Stability for binary scalar products

Andrey Kupavskii[1] and Dmitry Tsarev[2]

[1] *Moscow Institute of Physics and Technology, Moscow, Russia*
[1] *G-SCOP, CNRS, Grenoble, France*
[2] *University of Cambridge, Cambridge, United Kingdom*

Abstract

Recent results show that, in some sense, 2-level polytopes cannot simultaneously have many vertices and many facets. In this work we find the maximal possible product of the number of vertices and the number of facets in a 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope, resolving a strong form of the conjecture by Bohn, Faenza, Fiorini, Fisikopoulos, Macchia, and Pashkovich (2015). To do this we show the stability of Kupavskii's and Weltge's upper bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ with a property that $\forall a \in \mathcal{A}, b \in \mathcal{B}$ the scalar product $\langle a, b\rangle \in \{0, 1\}$.

1 Introduction

A polytope P is said to be 2-level if for every facet-defining hyperplane H there is a parallel hyperplane H' such that $H \cup H'$ contains all vertices of P. Basic examples of 2-level polytopes are simplices, hypercubes and cross-polytopes, but they also generalize a variety of interesting polytopes such as Birkhoff, Hanner, and Hansen polytopes, order polytopes and chain polytopes of posets, stable matching polytopes, and stable set polytopes of perfect graphs [1]. Combinatorial structure of two-level polytopes has also been studied in [3], and enumeration of such polytopes in [2] led to a beautiful conjecture about their vertex and facet count, which was proven in [5]:

Theorem 1. *If P is a d-dimensional 2-level polytope, it's number of vertices $f_0(P)$ and facets $f_{d-1}(P)$ satisfy*

$$f_0(P) \cdot f_{d-1}(P) \leq d2^{d+1}.$$

This bound is tight by considering P that is affinely isomorphic to the cube or the cross-polytope. Authors of [2] conjectured that those are the only instances where equality is attained (personal communication). In this paper, we prove this in a strong sense:

Theorem 2. *For $d > 1$ let P be a d-dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then*

$$f_0(P) \cdot f_{d-1}(P) \leq (d-1)\, 2^{d+1} + 8\,(d-1)\,.$$

Our main tool is going to be a stronger theorem regarding so-called families of vectors with binary scalar products:

Theorem 3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Furthermore, suppose $\mathcal{A}$ and $\mathcal{B}$ both have the size of at least $d + 2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$.*

In other words, our main tool is the stability of the bound in Theorem 4, which was the main result of [5]:

Theorem 4. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d + 1)2^d$.*

Notation In what follows, We will often treat vectors in $\mathbb{R}^d$ as points in an affine space, with dim always referring to the affine dimension while span referrs to linear span. The set of integers from 1 to n is denoted $[n]$.

Outline The next section lays out the proof of our main tool. In Section 3 we provide the proof of Theorem 2 and Section 4 contains proofs of claims from [5] that we use. Short but technical proofs of some statements used in the main sections are provided in Appendix A, as well as a conjecture that generalises Theorem 3.

2 Stability results

Let $\mathcal{A}, \mathcal{B}$ be families of vectors that both linearly span $\mathbb{R}^d$ and have binary scalar products, that is, $\langle a, b \rangle \in \{0, 1\}$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. We will use the following two simple observations a few times throughout our proofs. Let $a_1, \ldots, a_d$ be a basis of $\mathbb{R}^d$ contained in $\mathcal{A}$. Consider the dual basis $a_1^*, \ldots, a_d^*$:

$$\langle a_i, a_j^* \rangle = \begin{cases} 1, \; i = j \\ 0, \; i \neq j \end{cases}$$

and observe that elements of $\mathcal{B}$ have 0/1 coordinates when expressed in this dual basis, or, in other words, $\mathcal{B}$ is a subset of what we would call a cube:

$$\mathcal{B} \subseteq \left\{ \sum_{i=1}^{d} \delta_i a_i^*, \text{ where } \delta_i \text{ range over } \{0, 1\} \right\}.$$

Another observation is that projecting one family on the linear span of a subset of another preserves the binary scalar products property: if $\mathcal{A}' \subseteq \mathcal{A}$ and $\pi_{\mathcal{A}'} : \mathbb{R}^d \to \mathrm{span}(\mathcal{A}')$ is the orthogonal projection, then

$$\forall a \in \mathcal{A}', \, b \in \mathcal{B} : \; \langle a, \pi_{\mathcal{A}'}(b) \rangle = \langle a, b \rangle \in \{0, 1\}.$$

We will now introduce some notation and restate some claims proved in [5]. Proofs of those claims and inequalities are provided in Section 4 for completeness.
Since we are interested in bounding the product $|\mathcal{A}||\mathcal{B}|$ from above, we will assume that $\mathcal{A}$ and $\mathcal{B}$ are inclusion-wise maximal with respect to the property of having binary scalar products. Let $b_d \in \mathcal{B} \setminus \{\mathbf{0}\}$ be a vector with the maximum value of $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, where

$$\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\} \text{ for } i = 0, 1.$$

The choice of b_d among the vectors that maximise $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, in cases where it is important, will be specified at a later stage. We denote the orthogonal projection onto $U = b_d^{\perp}$ by $\pi : \mathbb{R}^d \to U$. We say that $X \subset \mathbb{R}^d$ does not contain opposite points if $\{x, -x\} \subseteq X$ is only possible if $x = \mathbf{0}$. Below, we state the claims and inequalities from [5].

Claim 1. *We may translate $\mathcal{A}$ and replace some points $b \in \mathcal{B}$ by the opposites $-b$ such that the following properties hold.*

(i) We (still) have $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, where $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$ for $i = 0, 1$ such that

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}$$

(ii) We have

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}$$

(iii) The set $\pi(\mathcal{B})$ does not contain opposite points.

Claim 2. *Every point in $\pi(\mathcal{B})$ has at most two preimages in $\mathcal{B}$.*

We denote the linear span of $\mathcal{A}_0$ by U_0 and define the orthogonal projection $\tau : U \to U_0$. Let $\mathcal{B}_* \subseteq \mathcal{B}$ be the set of $b \in \mathcal{B}$ for which $\pi(b)$ has exactly one preimage under projection onto U.

Inequality 1. $|\mathcal{A}|\,|\mathcal{B}| \leq 2\,|\mathcal{A}_0|\,|\pi(\mathcal{B})| + |\mathcal{A}_1|\,|\mathcal{B} \backslash \mathcal{B}_*|$

Claim 3. $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}\,|\tau(\pi(\mathcal{B}))|$.

Claim 4. *$\mathcal{B} \backslash \mathcal{B}_*$ can be partitioned as $\mathcal{B}_0 \sqcup \mathcal{B}_1$, with $\mathcal{B}_0, \mathcal{B}_1$ satisfying*

$$\forall b \in \mathcal{B}_i : |\{\langle a, b \rangle : a \in \mathcal{A}_i\}| = 1 \text{ for } i = 0, 1.$$

Inequality 2. $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1)\, 2^d + |\mathcal{A}_0|\,|\mathcal{B}_0| + |\mathcal{A}_1|\,|\mathcal{B}_1|$

Inequality 3. *For $i = 0, 1$ we have*

$$|\mathcal{A}_i| \leq 2^{\dim(\mathcal{A}_i)},\ |\mathcal{B}_i| \leq 2^{\dim(\mathrm{span}(\mathcal{B}_i))},\ \text{and } \dim(\mathcal{A}_i) + \dim\big(\mathrm{span}(\mathcal{B}_i)\big) \leq d$$

Claim 5. *For $i = 0, 1$, we have $|\mathcal{A}_i|\,|\mathcal{B}_i| \leq 2^d$.*

Looking at the definition of $\mathcal{B}_i$, we see that we can assume both $\mathbf{0}, b_d \in \mathcal{B}_0$ or $\mathbf{0}, b_d \in \mathcal{B}_1$. Therefore, claim 5 actually implies

$$|\mathcal{A}_1|\,|\mathcal{B}_1| \leq 2^d,\ |\mathcal{A}_0|\,(|\mathcal{B}_0| + 2) \leq 2^d, \tag{3}$$

assuming here and further that $\mathbf{0}, b_d \in \mathcal{B}_1$.
This outline of claims is sufficient to understand, under which conditions equality is achieved in Theorem 4. We prove it as a warm-up, and then use its proof as a carcass for the further analysis.

Theorem 5. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b\rangle \in \{0,1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then we only have $|\mathcal{A}| \cdot |\mathcal{B}| = (d+1)2^d$ if one of the families has size $d+1$ and the other is affinely isomorphic to $\{0,1\}^d$.*

Proof. Without loss of generality, we assume $|\mathcal{A}| \geq |\mathcal{B}|$. We will use induction on d, the statement is obvious in dimension 1. Assuming the statement holds for smaller dimensions, we prove it in dimension d. Consider two options for $\dim U_0$:

1. $\dim U_0 \leq d-2$. From Inequality 2 and (3), we get:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (d-1)\, 2^d + 2 \cdot 2^d - 2\,|\mathcal{A}_0| \leq (d+1)\, 2^d - |\mathcal{A}| < (d+1)\, 2^d$$

2. $\dim U_0 = d-1$. Note that since $\mathbf{0} \in \mathcal{A}_0$, the definition of $\mathcal{B}_0$ implies $\mathcal{B}_0 \subset U_0^\perp$, and thus here, assuming $\mathbf{0}, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \varnothing$. We consider two subcases:

   a) $\mathcal{B}_* \neq \varnothing$. Equality in Theorem 4 can only be achieved when Inequality 2 (and consequently Inequality 1) are tight, which is only the case when $|\mathcal{A}_0|\,|\pi(\mathcal{B})| = d2^{d-1}$ (and $|\mathcal{A}_0| = |\mathcal{A}_1|$). By the induction hypothesis, the former is possible in one of two cases:

      i) $\mathcal{A}_0$ is affinely isomorphic to $\{0,1\}^{d-1}$. Then, $|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = 2^d$, which is only possible if $\mathcal{A}$ is affinely isomorphic to $\{0,1\}^d$, and then $\mathcal{B}$ can only consist of a basis and the zero vector.

      ii) $|\mathcal{A}_0| = d$. Then, since $|\mathcal{B}| \leq |\mathcal{A}| = 2d$, $|\mathcal{A}| \cdot |\mathcal{B}| \leq 4d^2$, which is less than $(d+1)2^d$ for $d \geq 4$. For $d = 3$, the inequality $|\mathcal{B}| \cdot |\mathcal{A}| \leq 32$ cannot yield equality since $|\mathcal{A}| = 6$. Finally, for $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

   b) $\mathcal{B}_* = \varnothing$. Then, $\mathcal{B}_1 = \mathcal{B}$ and, consequently, $\dim(\operatorname{span}(\mathcal{B}_1)) = d$. In this case Inequality 3 implies $|\mathcal{A}_1| = 1$. Similarly to case a), Inequality 1 is only tight in one of the following cases:

      i) $|\mathcal{A}_0| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq (d+1)^2 < (d+1)\, 2^d$.

      ii) $|\mathcal{A}_0| = 2^{d-1}$, $|\pi(\mathcal{B})| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| = 2d\left(2^{d-1}+1\right)$, which is less than $(d+1)2^d$ for $d > 2$. For $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

□

We will improve the bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for families that differ from the extremal example. To do this, we will use an auxiliary

Inequality 4. *For an integer $2 \leq f \leq d$, we have:*

$$(d+f)(2^{d-1} + 2^{d-f}) \leq d2^d + 2d.$$

A short but technical proof of this inequality can be found in Appendix A.

Theorem 3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b\rangle \in \{0,1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Furthermore, suppose $\mathcal{A}$ and $\mathcal{B}$ both have the size of at least $d+2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$.*

Proof. As in the proof of Theorem 5, we will use induction on d, and without loss of generality assume that $|\mathcal{A}| \geq |\mathcal{B}|$. Note that we can also assume that $\mathcal{A}$ and $\mathcal{B}$ are inclusion-wise maximal with respect to the property of having binary scalar products. For $d < 3$, the estimate coincides with Theorem 4. Assuming validity for smaller dimensions, let us prove the statement for dimension d. We consider possible values of $\dim U_0$:

1. $\dim U_0 < d-2$. Then, from Inequality 2 and Claim 5, we have:
$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1)\, 2^d + 2^d + 2^d \leq d2^d \tag{4}$$

2. $\dim U_0 = d-2$. Applying the induction hypothesis to the families $\tau(\pi(\mathcal{B}))$ and $\mathcal{A}_0$, we have three cases:

   a) $|\tau(\pi(\mathcal{B}))| = d-1$. By maximality $\mathcal{B}$ contained $\mathbf{0}$, so $\tau(\pi(\mathcal{B}))$ consists of zero and the basis of U_0. Maximality of $\mathcal{A}$ now means that $\mathcal{A}_0$ is affinely isomorphic to $\{0,1\}^{d-2}$. From (3), it follows that $|\mathcal{B}_0| \leq 2$. If $b \in \mathcal{B}_0$, then both elements of $\pi^{-1}(\pi(b))$ can be assumed to be in $\mathcal{B}_0$, thus $|\mathcal{B}_0|$ is even and we have two scenarios:

      i) $|\mathcal{B}_0| = 0$. Then, from Inequality 1 and Claim 5, we obtain:
$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4\,(d-1)\, 2^{d-2} + 2^d = d2^d$$

      ii) $|\mathcal{B}_0| = 2$. $U_0^\perp \cap \mathcal{B}$ consists of $\mathbf{0}, b_d$ and two vectors from $\mathcal{B}_0$. Let $k+1$ vectors in $\tau(\pi(\mathcal{B}))$ have two preimages in $\pi(\mathcal{B})$ under the action of τ. Among these $k+1$, let t_2 be the number of those vectors with both preimages in $\pi(\mathcal{B}_1)$, and let t_1+1 be the number of those with exactly one preimage in $\pi(\mathcal{B}_1)$ (including zero). The remaining $k - t_1 - t_2$ have both preimages in $\pi(\mathcal{B}_*)$. Furthermore, let the vectors in $\tau(\pi(\mathcal{B}))$ with a single preimage under τ consist of q projections from $\pi(\mathcal{B}_1)$ and $d-2-k-q$ projections from $\pi(\mathcal{B}_*)$. We then have:
$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_*| + |\mathcal{B}_0| + |\mathcal{B}_1| \\ &= (k - t_1 - 2t_2 + d - 2 - q) + 2 + (2 + 4t_2 + 2t_1 + 2q) \\ &= d + k + q + t_1 + 2t_2 + 2 \end{aligned}$$

      First, consider the case when $t_2 > 0$. Then $\pi(\mathcal{B}_1)$ contains two elements that differ by a vector orthogonal to U_0, thus $U_0^\perp \subset \operatorname{span}(\mathcal{B}_1)$, which implies:
$$\dim(\operatorname{span}(\mathcal{B}_1)) = t_1 + t_2 + q + 2 \overset{3}{\Longrightarrow} |\mathcal{A}_1| \leq 2^{d-t_1-t_2-q-2},$$
$$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| \leq 2^{d-2} + 2^{d-2-t_1-t_2-q}$$
$$\begin{aligned} |\mathcal{A}| \cdot |\mathcal{B}| &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right)(d + k + q + t_1 + 2t_2 + 2) \\ &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right)(2d + t_1 + 2t_2) \qquad (5) \\ &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2-q}\right)(d + t_1 + t_2) \\ &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2}\right)(d + t_1 + t_2) \qquad (6) \\ &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2}\right)(d + t_1 + t_2 + 1) \\ &\leq d2^d + 2d. \qquad (7) \end{aligned}$$

