

# Stability of size bounds for families with binary scalar products

Andrey Kupavskii<sup>1</sup> and Dmitry Tsarev<sup>2</sup>

<sup>1</sup>*Moscow Institute of Physics and Technology, Moscow, Russia*

<sup>1</sup>*G-SCOP, CNRS, Grenoble, France*

<sup>2</sup>*University of Cambridge, Cambridge, United Kingdom*

## Abstract

Questions on possible number of vertices and facets in 2-level polytopes motivate the study of vector families  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  with a property that  $\forall a \in \mathcal{A}, b \in \mathcal{B}$  the scalar product  $\langle a, b \rangle \in \{0, 1\}$ . In this work we show the stability of Kupavskii's and Weltge's bound on  $|\mathcal{A}| \cdot |\mathcal{B}|$  for such  $\mathcal{A}$  and  $\mathcal{B}$ . We use this result to find the maximal possible product of the number of vertices and the number of facets of a 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope.

## 1 Introduction

A polytope  $P$  is said to be *2-level* if for every facet-defining hyperplane  $H$  there is a parallel hyperplane  $H'$  such that  $H \cup H'$  contains all vertices of  $P$ . Basic examples of 2-level polytopes are simplices, hypercubes and cross-polytopes, but they also generalize a variety of interesting polytopes such as Birkhoff, Hanner, and Hansen polytopes, order polytopes and chain polytopes of posets, stable matching polytopes, and stable set polytopes of perfect graphs [1]. Combinatorial structure of two-level polytopes has also been studied in [3], and enumeration of such polytopes in [2] led to a conjecture that the product of the number of facets and the number of vertices in a  $d$ -dimensional 2-level polytope is bounded by  $d2^{d+1}$ . It has been proven in [5] via a stronger theorem regarding so-called families of vectors with binary scalar products:

**Theorem 1.** *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  both linearly span  $\mathbb{R}^d$  such that  $\langle a, b \rangle \in \{0, 1\}$  holds for all  $a \in \mathcal{A}, b \in \mathcal{B}$ . Then we have  $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d + 1)2^d$ .*

In this paper, we prove several results regarding the stability of the bound in Theorem 1. . Our main result is the following

**Theorem 2.** *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  both linearly span  $\mathbb{R}^d$  such that  $\langle a, b \rangle \in \{0, 1\}$  holds for all  $a \in \mathcal{A}, b \in \mathcal{B}$ . Furthermore, suppose  $\mathcal{A}$  and  $\mathcal{B}$  both have the size of at least  $d + 2$ . Then  $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$ .*

We then use this to obtain the bound on the product of the vertex count  $f_0(P)$  and the facet count  $f_{d-1}(P)$  in a 2-level polytope  $P$  that is distinct from both the cube and the cross-polytope:

**Theorem 3.** *For  $d > 1$  let  $P$  be a  $d$ -dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then  $f_0(P)f_{d-1}(P) \leq (d-1)2^{d+1} + 8(d-1)$ .*

**Outline** The next section lays out the proof of main results. In Section 3 we provide the proof of Theorem 3 and Section 4 contains proofs of claims from [5] that we use. Short but technical proofs of some statements used in the main sections are provided in Appendix A, as well as a conjecture that generalises our main result.

## 2 Stability results

Let  $\mathcal{A}, \mathcal{B}$  both linearly span  $\mathbb{R}^d$  and have binary scalar products, that is,  $\langle a, b \rangle \in \{0, 1\}$  for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . We will use the following two simple observations a few times throughout our proofs. If  $a_1, \dots, a_d$  is a basis of  $\mathbb{R}^d$  contained in  $\mathcal{A}$ , take the dual basis  $a_1^*, \dots, a_d^*$ :

$$\langle a_i, a_j^* \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and observe that elements of  $\mathcal{B}$  have 0/1 coordinates when expressed in this dual basis, which means  $\mathcal{B}$  is a subset of what we would call a cube:

$$\mathcal{B} \subseteq \left\{ \sum_{i=1}^d \delta_i a_i^*, \text{ where } \delta_i \text{ range over } \{0, 1\} \right\}.$$

Another observation is that projecting one family on a linear span of a subset of another preserves the binary scalar products property: if  $\mathcal{A}' \subseteq \mathcal{A}$  and  $\pi_{\mathcal{A}'} : \mathbb{R}^d \rightarrow \text{span}(\mathcal{A}')$  is the orthogonal projection, then

$$\forall a \in \mathcal{A}', b \in \mathcal{B} : \langle a, \pi_{\mathcal{A}'}(b) \rangle = \langle a, b \rangle \in \{0, 1\}.$$

We will now introduce some notation and restate some claims proved in [5]. Proofs of those claims and inequalities are provided in Section 4 for completeness.

Since we are interested in bounding the product  $|\mathcal{A}||\mathcal{B}|$  from above, we will assume that  $\mathcal{A}$  and  $\mathcal{B}$  are inclusion-wise maximal with respect to the property of having binary scalar products. Let  $b_d \in \mathcal{B} \setminus \{0\}$  be a vector with the maximum value of  $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$ , where

$$\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\} \text{ for } i = 0, 1.$$

The choice of  $b_d$  among the vectors that maximise  $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$ , if important, will be specified at a later stage. We denote the orthogonal projection onto  $U = b_d^\perp$  as  $\pi : \mathbb{R}^d \rightarrow U$ .

**Claim 1.** *We may translate  $\mathcal{A}$  and replace some points in  $\mathcal{B}$  by their negatives such that the following holds.*

(i) *We can still write  $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ , where  $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$  for  $i = 0, 1$  such that*

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}$$

(ii) *We still have*

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}$$

(iii) The set  $\pi(\mathcal{B})$  does not contain opposite points.

**Claim 2.** Every point in  $\pi(\mathcal{B})$  has at most two preimages in  $\mathcal{B}$ .

**Inequality 1.**  $|\mathcal{A}| |\mathcal{B}| \leq 2 |\mathcal{A}_0| |\pi(\mathcal{B})| + |\mathcal{A}_1| |\mathcal{B} \setminus \mathcal{B}_*|$

We denote the linear span of  $\mathcal{A}_0$  as  $U_0$  and introduce the orthogonal projection  $\tau : U \rightarrow U_0$ . Let  $\mathcal{B}_* \subseteq \mathcal{B}$  be the set of  $b \in \mathcal{B}$  for which  $\pi(b)$  has exactly one pre-image under projection onto  $U$ .

**Claim 3.**  $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0} |\tau(\pi(\mathcal{B}))|$ .

**Claim 4.**  $\mathcal{B} \setminus \mathcal{B}_* = \mathcal{B}_0 \sqcup \mathcal{B}_1$  holds with

$$\forall b \in \mathcal{B}_i : |\{ \langle a, b \rangle : a \in \mathcal{A}_i \}| = 1 \text{ for } i = 0, 1$$

**Inequality 2.**  $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1) 2^d + |\mathcal{A}_0| |\mathcal{B}_0| + |\mathcal{A}_1| |\mathcal{B}_1|$

**Inequality 3.** For  $i = 0, 1$  we have

$$|\mathcal{A}_i| \leq 2^{\dim(\mathcal{A}_i)}, |\mathcal{B}_i| \leq 2^{\dim(\text{span } \mathcal{B}_i)}, \text{ and } \dim(\mathcal{A}_i) + \dim(\text{span } (\mathcal{B}_i)) \leq d$$

**Claim 5.** For  $i = 0, 1$ , we have  $|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^d$ .

Note that we can assume both  $\mathbf{0}, b_d \in \mathcal{B}_0$  or  $\mathbf{0}, b_d \in \mathcal{B}_1$ . Therefore, claim 5 actually implies

$$|\mathcal{A}_1| |\mathcal{B}_1| \leq 2^d, |\mathcal{A}_0| (|\mathcal{B}_0| + 2) \leq 2^d, \quad (3)$$

assuming here and further that  $\mathbf{0}, b_d \in \mathcal{B}_1$ .

We will now understand under which conditions equality is achieved in Theorem 1.

**Theorem 4.** Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  both linearly span  $\mathbb{R}^d$  such that  $\langle a, b \rangle \in \{0, 1\}$  holds for all  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ . Then we only have  $|\mathcal{A}| \cdot |\mathcal{B}| = (d+1)2^d$  if one of the families has size  $d+1$  and the other is affinely isomorphic to  $\{0, 1\}^d$ .

