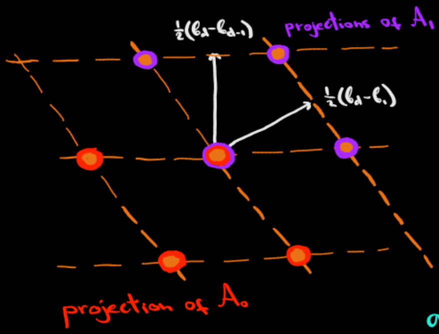
A useful projection of "cross-polytope" families

this should actually be named b_2, since e_0 doesn't exist, but I assume that is what was meant

$A = (\pm 1, \cdots, \pm 1, 1) \cup \{0\}$, $b_d = \frac{1}{2}(0, \cdots, -1, 1)$, $b_{d-1} = \frac{1}{2}(0, \cdots, -1, 0, 1)$, $b_1 = \frac{1}{2}(-1, 0, \cdots, 1)$

$\frac{1}{2}(b_d - b_{d-1}) = \frac{1}{4}(0, \cdots, 1, -1, 0)$; $\frac{1}{2}(b_d - b_1) = \frac{1}{4}(1, 0, \cdots, -1, 0)$. $A_0 = \{\pm 1, \cdots, 1, 1\}$

Possible values of $\langle a \overset{\in A}{,} \frac{1}{2}(b_d - b_{d-1}) \rangle$ — $\frac{1}{4} - \frac{1}{4} = 0$, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, $-\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$, and the same the same holds for dot products of A with $\frac{1}{2}(b_d - b_1)$. So the picture in the plane $\langle \frac{1}{2}(b_d - b_{d-1}), \frac{1}{2}(b_d - b_1) \rangle$ is something like this:



For each of the seven orange dots there is a vector in A that projects onto it. Projections of A_0 are circled red and projections of A_1 are circled purple.

There is an apparent bijection between projections of A_0 and A_1 (by a shift of $\sqrt{\frac{5}{3}} \cdot \frac{1}{2}(b_d - \frac{1}{2}b_{d-1} - \frac{1}{2}b_1)$, which is the vector $(1,1)$ in the coordinate system of the orange grid), but projecting along this vector doesn't put A_1 inside of A_0, which is easy to see as $b_d - \frac{1}{2}b_{d-1} - \frac{1}{2}b_1 = \frac{1}{4}(1, 0, \cdots, 0, 1, -1, 0)$.

I'm still confused by the vectors $b_d \pm b_{d-1} \pm b_1$, I don't see how they relate to the projections of A.

But I understand that there is a projection that put A_1 inside of A_0 — it is percisly the orthogonal projection along e_{d-1} ($A_0 = \{0\} \cup \{A_1 - 2e_d\}$).