Here, the second inequality follows from $k+q \leq d-2$, and the last one follows from Inequality 4. If $t_2 = 0$, we get a slightly weaker bound:

$$\dim(\operatorname{span}(\mathcal{B}_1)) \geq t_1 + t_2 + q + 1$$

With the same reasoning this means that (6) becomes $\left(2^{d-1} + 2^{d-t_1}\right)(d+t_1)$, which is still less than (7) when $t_1 \geq 2$ due to Inequality 4. Finally, when $t_2 = 0$ and $t_1 = 0, 1$, expression (5) yields a bound by $d2^d$ and $(2^{d-2}+2^{d-3})(2d+1) = d2^d - \left(d - \frac{3}{2}\right) 2^{d-2} \leq d2^d$, respectively.

b) $|\mathcal{A}_0| = d-1$. Then:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4|\mathcal{A}_0|^2 \leq 4(d-1)^2 \leq d2^d + 2d$$

c) Both $|\mathcal{A}_0|$ and $|\tau(\pi(\mathcal{B}))|$ are at least d. By induction this implies

$$|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq 2\,(d-2)\left(2^{d-3}+1\right).$$

Using Inequality 1, claim 3, and (3), we have

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d-2)\left(2^{d-2}+2\right) + 2 \cdot 2^d - 2\,|\mathcal{A}_0| = 2d(2^{d-1}+1) + 2\,(3d - 8 - |\mathcal{A}_0|)$$

This completes the proof when $|\mathcal{A}_0| \geq 3d-8$. Otherwise,

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4\,|\mathcal{A}_0|^2 \leq 4\,(3d-9)^2,$$

which is less than $d2^d + 2d$ for $d \geq 3$.

3. $\dim U_0 = d-1$. Again, applying the induction hypothesis to $\pi(\mathcal{B})$ and $\mathcal{A}_0$, we have three cases (recall that from the assumption $\mathbf{0}, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \varnothing$):

a) $|\pi(\mathcal{B})| = d$, which by maximality of $\mathcal{A}$ means that $\mathcal{A}_0$ is isomorphic to $\{0,1\}^{d-1}$.

i) $\dim \mathcal{B}_1 = 1$. In this case, $|\mathcal{B}| = d+1$, which does not satisfy the condition in the theorem's statement.

ii) $\dim \mathcal{B}_1 = k \geq 2$. Then $|\mathcal{B}_1| = 2k$, $|\mathcal{A}_1| \leq 2^{d-k}$ from Inequality 3, and we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (2^{d-1} + 2^{d-k})(d+k) \leq d2^d + 2d$$

by Inequality 4.

b) $|\mathcal{A}_0| = d$. Then $|\mathcal{A}||\mathcal{B}| \leq |\,|\mathcal{A}|^2 \leq 4|\mathcal{A}_0|^2 \leq 4d^2$, which is not larger than $d2^d + 2d$ for $d > 3$. For $d = 3$, $|\mathcal{A}|^2$ gives the desired bound when $|\mathcal{A}_1| \leq 2$, and finally $\mathcal{A}_1 = 3$ would by Inequality 3 imply

$$\dim \mathcal{A}_1 = 2 \Rightarrow |\mathcal{B}_1| = 2 \Rightarrow |\mathcal{B}| \leq 5 \Rightarrow |\mathcal{A}| \cdot |\mathcal{B}| \leq 3 \cdot 2^3 + 2 \cdot 3$$

c) Both $|\mathcal{A}_0|$ and $|\pi(\mathcal{B})|$ are at least $d+1$.

The remainder of the proof will deal with the case 3c). By the induction hypothesis,

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1)\left(2^{d-1}+2\right).$$

Therefore from the fact that $\mathcal{B}_0 = \varnothing$ and Claim 5, we have

$$\begin{aligned}
|\mathcal{A}| \cdot |\mathcal{B}| &= 2\,|\mathcal{A}_0|\,|\pi(\mathcal{B})| + |\mathcal{A}_1|\,|\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| && (8)\\
&\leq 2\,(d-1)\left(2^{d-1}+2\right) + |\mathcal{A}_1|\,|\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| && (9)\\
&\leq 2\,(d-1)\left(2^{d-1}+2\right) + 2^d - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \\
&= d2^d + 2d - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| + (2d-4). && (10)
\end{aligned}$$

Thus, it suffices to show, for example, that $(|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \geq 2d-4$.
Consider the case where $\dim \mathcal{A}_1 = d-1$: then $\mathcal{B}_1 = \{0, b_d\}$, and using

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}|\,|\pi(\mathcal{B})| + |\mathcal{A}| \cdot \frac{1}{2}\,|\mathcal{B}_1| \leq d2^d + 2d - 2^d + |\mathcal{A}| + (2d-4)$$

we obtain the desired inequality when $|\mathcal{A}| \leq 2^d - 2d + 4$. Note that $|\mathcal{A}| > 2^d - 2d + 4$ is indeed impossible, as that would imply

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| > (2^{d-1} - d + 2) \cdot (d+1) \geq (d-1)(2^{d-1}+2)$$

which contradicts the induction hypothesis. We may thus now assume $\dim A_1 < d-1$. Observe that, due to this, we can also assume that $|\mathcal{A}_0| > |\mathcal{A}_1|$, since in the case that $|\mathcal{A}_0| = |\mathcal{A}_1|$ no shifting was performed in Claim 1 and we can start by shifting the family $\mathcal{A}$ and changing the signs of some vectors in $\mathcal{B}$ so that all conditions remain in force and $\mathcal{A}_0$ with $\mathcal{A}_1$ switch places, reducing the situation to the case where $\dim U_0 < d-1$.

Consider the orthogonal projection $\pi_{\mathcal{B}_1} : \mathbb{R}^d \to \operatorname{span}(\mathcal{B}_1)$. By the definition of $\mathcal{A}_1$, we have $|\pi_{\mathcal{B}_1}(\mathcal{A}_1)| = 1$. Let $k = \dim(\operatorname{span}(\mathcal{B}_1))$. Since $\mathcal{B}$ contains a basis of $\mathbb{R}^d$, we have:

$$|\mathcal{B}_*| \geq d-k, \ \ (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \geq d-k \qquad (11)$$

We will now deal with possible values of k:

i) $k=1$, which means $\mathcal{B}_1 = \{0,\ b_d\}$. Since $\dim \mathcal{A}_1 < d-1$, from Inequality 3 it follows that $|\mathcal{A}_1| \leq 2^{d-2}$. Substituting this into (9), we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4 - 2^{d-1}) \leq d2^d + 2d$$

ii) $k=2$. From Inequality 3, it follows that $|\mathcal{B}_1| \leq 4$, and $|\mathcal{A}_1| \leq 2^{d-2}$. Due to (11), $|\mathcal{B}_*| \geq d-2$, so if $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2$, (10) yields the desired estimate. Similarly, (10) completes the proof if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| \geq 2d-4$. Finally, if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| < 2d-4$, then:

$$|\mathcal{A}| \cdot |\mathcal{B}| = (2\,|\mathcal{A}_1| + 1) \cdot (|\mathcal{B}_*| + |\mathcal{B}_1|) < \left(2^{d-1}+1\right) \cdot (2d-4+4) = d2^d + 2d$$

iii) $k = d$. Inequality 3 implies that $\mathcal{A}_1$ consists of only one point. Hence, (9) becomes

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2\,(d-1)\left(2^{d-1}+2\right) + |\mathcal{B}_1| - \left(|\mathcal{A}_0| - |\mathcal{A}_1|\right)|\mathcal{B}_*| \leq 2\,(d-1)\left(2^{d-1}+2\right) + |\mathcal{B}|\,,$$

which completes the proof when $|\mathcal{B}| \leq 2^d - 2d + 4$. The opposite is indeed impossible, as it would contradict Theorem 4:

$$|\mathcal{A}| \cdot |\mathcal{B}| \geq |\mathcal{B}|^2 \geq \left(2^d - 2d + 4\right)^2 > (d+1)\,2^d$$

Before proceeding with the last case in the proof, let us understand that when $k < d$, we can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$. Clearly $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k$ because $\mathbf{0} \in \mathcal{A}_0$, and

$$\operatorname{span}\bigl(\pi_{\mathcal{B}_1}(\mathcal{A}_0)\bigr) = \operatorname{span}\bigl(\pi_{\mathcal{B}_1}(\operatorname{span}(\mathcal{A}_0))\bigr) = \operatorname{span}(\mathcal{B}_1) \cap b_d^{\perp},$$

which means $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ contains $\mathbf{0}$ and a basis of an $(k-1)$-dimensional space. Since by replacing some vectors in $\mathcal{B}_1$ with their opposites (without affecting $|\mathcal{B}_1|$) we ensure it has binary scalar products with $\mathcal{A}$, by Theorem 4 we have, if $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k+1$,

$$|\mathcal{B}_1| \cdot |\pi_{\mathcal{B}_1}(\mathcal{A})| \leq (k+1)\,2^k,\ |\pi_{\mathcal{B}_1}(\mathcal{A})| \geq k+2 \Rightarrow |\mathcal{B}_1| \leq 2^k\left(1 - \frac{1}{k+2}\right) \Rightarrow$$
$$|\mathcal{A}_1|\,|\mathcal{B}_1| \leq 2^d\left(1 - \frac{1}{k+2}\right) \Rightarrow |\mathcal{A}| \cdot |\mathcal{B}| \overset{(9)}{\leq} d2^d + 2d + (2d-4) - \frac{2^d}{k+2} - (d-k)$$

This proves the required estimate, because for $d \geq 3$ and $k < d$

$$d + k - 4 - \frac{2^d}{k+2} \leq 2d - 5 - \frac{2^d}{d+1} = -\frac{1}{d+1}\left(2^d - (2d-5)(d+1)\right) \leq 0$$

We now can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$, meaning $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ consists of zero and a basis of $\operatorname{span}(\mathcal{B}_1) \cap b_d^{\perp}$, while $\pi_{\mathcal{B}_1}(\mathcal{A})$ consists of zero and a basis of $\operatorname{span}(\mathcal{B}_1)$. With those assumptions in place, we proceed to the final subcase:

iv) $2 < k < d$. Note that, due to (11), $\mathcal{B}_* \neq \varnothing$. Let's denote the elements of $\pi_{\mathcal{B}_1}(\mathcal{A})$ as $a_0 = 0, a_1, \ldots, a_k$, and their preimages in $\mathcal{A}$ as $\mathbb{A}_j = \pi_{\mathcal{B}_1}^{-1}(a_j) \cap \mathcal{A}$. We'll choose the numbering such that $\mathbb{A}_1 = \mathcal{A}_1$. Let $b_{11}, b_{12}, \ldots, b_{1k}$ be a basis of $\mathcal{B}_1$ that is dual to $a_1, \ldots, a_k$. For example, according to our choice of numbering, $b_{11} = b_d$. Note that, due to $\mathcal{B}$ being inclusion-wise maximal, all b_{1j} must belong to $\mathcal{B}_1$ (otherwise, they, along with $b_{1j} + b_d$ for $j > 1$, could be added to $\mathcal{B}$). If $\dim \mathcal{A}_1 < d - k$, we can follow a similar argument as in part i) to obtain $|\mathcal{A}_1| \leq 2^{d-2}$ and the desired estimate. Consequently, we can now assume that $\dim \mathcal{A}_1 = d - k$.

Our further plan is to write $\mathcal{A}$ in a particular basis to see that, due to $\dim \mathcal{A}_1 = d - k$, any of the b_{1j} could be initially chosen as b_d, and that a suitable choice would lead to the desired bound.

We will augment $\{b_{11}, \ldots, b_{1k}\}$ with elements from $\mathcal{B}_*$ to form a basis for $\mathbb{R}^d$ and represent $\mathcal{A}$ in the dual basis. Then vectors of $\mathcal{A}$, arranged as column-vectors, form a matrix of the following form:

$$\begin{array}{cccccc} & \mathbb{A}_0 & \mathbb{A}_1 & \mathbb{A}_2 & \cdots & \mathbb{A}_k \\ \mathcal{A} = & \left(\begin{array}{c|c|c|c|c} & 1\ 1 \cdots 1\ 1 & 0\ 0 \cdots 0\ 0 & & \\ \mathbf{0} & & 1\ 1 \cdots 1\ 1 & & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & & \\ & & & & 1\ 1 \cdots 1\ 1 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \ddots & \ddots & \ddots & & \ddots \end{array}\right) & & & & \begin{array}{l} \left.\right] k \\ \left.\right] d-k \end{array} \\ & & \underbrace{\qquad\qquad}_{\dim = d-k} & & & \end{array}$$

The rank of the highlighted block coincides with the affine dimension of $\mathbb{A}_1 = \mathcal{A}_1$, which is $d-k$. Therefore,

$$\forall j > 1 \colon\ d - 1 = \dim(\operatorname{span}(\mathcal{A} \setminus \mathbb{A}_j)) = \dim(\mathcal{A} \cap b_{1j}^{\perp}),$$

which means that, indeed, any of the b_{1j} could be set as b_d from the start. Choose b_{1j} with the smallest possible size of $\mathbb{A}_j$, and repeat all the same reasoning with it as b_d. Note that in this case, $|\mathcal{A} \setminus \mathbb{A}_j| > |\mathbb{A}_j|$, so there will be no need for translation of $\mathcal{A}$ that swaps $\mathcal{A}_0$ and $\mathcal{A}_1$ in Claim 1, and we can thus safely assume

$$\begin{gathered} \forall j > 1 \colon\ |\mathbb{A}_1| \leq |\mathbb{A}_j| \implies \\ |\mathcal{A}_0| - |\mathcal{A}_1| = |\mathbb{A}_0| + \sum_{j>1} |\mathbb{A}_j| \geq (k-1)\,|\mathcal{A}_1| \geq 2\,|\mathcal{A}_1| \end{gathered} \tag{12}$$

If $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2d - 4$, non-emptiness of $\mathcal{B}_*$ and (10) imply the desired estimate. Otherwise

$$\begin{gathered} |\mathcal{A}_0| - |\mathcal{A}_1| < 2d - 4 \xRightarrow{(12)} |\mathcal{A}| < 2 \cdot (2d-4) \implies \\ |\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 < (4d-8)^2 < d2^d + 2d, \end{gathered}$$

concluding the proof.

$\square$

Two examples that demonstrate tightness of the bound in Theorem 3 are

Example 1. *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*

$$\mathcal{A} = \left\{\sum_{i=2}^{d} \delta_i e_i\right\} \cup \{e_1\}, \ \mathcal{B} = \{\delta_1 e_1 + e_j\} \cup \{e_1, 0\}, \textit{ where } \delta_i \textit{ range over } \{0,1\} \textit{ and } j \textit{ over } [2,d].$$

Here $|\mathcal{A}| = 2^{d-1} + 1$ and $|\mathcal{B}| = 2d$.

Example 2. *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*

$$\mathcal{A} = \left\{e_d + \sum_{i=1}^{d-1} \varepsilon_i e_i\right\} \cup \{0\}, \ \mathcal{B} = \left\{\frac{1}{2}(e_d + \varepsilon_i e_i)\right\}, \textit{ where } \varepsilon_i \textit{ range over } \{-1,1\} \textit{ and } i \textit{ over } [d].$$

Just like in example 1, $|\mathcal{A}| = 2^{d-1} + 1$ and $|\mathcal{B}| = 2d$.

3 Application to 2-level polytopes

Our main application of Theorem 3 is the following

Theorem 2. *For $d > 1$ let P be a d-dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then*

$$f_0(P) \cdot f_{d-1}(P) \le (d-1)\, 2^{d+1} + 8\,(d-1)\,.$$

Before following with the proof let us make a simple observation, proof of which is given in Appendix A for completeness:

Lemma 1. *Let $\mathcal{S}$ be a family of subsets of $[d-1]$ such that $|\mathcal{S}| = d$ and*

$$\forall S_1, S_2 \in \mathcal{S} :\ |S_2 \setminus S_1| \le 1.$$

Then either $\mathcal{S} = \{S \subseteq [d-1] : |S| \ge d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \le 1\}$.

Proof of Theorem 2. The statement is trivial on the plane, so we assume $d > 2$. Let us denote $V = f_0(P)$ and $F = f_{d-1}(P)$ for conciseness. Shift P so that 0 is among it's vertices and let $\mathcal{A}$ denote the vertex set of P and $\mathcal{B}'$ denote the minimal set of vectors such that every facet of P lies in a hyperplane $\{x : \langle x, b\rangle = \delta\}$ for some $\delta \in \{0, 1\}$ and $b \in \mathcal{B}'$. Let $\mathcal{B} = \mathcal{B}' \cup \{0\}$. If every vector in $\mathcal{B}'$ defines one facet of P, we are done by Theorem 4:

$$V \cdot F < |\mathcal{A}| \cdot |\mathcal{B}| \le (d+1)2^d < (d-1)2^{d+1} + 8(d-1).$$

Otherwise, let $b_d \in \mathcal{B}'$ define two facets of P and consider the setting of the proof of Theorem 3. Note that we may assume $|\mathcal{A}_0| \ge |\mathcal{A}_1|$ if appropriate translation of P was made, so there will be no need for translation of $\mathcal{A}$ or inversions of vectors in $\mathcal{B}$. Since $\dim(\mathcal{A}_1) = d-1$, we have $\mathcal{B}_1 = \{0, b_d\}$ and $|\pi(\mathcal{B})| = |\mathcal{B}_*| + 1$, which means

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})| + |\mathcal{A}|\,. \tag{13}$$

Since every vector in $\mathcal{B}'$ defines at most two facets of P, $|\mathcal{B}| \ge \frac{F}{2} + 1$, thus from (13) we conclude

$$V \cdot F \le 2\,(|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})|) \le 4 \cdot |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \tag{14}$$

Consider three cases:

1. $|\mathcal{A}_0| > d$ and $|\pi(B)| > d$. By Theorem 3, we have

   $$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \le (d-1)2^{d-1} + 2(d-1)$$

   and with (14) we are done.

2. $|\pi(B)| = d$. Together with $\mathcal{B}_1 = \{0, b_d\}$, this means that $\mathcal{B}'$ is a basis of $\mathbb{R}^d$. Every vector in $\mathcal{B}'$ then has to define two facets of P, since otherwise P is unbounded. Thus P is affinely isomorphic to the cube.

3. $|\mathcal{A}_0| = d$. Note that as $|\mathcal{A}_1| \le |\mathcal{A}_0|$ and $\dim(\mathcal{A}_1) = d-1$, we also have $|\mathcal{A}_1| = d$. If $|\pi(\mathcal{B})| \le \frac{3}{4} \cdot 2^{d-1}$, then (14) implies $V \cdot F \le \frac{3}{4} d \cdot 2^{d+1} < (d-1)2^{d+1} + 8(d-1)$, so we may further assume
$$|\pi(\mathcal{B})| > \frac{3}{4} \cdot 2^{d-1}. \tag{15}$$
We will now make several observations about the structure of $\mathcal{A}$ and $\mathcal{B}$, after which it will become clear that P is affinely isomorphic to the cross-polytope. Let $a_0 = 0, a_1, \dots, a_{d-1}$ be the elements of $\mathcal{A}_0$ and $\{u_1, \dots, u_{d-1}\}$ be the basis of $\operatorname{span}(\mathcal{A}_0)$, dual to $\{a_1, \dots, a_{d-1}\}$. Note that for every $j \in [d-1]$ there is $b_{\{j\}} \in \mathcal{B}$ such that $\pi(b_{\{j\}}) = u_j$: $b_{\{j\}}$ is the vector orthogonal to the facet of P that contains vertices $\{a_0, \dots, a_{d-1}\} \setminus \{a_j\}$ and differs from $\mathcal{A}_0$. Given $S \subseteq [d-1]$ let us denote by b_S an element of $\mathcal{B}$ for which $\pi(b_S) = \sum_{j \in S} u_j$, if there is one, with $b_\varnothing = 0$ to avoid ambiguity. Observe that the basis of $\mathbb{R}^d$ dual to $\{b_{\{1\}}, b_{\{2\}}, \dots, b_{\{d-1\}}, b_d\}$ is $\{a_1, a_2, \dots, a_{d-1}, v\}$ for v that satisfies
$$\langle v, b_d \rangle = 1 \text{ and } \forall j \in [d-1] : \langle v, b_{\{j\}} \rangle = 0.$$
This means that
$$\mathcal{A}_1 = \{v + \sum_{j \in S} a_j : S \in \mathcal{S}\} \tag{16}$$
for some family $\mathcal{S}$ of subsets of $[d-1]$ with $|\mathcal{S}| = d$. Our goal is to show that $\mathcal{S} = \{S \subseteq [d-1] : |S| \ge d-2\}$, as then P is affinely isomorphic to the cross-polytope, and we would be done. For $T \subseteq [d-1]$ denote $\sigma_T = \sum_{j \in T} a_j$ and note that, given $b_S \in \mathcal{B}$,
$$\langle \sigma_T, b_S \rangle = \langle \sigma_T, \pi(b_S) \rangle = \Big\langle \sigma_T, \sum_{j \in S} \pi(b_{\{j\}}) \Big\rangle = \Big\langle \sum_{j \in T} a_j, \sum_{j \in S} b_{\{j\}} \Big\rangle = |T \cap S|.$$
Now assume, looking for a contradiction, that $\exists S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| > 1$. (15) means that there exists $b_S \in \mathcal{B}$ such that $|S \cap (S_2 \setminus S_1)| > 1$. (16) means that
$$\begin{aligned}\{-1, 0, 1\} \ni \langle v + \sum_{j \in S_2} a_j\, , \, b_S \rangle - \langle v + \sum_{j \in S_1} a_j\, , \, b_S \rangle &= \langle \sigma_{S_2} - \sigma_{S_1}\, , \, b_S \rangle \\ = |S_2 \cap S| - |S_1 \cap S| &= |(S_2 \setminus S_1) \cap S| > 1,\end{aligned}$$
a contradiction. Therefore, $\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \le 1$, which by Lemma 1 implies that either $\mathcal{S} = \{S \subseteq [d-1] : |S| \ge d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \le 1\}$. The latter is, however, impossible, since then P only has $d+1$ facets and $|\pi(\mathcal{B})| = d \le \frac{3}{4} 2^{d-1}$. (16) now shows that P is affinely isomorphic to the cross-polytope, and we are done.

$\square$

Two examples demonstrate tightness of the bound in Theorem 2:

Example 3 (Cross-polytope $\times$ segment)**.** *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*
$$P = \operatorname{Conv}\big(\{\varepsilon_i e_i + \varepsilon_d e_d\}_{i \le d-1}\big), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\} \text{ for } i \in [d].$$
Here $f_0(P) = 4(d-1)$ and $f_{d-1}(P) = 2 + 2^{d-1}$.

Example 4 (Suspension of a cube)**.** *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*

$$P = \mathrm{Conv}\left(\left\{\sum_{i=1}^{d-1} \varepsilon_i e_i\right\} \cup \{e_d, -e_d\}\right), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\}.$$

This is (up to coordinate scaling) the dual of the polytope in the previous example and, in particular, $f_0(P) = 2 + 2^{d-1}$ and $f_{d-1}(P) = 4(d-1)$.

4 Proofs of claims

In this section, we provide the proofs of the claims from [5] made at the beginning of Section 2.

Claim 1. *We may translate $\mathcal{A}$ and replace some points $b \in \mathcal{B}$ by the opposites $-b$ such that the following properties hold.*

(i) We (still) have $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, where $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d\rangle = i\}$ for $i = 0, 1$ such that

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}$$

(ii) We have

$$\langle a, b\rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}$$

(iii) The set $\pi(\mathcal{B})$ does not contain opposite points.

Proof. If $|\{a \in \mathcal{A} : \langle a, b_d\rangle = 0\}| \leq |\{a \in \mathcal{A} : \langle a, b_d\rangle = 1\}|$, then we can choose any $a_* \in \mathcal{A}$ with $\langle a_*, b_d\rangle = 1$ (which exists since $\mathcal{A}$ spans $\mathbb{R}^d$) and replace $\mathcal{A}$ by $\mathcal{A} - a_*$, $\mathcal{B}$ by $(\mathcal{B} \setminus \{b_d\}) \cup \{-b_d\}$, and b_d by $-b_d$. This yields (i).

After this replacement, for each $b \in \mathcal{B}$ there is some $\varepsilon_b \in \{\pm 1\}$ such that $\langle a, b\rangle \in \{0, \varepsilon_b\}$ holds for all $a \in \mathcal{A}$. Each b with $\{\langle a, b\rangle : a \in \mathcal{A}_0\} = \{0, -1\}$ is replaced by $-b$, which yields (ii).

Let $\mathcal{A}_1'$ be a translate of $\mathcal{A}_1$ such that $\mathbf{0} \in \mathcal{A}_1'$. Note that, for each $b \in \mathcal{B}$ we now have $\{\langle a, b\rangle : a \in \mathcal{A}_0\} = \{0, 1\}$ or $\{\langle a, b\rangle : a \in \mathcal{A}_0\} = \{0\}$. In the second case, we replace b by $-b$ if $\{\langle a, b\rangle : a \in \mathcal{A}_1'\} = \{0, -1\}$, otherwise we leave it as it is.

It remains to show that $\pi(\mathcal{B})$ does not contain opposite points after this transformation. To this end, let $b, b' \in \mathcal{B}$ such that $\pi(b) = \beta\pi(b')$ for some $\beta \neq 0$, where $\pi(b), \pi(b') \neq \mathbf{0}$. We have to show that $\beta = 1$. Note that for every $a \in \mathcal{A}_0 \cup \mathcal{A}_1' \subseteq U$ we have

$$\langle a, b\rangle = \langle a, \pi(b)\rangle = \beta\langle a, \pi(b')\rangle = \beta\langle a, b'\rangle.$$

Suppose first that $\{\langle a, b\rangle : a \in \mathcal{A}_0\} \neq \{0\}$. By (2) there exists some $a \in \mathcal{A}_0$ with $1 = \langle a, b\rangle = \beta\langle a, b'\rangle$. Thus, we have $\langle a, b'\rangle \neq 0$ and hence $\langle a, b'\rangle = 1$, again by (2). This yields $\beta = 1$.

Suppose now that $\{\langle a, b\rangle : a \in \mathcal{A}_0\} = \{0\}$. Note that this implies $\{\langle a, b'\rangle : a \in \mathcal{A}_0\} = \{0\}$. As $\mathcal{A}_0 \cup \mathcal{A}_1'$ spans U, we must have $\{\langle a, b\rangle : a \in \mathcal{A}_1'\} \neq \{0\}$ and hence there is some $a \in \mathcal{A}_1'$ with $\langle a, b\rangle = 1$. Moreover, we have $\beta\langle a, b'\rangle = 1$, and in particular $\langle a, b'\rangle \neq 0$. This implies $\langle a, b'\rangle = 1$ and hence $\beta = 1$. □

As in the previous proof, let $\mathcal{A}_1'$ be a translate of $\mathcal{A}_1$ such that $\mathbf{0} \in \mathcal{A}_1'$. Note that for each $b \in \mathcal{B}$ there are $\varepsilon_b, \gamma_b \in \{\pm 1\}$ such that

$$\langle a, b\rangle \in \{0, \varepsilon_b\} \text{ for each } a \in \mathcal{A} \text{ and} \tag{17}$$

$$\langle a, b\rangle \in \{0, \gamma_b\} \text{ for each } a \in \mathcal{A}_1'. \tag{18}$$

Inequality 1. $|\mathcal{A}|\,|\mathcal{B}| \leq 2\,|\mathcal{A}_0|\,|\pi(\mathcal{B})| + |\mathcal{A}_1|\,|\mathcal{B}\backslash\mathcal{B}_*|$

Proof. Claim 2 implies $|\mathcal{B}| = 2|\pi(\mathcal{B})| - |\mathcal{B}_*|$ or $2(|\pi(\mathcal{B})| - |\mathcal{B}_*|) = |\mathcal{B} \setminus \mathcal{B}_*|$. With $|\mathcal{A}_0| \geq |\mathcal{A}_1|$ this gives

$$\begin{aligned}|\mathcal{A}||\mathcal{B}| = (|\mathcal{A}_0| + |\mathcal{A}_1|)(2|\pi(\mathcal{B}_*)| - |\mathcal{B}_*|) &\leq 2|\mathcal{A}_0||\pi(\mathcal{B}_*)| + 2|\mathcal{A}_1||\pi(\mathcal{B})| - 2|\mathcal{A}_1||\pi(\mathcal{B})| \\ &= 2|\mathcal{A}_0||\pi(\mathcal{B}_*)| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*|\end{aligned}$$

□

The proofs of the subsequent claims rely on the following two lemmas.

Lemma 2. *Suppose that $X \subseteq \{0,1\}^d \cup \{0,-1\}^d$ does not contain opposite points. Then we have $|X| \leq 2^{\dim X}$.*

Proof. We prove the statement by induction on $d \geq 1$, and observe that it is true for $d = 1$. Now let $d \geq 2$. If $\dim X = d$, then we are also done. It remains to consider to case where X is contained in an affine hyperplane $H \subseteq \mathbb{R}^d$. Let $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$, $\delta \in \{0,1\}$ such that

$$H = \{x \in \mathbb{R}^d : \langle c, x\rangle = \delta\}.$$

For each $i \in \{1,\ldots,d\}$ let $\pi_i : H \to \mathbb{R}^{d-1}$ denote the projection that forgets the i-th coordinate, and let $e_i \in \mathbb{R}^d$ denote the i-th standard unit vector. Note that $\pi_{i^*}(X) \subseteq \{0,1\}^{d-1} \cup \{0,-1\}^{d-1}$.

Suppose there is some $i^* \in \{1,\ldots,d\}$ such that $\langle c, e_{i^*}\rangle \neq 0$ and $\pi_{i^*}(X)$ does not contain opposite points. By the induction hypothesis we obtain

$$|X| = |\pi_{i^*}(X)| \leq 2^{\dim \pi_{i^*}(X)} = 2^{\dim X},$$

where the first equality and the last equality hold since π_{i^*} is injective (due to $\langle c, e_{i^*}\rangle \neq 0$).

It remains to consider the case in which there is no such i^*. Consider any $i \in \{1,\ldots,d\}$. If $\langle c, e_i\rangle \neq 0$, then there exist $x = (x_1,\ldots,x_d), x' = (x'_1,\ldots,x'_d) \in X$, $x \neq x'$ such that $\pi_i(x) = -\pi_i(x')$. We may assume that $\pi_i(x) \in \{0,1\}^{d-1}$ and hence $\pi_i(x') \in \{0,-1\}^{d-1}$. As X does not contain opposite points, we must have $x_i = 1$ and $x'_i = 0$, or $x_i = 0$ and $x'_i = -1$. In the first case we obtain

$$\begin{aligned}2\delta = \langle c, x\rangle + \langle c, x'\rangle &= [\langle \pi_i(c), \pi_i(x)\rangle + c_i x_i] + [\langle \pi_i(c), \pi_i(x')\rangle + c_i x'_i] \\ &= [\langle \pi_i(c), \pi_i(x)\rangle + c_i] + [\langle \pi_i(c), \pi_i(x')\rangle] \\ &= c_i.\end{aligned}$$

Similarly, in the second case we obtain $2\delta = -c_i$.

If $\delta = 0$, this would imply that $c = \mathbf{0}$, a contradiction to the fact that $H \neq \mathbb{R}^d$. Otherwise, $\delta = 1$ and hence every nonzero coordinate of c is ± 2. Thus, for every $x \in \mathbb{Z}^d$ we see that $\langle c, x\rangle$ is an even number, in particular $\langle c, x\rangle \neq \delta$. This means that $X \subseteq \mathbb{Z}^d \cap H = \emptyset$, and we are done. □

A direct consequence of Lemma 2 that we will employ is

Lemma 3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ such that $\mathcal{A}$ spans $\mathbb{R}^d$, $\mathcal{B}$ does not contain opposite points, and for every $b \in \mathcal{B}$ there is some $\varepsilon_b \in \{\pm 1\}$ such that $\{\langle a, b\rangle : a \in \mathcal{A}\} \subseteq \{0, \varepsilon_b\}$. Then we have $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$.*

Proof. Let $a_1, \ldots, a_d \in \mathcal{A}$ be a basis of $\mathbb{R}^d$ and express elements of $\mathcal{B}$ in the dual basis, it then becomes a subset of $\{0,1\}^d \cup \{0,-1\}^d$ with no opposite points. By Lemma 2, $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$. $\square$

We are ready to continue with the proofs of the remaining claims.

Claim 2. *Every point in $\pi(\mathcal{B})$ has at most two preimages in $\mathcal{B}$.*

Proof. Let $y := \pi(b)$ for some $b \in \mathcal{B}$ and observe that $\pi^{-1}(y) = \{x \in \mathbb{R}^d : \pi(x) = y\}$ is a one-dimensional affine subspace. By (17) and Lemma 3 we obtain $|\mathcal{B} \cap \pi^{-1}(y)| \leq 2$. $\square$

Claim 3. $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0} |\tau(\pi(\mathcal{B}))|$.

Proof. Fix any $b \in \mathcal{B}$ and let $v := \pi(b)$. Consider the orthogonal complement $W \subseteq U$ of U_0 in U. As $\tau^{-1}(\tau(v)) = v + W$, it suffices to show that

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}$$

holds. To this end, consider the linear subspace $\Pi \subseteq U$ spanned by v and W and let $\sigma : U \to \Pi$ denote the orthogonal projection on Π.

First, suppose that $\sigma(\mathcal{A}_1')$ spans Π. For every $a \in \mathcal{A}_1' \subseteq U$ and every $b \in \mathcal{B}$ with $\pi(b) \in v + W \subseteq \Pi$ we have

$$\langle \sigma(a), \pi(b) \rangle = \langle a, \pi(b) \rangle = \langle a, b \rangle \in \{0, \gamma_b\}$$

by (18). Moreover, recall that $\pi(\mathcal{B})$ does not contain opposite points by Claim 1 (iii). Thus, the pair $\sigma(\mathcal{A}_1')$ and $(v + W) \cap \pi(\mathcal{B})$ satisfies the requirements of Lemma 3 (in Π), and hence we obtain

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{\dim(v+W)} = 2^{\dim W} = 2^{\dim U - \dim U_0} = 2^{d-1-\dim U_0}.$$

It remains to consider the case in which $\sigma(\mathcal{A}_1')$ does not span Π. Recall that we chose b_d as the nonzero vector in $\mathcal{B}$ with the maximal $\varphi(b_d) := \max\big(\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\big)$ for the corresponding $\mathcal{A}_0$ and $\mathcal{A}_1$. Unless $|(v+W) \cap \pi(\mathcal{B})| = 1$, we will identify points $b_1, b_2 \in \mathcal{B}$ with $\max\{\varphi(b_1), \varphi(b_2)\} > \varphi(b_d)$, a contradiction to the choice of b_d.

As $\mathcal{A}_0 \cup \mathcal{A}_1'$ spans U, we know that $\sigma(\mathcal{A}_0 \cup \mathcal{A}_1')$ spans Π. Since $\mathcal{A}_0$ is orthogonal to W, this means that $\sigma(\mathcal{A}_0)$ spans a line, and $\sigma(\mathcal{A}_1')$ spans a hyperplane H in Π. Note that we have $v \notin W$ (otherwise $W = \Pi$ and so $\sigma(\mathcal{A}_1')$ spans Π). Thus, every nonzero point in $\sigma(\mathcal{A}_0)$ has nonzero scalar product with v. Moreover, for every $a \in \mathcal{A}_0$ with $\sigma(a) \neq \mathbf{0}$ we have $\langle \sigma(a), v \rangle = \langle a, v \rangle = \langle a, b \rangle \in \{0, 1\}$ by (2). Thus, since the nonzero vectors in $\sigma(\mathcal{A}_0)$ are collinear, we obtain

$$\sigma(\mathcal{A}_0) \subseteq \{\mathbf{0}, \sigma(a_0)\}$$

for some $a_0 \in \mathcal{A}_0$. Since $\mathbf{0} \in H$, we have $\sigma(\mathcal{A}_0) \setminus H \subseteq \{\sigma(a_0)\}$ and further, since $\sigma(\mathcal{A}_0 \cup \mathcal{A}_1')$ spans Π, we have $\sigma(\mathcal{A}_0) \setminus H = \{\sigma(a_0)\}$. Let $c \in \Pi$ be a normal vector of H. As $\sigma(a_0) \notin H$, we may scale c so that $\langle \sigma(a_0), c \rangle = 1$. Let $a_* \in \mathcal{A}_1$ such that $\mathcal{A}_1' = \mathcal{A}_1 - a_*$. We define

$$b_1 := c - \delta_1 b_d \neq \mathbf{0},$$

where $\delta_1 := \langle a_*, c \rangle$. For every $a \in \mathcal{A}_0$ we have

$$\langle a, b_1 \rangle = \langle a, c \rangle = \langle \sigma(a), c \rangle \in \{\langle \mathbf{0}, c \rangle, \langle \sigma(a_0), c \rangle\} = \{0, 1\},$$

and for every $a \in \mathcal{A}_1$ we have

$$\langle a, b_1\rangle = \langle \underbrace{a - a_*}_{\in \mathcal{A}_1'}, b_1\rangle + \langle a_*, b_1\rangle = \langle a - a_*, c\rangle + \langle a_*, b_1\rangle = \langle \underbrace{\sigma(a - a_*)}_{\in H}, c\rangle + \langle a_*, b_1\rangle$$
$$= \langle a_*, b_1\rangle = \langle a_*, c\rangle - \delta_1\langle a_*, b_d\rangle = \langle a_*, c\rangle - \delta_1 = 0.$$

Thus, by the maximality of $\mathcal{B}$, (a scaling of) the vector b_1 is contained in $\mathcal{B}$. Since we assumed $\mathbf{0} \in \mathcal{A}_0$, we have $\varphi(b_1) \geq \dim(\mathcal{A}_1) + 1$.

In order to construct b_2, let us suppose that there is another point $b' \in \mathcal{B}$ with $v' := \pi(b') \neq v$ and $v' \in (v + W)$. If there is no such point, then the statement of the claim is true. Recall that $\sigma(a_0)$ is orthogonal to W, and let

$$\xi := \langle \sigma(a_0), v\rangle = \langle \sigma(a_0), \underbrace{v - v'}_{\in W}\rangle + \langle \sigma(a_0), v'\rangle = \langle \sigma(a_0), v'\rangle.$$

Choose $v'' \in \{v, v'\}$ such that $\xi c \neq v''$, and let $b'' \in \{b, b'\}$ such that $\pi(b'') = v''$. Define $\delta_2 := \langle a_*, v'' - \xi c\rangle$ and note that

$$b_2 := v'' - \xi c - \delta_2 b_d$$

is nonzero since $v'' - \xi c \in U \setminus \{\mathbf{0}\}$. For every $a \in \mathcal{A}_0$ we have

$$\langle a, b_2\rangle = \langle a, \underbrace{v'' - \xi c}_{\in \Pi}\rangle = \langle \sigma(a), v'' - \xi c\rangle,$$

which is zero if $\sigma(a) = \mathbf{0}$. Otherwise, $\sigma(a) = \sigma(a_0)$ and we obtain

$$\langle a, b_2\rangle = \langle \sigma(a_0), v''\rangle - \xi\langle \sigma(a_0), c\rangle = \langle \sigma(a_0), v''\rangle - \xi = 0.$$

Thus, b_2 is orthogonal to $\mathcal{A}_0$. Moreover, note that

$$\langle a_*, b_2\rangle = \langle a_*, v'' - \xi c\rangle - \delta_2 \underbrace{\langle a_*, b_d\rangle}_{=1} = 0.$$

