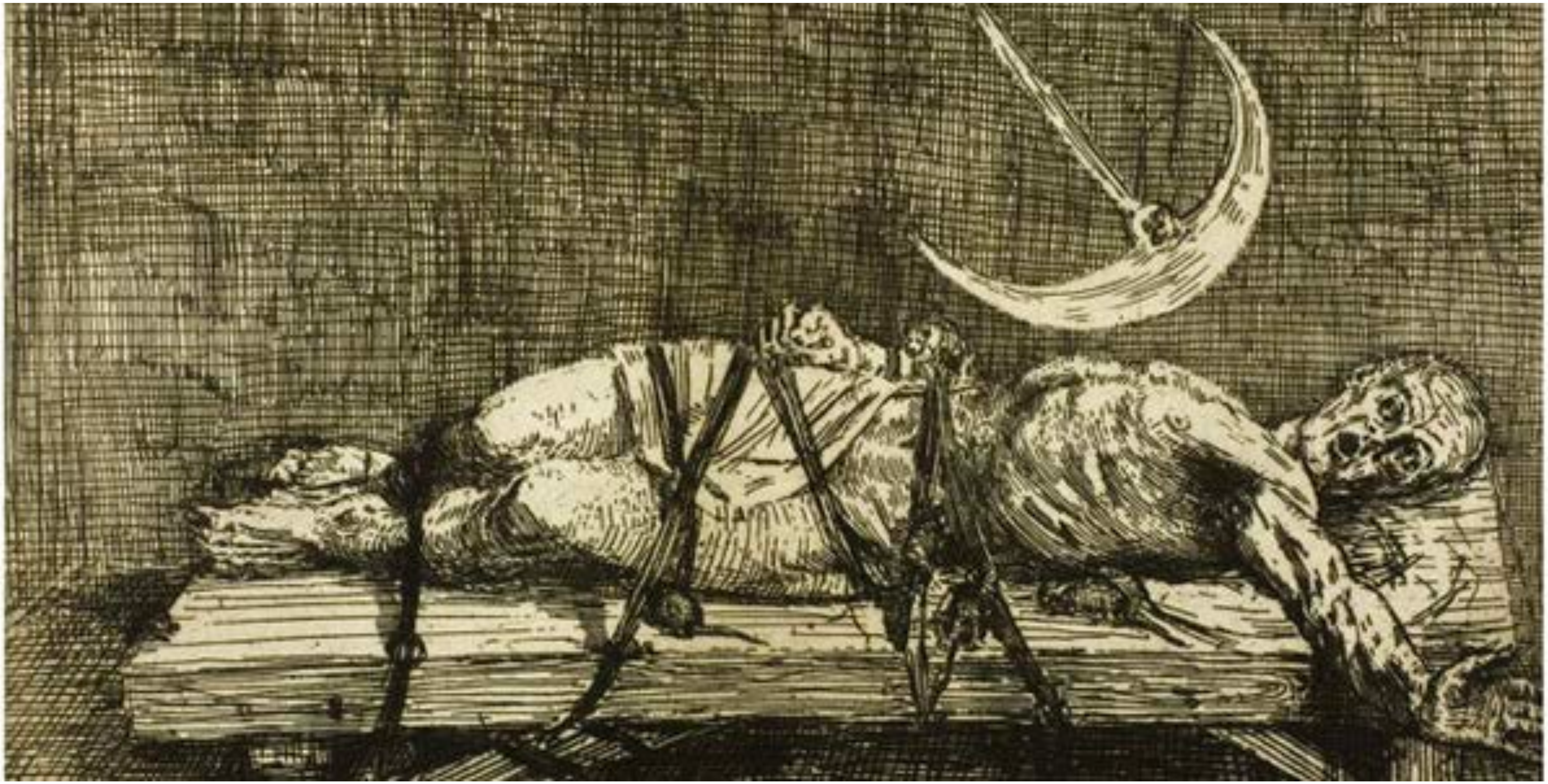
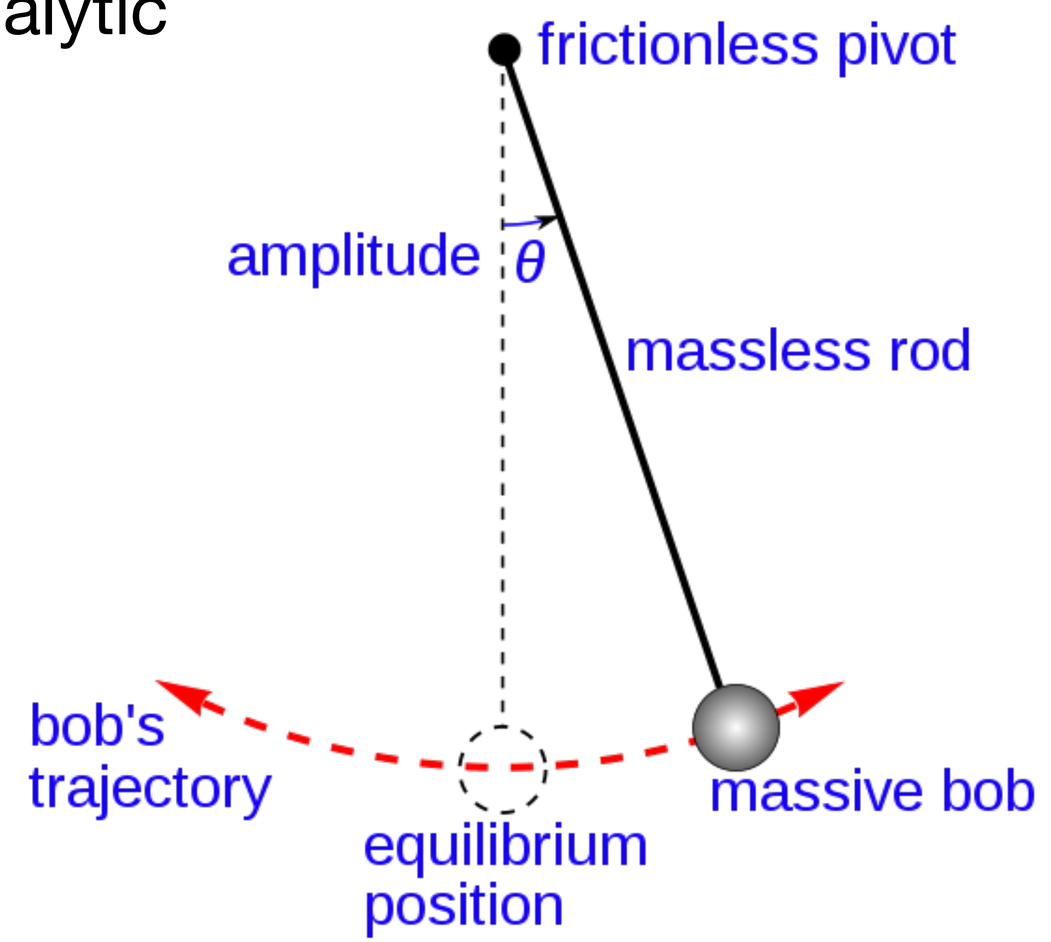


