have a simple description. In particular, for a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, any codimension 1 projective hyperplane in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ that intersects Ω is a codimension 1 TGS of Ω in the d_Ω-metric. Thus, I adopt the viewpoint of studying discrete groups in higher rank via studying TGS-s in properly convex domains. Under this broad research umbrella of understanding TGS-s in higher rank, I will discuss two focused projects that I am currently invested in.

• **Haagerup property or Gromov's a-T-menability:** My goal here is to use TGS-s to establish the Haagerup property for a large class of discrete subgroups in higher rank. This question is due to Benoist[Ben12]. As a starting point, we consider discrete groups $\Gamma \leq \mathrm{SL}_d(\mathbb{R})$ that preserve and act co-compactly on trictly convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. In ongoing joint work with T. Nguyen [IN], we are proving that any such Γ is a-T-menable, provided its critical exponent is at least 1.

To discuss our strategy, it is useful to recall the proof of a-T-menability for any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. First, one constructs the so-called Crofton measure on the space of all codimension 1 TGS-s of $\mathbb{H}^2$. Then the Crofton formula provides a recipe for recovering the Riemannian distance $d_{\mathbb{H}^2}$ solely from the Crofton measure. This Crofton formula becomes the key to proving a-T-menability for Fuchsian groups [CCJ+01]. This inspires our method in [IN]. We consider $\mathcal{H}$, the set of codimesion 1 projective hyperplanes in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ that intersect Ω. As noted above, $\mathcal{H}$ is precisely the set of such codimension 1 TGS-s in Ω for the Hilbert metric. We introduce a Crofton-like measure on $\mathcal{H}$ and establish several properties that are reminiscent of the Crofton formula. While we do not get an exact Crofton formula, the properties of our measure are strong enough to imply a-T-menability.

• **Rigidity and homogeneity:** This is a project that I want to pursue in the immediate future. In brief, my goal is to show that plenty of closed TGS-s force the Hilbert metric to become a homogeneous Riemannian metric, akin to the situation in [FFL24] alluded to above. However, in contrast to the negative curvature (i.e. rank one) setting of [FFL24], my work will be in the context of higher rank groups, thus introducing some unique challenges.

To state my target theorem more precisely, consider $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$ that preserves a strictly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ and acts co-compactly on Ω. Then $M := \Omega/\Gamma$ is a closed strictly convex projective manifold of dimension d. Assume that M has infinitely many codimension 1 closed TGS-s $\{N_1, N_2, \dots\}$ where each N_i is 'hyperbolic', i.e. the lift $\widetilde{N_i}$ of N_i in the universal cover Ω is an open $(d-1)$ ball so that $\widetilde{N_i}$ is isometric to $\mathbb{H}^{d-1}$. My conjecture is that this will force Ω itself to be a projective d-ball, i.e. Ω is isometric to $\mathbb{H}^d$. In particular, the result of [BFMS21] would then imply that the group Γ that we started with is in fact an arithmetic (uniform) lattice inside some conjugate of $\mathrm{SO}(d,1)$.

To prove this conjecture, my strategy is to import tools from homogeneous dynamics into convex projective geometry. First consider a sequence of measures $\{\mu_i\}$ supported on $\{N_i\}$ and consider an accumulation point μ_∞ of these measures. The compactness of M would enable us to prevent any mass-escape phenomenon. The idea is to then use a Ratner-like theorem [Rat91, MS95] to get a homogeneous action on the support of the measure μ_∞, and hence on some subset of Ω. The next step would be to use projective geometry and upgrade this to a homogeneous action on Ω.

5.2. **Geometric finiteness in projective geometry and representations beyond Anosov.** In hyperbolic geometry, *geometrically finite(GF) subgroups of* $\mathrm{SO}(d,1)$ encompass the broadest class of discrete subgroups that can be studied well using tools from geometry and dynamics. To get a sense of such groups, let us look at $\mathrm{SO}(2,1)$. A discrete subgroup $\Gamma \leq \mathrm{SO}(2,1)$ is GF if the surface $\mathbb{H}^2/\Gamma$ is a geometrically finite, i.e. it decomposes as the union of a compact surface with boundary and a union of cusps and funnels (see Figure). This topological decomposition has an equivalent, albeit dynamical, reinterpretation [Bow93]. The dynamical definition says, Γ is GF if it has convergence dynamics (a generalization of 'north-south' dynamics) on its limit set $\Lambda_\Gamma \subset \partial\,\mathbb{H}^2$, and each point in Λ_Γ is either bounded parabolic (cusp point) or conical (endpoint of a geodesic lying in the compact part) [Yam04].

Context. In stark contrast to the rank one case above, the notion of geometrical finiteness in higher rank groups (e.g. $\mathrm{SL}_d(\mathbb{R})$ with $d \geq 3$) has remained elusive. In an ongoing joint research project [FIZ24] (with B. Fléchelles and F. Zhu; scheduled to be finished by the end of 2024), I am developing a theory of geometrically finite subgroups of $\mathrm{SL}_d(\mathbb{R})$ and fill this gap. A major challenge in this work is to control the behavior near cusp points. In fact, as Bowditch showed in [Bow95], cusp points already prove to be a delicate issue even in variable negative curvature. Our goal is to use convex projective geometry to tame the cusp regions and get a good theory of geometric finiteness in higher rank.

The importance of our work [FIZ24] is that we develop geometric tools to study *rel-Anosov representations*, a broad generalization of Anosov representations. While Anosov representations dealt exclusively with Gromov hyperbolic groups, current research has pushed the boundary to relatively hyperbolic groups. For relatively hyperbolic groups, researchers have identified several such interesting families of representations that I will informally club together under *rel-Anosov representations* [KL18, Zhu22, ZZ23, Wei23b]. On the other hand, Anosov representations are now widely regarded as a good (higher rank) generalization of convex co-compact subgroups of $\mathrm{SO}(2,1)$. So, it is natural to expect that these *rel-Anosov representations* generalize geometrically finite subgroups of $\mathrm{SO}(2,1)$. However, in my opinion, this geometric perspective on *rel-Anosov representations* is completely missing in all of the current approaches that I mentioned above. For instance, there is no topological decomposition result characterizing rel-Anosov representations akin to the $\mathrm{SO}(2,1)$ case discussed at the start of this subsection. Our work [FIZ24] fills this gap in current understanding.

Research Project. Our project [FIZ24] has two phases: first, we introduce a notion of geometric finiteness for subgroups of $\mathrm{SL}_d(\mathbb{R})$ (using convex projective geometry), and second, we identify a new family of discrete groups in $\mathrm{SL}_d(\mathbb{R})$ that encompass the geometrically finite ones. We christen this new family as *asymmetrically Anosov subgroups* and propose them as the right candidate for being *rel-Anosov*, owing to their geometric origin. We can already prove that *asymmetrically Anosov* subgroups are stable under certain small deformations (with some mild conditions on deformations of the 'cusps'). The broadest class of deformations that they can admit is the object of our current investigation.

To discuss our result in more details, consider a discrete group $\Gamma < \mathrm{SL}_d(\mathbb{R})$ that preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and whose limit set $\Lambda_\Gamma \subset \partial\Omega$ is visible (i.e. for any $x \neq y \in \Lambda_\Gamma$, the projective line segment joining x and y lies in Ω). Further, assume that the group Γ is relatively hyperbolic with respect to parabolic subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$. Then, following [Bow95], we say that Γ is *geometrically finite* if it acts on Λ_Γ as a convergence group and every limit point in Λ_Γ is bounded parabolic or conical. We discover an interesting distinction between conical and parabolic points: while a conical point is C^1 (i.e. there is a unique supporting hyperplane of the domain Ω at that point), the parabolic points often have non-unique supporting hyperplanes. This distinction necessitates the introduction of a new class of *rel-Anosov representations* that is not yet in the literature. These are our *asymmetrically Anosov representations* and I will now discuss them.

