Stability of size bounds for families with binary scalar products

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Abstract

Questions on possible number of vertices and facets in two-level polytopes motivate the study of vector families $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ with a property that $\forall a \in \mathcal{A}, b \in \mathcal{B}$ the scalar product $\langle a, b\rangle \in \{0, 1\}$. In this work we show the stability of Kupavskii's and Weltge's bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for such $\mathcal{A}$ and $\mathcal{B}$. We use this result to find the maximal possible product of the number of vertices and the number of facets of a two-level polytope that is not affinely isomorphic to a cube or a cross-polytope.

1 Introduction

A polytope P is said to be *2-level* if for every facet-defining hyperplane H there is a parallel hyperplane H' such that $H \cup H'$ contains all vertices of P. Basic examples of 2-level polytopes are simplices, hypercubes and cross-polytopes, but they also generalize a variety of interesting polytopes such as Birkhoff, Hanner, and Hansen polytopes, order polytopes and chain polytopes of posets, stable matching polytopes, and stable set polytopes of perfect graphs [1]. Combinatorial structure of two-level polytopes has also been studied in [3], and enumeration of such polytopes in [2] led to a conjecture that the product of the number of facets and the number of vertices in a d-dimensional 2-level polytope is bounded by $d2^{d+1}$. It has been proven in [5] via a stronger theorem regarding so-called families of vectors with binary scalar products:

Theorem 1. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b\rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then we have $|\mathcal{A}| \cdot |\mathcal{B}| \le (d + 1)2^d$.*

In this paper we prove several bounds regarding the stability of the bound in Theorem 1. Our main result is the following

Theorem 2. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b\rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Furthermore, suppose $\mathcal{A}$ and $\mathcal{B}$ are inclusion-wise maximal with respect to this property and both have the size of at least $d + 2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \le d2^d + 2d$.*

We then use this to improve the bound on the product of the vertex count $f_0(P)$ and the facet count $f_{d-1}(P)$ in a 2-level polytope P that is distinct from both the cube and the cross-polytope:

Theorem 3. *For $d > 1$ let P be a d-dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then $f_0(P)f_{d-1}(P) \leq (d-1)\,2^{d+1} + 8\,(d-1)$.*

Outline The next section lays out the proof of main results. In Section 3 we provide the proof of Theorem 3. Technical proofs of some claims used in the main section, as well as an enumeration approach used to verify statements in low dimensions are provided in Appendix A.

2 Stability results

We will first introduce some notation and restate some claims proved in [5].
Let $b_d \in \mathcal{B}$ be a vector with the maximum value of $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, where

$$\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\} \text{ for } i = 0, 1.$$

The choice of b_d among the vectors that maximise $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, if important, will be specified at a later stage. We denote the orthogonal projection onto $U = b_d^{\perp}$ as $\pi : \mathbb{R}^d \to U$.

Claim 1. *We may translate $\mathcal{A}$ and replace some points in $\mathcal{B}$ by their negatives such that the following holds.*

(i) We can write $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, where $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$ for $i = 0, 1$ such that

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}$$

(ii) We still have

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}$$

(iii) The set $\pi(\mathcal{B})$ does not contain opposite points.

Claim 2. *Every point in $\pi(\mathcal{B})$ has at most two preimages in $\mathcal{B}$.*

Inequality 1. $|\mathcal{A}|\,|\mathcal{B}| \leq 2\,|\mathcal{A}_0|\,|\pi(\mathcal{B})| + |\mathcal{A}_1|\,|\mathcal{B}\backslash\mathcal{B}_*|$

We denote the linear span of $\mathcal{A}_0$ as U_0 and introduce the orthogonal projection $\tau : U \to U_0$. Let $\mathcal{B}_* \subseteq \mathcal{B}$ be the set of $b \in \mathcal{B}$ for which $\pi(b)$ has exactly one pre-image under projection onto U.

Claim 3. $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}\,|\tau(\pi(\mathcal{B}))|$.

Claim 4. *$\mathcal{B}\backslash\mathcal{B}_* = \mathcal{B}_0 \sqcup \mathcal{B}_1$ holds with $i = 0, 1$ satisfying:*

$$\forall b \in \mathcal{B}_i : |\{\langle a, b \rangle : a \in \mathcal{A}_i\}| = 1$$

Claim 5. *For $i = 0, 1$, we have $|\mathcal{A}_i|\,|\mathcal{B}_i| \leq 2^d$.*

Inequality 2. $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1)\, 2^d + |\mathcal{A}_0|\, |\mathcal{B}_0| + |\mathcal{A}_1|\, |\mathcal{B}_1|$

We will now understand under which conditions equality is achieved in Theorem 1. Without loss of generality, we assume $|\mathcal{A}| \geq |\mathcal{B}|$.

Theorem 4. $|\mathcal{A}| \cdot |\mathcal{B}| = (d+1)2^d$ *only if* $|\mathcal{B}| = d+1$*, and* $\mathcal{A}$ *is affinely isomorphic to* $\{0,1\}^d$.

Proof. We will use induction on d, the statement is obvious in dimension 1. Assuming the statement holds for smaller dimensions, we prove it for dimension d. Consider the possible values of $\dim U_0$:

1. $\dim U_0 < d - 2$. Then, from inequality 2 and claim 5, we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 3)\, 2^d \leq d2^d \tag{3}$$

2. $\dim U_0 = d - 2$. Notice that we can assume without loss of generality that $0, b_d \in \mathcal{B}_0$ or $0, b_d \in \mathcal{B}_1$. Therefore, from the proof of claim 5, we have:

$$|\mathcal{A}_1|\, |\mathcal{B}_1| \leq 2^d,\ |\mathcal{A}_0|\, (|\mathcal{B}_0| + 2) \leq 2^d$$

   Thus, from inequality 2 and claim 5, we get:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (d-1)\, 2^d + 2 \cdot 2^d - 2\, |\mathcal{A}_0| \leq (d+1)\, 2^d - |\mathcal{A}| < (d+1)\, 2^d$$

3. $\dim U_0 = d - 1$. In this case, assuming $0, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \varnothing$. We consider two subcases:

   a) $\mathcal{B}_* \neq \varnothing$. Then inequality 1 is only tight when $|\mathcal{A}_0| = |\mathcal{A}_1|$, and inequality 2 is only tight when $|\mathcal{A}_0|\, |\pi(\mathcal{B})| = d2^{d-1}$. By the induction hypothesis, the latter is possible in one of two cases:

      i) $\mathcal{A}_0$ is affinely isomorphic to $\{0,1\}^{d-1}$. Then, $|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = 2^d$, which is only possible if $\mathcal{A}$ is affinely isomorphic to $\{0,1\}^d$, and $\mathcal{B}$ can consist only of a basis and zero vector.

      ii) $|\mathcal{A}_0| = d$. Then, $|\mathcal{B}| \leq |\mathcal{A}| = 2d$, and $|\mathcal{B}| \cdot |\mathcal{A}| \leq 4d^2$, which is less than $(d+1)2^d$ for $d \geq 4$. For $d = 3$, the inequality $|\mathcal{B}| \cdot |\mathcal{A}| \leq 32$ cannot yield equality since $|\mathcal{A}| = 6$. Finally, for $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

   b) $\mathcal{B}_* = \varnothing$. Then, $\mathcal{B}_1 = \mathcal{B}$ and, consequently, $\dim(\operatorname{span}(\mathcal{B}_1)) = d$. In this case:

$$(\forall b \in \mathcal{B}_1\ \exists \xi : \forall a \in \mathcal{A}_1\ \langle a, b \rangle = \xi) \Rightarrow \dim(\mathcal{A}_1) \leq d - \dim(\operatorname{span}(\mathcal{B}_1)) = 0 \Rightarrow |\mathcal{A}_1| = 1$$

   Similarly to case a), inequality 1 is only tight in one of the following cases:

      i) $|\mathcal{A}_0| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1)^2 < (d+1)\, 2^d$.

      ii) $|\mathcal{A}_0| = 2^{d-1}$, $|\pi(\mathcal{B})| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| = 2d\left(2^{d-1} + 1\right)$, which is less than $(d+1)2^d$ for $d > 2$. For $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

□

We will improve the bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for families that differ from the extremal example. To do this, we will use an auxiliary

Inequality 3. *For an integer $2 \leq f \leq d$, we have:*

$$(d+f)(2^{d-1}+2^{d-f}) \leq d2^d + 2d.$$

A short but technical proof of this inequality can be found in Appendix A.

Theorem 5. *Let both $\mathcal{A}$ and $\mathcal{B} \subseteq \mathbb{R}^d$ contain a basis for $\mathbb{R}^d$, and $\langle a, b\rangle \in \{0,1\}$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$. If, in addition, both families are inclusion-wise maximal with respect to this property and have size of at least $d+2$, then $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d + \lambda(d))\, 2^d$, where $0 < \lambda(d) \leq 1$ is a (non-strictly) decreasing function.*

Proof. We will denote $\lambda(d)$ as λ_d. As in the proof of Theorem 4, we will use induction on d. For the base case, we can choose $\lambda = 1$. Assuming validity for smaller dimensions, let us prove the statement for dimension d. We consider possible values of $\dim U_0$:

1. $\dim U_0 < d-2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 3)\, 2^d \leq d2^d$.

2. $\dim U_0 = d-2$. Applying the induction hypothesis and Theorem 4 to the families $\tau(\pi(\mathcal{B}))$ and $\mathcal{A}_0$, we have three cases:

   a) $\mathcal{A}_0$ is affine isomorphic to $\{0,1\}^{d-2}$, and $\tau(\pi(\mathcal{B}))$ consists of zero and the basis of U_0. From claim 5 and the assumption that 0, $b_d \in \mathcal{B}_1$, it follows that $|\mathcal{B}_0| \leq 2$. Since $|\mathcal{B}_0|$ is even, we have two scenarios:

      i) $|\mathcal{B}_0| = 0$. Then, from inequality 1 and statement 5, we obtain:

      $$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4\,(d-1)\,2^{d-2} + 2^d = d2^d$$

      ii) $|\mathcal{B}_0| = 2$. Let $k+1$ vectors have two pre-images under the action of τ ($k \geq 0$, as $\mathcal{B}_0 \subset U_0^{\perp}$ is not empty). Among these $k+1$, let t_2 be the number of those vectors with both pre-images in $\pi(\mathcal{B}_1)$, and let t_1 be the number of those with exactly one pre-image in $\pi(\mathcal{B}_1)$. The remaining $k-t_1-t_2$ have both pre-images in $\pi(\mathcal{B}_*)$. Suppose that the vectors in $\tau(\pi(\mathcal{B}))$ with a single pre-image under τ consist of q projections onto $\pi(\mathcal{B}_1)$ and $d-2-k-q$ projections onto $\pi(\mathcal{B}_*)$. We then have:

      $$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_*| + |\mathcal{B}_0| + |\mathcal{B}_1| \\ &= (k - t_1 - 2t_2 + d - 2 - q) + 2 + (2 + 4t_2 + 2t_1 + 2q) \\ &= d + k + q + t_1 + 2t_2 + 2 \end{aligned}$$

      First, consider the case when $t_2 > 0$. Then, $U_0^{\perp} \subset \operatorname{span}(\mathcal{B}_1)$, which implies:

      $$\dim(\operatorname{span}(\mathcal{B}_1)) = t_1 + t_2 + q + 2 \implies |\mathcal{A}_1| \leq 2^{d-t_1-t_2-q-2},$$
      $$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| \leq 2^{d-2} + 2^{d-2-t_1-t_2-q}$$

      $$\begin{aligned} |\mathcal{A}| \cdot |\mathcal{B}| &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right)(d + k + q + t_1 + 2t_2 + 2) \\ &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right)(2d + t_1 + 2t_2) && (4) \\ &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2-q}\right)(d + t_1 + t_2) && (5) \\ &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2}\right)(d + t_1 + t_2 + 1) \\ &\leq d2^d + 2d && (6) \end{aligned}$$

Here, the second inequality follows from $k+q \leq d-2$, and the last one follows from inequality 3. If $t_2 = 0$, we get a slightly weaker bound:

$$\dim(\operatorname{span}(\mathcal{B}_1)) \geq t_1 + t_2 + q + 1$$

This means (5) becomes $\left(2^{d-1} + 2^{d-t_1}\right)(d+t_1)$, which is still less than (6) when $t_1 \geq 2$ according to inequality 3. Finally, when $t_2 = 0$ and $t_1 = 0, 1$, expression (4) yields estimates of $d2^d$ and $(2^{d-2} + 2^{d-3})(2d+1) = d2^d - \left(d - \frac{3}{2}\right) 2^{d-2} \leq d2^d$, respectively.

b) $\mathcal{A}_0$ consists of zero and the basis of U_0. Then:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4(d-1)^2 \leq d2^d + 2d$$

c) $|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq (d - 2 + \lambda_{d-2}) \, 2^{d-2}$. Using inequalities 1, 2, and statement 3, we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d - 2 + \lambda_{d-2}) \, 2^{d-2} + 2 \cdot 2^d = (d + \lambda_{d-2}) \, 2^d$$

3. $\dim U_0 = d - 1$. Again, applying the induction hypothesis to $\pi(\mathcal{B})$ and $\mathcal{A}_0$, we have three cases (recall that from the assumption $0, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \varnothing$):

a) $\mathcal{A}_0$ is isomorphic to $\{0,1\}^{d-1}$, and $\pi(\mathcal{B})$ is the basis with the zero vector.

i) $\dim \mathcal{B}_1 = 1$. In this case, $|\mathcal{B}| = d+1$, which does not satisfy the condition in the theorem's statement.

ii) $\dim \mathcal{B}_1 = k \geq 2$. Then $|\mathcal{B}_1| = 2k$, $|\mathcal{A}_1| \leq 2^{d-k}$, and we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (2^{d-1} + 2^{d-k})(d+k) \leq d2^d + 2d$$

based on inequality 3.

