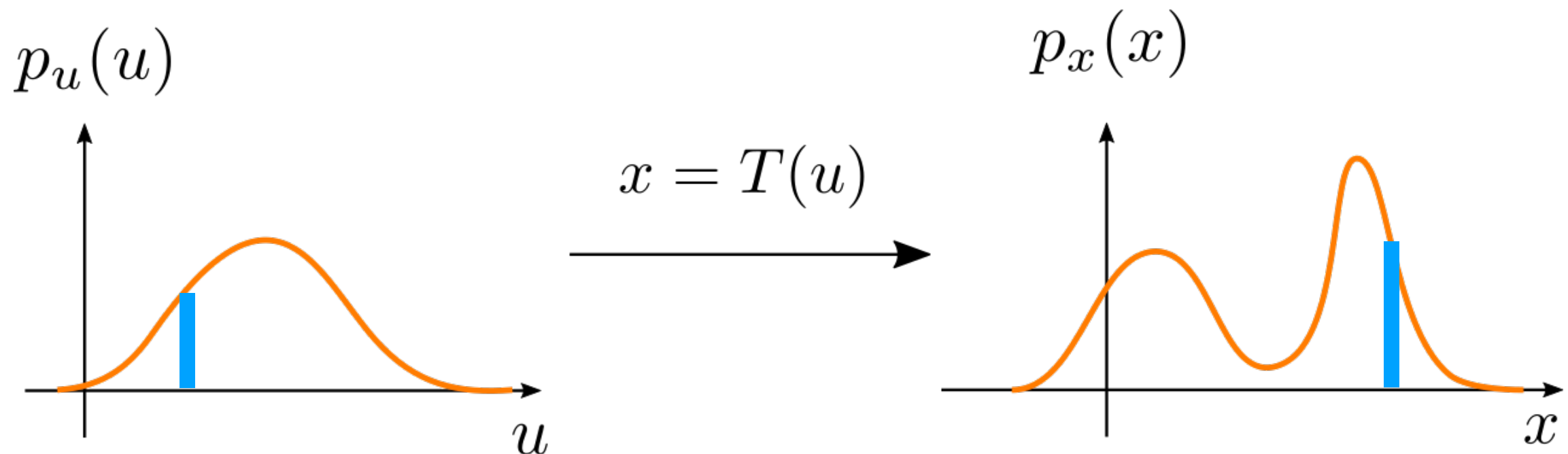# Lecture 18:
# Monte Carlo methods

# Monaco

# Monte Carlo(MC)

- Have been seeing Monte Carlo methods throughout class

  - Any time we randomly sample thats an MC method

  - Effectively we are just rolling the die
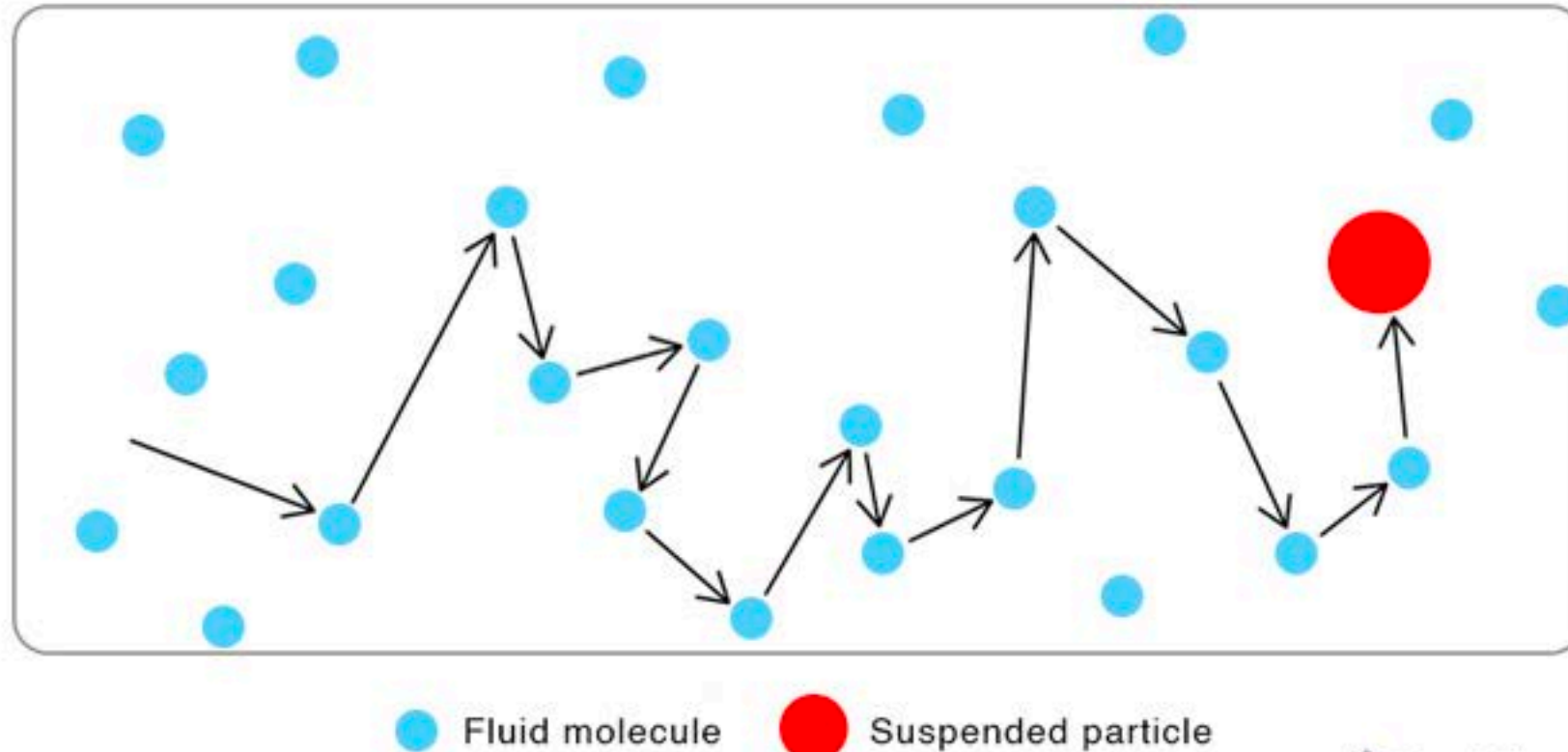
# Monte Carlo vs Integration

- Monte Carlo is a form of integrator

  - However non-deterministic and varies over distribution

$$p_u(u) \qquad\qquad x = T(u) \qquad\qquad p_x(x)$$



- Monte Carlo typically used when

  - we can't model things analytically any more

  - Replace a whole distribution with just an event (small region

# Brownian Motion



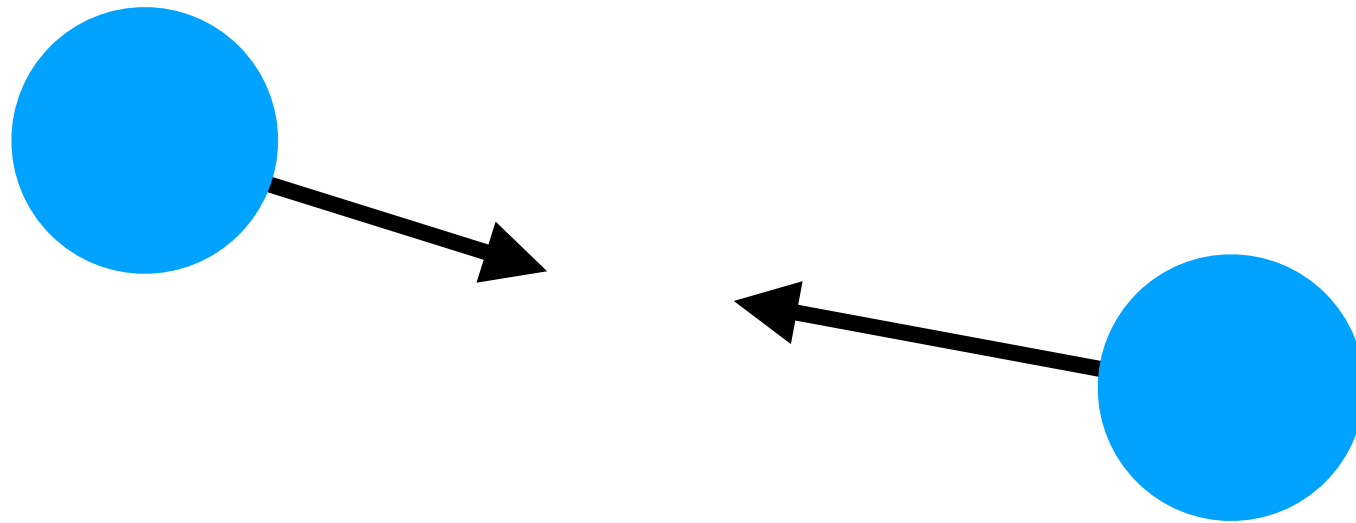$$f(v_x, v_y, v_z) = \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-m(v_x^2+v_y^2+v_z^2)/2kT}$$

$$= \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-mv^2/2kT}$$

using $\quad v^2 = v_x^2 + v_y^2 + v_z^2$

- At each step

    - We just randomly sample the velocity from a Gaussian

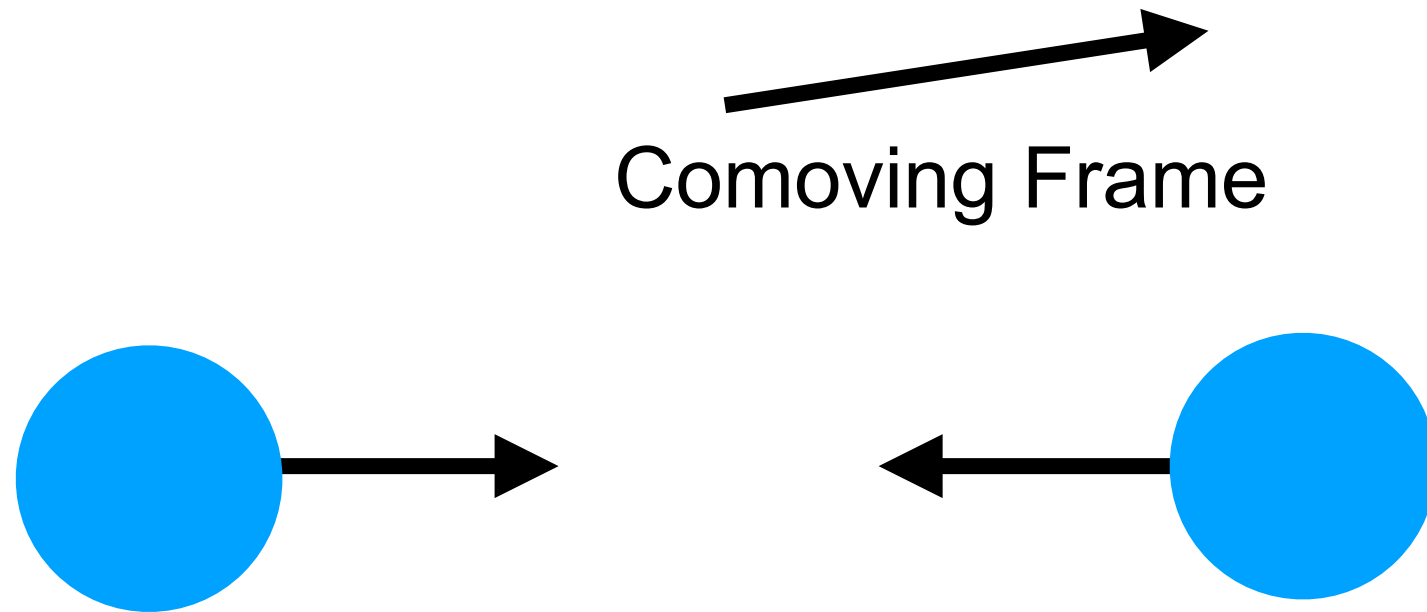    - We can do this many times to look at overall motion
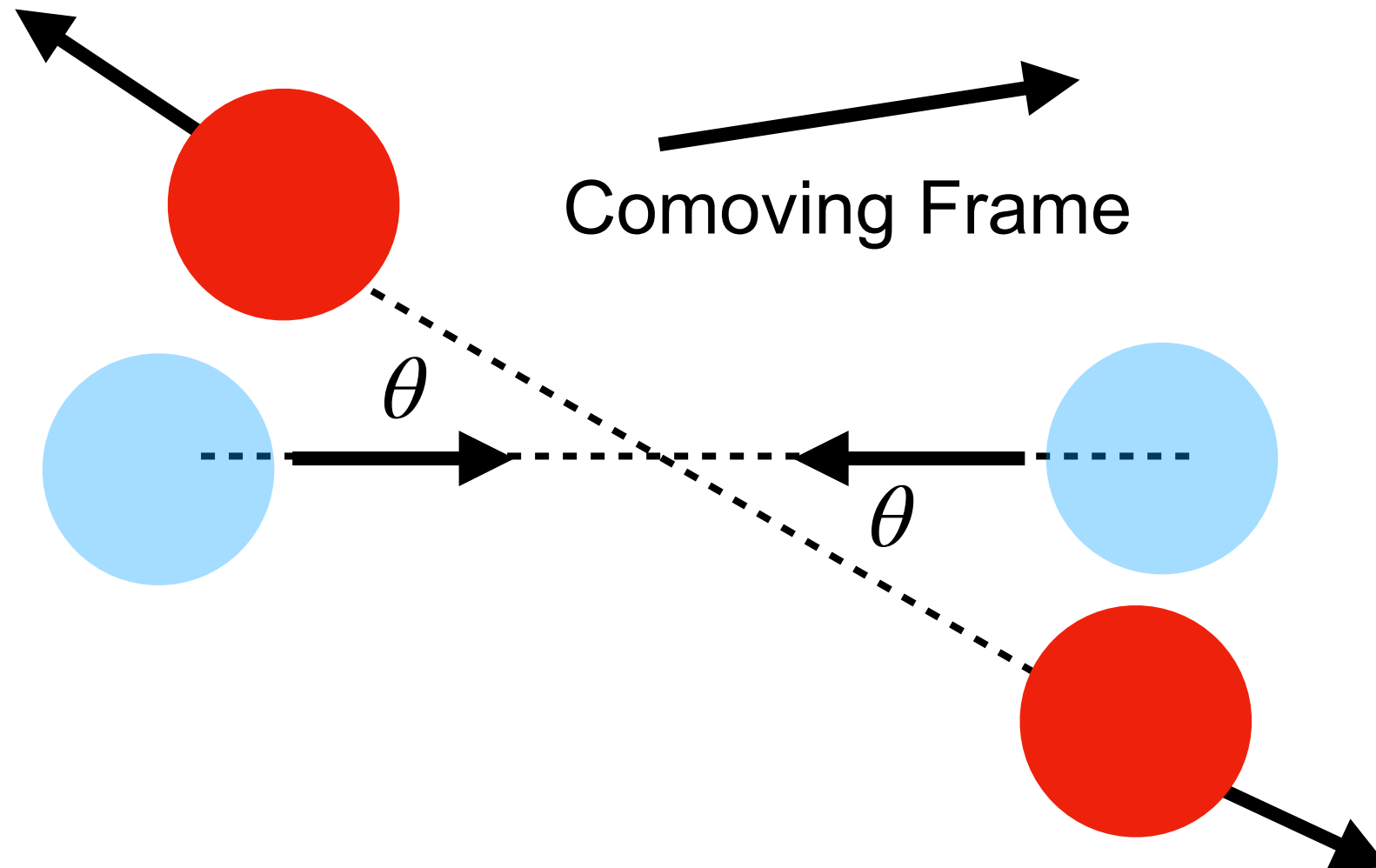
# The motion at each step

## Elastic Collision

- Just sample particle collisions at each step

# The motion at each step

Comoving Frame

Elastic Collision
In COM Frame

# The motion at each step


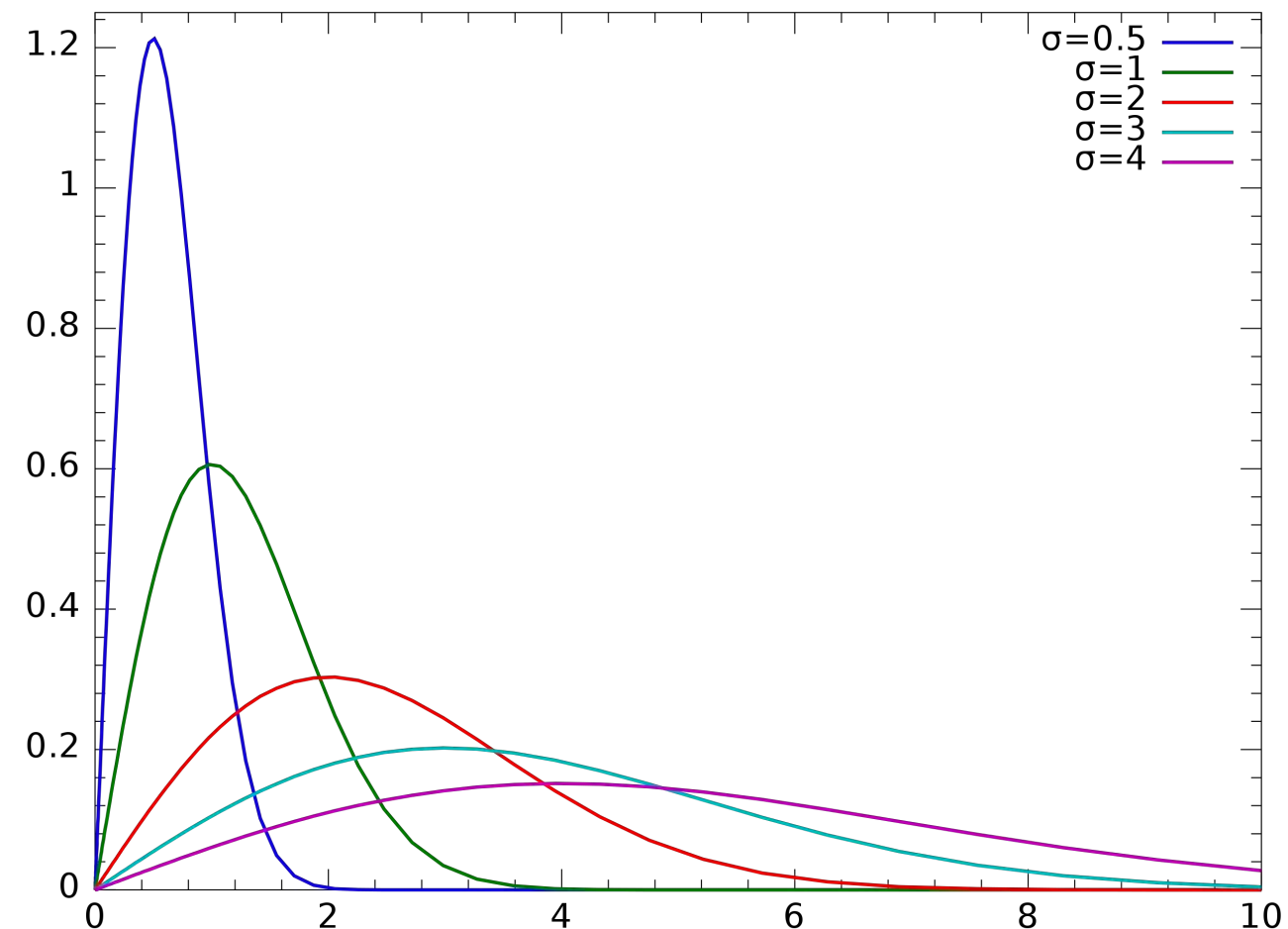
Comoving Frame

$\theta$

$\theta$

Elastic Collision
In COM Frame

# Rayleigh Distribution

Rayleigh is a distribution of the radius in a 2D Gaussian

$$f_U(x;\sigma) = f_V(x;\sigma) = \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}.$$

$$f(x;\sigma) = \frac{x}{\sigma^2}e^{-x^2/(2\sigma^2)}, \quad x \geq 0,$$

$$F_X(x;\sigma) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^x re^{-r^2/(2\sigma^2)}\,dr\,d\theta = \frac{1}{\sigma^2}\int_0^x re^{-r^2/(2\sigma^2)}\,dr.$$
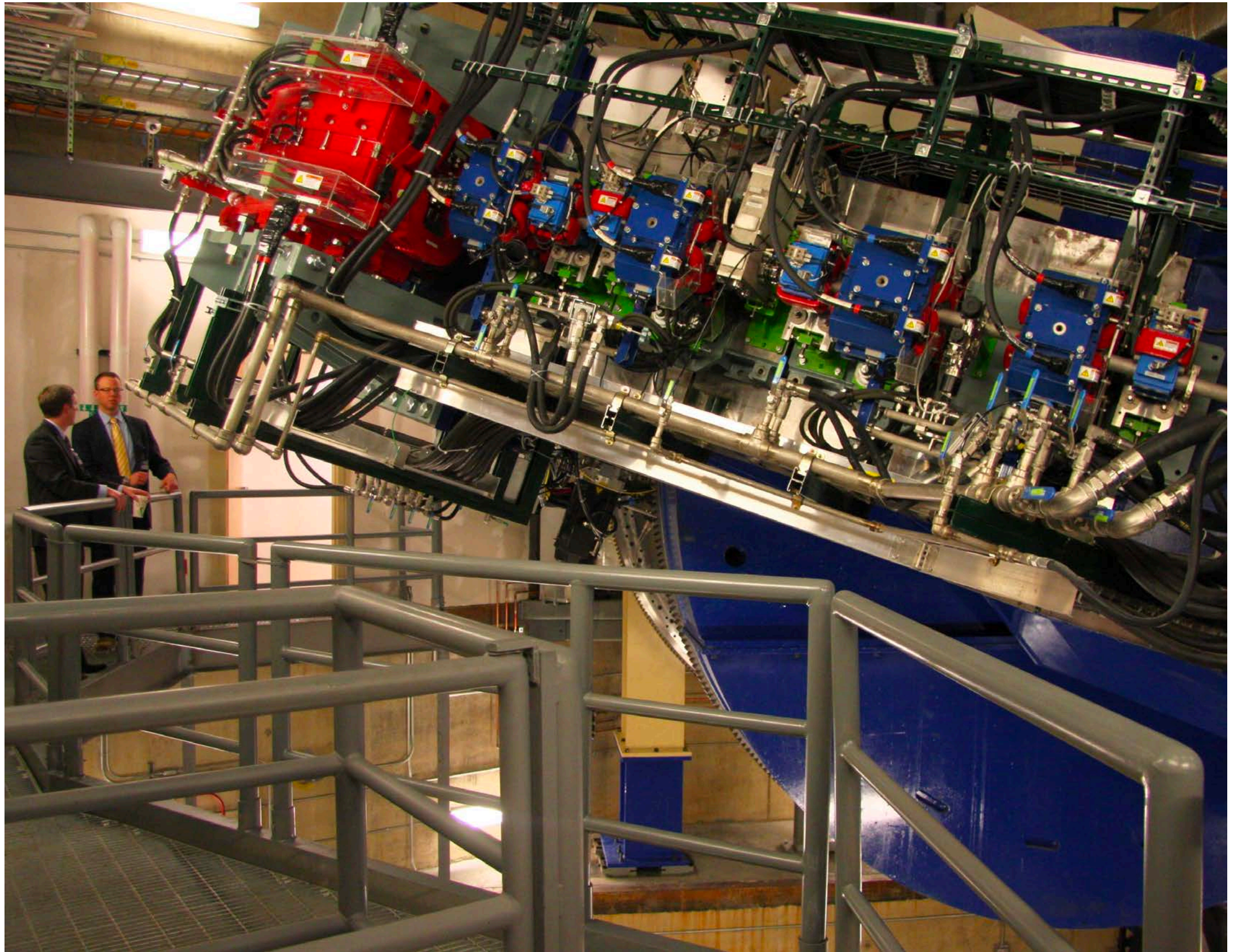
# Proton Therapy



Proton Therapy Center at MGH
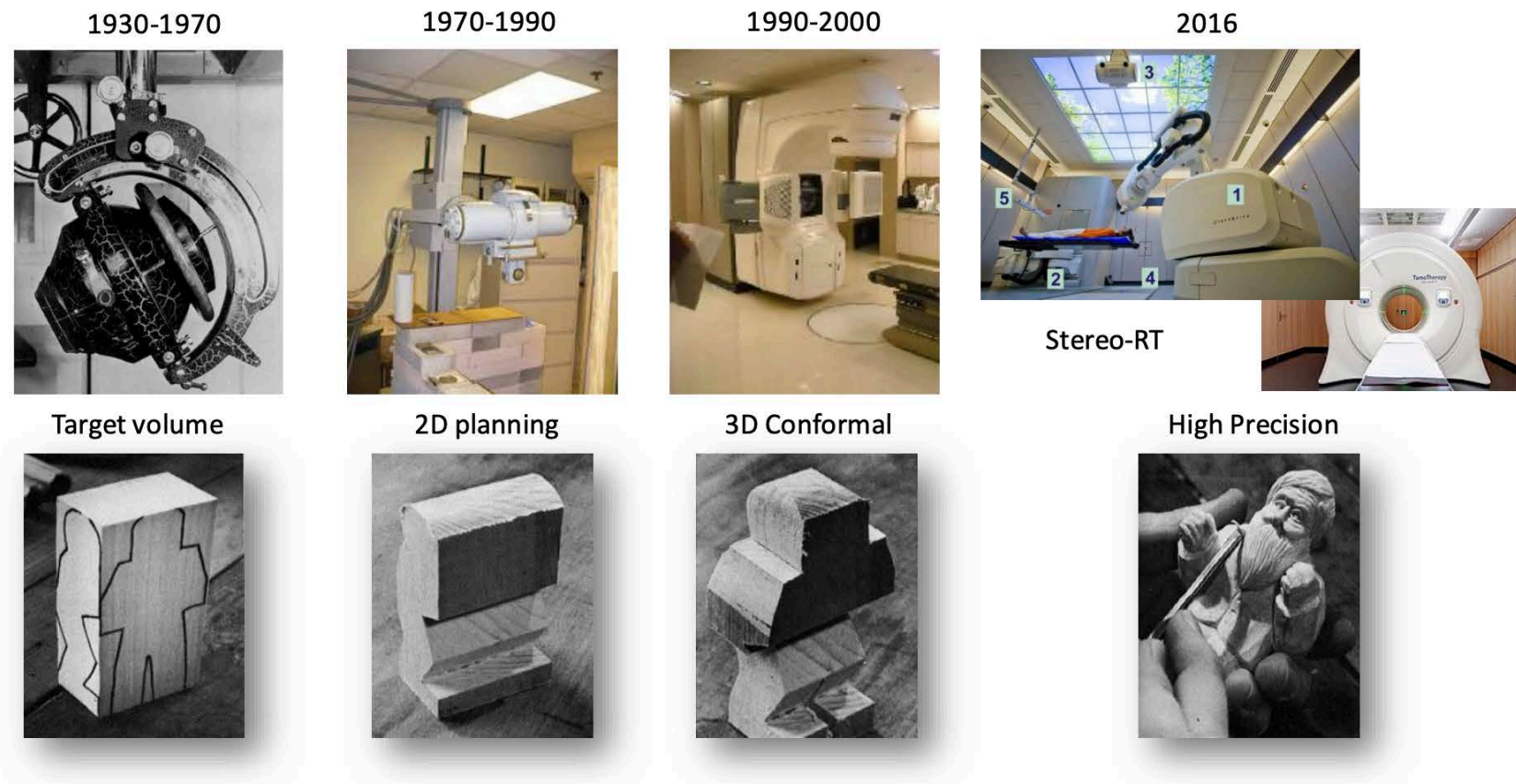
# Typical Device



**Particle Therapy Centre**

# Mayo Clinic

# Radiation Therapy

**Fractionation and Enhanced precision**

| 1930-1970 | 1970-1990 | 1990-2000 | 2016 |
|---|---|---|---|

Stereo-RT

| Target volume | 2D planning | 3D Conformal | High Precision |
|---|---|---|---|



- To fight Cancer

  - Radiation therapy has had a long history of usage

  - Radiation is sent to a tumor to kill it
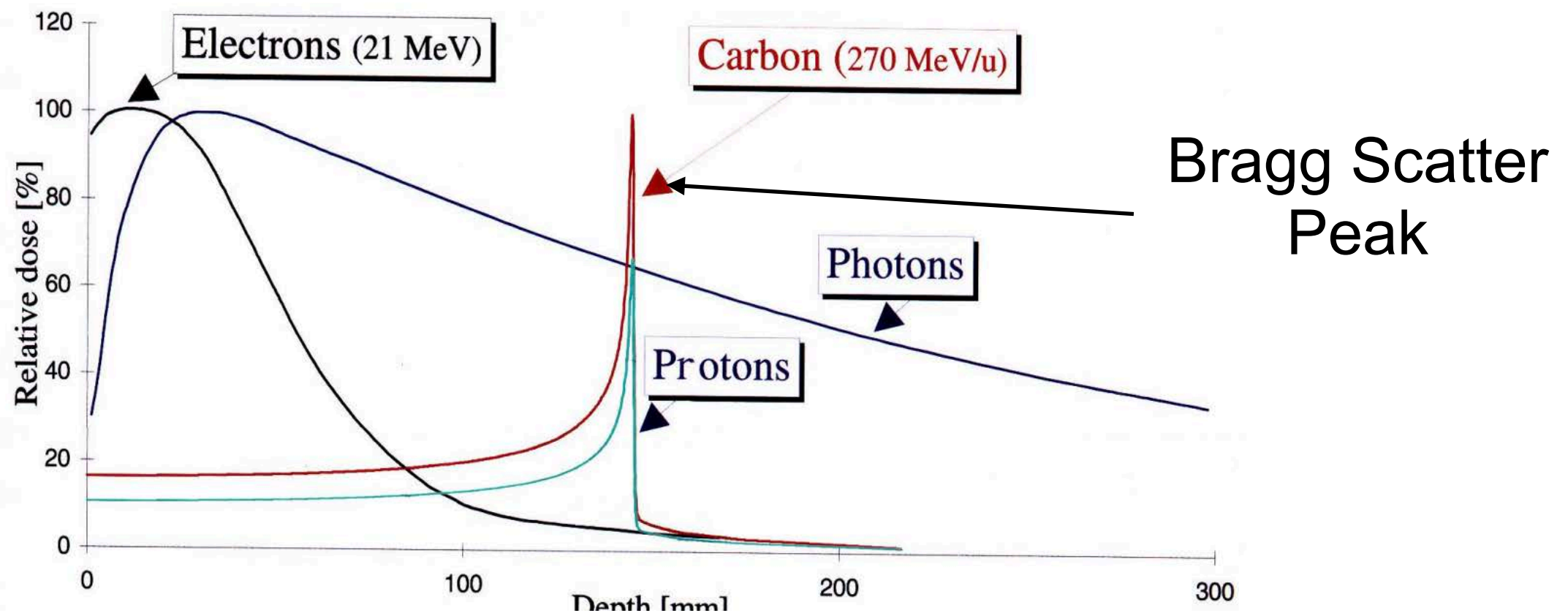
  - Critical when you can't cut the tumor out

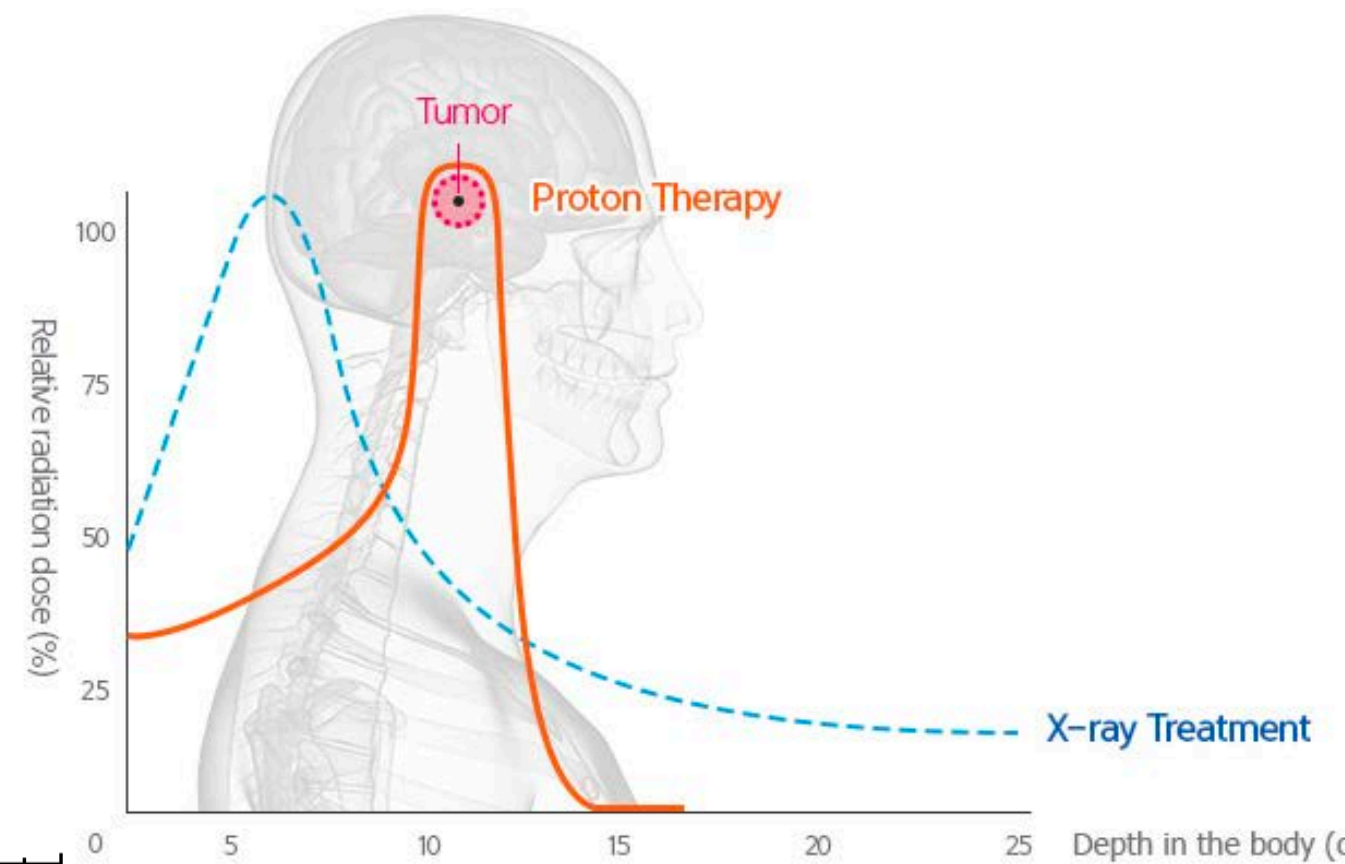# Classical Radiotherapy with X-rays

single beam
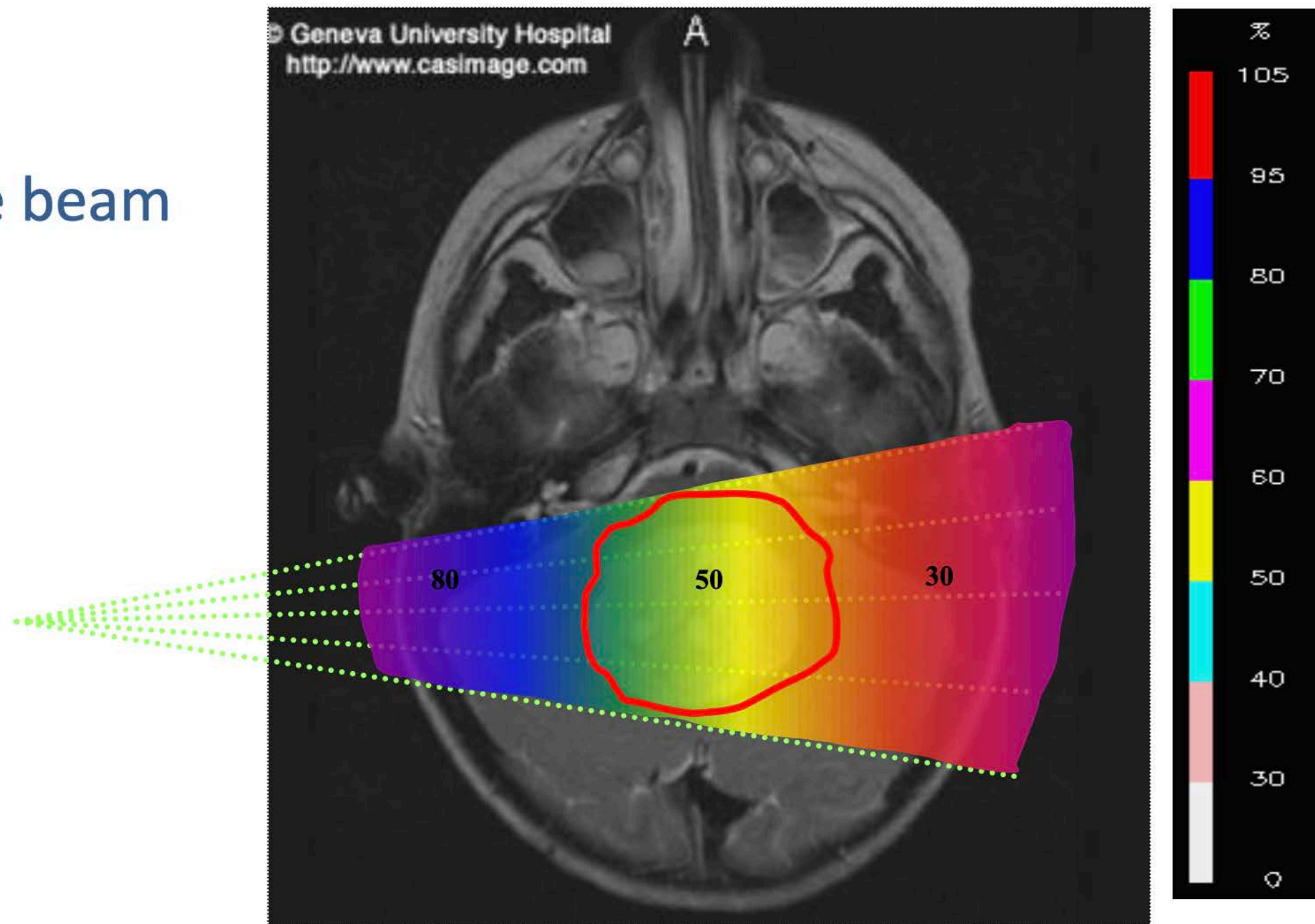
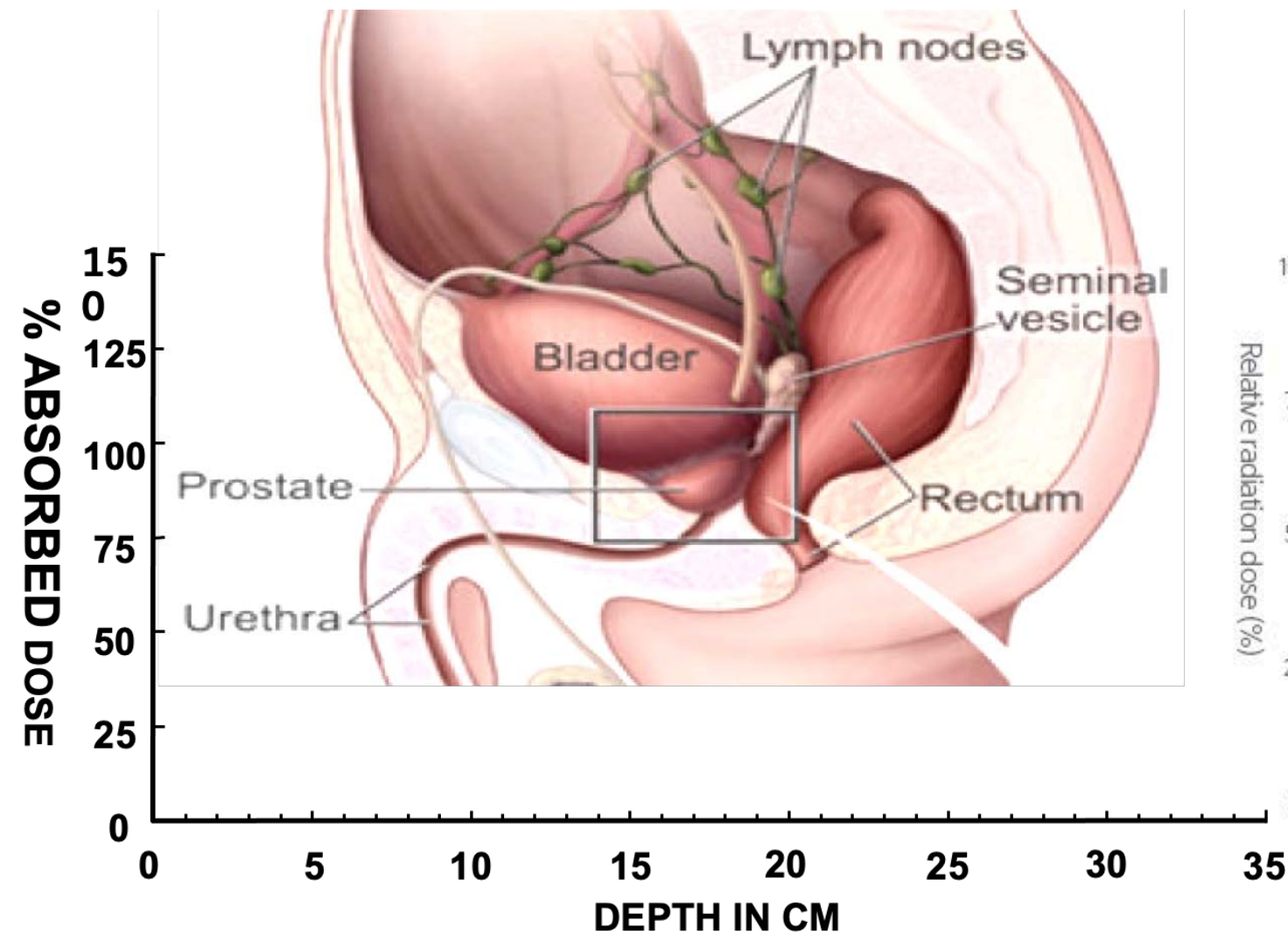# Hadron Therapy

- Therapy

  - Hadrons allow you to control deposit

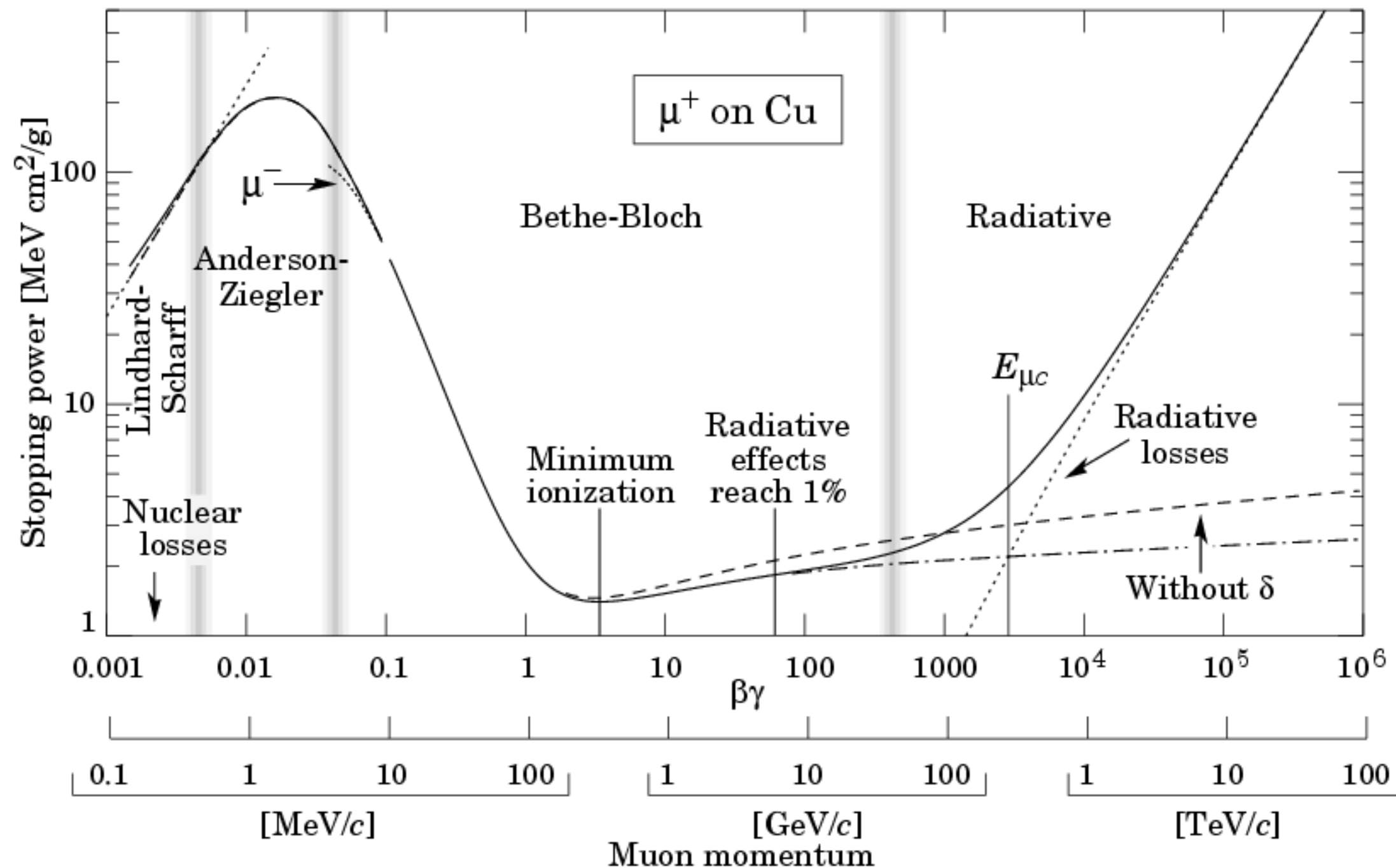  - Can vary the depth of the hadrons through Bragg scatter

Electrons (21 MeV)

Carbon (270 MeV/u)

Bragg Scatter Peak

Photons

Protons

Relative dose [%]

Depth [mm]

# Proton Therapy

# Bethe-Bloch Equation

- Charged Particles in matter are goverened by this equation

# Protons Governed

$$-\left\langle \frac{dE}{dx} \right\rangle = Kz^2 \frac{Z}{A} \frac{1}{\beta^2} \left[ \frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{\max}}{I^2} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

$[\cdot \rho]$

density

$K \quad = 4\pi N_A r_e^2 m_e c^2 = 0.307 \text{ MeV g}^{-1} \text{ cm}^2$

$T_{max} = 2m_e c^2 \beta^2 \gamma^2 / (1 + 2\gamma \, m_e/M + (m_e/M)^2)$
[Max. energy transfer in single collision]

$N_A = 6.022 \cdot 10^{23}$
[Avogardo's number]

$r_e \quad = e^2/4\pi\varepsilon_0 m_e c^2 = 2.8 \text{ fm}$
[Classical electron radius]

$m_e = 511 \text{ keV}$
[Electron mass]

z   :   Charge of incident particle

M   :   Mass of incident particle

$\beta \quad = v/c$
[Velocity]

Z   :   Charge number of medium

A   :   Atomic mass of medium

$\gamma \quad = (1-\beta^2)^{-2}$
[Lorentz factor]

Validity:

I   :   Mean excitation energy of medium

$\delta$   :   Density correction [transv. extension of electric field]
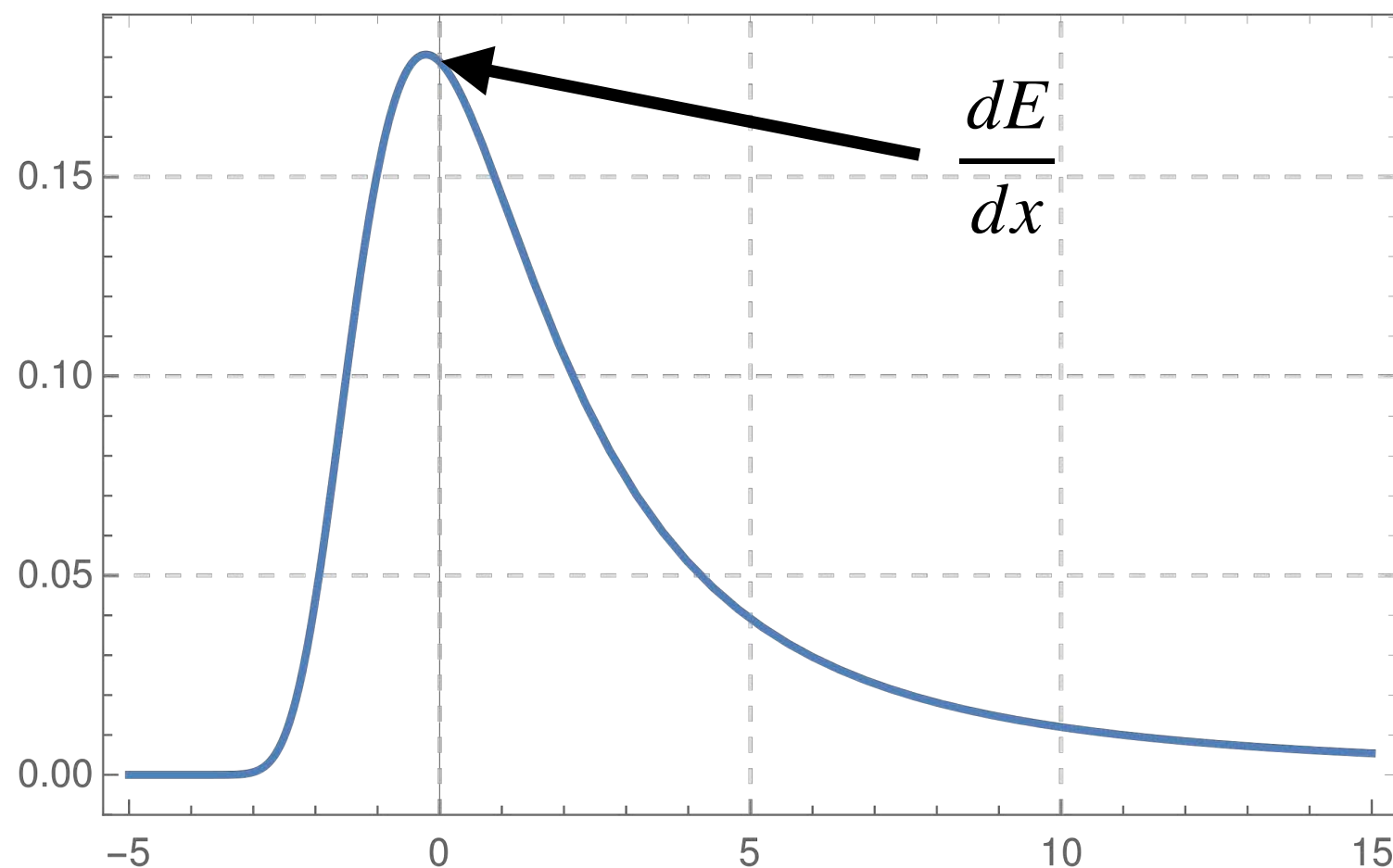
$.05 < \beta\gamma < 500$

$M > m_\mu$

# Actual Energy Loss

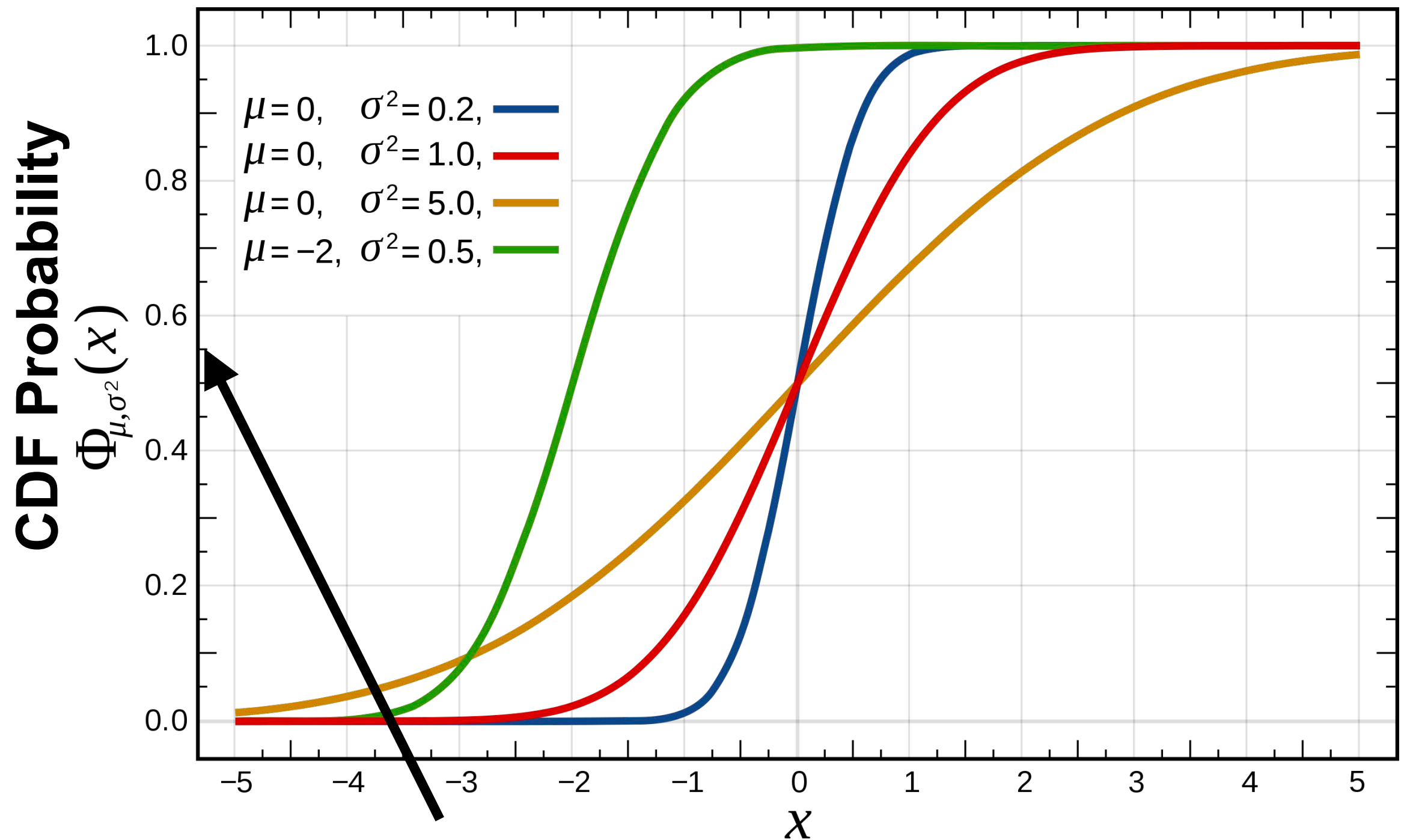- As we step along we lose energy by the Landau distribution

$$p(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{s \log(s) + xs}\, ds,$$