Lecture 15: Numerical ODEs

The pendulum

- While seemingly simple the solution is not analytic
  - $m\ell\ddot{\theta} = -mg\sin(\theta)$
  - $\frac{1}{2}\dot{\theta}^2 = \frac{g}{\ell}\left(\cos\theta - \cos\theta_0\right)$
  - $\int \frac{d\theta}{\sqrt{\left(\cos\theta - \cos\theta_0\right)}} = 2\int \frac{g}{\ell}dt$



Elliptic Integral : This is what actually

Numerical Simulation

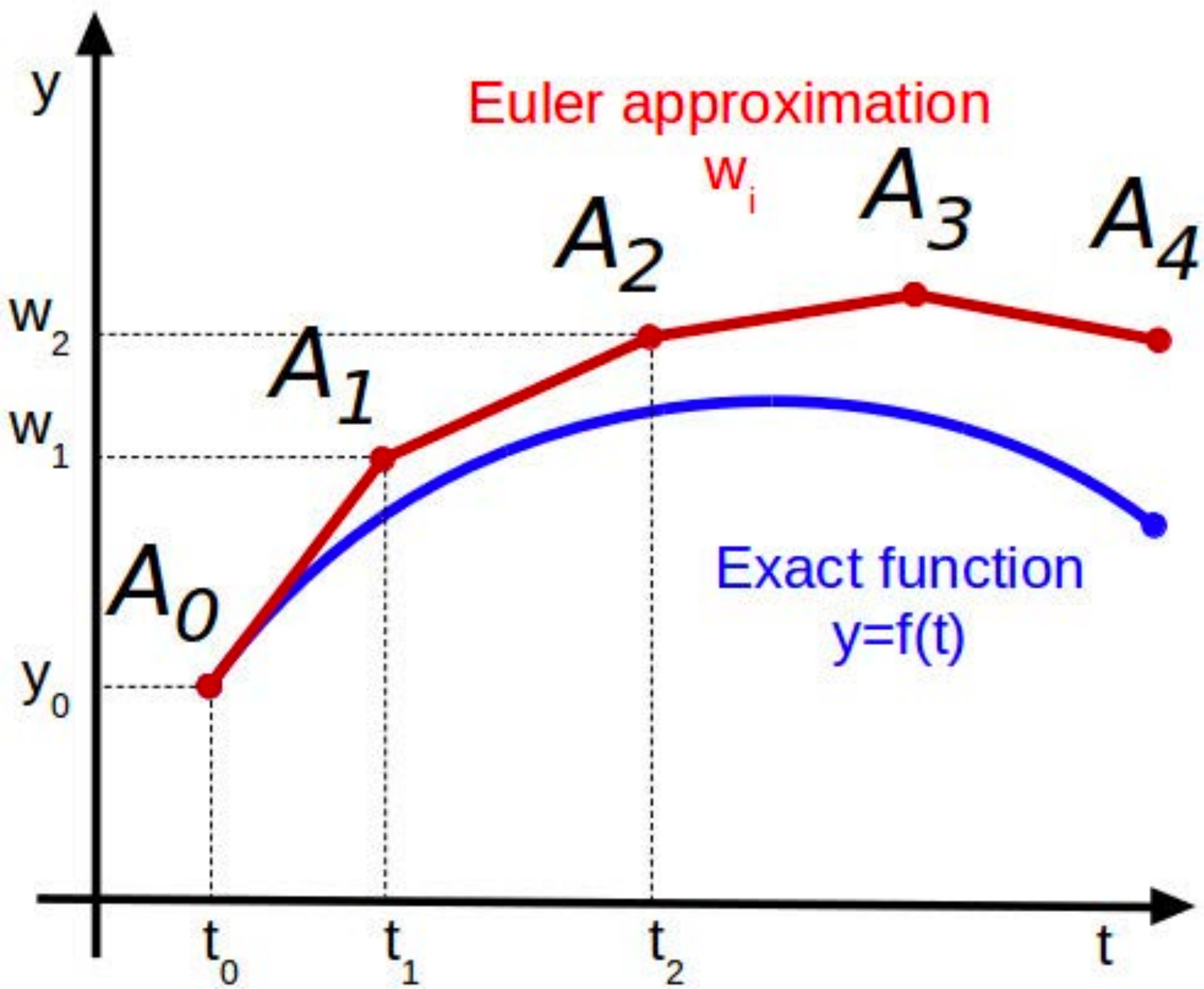
- This part of the class will cover numerical simulation
    - Typically this involves stepping through a simulation
    - Simplest stepping involves computing velocity/acceleration
