Groups and cubulations
GGT Seminar January 2026

1 Introduction to CAT(0) cube complexes

Basic definition of cube complexes, criterion for when they are CAT(0), basic examples.

Hyperplanes, spaces with walls.

Right-angled Artin and Coxeter groups.

References:
– Bridson-Haefliger
– Sageev's notes at: https://www.math.utah.edu/pcmi12/lecture_notes/sageev.pdf

2 Cubulations and partially ordered sets/spaces with walls

The idea is to go from an action on a space with walls to an action on a "dual" CAT(0) cube comples. I believe Sageev calls this space with walls a pocset in his lecture notes. We could read this stuff:

– either from Lecture 1 (Sec 3) and Lecture 2 (Sec 1,2) of Sageev's notes. Discuss examples from Lecture 2 Section 3.

– alternatively, we could follow the paper of Chatterji-Niblo titled , available at https://arxiv.org/abs/math/0309036.

3 Hyperbolic 3-manifolds, quasi-convex surface subgroups, subgroup separability

This could roughly be based on Lecture 2 of Sageev's notes cited above. The focus will be on Section 4.

Emphasize on the case of fundamental groups of compact hyperbolic 3-manifolds. Discuss quasi-convex surface subgroups in such hyperbolic 3-manifold groups. Explain with examples. State Kahn-Markovic's result that gives plenty of quasi-convex surface subgroups.

A good case to see could be hyperbolic 3 manifolds fibering over the circle. Give examples of non-quasi-convex surface subgroups.

Discuss the notions of subgroup separability and some related properties of hyperbolic 3 manifolds.

References: TBD.

4 Arithmetic hyperbolic manifolds

Introduce the notion of arithmetic lattices of $\mathsf{SO}(n,1)$. Discuss some explicit constructions. Discuss the ways to detect when an arithmetic lattice is uniform vs non-uniform.

Possible references: The book by Maclachlan-Reid titled 'The Arithmetic of Hyperbolic 3-manifolds'. A secondary reference could be some specific sections of Dave Witte Morris's book on Arithmetic Groups, where he discusses some examples of arithmetic lattices in $\mathsf{SO}(Q)$.

5 Cubulating arithmetic hyperbolic manifolds

This talk will be based on: *Hyperplane sections in arithmetic hyperbolic manifolds* of Bergeron-Haglund-Wise. The url is: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/jlms/jdq082

The first goal is Theorem 1.10. In particular, the Step 1 in the proof Theorem 1.10 (see the remarks after Theorem 1.10). The second goal is to discuss how Theorem 1.10 gives a host of other conclusion about the homology of the underlying manifold and the separability properties.

6 Gromov Thurston manifolds

These are examples of negatively curved manifolds that cannot carry hyperbolic metrics. Classical reference is Gromov-Thurston's original paper titled *Pinching constants for hyperbolic manifolds*.

Available at https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/755.pdf.

7 Cubulating Gromov-Thurston manifolds

Giralt's proof that Gromov-Thurston manifolds are cubulated.

Reference: https://ems.press/content/serial-article-files/29822.

8 Geometric condition for cubulation

Discuss Bergeron-Wise's geometric condition on cubulability via quasiconvex subgroups. The paper is *A boundary criterion for cubulation*, available at: https://www.jstor.org/stable/23240564.

9 Markovic's paper on a criterion for Canon conjecture

Goal is to read Markovic's paper connecting Canon conjecture to abundance of quasi-convex surface subgroups.

Reference: https://link.springer.com/article/10.1007/s00039-013-0228-5.

Below are some optional topics and directions we could take after we have learnt the basics as above.

10 Definitions of special cube complex and virtual specialness

Possbly, Lecture 4 of Sageev's notes.

11 Sketch of Agol-Wise's proof of Virtual fibering theorem

A good starting point maybe Lorenzo Ruffoni's Master's thesis: https://amslaurea.unibo.it/id/eprint/5637/1/ruffoni_lorenzo_tesi.pdf.

12 Cubulation and obstruction to Property T

The introductory case could come from Bekka-de la Harpe's book of Property T. Discuss the Crofton formula of H^n.

Then discuss how the hyperplane structure for CAT(0) cube complexes aid in a coarse version of a Crofton formula. This second part could be based on https://chatterj.perso.math.cnrs.fr/papers/CDHMedian.pdf.

13 Boundaries of cube complexes

Hagen's paper comparing the simplicial boundary to the other ones, like Roller, etc.: https://www.wescac.net/SimplicialBoundaryRevisedAGT_318fix.pdf

14 Discussion on November 20

'3-manifold groups' by Aschenbrenner-Friedl-Wilton https://ems.press/content/book-files/23406 : Focusses on 3-manifod *groups* but has a nice outline with references of getting started on 3-manifolds and then move onto Agol-Wise Virtual Fibering.

Also these notes by Hatcher seem to be a good place to read proofs of decomposition thms for 3-manifolds (including JSJ) and the Loop and Sphere theorem. https://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf