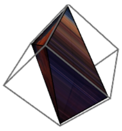
In[ ]:=
```
Quit[]
```

In[ ]:=
```
With[{pts = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {1, 0, 1}, {0, 1, 1}, {1, 1, 1}}, fcs =
   {{1, 2, 3}, {4, 5, 6}, {1, 2, 4}, {1, 3, 5}, {2, 3, 6}, {1, 4, 5}, {3, 5, 6}, {2, 4, 6}}},
 Graphics3D[{Opacity[0.9], Texture[
```

```
], Polyhedron[pts[[Sequence@#]] & /@ fcs,
    VertexTextureCoordinates → (pts[[Sequence@#]] & /@ fcs)]}]]
```

Out[ ]=



In[ ]:=
```
With[{pts = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {1, 0, 1}, {0, 1, 1}, {1, 1, 1}},
  fcs = {{1, 2, 3}, {4, 5, 6}, {1, 2, 4}, {1, 3, 5}, {2, 3, 6},
    {1, 4, 5}, {3, 5, 6}, {2, 4, 6}}}, pts[[Sequence@#]] & /@ fcs]
```

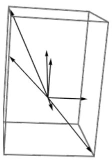
Out[ ]=
```
{{{0, 0, 0}, {1, 0, 0}, {0, 1, 0}}, {{1, 0, 1}, {0, 1, 1}, {1, 1, 1}},
 {{0, 0, 0}, {1, 0, 0}, {1, 0, 1}}, {{0, 0, 0}, {0, 1, 0}, {0, 1, 1}},
 {{1, 0, 0}, {0, 1, 0}, {1, 1, 1}}, {{0, 0, 0}, {1, 0, 1}, {0, 1, 1}},
 {{0, 1, 0}, {0, 1, 1}, {1, 1, 1}}, {{1, 0, 0}, {1, 0, 1}, {1, 1, 1}}}
```

In[ ]:=
```
Inverse[{{1, 0, 1}, {0, 1, 1}, {1, 1, 1}}ᵀ]
```

Out[ ]=
```
{{0, -1, 1}, {-1, 0, 1}, {1, 1, -1}}
```

In[ ]:=
```
Graphics3D[Arrow[{{0, 0, 0}, #}] & /@ (Tuples[{0, 1}, 3].%32)]
```

Out[ ]=



Are there any arrangement acheiving maximum product except for cube-with-basis? achNumber counts number of such arrangements given a non-degenerate 0-1 matrix -- basis of the second set in the coordinates dual to some basis of the second set. If maximum is only acheived with cube+basis, the function will return 2

In[ ]:=
```
$HistoryLength = 1
```

Out[ ]=
```
1
```

In[ ]:=
```
achNumber[B01_List] := With[{Cpl = Complement, Len = Length,
   d = Length@B01, cube = Tuples[{0, 1}, Length@B01], inv = Inverse[B01]},
  With[{acan = Select[Subsets[Cpl[cube, B01]], Divisible[2^d (d + 1), (Len@# + d)] &],
    bcan = Select[Subsets[Cpl[cube, B01^T]], Divisible[2^d (d + 1), (Len@# + d)] &]},
   Total@Flatten@Table[Boole[Cpl[Flatten[Join[B01, a].inv.(Join[B01^T, b]^T)],
        {0, 1}] == {}], {a, acan}, {b, Select[bcan, Len@# == (2^d (d + 1))/(d + Len@a) - d &]}]]]
```

Test on all invertible 2 x2 matrices :

In[ ]:=
```
Tally[achNumber /@ Select[Tuples[{0, 1}, {2, 2}], MatrixRank@# == 2 &]] // AbsoluteTiming
```
Out[ ]=
```
{0.0036872, {{2, 6}}}
```

Test on all invertible 3x3 matrices:

In[ ]:=
```
Tally[achNumber /@ Select[Tuples[{0, 1}, {3, 3}], MatrixRank@# == 3 &]] // AbsoluteTiming
```
Out[ ]=
```
{0.140387, {{2, 174}}}
```

Number of 4x4 basises ignoring order:

In[ ]:=
```
Select[DeleteDuplicates[Sort /@ Tuples[{0, 1}, {4, 4}]], MatrixRank@# == 4 &] // Length
```
Out[ ]=
```
940
```

In[ ]:=
```
achNumber[IdentityMatrix[4]] // AbsoluteTiming
```
Out[ ]=
```
{15.8582, 2}
```

In[ ]:=
$$35 * \frac{940}{3600.}$$
Out[ ]=
```
9.13889
```

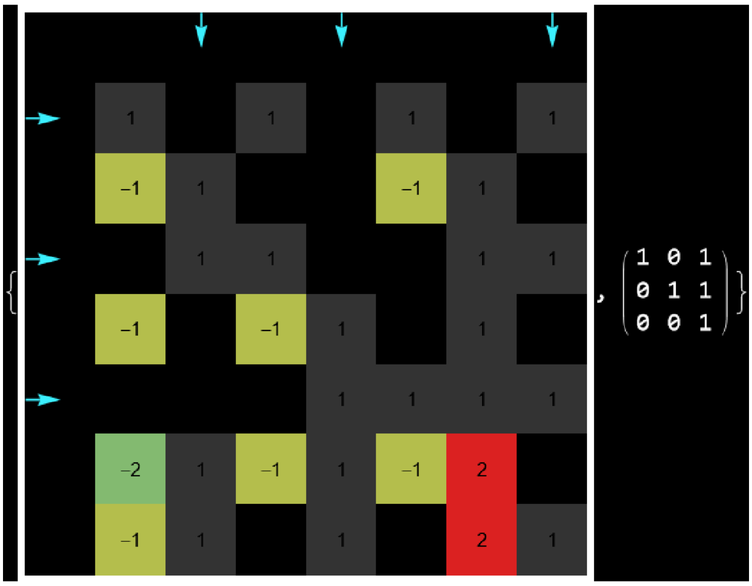
In[ ]:=
```
SeedRandom[42];
fourByFourSample = Transpose /@ RandomSample[
   Select[DeleteDuplicates[Sort /@ Tuples[{0, 1}, {4, 4}]], MatrixRank@# == 4 &], 4]
```
Out[ ]=
```
{{{0, 0, 1, 1}, {0, 1, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}},
 {{0, 0, 0, 1}, {0, 1, 1, 0}, {0, 0, 1, 1}, {1, 1, 0, 0}},
 {{0, 0, 0, 1}, {0, 0, 1, 1}, {0, 1, 0, 0}, {1, 1, 1, 0}},
 {{0, 0, 1, 1}, {0, 1, 0, 1}, {0, 1, 1, 0}, {1, 1, 1, 0}}}
```

In[ ]:=
```
(achNumber /@ fourByFourSample) // AbsoluteTiming
```
Out[ ]=
```
{565.683, {2, 2, 2, 2}}
```

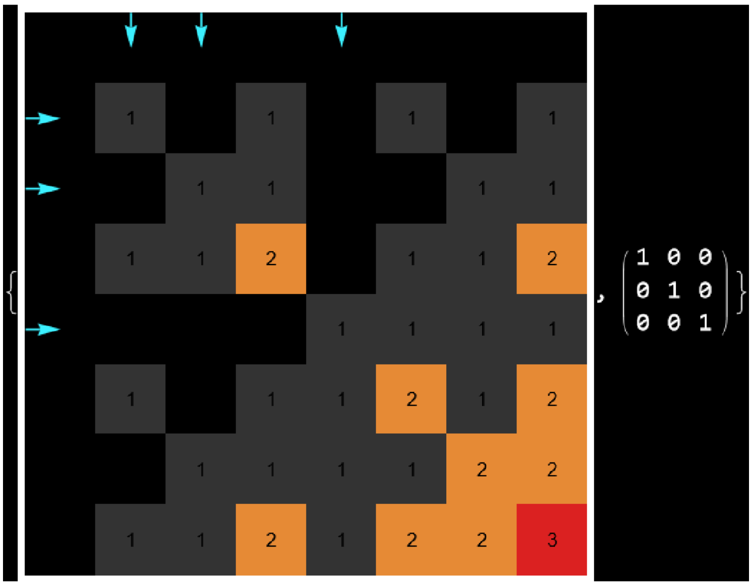
In[ ]:=
```
matrixPicture[B01_] :=
 Module[{d = Length@B01, cube, final, pos1, pos2}, cube = Tuples[{0, 1}, d];
  pos1 = Flatten[Position[cube, #] & /@ (B01^T)];
  pos2 = Flatten[Position[cube, #] & /@ B01];
  final = cube.Inverse@B01.(cube^T);
  Style[{ArrayPlot[final, ColorRules → {0 → ■, 1 → ■},
      ColorFunction → (ColorData["Rainbow", (# + 1)/2] &),
      Epilog → {Black, MapIndexed[Text[#1, Reverse[#2 - 0.5]] &, Reverse[final], {2}],
        ■, Arrow[{{# - 0.5, 2^d}, {# - 0.5, 2^d - 1/2}}] & /@ pos1,
        Arrow[{{0, 2^d - # + 0.5}, {0.5, 2^d - # + 0.5}}] & /@ pos2},
      ImageSize → 300, Frame → None, Background → White, PlotRangePadding → 0.1],
    Style[B01 // MatrixForm, White]}, White, Background → Black]]
```

In[ ]:=
```
matrixPicture[{{1, 0, 1}, {0, 1, 1}, {0, 0, 1}}] // Rasterize
```

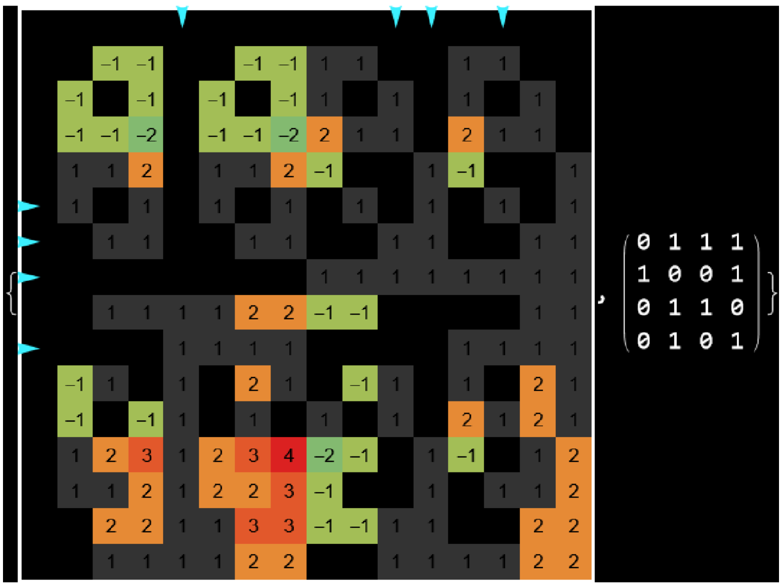
Out[ ]=



```
In[ ]:= matrixPicture[IdentityMatrix@3] // Rasterize
Out[ ]=
```



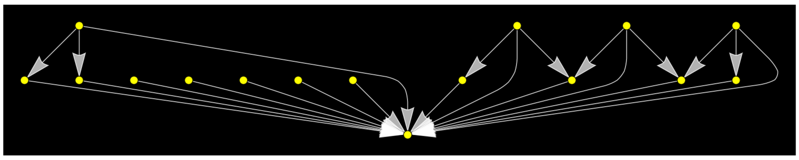

```
In[ ]:= matrixPicture@First[
          RandomChoice[Select[Tuples[{0, 1}, {3, 3}], MatrixRank@# == 3 &], 1]] // Rasterize

        imlist = Rasterize /@ matrixPicture /@ Select[Tuples[{0, 1}, {3, 3}], MatrixRank@# == 3 &];

In[ ]:= matrixPicture@IdentityMatrix[4] // Rasterize

In[ ]:= NestWhile[RandomChoice[{0, 1}, {4, 4}] &, {{0}}, MatrixRank[#] ≠ 4 &]
Out[ ]=
        {{0, 1, 1, 1}, {1, 0, 0, 1}, {0, 1, 1, 0}, {0, 1, 0, 1}}

In[ ]:= matrixPicture@% // Rasterize
Out[ ]=
```




```
In[ ]:= Inverse@{{0, 1, 1, 1}, {1, 0, 0, 1}, {0, 1, 1, 0}, {0, 1, 0, 1}}
Out[ ]=
        {{-1, 1, 1, 0}, {-1, 0, 1, 1}, {1, 0, 0, -1}, {1, 0, -1, 0}}
```