    - Stepping through the forces :
        - $$\frac{d\vec{x}}{dt} = \vec{v}(t) \rightarrow \vec{x}(t) = \int d\vec{x} = \int \vec{v}(t)dt$$
        - $$\frac{d\vec{v}}{dt} = \vec{a}(t) \rightarrow \vec{v}(t) = \int d\vec{v} = \int \frac{\overrightarrow{F}(t)}{m}dt$$

What can we do to step

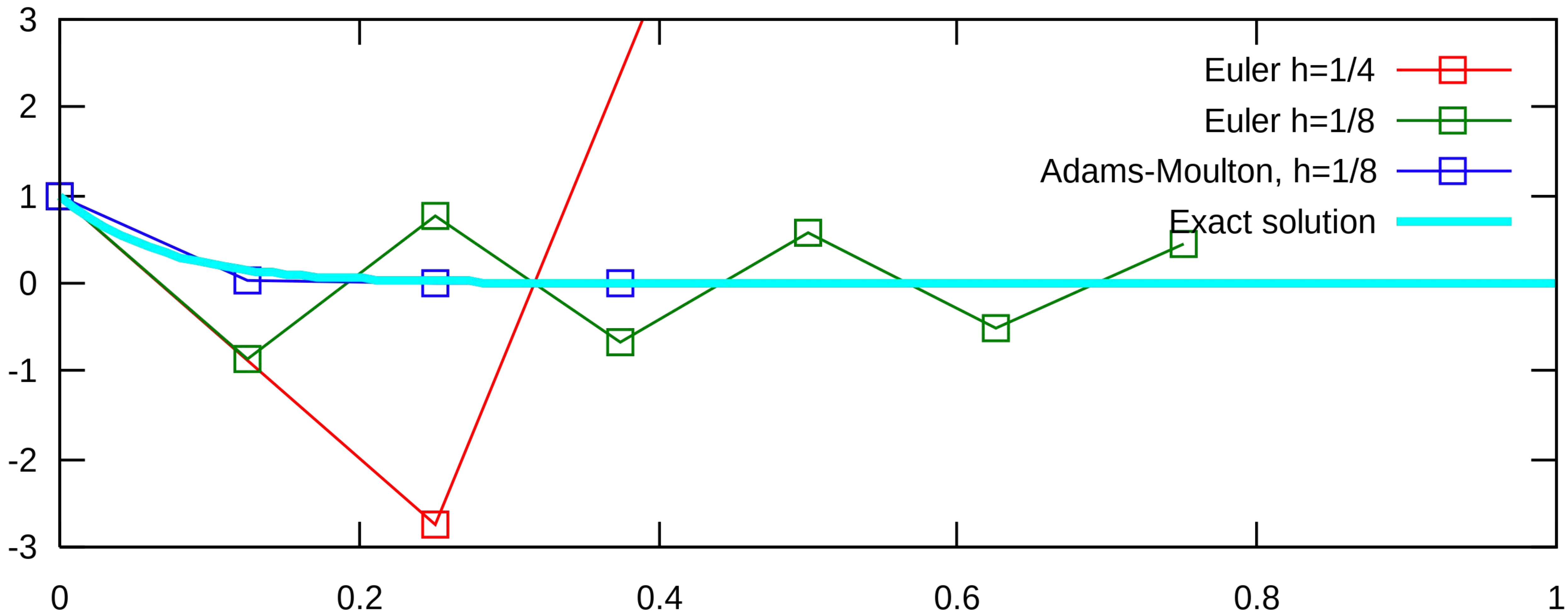
- For some time interval Δt, we can assume that
- $\vec{v}(t) \approx v_t$ (a constant for a short time)
- $\vec{a}(t) \approx a_t$ (a constant for a short time)
- From this base assumption, we can start to approximate
- These lead to a model

Tiers of approximation

- Strategy to linearize
  - Rely on Slope take appropriate timesteps

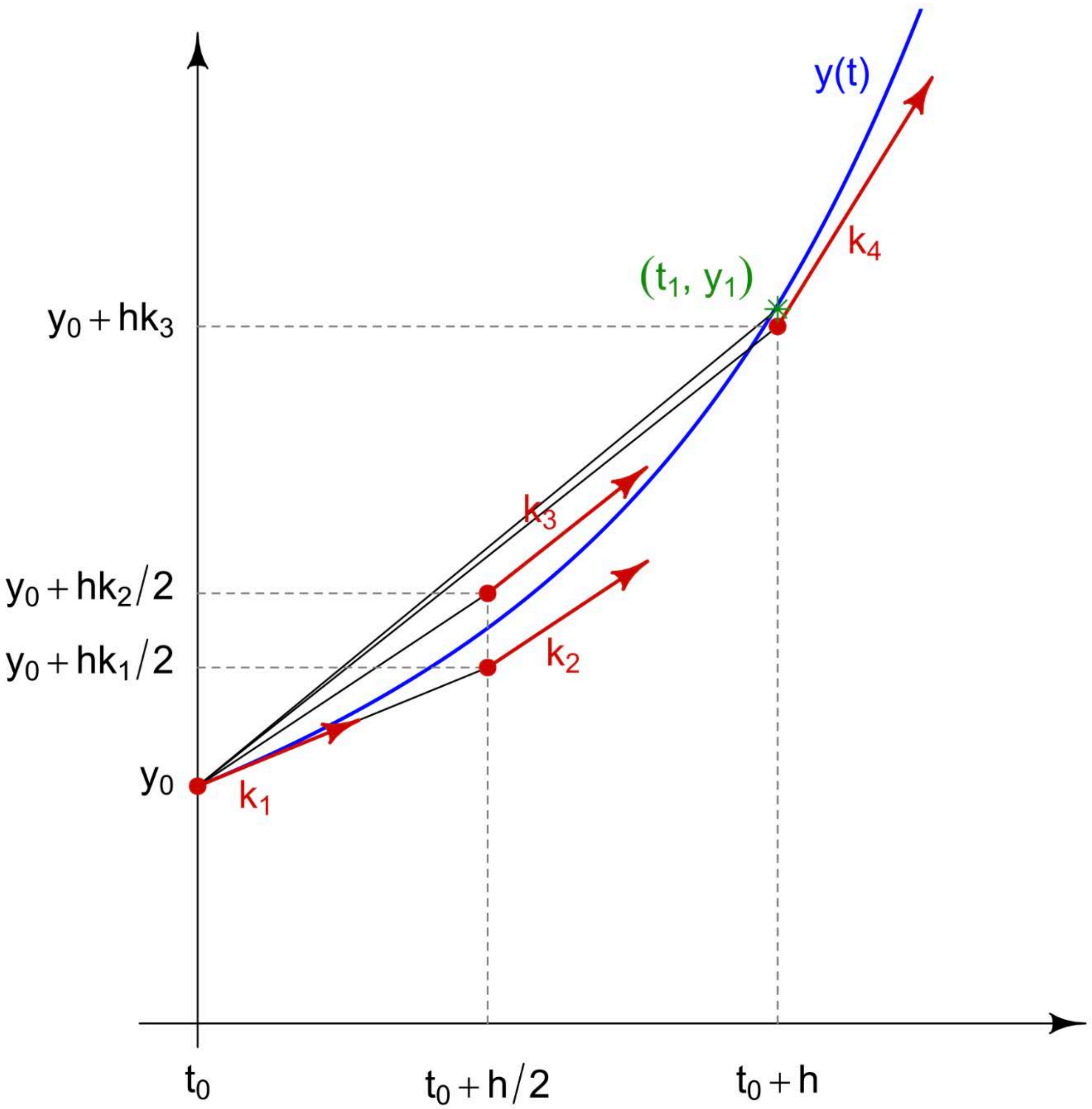


ODE Stiffnesss



- Stiff ODEs breakdown when step size too large
  - Stiffness is a sign of a difficult ODE

Runge-Kutta



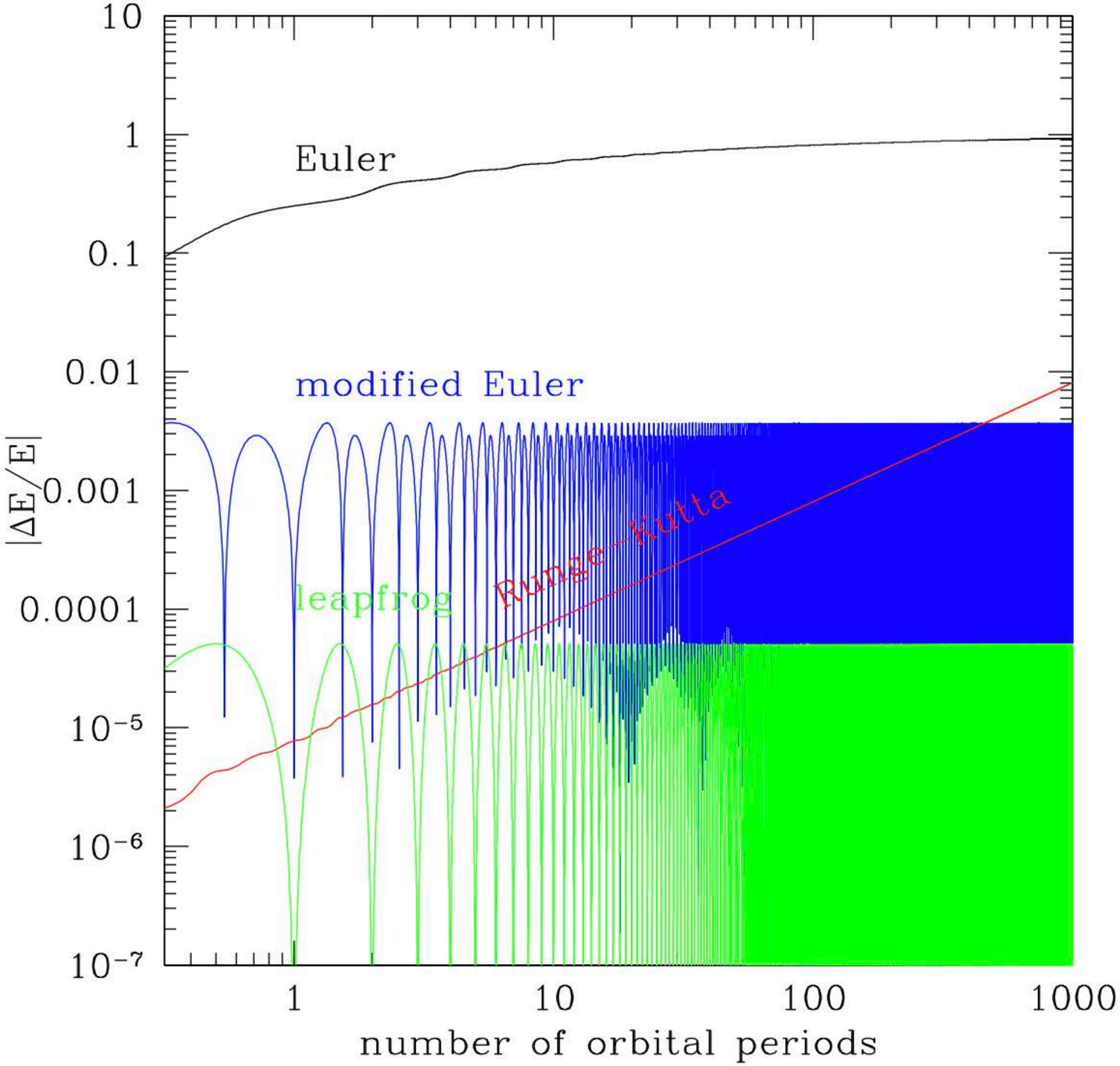
$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h,$$
$$t_{n+1} = t_n + h$$

for $n = 0, 1, 2, 3, \ldots$, using[3]

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), \\ k_4 &= f(t_n + h, y_n + hk_3). \end{aligned}$$

- Construct 4 or more steps to get to the next one
  - For Pendulum we have to intertwine velocity and position

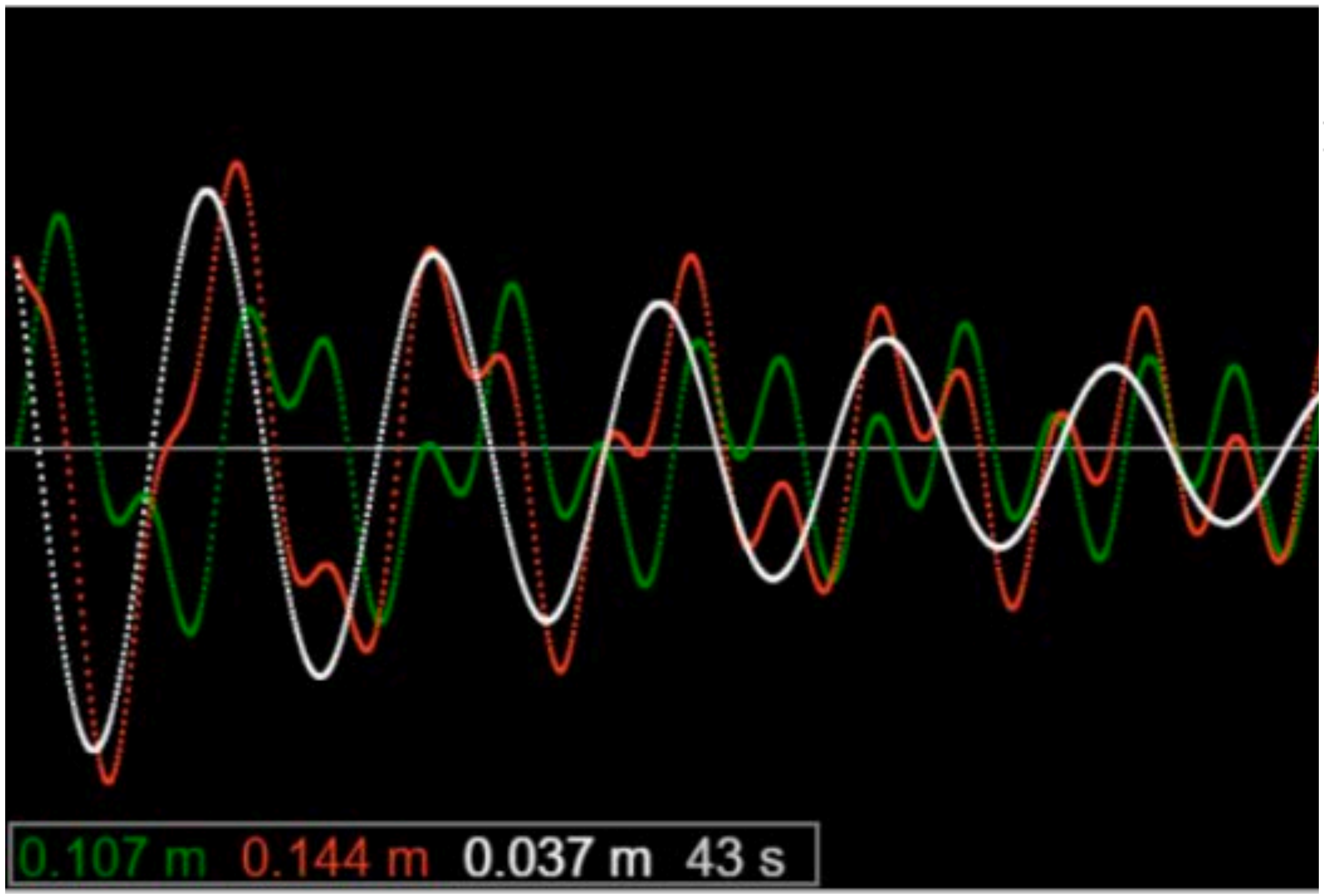
Precision

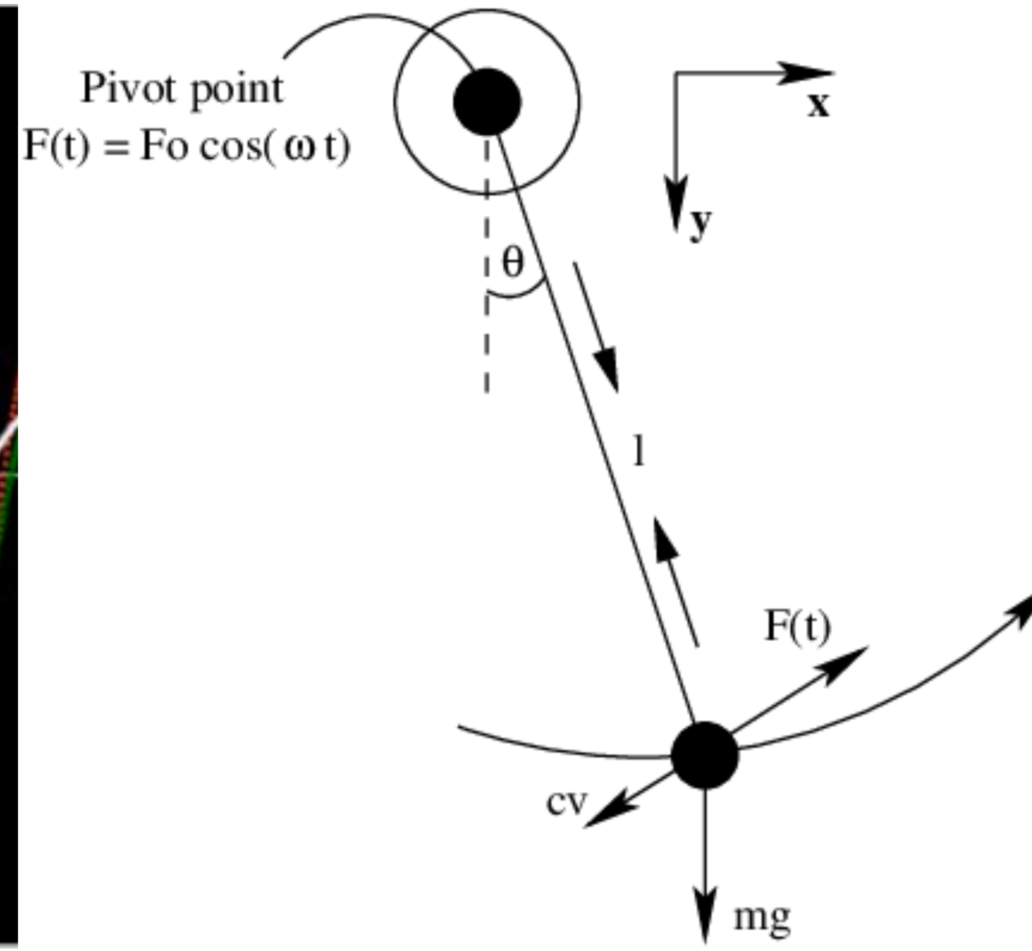


Steps (planetary simulation)

- Each step has its own benefits and limitations
- Can see this from precision over time for the left approximations

Damped Driven Harmonic Oscillator

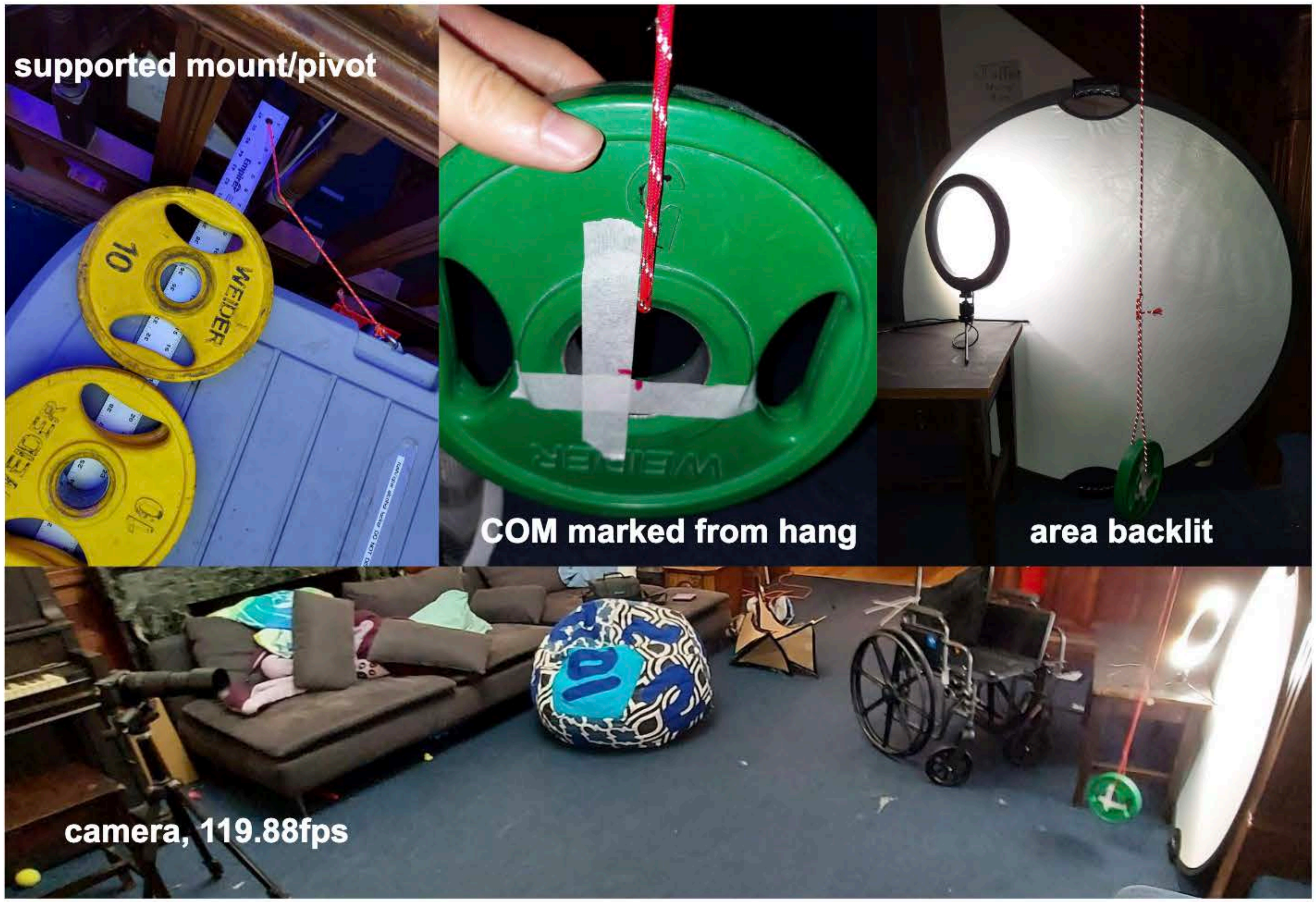




- We can extend our simulation towards damped driven HO
  - Dynamics here are fun and interesting
  - But we need a good integrator to understand it