b) $|\mathcal{A}_0| = d$. Then $|\mathcal{A}|^2 \leq 4d^2$, which is not larger than $d2^d + 2d$ for $d > 3$.

c) $|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d - 1 + \lambda_{d-1}) \, 2^{d-1}$. For the final time, from inequalities 1 and 2 and statement 5, we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2 \cdot (d - 1 + \lambda_{d-1}) \, 2^{d-1} + 2^d = (d + \lambda_{d-1}) \, 2^d.$$

□

Now, let us find the optimal value of $\lambda(d)$ from Lemma 5:

Theorem 6. *Suppose both $\mathcal{A}$ and $\mathcal{B} \subseteq \mathbb{R}^d$ contain a basis for $\mathbb{R}^d$, and $\langle a, b\rangle \in \{0,1\}$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$. If both families are maximal with respect to this property and have size of at least $d+2$, then $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$.*

Proof. We will again proceed with induction on d, and without loss of generality, assume that $|\mathcal{A}| \geq |\mathcal{B}|$. For $d < 3$, the estimate coincides with Theorem 1. The desired estimate has already been obtained in all subcases of the proof of Lemma 5, except for two induction steps, 2c) and 3c). Thus, it suffices to obtain the required estimate in these cases:

2c') $|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq 2\,(d-2)\left(2^{d-3}+1\right)$. Using inequalities 1, claim 3, and (3), we have

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d-2)\left(2^{d-2}+2\right) + 2 \cdot 2^d - 2\,|\mathcal{A}_0| = 2d(2^{d-1}+1) + 2\,(3d-8-|\mathcal{A}_0|)$$

This completes the proof when $|\mathcal{A}_0| \geq 3d-8$. Otherwise:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4\,|\mathcal{A}_0|^2 \leq 4\,(3d-8)^2$$

This is less than $d2^d + 2d$ for all d except $d = 5, 6$, for which the desired estimate can be obtained by computer enumeration.

3c') Both $|\mathcal{A}_0|$ and $|\pi(\mathcal{B})|$ are at least $d+1$. By the induction hypothesis,

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1)\left(2^{d-1}+2\right).$$

Then, from statements 2, 5, and the fact that $\mathcal{B}_0 = \varnothing$, we have

$$\begin{aligned}
|\mathcal{A}| \cdot |\mathcal{B}| &= 2\,|\mathcal{A}_0|\,|\pi(\mathcal{B})| + |\mathcal{A}_1|\,|\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| && (7)\\
&\leq 2\,(d-1)\left(2^{d-1}+2\right) + |\mathcal{A}_1|\,|\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| && (8)\\
&\leq 2\,(d-1)\left(2^{d-1}+2\right) + 2^d - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \\
&= d2^d + 2d - (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| + (2d-4) && (9)
\end{aligned}$$

Thus, it suffices to show, for example, that $(|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \geq 2d-4$.
Consider the case where $\dim A_1 = d-1$: then $\mathcal{B}_1 = \{0, b_d\}$, and using

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}|\,|\pi(\mathcal{B})| + |\mathcal{A}| \cdot \frac{1}{2}\,|\mathcal{B}_1| \leq d2^d + 2d - 2^d + |\mathcal{A}| + (2d-4)$$

we obtain the desired inequality when $|\mathcal{A}| \leq 2^d - 2d + 4$. Note that $|\mathcal{A}| > 2^d - 2d + 4$, is indeed impossible, as that would imply

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| > (2^{d-1} - d + 2) \cdot (d+1) \geq (d-1)(2^{d-1}+2)$$

which contradicts the induction hypothesis. We may thus now assume $\dim A_1 < d-1$. Note that due to this, we can also assume that $|\mathcal{A}_0| > |\mathcal{A}_1|$, since in the case that $|\mathcal{A}_0| = |\mathcal{A}_1|$ we can start by shifting the family $\mathcal{A}$ and changing the signs of some vectors in $\mathcal{B}$ so that all conditions remain in force, and $\mathcal{A}_0$ and $\mathcal{A}_1$ switch places, reducing the situation to the case where $\dim U_0 < d-1$.

Consider the orthogonal projection $\pi_{\mathcal{B}_1} : \mathbb{R}^d \to \operatorname{span}(\mathcal{B}_1)$. By the definition of $\mathcal{A}_1$, we have $|\pi_{\mathcal{B}_1}(\mathcal{A}_1)| = 1$. Let $k = \dim(\operatorname{span}(\mathcal{B}_1))$. Since $\mathcal{B}$ contains a basis of $\mathbb{R}^d$, we have:

$$|\mathcal{B}_*| \geq d-k,\ \ (|\mathcal{A}_0| - |\mathcal{A}_1|)\,|\mathcal{B}_*| \geq d-k \qquad (10)$$

Before proceeding to a slightly more systematic case analysis, let us understand that we can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$: first, $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k$ because $0 \in \mathcal{A}_0$, and $\operatorname{span}(\pi_{\mathcal{B}_1}(\mathcal{A}_0)) = \operatorname{span}(\pi_{\mathcal{B}_1}(\operatorname{span}(\mathcal{A}_0))) = \operatorname{span}(\mathcal{B}_1) \cap b_d^{\perp}$, which means $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ contains 0 and a basis of an $(k-1)$-dimensional space. Second, if $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k+1$, then applying Theorem 1 to $\mathcal{B}_1$ and $\pi_{\mathcal{B}_1}(\mathcal{A})$ yields:

$$|\mathcal{B}_1| \cdot |\pi_{\mathcal{B}_1}(\mathcal{A})| \leq (k+1)\, 2^k,\ |\pi_{\mathcal{B}_1}(\mathcal{A})| \geq k+2 \Rightarrow |\mathcal{B}_1| \leq 2^k \left(1 - \frac{1}{k+2}\right) \Rightarrow$$
$$|\mathcal{B}_1|\,|\mathcal{A}_1| \leq 2^d \left(1 - \frac{1}{k+2}\right) \Rightarrow |\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d-4) - \frac{2^d}{k+2} - (d-k)$$

This proves the required estimate for all $d \notin \{3,4,5\}$, because for $d \geq 6$

$$d + k - 4 - \frac{2^d}{k+2} \leq 2d - 4 - \frac{2^d}{d+2} = -\frac{2}{d+2}\left(2^{d-1} - (d+2)(d-2)\right) \leq 0$$

For $d \in \{3,4,5\}$, computer enumeration is carried out.

Now we can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$, meaning $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ consists of zero and a basis of $\operatorname{span}(\mathcal{B}_1) \cap b_d^{\perp}$, while $\pi_{\mathcal{B}_1}(\mathcal{A})$ consists of zero and a basis of $\operatorname{span}(\mathcal{B}_1)$. We will now deal with possible values of k:

i) $k=1$, which means $\mathcal{B}_1 = \{0,\ b_d\}$. Since $\dim \mathcal{A}_1 < d-1$, from the proof of claim 5, it follows that $|\mathcal{A}_1| \leq 2^{d-2}$. Substituting this into (8), we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4 - 2^{d-1}) \leq d2^d + 2d$$

ii) $k = 2$. From the proof of claim 5, it follows that $|\mathcal{B}_1| \leq 4$, and $|\mathcal{A}_1| \leq 2^{d-2}$. Due to (10), $|\mathcal{B}_*| \geq d-2$, so if $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2$, (9) yields the desired estimate. Similarly, (9) completes the proof if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| \geq 2d-4$. Finally, if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| < 2d-4$, then:

$$|\mathcal{A}| \cdot |\mathcal{B}| = (2\,|\mathcal{A}_1| + 1) \cdot (|\mathcal{B}_*| + |\mathcal{B}_1|) < \left(2^{d-1} + 1\right) \cdot (2d - 4 + 4) = d2^d + 2d$$

iii) $k = d$ and $\mathcal{B}_* = \varnothing$. Note that, due to (10), $\mathcal{B}_* = \varnothing$ is not possible for other values of k. The definition of $\mathcal{B}_1$ and our assumption that it has full dimension imply that $\mathcal{A}_1$ consists of only one point. Hence, (8) becomes:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2\,(d-1)\left(2^{d-1} + 2\right) + |\mathcal{B}|$$

which completes the proof when $|\mathcal{B}| \leq 2^d - 2d + 4$. The opposite is indeed impossible, as it would contradict Theorem 1:

$$|\mathcal{A}| \cdot |\mathcal{B}| \geq |\mathcal{B}|^2 \geq \left(2^d - 2d + 4\right)^2 > (d+1)\, 2^d$$

iv) $2 < k \leq d$ and $\mathcal{B}_* \neq \varnothing$. Let's denote the elements of $\pi_{\mathcal{B}_1}(\mathcal{A})$ as $a_0 = 0, a_1, \ldots, a_k$, and their preimages in $\mathcal{A}$ as $\mathbb{A}_j = \pi_{\mathcal{B}_1}^{-1}(a_j) \cap \mathcal{A}$. We'll choose the numbering such that $\mathbb{A}_1 = \mathcal{A}_1$. Let $b_{11}, b_{12}, \ldots, b_{1k}$ be a basis of $\mathcal{B}_1$ that is dual to $a_1, \ldots, a_k$. For example, according to our choice of numbering, $b_{11} = b_d$. Note that, due to $\mathcal{B}$ being inclusion-wise maximal, all b_{1j} must belong to $\mathcal{B}_1$ (otherwise, they, along with $b_{1j} + b_d$ for $j > 1$, could be added to $\mathcal{B}$). If $\dim \mathcal{A}_1 < d - k$, we can follow a similar argument as in part i) to obtain $|\mathcal{A}_1| \le 2^{d-2}$ and the desired estimate. Consequently, we can now assume that $\dim \mathcal{A}_1 = d - k$.