*Proof.* Without loss of generality, we assume  $|\mathcal{A}| \geq |\mathcal{B}|$ . We will use induction on  $d$ , the statement is obvious in dimension 1. Assuming the statement holds for smaller dimensions, we prove it in dimension  $d$ . Consider the possible values of  $\dim U_0$ :

1.  $\dim U_0 \leq d-2$ . From inequality 2 and (3), we get:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (d-1) 2^d + 2 \cdot 2^d - 2 |\mathcal{A}_0| \leq (d+1) 2^d - |\mathcal{A}| < (d+1) 2^d$$

2.  $\dim U_0 = d-1$ . In this case, assuming  $\mathbf{0}, b_d \in \mathcal{B}_1$ , we have  $\mathcal{B}_0 = \emptyset$ . We consider two subcases:

- a)  $\mathcal{B}_* \neq \emptyset$ . Equality in Theorem 1 can only be achieved when inequalities 1 and 2 are tight, which is only the case when  $|\mathcal{A}_0| = |\mathcal{A}_1|$  and  $|\mathcal{A}_0| |\pi(\mathcal{B})| = d2^{d-1}$ . By the induction hypothesis, the latter is possible in one of two cases:

- i)  $\mathcal{A}_0$  is affinely isomorphic to  $\{0, 1\}^{d-1}$ . Then,  $|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = 2^d$ , which is only possible if  $\mathcal{A}$  is affinely isomorphic to  $\{0, 1\}^d$ , and then  $\mathcal{B}$  can only consist of a basis and the zero vector.
- ii)  $|\mathcal{A}_0| = d$ . Then, since  $|\mathcal{B}| \leq |\mathcal{A}| = 2d$ ,  $|\mathcal{B}| \cdot |\mathcal{A}| \leq 4d^2$ , which is less than  $(d+1)2^d$  for  $d \geq 4$ . For  $d = 3$ , the inequality  $|\mathcal{B}| \cdot |\mathcal{A}| \leq 32$  cannot yield equality since  $|\mathcal{A}| = 6$ . Finally, for  $d = 2$ , we have  $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$ .
- b)  $\mathcal{B}_* = \emptyset$ . Then,  $\mathcal{B}_1 = \mathcal{B}$  and, consequently,  $\dim(\text{span}(\mathcal{B}_1)) = d$ . In this case:

$$(\forall b \in \mathcal{B}_1 \exists \xi : \forall a \in \mathcal{A}_1 \langle a, b \rangle = \xi) \Rightarrow \dim(\mathcal{A}_1) \leq d - \dim(\text{span}(\mathcal{B}_1)) = 0 \Rightarrow |\mathcal{A}_1| = 1$$

Similarly to case a), inequality 1 is only tight in one of the following cases:

- i)  $|\mathcal{A}_0| = d$ . Then,  $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1)^2 < (d+1)2^d$ .
- ii)  $|\mathcal{A}_0| = 2^{d-1}$ ,  $|\pi(\mathcal{B})| = d$ . Then,  $|\mathcal{A}| \cdot |\mathcal{B}| = 2d(2^{d-1} + 1)$ , which is less than  $(d+1)2^d$  for  $d > 2$ . For  $d = 2$ , we have  $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$ .

□

We will improve the bound on  $|\mathcal{A}| \cdot |\mathcal{B}|$  for families that differ from the extremal example. To do this, we will use an auxiliary

**Inequality 4.** *For an integer  $2 \leq f \leq d$ , we have:*

$$(d+f)(2^{d-1} + 2^{d-f}) \leq d2^d + 2d.$$

A short but technical proof of this inequality can be found in Appendix A.

**Theorem 2.** *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  both linearly span  $\mathbb{R}^d$  such that  $\langle a, b \rangle \in \{0, 1\}$  holds for all  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ . Furthermore, suppose  $\mathcal{A}$  and  $\mathcal{B}$  both have the size of at least  $d+2$ . Then  $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$ .*

*Proof.* As in the proof of Theorem 4, we will use induction on  $d$ , and without loss of generality assume that  $|\mathcal{A}| \geq |\mathcal{B}|$ . Note that we can also assume that  $\mathcal{A}$  and  $\mathcal{B}$  are inclusion-wise maximal with respect to the property of having binary scalar products. For  $d < 3$ , the estimate coincides with Theorem 1. Assuming validity for smaller dimensions, let us prove the statement for dimension  $d$ . We consider possible values of  $\dim U_0$ :

1.  $\dim U_0 < d - 2$ . Then, from inequality 2 and claim 5, we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1)2^d + 2^d + 2^d \leq d2^d \tag{4}$$

2.  $\dim U_0 = d - 2$ . Applying the induction hypothesis to the families  $\tau(\pi(\mathcal{B}))$  and  $\mathcal{A}_0$ , we have three cases:

- a)  $|\tau(\pi(\mathcal{B}))| = d - 1$ . By maximality  $\mathcal{B}$  contained  $\mathbf{0}$ , so  $\tau(\pi(\mathcal{B}))$  consists of zero and the basis of  $U_0$ . Maximality of  $\mathcal{A}$  now means that  $\mathcal{A}_0$  is affine isomorphic to  $\{0, 1\}^{d-2}$ . From (3), it follows that  $|\mathcal{B}_0| \leq 2$ . Since  $|\mathcal{B}_0|$  is even, we have two scenarios:

i)  $|\mathcal{B}_0| = 0$ . Then, from inequality 1 and claim 5, we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4(d-1)2^{d-2} + 2^d = d2^d$$

ii)  $|\mathcal{B}_0| = 2$ . Let  $k+1$  vectors have two preimages under the action of  $\tau$  ( $k \geq 0$ , as  $\mathcal{B}_0 \subset U_0^\perp$  is not empty). Among these  $k+1$ , let  $t_2$  be the number of those vectors with both preimages in  $\pi(\mathcal{B}_1)$ , and let  $t_1$  be the number of those with exactly one preimage in  $\pi(\mathcal{B}_1)$ . The remaining  $k - t_1 - t_2$  have both preimages in  $\pi(\mathcal{B}_*)$ . Suppose that the vectors in  $\tau(\pi(\mathcal{B}))$  with a single preimage under  $\tau$  consist of  $q$  projections from  $\pi(\mathcal{B}_1)$  and  $d-2-k-q$  projections from  $\pi(\mathcal{B}_*)$ . We then have:

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_*| + |\mathcal{B}_0| + |\mathcal{B}_1| \\ &= (k - t_1 - 2t_2 + d - 2 - q) + 2 + (2 + 4t_2 + 2t_1 + 2q) \\ &= d + k + q + t_1 + 2t_2 + 2 \end{aligned}$$

First, consider the case when  $t_2 > 0$ . Then,  $U_0^\perp \subset \text{span}(\mathcal{B}_1)$ , which implies:

$$\begin{aligned} \dim(\text{span}(\mathcal{B}_1)) &= t_1 + t_2 + q + 2 \implies |\mathcal{A}_1| \leq 2^{d-t_1-t_2-q-2}, \\ |\mathcal{A}| &= |\mathcal{A}_0| + |\mathcal{A}_1| \leq 2^{d-2} + 2^{d-2-t_1-t_2-q} \end{aligned}$$

$$\begin{aligned} |\mathcal{A}| \cdot |\mathcal{B}| &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right) (d + k + q + t_1 + 2t_2 + 2) \\ &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right) (2d + t_1 + 2t_2) \end{aligned} \quad (5)$$

$$\leq \left(2^{d-1} + 2^{d-1-t_1-t_2-q}\right) (d + t_1 + t_2) \quad (6)$$

$$\begin{aligned} &\leq \left(2^{d-1} + 2^{d-1-t_1-t_2}\right) (d + t_1 + t_2 + 1) \\ &\leq d2^d + 2d \end{aligned} \quad (7)$$

Here, the second inequality follows from  $k+q \leq d-2$ , and the last one follows from inequality 4. If  $t_2 = 0$ , we get a slightly weaker bound:

$$\dim(\text{span}(\mathcal{B}_1)) \geq t_1 + t_2 + q + 1$$

This means (6) becomes  $(2^{d-1} + 2^{d-t_1})(d + t_1)$ , which is still less than (7) when  $t_1 \geq 2$  according to inequality 4. Finally, when  $t_2 = 0$  and  $t_1 = 0, 1$ , expression (5) yields estimates of  $d2^d$  and  $(2^{d-2} + 2^{d-3})(2d + 1) = d2^d - (d - \frac{3}{2})2^{d-2} \leq d2^d$ , respectively.

b)  $|\mathcal{A}_0| = d-1$ . Then:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4(d-1)^2 \leq d2^d + 2d$$

c) Both  $|\mathcal{A}_0|$  and  $|\tau(\pi(\mathcal{B}))|$  are at least  $d$ . By induction this implies

$$|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq 2(d-2) \left(2^{d-3} + 1\right).$$

Using inequalities 1, claim 3, and (4), we have

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d-2) \left( 2^{d-2} + 2 \right) + 2 \cdot 2^d - 2 |\mathcal{A}_0| = 2d(2^{d-1} + 1) + 2(3d - 8 - |\mathcal{A}_0|)$$

This completes the proof when  $|\mathcal{A}_0| \geq 3d - 8$ . Otherwise:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4 |\mathcal{A}_0|^2 \leq 4(3d - 9)^2,$$

which is less than  $d2^d + 2d$  for  $d \geq 3$ .

3.  $\dim U_0 = d - 1$ . Again, applying the induction hypothesis to  $\pi(\mathcal{B})$  and  $\mathcal{A}_0$ , we have three cases (recall that from the assumption  $0, b_d \in \mathcal{B}_1$ , we have  $\mathcal{B}_0 = \emptyset$ ):

- a)  $|\pi(\mathcal{B})| = d$ , which just like in 2a) means that  $\mathcal{A}_0$  is isomorphic to  $\{0, 1\}^{d-1}$ .
  - i)  $\dim \mathcal{B}_1 = 1$ . In this case,  $|\mathcal{B}| = d + 1$ , which does not satisfy the condition in the theorem's statement.
  - ii)  $\dim \mathcal{B}_1 = k \geq 2$ . Then  $|\mathcal{B}_1| = 2k$ ,  $|\mathcal{A}_1| \leq 2^{d-k}$ , and we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (2^{d-1} + 2^{d-k})(d + k) \leq d2^d + 2d$$

based on inequality 4.

- b)  $|\mathcal{A}_0| = d$ . Then  $|\mathcal{A}|^2 \leq 4d^2$ , which is not larger than  $d2^d + 2d$  for  $d > 3$ .
- c) Both  $|\mathcal{A}_0|$  and  $|\pi(\mathcal{B})|$  are at least  $d + 1$ .

The remainder of the proof will deal with the case 3c). By the induction hypothesis,

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1) \left( 2^{d-1} + 2 \right).$$

Then, from claims 2, 5, and the fact that  $\mathcal{B}_0 = \emptyset$ , we have

$$|\mathcal{A}| \cdot |\mathcal{B}| = 2 |\mathcal{A}_0| |\pi(\mathcal{B})| + |\mathcal{A}_1| |\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \tag{8}$$

$$\leq 2(d-1) \left( 2^{d-1} + 2 \right) + |\mathcal{A}_1| |\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \tag{9}$$

$$\begin{aligned} &\leq 2(d-1) \left( 2^{d-1} + 2 \right) + 2^d - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \\ &= d2^d + 2d - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| + (2d - 4) \end{aligned} \tag{10}$$

Thus, it suffices to show, for example, that  $(|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \geq 2d - 4$ .

Consider the case where  $\dim \mathcal{A}_1 = d - 1$ : then  $\mathcal{B}_1 = \{0, b_d\}$ , and using

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}| |\pi(\mathcal{B})| + |\mathcal{A}| \cdot \frac{1}{2} |\mathcal{B}_1| \leq d2^d + 2d - 2^d + |\mathcal{A}| + (2d - 4)$$

we obtain the desired inequality when  $|\mathcal{A}| \leq 2^d - 2d + 4$ . Note that  $|\mathcal{A}| > 2^d - 2d + 4$  is indeed impossible, as that would imply

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| > (2^{d-1} - d + 2) \cdot (d + 1) \geq (d - 1)(2^{d-1} + 2)$$

which contradicts the induction hypothesis. We may thus now assume  $\dim \mathcal{A}_1 < d - 1$ . Observe that due to this, we can also assume that  $|\mathcal{A}_0| > |\mathcal{A}_1|$ , since in the case that  $|\mathcal{A}_0| = |\mathcal{A}_1|$  we

can start by shifting the family  $\mathcal{A}$  and changing the signs of some vectors in  $\mathcal{B}$  so that all conditions remain in force, and  $\mathcal{A}_0$  and  $\mathcal{A}_1$  switch places, reducing the situation to the case where  $\dim U_0 < d - 1$ .

Consider the orthogonal projection  $\pi_{\mathcal{B}_1} : \mathbb{R}^d \rightarrow \text{span}(\mathcal{B}_1)$ . By the definition of  $\mathcal{A}_1$ , we have  $|\pi_{\mathcal{B}_1}(\mathcal{A}_1)| = 1$ . Let  $k = \dim(\text{span}(\mathcal{B}_1))$ . Since  $\mathcal{B}$  contains a basis of  $\mathbb{R}^d$ , we have:

$$|\mathcal{B}_*| \geq d - k, \quad (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \geq d - k \quad (11)$$

We will now deal with possible values of  $k$ :

- i)  $k = 1$ , which means  $\mathcal{B}_1 = \{0, b_d\}$ . Since  $\dim \mathcal{A}_1 < d - 1$ , from Inequality 3, it follows that  $|\mathcal{A}_1| \leq 2^{d-2}$ . Substituting this into (9), we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4 - 2^{d-1}) \leq d2^d + 2d$$

- ii)  $k = 2$ . From Inequality 3, it follows that  $|\mathcal{B}_1| \leq 4$ , and  $|\mathcal{A}_1| \leq 2^{d-2}$ . Due to (11),  $|\mathcal{B}_*| \geq d - 2$ , so if  $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2$ , (10) yields the desired estimate. Similarly, (10) completes the proof if  $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$  and  $|\mathcal{B}_*| \geq 2d - 4$ . Finally, if  $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$  and  $|\mathcal{B}_*| < 2d - 4$ , then:

$$|\mathcal{A}| \cdot |\mathcal{B}| = (2|\mathcal{A}_1| + 1) \cdot (|\mathcal{B}_*| + |\mathcal{B}_1|) < (2^{d-1} + 1) \cdot (2d - 4 + 4) = d2^d + 2d$$

- iii)  $k = d$ . The definition of  $\mathcal{B}_1$  and our assumption that it has full dimension imply that  $\mathcal{A}_1$  consists of only one point. Hence, (9) becomes:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2(d - 1) \left( 2^{d-1} + 2 \right) + |\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \leq 2(d - 1) \left( 2^{d-1} + 2 \right) + |\mathcal{B}|$$

which completes the proof when  $|\mathcal{B}| \leq 2^d - 2d + 4$ . The opposite is indeed impossible, as it would contradict Theorem 1:

$$|\mathcal{A}| \cdot |\mathcal{B}| \geq |\mathcal{B}|^2 \geq \left( 2^d - 2d + 4 \right)^2 > (d + 1) 2^d$$

Before proceeding with the last case in the proof, let us understand that when  $k < d$ , we can assume  $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$ . Clearly  $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k$  because  $0 \in \mathcal{A}_0$ , and  $\text{span}(\pi_{\mathcal{B}_1}(\mathcal{A}_0)) = \text{span}(\pi_{\mathcal{B}_1}(\text{span}(\mathcal{A}_0))) = \text{span}(\mathcal{B}_1) \cap b_d^\perp$ , which means  $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$  contains 0 and a basis of an  $(k - 1)$ -dimensional space. Since by replacing some vectors in  $\mathcal{B}_1$  with their opposites (without affecting  $|\mathcal{B}_1|$ ) we ensure it has binary scalar products with  $\mathcal{A}$ , by Theorem 1 we have, if  $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k + 1$ ,

$$\begin{aligned} |\mathcal{B}_1| \cdot |\pi_{\mathcal{B}_1}(\mathcal{A})| &\leq (k + 1) 2^k, \quad |\pi_{\mathcal{B}_1}(\mathcal{A})| \geq k + 2 \Rightarrow |\mathcal{B}_1| \leq 2^k \left( 1 - \frac{1}{k + 2} \right) \Rightarrow \\ |\mathcal{B}_1| |\mathcal{A}_1| &\leq 2^d \left( 1 - \frac{1}{k + 2} \right) \Rightarrow |\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4) - \frac{2^d}{k + 2} - (d - k) \end{aligned}$$

This proves the required estimate, because for  $d \geq 3$  and  $k < d$

$$d + k - 4 - \frac{2^d}{k + 2} \leq 2d - 5 - \frac{2^d}{d + 1} = -\frac{1}{d + 1} \left( 2^d - (2d - 5)(d + 1) \right) \leq 0$$

We now can assume  $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$ , meaning  $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$  consists of zero and a basis of  $\text{span}(\mathcal{B}_1) \cap b_d^\perp$ , while  $\pi_{\mathcal{B}_1}(\mathcal{A})$  consists of zero and a basis of  $\text{span}(\mathcal{B}_1)$ . With those assumptions in place, we proceed to the final subcase:

iv)  $2 < k < d$ . Note that, due to (11),  $\mathcal{B}_* \neq \emptyset$ . Let's denote the elements of  $\pi_{\mathcal{B}_1}(\mathcal{A})$  as  $a_0 = 0, a_1, \dots, a_k$ , and their preimages in  $\mathcal{A}$  as  $\mathbb{A}_j = \pi_{\mathcal{B}_1}^{-1}(a_j) \cap \mathcal{A}$ . We'll choose the numbering such that  $\mathbb{A}_1 = \mathcal{A}_1$ . Let  $b_{11}, b_{12}, \dots, b_{1k}$  be a basis of  $\mathcal{B}_1$  that is dual to  $a_1, \dots, a_k$ . For example, according to our choice of numbering,  $b_{11} = b_d$ . Note that, due to  $\mathcal{B}$  being inclusion-wise maximal, all  $b_{1j}$  must belong to  $\mathcal{B}_1$  (otherwise, they, along with  $b_{1j} + b_d$  for  $j > 1$ , could be added to  $\mathcal{B}$ ). If  $\dim \mathcal{A}_1 < d - k$ , we can follow a similar argument as in part i) to obtain  $|\mathcal{A}_1| \leq 2^{d-2}$  and the desired estimate. Consequently, we can now assume that  $\dim \mathcal{A}_1 = d - k$ .

Our further plan is to write  $\mathcal{A}$  in a particular basis to see that, due to  $\dim \mathcal{A}_1 = d - k$ , any of the  $b_{1j}$  could be initially chosen as  $b_d$ , and that a suitable choice would lead to the desired bound.

We will augment  $\{b_{11}, \dots, b_{1k}\}$  with elements from  $\mathcal{B}_*$  to form a basis for  $\mathbb{R}^d$  and represent  $\mathcal{A}$  in the dual basis. Then vectors of  $\mathcal{A}$ , arranged as column-vectors, form a matrix of the following form:

$$\mathcal{A} = \left( \begin{array}{c|c|c|c|c|c} \mathbb{A}_0 & \mathbb{A}_1 & \mathbb{A}_2 & \dots & \mathbb{A}_k & \\ \hline \mathbf{0} & \begin{array}{c} 1 \ 1 \ \dots \ 1 \ 1 \\ \mathbf{0} \end{array} & \begin{array}{c} 0 \ 0 \ \dots \ 0 \ 0 \\ 1 \ 1 \ \dots \ 1 \ 1 \\ \mathbf{0} \end{array} & \dots & \mathbf{0} & \\ \hline \dots & \dots & \dots & \dots & \dots & \\ \hline \vdots & \underbrace{\begin{array}{c} \vdots \\ \vdots \end{array}}_{\dim = d - k} & \vdots & \dots & \begin{array}{c} 1 \ 1 \ \dots \ 1 \ 1 \\ \vdots \end{array} & \end{array} \right) \begin{array}{l} k \\ d - k \end{array}$$

The rank of the highlighted block coincides with the affine dimension of  $\mathbb{A}_1 = \mathcal{A}_1$ , which is  $d - k$ . Therefore,

$$\forall j > 1: d - 1 = \dim(\text{span}(\mathcal{A} \setminus \mathbb{A}_j)) = \dim(\mathcal{A} \cap b_{1j}^\perp)$$

Which means that, indeed, any of the  $b_{1j}$  could be set as  $b_d$  from the start. Choose  $b_{1j}$  with the smallest possible size of  $\mathbb{A}_j$ , and repeat all the same reasoning with it as  $b_d$ . Note that in this case,  $|\mathcal{A} \setminus \mathbb{A}_j| > |\mathbb{A}_j|$ , so there will be no need for translation of  $\mathcal{A}$  that swaps  $\mathcal{A}_0$  and  $\mathcal{A}_1$  in claim 1, and we can thus safely assume

$$\begin{aligned} \forall j > 1: |\mathbb{A}_1| \leq |\mathbb{A}_j| &\implies \\ |\mathcal{A}_0| - |\mathcal{A}_1| = |\mathbb{A}_0| + \sum_{j>1} |\mathbb{A}_j| &\geq (k - 1) |\mathcal{A}_1| \geq 2 |\mathcal{A}_1| \end{aligned} \quad (12)$$

If  $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2d - 4$ , non-emptiness of  $\mathcal{B}_*$  and (10) imply the desired estimate. Otherwise

$$\begin{aligned} |\mathcal{A}_0| - |\mathcal{A}_1| < 2d - 4 &\stackrel{(12)}{\implies} |\mathcal{A}| < 2 \cdot (2d - 4) \implies \\ |\mathcal{A}| \cdot |\mathcal{B}| &\leq |\mathcal{A}|^2 < (4d - 8)^2 < d2^d + 2d, \end{aligned}$$



concluding the proof. □

Two examples that demonstrate tightness of the bound in Theorem 2 are

**Example 1.** Let  $\{e_i\}$  be the standard basis of  $\mathbb{R}^d$ ,

$$\mathcal{A} = \left\{ \sum_{i=2}^d \delta_i e_i \right\} \cup \{e_1\}, \mathcal{B} = \{\delta_1 e_1 + e_j\} \cup \{e_1, 0\}, \text{ where } \delta_i \text{ range over } \{0, 1\} \text{ and } j > 1.$$

Here  $|\mathcal{A}| = 2^{d-1} + 1$  and  $|\mathcal{B}| = 2d$ .

**Example 2.** Let  $\{e_i\}$  be the standard basis of  $\mathbb{R}^d$ ,

$$\mathcal{A} = \left\{ e_d + \sum_{i=1}^{d-1} \varepsilon_i e_i \right\} \cup \{0\}, \mathcal{B} = \left\{ \frac{1}{2} (e_d + \varepsilon_i e_i) \right\}, \text{ where } \varepsilon_i \text{ range over } \{-1, 1\}.$$

Just like in example 1,  $|\mathcal{A}| = 2^{d-1} + 1$  and  $|\mathcal{B}| = 2d$ .

### 3 Application to 2-level polytopes

Our main application of Theorem 2 is the following

**Theorem 3.** For  $d > 1$  let  $P$  be a  $d$ -dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then  $f_0(P)f_{d-1}(P) \leq (d-1)2^{d+1} + 8(d-1)$ .

Before following with the proof let us make a simple observation, proof of which is given in Appendix A for completeness:

**Lemma 1.** Let  $\mathcal{S}$  be a family of subsets of  $[d-1]$  such that  $|\mathcal{S}| = d$  and

$$\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1.$$

Then either  $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$  or  $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$ .

*Proof of Theorem 3.* The statement is trivial on the plane, so we assume  $d > 2$ . Let us denote  $V = f_0(P)$  and  $F = f_{d-1}(P)$  for conciseness. Shift  $P$  so that 0 is among its vertices and let  $\mathcal{A}$  denote the vertex set of  $P$  and  $\mathcal{B}'$  denote the minimal set of vectors such that every facet of  $P$  lies in a hyperplane  $\{x : \langle x, b \rangle = \delta\}$  for some  $\delta \in \{0, 1\}$  and  $b \in \mathcal{B}'$ . Let  $\mathcal{B} = \mathcal{B}' \cup \{0\}$ . If every vector in  $\mathcal{B}'$  defines one facet of  $P$ , we are done by Theorem 1:

$$V \cdot F < |\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1)2^d < (d-1)2^{d+1} + 8(d-1).$$

Otherwise, let  $b_d \in \mathcal{B}'$  define two facets of  $P$  and consider the setting of the proof of Theorem 2. Note that we may assume  $|\mathcal{A}_0| \geq |\mathcal{A}_1|$  if appropriate translation of  $P$  was made, so there will be no need for translation of  $\mathcal{A}$  or inversions of vectors in  $\mathcal{B}$ . Since  $\dim(\mathcal{A}_1) = d-1$ , we have  $\mathcal{B}_1 = \{0, b_d\}$  and  $|\pi(\mathcal{B})| = |\mathcal{B}_*| + 1$ , which means

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})| + |\mathcal{A}|. \quad (13)$$

Since every vector in  $\mathcal{B}'$  defines at most two facets of  $P$ ,  $|\mathcal{B}| \geq \frac{F}{2} + 1$ , thus from (13) we conclude

$$V \cdot F \leq 2(|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})|) \leq 4 \cdot |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \quad (14)$$

Consider three cases:

1.  $|\mathcal{A}_0| > d$  and  $|\pi(\mathcal{B})| > d$ . By Theorem 2 we have

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1)2^{d-1} + 2(d-1)$$

and with (14) we are done.