Thus, for every $a \in \mathcal{A}_1$ we have

$$\langle a, b_2\rangle = \langle a - a_*, b_2\rangle + \langle a_*, b_2\rangle = \langle a - a_*, b_2\rangle = \langle a - a_*, v''\rangle - \xi \underbrace{\langle a - a_*, c\rangle}_{=0} - \delta_2 \underbrace{\langle a - a_*, b_d\rangle}_{=0}$$
$$= \langle a - a_*, v''\rangle = \langle a - a_*, b''\rangle \in \{0, \gamma_{b''}\}$$

by (18). Thus, again by the maximality of $\mathcal{B}$, (a scaling of) the vector b_2 is contained in $\mathcal{B}$, and since b_2 is orthogonal to $\mathcal{A}_0$ and $a_* \in \mathcal{A}_1$, we have $\varphi(b_2) \geq \dim(\mathcal{A}_0) + 1$. However, by the choice of b_d we must have

$$\max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\} + 1 \leq \max\{\varphi(b_1), \varphi(b_2)\} \leq \varphi(b_d) = \max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\},$$

a contradiction. $\square$

Claim 4. *$\mathcal{B}\backslash\mathcal{B}_*$ can be partitioned as $\mathcal{B}_0 \sqcup \mathcal{B}_1$, with $\mathcal{B}_0, \mathcal{B}_1$ satisfying*

$$\forall b \in \mathcal{B}_i : |\{\langle a, b\rangle : a \in \mathcal{A}_i\}| = 1 \textit{ for } i = 0, 1.$$

Proof. Let $b \in \mathcal{B} \setminus \mathcal{B}_*$ and, for the sake of contradiction, suppose that $|\{\langle a, b\rangle : a \in \mathcal{A}_0\}| = |\{\langle a, b\rangle : a \in \mathcal{A}_1\}| = 2$. Let $b' \in \mathcal{B} \setminus \{b\}$ such that $\pi(b) = \pi(b')$. In other words, we have $b' = b + \gamma b_d$ for some $\gamma \neq 0$. Then, by (2) we have

$$\{\langle a, b'\rangle : a \in \mathcal{A}_0\} = \{\langle a, b\rangle : a \in \mathcal{A}_0\} = \{0, 1\}$$

and hence we obtain $\varepsilon_b = \varepsilon_{b'} = 1$ by (17). Again by (17) we see

$$\{0, 1\} \supseteq \{\langle a, b'\rangle : a \in \mathcal{A}_1\} = \{\langle a, b\rangle : a \in \mathcal{A}_1\} + \gamma = \{0, 1\} + \gamma = \{\gamma, 1 + \gamma\},$$

which implies $\gamma = 0$, a contradiction. $\square$

Inequality 2. $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1)\, 2^d + |\mathcal{A}_0|\, |\mathcal{B}_0| + |\mathcal{A}_1|\, |\mathcal{B}_1|$

Proof. $\tau(\pi(\mathcal{B}))$ and $\mathcal{A}_0$ are both spanning U_0 and have binary scalar products, so by Theorem 4 (or by the induction hypothesis, in the context of the proof of Theorem 4 in [5])

$$|\tau(\pi(\mathcal{B}))||\mathcal{A}_0| \leq (\dim U_0 + 1) 2^{\dim U_0}$$

Combining this with Claim 3 and Inequality 1 we get

$$|\mathcal{A}||\mathcal{B}| \leq 2 \cdot (\dim(U_0) + 1) 2^{d-1} + |\mathcal{A}_1|(|\mathcal{B}_0| + |\mathcal{B}_1|) \leq (\dim U_0 + 1)\, 2^d + |\mathcal{A}_0||\mathcal{B}_0| + |\mathcal{A}_1||\mathcal{B}_1|,$$

where the second inequality is due to $|\mathcal{A}_0| \geq |\mathcal{A}_1|$. $\square$

Inequality 3. *For $i = 0, 1$ we have*

$$|\mathcal{A}_i| \leq 2^{\dim(\mathcal{A}_i)},\ |\mathcal{B}_i| \leq 2^{\dim(\mathrm{span}(\mathcal{B}_i))},\ \textit{and}\ \dim(\mathcal{A}_i) + \dim\bigl(\mathrm{span}(\mathcal{B}_i)\bigr) \leq d$$

Proof. The first (and second) inequality is a direct consequence of Lemma 3 after writing $\mathcal{A}$ (or $\mathcal{B}$) in the basis, dual to a basis bound in $\mathcal{B}$ (or $\mathcal{A}$). The last inequality follows from the definition of $\mathcal{B}_i$: for each $b \in \mathcal{B}_i$ there is ξ_b such that

$$\mathcal{A}_i \subset W_i, \text{ where } W_i = \{x \in \mathbb{R}^d : \langle x, b\rangle = \xi_b \text{ for all } b \in \mathcal{B}_i\},$$

and clearly $\dim(W_i) \leq d - \dim\bigl(\mathrm{span}(\mathcal{B}_i)\bigr)$. $\square$

Claim 5. *For $i = 0, 1$, we have $|\mathcal{A}_i|\, |\mathcal{B}_i| \leq 2^d$.*

Proof. By Inequality 3,

$$|\mathcal{A}_i||\mathcal{B}_i| \leq 2^{\dim(\mathcal{A}_i)} \cdot 2^{\dim(\mathrm{span}(\mathcal{B}_i))} \leq 2^d$$

$\square$

A Appendix

Inequality 4. *For an integer $2 \le f \le d$, we have:*

$$(d+f)(2^{d-1}+2^{d-f}) \le d2^d + 2d.$$

Proof. We will prove this by induction on d: when $d = f$, the equality is satisfied. Let's perform the induction step from d to $d+1$. Denoting the left and right sides of the inequality as $l(d,f)$ and $r(d)$, respectively, we have

$$\begin{aligned} r(d+1) - l(d+1,f) &\ge (r(d+1) - r(d)) - (l(d+1,f) - l(d,f)) \\ &= \left(d2^d + 2^{d+1} + 2\right) - (d+f+2)\left(2^{d-1} + 2^{d-f}\right) \\ &= 2^{d-f}(d-f+2)\left(2^{f-1} - 1 - \frac{2f}{d-f+2}\right) + 2 \\ &\ge 2^{d-f}(d-f+2)\left(2^{f-1} - 1 - f\right) \end{aligned}$$

The obtained expression is non-negative for $f > 2$. For $f = 2$ and $d \ge 4$, we have $2^{f-1} - 1 - \frac{2f}{d-f+2} \ge 0$, and for $f = 2$ and $d = 2, 3$, the initial inequality is checked explicitly. □

Lemma 1. *Let $\mathcal{S}$ be a family of subsets of $[d-1]$ such that $|\mathcal{S}| = d$ and*

$$\forall S_1, S_2 \in \mathcal{S} : \ |S_2 \setminus S_1| \le 1.$$

Then either $\mathcal{S} = \{S \subseteq [d-1] : |S| \ge d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \le 1\}$.

Proof. We assume $d > 2$ as the statement is trivial otherwise. $|\mathcal{S}| > 2$ and clearly $\mathcal{S}$ contains sets of at most two different sizes (that differ by one), so let $U, V \in \mathcal{S}$ both be of size $k \in [d-2]$. Observe that there are now only four options for sets in $\mathcal{S}$:

(a) $U \cup V$ of size $k+1$.

(b) Sets of size k that are contained in $U \cup V$.

(c) Sets of size k that contain $U \cap V$ as a subset.

(d) $U \cap V$ of size $k-1$.

(a) and (d) are not possible simultaneously, neither are (b) and (c) with the exception of U and V. There are $k+1$ and $d-k$ sets satisfying (b) and (c) respectively, so $|\mathcal{S}| = d$ is only possible if $k = d-2$ or $k = 1$ with $\mathcal{S} = \{S \subseteq [d-1] : |S| \ge d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \le 1\}$ respectively. □

We finish with a conjecture that generalises Theorem 3:

Conjecture 1. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b\rangle \in \{0,1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Furthermore, suppose $|\mathcal{A}|$ and $|\mathcal{B}|$ are both strictly larger then $2^{k-1}(d-k+2)$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \le (2^{d-k}+k)2^k(d-k+1)$.*

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