Our further plan is to write $\mathcal{A}$ in a particular basis to see that, due to $\dim \mathcal{A}_1 = d-k$, any of the b_{1j} could be initially chosen as b_d, and that a suitable choice would lead to the desired bound.

We will augment $\{b_{11}, \dots, b_{1k}\}$ with elements from $\mathcal{B}_*$ to form a basis for $\mathbb{R}^d$ and represent $\mathcal{A}$ in the dual basis. Then vectors of $\mathcal{A}$, arranged as column-vectors, form a matrix of the following form:

$$\begin{array}{ccc}
 & \begin{array}{ccccc} \quad\mathbb{A}_0\quad & \quad\mathbb{A}_1\quad & \quad\mathbb{A}_2\quad & \quad\cdots\quad & \quad\mathbb{A}_k\quad \end{array} & \\
\mathcal{A} = & \left(\begin{array}{c|c|c|c|c}
 & 1\ 1\cdots 1\ 1 & 0\ 0\cdots 0\ 0 & & \\
\mathbf{0} & & 1\ 1\cdots 1\ 1 & & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & & \\
 & & & \cdots & 1\ 1\cdots 1\ 1 \\
\hline
\ddots & \ddots & \ddots & & \ddots
\end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right] k \\ \left.\vphantom{\begin{array}{c}1\end{array}}\right] d-k \end{array} \\
 & \begin{array}{ccccc} \qquad\quad & \underbrace{\qquad\qquad}_{\dim = d-k} & \qquad\quad & \qquad & \qquad \end{array} &
\end{array}$$

The rank of the highlighted block coincides with the affine dimension of $\mathbb{A}_1 = \mathcal{A}_1$, which is $d - k$. Therefore,

$$\forall j > 1\colon\ d - 1 = \dim(\operatorname{span}(\mathcal{A} \setminus \mathbb{A}_j)) = \dim(\mathcal{A} \cap b_{1j}^{\perp})$$

Which means that, indeed, any of the b_{1j} could be set as b_d from the start. Choose b_{1j} with the smallest dimension of $\mathbb{A}_j$, and repeat all the same reasoning with it as b_d. Note that in this case, $|\mathcal{A} \setminus \mathbb{A}_j| > |\mathbb{A}_j|$, so there will be no need for translation of $\mathcal{A}$ that swaps $\mathcal{A}_0$ and $\mathcal{A}_1$ in claim 1, and we can thus safely assume

$$\begin{gathered} \forall j > 1\colon\ |\mathbb{A}_1| \le |\mathbb{A}_j| \implies \\ |\mathcal{A}_0| - |\mathcal{A}_1| = |\mathbb{A}_0| + \sum_{j>1} |\mathbb{A}_j| \ge (k-1)\,|\mathcal{A}_1| \ge 2\,|\mathcal{A}_1| \end{gathered} \tag{11}$$

If $|\mathcal{A}_0| - |\mathcal{A}_1| \ge 2d - 4$, non-emptiness of $\mathcal{B}_*$ and (9) imply the desired estimate. Otherwise

$$\begin{aligned} |\mathcal{A}_0| - |\mathcal{A}_1| < 2d - 4 \overset{(11)}{\implies} |\mathcal{A}| < 2 \cdot (2d-4) \implies \\ |\mathcal{A}| \cdot |\mathcal{B}| \le |\mathcal{A}|^2 < (4d-8)^2 < d2^d + 2d, \end{aligned}$$

concluding the proof.

$\square$

Two examples that demonstrate tightness of the bound in Theorem 6 are

Example 1. *Let* $\{e_i\}$ *be the standard basis of* $\mathbb{R}^d$*,*

$$\mathcal{A} = \left\{ \sum_{i=2}^{d} \delta_i e_i \right\} \cup \{e_1\}, \; \mathcal{B} = \{\delta_1 e_1 + e_j\} \cup \{e_1, 0\}, \; \textit{where } \delta_i \textit{ range over } \{0,1\} \textit{ and } j > 1.$$

Here $|\mathcal{A}| = 2^{d-1} + 1$ *and* $|\mathcal{B}| = 2d$*.*

Example 2. *Let* $\{e_i\}$ *be the standard basis of* $\mathbb{R}^d$*,*

$$\mathcal{A} = \left\{ e_d + \sum_{i=1}^{d-1} \varepsilon_i e_i \right\} \cup \{0\}, \; \mathcal{B} = \left\{ \frac{1}{2} (e_d + \varepsilon_i e_i) \right\}, \; \textit{where } \varepsilon_i \textit{ range over } \{-1,1\}.$$