2.  $|\pi(\mathcal{B})| = d$ . Together with  $\mathcal{B}_1 = \{0, b_d\}$ , this means that  $\mathcal{B}'$  is a basis of  $\mathbb{R}^d$ . Every vector in  $\mathcal{B}'$  then has to define two facets of  $P$ , since otherwise  $P$  is unbounded. Thus  $P$  is affinely isomorphic to the cube.
3.  $|\mathcal{A}_0| = d$ . Note that as  $|\mathcal{A}_1| \leq |\mathcal{A}_0|$  and  $\dim(\mathcal{A}_1) = d-1$ , we also have  $|\mathcal{A}_1| = d$ . If  $|\pi(\mathcal{B})| \leq \frac{3}{4} \cdot 2^{d-1}$ , (14) implies  $V \cdot F \leq \frac{3}{4}d \cdot 2^{d+1} < (d-1)2^{d+1} + 8d$ , so we may further assume

$$|\pi(\mathcal{B})| > \frac{3}{4} \cdot 2^{d-1}. \quad (15)$$

We will now make several observations about the structure of  $\mathcal{A}$  and  $\mathcal{B}$ , after which it will become clear that  $P$  is affinely isomorphic to the cross-polytope. Let  $a_0 = 0, a_1, \dots, a_{d-1}$  be the elements of  $\mathcal{A}_0$  and  $\{u_1, \dots, u_{d-1}\}$  be the basis of  $\text{span}(\mathcal{A}_0)$ , dual to  $\{a_1, \dots, a_{d-1}\}$ . Note that for every  $j \in [d-1]$  there is  $b_{\{j\}} \in \mathcal{B}$  such that  $\pi(b_{\{j\}}) = u_j$ :  $b_{\{j\}}$  is the vector orthogonal to the facet of  $P$  that contains vertices  $\{a_0, \dots, a_{d-1}\} \setminus \{a_j\}$  and differs from  $\mathcal{A}_0$ . Given  $S \subseteq [d-1]$  let us denote by  $b_S$  an element of  $\mathcal{B}$  for which  $\pi(b_S) = \sum_{j \in S} u_j$ , if there is one, with  $b_\emptyset = 0$  to avoid ambiguity. Observe that the basis of  $\mathbb{R}^d$  dual to  $\{b_{\{1\}}, b_{\{2\}}, \dots, b_{\{d-1\}}, b_d\}$  is  $\{a_1, a_2, \dots, a_{d-1}, v\}$  for  $v$  that satisfies

$$\langle v, b_d \rangle = 1 \text{ and } \forall j \in [d-1] : \langle v, b_{\{j\}} \rangle = 0.$$

This means that

$$\mathcal{A}_1 = \{v + \sum_{j \in S} a_j : S \in \mathcal{S}\} \quad (16)$$

for some family  $\mathcal{S}$  of subsets of  $[d-1]$  with  $|\mathcal{S}| = d$ . Our goal is to show that  $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ , as then  $P$  is affinely isomorphic to the cross-polytope, and we would be done. For  $T \subseteq [d-1]$  denote  $\sigma_T = \sum_{j \in T} a_j$  and note that, given  $b_S \in \mathcal{B}$ ,

$$\langle \sigma_T, b_S \rangle = \langle \sigma_T, \pi(b_S) \rangle = \left\langle \sigma_T, \sum_{j \in S} \pi(b_{\{j\}}) \right\rangle = \left\langle \sum_{j \in T} a_j, \sum_{j \in S} b_{\{j\}} \right\rangle = |T \cap S|.$$

Now assume, looking for a contradiction, that  $\exists S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| > 1$ . (15) means there exists  $b_S \in \mathcal{B}$  such that  $|S \cap (S_2 \setminus S_1)| > 1$ . (16) means that

$$\begin{aligned} \{-1, 0, 1\} &\ni \langle v + \sum_{j \in S_2} a_j, b_S \rangle - \langle v + \sum_{j \in S_1} a_j, b_S \rangle = \langle \sigma_{S_2} - \sigma_{S_1}, b_S \rangle \\ &= |S_2 \cap S| - |S_1 \cap S| = |(S_2 \setminus S_1) \cap S| > 1, \end{aligned}$$

a contradiction. Therefore,  $\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1$ , which by Lemma 1 implies that either  $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$  or  $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$ . The latter is, however, impossible, since then  $P$  only has  $d+1$  facets and  $|\pi(\mathcal{B})| = d \leq \frac{3}{4}2^{d-1}$ . (16) now shows that  $P$  is affinely isomorphic to the cross-polytope, and we are done.  $\square$

Two examples demonstrate tightness of the bound in Theorem 3:

**Example 3** (Cross-polytope  $\times$  segment). *Let  $\{e_i\}$  be the standard basis of  $\mathbb{R}^d$ ,*

$$P = \text{Conv}(\{\varepsilon_i e_i + \varepsilon_d e_d\}_{i \leq d-1}), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\} \text{ for } i \in [d].$$

*Here  $f_0(P) = 4(d-1)$  and  $f_{d-1}(P) = 2 + 2^{d-1}$ .*

**Example 4** (Suspension of a cube). *Let  $\{e_i\}$  be the standard basis of  $\mathbb{R}^d$ ,*

$$P = \text{Conv}\left(\left\{\sum_{i=1}^{d-1} \varepsilon_i e_i\right\} \cup \{e_d, -e_d\}\right), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\}.$$

*This is (up to coordinate scaling) the dual of the polytope in the previous example, in particular  $f_0(P) = 2 + 2^{d-1}$  and  $f_{d-1}(P) = 4(d-1)$ .*

## 4 Proofs of claims

In this section, we provide the proofs of the claims from [5] made at the beginning of section 2.

**Claim 1.** *We may translate  $\mathcal{A}$  and replace some points in  $\mathcal{B}$  by their negatives such that the following holds.*

(i) *We can still write  $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ , where  $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$  for  $i = 0, 1$  such that*

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \quad (1)$$

(ii) *We still have*

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \quad (2)$$

(iii) *The set  $\pi(\mathcal{B})$  does not contain opposite points.*

*Proof.* If  $|\{a \in \mathcal{A} : \langle a, b_d \rangle = 0\}| \leq |\{a \in \mathcal{A} : \langle a, b_d \rangle = 1\}|$ , then we can choose any  $a_* \in \mathcal{A}$  with  $\langle a_*, b_d \rangle = 1$  (which exists since  $\mathcal{A}$  spans  $\mathbb{R}^d$ ) and replace  $\mathcal{A}$  by  $\mathcal{A} - a_*$ ,  $\mathcal{B}$  by  $(\mathcal{B} \setminus \{b_d\}) \cup \{-b_d\}$ , and  $b_d$  by  $-b_d$ . This yields (i).

After this replacement, for each  $b \in \mathcal{B}$  there is some  $\varepsilon_b \in \{\pm 1\}$  such that  $\langle a, b \rangle \in \{0, \varepsilon_b\}$  holds for all  $a \in \mathcal{A}$ . Each  $b$  with  $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, -1\}$  is replaced by  $-b$ , which yields (ii).

Let  $\mathcal{A}'_1$  be a translate of  $\mathcal{A}_1$  such that  $\mathbf{0} \in \mathcal{A}'_1$ . Note that, for each  $b \in \mathcal{B}$  we now have  $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, 1\}$  or  $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0\}$ . In the second case, we replace  $b$  by  $-b$  if  $\{\langle a, b \rangle : a \in \mathcal{A}'_1\} = \{0, -1\}$ , otherwise we leave it as it is.

