PROSEMINAR/ SEMINAR: MATRIX (LIE) GROUPS

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Meeting Time: TBD
Language: The talks are supposed to be in English.

Abstract

This is a seminar targeted at mid-level as well as advanced undergraduate students. The goal of the course is to introduce some basic ideas in Lie theory through matrix groups. There will be a strong emphasis on looking at plenty of examples of matrix groups.

The first part of the course will be a fairly general introduction to matrix groups and their Lie algebras - both as algebraic objects and as tangent spaces at identity to the respective groups. In the second part, we will focus on compact matrix groups and discuss their structure theory. At the end, we will cover the classification theorem of compact semi-simple Lie groups and look at some concrete examples.

Recommended Pre-requisite

Linear Algebra, Analysis (Basic topology in $\mathbb{R}^n$ and the notion of derivatives), Basic abstract algebra (group theory).

Roughly speaking, if you feel comfortable with the first three chapters of Reference (1) below, this class is well-suited for you.

Key References

(1) Matrix Groups for undergraduates, Kristopher Tapp (AMS). (**for talks 1-6, 9**)
(2) Matrix Groups: an Introduction to Lie Group Theory, Andrew Baker, Springer, 2002. (**for talks 7,-12**)

LIST OF TALKS

1. Introduction to Matrix Groups and Examples

We will introduce families of matrix groups which will serve as the main examples throughout the course. Some of the main classes of examples that we will discuss are: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$, orthogonal groups $O(n), U(n), SO(n), SU(n)$. We will also discuss some low-dimensional examples: geometric visualizations, accidental group isomorphisms in low dimensions, etc.

Give examples to show that not all groups are matrix groups!

2. Topology of Matrix Groups

Discuss the topology on matrix groups - open sets, closed sets, limit points, homeomorphisms, connectedness, compactness. Give examples for each of these in the context of matrix groups.

Date: February 2023.

3. Lie Algebra

Define the Lie algebra of a matrix group. Give some examples. Then define of tangent vectors and tangent space for open subsets of Euclidean spaces. Discuss how lie algebra of a matrix group is the tangent space at identity to a matrix group.

Explain how the Lie algebras can be thought of as the space of left-invariant vector fields.

Compute Lie algebras of key examples.

4. Matrix Exponentiation

Define exponential map for matrices. Show that this provides a way of going from the Lie algebra to the lie group.

Discuss the application of matrix exponentiation in solving system of linear differential equations.

5. Matrix groups as manifolds

Lecture given by instructor

6. Lie Bracket

Define Lie brackets. Explain examples of Lie brackets in the previous examples of Lie algebras. Give the abstract definition of a Lie algebra.

Define the adjoint representation and show examples. Discuss the surjectivity of this map.

Discuss the case of $Sp(1)$ double cover of $SO(3)$.

7. Lie groups and Homogeneous Spaces

Introduce the notion of matrix group actions on spaces and define homogeneous spaces.

As a first example, define real and complex projective spaces and realize them as projective spaces.

As a second example, consider the full flag variety and realize it as a homogeneous space.

As the third example, define grassmanians and realize them as homogeneous spaces.

Compute tangent space to homogeneous spaces in these examples and show that they are subsets of Lie algebras but not necessarily Lie algebras.

8. Some Decomposition Theorems

Singular value decomposition. Applications: Principal component analysis. Mention that this generalizes to the Cartan decomposition.

QR decomposition. Application: Gram-Schmidt orthogonalization. Mention that this generalizes to the Iwasawa decomposition.

9. Maximal Tori

Introduce the maximal tori in a compact Lie group. Compute examples of tori and center of group in the key examples of orthogonal groups.

Explain what happens to maximal tori under conjugation and discuss the uniqueness of such tori.

Define the notion of rank.

10. Semi-simple factorisation

Define an invariant inner product and compute the root space decomposition of the adjoint representation of $SO(4)$. Then state the general result about the structure of adjoint representations.

11. Classification of compact Lie groups

Discuss the root systems A_1, A_2, $A_1 \times A_1$. Discuss the Weyl groups and draw the Dynkin diagram. Explain how these connect to Lie algebras. Explain how these are Lie algebras of some well-known groups.

At the end, show the classification table of compact simple Lie groups. And say a few words about how this gives a complete list.

12. What lies beyond - Lie groups and symmetric spaces

Define Lie groups abstractly.