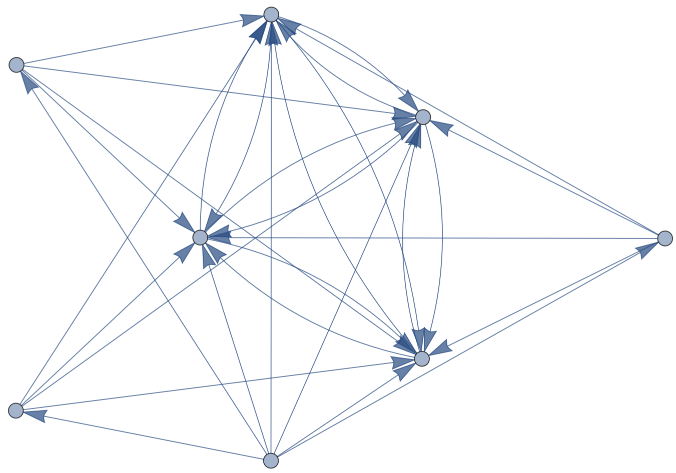
In[ ]:=
```
Graph[graphPicture@{{0, 1, 1, 1}, {1, 0, 0, 1}, {0, 1, 1, 0}, {0, 1, 0, 1}},
 Background → Black, VertexStyle → Yellow, EdgeStyle → White]
```

Out[ ]=



```
Graph[graphFullSym@IdentityMatrix[3]]
```

Out[ ]=



Check wether sets of “skewed dot products” coincide with all nonbinary inputs treated as same and no distinction between 0 and 1:

In[ ]:=
```
coincidingSets[B01_] := Module[{d = Length@B01, cube, final}, cube = Tuples[{0, 1}, d];
  final = cube.Inverse@B01.(cubeᵀ) /. {1 → 0, x_ /; x ≠ 0 → 2};
  Map[Sort, final, {0, -2}] == Map[Sort, finalᵀ, {0, -2}]]
```

Check on invertible d*d matrices for d from 2 to 4:

In[ ]:=
```
Table[
 And @@ coincidingSets /@ Select[Tuples[{0, 1}, {d, d}], MatrixRank@# == d &], {d, 2, 4}]
```

Out[ ]=
```
{True, True, True}
```

In[ ]:=
```
randomInvertible[d_] :=
  NestWhile[RandomChoice[{0, 1}, {d, d}] &, {{0}}, MatrixRank[#] ≠ d &]
```

In[ ]:=
```
randomInvertible[5]
```

Out[ ]=
```
{{1, 1, 1, 0, 1}, {1, 0, 0, 1, 0}, {0, 1, 0, 0, 0}, {1, 0, 0, 1, 1}, {1, 0, 0, 0, 1}}
```

In[ ]:=
```
NotebookDelete[temp]
```

Out[ ]=
```
NotebookDelete[temp]
```

In[ ]:=
```
cnt = 0;
SeedRandom[42];
NestWhile[If[Mod[cnt, 1] == 0, NotebookDelete[temp];
   temp = PrintTemporary[cnt]];
  cnt++;
  randomInvertible[4] &, {{1}}, coincidingSets[#] &]
```

Out[ ]=
```
$Aborted
```

In[ ]:=
```
counterexample5 =
  {{0, 0, 1, 1, 1}, {0, 1, 0, 0, 0}, {1, 0, 0, 1, 1}, {1, 1, 0, 1, 0}, {1, 0, 0, 0, 0}}
```

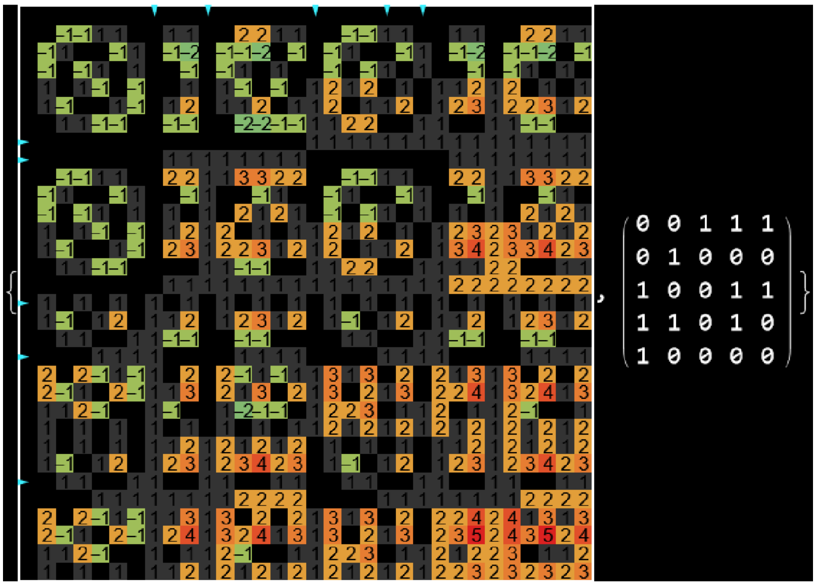
Out[ ]=
```
{{0, 0, 1, 1, 1}, {0, 1, 0, 0, 0}, {1, 0, 0, 1, 1}, {1, 1, 0, 1, 0}, {1, 0, 0, 0, 0}}
```

In[ ]:=
```
coincidingSets@counterexample5
```

Out[ ]=
```
False
```

In[ ]:=
```
matrixPicture@counterexample5 // Rasterize
```

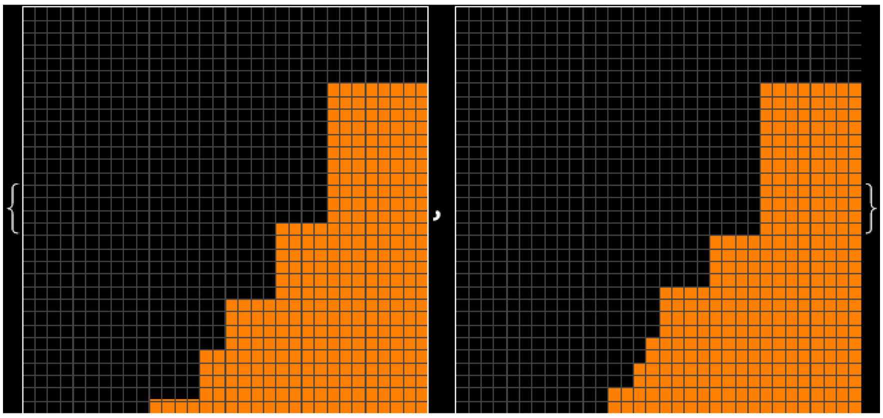
Out[ ]=



In[ ]:=
```
cube01[d_] := Tuples[{0, 1}, d]
```

In[∘]:=
```
Style[With[{cube = Tuples[{0, 1}, 5]},
    ArrayPlot[Map[Sort, (cube.Inverse[#].(cubeᵀ)) /. {1 → 0, x_ /; x ≠ 0 → 2}, {0, -2}],
     Frame → True, FrameStyle → White, Background → Black, PlotRangePadding → None,
     Mesh → True, MeshStyle → Directive[■, Thickness[10^-4]],
     ColorRules → {0 → Black, 2 → Orange}]] & /@
   {counterexample5, counterexample5ᵀ}, White, Background → Black] // Rasterize
```

Out[∘]=



In[∘]:=
```
Inverse@counterexample5
```

Out[∘]=
```
{{0, 0, 0, 0, 1}, {0, 1, 0, 0, 0}, {1, 0, -1, 0, 1}, {0, -1, 0, 1, -1}, {0, 1, 1, -1, 0}}
```

Function to produce graph of allowable relations given symmetric invertible 01 matrix:

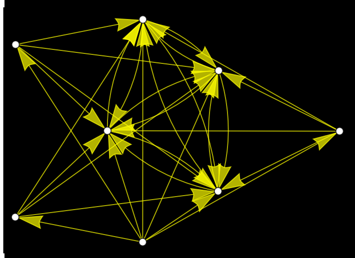
In[∘]:=
```
graphFullSym[B01_] := Module[{d = Length@B01, cube, matr}, cube = Tuples[{0, 1}, d];
  matr = (cube.Inverse@B01.(cubeᵀ)) /. {1 → 0, x_ /; x ≠ 0 → 2};
  Flatten@Table[If[i ≠ j && And @@ NonNegative[matr⟦i⟧ - matr⟦j⟧],
     cube⟦i⟧ ↔ cube⟦j⟧, Nothing], {i, 2^d}, {j, 2^d}]]
```

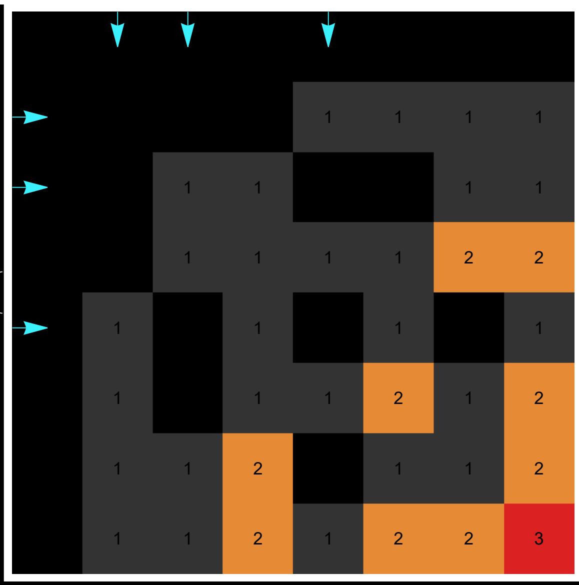
All symmetric invertible (without row permutations!) :

In[∘]:=
```
allSymInvertible[d_] :=
 DeleteDuplicatesBy[Select[(# + LowerTriangularize[#ᵀ, -1]) & /@
    PadLeft /@ (Internal`PartitionRagged[#, Range[d, 1, -1]] &) /@
     Tuples[{0, 1}, d (d + 1)/2], MatrixRank@# == d &], Sort]
```

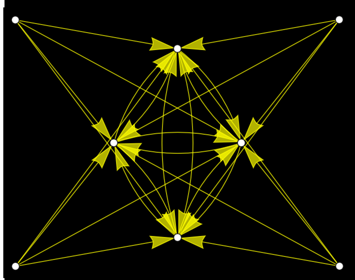
In[∘]:=
```
Style[
 Grid@DeleteDuplicates[{Graph[graphFullSym@#, Background → Black, VertexStyle → White,
       EdgeStyle → Yellow], matrixPicture@#} & /@ allSymInvertible[3],
   IsomorphicGraphQ[#1⟦1⟧, #2⟦1⟧] &], Background → Black, White]
```

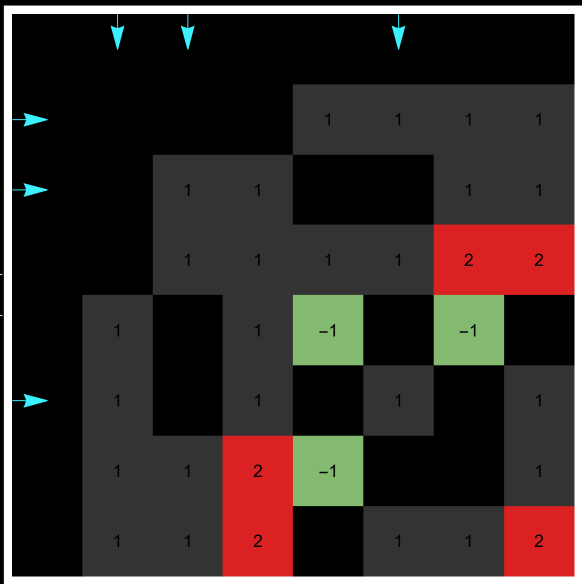
Out[ ]=



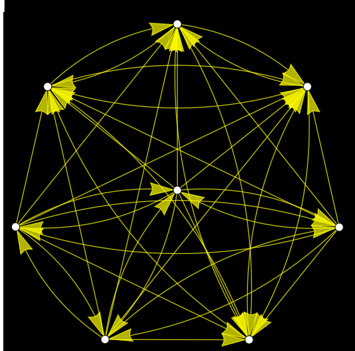


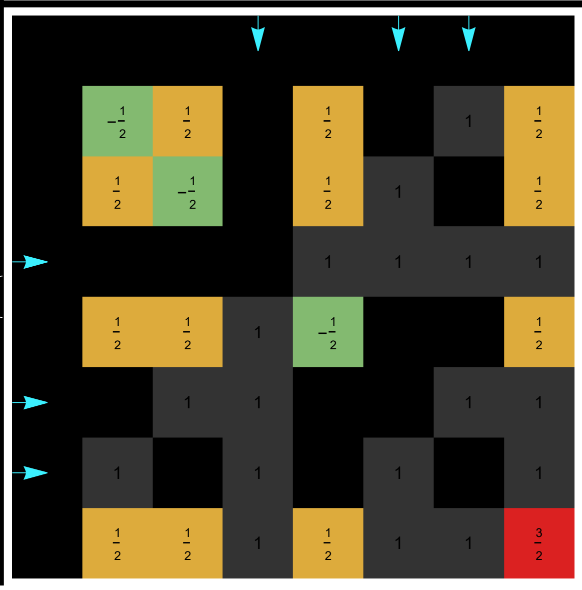
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$





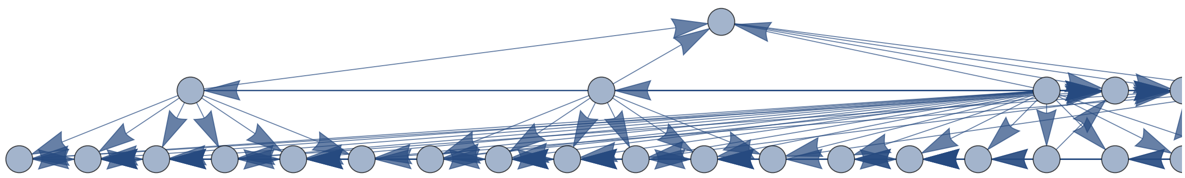
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



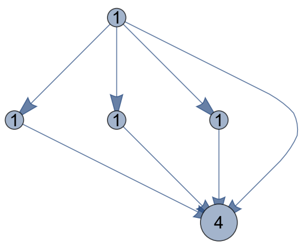


$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

```
In[ ]:= Inverse@ (0 0 1
                  0 1 0
                  1 0 1)
Out[ ]=
{{-1, 0, 1}, {0, 1, 0}, {1, 0, 0}}

In[ ]:= Style[
         Grid@DeleteDuplicates[{Graph[graphFullSym@#, Background → Black, VertexStyle → White,
                EdgeStyle → Yellow], matrixPicture@#} & /@ allSymInvertible[4],
           IsomorphicGraphQ[#1[[1]], #2[[1]]] &], Background → Black, White]

In[ ]:= {Graph[graphFullSym[counterexample5]], Graph[graphFullSym[counterexample5ᵀ]]}

In[ ]:= graphSymNoSinks[B01_] := Module[{d = Length@B01, filtcube, matr, pos},
          filtcube = Rest@DeleteCases[Tuples[{0, 1}, d], Alternatives @@ B01];
          matr = (filtcube.Inverse@B01.(filtcubeᵀ)) /. {1 → 0, x_ /; x ≠ 0 → 2};
          Flatten@Table[If[i ≠ j && And @@ NonNegative[matr[[i]] - matr[[j]]],
              filtcube[[i]] ↔ filtcube[[j]], Nothing], {i, 2^d - d - 1}, {j, 2^d - d - 1}]]

In[ ]:= Graph[graphSymNoSinks@IdentityMatrix@5, GraphLayout → "LayeredEmbedding"]
Out[ ]=
```



```
In[ ]:= graphSymFactorizedAndSzs[B01_] :=
         Module[{d = Length@B01, cube, grps, filt = (# /. {1 → 0, x_ /; x ≠ 0 → 2} &)},
          cube = Tuples[{0, 1}, d];
          grps = {#, filt[#[[1]]]} & /@ GatherBy[cube.Inverse@B01.(cubeᵀ), filt];
          {Flatten@Table[If[i ≠ j && And @@ NonNegative[grps[[i, 2]] - grps[[j, 2]]],
               grps[[i, 1]] ↔ grps[[j, 1]], Nothing], {i, Length@grps}, {j, Length@grps}],
           Normal@AssociationMap[Length, grps[[All, 1]]]}]

In[ ]:= With[{gsf = graphSymFactorizedAndSzs[IdentityMatrix[3]]},
         Graph[First@gsf, VertexLabels → MapAt[Placed[#, Center] &, gsf[[2]], {All, 2}],
          VertexSize → MapAt[{"Scaled", Sqrt[#]/20} &, gsf[[2]], {All, 2}]]]
Out[ ]=
```




```
In[ ]:= Get["https://raw.githubusercontent.com/szhorvat/IGraphM/master/IGInstaller.m"]
```

The currently installed versions of IGraph/M are: {}

Installing IGraph/M is complete: PacletObject[ Name: IGraphM Version: 0.6.5].

It can now be loaded using the command << IGraphM`

In[∘]:=
```
<< IGraphM`
```

Out[∘]=

IGraph/M 0.6.5 (December 21, 2022)
Evaluate IGDocumentation[] to get started.

Remove[Global`IGVertexMap]
Remove[Global`IGVertexProp]

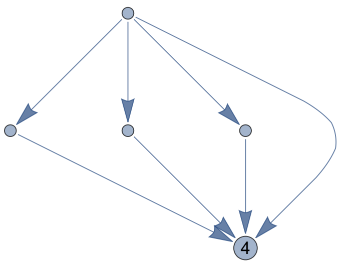
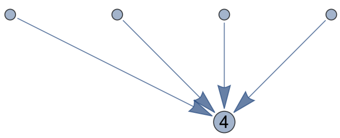
In[∘]:=
```
graphSymFactorizedRaw[B01_] :=
 Module[{d = Length@B01, cube, grps, filt = (# /. {1 → 0, x_ /; x ≠ 0 → 2} &)},
  cube = Tuples[{0, 1}, d];
  grps = {#, filt[#[[1]]]} & /@ GatherBy[cube.Inverse@B01.(cubeᵀ), filt];
  Flatten@Table[If[i ≠ j && And @@ NonNegative[grps[[i, 2]] - grps[[j, 2]]],
     grps[[i, 1]] ⟷ grps[[j, 1]], Nothing], {i, Length@grps}, {j, Length@grps}]]
hasseDiagram[gr_] := With[{f = EdgeQ[gr, #1 → #2] &, s = VertexList[gr]},
  ResourceFunction["HasseDiagram"][f, s, PerformanceGoal → "Quality"]]
```

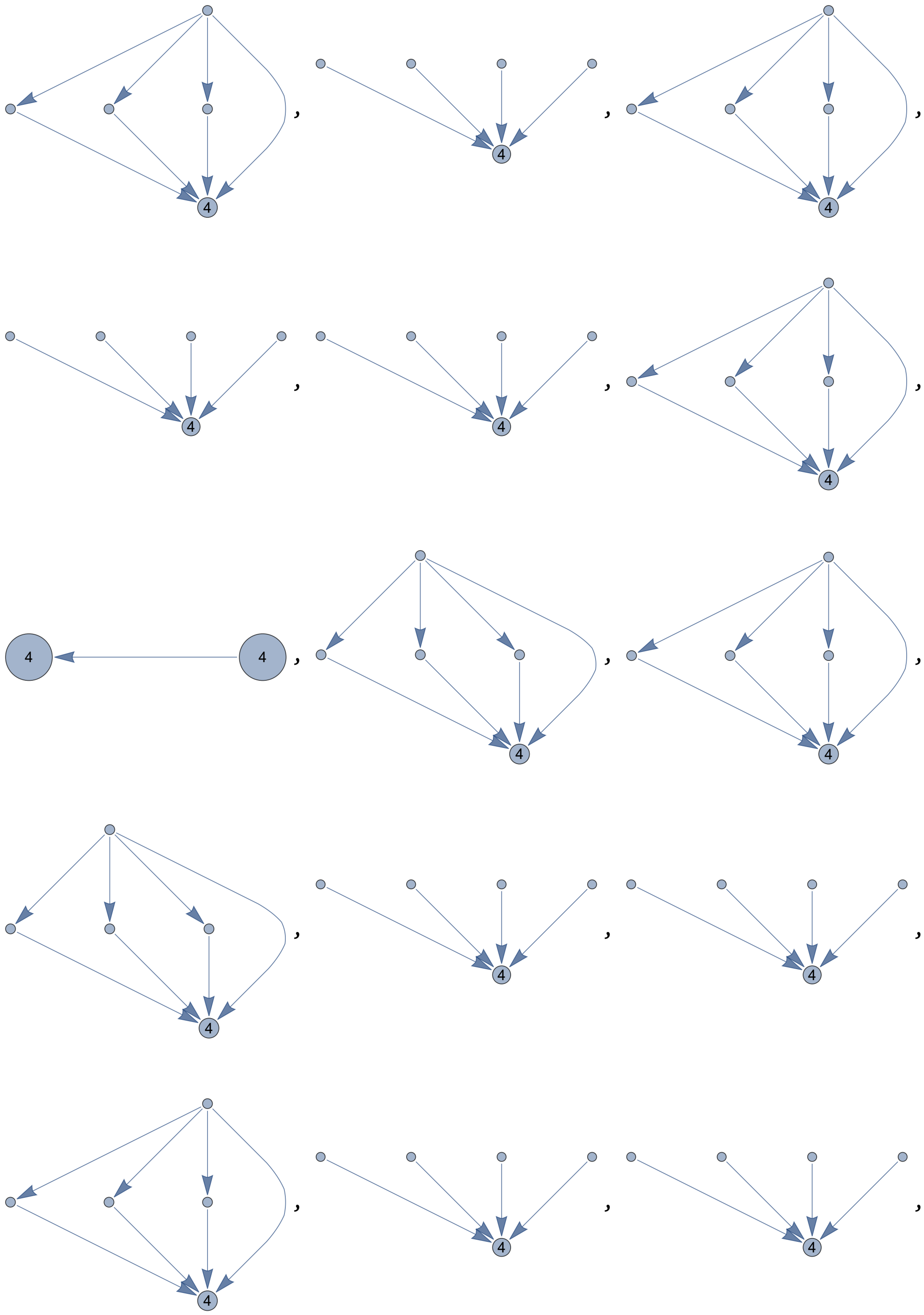
In[∘]:=
```
addGraphProps[gr_] := Module[{filt = (# /. {1 → 0, x_ /; x ≠ 0 → 2} &)},
  gr // IGVertexMap[Sqrt@*Length, "SqLength" → VertexList] /*
    IGVertexMap[Length, "Length" → VertexList] /* IGVertexMap[filt[#[[1]]] &,
     "Filter" → VertexList] /* IGVertexMap[MatrixForm[#ᵀ] &, Tooltip → VertexList]]
grSymFact[B01_] :=
 addGraphProps[Graph[graphSymFactorizedRaw[B01], PerformanceGoal → "Quality"]]
hasseSymFact[B01_] := addGraphProps[
  hasseDiagram@Graph[graphSymFactorizedRaw[B01], PerformanceGoal → "Speed"]]
```

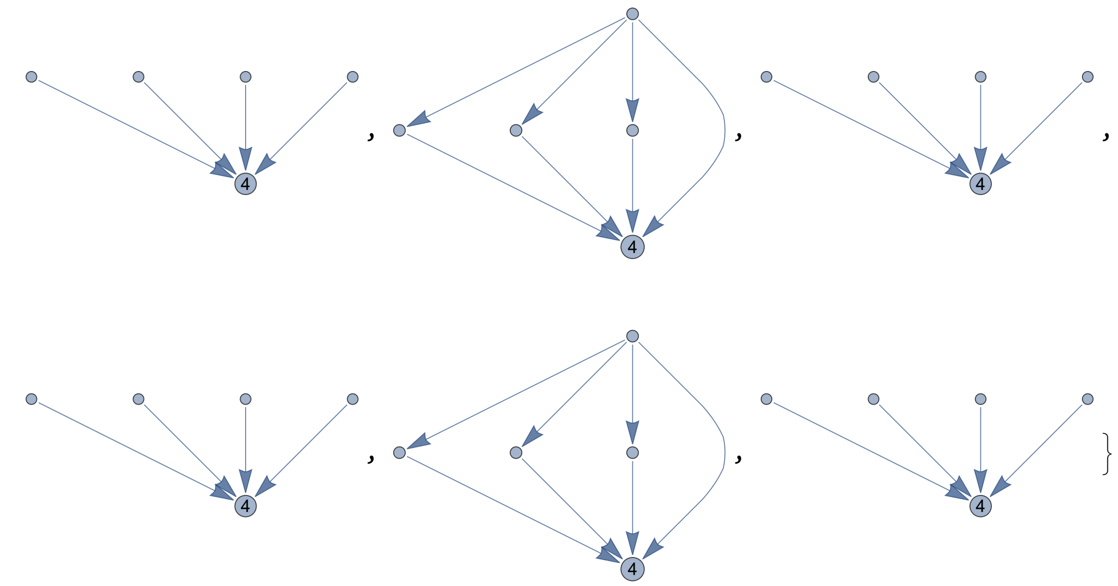
In[∘]:=
```
dispFact[gr_] :=
 gr // IGVertexMap[0.1 # &, VertexSize → IGVertexProp["SqLength"]] /* IGVertexMap[
    Placed[If[# == 1, "", #], Center] &, VertexLabels → IGVertexProp["Length"]]
```