It remains to show that  $\pi(\mathcal{B})$  does not contain opposite points after this transformation. To this end, let  $b, b' \in \mathcal{B}$  such that  $\pi(b) = \beta\pi(b')$  for some  $\beta \neq 0$ , where  $\pi(b), \pi(b') \neq \mathbf{0}$ . We have to show that  $\beta = 1$ . Note that for every  $a \in \mathcal{A}_0 \cup \mathcal{A}'_1 \subseteq U$  we have

$$\langle a, b \rangle = \langle a, \pi(b) \rangle = \beta \langle a, \pi(b') \rangle = \beta \langle a, b' \rangle.$$

Suppose first that  $\{\langle a, b \rangle : a \in \mathcal{A}_0\} \neq \{0\}$ . By (2) there exists some  $a \in \mathcal{A}_0$  with  $1 = \langle a, b \rangle = \beta \langle a, b' \rangle$ . Thus, we have  $\langle a, b' \rangle \neq 0$  and hence  $\langle a, b' \rangle = 1$ , again by (2). This yields  $\beta = 1$ .

Suppose now that  $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0\}$ . Note that this implies  $\{\langle a, b' \rangle : a \in \mathcal{A}_0\} = \{0\}$ . As  $\mathcal{A}_0 \cup \mathcal{A}'_1$  spans  $U$ , we must have  $\{\langle a, b \rangle : a \in \mathcal{A}'_1\} \neq \{0\}$  and hence there is some  $a \in \mathcal{A}'_1$  with  $\langle a, b \rangle = 1$ . Moreover, we have  $\beta \langle a, b' \rangle = 1$ , and in particular  $\langle a, b' \rangle \neq 0$ . This implies  $\langle a, b' \rangle = 1$  and hence  $\beta = 1$ .  $\square$

As in the previous proof, let  $\mathcal{A}'_1$  be a translate of  $\mathcal{A}_1$  such that  $\mathbf{0} \in \mathcal{A}'_1$ . Note that for each  $b \in \mathcal{B}$  there are  $\varepsilon_b, \gamma_b \in \{\pm 1\}$  such that

$$\langle a, b \rangle \in \{0, \varepsilon_b\} \text{ for each } a \in \mathcal{A} \text{ and} \quad (17)$$

$$\langle a, b \rangle \in \{0, \gamma_b\} \text{ for each } a \in \mathcal{A}'_1. \quad (18)$$

**Inequality 1.**  $|\mathcal{A}||\mathcal{B}| \leq 2|\mathcal{A}_0||\pi(\mathcal{B})| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*|$

*Proof.* Claim 2 implies  $|\mathcal{B}| = 2|\pi(\mathcal{B})| - |\mathcal{B}_*|$  or  $2(|\pi(\mathcal{B})| - |\mathcal{B}_*|) = |\mathcal{B} \setminus \mathcal{B}_*|$ . With  $|\mathcal{A}_0| \geq |\mathcal{A}_1|$  this gives

$$\begin{aligned} |\mathcal{A}||\mathcal{B}| &= (|\mathcal{A}_0| + |\mathcal{A}_1|)(2|\pi(\mathcal{B}_*)| - |\mathcal{B}_*|) \leq 2|\mathcal{A}_0||\pi(\mathcal{B}_*)| + 2|\mathcal{A}_1||\pi(\mathcal{B})| - 2|\mathcal{A}_1||\pi(\mathcal{B})| \\ &= 2|\mathcal{A}_0||\pi(\mathcal{B}_*)| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*| \end{aligned}$$

$\square$

The proofs of the subsequent claims rely on the following two lemmas.

**Lemma 2.** *Suppose that  $X \subseteq \{0, 1\}^d \cup \{0, -1\}^d$  does not contain opposite points. Then we have  $|X| \leq 2^{\dim X}$ .*

*Proof.* We prove the statement by induction on  $d \geq 1$ , and observe that it is true for  $d = 1$ . Now let  $d \geq 2$ . If  $\dim X = d$ , then we are also done. It remains to consider the case where  $X$  is contained in an affine hyperplane  $H \subseteq \mathbb{R}^d$ . Let  $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ ,  $\delta \in \{0, 1\}$  such that

$$H = \{x \in \mathbb{R}^d : \langle c, x \rangle = \delta\}.$$

For each  $i \in \{1, \dots, d\}$  let  $\pi_i : H \rightarrow \mathbb{R}^{d-1}$  denote the projection that forgets the  $i$ -th coordinate, and let  $e_i \in \mathbb{R}^d$  denote the  $i$ -th standard unit vector. Note that  $\pi_{i^*}(X) \subseteq \{0, 1\}^{d-1} \cup \{0, -1\}^{d-1}$ .

Suppose there is some  $i^* \in \{1, \dots, d\}$  such that  $\langle c, e_{i^*} \rangle \neq 0$  and  $\pi_{i^*}(X)$  does not contain opposite points. By the induction hypothesis we obtain

$$|X| = |\pi_{i^*}(X)| \leq 2^{\dim \pi_{i^*}(X)} = 2^{\dim X},$$

where the first equality and the last equality hold since  $\pi_{i^*}$  is injective (due to  $\langle c, e_{i^*} \rangle \neq 0$ ).

It remains to consider the case in which there is no such  $i^*$ . Consider any  $i \in \{1, \dots, d\}$ . If  $\langle c, e_i \rangle \neq 0$ , then there exist  $x = (x_1, \dots, x_d), x' = (x'_1, \dots, x'_d) \in X$ ,  $x \neq x'$  such that  $\pi_i(x) = -\pi_i(x')$ . We may assume that  $\pi_i(x) \in \{0, 1\}^{d-1}$  and hence  $\pi_i(x') \in \{0, -1\}^{d-1}$ . As  $X$  does not contain opposite points, we must have  $x_i = 1$  and  $x'_i = 0$ , or  $x_i = 0$  and  $x'_i = -1$ . In the first case we obtain

$$\begin{aligned} 2\delta &= \langle c, x \rangle + \langle c, x' \rangle = [\langle \pi_i(c), \pi_i(x) \rangle + c_i x_i] + [\langle \pi_i(c), \pi_i(x') \rangle + c_i x'_i] \\ &= [\langle \pi_i(c), \pi_i(x) \rangle + c_i] + [\langle \pi_i(c), \pi_i(x') \rangle] \\ &= c_i. \end{aligned}$$

Similarly, in the second case we obtain  $2\delta = -c_i$ .

If  $\delta = 0$ , this would imply that  $c = \mathbf{0}$ , a contradiction to the fact that  $H \neq \mathbb{R}^d$ . Otherwise,  $\delta = 1$  and hence every nonzero coordinate of  $c$  is  $\pm 2$ . Thus, for every  $x \in \mathbb{Z}^d$  we see that  $\langle c, x \rangle$  is an even number, in particular  $\langle c, x \rangle \neq \delta$ . This means that  $X \subseteq \mathbb{Z}^d \cap H = \emptyset$ , and we are done.  $\square$

A direct consequence of Lemma 2 that we will employ is

**Lemma 3.** *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  such that  $\mathcal{A}$  spans  $\mathbb{R}^d$ ,  $\mathcal{B}$  does not contain opposite points, and for every  $b \in \mathcal{B}$  there is some  $\varepsilon_b \in \{\pm 1\}$  such that  $\{\langle a, b \rangle : a \in \mathcal{A}\} \subseteq \{0, \varepsilon_b\}$ . Then we have  $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$ .*

*Proof.* Let  $a_1, \dots, a_d \in \mathcal{A}$  be a basis of  $\mathbb{R}^d$  and express elements of  $\mathcal{B}$  in the dual basis, it then becomes a subset of  $\{0, 1\}^d \cup \{0, -1\}^d$  with no opposite points. By Lemma 2,  $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$ .  $\square$

We are ready to continue with the proofs of the remaining claims.

**Claim 2.** *Every point in  $\pi(\mathcal{B})$  has at most two preimages in  $\mathcal{B}$ .*

*Proof.* Let  $y := \pi(b)$  for some  $b \in \mathcal{B}$  and observe that  $\pi^{-1}(y) = \{x \in \mathbb{R}^d : \pi(x) = y\}$  is a one-dimensional affine subspace. By (17) and Lemma 3 we obtain  $|\mathcal{B} \cap \pi^{-1}(y)| \leq 2$ .  $\square$

**Claim 3.**  $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0} |\tau(\pi(\mathcal{B}))|$ .

*Proof.* Fix any  $b \in \mathcal{B}$  and let  $v := \pi(b)$ . Consider the orthogonal complement  $W \subseteq U$  of  $U_0$  in  $U$ . As  $\tau^{-1}(\tau(v)) = v + W$ , it suffices to show that

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}$$

holds. To this end, consider the linear subspace  $\Pi \subseteq U$  spanned by  $v$  and  $W$  and let  $\sigma : U \rightarrow \Pi$  denote the orthogonal projection on  $\Pi$ .