Since the group Γ above is relatively hyperbolic, it has an abstract Bowditch boundary denoted by $\partial_{\mathcal{P}}\Gamma$. By results of [Yam04], we have a 'limit map' $\xi : \partial_{\mathcal{P}}\Gamma \to \Lambda_\Gamma \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, an equivariant homeomorphism that identifies the Bowditch boundary with the limit set. The duality relation, i.e. taking supporting hyperplane to Ω at a point on $\partial\Omega$, then produces a 'dual limit map' ξ^* taking values in $\mathbb{P}(\mathbb{R}^{d*})$. However, only conical points have unique supporting hyperplanes. Thus the 'dual limit map' is only partly well-defined, i.e. $\xi^* : (\partial_{\mathcal{P}}\Gamma)_{con} \to \mathbb{P}(\mathbb{R}^{d*})$ is defined only at conical points. This is a marked departure from the case of Anosov and many of the other contending definitions of rel-Anosov representation; they always work with both limit maps defined everywhere. Our definition of *asymmetrically Anosov representations* is an abstraction of this picture above. We have a limit map (identifying the limit set) and a dual limit map defined only at conical limit points, with appropriate compatibility conditions.

5.3. **Higher rank hyperbolicity in projective geometry.** My work [Isl] shows that rank one properly convex domains have a lot of 'hyperbolicity'. However, there are plenty of domains that are not rank one and hence, lack all 'hyperbolicity' in a strong sense. To digest this point, consider Pos_n, the space of positive definite n-by-n matrices with trace 1 and $n \geq 3$. This space Pos_n is the projective model of the symmetric space $X_n := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. As is well-known, X_n is very 'non-hyperbolic' for any $n \geq 3$.

However, it has recently been discovered that even when $n \geq 3$, the higher rank globally symmetric space X_n – equipped with its homogeneous Riemannian metric – exhibit a host properties that are now being popularly called 'higher rank hyperbolicity' [KL20]. An informal way to explain this notion is that, X_n satisfies a higher-rank Morse lemma above the dimension $(n-1)$, where $(n-1)$ is the maximal dimension of any flat in X_n. That is, any quasi-isometric embedding of $\mathbb{R}^{n-1}$ into X_n stays close to an actual flat (much like a quasi-geodesic in $\mathbb{H}^2$ staying close to a geodesic).

Inspired by this higher rank hyperbolicity phenomena, we asked: does Pos_n – with its Hilbert metric – also satisfy some notion of higher rank hyperbolicity? In spite the amazing results of [KL20] at our disposal, this is a complicated question. The Hilbert geometry of Pos_n is starkly different from the Riemannian geometry of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ when $n \geq 3$. In particular, much of the technology in [KL20] (and related work) uses some form of convexity of the distance function and the distance function d_Ω is far from being convex.

Research Project. In ongoing joint work with G. Raggo [IR], we have recently discovered a geometric approach to answering this question. In fact, we can address an even more general question: do 'higher rank' convex projective domains [Zim20] satisfy some notion of higher rank hyperbolicity? Our idea is to consider a higher dimensional "slim k-simplex" condition to charaterize higher rank hyperbolicity. The notion of slim simplices is inspired by results in [KL20, GL23] and is a higher dimensional generalization of Gromov's slim triangle condition, used in the theory of Gromov hyperbolic metric spaces [BH99]. The slim simplex condition proves to be extremely fruitful in convex projective geometry. To be more precise, in [IR], we consider a properly convex domain Ω such that Ω/Γ is compact and let m be the maximal dimension of any projective simplex (properly embedded) in Ω. Then, we prove that any projective $(m+1)$-simplex in Ω is D-thin for some constant D, independent of the simplex [IR].

Our immediate next goal is to use the slimness of $(m+1)$-simplices to define a *coarse median* for $(m+1)$-simplices. Using this as a motivation, we plan to introduce the notion *coarse k-median spaces* generalizing the notion of *coarse median spaces* that works for triples [Bow13, NWZ19, BL23]. Our work would imply that the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ has the structure of a coarse $(n-1)$-median space. This will answer a question raised in [BL23], where the authors prove a 2-median property for CAT(0) polygonal 2-complexes.

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