In[∘]:=
```
dispFact[grSymFact[#]] & /@ allSymInvertible[3]
```

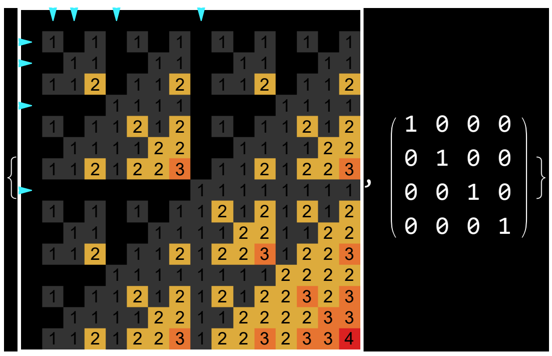
Out[∘]=

{  ,  ,

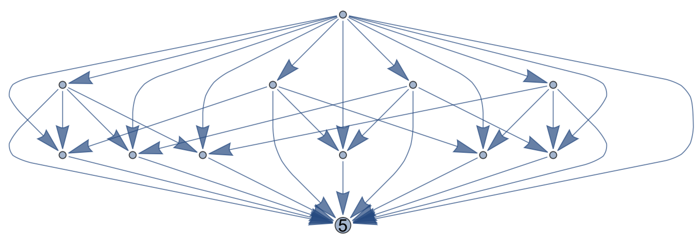




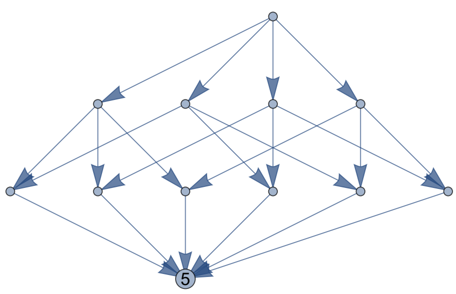
In[∘]:=
```
matrixPicture@IdentityMatrix[4]
```
Out[∘]=



In[∘]:=
```
dispFact[grSymFact[IdentityMatrix[4]]]
```
Out[∘]=



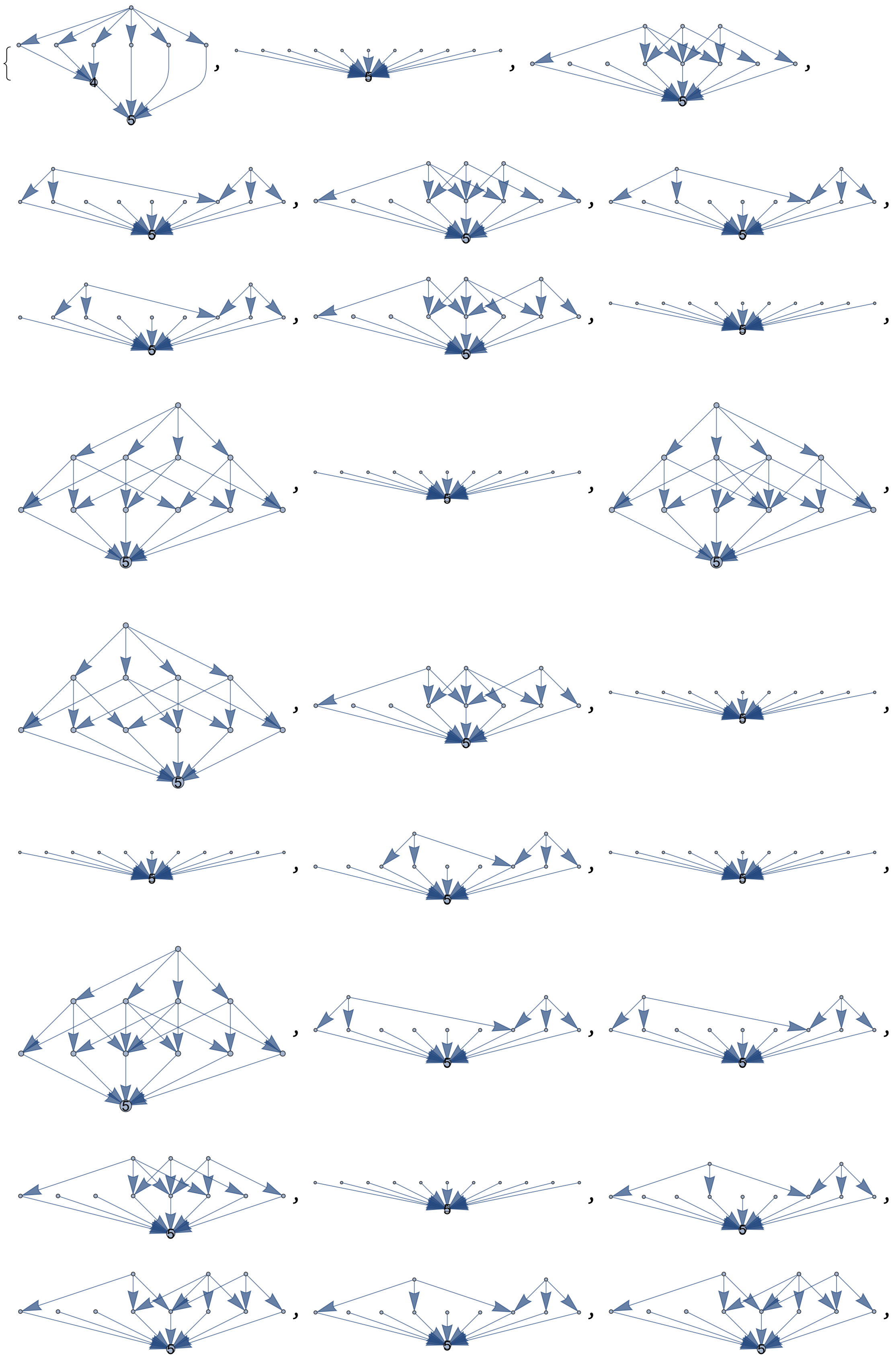
In[∘]:=
```
dispFact[hasseSymFact[IdentityMatrix@4]]
```
Out[∘]=

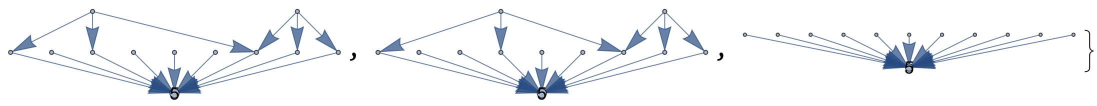


In[ ]:=

```
(dispFact@*hasseSymFact) /@ RandomSample[allSymInvertible[4], 30]
```

Out[ ]=



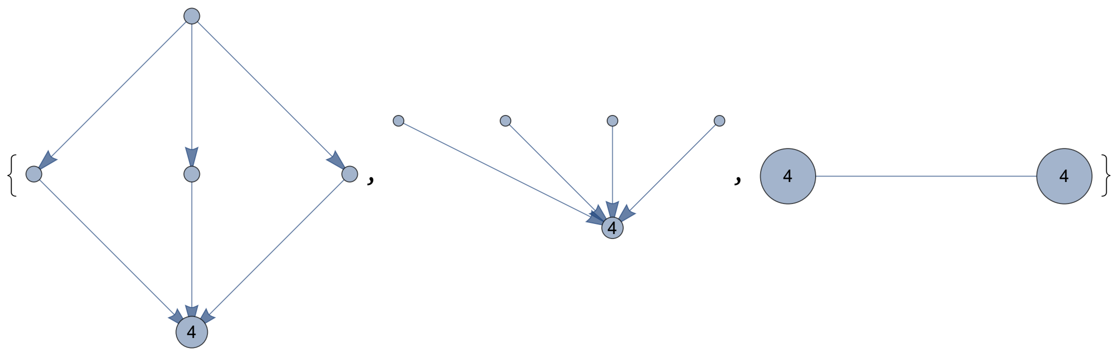


In[∘]:=
```
deepSort[expr_] := Map[Sort, expr, {0, -2}]
```

In[∘]:=
```
allSymInvertibleDiffGraph[d_] :=
 First /@ DeleteDuplicates[{#, Graph@graphFullSym@#} & /@ allSymInvertible[d],
   IsomorphicGraphQ[#1⟦2⟧, #2⟦2⟧] &]
```

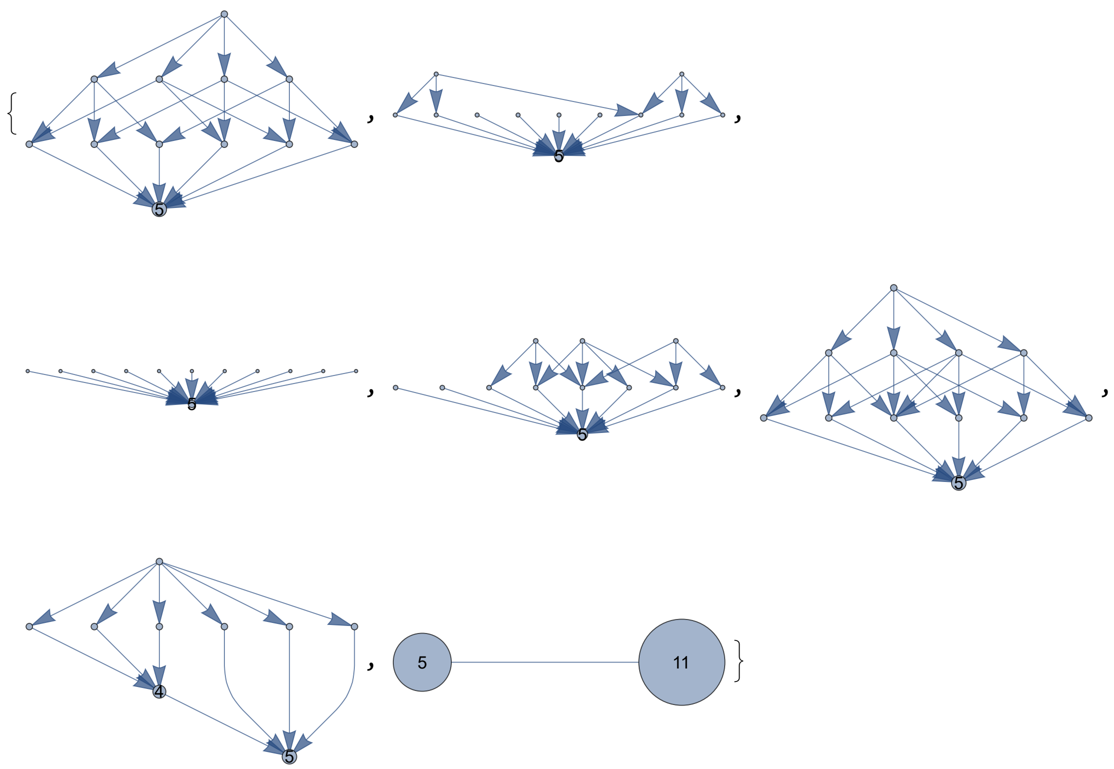
In[∘]:=
```
allThreeHasse = dispFact /@ hasseSymFact /@ allSymInvertibleDiffGraph[3]
```

Out[∘]=



In[∘]:=
```
allFourHasse = dispFact /@ hasseSymFact /@ allSymInvertibleDiffGraph[4]
```

Out[∘]=



In[∘]:=
```
setGraphStyle[graph_Graph] := Graph[graph, EdgeStyle → Yellow,
  VertexStyle → White, Background → Black, PerformanceGoal → "Quality"]
```

```
Grid[ArrayReshape[setGraphStyle /@ (allFourHasse ~ Join ~ allThreeHasse), {5, 2}],
 Background → Black]
```

```
In[ ]:= Export[FileNameJoin[{NotebookDirectory[], "All-Hasse-3-4-diagrams.jpg"}],
```

```
In[ ]:= allHasse5withDup = ParallelMap[hasseSymFact, allSymInvertible[5]];
```

```
In[ ]:= allFiveHasse = DeleteDuplicates[allHasse5withDup, IsomorphicGraphQ];
```

```
Grid[ArrayReshape[setGraphStyle /@ (allFourHasse ~ Join ~ allThreeHasse), {5, 2}],
 Background → Black]
```