First, suppose that  $\sigma(\mathcal{A}'_1)$  spans  $\Pi$ . For every  $a \in \mathcal{A}'_1 \subseteq U$  and every  $b \in \mathcal{B}$  with  $\pi(b) \in v + W \subseteq \Pi$  we have

$$\langle \sigma(a), \pi(b) \rangle = \langle a, \pi(b) \rangle = \langle a, b \rangle \in \{0, \gamma_b\}$$

by (18). Moreover, recall that  $\pi(\mathcal{B})$  does not contain opposite points by Claim 1 (iii). Thus, the pair  $\sigma(\mathcal{A}'_1)$  and  $(v + W) \cap \pi(\mathcal{B})$  satisfies the requirements of Lemma 3 (in  $\Pi$ ), and hence we obtain

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{\dim(v+W)} = 2^{\dim W} = 2^{\dim U - \dim U_0} = 2^{d-1-\dim U_0}.$$

It remains to consider the case in which  $\sigma(\mathcal{A}'_1)$  does not span  $\Pi$ . Recall that we chose  $b_d$  as the nonzero vector in  $\mathcal{B}$  with the maximal  $\varphi(b_d) := \max(\dim(\mathcal{A}_0), \dim(\mathcal{A}_1))$  for the corresponding  $\mathcal{A}_0$  and  $\mathcal{A}_1$ . Unless  $|(v+W) \cap \pi(\mathcal{B})| = 1$ , we will identify points  $b_1, b_2 \in \mathcal{B}$  with  $\max\{\varphi(b_1), \varphi(b_2)\} > \varphi(b_d)$ , a contradiction to the choice of  $b_d$ .

As  $\mathcal{A}_0 \cup \mathcal{A}'_1$  spans  $U$ , we know that  $\sigma(\mathcal{A}_0 \cup \mathcal{A}'_1)$  spans  $\Pi$ . Since  $\mathcal{A}_0$  is orthogonal to  $W$ , this means that  $\sigma(\mathcal{A}_0)$  spans a line, and  $\sigma(\mathcal{A}'_1)$  spans a hyperplane  $H$  in  $\Pi$ . Note that we have  $v \notin W$  (otherwise  $W = \Pi$  and so  $\sigma(\mathcal{A}'_1)$  spans  $\Pi$ ). Thus, every nonzero point in  $\sigma(\mathcal{A}_0)$  has nonzero scalar product with  $v$ . Moreover, for every  $a \in \mathcal{A}_0$  with  $\sigma(a) \neq \mathbf{0}$  we have  $\langle \sigma(a), v \rangle = \langle a, v \rangle = \langle a, b \rangle \in \{0, 1\}$  by (2). Thus, since the nonzero vectors in  $\sigma(\mathcal{A}_0)$  are collinear, we obtain

$$\sigma(\mathcal{A}_0) \subseteq \{\mathbf{0}, \sigma(a_0)\}$$

for some  $a_0 \in \mathcal{A}_0$ . Since  $\mathbf{0} \in H$ , we have  $\sigma(\mathcal{A}_0) \setminus H \subseteq \{\sigma(a_0)\}$  and further, since  $\sigma(\mathcal{A}_0 \cup \mathcal{A}'_1)$  spans  $\Pi$ , we have  $\sigma(\mathcal{A}_0) \setminus H = \{\sigma(a_0)\}$ . Let  $c \in \Pi$  be a normal vector of  $H$ . As  $\sigma(a_0) \notin H$ , we may scale  $c$  so that  $\langle \sigma(a_0), c \rangle = 1$ . Let  $a_* \in \mathcal{A}_1$  such that  $\mathcal{A}'_1 = \mathcal{A}_1 - a_*$ . We define

$$b_1 := c - \delta_1 b_d \neq \mathbf{0},$$

where  $\delta_1 := \langle a_*, c \rangle$ . For every  $a \in \mathcal{A}_0$  we have

$$\langle a, b_1 \rangle = \langle a, c \rangle = \langle \sigma(a), c \rangle \in \{\langle \mathbf{0}, c \rangle, \langle \sigma(a_0), c \rangle\} = \{0, 1\},$$

and for every  $a \in \mathcal{A}_1$  we have

$$\begin{aligned} \langle a, b_1 \rangle &= \underbrace{\langle a - a_*, b_1 \rangle}_{\in \mathcal{A}'_1} + \langle a_*, b_1 \rangle = \langle a - a_*, c \rangle + \langle a_*, b_1 \rangle = \underbrace{\langle \sigma(a - a_*), c \rangle}_{\in H} + \langle a_*, b_1 \rangle \\ &= \langle a_*, b_1 \rangle = \langle a_*, c \rangle - \delta_1 \langle a_*, b_d \rangle = \langle a_*, c \rangle - \delta_1 = 0. \end{aligned}$$

Thus, by the maximality of  $\mathcal{B}$ , (a scaling of) the vector  $b_1$  is contained in  $\mathcal{B}$ . Since we assumed  $\mathbf{0} \in \mathcal{A}_0$ , we have  $\varphi(b_1) \geq \dim(\mathcal{A}_1) + 1$ .

In order to construct  $b_2$ , let us suppose that there is another point  $b' \in \mathcal{B}$  with  $v' := \pi(b') \neq v$  and  $v' \in (v + W)$ . If there is no such point, then the statement of the claim is true. Recall that  $\sigma(a_0)$  is orthogonal to  $W$ , and let

$$\xi := \langle \sigma(a_0), v \rangle = \langle \sigma(a_0), \underbrace{v - v'}_{\in W} \rangle + \langle \sigma(a_0), v' \rangle = \langle \sigma(a_0), v' \rangle.$$

Choose  $v'' \in \{v, v'\}$  such that  $\xi c \neq v''$ , and let  $b'' \in \{b, b'\}$  such that  $\pi(b'') = v''$ . Define  $\delta_2 := \langle a_*, v'' - \xi c \rangle$  and note that

$$b_2 := v'' - \xi c - \delta_2 b_d$$

is nonzero since  $v'' - \xi c \in U \setminus \{\mathbf{0}\}$ . For every  $a \in \mathcal{A}_0$  we have

$$\langle a, b_2 \rangle = \langle a, \underbrace{v'' - \xi c}_{\in \Pi} \rangle = \langle \sigma(a), v'' - \xi c \rangle,$$

which is zero if  $\sigma(a) = \mathbf{0}$ . Otherwise,  $\sigma(a) = \sigma(a_0)$  and we obtain

$$\langle a, b_2 \rangle = \langle \sigma(a_0), v'' \rangle - \xi \langle \sigma(a_0), c \rangle = \langle \sigma(a_0), v'' \rangle - \xi = 0.$$

Thus,  $b_2$  is orthogonal to  $\mathcal{A}_0$ . Moreover, note that

$$\langle a_*, b_2 \rangle = \langle a_*, v'' - \xi c \rangle - \delta_2 \underbrace{\langle a_*, b_d \rangle}_{=1} = 0.$$

Thus, for every  $a \in \mathcal{A}_1$  we have

$$\begin{aligned} \langle a, b_2 \rangle &= \langle a - a_*, b_2 \rangle + \langle a_*, b_2 \rangle = \langle a - a_*, b_2 \rangle = \langle a - a_*, v'' \rangle - \xi \underbrace{\langle a - a_*, c \rangle}_{=0} - \delta_2 \underbrace{\langle a - a_*, b_d \rangle}_{=0} \\ &= \langle a - a_*, v'' \rangle = \langle a - a_*, b'' \rangle \in \{0, \gamma_{b''}\} \end{aligned}$$

by (18). Thus, again by the maximality of  $\mathcal{B}$ , (a scaling of) the vector  $b_2$  is contained in  $\mathcal{B}$ , and since  $b_2$  is orthogonal to  $\mathcal{A}_0$  and  $a_* \in \mathcal{A}_1$ , we have  $\varphi(b_2) \geq \dim(\mathcal{A}_0) + 1$ . However, by the choice of  $b_d$  we must have

$$\max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\} + 1 \leq \max\{\varphi(b_1), \varphi(b_2)\} \leq \varphi(b_d) = \max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\},$$

a contradiction.  $\square$

**Claim 4.**  $\mathcal{B} \setminus \mathcal{B}_* = \mathcal{B}_0 \sqcup \mathcal{B}_1$  holds with

$$\forall b \in \mathcal{B}_i : |\{\langle a, b \rangle : a \in \mathcal{A}_i\}| = 1 \text{ for } i = 0, 1$$