High quality Pendulum data

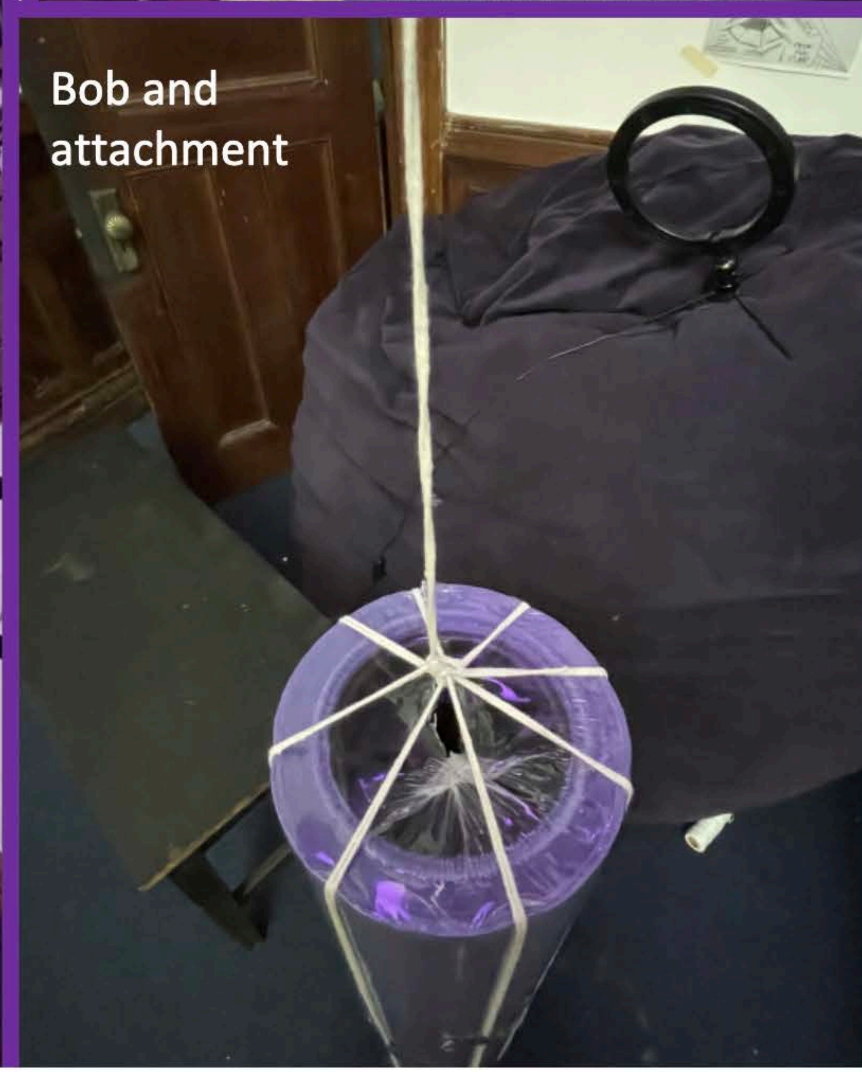
Apparatus



Pendulum designed to make length measurement more repeatable, improve small angle approx. and facilitate timing via computer vision, with low damping.

length approx. 4 m
displacement < 1.5°
mass approx. 5 lbs

High quality Pendulum data



Length:
$10.7886 \pm 0.0032\ m$

Period measurement:
phone camera + Jade’s computer vision program
30fps $\rightarrow \sigma = 0.0096\ s$

Small angle approximation:
1.06°: $T_{corr} = 0.9999 T_{meas}$

Procedure:
2 minutes damping time
60s recording
Video analysis

Image Sources

pit and pendulum

link: https://www.artic.edu/artworks/104911/the-pit-and-the-pendulum-second-plate

attribution: The Pit and the Pendulum, second Plate, Alphonse Legros

pendulum from wiki

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ODE stiffness plot

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Runge-Kutta plot

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damped-driven oscillator plot

link: https://galileoandeinstein.phys.virginia.edu/7010/CM_22a_Period_Doubling_Chaos.html

attribution: Michael Fowler, UVa

Image Sources

damped-driven oscillator diagram

link: https://www.researchgate.net/figure/Driven-damped-pendulum_fig2_341399839

attribution: Dynamics of multiple pendula, Wojciech Szumiński, DOI:10.13140/RG.2.2.32980.22406

pendulum experimental setup images

attribution: Kiran & Jade