In[ ]:=
```
DumpSave[FileNameJoin[{NotebookDirectory[], "all-sym-hasse-3.mx"}], allThreeHasse];
DumpSave[FileNameJoin[{NotebookDirectory[], "all-sym-hasse-4.mx"}], allFourHasse];
DumpSave[FileNameJoin[{NotebookDirectory[], "all-sym-hasse-5.mx"}], allFiveHasse];
```

In[ ]:=
```
MapIndexed[Export[FileNameJoin[{NotebookDirectory[],
      "all-sym-hasse-5-pictures", ToString[#2[[1]]] <> ".jpg"}], #1] &,
  setGraphStyle[Graph[dispFact@#, ImageSize → 800]] & /@ allFiveHasse];
```

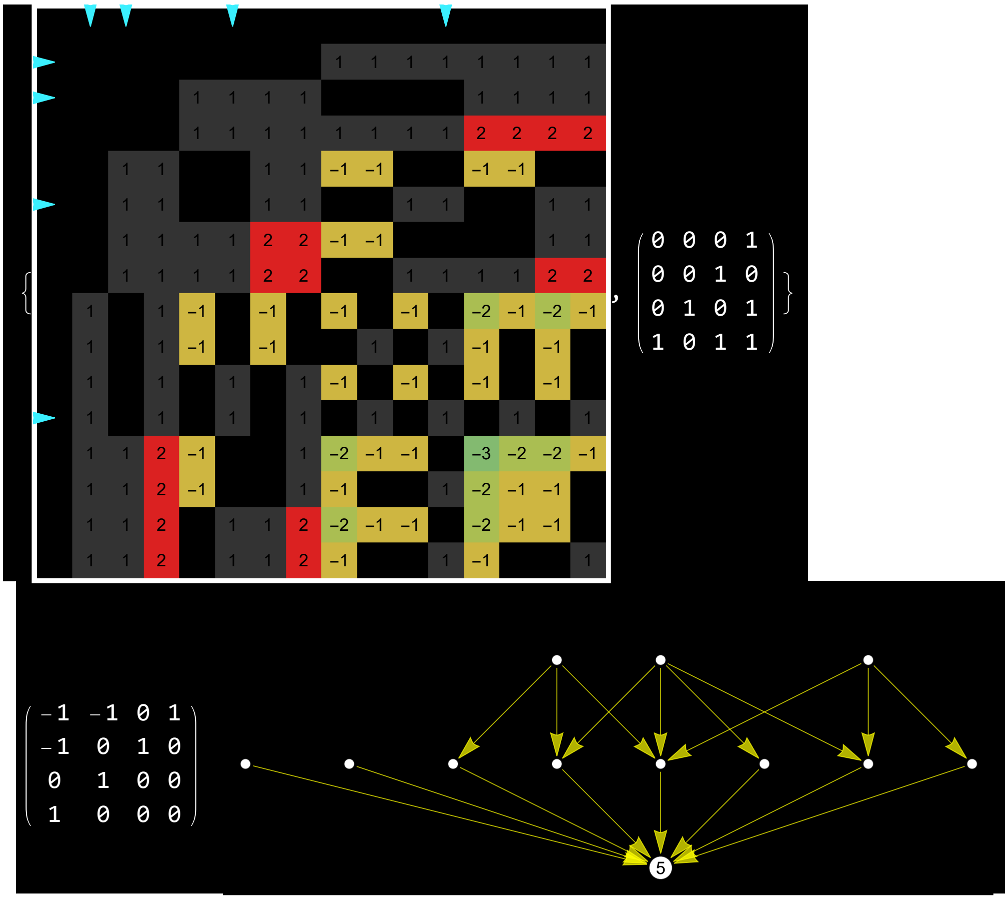
There is an example that is not even a semilattice among 4 by 4 matrices

In[ ]:=
```
nonSemilatticecCounter = allSymInvertibleDiffGraph[4][[4]]
```

Out[ ]=
```
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 1}, {1, 0, 1, 1}}
```

In[ ]:=
```
Style[{matrixPicture[nonSemilatticecCounter],
  Style[MatrixForm@Inverse@nonSemilatticecCounter, White],
  Graph[setGraphStyle@dispFact@hasseSymFact@nonSemilatticecCounter,
   ImageSize → 400]}, Background → Black]
```

Out[ ]=



In[ ]:=
```
Export[FileNameJoin[{NotebookDirectory[], "non-semilattice-4.jpg"}], %];
```

In[ ]:=
```
With[{c = cube01[4], ctr = Inverse@nonSemilatticecCounter},
 {MatrixForm[c.ctr.(cᵀ)], MatrixForm@c, MatrixForm@ctr, MatrixForm@(cᵀ)}]
```

Unlike the simple cube case there might be more then d+1 vectors that can lie in intersection (meet) of A and B; the following function counts the number of such vectors.

```
In[ ]:= selfCompatibleCount[B01_] :=
         With[{c = Tuples[{0, 1}, Length@B01]}, Count[Diagonal[c.Inverse[B01].(cᵀ)], 0 | 1]]
```

```
In[ ]:= Union[selfCompatibleCount /@ allSymInvertible[3]]
Out[ ]=
        {4, 6}
```

```
In[ ]:= Union[selfCompatibleCount /@ allSymInvertible[4]]
Out[ ]=
        {5, 6, 8, 9, 10}
```

```
In[ ]:= Union[selfCompatibleCount /@ allSymInvertible[5]]
Out[ ]=
        {6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24}
```

In case of 4 by 4 matrices numbers of possible self-compatible vectors reduces when only considering matrices that produce non-isomorphic graphs. So there are matrices producing isomorphic graphs but having different-sized self-compatible sets.

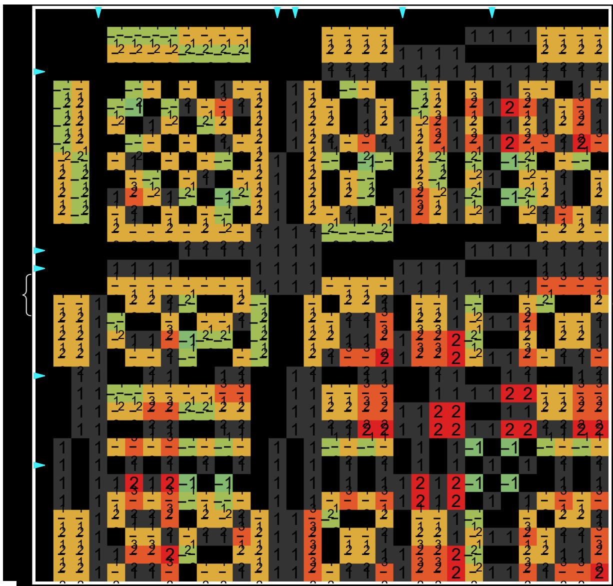
A matrix with many self - compatible vecrors:

```
In[ ]:= manySelfCompatibleExample =
         Select[allSymInvertible[5], selfCompatibleCount@# == 24 &, 1] // First
Out[ ]=
        {{0, 0, 0, 1, 1}, {0, 1, 1, 0, 1}, {0, 1, 1, 1, 0}, {1, 0, 1, 0, 0}, {1, 1, 0, 0, 1}}
```

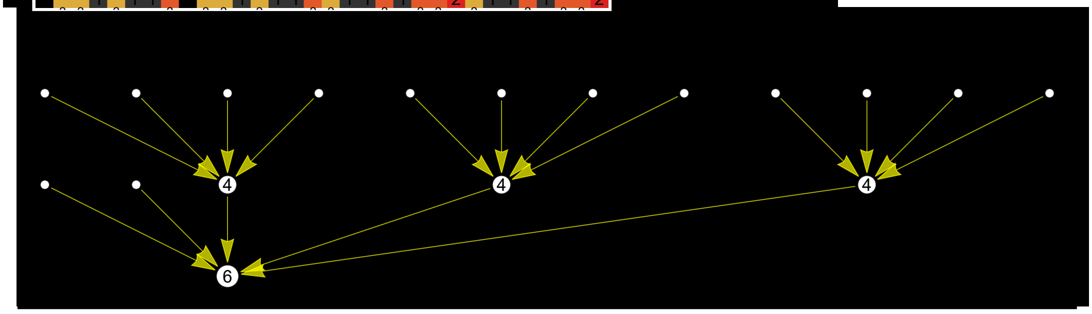
In[ ]:=
```
Labeled[Style[{matrixPicture[manySelfCompatibleExample], Graph[
    setGraphStyle@dispFact@hasseSymFact@manySelfCompatibleExample, ImageSize → 550]},
  Background → Black], Style["5×5 matrix producing 24 self-compatible vectors",
  White, Background → Black], Top]
```

Out[ ]=

5×5 matrix producing 24 self-compatible vectors



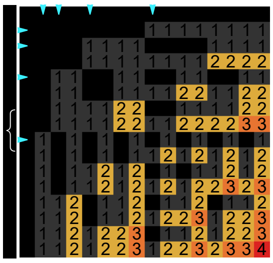
$$\left\{\ ,\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}\right\}$$



In[ ]:=
```
Export[FileNameJoin[{NotebookDirectory[], "many-self-compatible-example.jpg"}],
  %345, Background → Black];
```

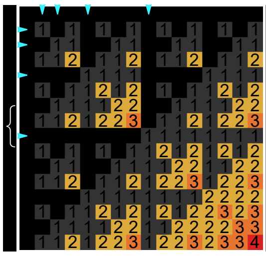
In[ ]:=
```
matrixPicture@allSymInvertible[4][[1]]
```

Out[ ]=



$$\left\{\ ,\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right\}$$

In[∘]:=
```
matrixPicture@IdentityMatrix[4]
```
Out[∘]=

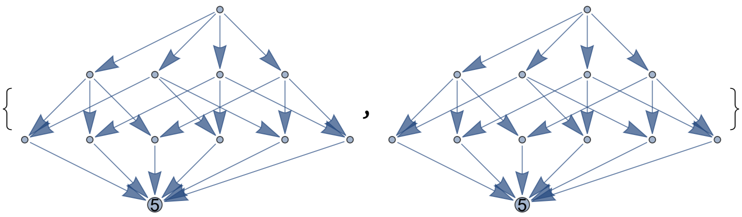


$$\left\{ \ldots, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

In[∘]:=
```
IsomorphicSubgraphQ @@
  (Graph /@ graphFullSym /@ {IdentityMatrix[4], Reverse[IdentityMatrix[4]]})
```
Out[∘]=
```
True
```

In[∘]:=
```
allSymPermutations[d_] :=
 Select[PermutationMatrix /@ Permutations[Range@d], SymmetricMatrixQ]
```

In[∘]:=
```
dispFact@addGraphProps@hasseSymFact@# & /@
 {IdentityMatrix[4], Reverse[IdentityMatrix[4]]}
```
Out[∘]=




```
posSelfComp3 = With[{as3p = allSymPermutations[3], as3in = allSymInvertible[3]},
   (m ↦ Sort[selfCompatibleCount /@ (m.# & /@ as3p)]) /@ as3in];

possibleSelfCompCountsGrouped3 = SortBy[Tally[posSelfComp3], First]
```
Out[∘]=
```
{{{4, 4, 4, 4}, 1}, {{4, 6, 7, 7}, 4}, {{5, 6, 6, 7}, 12}, {{5, 6, 7, 8}, 6}}
```

In[∘]:=
```
posSelfComp4 = With[{as4p = allSymPermutations[4], as4in = allSymInvertible[4]},
   (m ↦ Sort[selfCompatibleCount /@ (m.# & /@ as4p)]) /@ as4in];
```

```
In[ ]:= possibleSelfCompCountsGrouped4 = SortBy[Tally[posSelfComp4], First]
Out[ ]=
{{{5, 5, 5, 5, 5, 5, 5, 5, 5, 5}, 1}, {{5, 5, 5, 5, 8, 8, 8, 8, 8, 8}, 4},
 {{5, 5, 6, 7, 7, 7, 7, 8, 8, 8}, 4}, {{5, 9, 9, 9, 9, 9, 11, 11, 11, 11}, 1},
 {{5, 9, 9, 9, 10, 10, 10, 12, 12, 12}, 4}, {{6, 6, 6, 8, 10, 10, 10, 12, 12, 12}, 4},
 {{6, 6, 7, 7, 8, 10, 10, 11, 11, 12}, 24}, {{6, 6, 8, 8, 8, 10, 12, 12, 12, 12}, 12},
 {{6, 10, 10, 10, 10, 12, 12, 12, 12, 12}, 12}, {{6, 10, 10, 12, 12, 13, 14, 14, 15, 16}, 24},
 {{7, 9, 9, 10, 10, 10, 11, 11, 12, 13}, 24}, {{7, 9, 9, 10, 10, 11, 11, 11, 14, 14}, 3},
 {{7, 9, 9, 11, 11, 11, 12, 12, 12, 12}, 9}, {{7, 9, 10, 12, 12, 12, 13, 13, 14, 14}, 24},
 {{7, 10, 10, 10, 10, 11, 12, 12, 14, 14}, 9}, {{7, 10, 10, 11, 12, 12, 12, 12, 12, 12}, 3},
 {{8, 9, 10, 10, 10, 11, 12, 12, 14, 14}, 24}, {{8, 10, 10, 10, 11, 12, 12, 13, 14, 14}, 24},
 {{8, 10, 10, 11, 11, 12, 14, 14, 14, 14}, 24}, {{9, 9, 9, 10, 10, 10, 11, 11, 11, 12}, 48},
 {{9, 9, 10, 10, 10, 10, 11, 11, 11, 11}, 12}, {{9, 9, 10, 10, 10, 11, 11, 12, 12, 12}, 24},
 {{9, 9, 10, 10, 10, 11, 11, 12, 12, 14}, 6}, {{9, 9, 10, 10, 11, 11, 11, 11, 12, 14}, 24},
 {{10, 10, 10, 10, 10, 10, 11, 11, 12, 12}, 12},
 {{10, 10, 10, 10, 11, 11, 12, 14, 14, 14}, 12}}

In[ ]:= posSelfComp5 = With[{as5p = allSymPermutations[5], as5in = allSymInvertible[5]},
   ParallelMap[m ↦ Sort[selfCompatibleCount /@ (m.# & /@ as5p)], as5in]];

In[ ]:= possibleSelfCompCountsGrouped5 = SortBy[Tally[posSelfComp5], First]

In[ ]:= DumpSave[
  FileNameJoin[{NotebookDirectory[], "possible-self-comp-counts-grouped-3-to-5.mx"}],
  {possibleSelfCompCountsGrouped3,
   possibleSelfCompCountsGrouped4, possibleSelfCompCountsGrouped5}];
```

Best definitely achieveable by symmetric permutation size of self-comp set:

```
In[ ]:= Max[#[[All, 1, 1]]] & /@ {possibleSelfCompCountsGrouped3,
  possibleSelfCompCountsGrouped4, possibleSelfCompCountsGrouped5}
Out[ ]=
{5, 10, 16}

In[ ]:= only5selfcomp = Select[allSymInvertible[4],
  m ↦ (Union[selfCompatibleCount /@ (m.# & /@ allSymPermutations[4])] == {5})]
Out[ ]=
{{{0, 1, 1, 1}, {1, 0, 1, 1}, {1, 1, 0, 1}, {1, 1, 1, 0}}}

In[ ]:= MatrixForm@First@only5selfcomp
Out[ ]//MatrixForm=
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

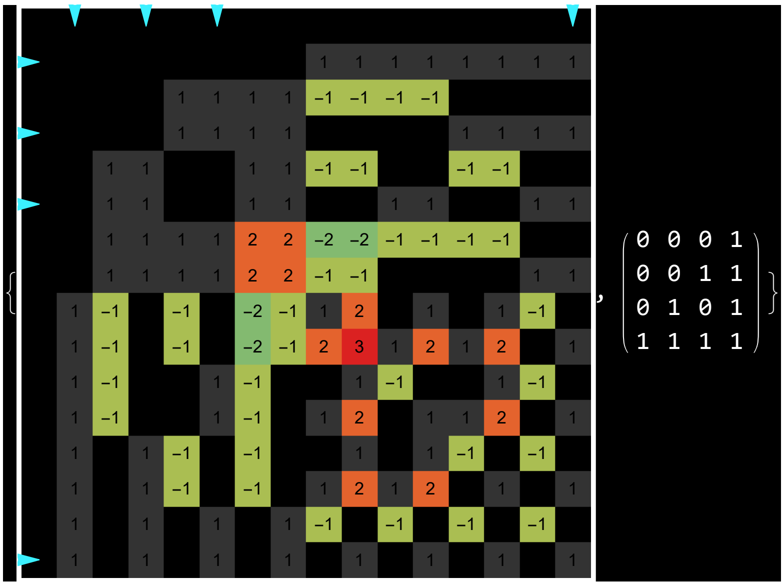
```
In[ ]:= With[{missing5 = Table[1, 5, 5] - IdentityMatrix[5]},
  Sort[selfCompatibleCount[missing5.#] & /@ allSymPermutations[5]]]
Out[ ]=
{6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12,
 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14}
```

In[∘]:=
```
unavoidablyManySelfComp4 = Select[allSymInvertible[4],
   m ↦ (Min[selfCompatibleCount /@ (m.# & /@ allSymPermutations[4])] == 10)] // First
```

Out[∘]=
```
{{0, 0, 0, 1}, {0, 0, 1, 1}, {0, 1, 0, 1}, {1, 1, 1, 1}}
```

In[∘]:=
```
matrixPicture[unavoidablyManySelfComp4]
```

Out[∘]=



Shit, product of symmetric permutations is only symmetric if they commute. The following function gives actual symmetric permutations of rows of a given (presumably symmetric) matrix:

In[∘]:=
```
symPermutations[B01_] := Select[
  B01.PermutationMatrix[#] & /@ Permutations[Range@Length@B01], SymmetricMatrixQ]
```

In[∘]:=
```
(selfCompatibleCount /@ symPermutations[#]) & /@ allSymInvertible[3]
```

Out[∘]=
```
{{6, 6, 6, 4}, {6}, {6}, {6}, {6}, {6}, {6}, {6}, {4, 4, 4, 4}, {4, 6},
 {6}, {4, 6}, {6}, {6}, {6}, {6}, {6}, {6}, {6}, {6}, {6}, {4, 6}, {6}}
```

In[∘]:=
```
(selfCompatibleCount /@ symPermutations[#]) & /@ allSymInvertible[2]
```

Out[∘]=
```
{{3, 3}, {3}, {3}}
```

In[∘]:=
```
(selfCompatibleCount /@ symPermutations[#]) & /@ allSymInvertible[1]
```

Out[∘]=
```
{{2}}
```

In[∘]:=
```
SortBy[Tally[Sort[selfCompatibleCount /@ symPermutations[#]] & /@ allSymInvertible[4]],
 #[[1, 1]] &]
```

Out[∘]=
```
{{{5, 5, 5, 5}, 4}, {{5, 9, 9, 9}, 4}, {{5, 5, 5, 5, 5, 5, 5, 5, 5, 5}, 1},
 {{5, 9, 9, 9, 9, 9, 9, 9, 9, 9}, 1}, {{6}, 24}, {{6, 8}, 12},
 {{6, 6, 6, 8}, 8}, {{9}, 144}, {{9, 9}, 30}, {{10}, 96}, {{10, 10}, 48}}
```

In[ ]:=
```
correctSymPermSelfCompCount5 =
 SortBy[Tally[ParallelMap[Sort[selfCompatibleCount /@ symPermutations[#]] &,
    allSymInvertible[5]]], #[[1, 1]] &]
```
Out[ ]=
```
{{{6, 6}, 30}, {{6, 8}, 120}, {{6, 6, 6, 6}, 20},
 {{6, 8, 8, 8, 8, 8}, 12}, {{6, 10, 10, 10, 10, 10}, 12},
 {{6, 6, 6, 6, 6, 6, 6, 6}, 10}, {{6, 6, 6, 8, 8, 8, 8, 8, 8, 10}, 10},
 {{6, 12, 12, 12, 12, 12, 12, 18, 18, 18}, 5}, {{6, 12, 12, 12, 12, 12, 12, 12,
   12, 12, 12, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14}, 1},
 {{6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 18, 18, 18, 18, 18, 18, 18, 18, 18,
   18, 18, 18, 18, 18, 18}, 1}, {{7, 9}, 60}, {{7, 9, 9, 9}, 60}, {{8}, 240},
 {{8, 12}, 240}, {{8, 8, 8, 10}, 20}, {{8, 8, 8, 12}, 20}, {{8, 12, 12, 12}, 30},
 {{8, 8, 8, 12, 12, 12, 12, 12}, 10}, {{9}, 120}, {{10}, 120}, {{10, 10}, 60},
 {{10, 12}, 240}, {{11}, 720}, {{12}, 1320}, {{12, 12}, 420}, {{12, 14}, 120},
 {{12, 16}, 240}, {{12, 18}, 240}, {{12, 24}, 60}, {{12, 12, 12, 12}, 80},
 {{12, 18, 18, 18}, 20}, {{12, 12, 12, 12, 14, 14, 14, 14, 14, 14}, 5}, {{13}, 840},
 {{13, 17}, 240}, {{14}, 240}, {{15}, 120}, {{15, 19}, 300}, {{15, 19, 19, 19}, 60},
 {{16}, 120}, {{16, 20}, 120}, {{16, 16, 20, 20}, 30}, {{17}, 1080}, {{17, 17}, 60},
 {{18}, 2040}, {{18, 18}, 210}, {{19}, 3120}, {{20}, 840}, {{20, 20}, 120}}
```

Guarantied lower bound from permutations looks (below) look line Binomial[n+1,Floor[(n+1)/2]]

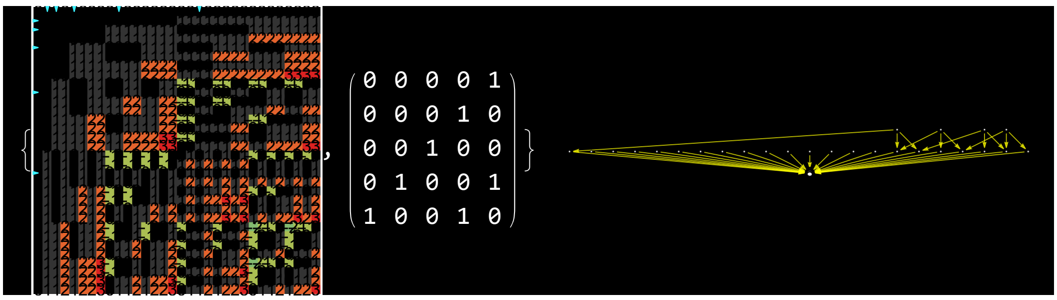
In[ ]:=
```
{1 → 2, 2 → 3, 3 → 6, 4 → 10, 5 → 20}
```

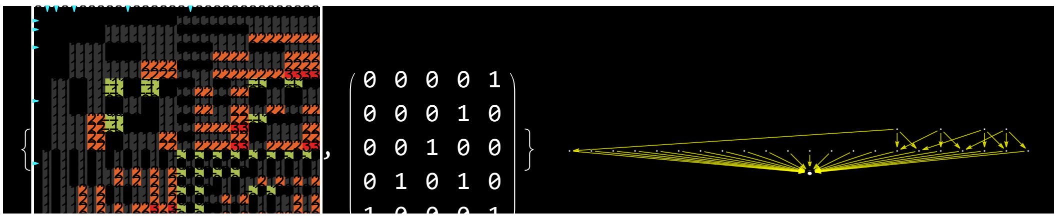
In[ ]:=
```
Table[Binomial[n, Floor[n / 2]], {n, 1, 7}]
```
Out[ ]=
```
{1, 2, 3, 6, 10, 20, 35}
```

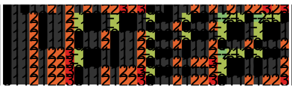
In[ ]:=
```
someAnavoidablyLargeSelfCount5 =
 Select[allSymInvertible[5], selfCompatibleCount /@ symPermutations[#] == {20, 20} &, 10]
```

In[ ]:=
```
Labeled[Column[Style[{matrixPicture[#] /. (ImageSize → _) → (ImageSize → 150),
      Graph[setGraphStyle@dispFact@hasseSymFact@#, ImageSize → 250]},
     Background → Black] & /@ someAnavoidablyLargeSelfCount5],
 Style["Some 5×5 matrices unavoidably producing 20 self-compatible
   vectors (two sym perms for each)", White, Background → Black], Top]
```
Out[ ]=

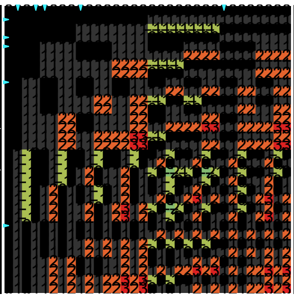
Some 5×5 matrices unavoidably producing
20 self-compatible vectors (two sym perms for each)



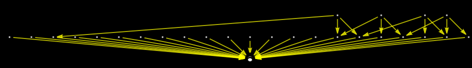


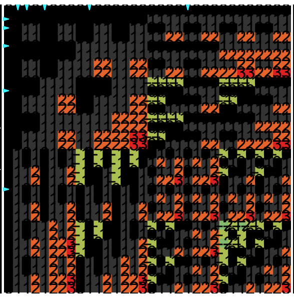


$\begin{matrix}1 & 0 & 0 & 0 & 1\end{matrix}$

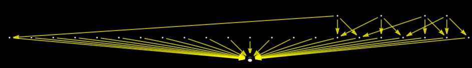


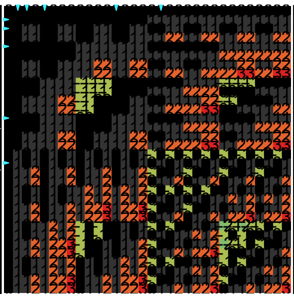
$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&0&0\\0&1&0&0&0\\1&1&0&0&0\end{pmatrix}\right\}$$



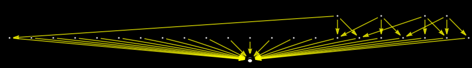


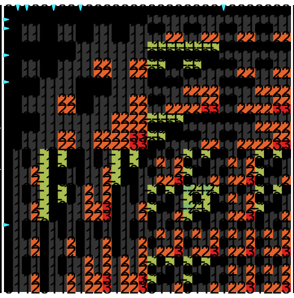
$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&0&1&0&0\\0&1&0&0&1\\0&0&0&1&0\\1&0&1&0&0\end{pmatrix}\right\}$$



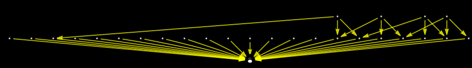


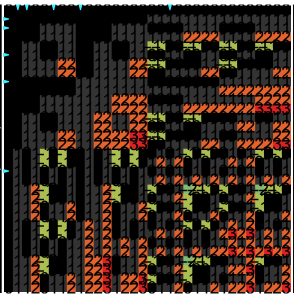
$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&0&1&0&0\\0&1&1&0&0\\0&0&0&1&0\\1&0&0&0&1\end{pmatrix}\right\}$$



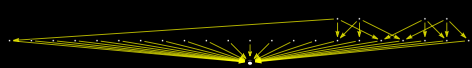


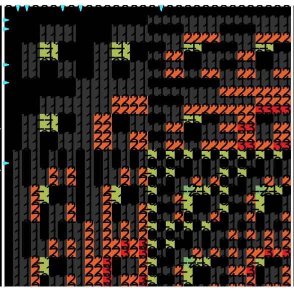
$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&0&1&0&1\\0&1&0&0&0\\0&0&0&1&0\\1&1&0&0&0\end{pmatrix}\right\}$$



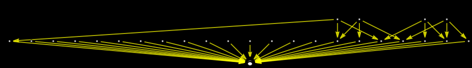


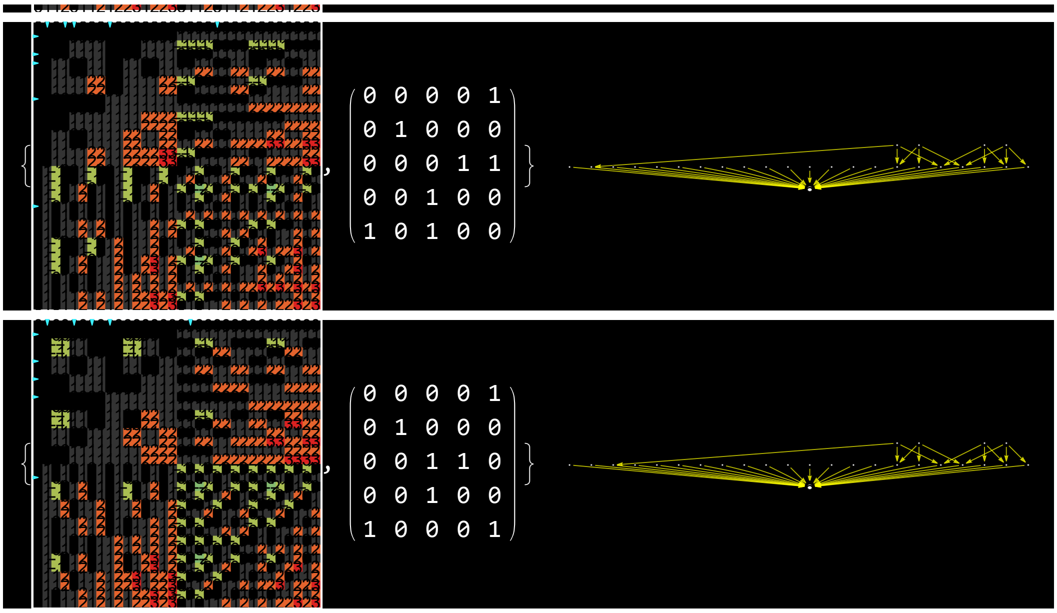
$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&1\\1&0&0&1&0\end{pmatrix}\right\}$$





$$\left\{\ ,\ \begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&1&0\\1&0&0&0&1\end{pmatrix}\right\}$$





In[ ]:=
```
Export[FileNameJoin[{NotebookDirectory[], "some-unavoidably-large-self-comp-5.jpg"}],
  %, Background → Black];
```

The last two are quite dissapointing:

In[ ]:=
```
Table[Sort[selfCompatibleCount /@ symPermutations[m]],
 {m, {{{0,0,1},{0,1,0},{1,0,1}},
      {{0,0,0,1},{0,0,1,0},{0,1,1,0},{1,0,0,1}},
      {{0,0,0,0,1},{0,0,0,1,0},{0,0,1,0,0},{0,1,0,1,0},{1,0,0,0,1}},
      {{0,0,0,0,0,1},{0,0,0,0,1,0},{0,0,0,1,0,0},{0,0,1,1,0,0},{0,1,0,0,1,0},{1,0,0,0,0,1}},
      {{0,0,0,0,0,0,1},{0,0,0,0,0,1,0},{0,0,0,0,1,0,0},{0,0,0,1,0,0,0},{0,0,1,0,1,0,0},{0,1,0,0,0,1,0},{1,0,0,0,0,0,1}},
      {{0,0,0,0,0,0,0,1},{0,0,0,0,0,0,1,0},{0,0,0,0,0,1,0,0},{0,0,0,0,1,0,0,0},{0,0,0,1,1,0,0,0},{0,0,1,0,0,1,0,0},{0,1,0,0,0,0,1,0},{1,0,0,0,0,0,0,1}}}}]
```

Out[ ]=
```
{{6}, {10, 10}, {20, 20}, {35, 35, 35, 35},
 {66, 66, 66, 70}, {110, 110, 110, 126, 126, 126, 126, 126, 126, 126}}
```

In[ ]:=
```
Table[Sort[selfCompatibleCount /@ symPermutations[m]], {m, {{{0,0,0,0,0,0,1},{0,0,0,0,0,1,1},{0,0,0,0,1,0,0},{0,0,0,1,0,0,0},{0,0,1,0,0,0,0},{0,1,0,0,0,0,0},{1,1,0,0,0,0,0}}}}]
```

Out[ ]=
```
{{44, 56, 62, 62, 62, 66, 66, 66}}
```

Trying to find the structure of a matrix that unavoidably produces Binomial[n,n/2] self compatible 0-1 vectors

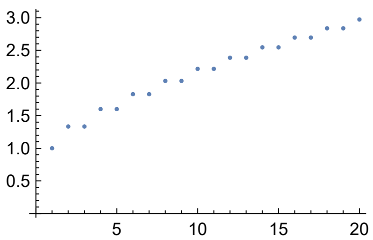
In[ ]:=
```
manydiagMatr[sz_, diagPos_ /; AllTrue[diagPos, NonNegative]] :=
 Total[Table[Reverse[DiagonalMatrix[Table[1, sz - p], p]], {p, diagPos}]]
```

In[ ]:=
```
Select[manydiagMatr[7, #] & /@ (Prepend[0] /@ Subsets[Range[6]]),
 selfCompatibleCount /@ symPermutations[#] == {70} &]
```
Out[ ]=
```
{{{0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 1, 0, 1}, {0, 0, 0, 1, 0, 1, 0},
  {0, 0, 1, 0, 1, 0, 1}, {0, 1, 0, 1, 0, 1, 0}, {1, 0, 1, 0, 1, 0, 1}},
 {{0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 1, 1, 1}, {0, 0, 0, 1, 1, 1, 1},
  {0, 0, 1, 1, 1, 1, 1}, {0, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}}}
```

In[ ]:=
```
MatrixForm /@ %
```
Out[ ]=

$$\left\{\begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&1&0&1&0\\ 0&0&1&0&1&0&1\\ 0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 1&1&1&1&1&1&1 \end{pmatrix}\right\}$$

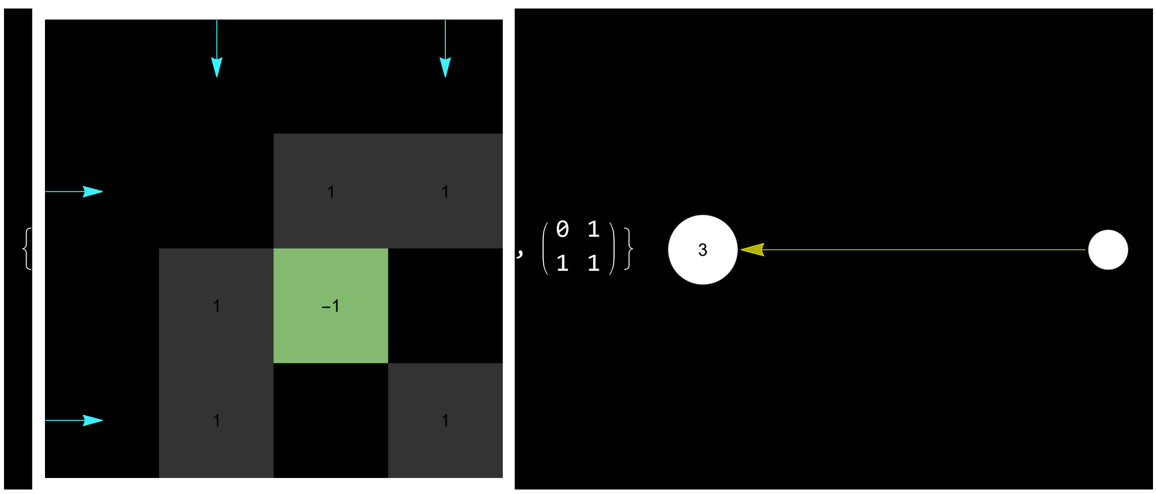
In[ ]:=
```
ListPlot[Table[2^n / Binomial[n + 1, Floor[(n + 1) / 2]], {n, 1, 20}]]
```
Out[ ]=

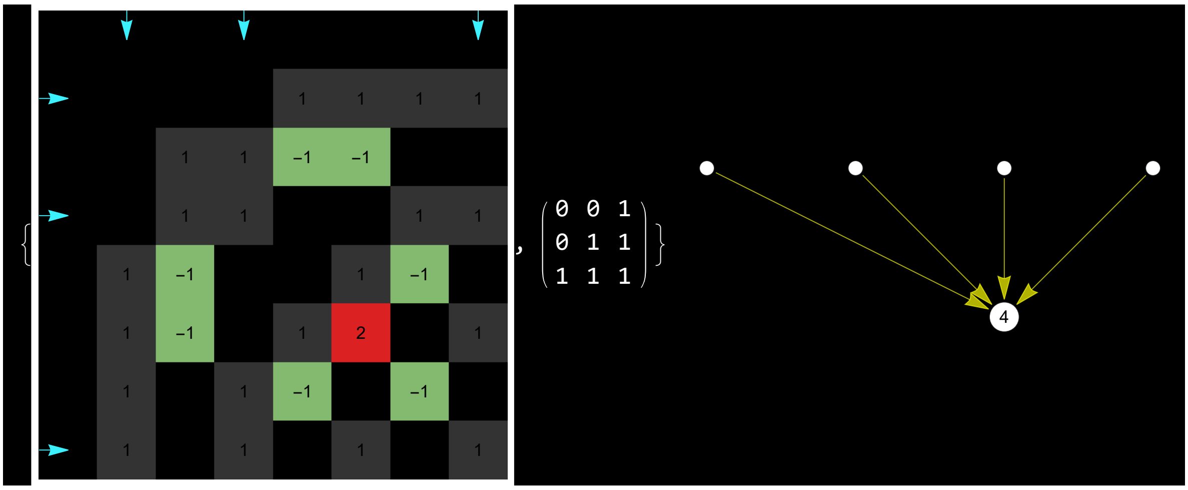


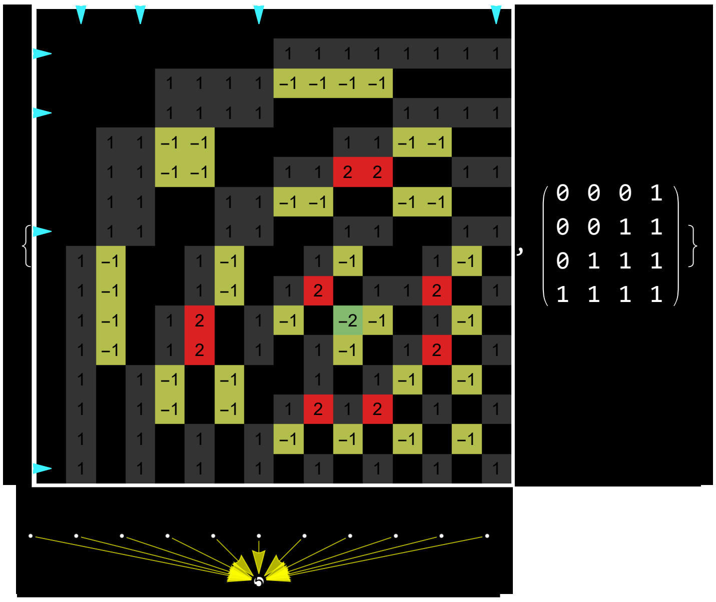
In[ ]:=
```
(selfCompatibleCount /@ symPermutations[#]) & /@
 ((manydiagMatr[#, Range[0, # - 1]] &) /@ Range[8])
```
Out[ ]=
```
{{2}, {3}, {6}, {10}, {20}, {35}, {70}, {126}}
```

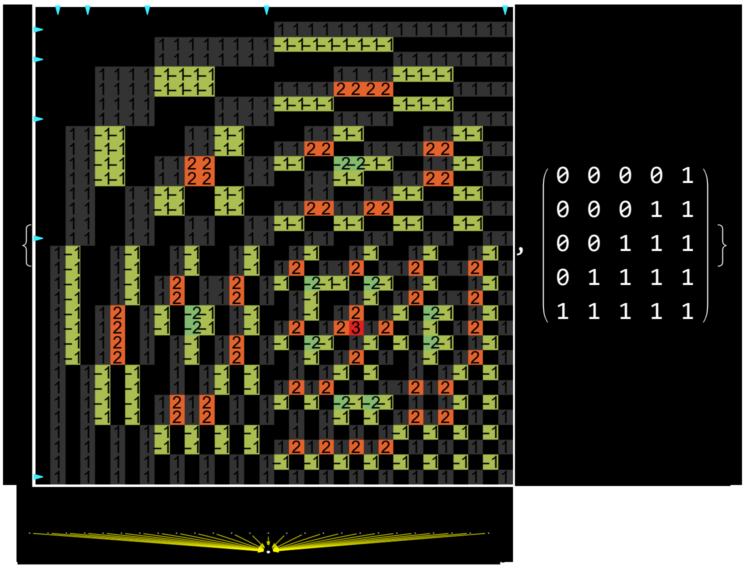
In[ ]:=
```
exportBlackBack[file_, expr_] :=
 Export[FileNameJoin@{NotebookDirectory[], file}, expr, Background → Black]
```

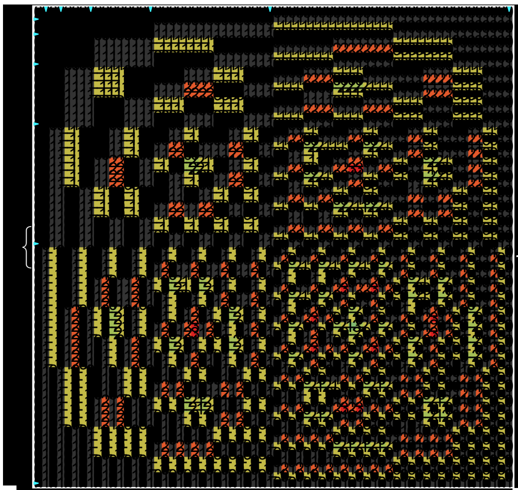
In[ ]:=
```
With[{halfmatr = (manydiagMatr[#, Range[0, # - 1]] &) /@ Range[2, 6]},
 Column[Style[{matrixPicture[#] /. (ImageSize → _) → (ImageSize → 250),
      Graph[setGraphStyle@dispFact@hasseSymFact@#, ImageSize → 250]},
     Background → Black] & /@ halfmatr]]
```
Out[ ]=



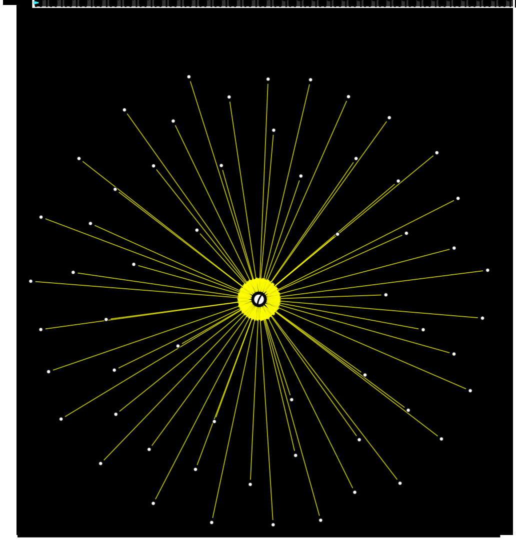

0 0 1
0 1 1
1 1 1
4


0 0 0 1
0 0 1 1
0 1 1 1
1 1 1 1


0 0 0 0 1
0 0 0 1 1
0 0 1 1 1
0 1 1 1 1
1 1 1 1 1



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

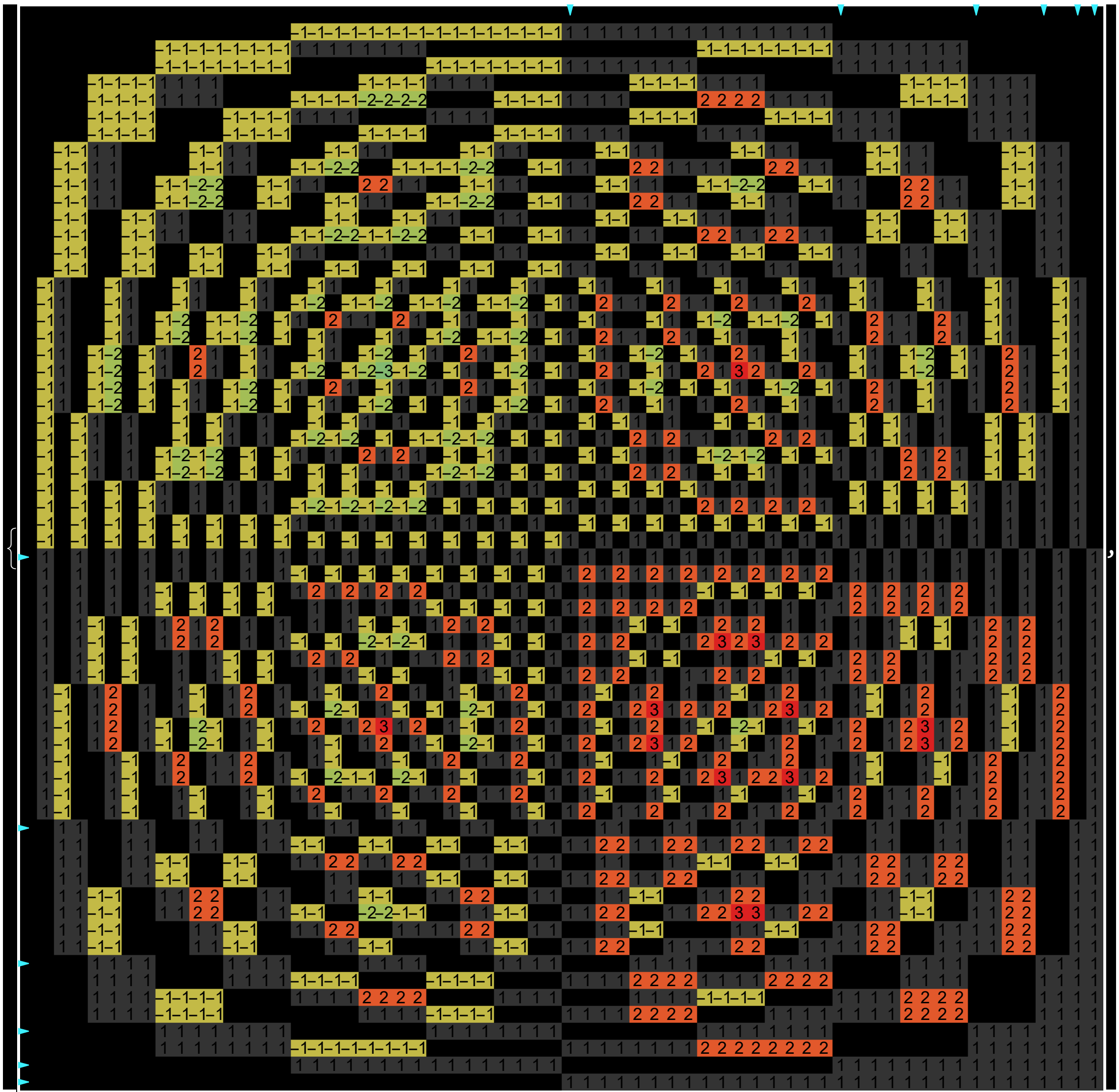


In[ ]:=