Just like in example 1, $|\mathcal{A}| = 2^{d-1} + 1$ *and* $|\mathcal{B}| = 2d$*.*

3 Application to 2-level polytopes

Our main application of Theorem 6 is the following

Theorem 3. *For* $d > 1$ *let* P *be a* d*-dimensional* 2*-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then* $f_0(P) f_{d-1}(P) \leq (d-1)\, 2^{d+1} + 8\,(d-1)$*.*

Before following with the proof let us make a simple observation, proof of which is given in Appendix A for completeness:

Lemma 1. *Let* $\mathcal{S}$ *be a family of subsets of* $[d-1]$ *such that* $|\mathcal{S}| = d$ *and*

$$\forall S_1, S_2 \in \mathcal{S} : \; |S_2 \setminus S_1| \leq 1.$$

Then either $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ *or* $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$*.*

Proof of Theorem 3. The statement is trivial on the plane, so we assume $d > 2$. Let us denote $V = f_0(P)$ and $F = f_{d-1}(P)$ for conciseness. Shift P so that 0 is among it's vertices and let $\mathcal{A}$ denote the vertex set of P and $\mathcal{B}'$ denote the minimal set of vectors such that every facet of P lies in a hyperplane $\{x : \langle x, b \rangle = \delta\}$ for some $\delta \in \{0,1\}$ and $b \in \mathcal{B}'$. Let $\mathcal{B} = \mathcal{B}' \cup \{0\}$. If every vector in $\mathcal{B}'$ defines one facet of P, we are done by Theorem 1:

$$V \cdot F < |\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1) 2^d < (d-1) 2^{d+1} + 8(d-1).$$

Otherwise, let $b_d \in \mathcal{B}'$ define two facets of P and consider the setting of the proof of Theorem 6. Note that we may assume $|\mathcal{A}_0| \geq |\mathcal{A}_1|$ if appropriate translation of P was made. Since $\dim(\mathcal{A}_1) = d-1$, $\mathcal{B}_1 = \{0, b_d\}$ and $|\pi(\mathcal{B})| = |\mathcal{B}_*| + 1$, which means

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})| + |\mathcal{A}|. \tag{12}$$

Since every vector in $\mathcal{B}'$ defines at most two facets of P, $|\mathcal{B}| \geq \frac{F}{2} + 1$, thus from 12 we conclude

$$V \cdot F \leq 2\left(|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})|\right) \leq 4 \cdot |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \tag{13}$$

Consider two cases:

1. $|\mathcal{A}_0| \geq |\pi(\mathcal{B})|$. If $|\pi(B)| > d$, by Theorem 6 we have
$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1)2^{d-1} + 2(d-1)$$
and with (13) we are done. Otherwise $|\pi(B)| = d$ which, together with $\mathcal{B}_1 = \{0, b_d\}$, means that $\mathcal{B}'$ is a basis of $\mathbb{R}^d$. Every vector in $\mathcal{B}'$ then has to define two facets of P, since otherwise P is unbounded. Thus P is affinely isomorphic to a cube.

2. $|\mathcal{A}_0| < |\pi(\mathcal{B})|$. If $|\mathcal{A}_0| > d$ or $\pi(\mathcal{B})$ is not affinely isomorphic to $\{0,1\}^{d-1}$, Theorem 6 and (13) imply the desired bound just like in case 1. Otherwise, $|\mathcal{A}_0| = |\mathcal{A}_1| = d$, so let $a_0 = 0, a_1, \ldots, a_{d-1}$ be the elements of $\mathcal{A}_0$. Using the affine isomorphism between $\pi(\mathcal{B})$ and $\{0,1\}^{d-1}$ let us index elements of $\{0\} \cup \mathcal{B}_*$ by subsets of $[d-1]$, for example $b_\varnothing = 0$ and $\{b_{\{1\}}, b_{\{2\}}, \ldots, b_{\{d-1\}}, b_d\}$ form a basis of $\mathbb{R}^d$. Observe that the dual basis of $\mathbb{R}^d$ is $\{a_1, a_2, \ldots, a_{d-1}, v\}$ for v that satisfies
$$\langle v, b_d \rangle = 1 \text{ and } \forall j \in [d-1] :\ \langle v, b_{\{j\}} \rangle = 0.$$
This means that
$$\mathcal{A}_1 = \{v + \sum_{j \in S} a_j :\ S \in \mathcal{S}\} \tag{14}$$
for some family $\mathcal{S}$ of subsets of $[d-1]$ with $|\mathcal{S}| = d$. Our goal is to show that $\mathcal{S} = \{S \subseteq [d-1] :\ |S| \geq d-2\}$, as then P is affinely isomorphic to the cross-polytope, and we would be done. For $T, S \subseteq [d-1]$ denote $\sigma_T = \sum_{j \in T} a_j$ and note that
$$\langle \sigma_T, b_S \rangle = \langle \sigma_T, \pi(b_S) \rangle = \Big\langle \sigma_T, \sum_{j \in S} \pi(b_{\{j\}}) \Big\rangle = \Big\langle \sum_{j \in T} a_j, \sum_{j \in S} b_{\{j\}} \Big\rangle = |T \cap S|.$$
Now (14) means that for $S_1, S_2 \in \mathcal{S}$
$$\begin{aligned} \{-1,0,1\} \ni \langle v + \sum_{j \in S_2} a_j\ ,\ b_{S_2 \setminus S_1} \rangle - \langle v + \sum_{j \in S_1} a_j\ ,\ b_{S_2 \setminus S_1} \rangle = \langle \sigma_{S_2} - \sigma_{S_1}\ ,\ b_{S_2 \setminus S_1} \rangle \\ = |S_2 \cap (S_2 \setminus S_1)| - |S_1 \cap (S_1 \setminus S_2)| = |S_2 \setminus S_1| \end{aligned}$$
Which by Lemma 1 implies that either $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$. The latter is, however, impossible, since then P only has $d+1$ facets and $|\pi(\mathcal{B})| = d < 2^{d-1}$. (14) now shows that P is affinely isomorphic to the cross-polytope, and we are done.