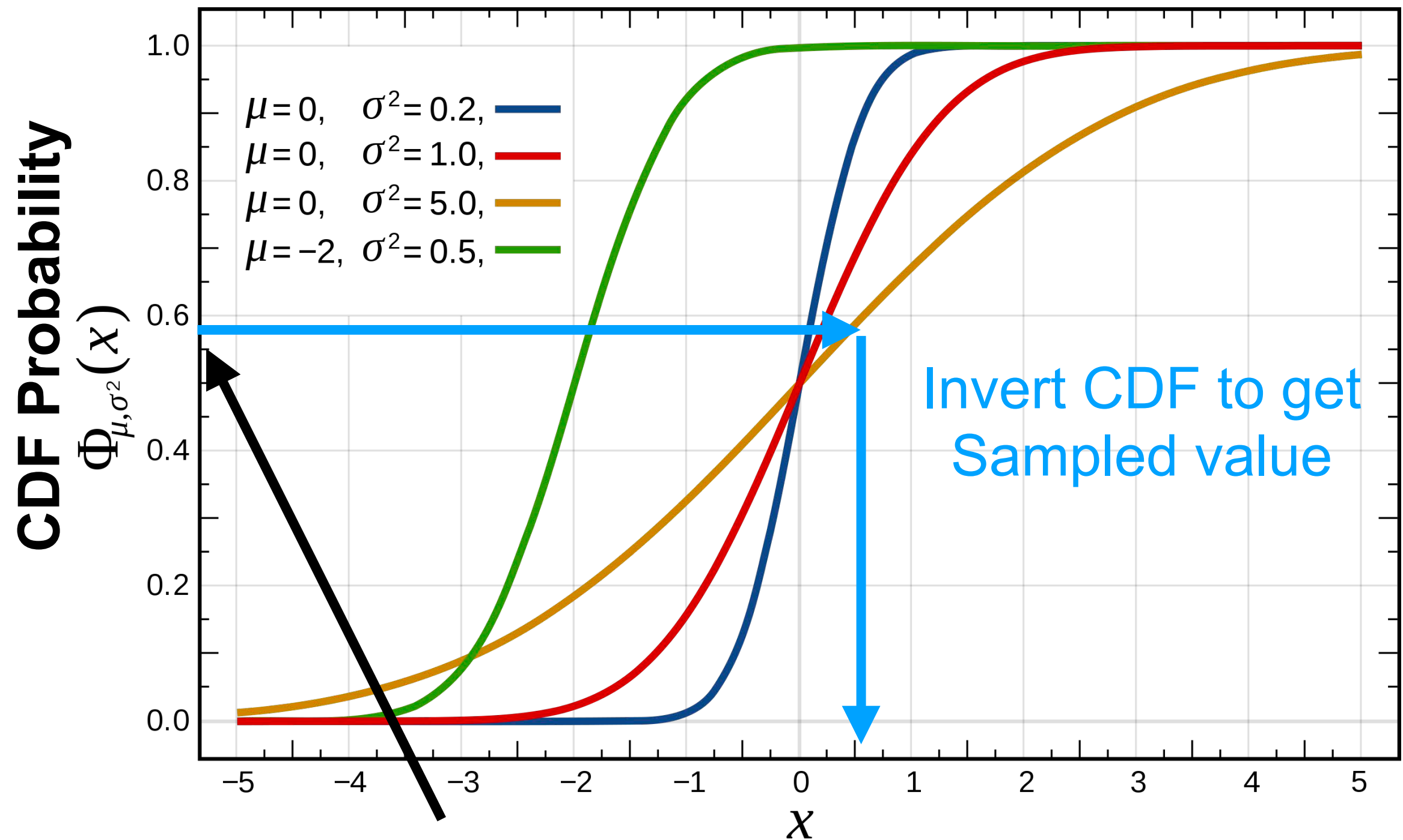$$\frac{dE}{dx}$$

Average of this distribution gives Bethe-Bloch
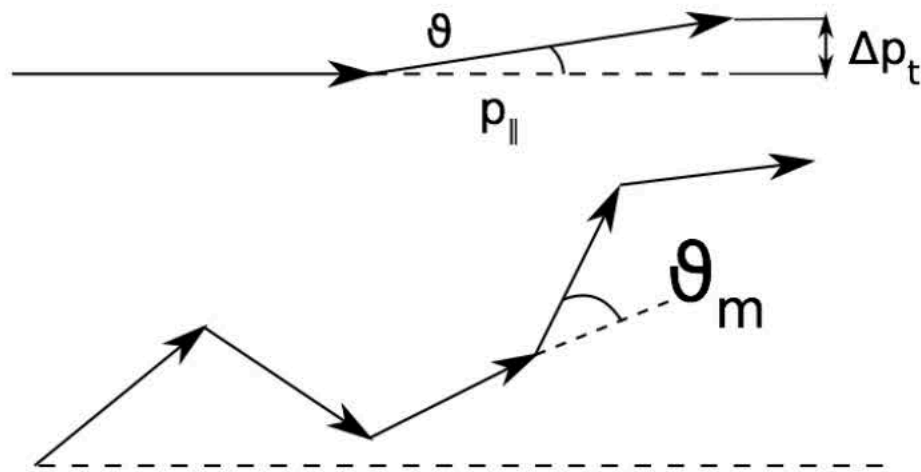
We can sample this At each step

# Sampling a Distribution



Sample from a p-value from 0 to 1 (flat 0 to 1)

# Sampling a Distribution



Legend:
$\mu = 0, \quad \sigma^2 = 0.2,$
$\mu = 0, \quad \sigma^2 = 1.0,$
$\mu = 0, \quad \sigma^2 = 5.0,$
$\mu = -2, \quad \sigma^2 = 0.5,$

CDF Probability $\Phi_{\mu,\sigma^2}(x)$

$x$

Invert CDF to get Sampled value

Sample from a p-value from 0 to 1 (flat 0 to 1)

# Multiple Scatter Particles



$$\theta \simeq \frac{\Delta p_\perp}{p_\parallel} \simeq \frac{\Delta p_\perp}{p}$$

$$= \frac{2Zze^2}{b}\frac{1}{pv}$$

after k collisions

$$\langle\theta_k^2\rangle = \sum_{m=1}^{k}\theta_m^2 = k\langle\theta^2\rangle$$

- Single collision (thin absorber): Rutherford scattering $d\sigma/d\Omega \propto \sin^{-4}\theta/2$
- Few collisions: difficult problem
- Many (>20) collisions: statistical treatment "Molière theory"

# Multiple Scatter Particles

$$\theta \simeq \frac{\Delta p_\perp}{} \simeq \frac{\Delta p_\perp}{}$$

Obtain the mean deflection angle in a plane by averaging over many collisions and integrating over b:

$$\sqrt{\langle \theta^2(x) \rangle} = \theta_{rms}^{plane} = \frac{13.6 \text{ MeV}}{\beta pc} z \sqrt{\frac{x}{X_0}} \left(1 + 0.038 \ln \frac{x}{X_0}\right)$$

- Material constant $X_0$: radiation length
- $\propto \sqrt{x} \rightarrow$ use thin detectors
- $\propto 1/\sqrt{X_0} \rightarrow$ use light detectors
- $\propto 1/\beta p \rightarrow$ serious problem at low momenta

In 3 dimensions: $\theta_{rms}^{space} = \sqrt{2} \, \theta_{rms}^{plane}$    $13.6 \rightarrow 19.2$

# Image Sources

**Monte Carlo**

link: https://frenchriviera.travel/wp-content/uploads/2018/03/Monte-Carlo-Casino1.jpg

attribution: French Riviera travel

**Monaco**

link: https://commons.wikimedia.org/wiki/File:Panorama_von_Monaco-La_Turbie.jpg

attribution: Tobi 87, CC BY-SA 3.0 <https://creativecommons.org/licenses/by-sa/3.0>, via Wikimedia Commons

**James Bond in Thunderball**

link: https://www.007.com/the-films/thunderball/

attribution: www.007.com

**Brownian motion**

link: https://www.sciencefacts.net/brownian-motion.html

attribution: © 2024 Science Facts

**Rayleigh distribution**

link: https://commons.wikimedia.org/wiki/File:Rayleigh_distributionPDF.svg

attribution: Krishnavedala, CC0, via Wikimedia Commons

# Image Sources

**proton therapy machine**

link: https://www.itnonline.com/content/mass-general-cancer-center-researchers-share-insights-evolution-proton-radiotherapy

attribution: Massachusetts General Hospital

**particle therapy center**

link: https://seeiist.eu/expression-for-interest/

attribution: Virtual center, source http://enlight.web.cern.ch/media

**Mayo clinic**

link: https://commons.wikimedia.org/wiki/File:MayoProton.jpg

attribution: Jonathunder, GFDL 1.2 <http://www.gnu.org/licenses/old-licenses/fdl-1.2.html>, via Wikimedia Commons

radiation therapy images; classical radiotherapy

attribution: origin unknown

# Image Sources