An Introduction to Lattice QCD

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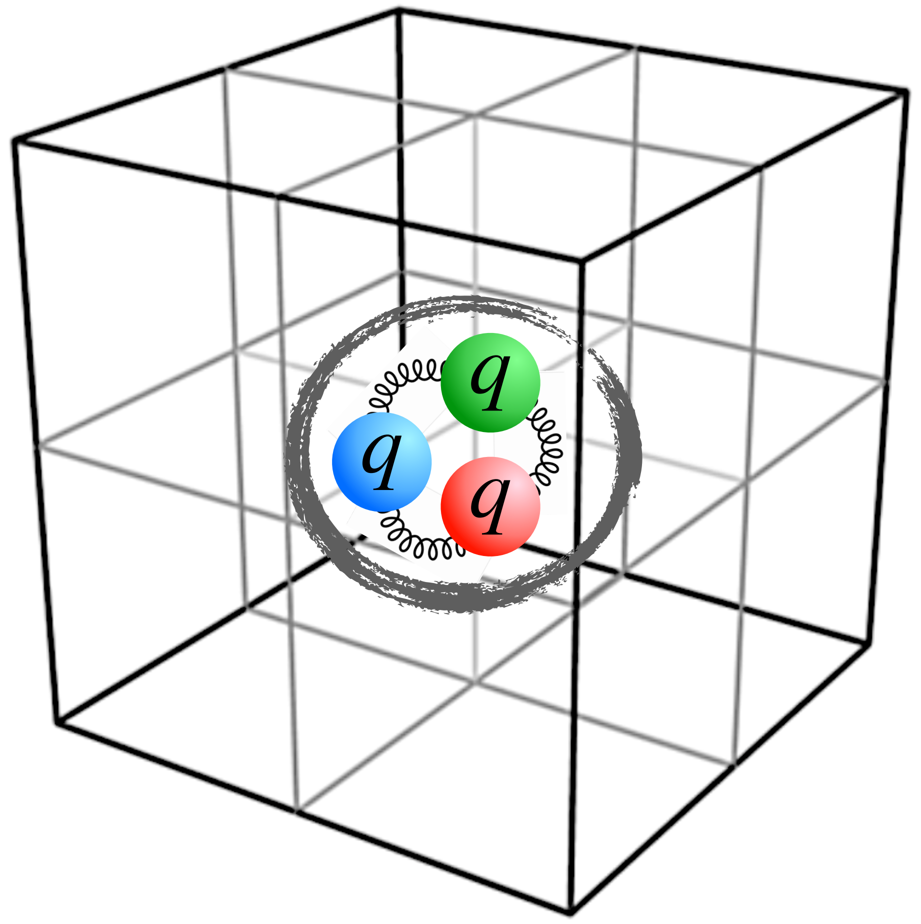
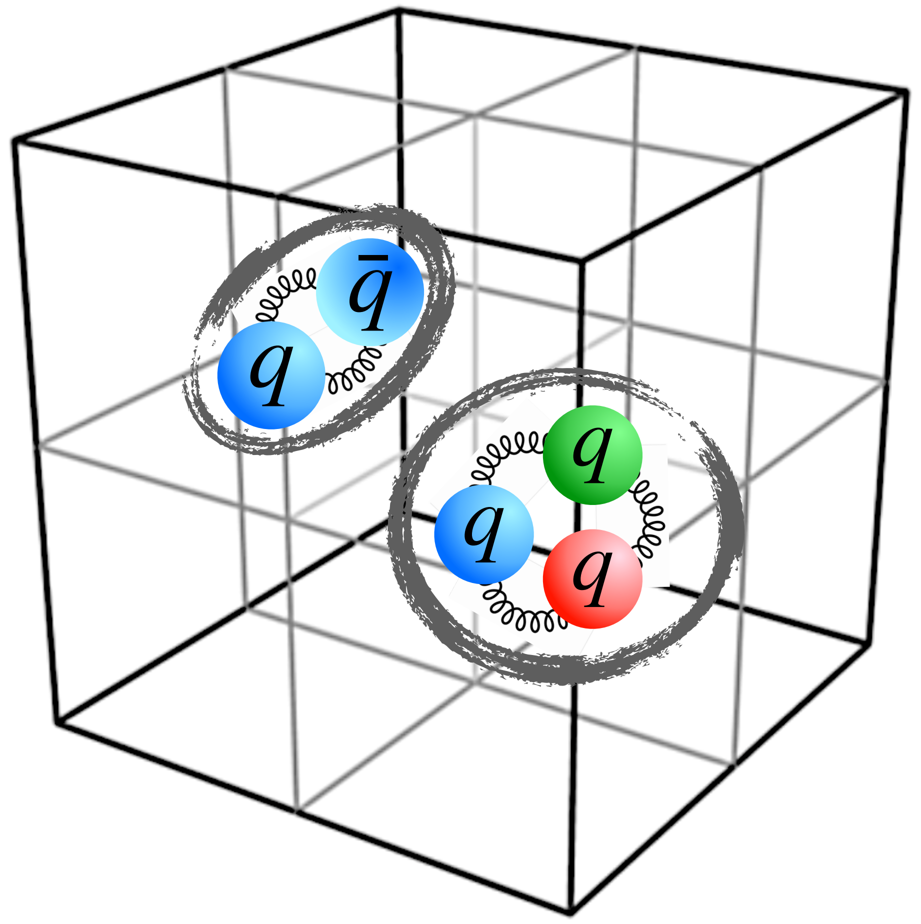


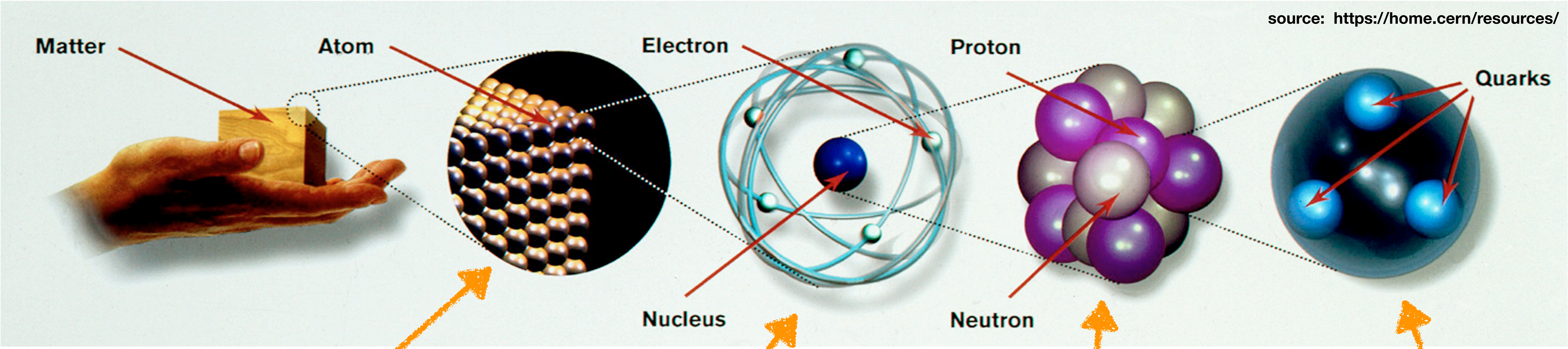


Outline

1. The strong interaction
2. Lattice Field Theory
3. Sampling in LFT
4. The QCD spectrum
5. Lattice QCD for SM tests

The strong interaction in the standard model

From atoms to quarks



Atoms bound together by van der Waals forces (electromagnetic)

Electron-nucleus attraction by Coulomb forces (electromagnetic)

protons and neutrons bound by the strong force

quarks confined in protons by the strong force

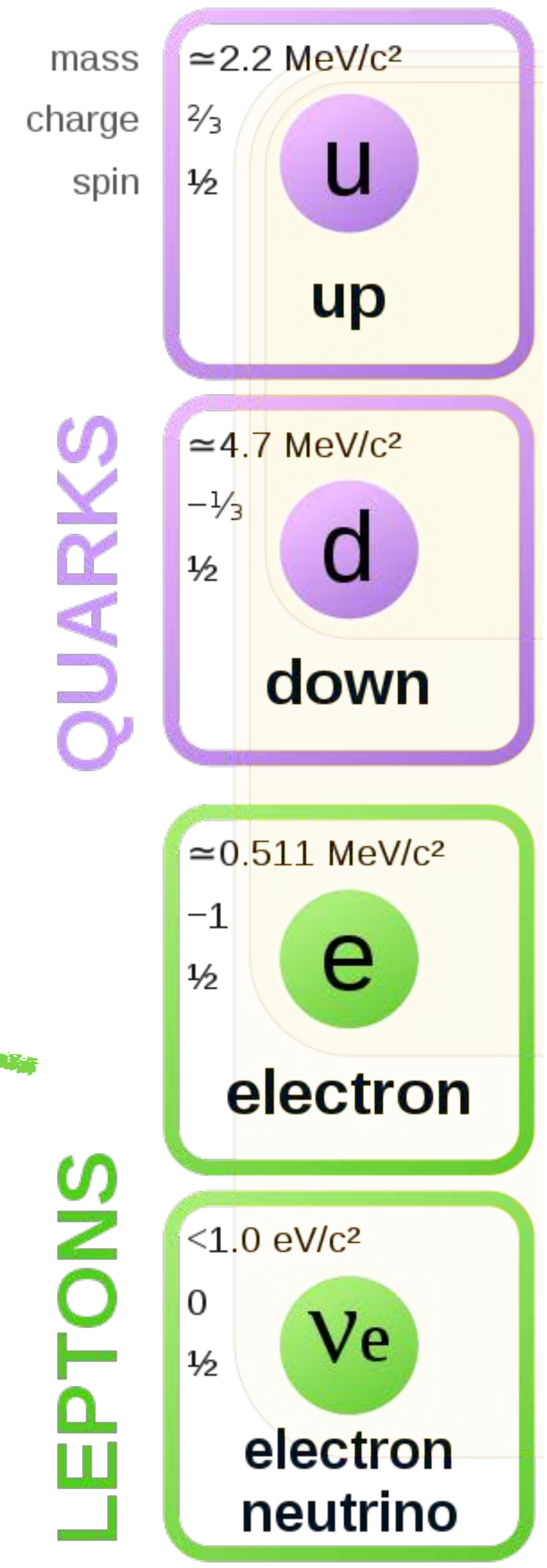
Standard Model of Particle Physics

very successful description of the subatomic world

protons & neutrons

atoms

appears in
β radiation



(including antiparticles)

[source: https://en.wikipedia.org/wiki/Standard_Model]

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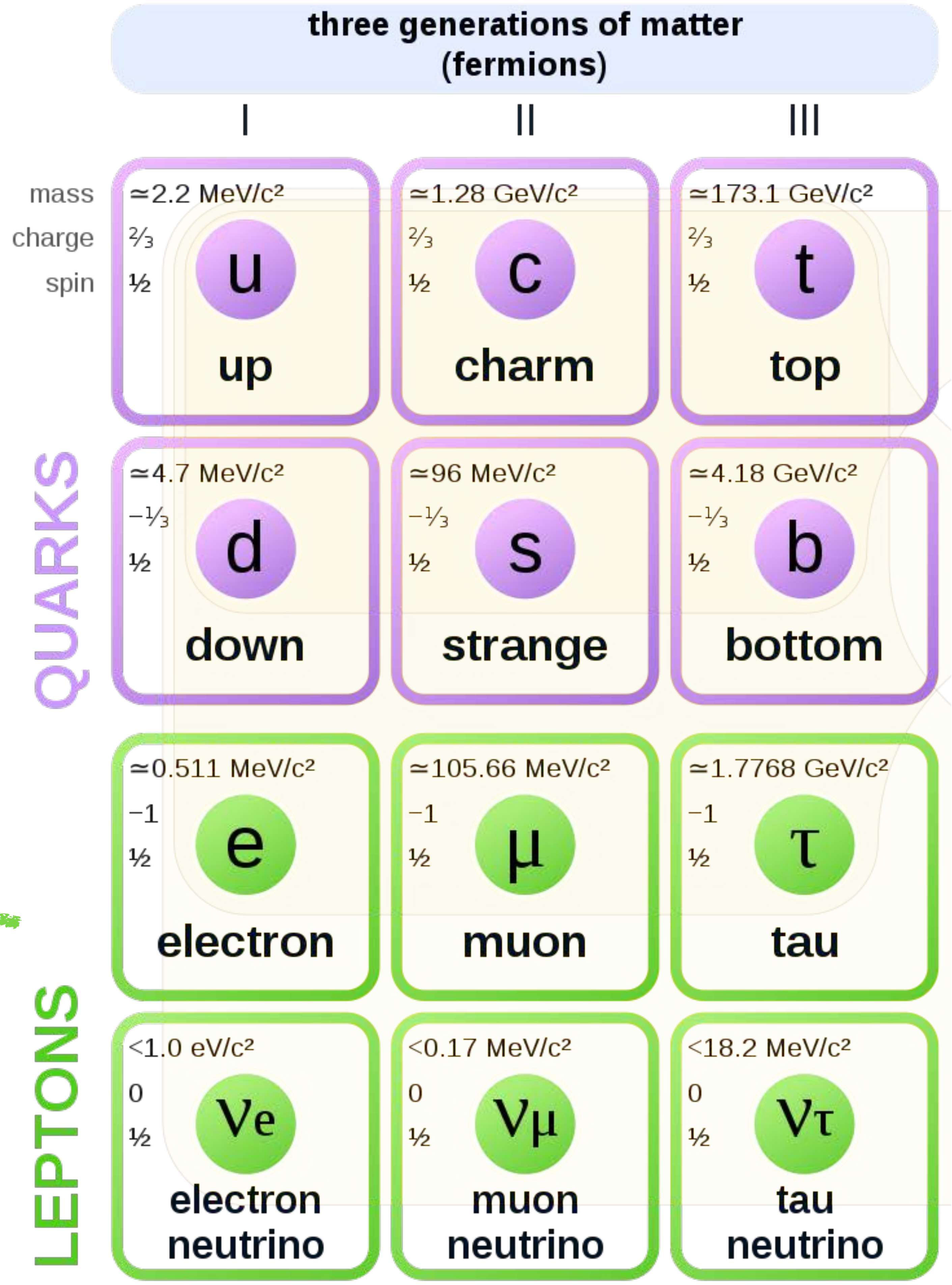

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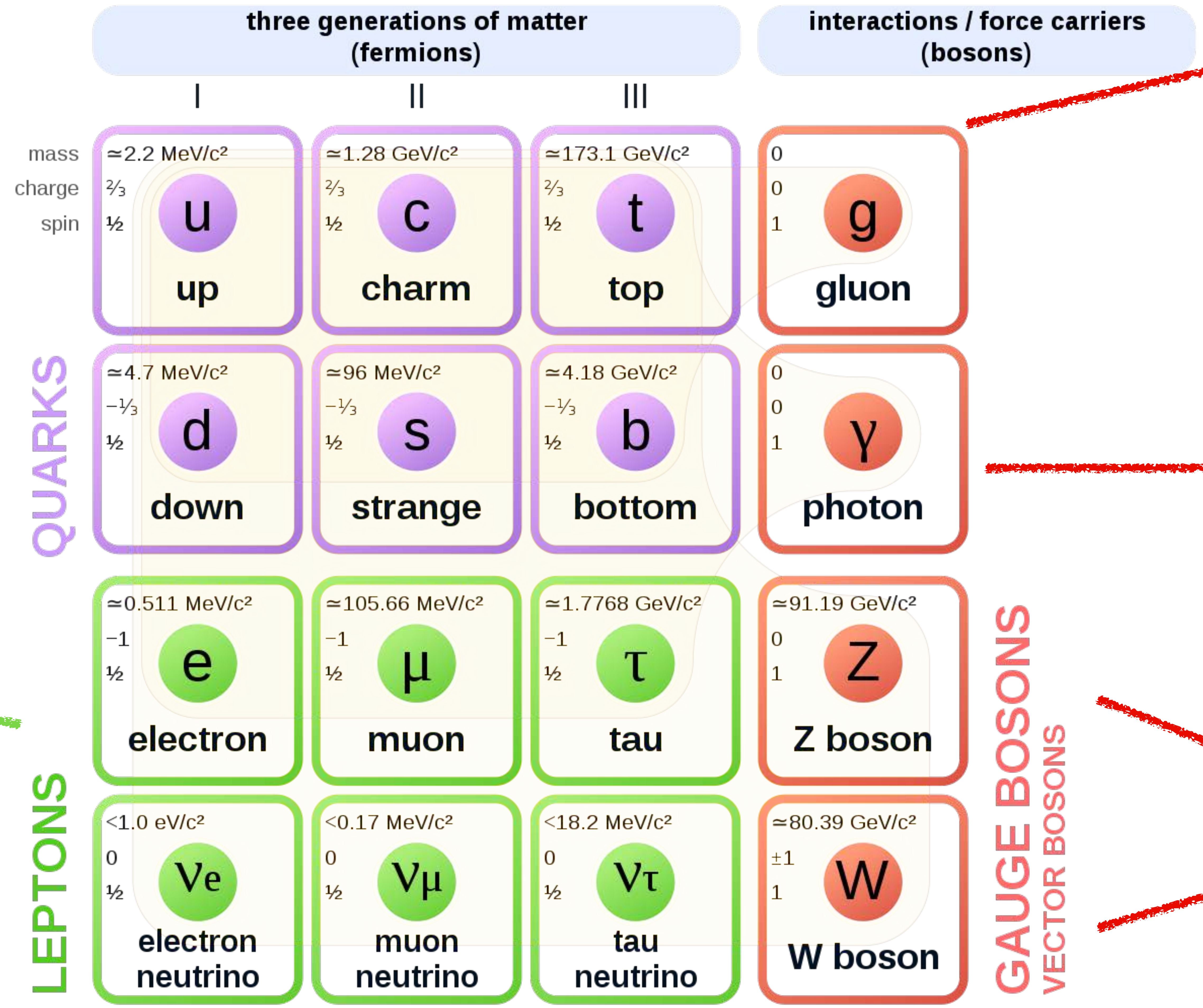

strong force

protons & neutrons

electromagnetism

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weak force

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(no gravity)

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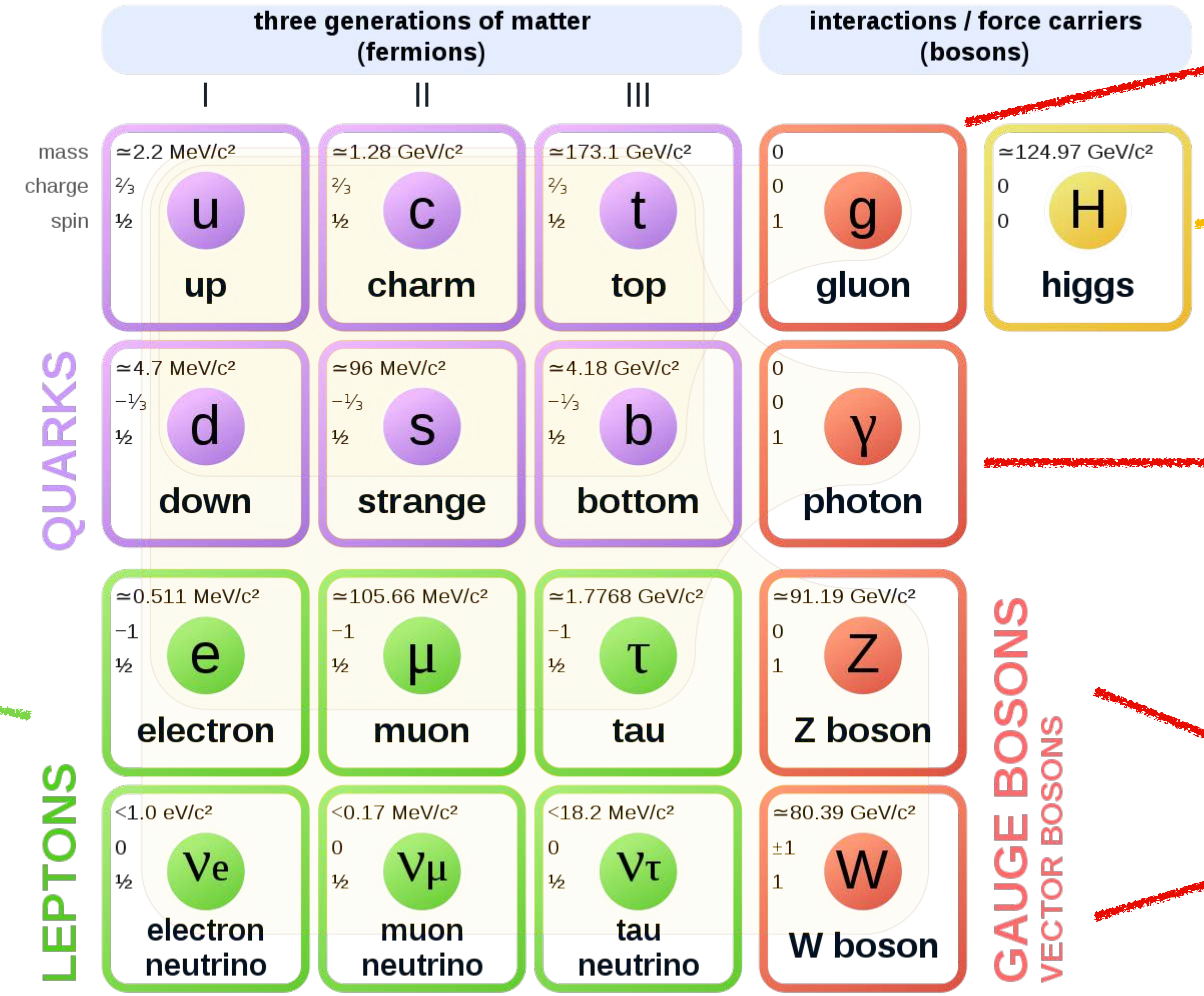

strong force

gives mass to elementary particles

electromagnetism

weak force

(no gravity)

protons & neutrons

atoms

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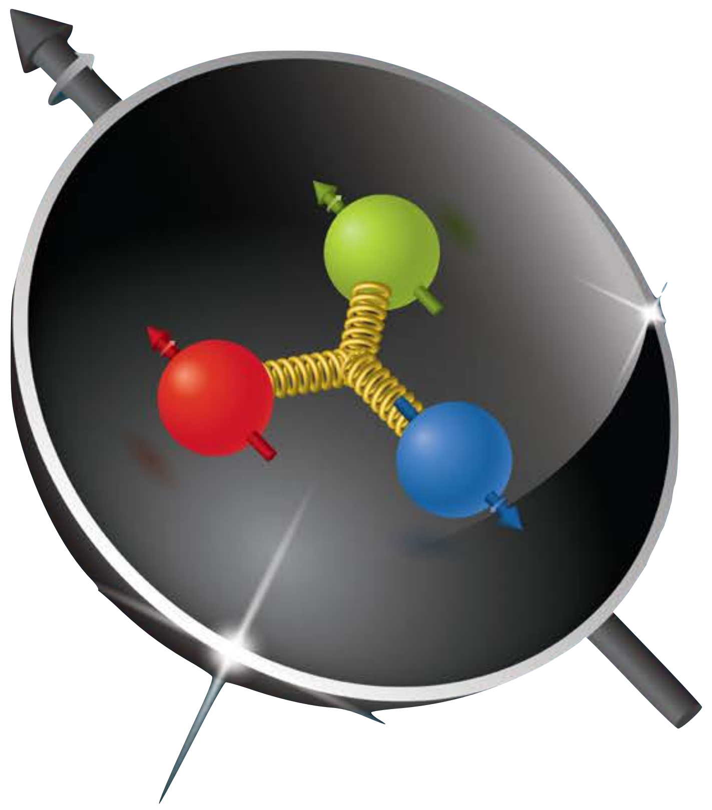
A successful theory

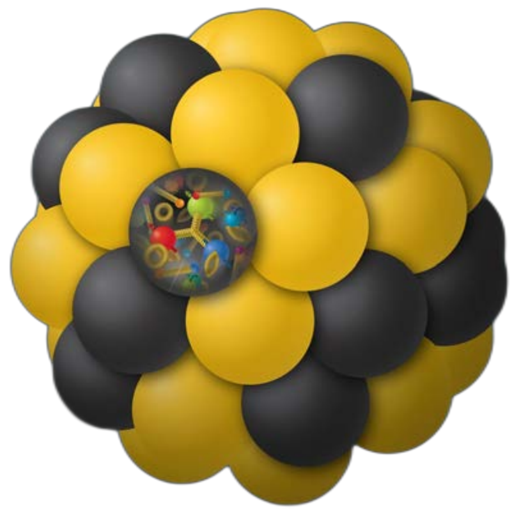
proton



source: EIC

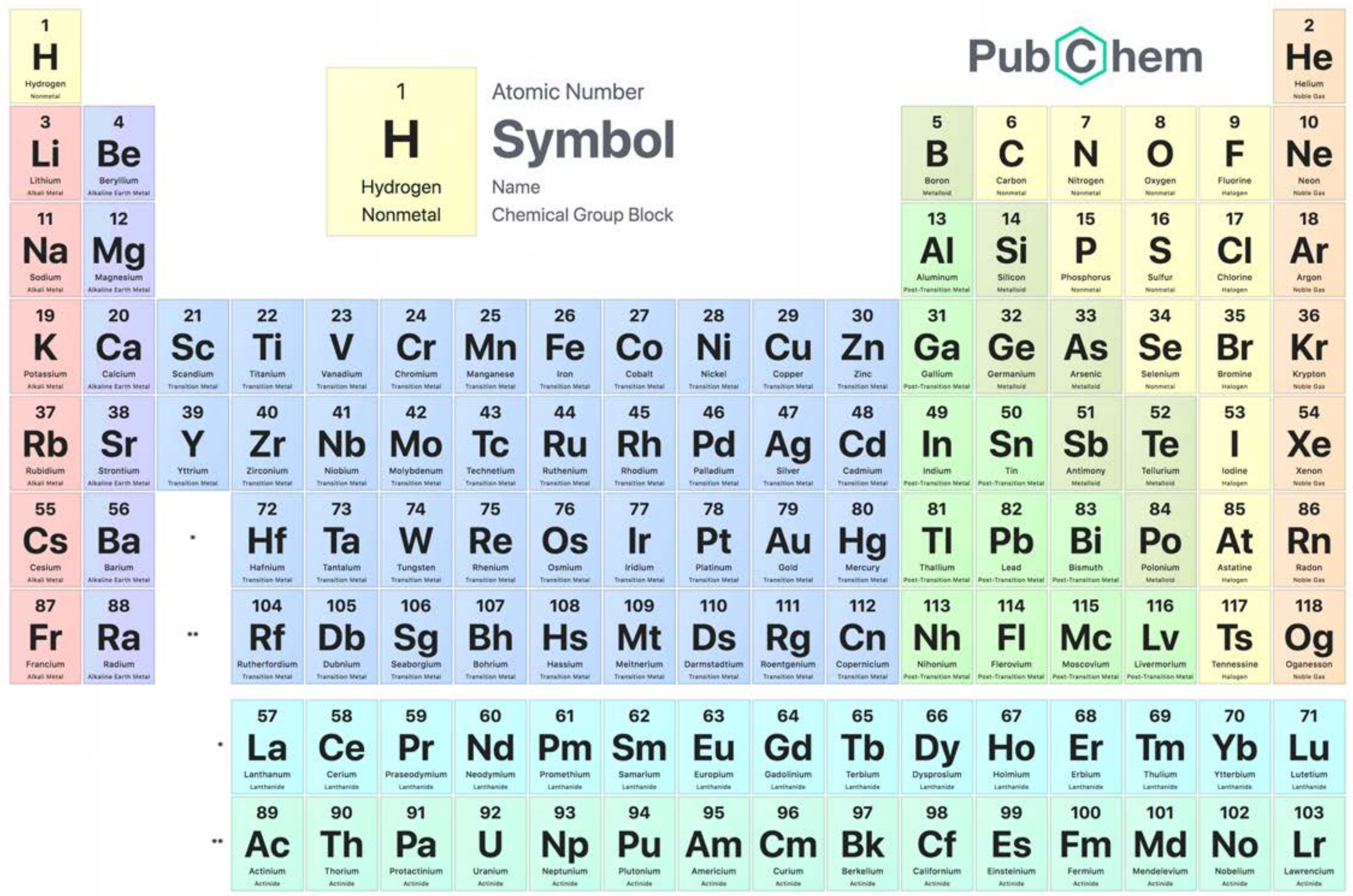
Standard Model

nuclei



Standard Model?

periodic table of elements



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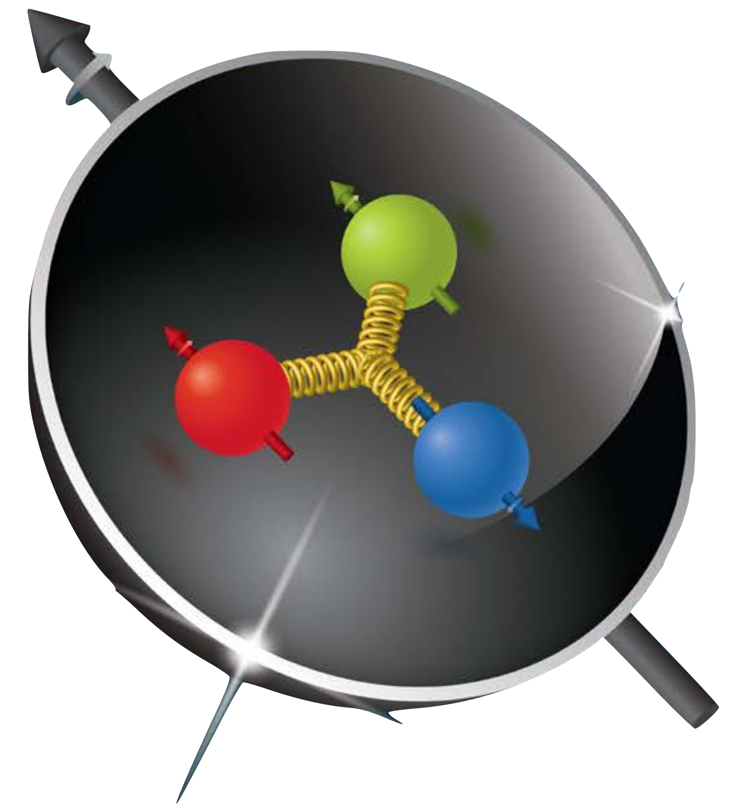

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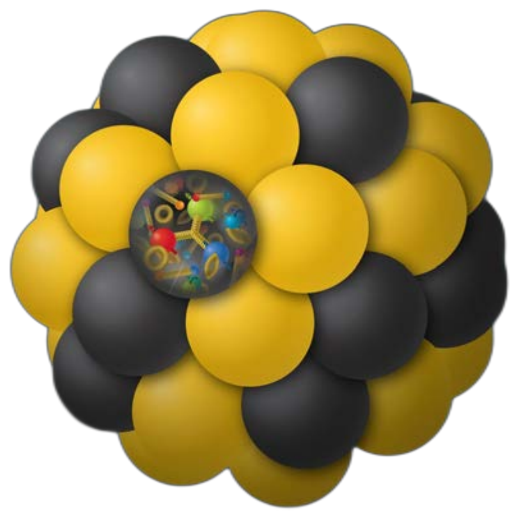

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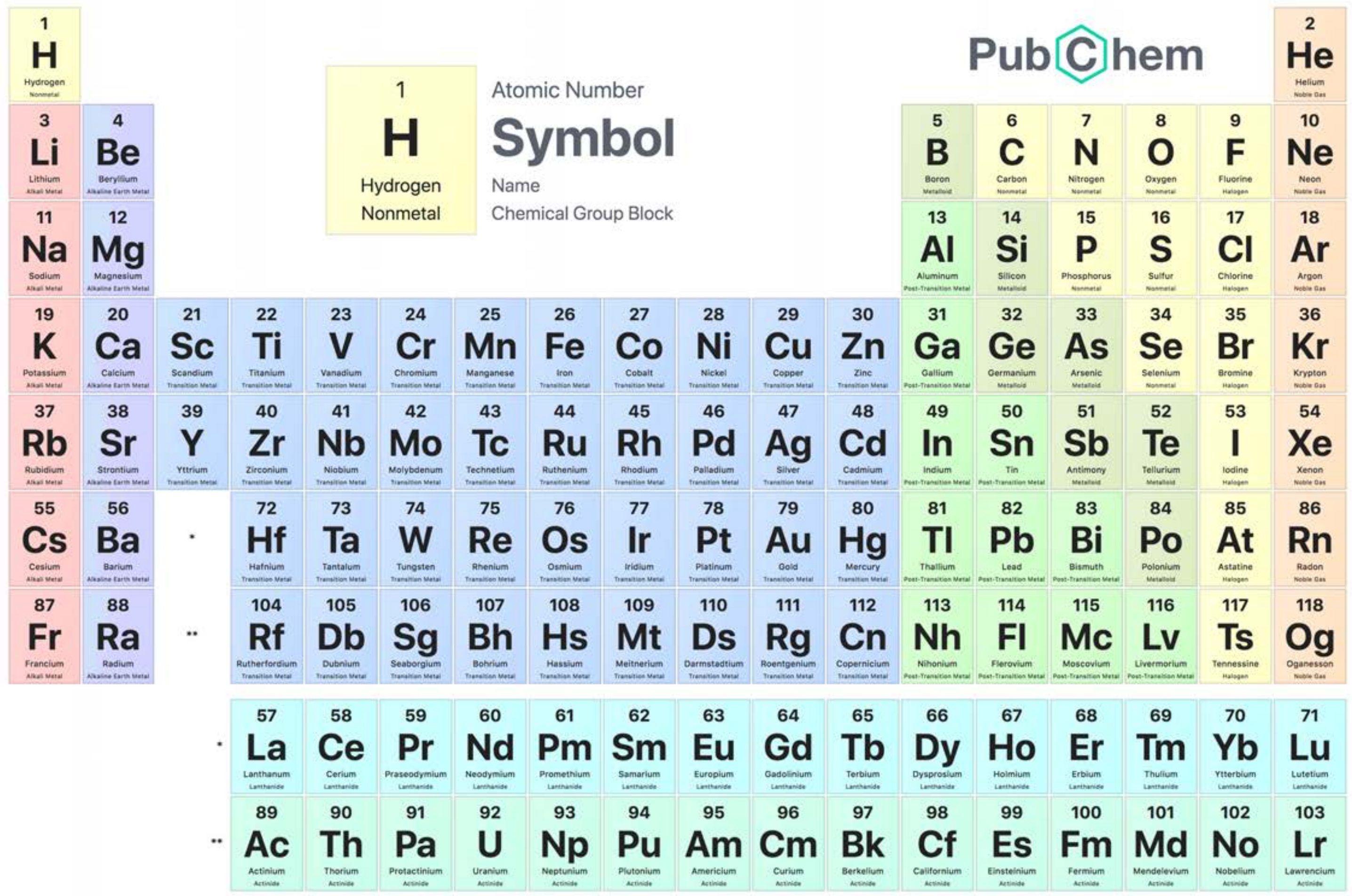


However, it is incomplete!

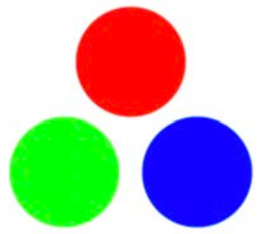
- Dark matter and dark energy
- Matter-antimatter asymmetry
- Neutrino masses
- Gravity

…

Confinement & hadrons

Our understanding of the SM is limited by the complexity of the strong force

- **Quarks and gluons carry the strong charge: the so-called "color"**



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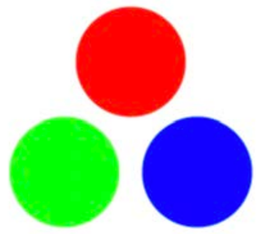
- Quarks and gluons carry the strong charge: the so-called "color" 

Confinement

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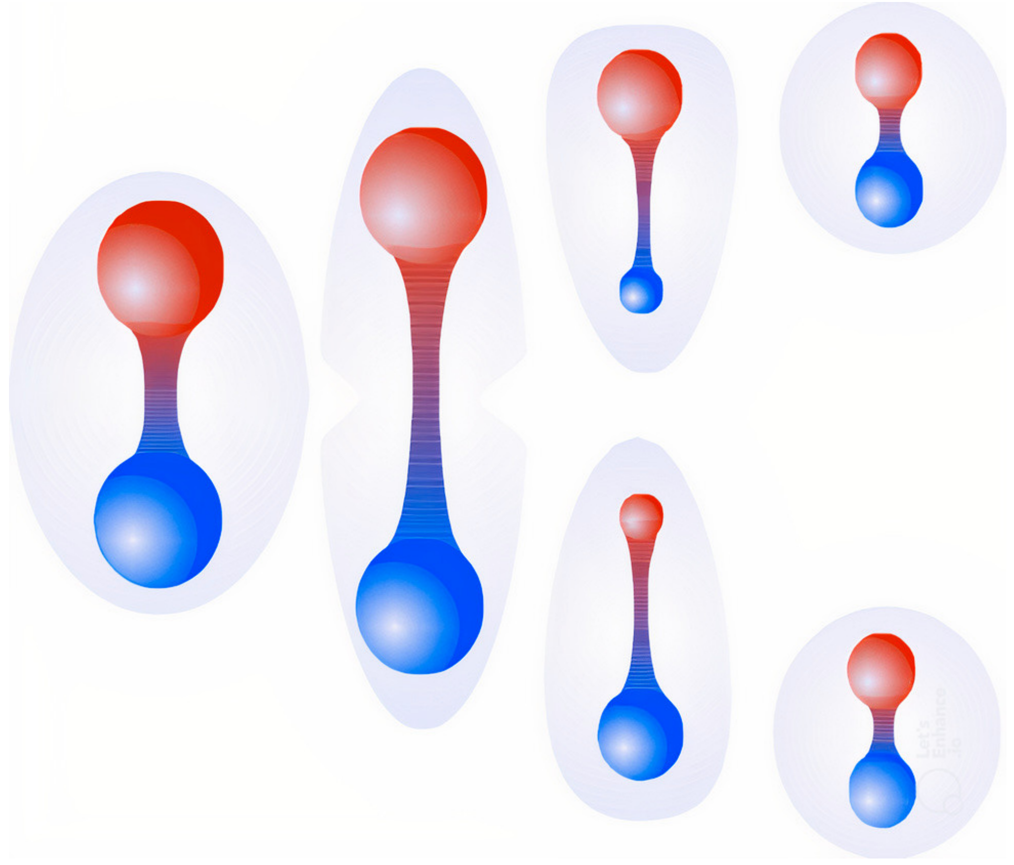

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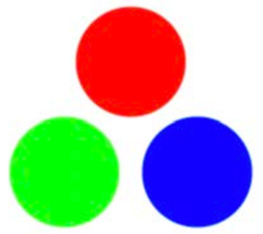

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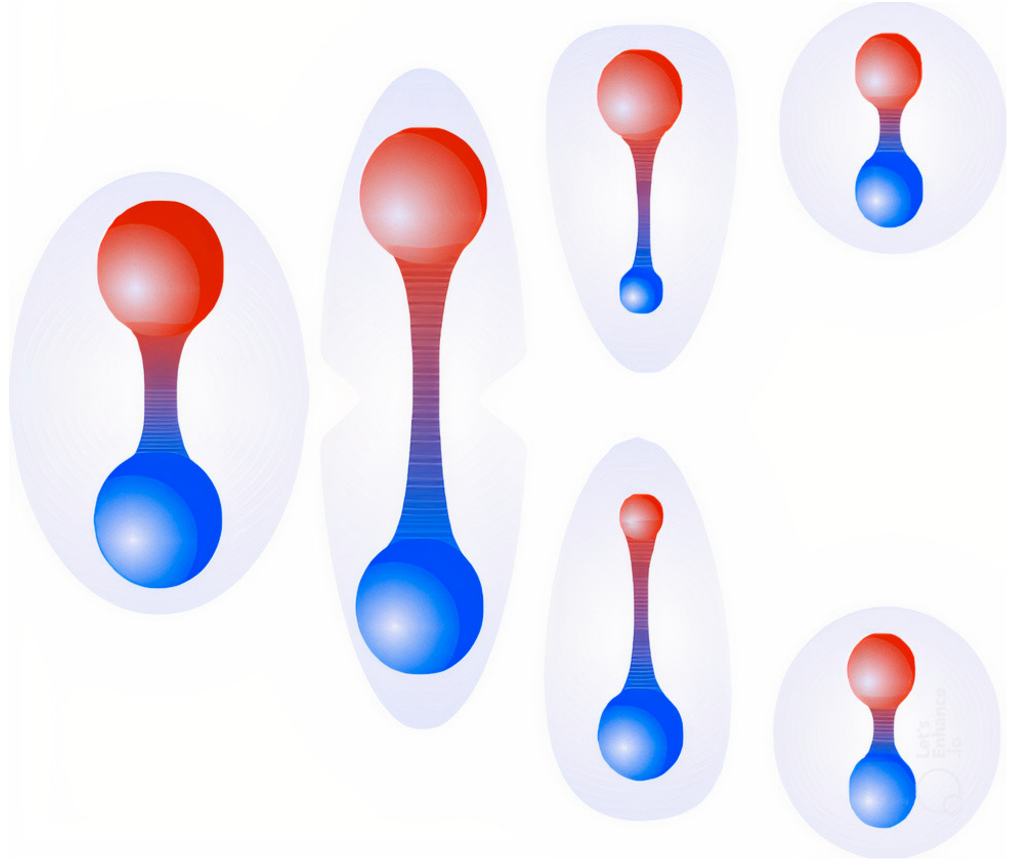

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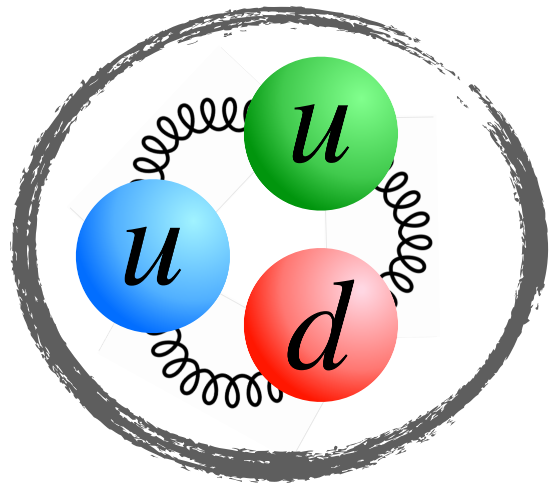
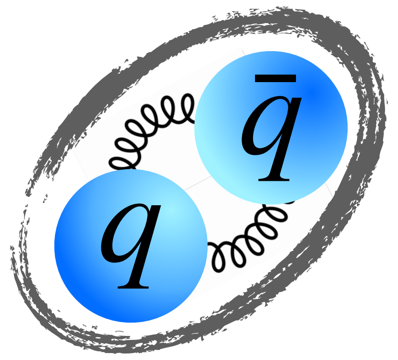
baryons
e.g. proton



mesons
e.g. π, K



standard hadrons

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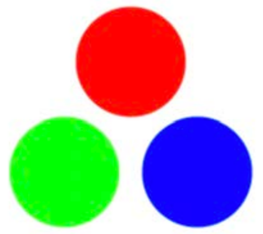

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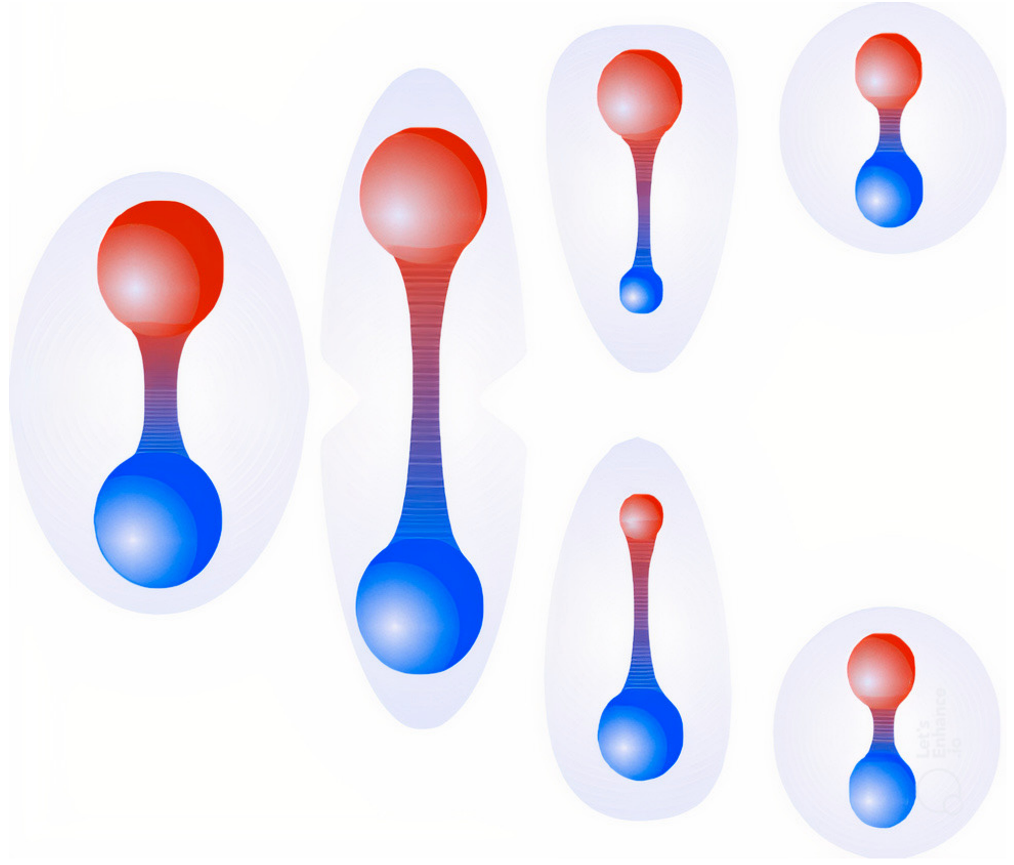

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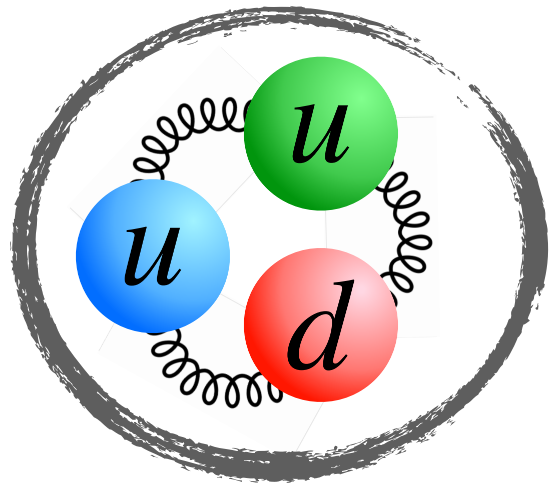

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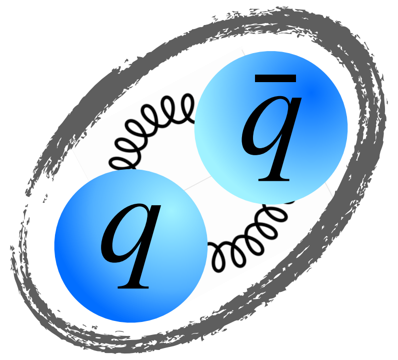
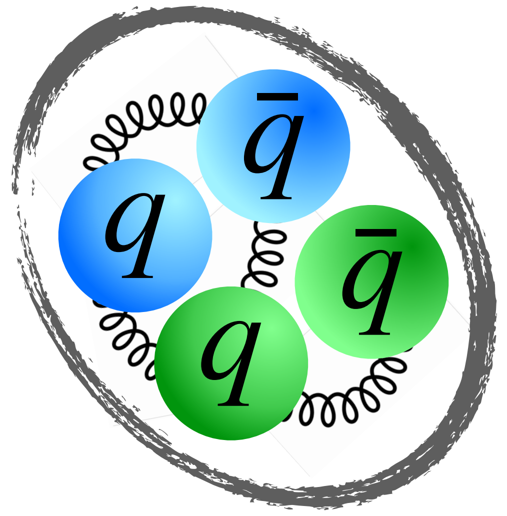


standard hadrons

tetraquarks



+ **other structures**
e.g. pentaquarks

“exotic” hadrons

Quantum Chromodynamics

> Quantum chromodynamics is conceptually simple. Its realization in nature, however, is usually very complex.
>
> Frank Wilczek

[https://physicstoday.scitation.org/doi/10.1063/1.1310117]

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$$\mathscr{L}_{QCD} = \bar{q}\left(D_\mu \gamma^\mu + m_q\right) q + \frac{1}{4g_s^2} G^a_{\mu\nu} G^{\mu\nu}_a$$

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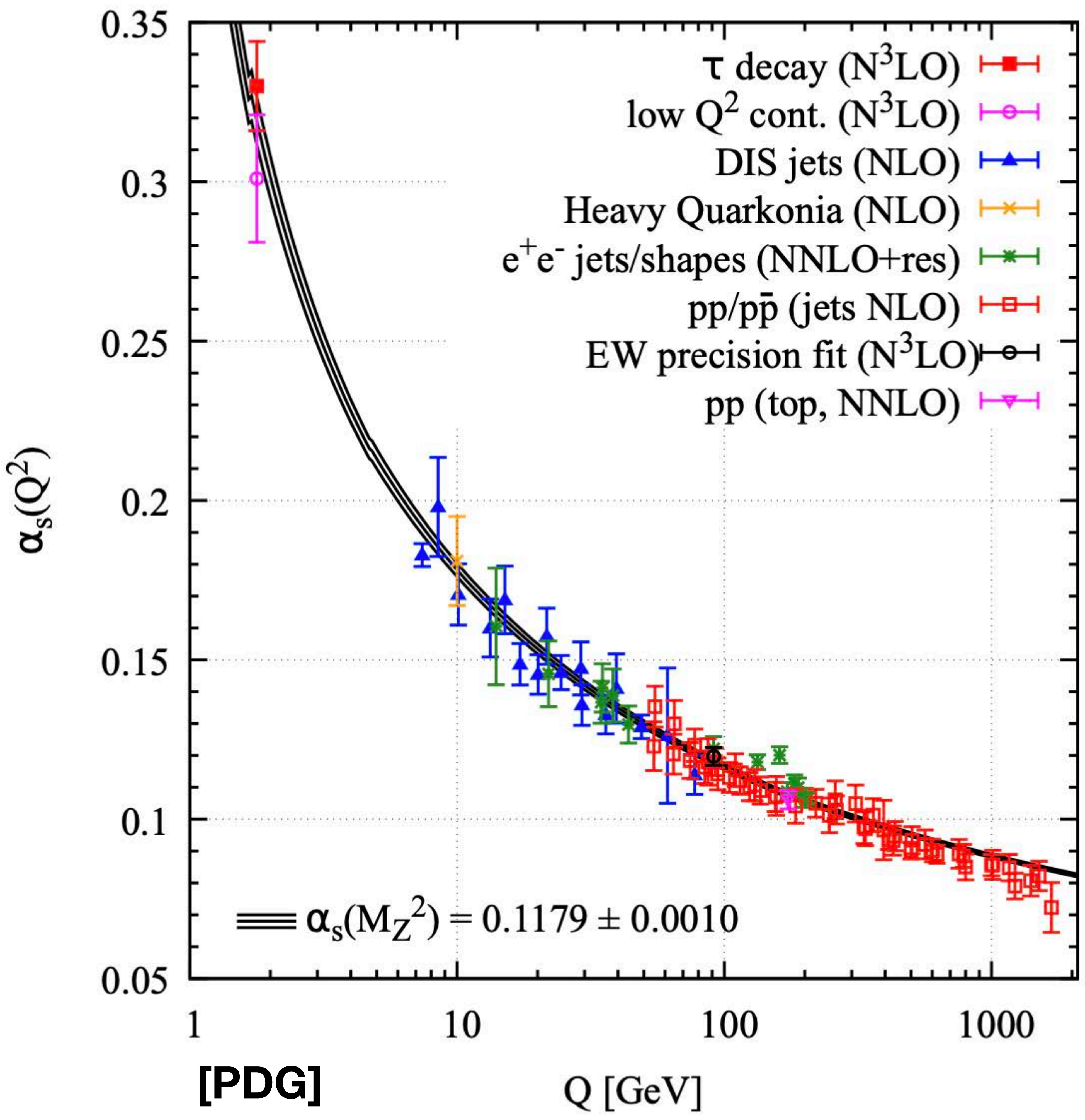
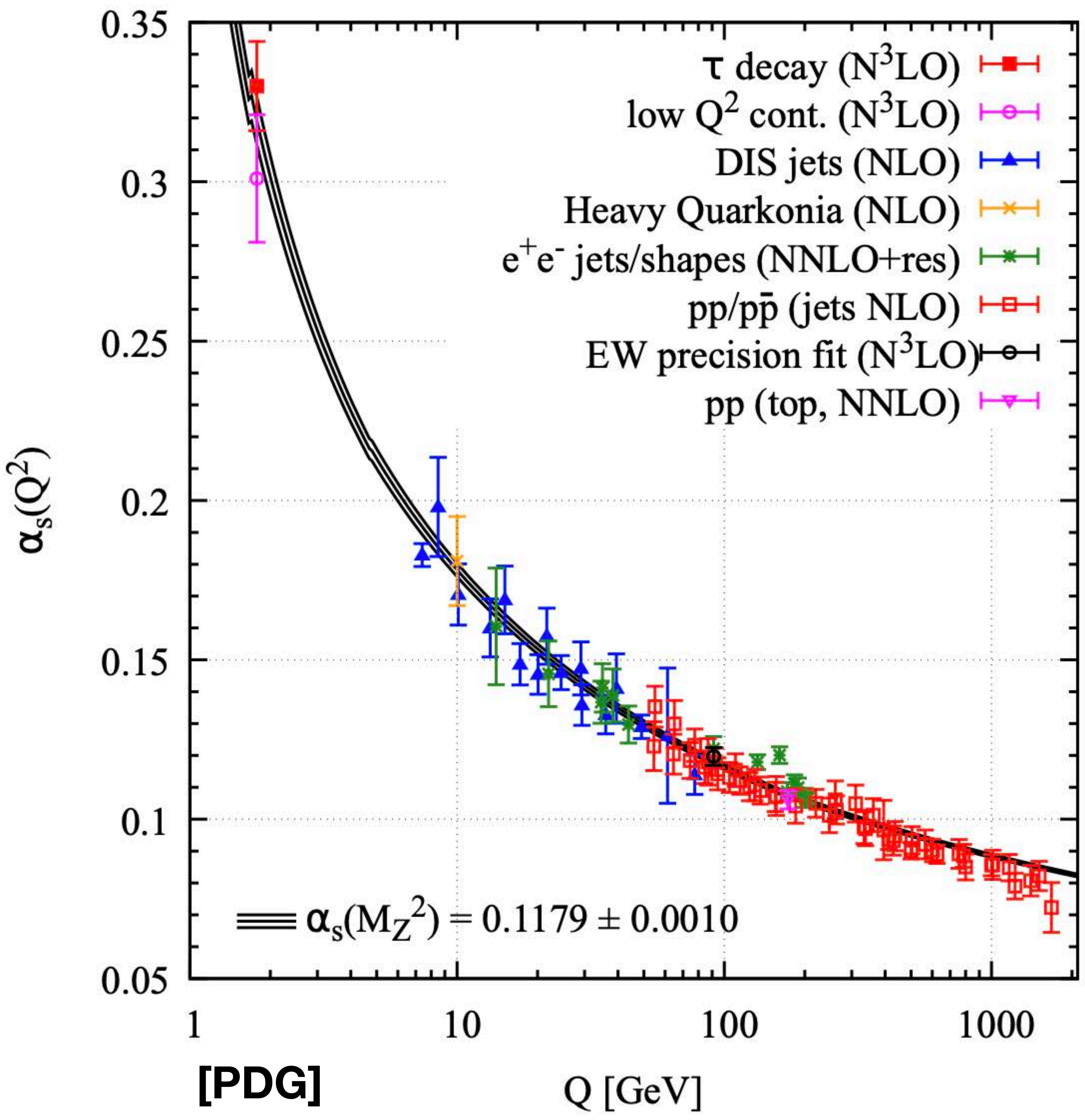
The QCD coupling



[PDG]

The QCD coupling



[PDG]

Asymptotic freedom

[Gross, Politzer, Wilczek, Nobel 2004]

Perturbation theory:

$$\mathcal{O} = \mathcal{O}_0 + \alpha_s \mathcal{O}_1 + \alpha_s^2 \mathcal{O}_2 + \ldots$$

The QCD coupling

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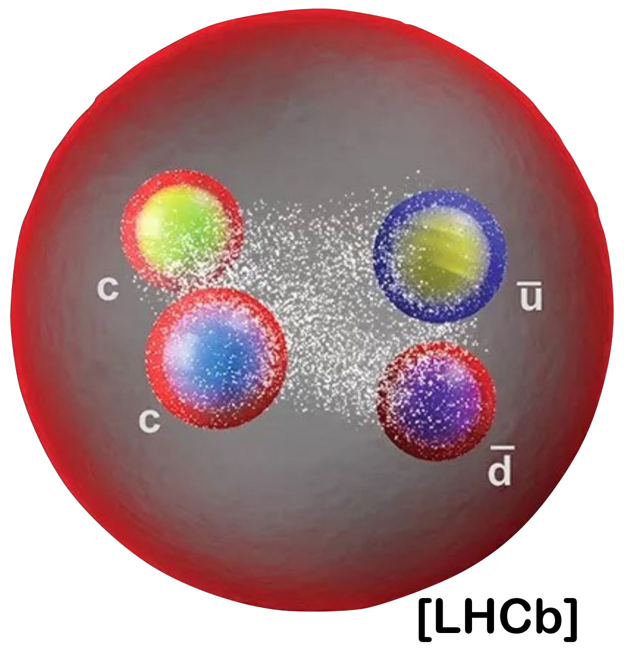
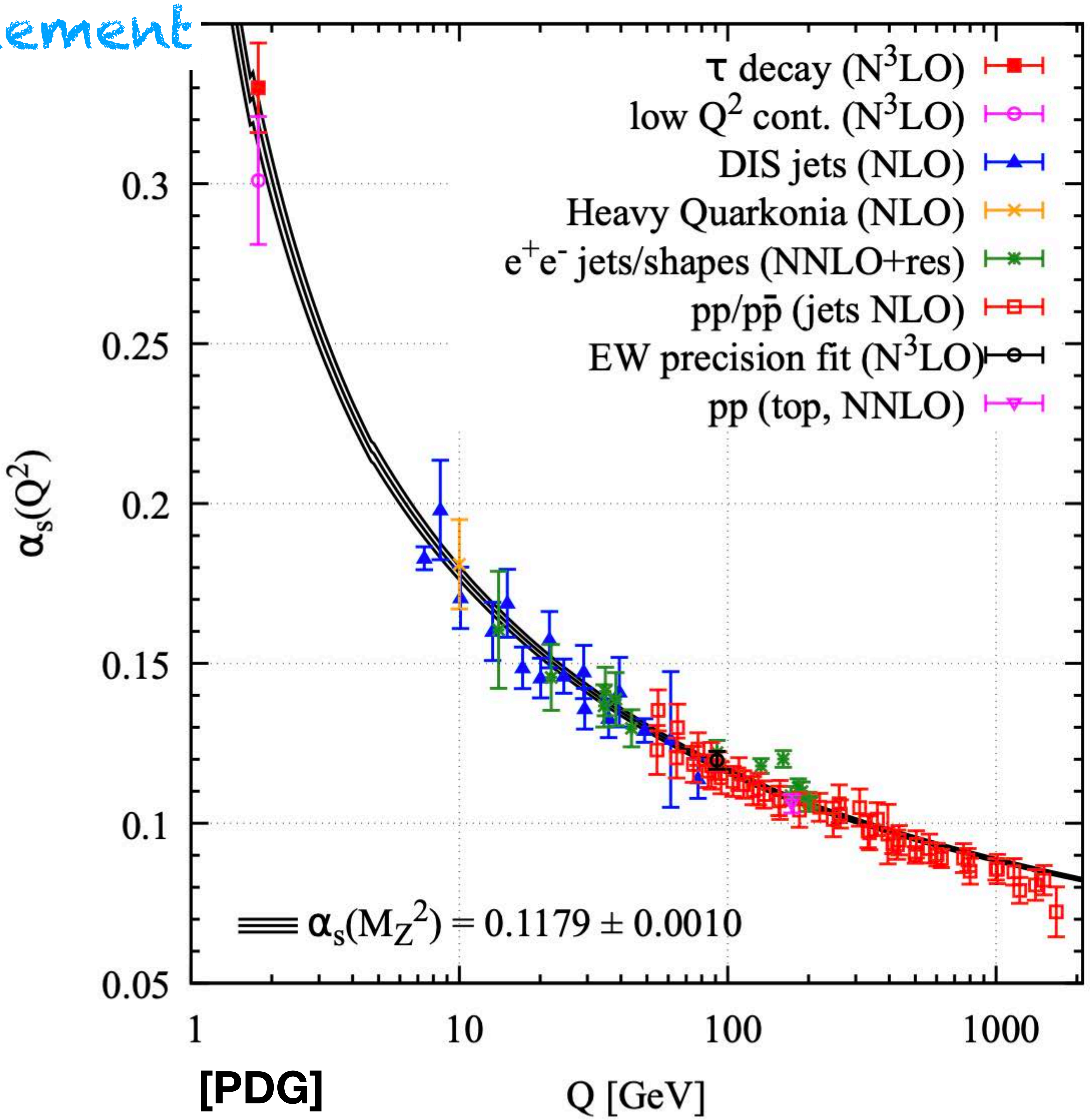


[LHCb]



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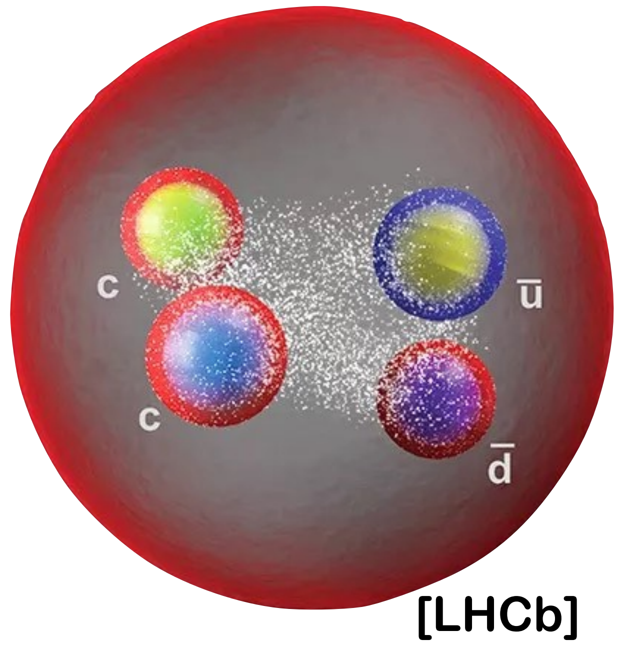
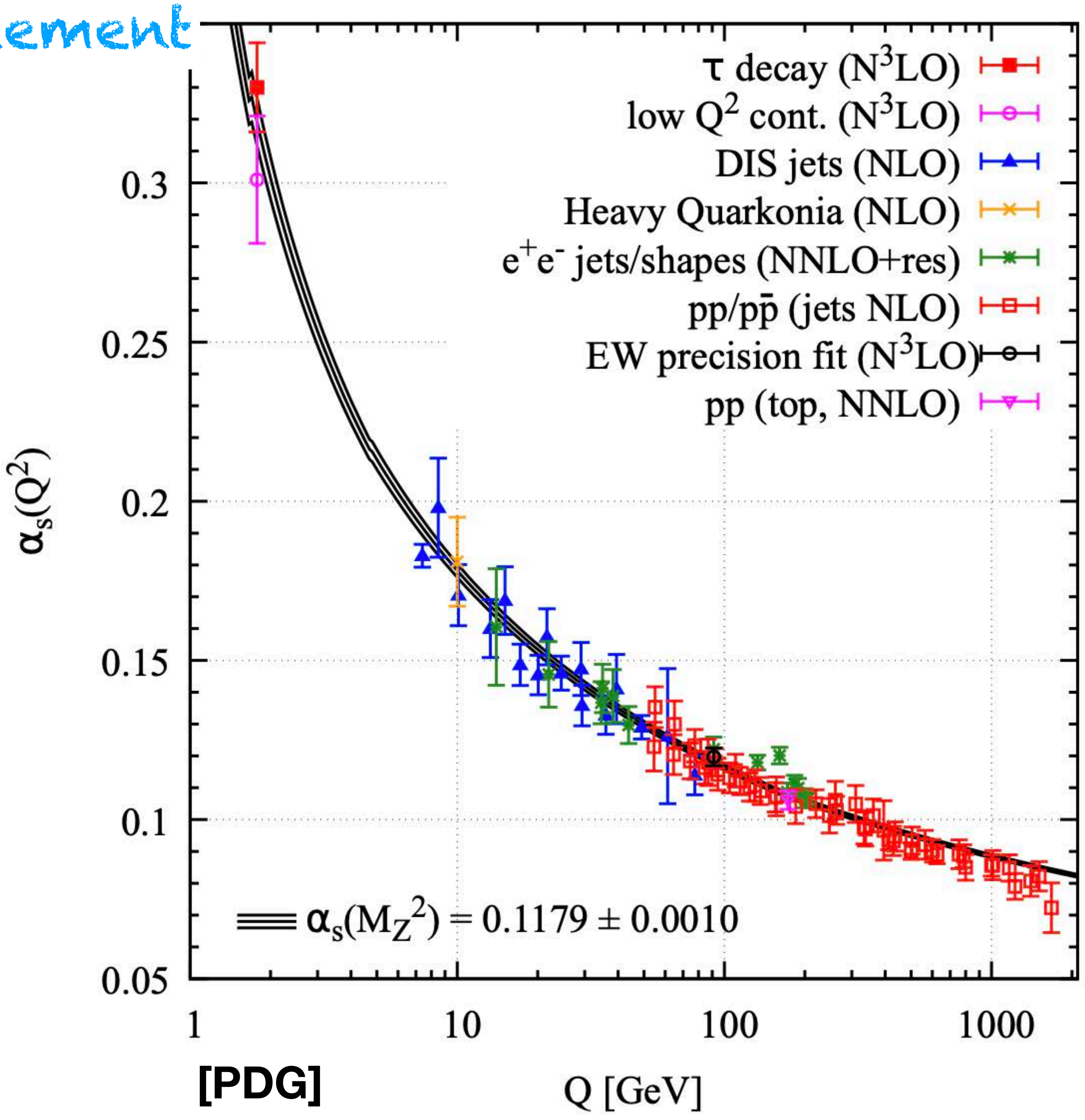


[LHCb]

Non-perturbative
at low energies

Lattice QCD



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Lattice Field Theory

Path Integral Formalism (I)

- QFTs can be formulated using the Feynman path integral formalism

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi)$$

Action

Lagrangian

Quantum Fields

$$\mathcal{Z} = \int D\phi e^{iS[\phi]}$$

Partition function

resembles partition function in statistical mechanics

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$$\underbrace{S[\phi]}_{\text{Action}} = \int d^4x \, \underbrace{\mathcal{L}}_{\text{Lagrangian}}(\underbrace{\phi}_{\text{Quantum Fields}})$$

$$\underbrace{\mathcal{Z}}_{\text{Partition function}} = \int D\phi e^{iS[\phi]}$$

resembles partition function in statistical mechanics

- Compute n-point correlation functions:

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = \frac{\int D\phi e^{iS[\phi]} \phi(x_1) \ldots \phi(x_n)}{\mathcal{Z}}$$

contain information about interactions n-particle interactions

Path Integral Formalism (II)

- Partition function almost looks like the normalization of some probability distribution

$$\mathcal{Z} = \int D\phi e^{-iS(\phi)}$$

  - But complex numbers
  - Infinite number variables

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- Perform Wick rotation to Euclidean spacetime:

$$x^0 \to -ix_E^0, \qquad \int d^4x \to i\int d^4x_E \qquad \partial_\mu\partial^\mu \to -\partial_\mu\partial_\mu$$

- Path integral in Euclidean or imaginary time: statistical meaning

$$\mathcal{Z} = \int D\phi\, e^{-S_E(\phi)}$$, where $S_E(\phi) = \int d^4x_E \mathscr{L}_E(\phi)$ Euclidean action

Boltzmann factor

Lattice Field Theory

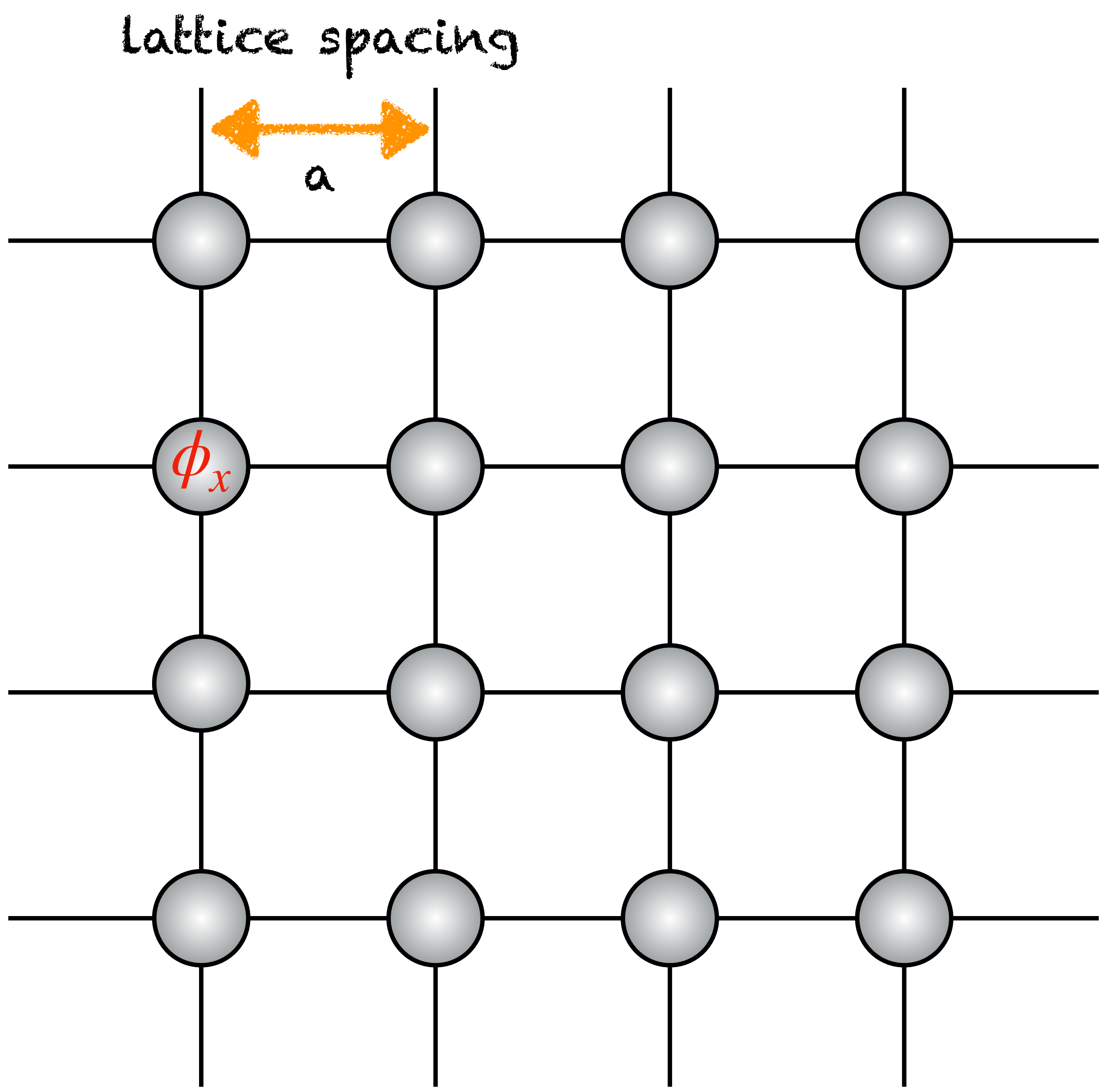
- LFT is the first-principles treatment of the generic QFT
  - Path integral in Euclidean or imaginary time: statistical meaning

$$\mathcal{Z} = \int D\phi \, e^{-S_E(\phi)}$$, where $S_E(\phi) = \int d^4x_E \mathscr{L}_E(\phi)$

Euclidean action

"Boltzmann factor"

  - Discretize quantum fields (real scalars):



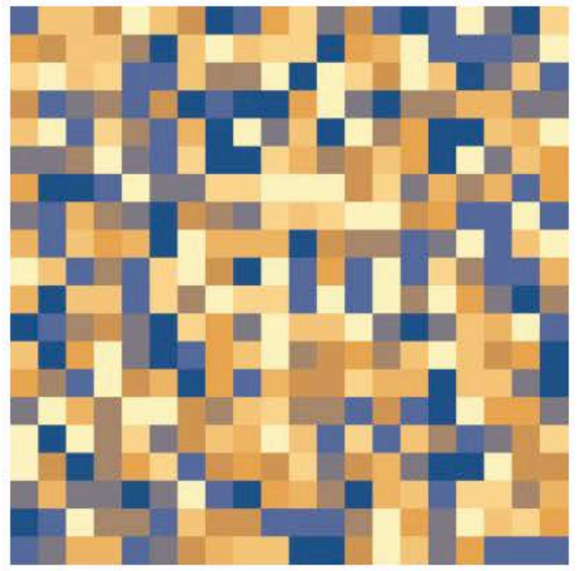
Continuum: $$S_E = \int d^4x \left[\frac{1}{2}\partial_\mu\phi(x)^2 + \frac{m^2}{2}\phi(x)^2 + \lambda\phi(x)^4\right]$$

Lattice: $$S_E = a^4 \sum_x \frac{1}{2a^2}(\phi_{x+\mu} - \phi_x)^2 + \frac{m^2}{2}\phi_x^2 + \lambda\phi_x^4$$

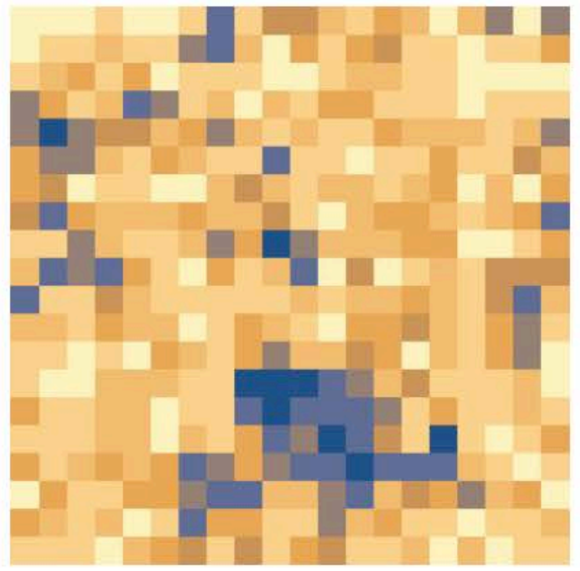
Test case: scalar theory

- A **real scalar field** per lattice site in a 2D lattice. $\phi(x) \in (-\infty, +\infty)$

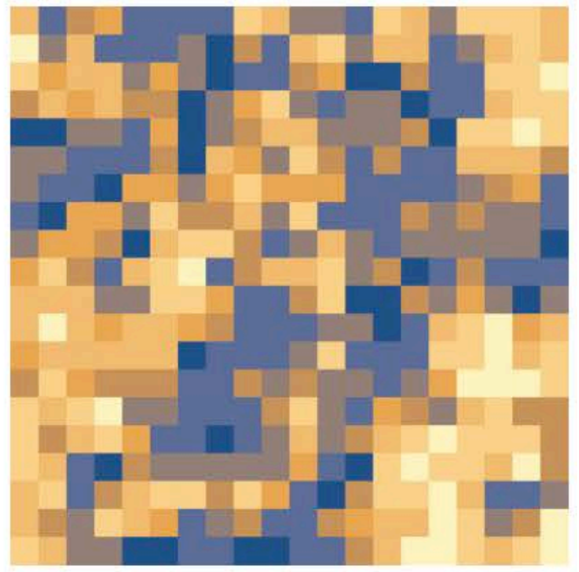
A real number in each point of space time in the lattice



unlikely
(log prob = -6107)



likely
(log prob = 22)



likely
(log prob = 5)

long-range correlations

no structure, i.e. looks like random noise

Field configurations with probability:

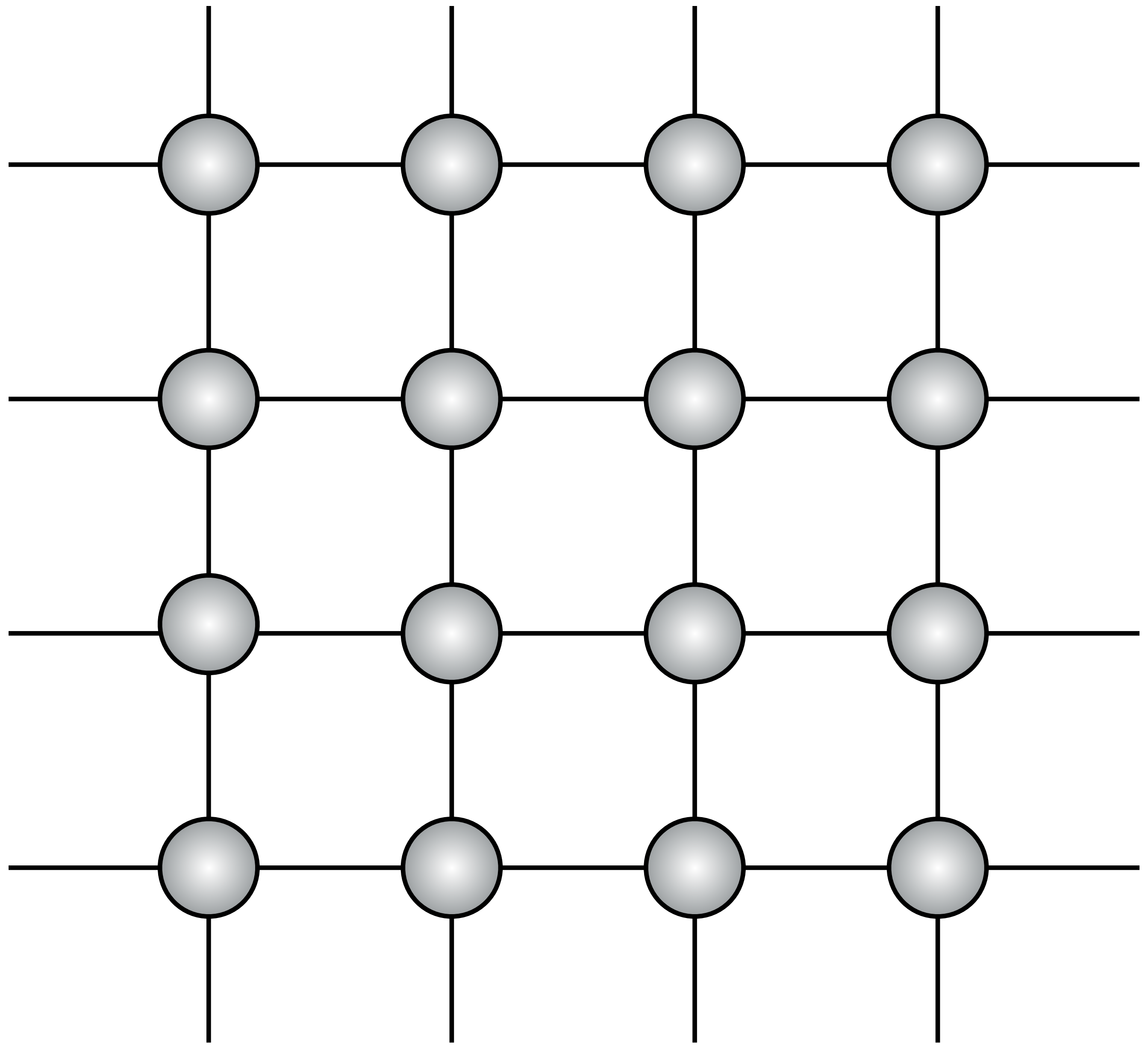
$$P[\phi(x)] \sim e^{-S[\phi(x)]}$$

- Lattice action:

$$S = a^4 \sum_x \frac{1}{2a^2}(\phi_{x+\mu} - \phi_x)^2 + \frac{m^2}{2}\phi_x^2 + \lambda\phi_x^4$$

Lattice QCD Basics

- Lattice QCD is the state-of-the-art treatment of the strong interaction at hadronic energies



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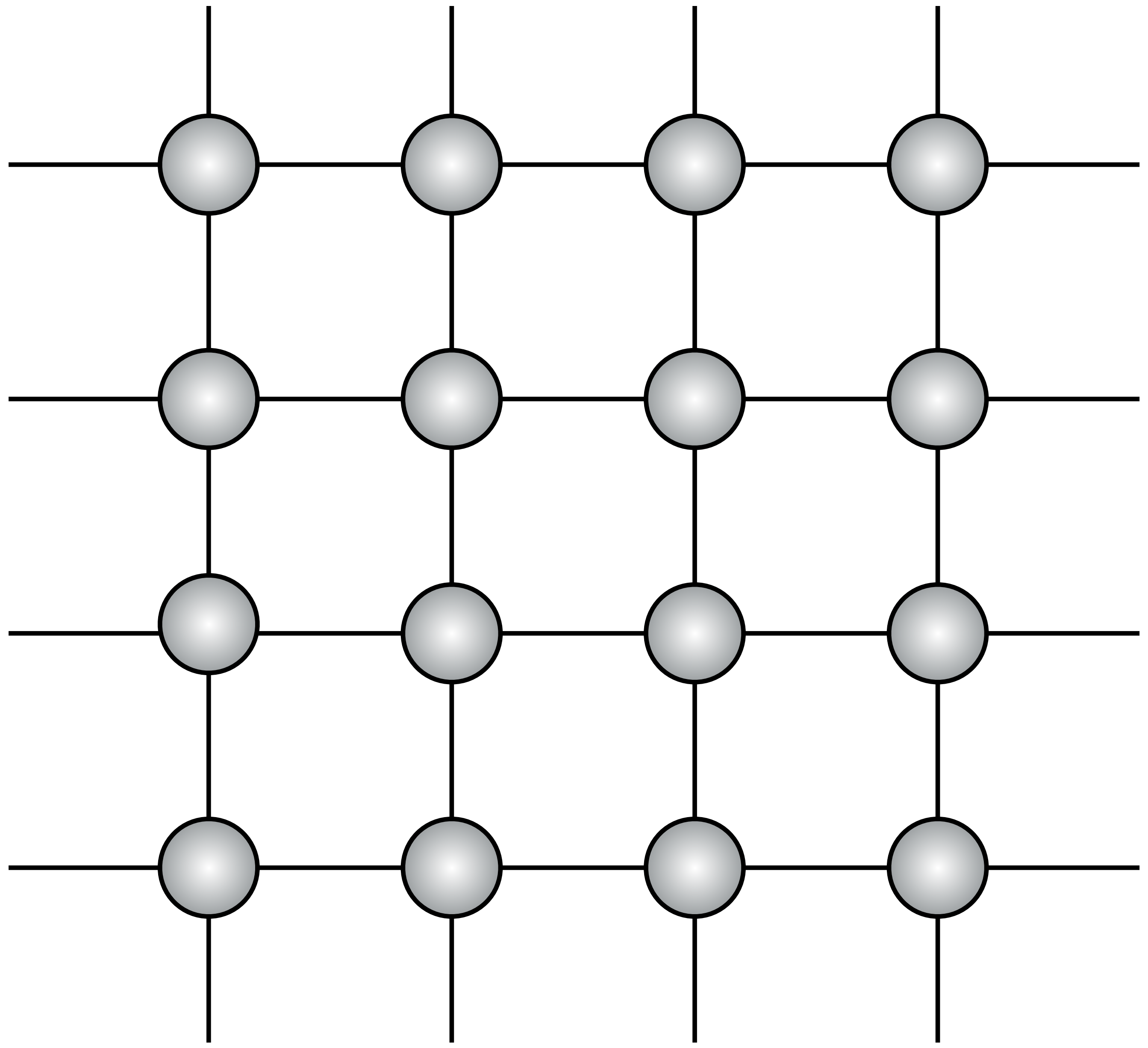
- Euclidean space-time: action has statistical meaning

$$\mathcal{Z} = \int D\psi\, D\bar{\psi}\, DA e^{-S_E(\psi,\bar{\psi},A_\mu)}$$

"Boltzmann factor"



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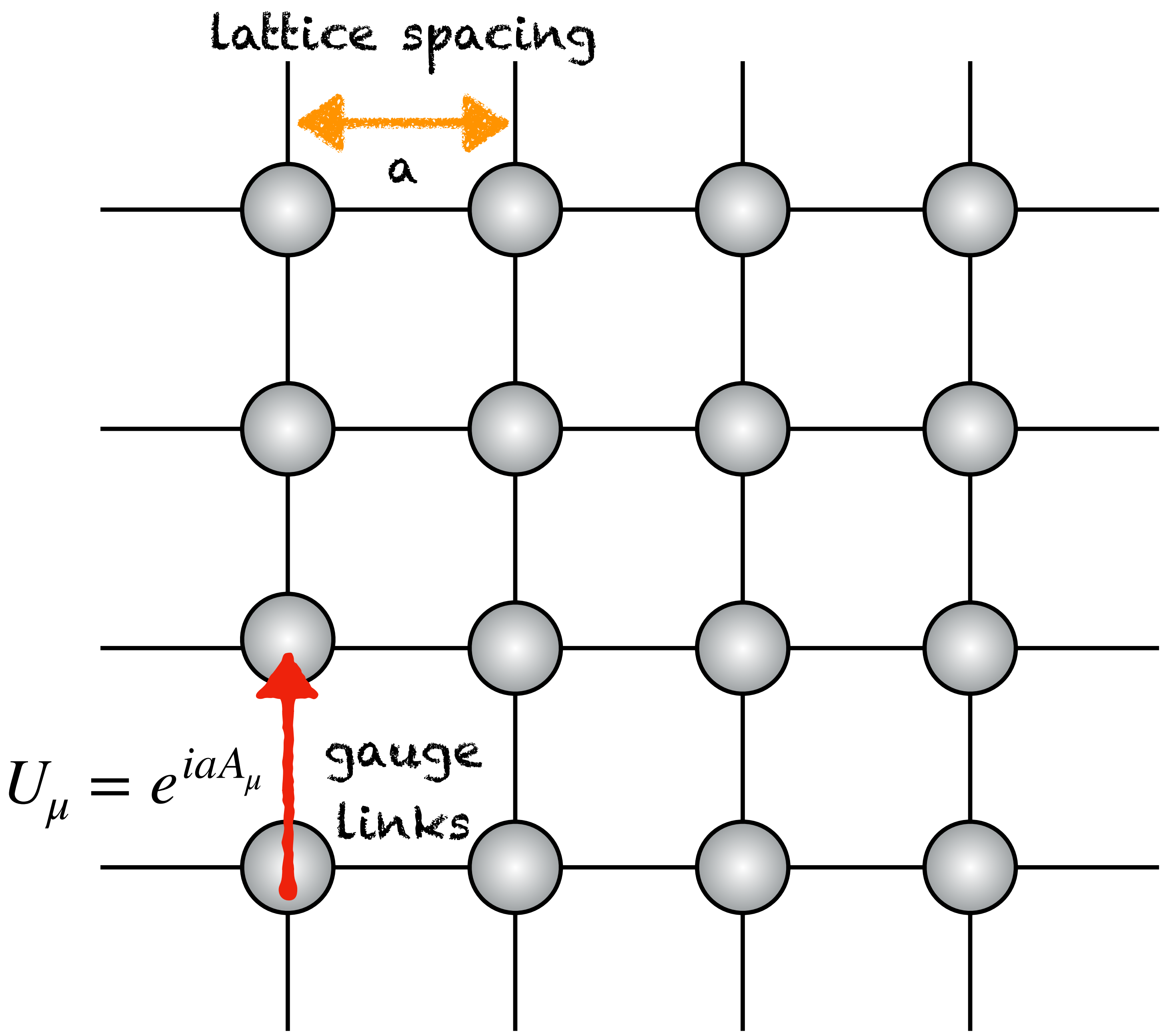
  →Under control but technical

  →Need continuum limit



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"Boltzmann factor"

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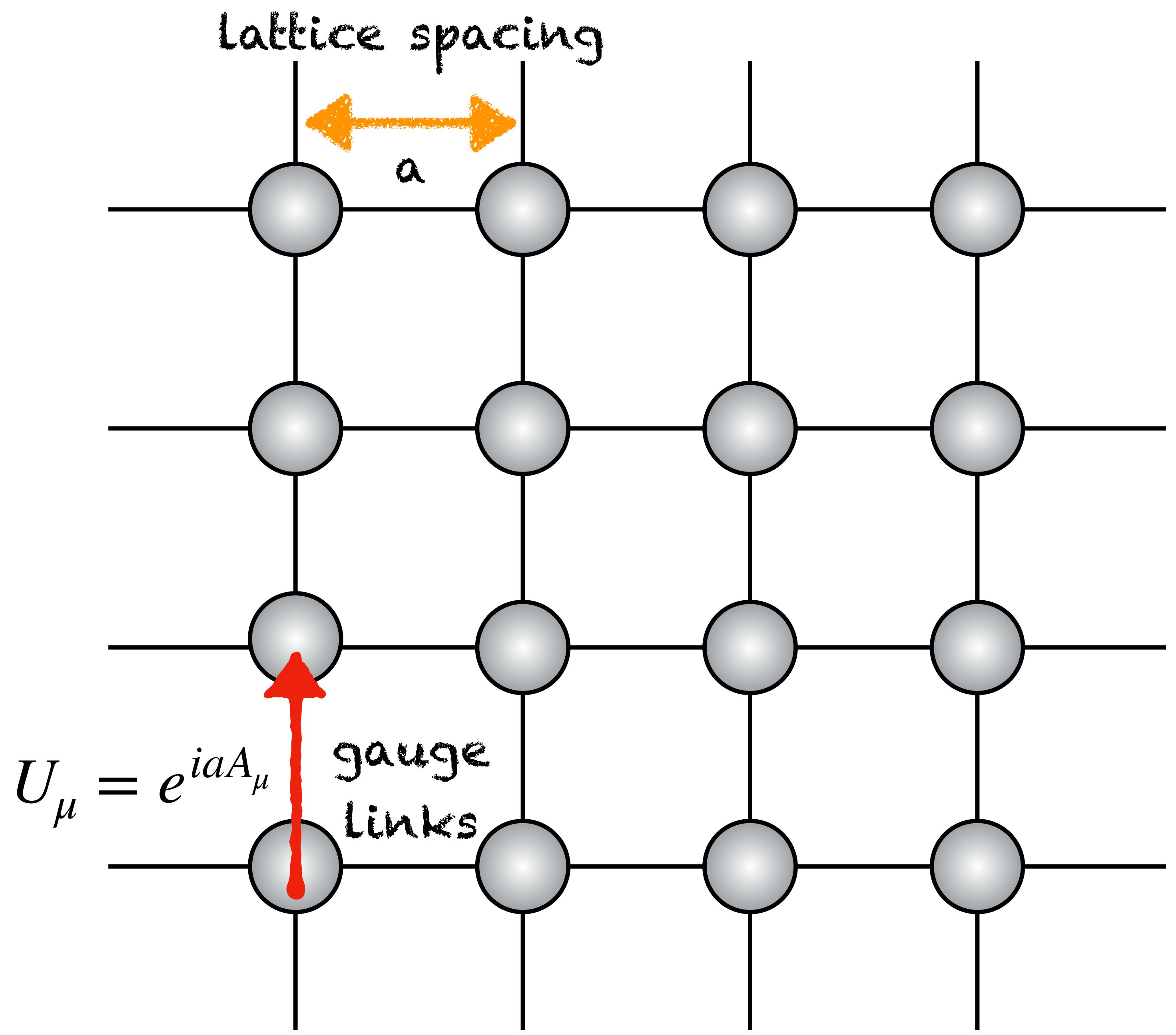
- Compute observables as expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int D\psi D\bar{\psi} DA\, \mathcal{O}\, e^{-S_E(\psi,\bar{\psi},A_\mu)}$$



An integral over many variables:
$2 \times 3^2 \times 4 \times L^4 \simeq 10^{10}$

Sampling in Lattice Field Theory

Markov Chain Monte Carlo

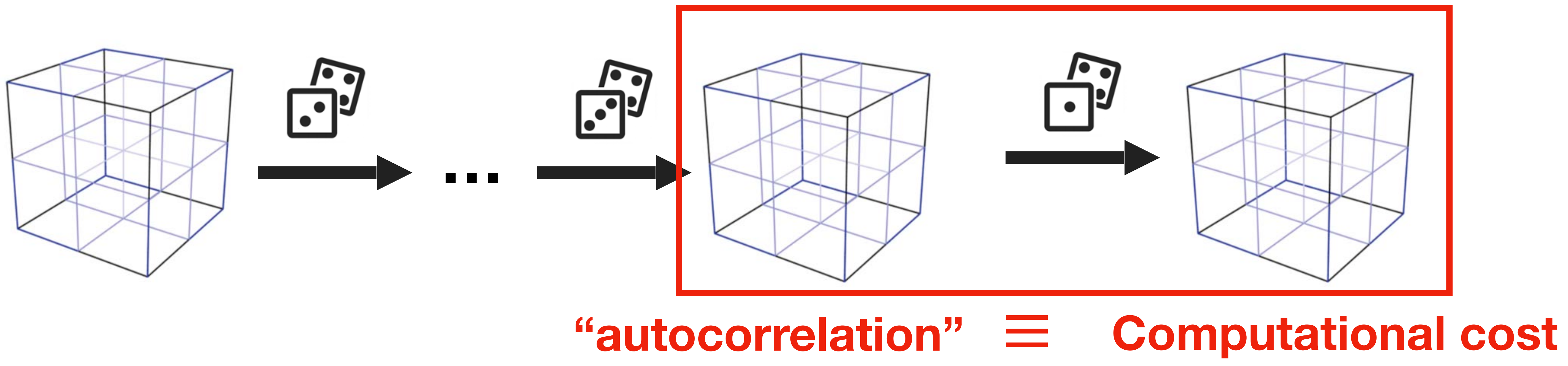
Lattice QCD is sampling problem over a very large number of variables

Markov Chain Monte Carlo

Lattice QCD is sampling problem over a very large number of variables

STEP 1: Generate field configurations:

$$\{U_i\} \sim p(U) = \frac{e^{-S_E(U)}}{\mathcal{Z}}$$



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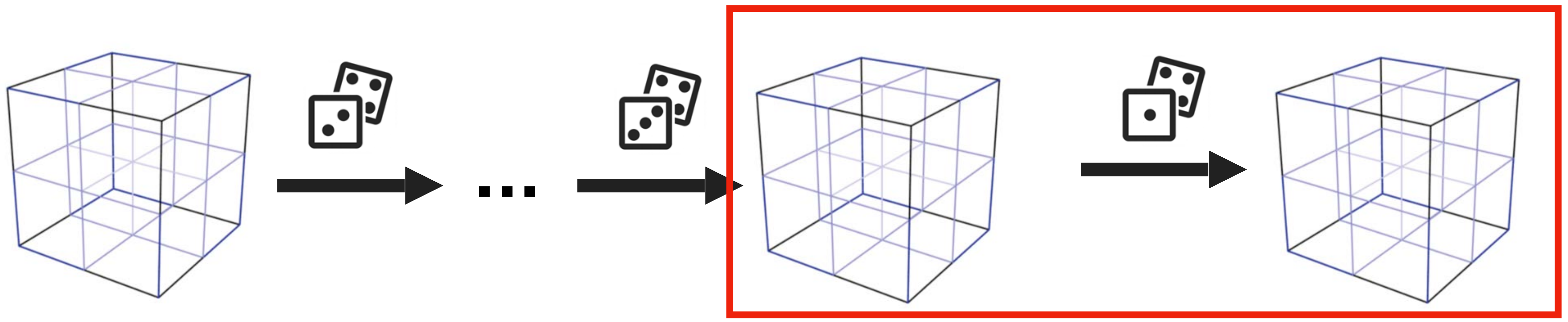


"autocorrelation" $\equiv$ Computational cost

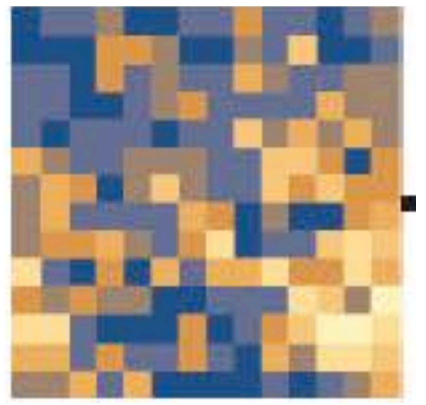
STEP 2: Compute observables: $\longrightarrow$ **For instance, the proton mass!**

$$\langle \mathcal{O} \rangle \simeq \sum_{i=0}^{N_{conf}} O(U_i)$$

Standard algorithms

- The simplest sampling algorithm is the Metropolis-Hastings algorithm

initial config



$\{\phi_0(x)\}$

$S[\phi_0]$

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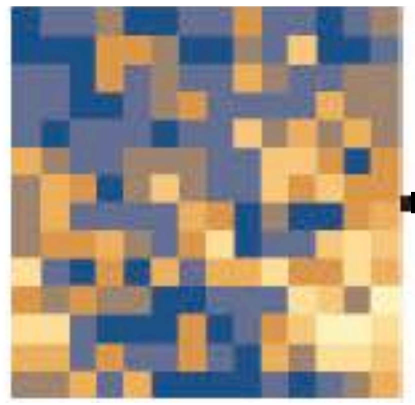
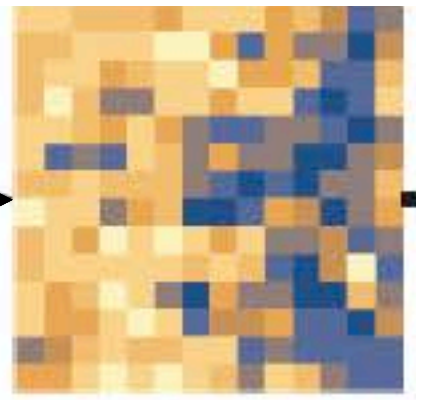


$\phi_1 = \phi_0 + \Delta\phi$

proposal



$\{\phi_0(x)\}$ $\quad$ $\{\phi_1(x)\}$

$S[\phi_0]$ $\quad$ $S[\phi_1]$

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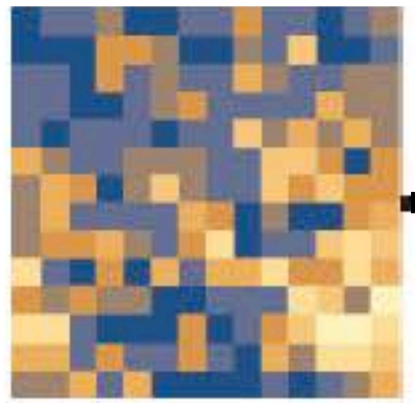

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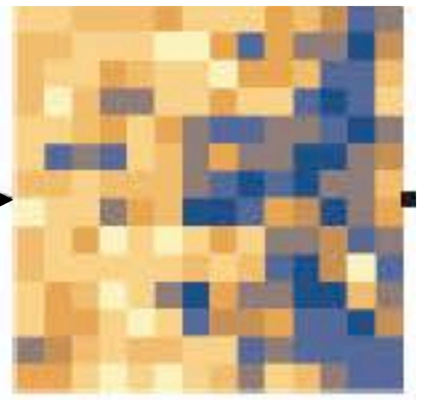
proposal



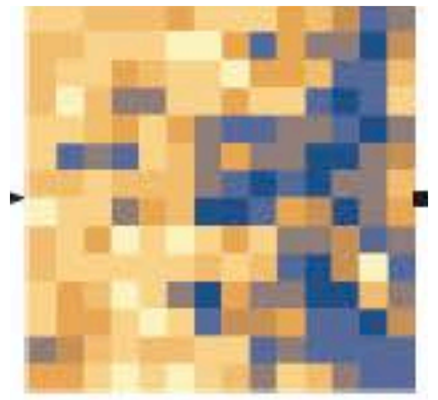
$\{\phi_1(x)\}$

$S[\phi_1]$

If $S[\phi_1] < S[\phi_0]$**:**

accept $\{\phi_1(x)\}$

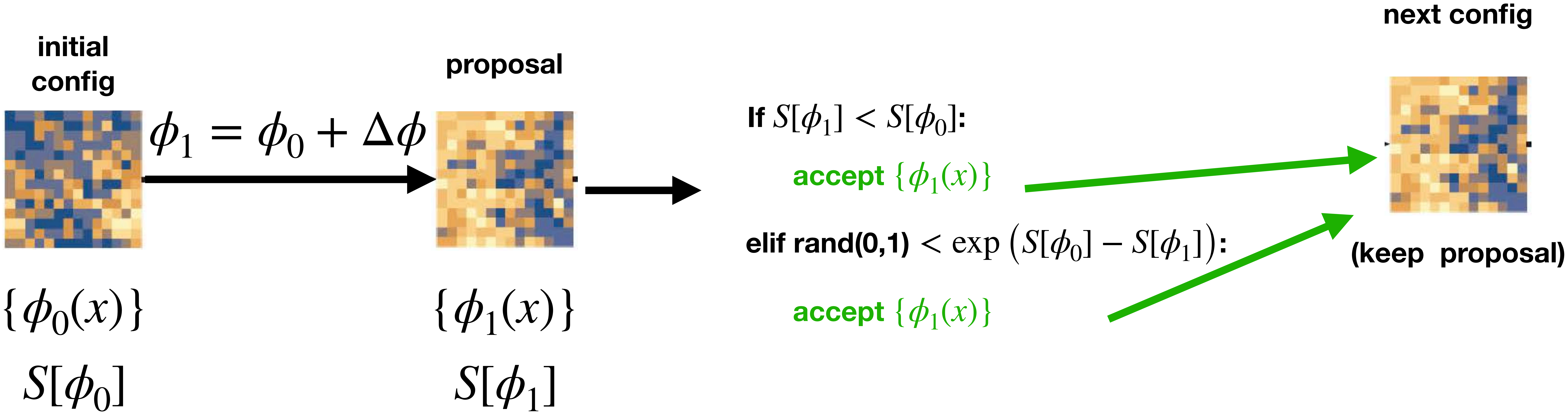
next config



(keep proposal)

Standard algorithms

- The simplest sampling algorithm is the Metropolis-Hastings algorithm



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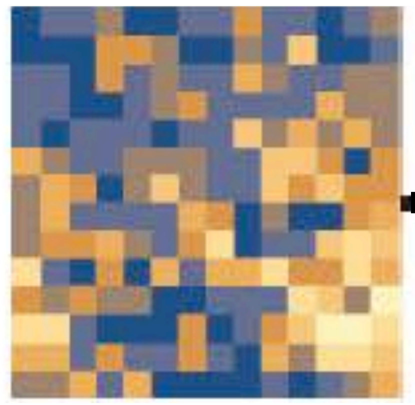
initial config

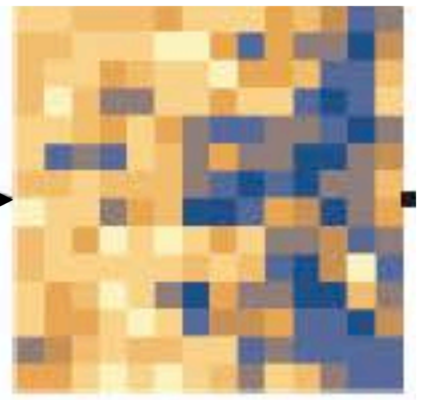


$\{\phi_0(x)\}$

$S[\phi_0]$

$$\phi_1 = \phi_0 + \Delta\phi$$

proposal



$\{\phi_1(x)\}$

$S[\phi_1]$

If $S[\phi_1] < S[\phi_0]$**:**

accept $\{\phi_1(x)\}$

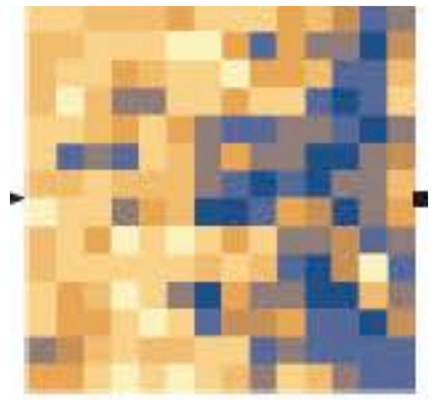
elif rand(0,1) $< \exp\left(S[\phi_0] - S[\phi_1]\right)$**:**

accept $\{\phi_1(x)\}$

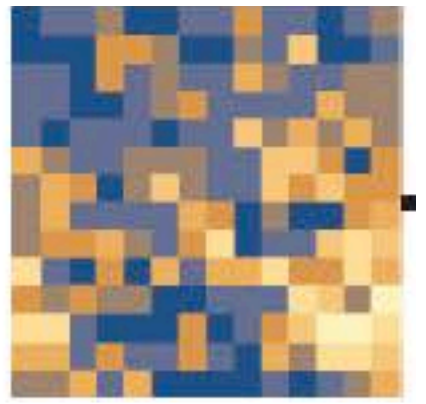
else:

reject $\{\phi_1(x)\}$ **and keep** $\{\phi_0(x)\}$

next config



(keep proposal)

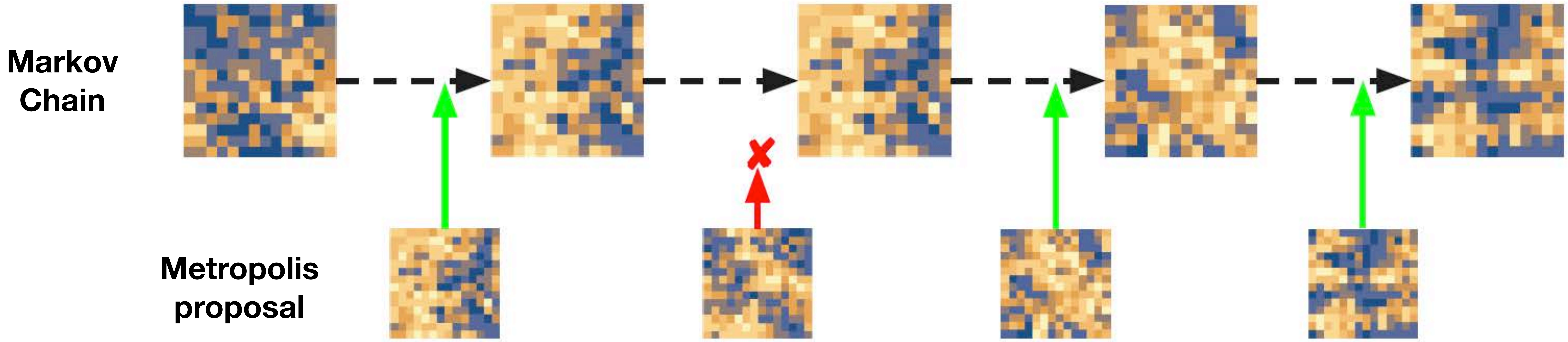


(same as initial)

Fernando Romero-López, MIT

Markov Chain Monte Carlo

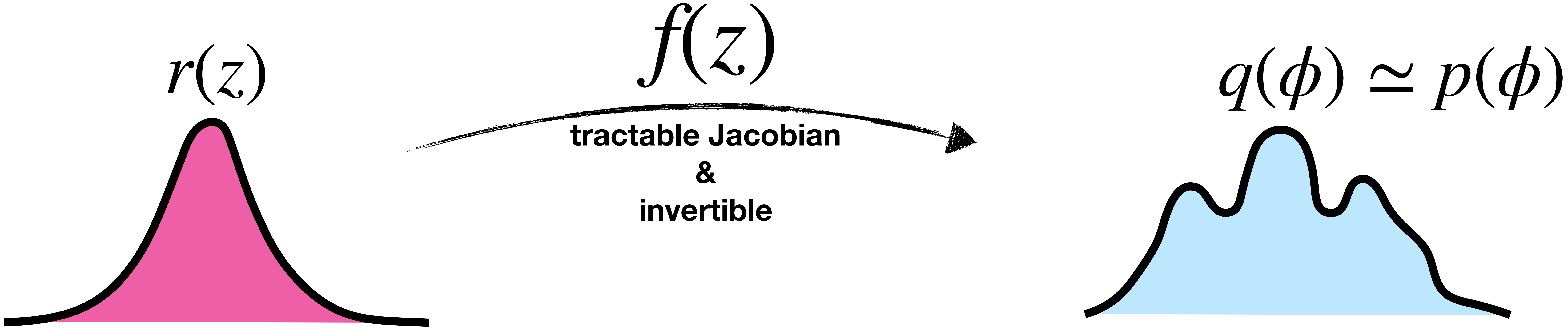
- "Ensembles" of configurations are generated by building a Markov Chain



- State-of-the-art calculations use the Hybrid Monte Carlo (HMC) algorithm
  - Based on Hamilton equations of motion

Generative flow models

[Rezende, Mohamed, 1505.05770]



easily sampled distribution (prior)

approximates target distribution (model)

“Trainable change of variables”

Model probability

$$q(\phi) = r(z) \left| \det \frac{\partial f(z)}{\partial z} \right|^{-1}$$

Generative flow models

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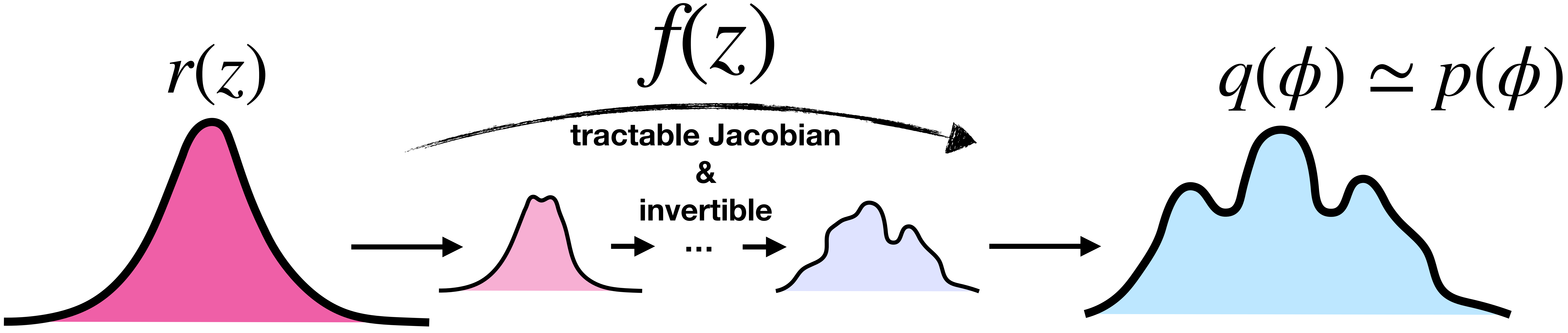

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parametrized by neural networks (trainable and expressive) → **composed of many simple layers**

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Generative flow models

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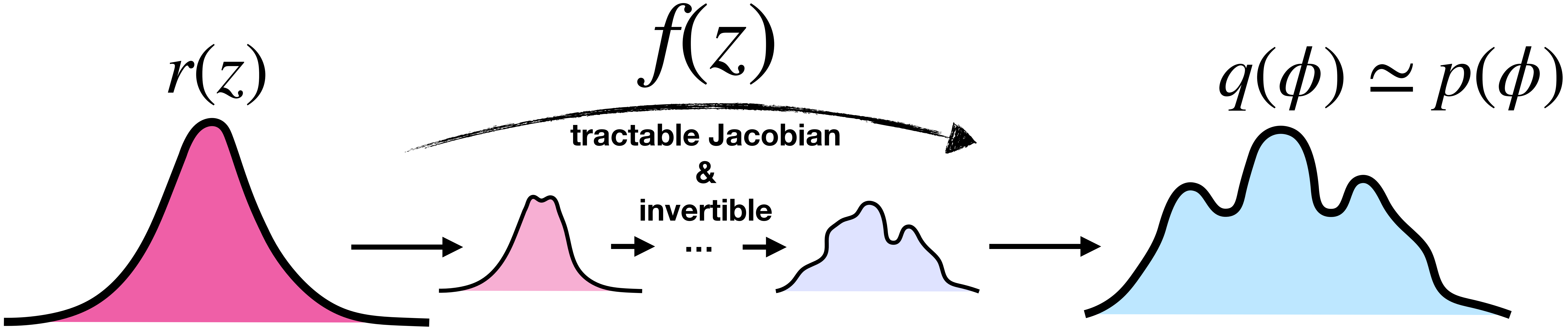

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"Trainable change of variables"

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$$q(\phi) = r(z)\left|\det \frac{\partial f(z)}{\partial z}\right|^{-1}$$

❗ Trained models are not perfect.

☑ But exact sampling can be recovered via Markov Chain

Application: Schwinger Model

- **QED in 1+1 dimesions: a toy model for QCD**
  - **Confinement, topology, symmetry breaking**
  - **Standard algorithms affected by topology freezing**

MCMC with traditional algorithms

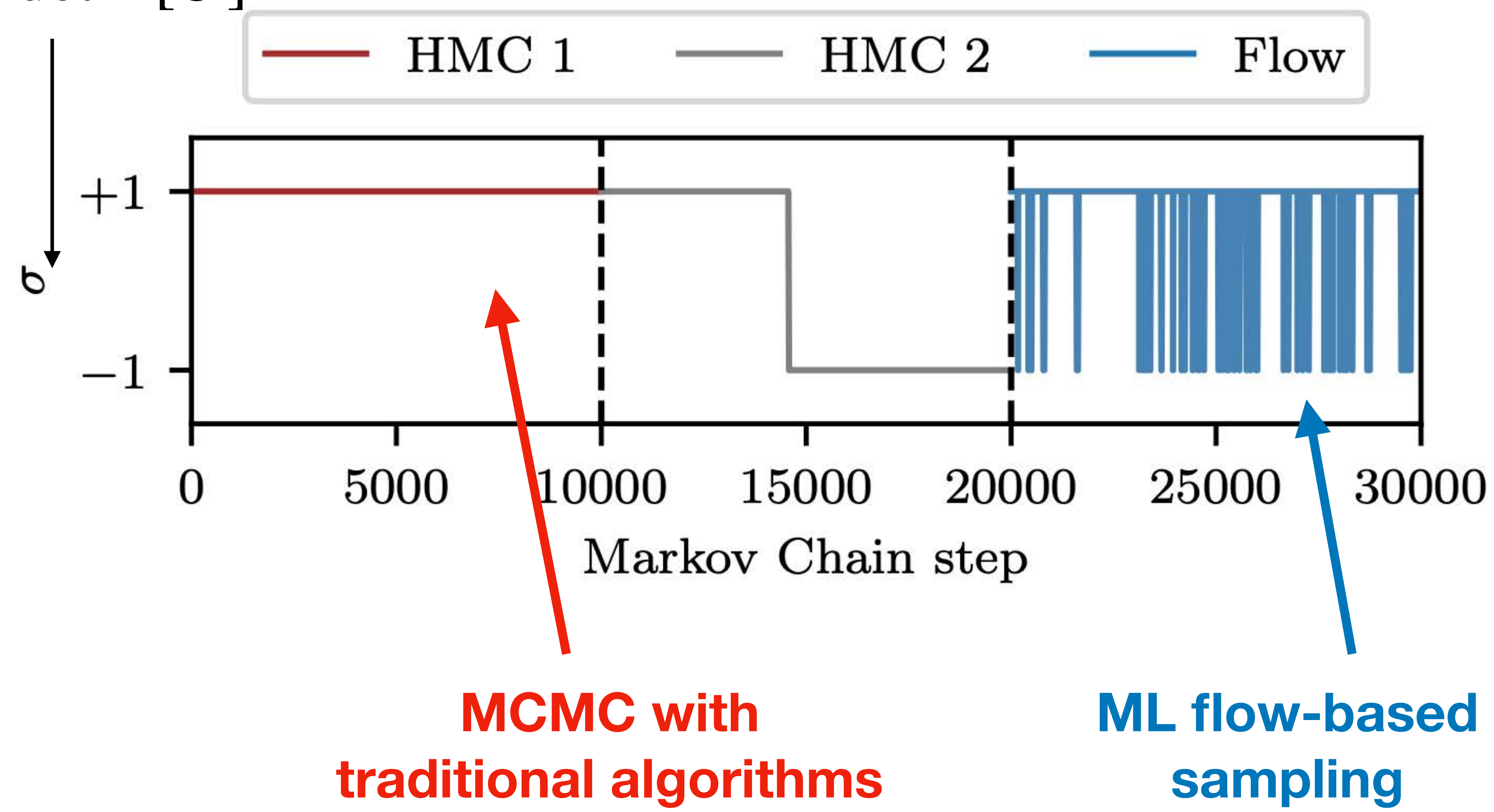
ML flow-based sampling

[Albergo, Boyda, Cranmer, Hackett, Kanwar, Racanière, Rezende, FRL, Shanahan, Urban 2202.11712]

Application: Schwinger Model

- QED in 1+1 dimesions: a toy model for QCD
  - Confinement, topology, symmetry breaking
  - Standard algorithms affected by topology freezing
  - ✓ Flow-based sampling mitigates topology freezing

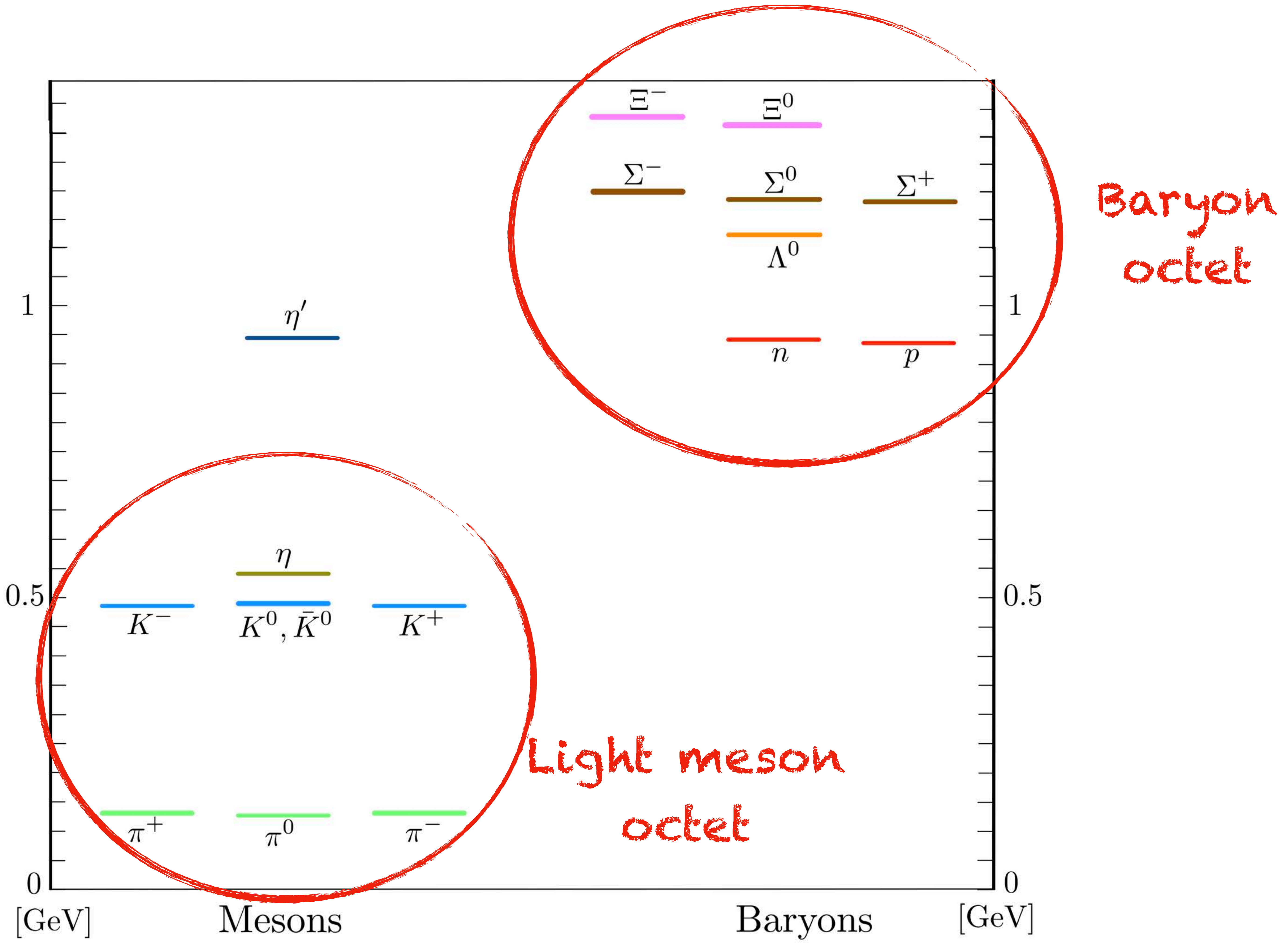
(topological observable)
sign of $\det D[U]$



[Albergo, Boyda, Cranmer, Hackett, Kanwar, Racanière, Rezende, FRL, Shanahan, Urban 2202.11712]

The QCD spectrum from lattice QCD

The QCD spectrum



[credits: Z. Davoudi]

Lattice QCD

- We measure energy levels and matrix elements:

"Spectral decomposition"

$$C(t) = \langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle = \sum_n \langle 0 | \mathcal{O}^\dagger(t) | n\rangle\langle n | \mathcal{O}(0) | 0\rangle$$

$$= \sum_n \langle 0 | e^{Ht}\mathcal{O}^\dagger(0)e^{-Ht} | n\rangle\langle n | \mathcal{O}(0) | 0\rangle$$

$$= \sum_n \left| \langle 0 | \mathcal{O}(0) | n\rangle \right|^2 e^{-E_n t}$$

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Ground state

$$\lim_{t\to\infty} C(t) = A_0 e^{-E_0 t}$$

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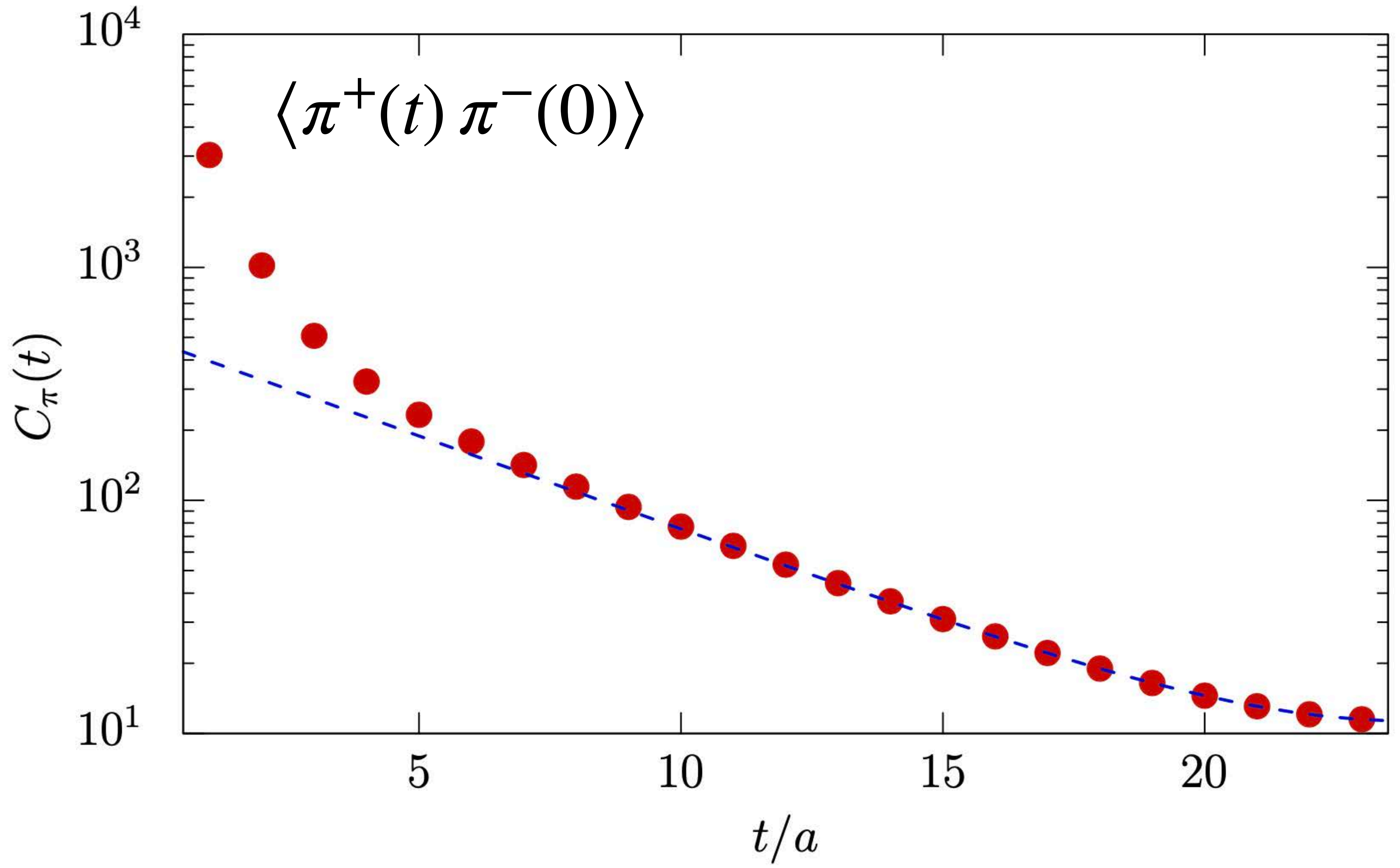
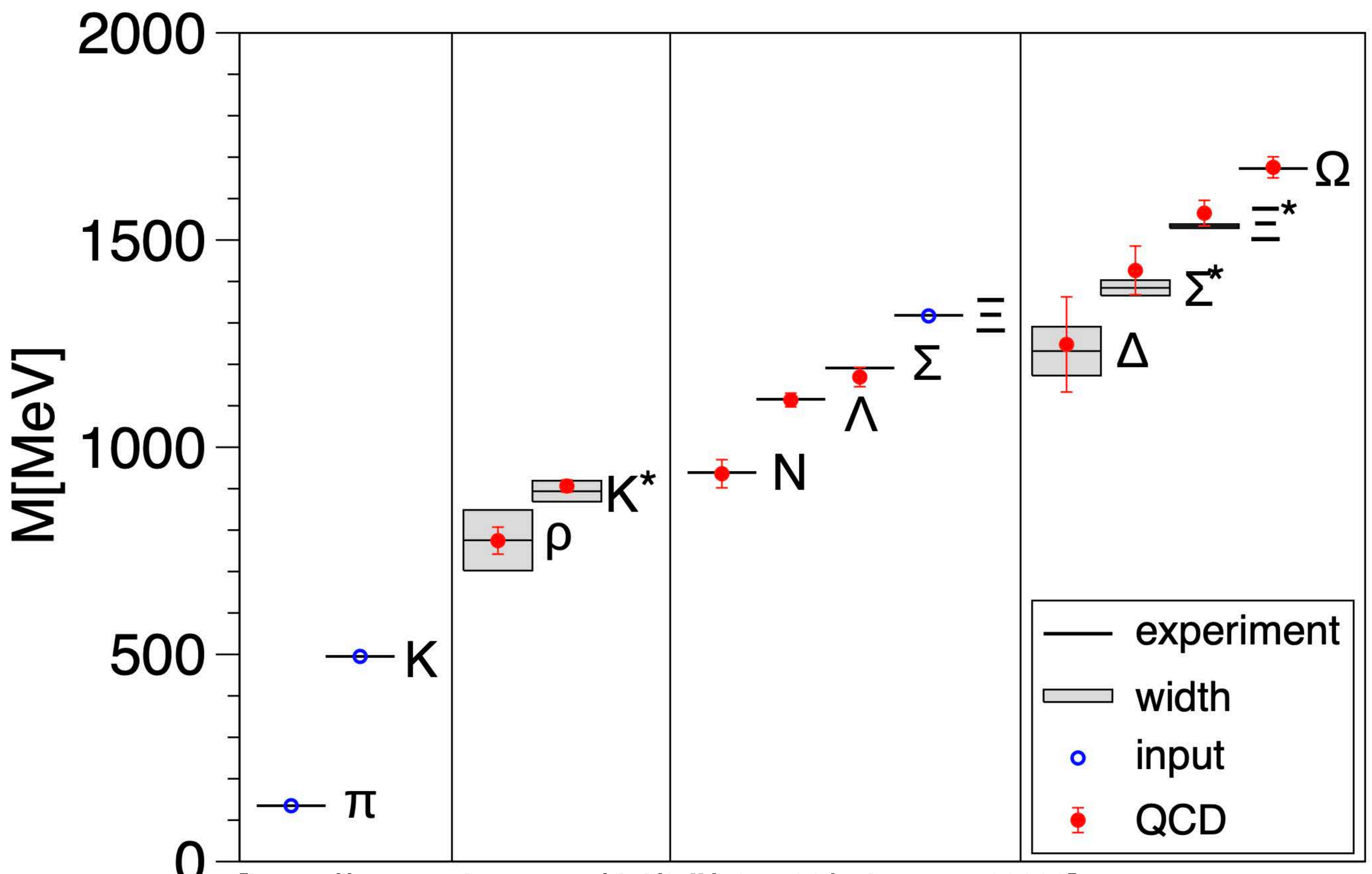


Ground state

$$\lim_{t\to\infty} C(t) = A_0 e^{-E_0 t}$$

The QCD spectrum (I)

- Consider isospin-symmetric QCD ($m_u = m_d$)
- Neglect QED effects in hadrons
- Need a few inputs to fix quark masses
- Reproduce the lowest-lying hadrons!

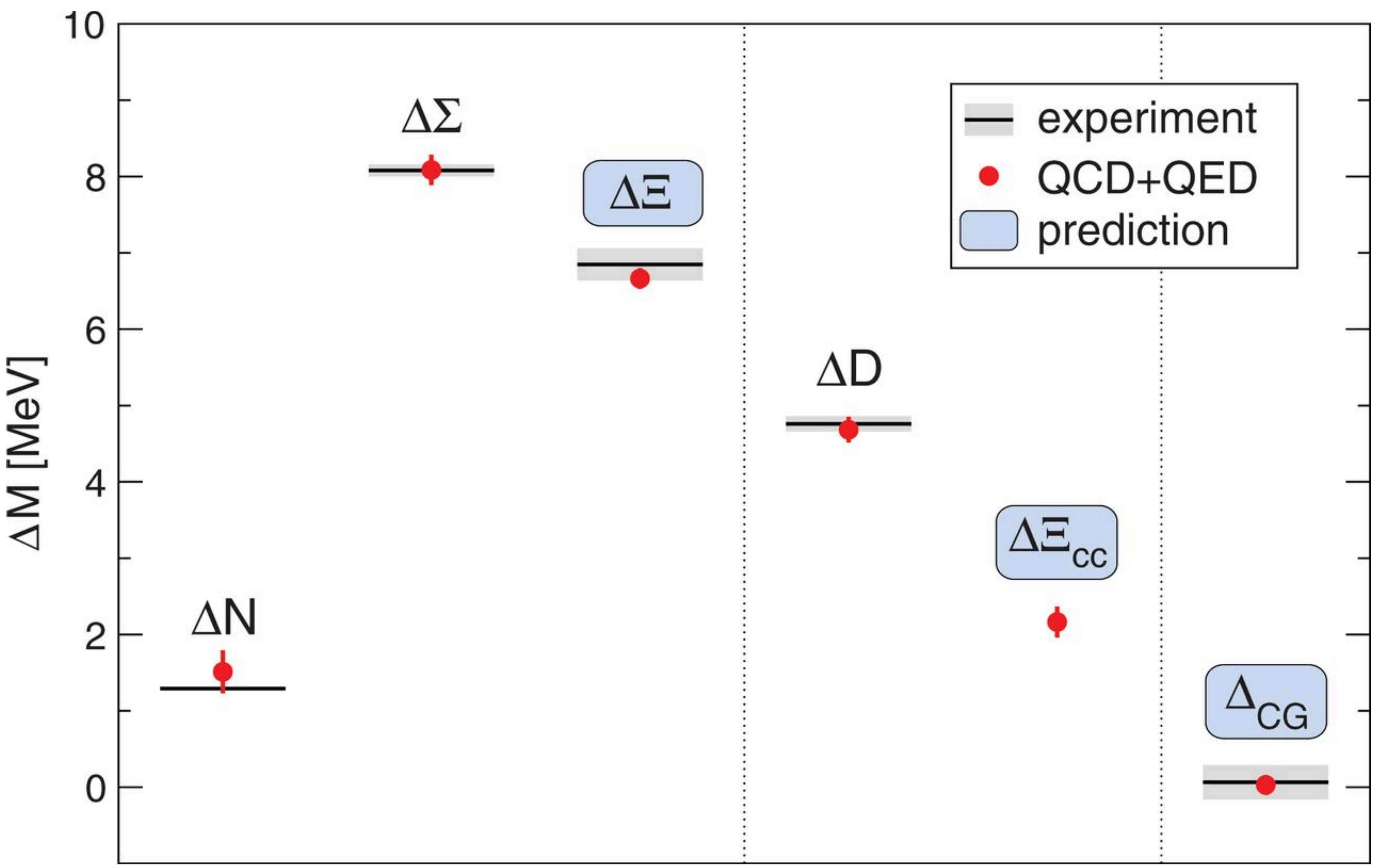


[https://www.science.org/doi/full/10.1126/science.1163233]

[BMW collaboration, 2008]

The QCD spectrum (II)

- More recently: add QED and $m_u \neq m_d$
- ☑ Can reproduce neutron/proton mass difference
- ☑ More precise than some experimental results



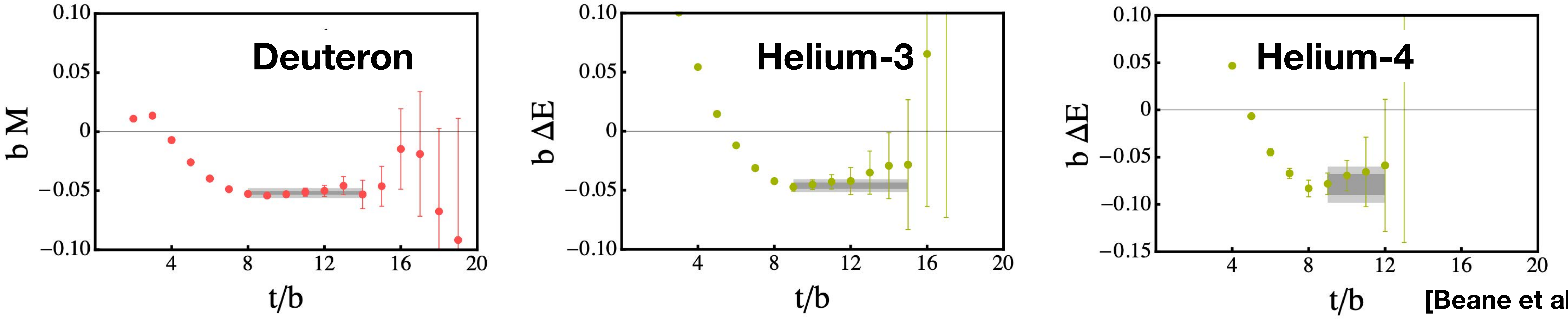
[BMW collaboration (2015)]

[https://www.science.org/doi/full/10.1126/science.1257050]

Nuclear Physics from LQCD

- Enormous progress in computing properties of nuclei

- Signal-to-noise problem grows rapidly with the baryon number



[Beane et al (NPLQCD), 2012]

- However, computational cost of Wick contractions grows (naively) factorially

  # contractions ~ (3A)!

Lattice QCD for precision SM tests

Example: muon g-2

- Muon magnetic moment marks the interaction of a muon with a magnetic field

$$\underbrace{H_I}_{\text{Interacting Hamiltonian}} \propto \overbrace{g_\mu}^{\text{Magnetic moment}} \underbrace{\overrightarrow{B}}_{\text{external magnetic field}} \cdot \underbrace{\vec{S}}_{\text{muon spin}}$$

$$\overbrace{a_\mu}^{\text{“anomalous magnetic moment”}} = \frac{1}{2}\left(g_\mu - 2\right)$$

Example: muon g-2

- Muon magnetic moment marks the interaction of a muon with a magnetic field

Magnetic moment

$$H_I \propto g_\mu \overrightarrow{B} \cdot \vec{S}$$

Interacting Hamiltonian

external magnetic field

muon spin

"anomalous magnetic moment"

$$a_\mu = \frac{1}{2}\left(g_\mu - 2\right)$$

- Dirac limit

$$a_\mu = 0$$

- First QFT correction [J. Schwinger (1948)]

$$a_\mu = \frac{\alpha_{em}}{2\pi} \simeq 0.0011614$$

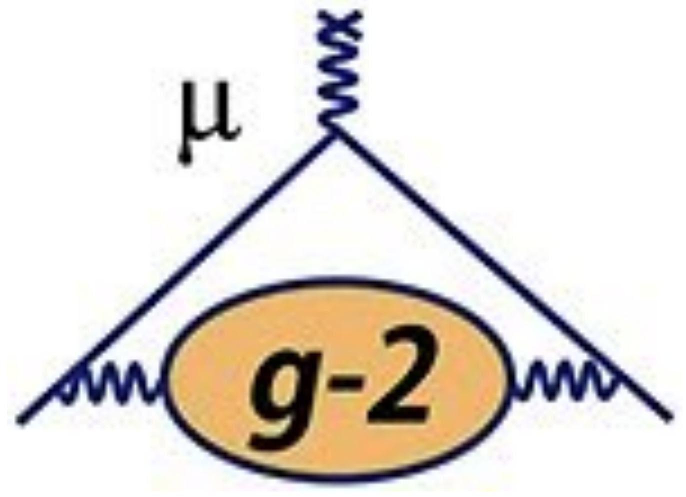
- Most precise number in nature

$$a_\mu(\text{exp}) = 116592061(41) \cdot 10^{-11}$$
$$a_\mu(\text{theory}) = 116591810(43) \cdot 10^{-11}$$

Example: muon g-2

Contribution	Section	Equation	Value $\times 10^{11}$
Experiment (E821)		Eq. (8.13)	116 592 089(63)
HVP LO (e^+e^-)	Sec. 2.3.7	Eq. (2.33)	6931(40)
HVP NLO (e^+e^-)	Sec. 2.3.8	Eq. (2.34)	−98.3(7)
HVP NNLO (e^+e^-)	Sec. 2.3.8	Eq. (2.35)	12.4(1)
HVP LO (lattice, $udsc$)	Sec. 3.5.1	Eq. (3.49)	7116(184)
HLbL (phenomenology)	Sec. 4.9.4	Eq. (4.92)	92(19)
HLbL NLO (phenomenology)	Sec. 4.8	Eq. (4.91)	2(1)
HLbL (lattice, uds)	Sec. 5.7	Eq. (5.49)	79(35)
HLbL (phenomenology + lattice)	Sec. 8	Eq. (8.10)	90(17)
QED	Sec. 6.5	Eq. (6.30)	116 584 718.931(104)
Electroweak	Sec. 7.4	Eq. (7.16)	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	Sec. 8	Eq. (8.5)	6845(40)
HLbL (phenomenology + lattice + NLO)	Sec. 8	Eq. (8.11)	92(18)
Total SM Value	Sec. 8	Eq. (8.12)	116 591 810(43)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	Sec. 8	Eq. (8.14)	279(76)

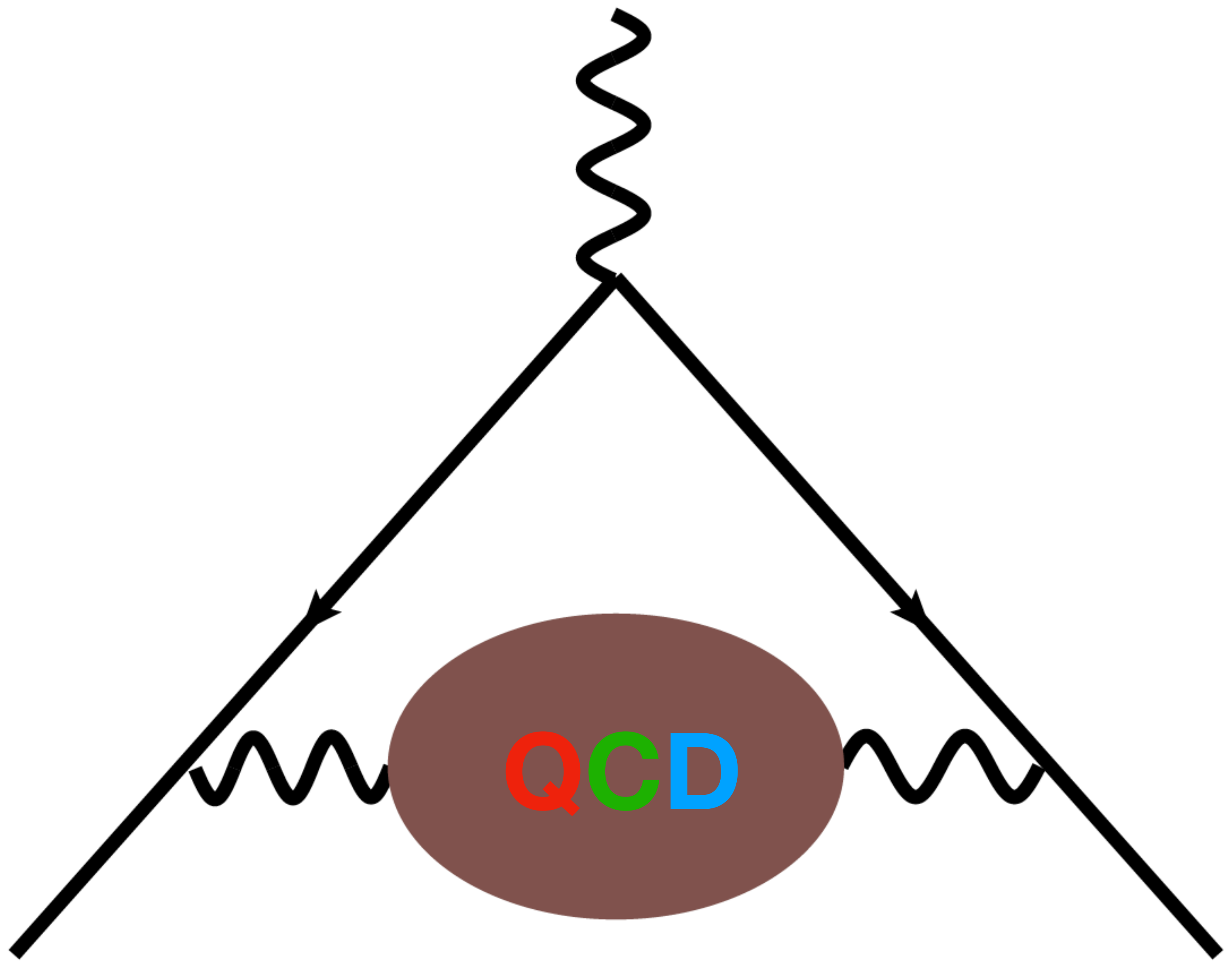
[https://arxiv.org/pdf/2006.04822.pdf]



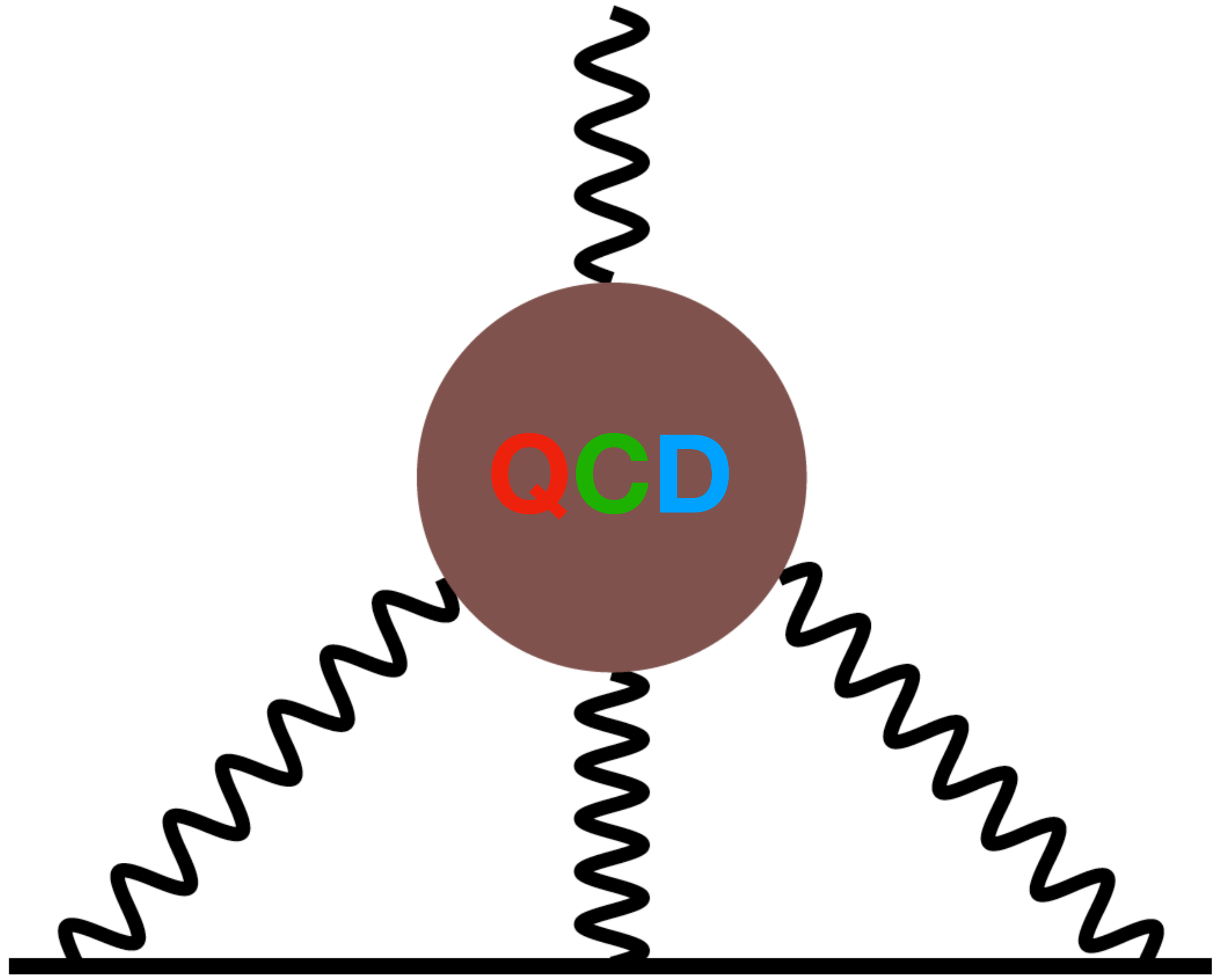
[https://muon-g-2.fnal.gov]

Example: muon g-2

- Lattice QCD can contribute in the computation of diagrams involving hadrons



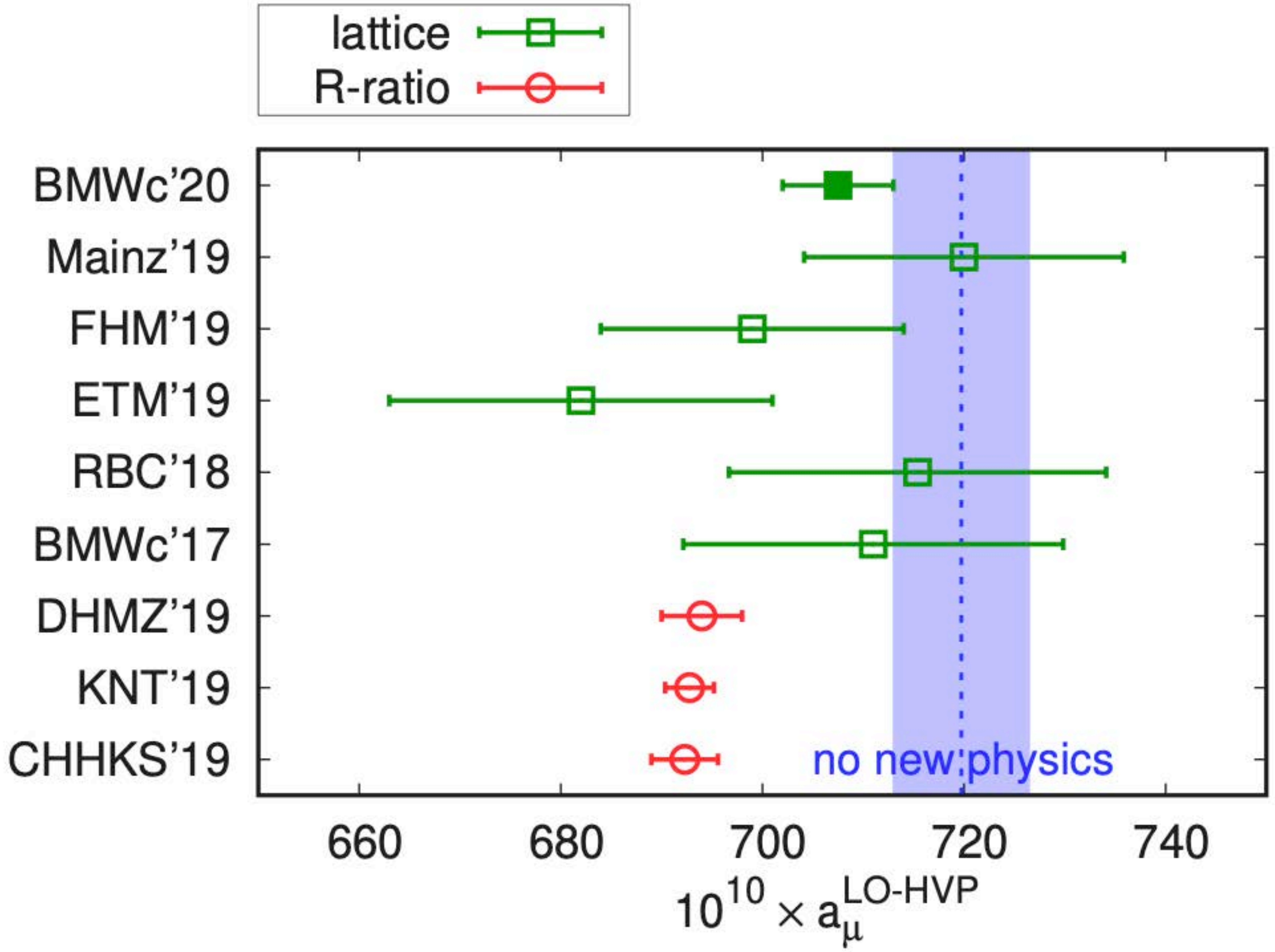
Hadronic Vacuum Polarization (HVP)



Hadronic Light-by-light scattering (HLbL)

Lattice QCD results for g-2 HVP

- Competitive precision from Lattice QCD
- Discrepancy between LQCD and data driven approach
- Essential calculation for the search of BSM Physics



Summary

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- Quantum Chromodynamics is the theory of the strong interaction
- Lattice QCD is the first-principle treatment of the strong interaction at hadronic energies
- Lattice Field Theory is a sampling problem over a very large number of variables
- Markov-Chain Monte Carlo approaches can be used for efficient sampling (even including ML!)
- Great success in computing important quantities for hadronic, nuclear and particle physics.
    - But many challenges remain: nuclear physics, SM tests…

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Thanks!

Image Sources

image

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