```
exportBlackBack["half-matrix-demo.jpg", %];
```

In[ ]:= `matrixPicture[manydiagMatr[6, Range[0, 5]][[-1 ;; 1 ;; -1, -1 ;; 1 ;; -1]]]`

Out[ ]=



$$\left\{ \ldots, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

In[ ]:=
```
SeedRandom[42];
With[{p = PermutationMatrix[RandomPermutation[5]],
  utr = manydiagMatr[5, Range[0, 4]][[-1 ;; 1 ;; -1, -1 ;; 1 ;; -1]], m = MatrixForm},
 {p, utr // MatrixForm, p.utr // m, utr.p // m, p.utr.p // m, p.utr.(pᵀ) // m}]
```

Out[ ]=

{PermutationMatrix[Dimensions: {5, 5}], $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$}

Since conjugation by a permutation doesn’t effect the order and the self compatible count,

In[ ]:=
```
allSymInvDifCongClass[d_] := With[{prms = PermutationMatrix /@ Permutations[Range[d]]},
  DeleteDuplicatesBy[Select[(# + LowerTriangularize[#ᵀ, -1]) & /@ PadLeft /@
      (Internal`PartitionRagged[#, Range[d, 1, -1]] &) /@ Tuples[{0, 1}, d (d + 1)/2],
    MatrixRank@# == d &], First@MinimalBy[Table[p.#.(pᵀ), {p, prms}], Identity] &]]

allSymInvDifClassDifSort[d_] :=
 With[{prms = PermutationMatrix /@ Permutations[Range[d]]},
  DeleteDuplicatesBy[DeleteDuplicatesBy[allSymInvertible[d], Sort[#ᵀ] &],
   First@MinimalBy[Table[p.#.(pᵀ), {p, prms}], Identity] &]]
  (*deleting duplicates by sort of transpose is likely not needed*)
```

In[ ]:=
```
Length[allSymInvDifCongClass[#]] & /@ Range[4]
```

Out[ ]=
```
{1, 3, 9, 40}
```

In[ ]:=
```
Length[allSymInvertible@#] & /@ Range[5]
```

Out[ ]=
```
{1, 3, 23, 372, 14206}
```

In[ ]:=
```
Length[allSymInvDifClassDifSort@#] & /@ Range[4]
```

Out[ ]=
```
{1, 2, 6, 27}
```

In[ ]:=
```
Length /@ {{1}, DeleteDuplicates[allSymInvertible[2],
    IsomorphicGraphQ[hasseSymFact@#1, hasseSymFact@#2] &], allThreeHasse, allFourHasse}
```

Out[ ]=
```
{1, 1, 3, 7}
```