□

Two examples demonstrate tightness of the bound in Theorem 3:

Example 3 (Cross-polytope × segment). *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*
$$P = \mathrm{Conv}\big(\{\varepsilon_i e_i + \varepsilon_d e_d\}_{i \leq d-1}\big), \text{ where } \varepsilon_i \text{ range over } \{-1,1\} \text{ for } i \in [d].$$
Here $f_0(P) = 4(d-1)$ and $f_{d-1}(P) = 2 + 2^{d-1}$.

Example 4 (Suspension of a cube). *Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$,*
$$P = \mathrm{Conv}\left(\left\{\sum_{i=1}^{d-1} \varepsilon_i e_i\right\} \cup \{e_d, -e_d\}\right), \text{ where } \varepsilon_i \text{ range over } \{-1,1\}.$$
This is (up to coordinate scaling) the dual of the polytope in the previous example, in particular $f_0(P) = 2 + 2^{d-1}$ and $f_{d-1}(P) = 4(d-1)$.

A Appendix

Inequality 3. *For an integer $2 \le f \le d$, we have:*

$$(d+f)(2^{d-1}+2^{d-f}) \le d2^d + 2d.$$

Proof. We will prove this by induction on d: when $d = k$, the equality is satisfied. Let's perform the induction step from d to $d+1$. Denoting the left and right sides of the inequality as $l(d,f)$ and $r(d,f)$, respectively, we have

$$\begin{aligned} r(d+1,f) - l(d+1,f) &\ge (r(d+1,f) - r(d,f)) - (l(d+1,f) - l(d,f)) \\ &= \left(d2^d + 2^{d+1} + 2\right) - (d+f+2)\left(2^{d-1} + 2^{d-f}\right) \\ &= 2^{d-f}(d-f+2)\left(2^{f-1} - 1 - \frac{2f}{d-f+2}\right) + 2 \\ &\ge 2^{d-f}(d-f+2)\left(2^{f-1} - 1 - f\right) \end{aligned}$$

The obtained expression is non-negative for $f > 2$. For $f = 2$ and $d \ge 4$, we have $2^{f-1} - 1 - \frac{2f}{d-f+2} \ge 0$, and for $f = 2$ and $d = 2, 3$, the initial inequality is checked explicitly. □

References

[1] M. Aprile, A. Cevallos, and Y. Faenza. On 2-level polytopes arising in combinatorial settings. *SIAM Journal on Discrete Mathematics*, 32(3):1857–1886, 2018.

[2] A. Bohn, Y. Faenza, S. Fiorini, V. Fisikopoulos, M. Macchia, and K. Pashkovich. Enumeration of 2-level polytopes. *Mathematical Programming Computation*, 11, 2018.

[3] S. Fiorini, V. Fisikopoulos, and M. Macchia. Two-level polytopes with a prescribed facet. In R. Cerulli, S. Fujishige, and A. R. Mahjoub, editors, *Combinatorial Optimization*, pages 285–296, Cham, 2016. Springer International Publishing.

[4] A. Kupavskii and F. Noskov. Octopuses in the boolean cube: Families with pairwise small intersections, part i. *Journal of Combinatorial Theory, Series B*, 2023.

[5] A. Kupavskii and S. Weltge. Binary scalar products. *Journal of Combinatorial Theory, Series B*, 156, 2022.