*Proof.* Let  $b \in \mathcal{B} \setminus \mathcal{B}_*$  and, for the sake of contradiction, suppose that  $|\{\langle a, b \rangle : a \in \mathcal{A}_0\}| = |\{\langle a, b \rangle : a \in \mathcal{A}_1\}| = 2$ . Let  $b' \in \mathcal{B} \setminus \{b\}$  such that  $\pi(b) = \pi(b')$ . In other words, we have  $b' = b + \gamma b_d$  for some  $\gamma \neq 0$ . Then, by (2) we have

$$\{\langle a, b' \rangle : a \in \mathcal{A}_0\} = \{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, 1\}$$

and hence we obtain  $\varepsilon_b = \varepsilon_{b'} = 1$  by (17). Again by (17) we see

$$\{0, 1\} \supseteq \{\langle a, b' \rangle : a \in \mathcal{A}_1\} = \{\langle a, b \rangle : a \in \mathcal{A}_1\} + \gamma = \{0, 1\} + \gamma = \{\gamma, 1 + \gamma\},$$

which implies  $\gamma = 0$ , a contradiction.  $\square$

**Inequality 2.**  $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1) 2^d + |\mathcal{A}_0| |\mathcal{B}_0| + |\mathcal{A}_1| |\mathcal{B}_1|$

*Proof.*  $\tau(\pi(\mathcal{B}))$  and  $\mathcal{A}_0$  are both spanning  $U_0$  and have binary scalar products, so by Theorem 1 (or by the induction hypothesis, in the context of the proof of Theorem 1 in [5])

$$|\tau(\pi(\mathcal{B}))| |\mathcal{A}_0| \leq (\dim(U_0) + 1) 2^{\dim(U_0)}$$

Combining this with Claim 3 and Inequality 1 we get

$$|\mathcal{A}| |\mathcal{B}| \leq 2 \cdot (\dim(U_0) + 1) 2^{d-1} + |\mathcal{A}_1| (|\mathcal{B}_0| + |\mathcal{B}_1|) \leq (\dim(U_0) + 1) 2^d + |\mathcal{A}_0| |\mathcal{B}_0| + |\mathcal{A}_1| |\mathcal{B}_1|,$$

where the second inequality is due to  $|\mathcal{A}_0| \geq |\mathcal{A}_1|$ .  $\square$

**Inequality 3.** For  $i = 0, 1$  we have

$$|\mathcal{A}_i| \leq 2^{\dim(\mathcal{A}_i)}, |\mathcal{B}_i| \leq 2^{\dim(\text{span } \mathcal{B}_i)}, \text{ and } \dim(\mathcal{A}_i) + \dim(\text{span } (\mathcal{B}_i)) \leq d$$

*Proof.* The first (and second) inequality is a direct consequence of Lemma 3 after writing  $\mathcal{A}$  (or  $\mathcal{B}$ ) in the basis, dual to a basis bound in  $\mathcal{B}$  (or  $\mathcal{A}$ ). The last inequality follows from the definition of  $\mathcal{B}_i$ : for each  $b \in \mathcal{B}_i$  there is  $\xi_b$  so that

$$\mathcal{A}_i \subset W_i, \text{ where } W_i = \{x \in \mathbb{R}^d : \langle x, b \rangle = \xi_b \text{ for all } b \in \mathcal{B}_i\}$$

and clearly  $\dim(W_i) \leq d - \dim(\text{span}(\mathcal{B}_i))$ . □

**Claim 5.** For  $i = 0, 1$ , we have  $|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^d$ .

*Proof.* By Inequality 3,

$$|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^{\dim(\mathcal{A}_i)} \cdot 2^{\dim(\text{span}(\mathcal{B}_i))} \leq 2^d$$

□

## A Appendix

**Inequality 4.** For an integer  $2 \leq f \leq d$ , we have:

$$(d + f)(2^{d-1} + 2^{d-f}) \leq d2^d + 2d.$$

*Proof.* We will prove this by induction on  $d$ : when  $d = k$ , the equality is satisfied. Let's perform the induction step from  $d$  to  $d + 1$ . Denoting the left and right sides of the inequality as  $l(d, f)$  and  $r(d, f)$ , respectively, we have

$$\begin{aligned} r(d + 1, f) - l(d + 1, f) &\geq (r(d + 1, f) - r(d, f)) - (l(d + 1, f) - l(d, f)) \\ &= \left(d2^d + 2^{d+1} + 2\right) - (d + f + 2) \left(2^{d-1} + 2^{d-f}\right) \\ &= 2^{d-f} (d - f + 2) \left(2^{f-1} - 1 - \frac{2f}{d - f + 2}\right) + 2 \\ &\geq 2^{d-f} (d - f + 2) \left(2^{f-1} - 1 - f\right) \end{aligned}$$

The obtained expression is non-negative for  $f > 2$ . For  $f = 2$  and  $d \geq 4$ , we have  $2^{f-1} - 1 - \frac{2f}{d-f+2} \geq 0$ , and for  $f = 2$  and  $d = 2, 3$ , the initial inequality is checked explicitly. □

**Lemma 1.** Let  $\mathcal{S}$  be a family of subsets of  $[d - 1]$  such that  $|\mathcal{S}| = d$  and

$$\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1.$$

Then either  $\mathcal{S} = \{S \subseteq [d - 1] : |S| \geq d - 2\}$  or  $\mathcal{S} = \{S \subseteq [d - 1] : |S| \leq 1\}$ .

*Proof.* We assume  $d > 2$  as the statement is trivial otherwise.  $|\mathcal{S}| > 2$  and clearly  $\mathcal{S}$  contains sets of at most two different sizes (that differ by one), so let  $U, V \in \mathcal{S}$  both be of size  $k \in [d - 2]$ . Observe that there are now only four options for sets in  $\mathcal{S}$ :

- (a)  $U \cup V$  of size  $k + 1$ .
- (b) Sets of size  $k$  that are contained in  $U \cup V$ .



(c) Sets of size  $k$  that contain  $U \cap V$  as a subset.

(d)  $U \cap V$  of size  $k - 1$ .

(a) and (d) are not possible simultaneously, neither are (b) and (c) with the exception of  $U$  and  $V$ . There are  $k + 1$  and  $d - k$  sets satisfying (b) and (c) respectively, so  $|\mathcal{S}| = d$  is only possible if  $k = d - 2$  or  $k = 1$  with  $\mathcal{S} = \{S \subseteq [d - 1] : |S| \geq d - 2\}$  or  $\mathcal{S} = \{S \subseteq [d - 1] : |S| \leq 1\}$  respectively.  $\square$

We finish with a conjecture that generalises our main result:

**Conjecture 1.** *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$  both linearly span  $\mathbb{R}^d$  such that  $\langle a, b \rangle \in \{0, 1\}$  holds for all  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ . Furthermore,  $|\mathcal{A}|$  and  $|\mathcal{B}|$  are both strictly larger than  $2^{k-1}(d - k + 2)$ . Then  $|\mathcal{A}| \cdot |\mathcal{B}| \leq (2^{d-k} + k)2^k(d - k + 1)$ .*

## References

- [1] M. Aprile, A. Cevallos, and Y. Faenza. On 2-level polytopes arising in combinatorial settings. *SIAM Journal on Discrete Mathematics*, 32(3):1857–1886, 2018.
- [2] A. Bohn, Y. Faenza, S. Fiorini, V. Fisikopoulos, M. Macchia, and K. Pashkovich. Enumeration of 2-level polytopes. *Mathematical Programming Computation*, 11, 2018.
- [3] S. Fiorini, V. Fisikopoulos, and M. Macchia. Two-level polytopes with a prescribed facet. In R. Cerulli, S. Fujishige, and A. R. Mahjoub, editors, *Combinatorial Optimization*, pages 285–296, Cham, 2016. Springer International Publishing.
- [4] A. Kupavskii and F. Noskov. Octopuses in the boolean cube: Families with pairwise small intersections, part i. *Journal of Combinatorial Theory, Series B*, 2023.
- [5] A. Kupavskii and S. Weltge. Binary scalar products. *Journal of Combinatorial Theory, Series B*, 156, 2022.