

Stability of size bounds for families with binary scalar products

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Abstract

Questions on possible number of vertices and facets in 2-level polytopes motivate the study of vector families $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ with a property that $\forall a \in \mathcal{A}, b \in \mathcal{B}$ the scalar product $\langle a, b \rangle \in \{0, 1\}$. In this work we show the stability of Kupavskii's and Weltge's bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for such \mathcal{A} and \mathcal{B} . We use this result to find the maximal possible product of the number of vertices and the number of facets of a 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope.

1 Introduction

A polytope P is said to be *2-level* if for every facet-defining hyperplane H there is a parallel hyperplane H' such that $H \cup H'$ contains all vertices of P . Basic examples of 2-level polytopes are simplices, hypercubes and cross-polytopes, but they also generalize a variety of interesting polytopes such as Birkhoff, Hanner, and Hansen polytopes, order polytopes and chain polytopes of posets, stable matching polytopes, and stable set polytopes of perfect graphs [1]. Combinatorial structure of two-level polytopes has also been studied in [3], and enumeration of such polytopes in [2] led to a conjecture that the product of the number of facets and the number of vertices in a d -dimensional 2-level polytope is bounded by $d2^{d+1}$. It has been proven in [5] via a stronger theorem regarding so-called families of vectors with binary scalar products:

Theorem 1. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span \mathbb{R}^d such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}, b \in \mathcal{B}$. Then we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d + 1)2^d$.*

In this paper we prove several bounds regarding the stability of the bound in Theorem 1. Our main result is the following

Theorem 2. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ both linearly span \mathbb{R}^d such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in \mathcal{A}, b \in \mathcal{B}$. Furthermore, suppose \mathcal{A} and \mathcal{B} are inclusion-wise maximal with respect to this property and both have the size of at least $d + 2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$.*

We then use this to improve the bound on the product of the vertex count $f_0(P)$ and the facet count $f_{d-1}(P)$ in a 2-level polytope P that is distinct from both the cube and the cross-polytope:

Theorem 3. *For $d > 1$ let P be a d -dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then $f_0(P)f_{d-1}(P) \leq (d-1)2^{d+1} + 8(d-1)$.*

Outline The next section lays out the proof of main results. In Section 3 we provide the proof of Theorem 3 and section 4 contains proofs of claims from [5] that we use. Technical proofs of some statements used in the main section are provided in Appendix A.

2 Stability results

We will first introduce some notation and restate some claims proved in [5]. Proofs of claims are provided in section 4 for completeness.

Let $b_d \in \mathcal{B}$ be a vector with the maximum value of $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, where

$$\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\} \text{ for } i = 0, 1.$$

The choice of b_d among the vectors that maximise $\max(\dim \mathcal{A}_0, \dim \mathcal{A}_1)$, if important, will be specified at a later stage. We denote the orthogonal projection onto $U = b_d^\perp$ as $\pi : \mathbb{R}^d \rightarrow U$.

Claim 1. *We may translate \mathcal{A} and replace some points in \mathcal{B} by their negatives such that the following holds.*

(i) *We can write $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, where $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$ for $i = 0, 1$ such that*

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}$$

(ii) *We still have*

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}$$

(iii) *The set $\pi(\mathcal{B})$ does not contain opposite points.*

Claim 2. *Every point in $\pi(\mathcal{B})$ has at most two preimages in \mathcal{B} .*

Inequality 1. $|\mathcal{A}| |\mathcal{B}| \leq 2 |\mathcal{A}_0| |\pi(\mathcal{B})| + |\mathcal{A}_1| |\mathcal{B} \setminus \mathcal{B}_*|$

We denote the linear span of \mathcal{A}_0 as U_0 and introduce the orthogonal projection $\tau : U \rightarrow U_0$. Let $\mathcal{B}_* \subseteq \mathcal{B}$ be the set of $b \in \mathcal{B}$ for which $\pi(b)$ has exactly one pre-image under projection onto U .

Claim 3. $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0} |\tau(\pi(\mathcal{B}))|$.

Claim 4. $\mathcal{B} \setminus \mathcal{B}_* = \mathcal{B}_0 \sqcup \mathcal{B}_1$ holds with $i = 0, 1$ satisfying:

$$\forall b \in \mathcal{B}_i : |\{ \langle a, b \rangle : a \in \mathcal{A}_i \}| = 1$$

Claim 5. For $i = 0, 1$, we have $|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^d$.

Inequality 2. $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 1) 2^d + |\mathcal{A}_0| |\mathcal{B}_0| + |\mathcal{A}_1| |\mathcal{B}_1|$

We will now understand under which conditions equality is achieved in Theorem 1. Without loss of generality, we assume $|\mathcal{A}| \geq |\mathcal{B}|$.

Theorem 4. $|\mathcal{A}| \cdot |\mathcal{B}| = (d+1)2^d$ only if $|\mathcal{B}| = d+1$, and \mathcal{A} is affinely isomorphic to $\{0, 1\}^d$.

Proof. We will use induction on d , the statement is obvious in dimension 1. Assuming the statement holds for smaller dimensions, we prove it for dimension d . Consider the possible values of $\dim U_0$:

1. $\dim U_0 < d - 2$. Then, from inequality 2 and claim 5, we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 3) 2^d \leq d 2^d \quad (3)$$

2. $\dim U_0 = d - 2$. Notice that we can assume without loss of generality that $0, b_d \in \mathcal{B}_0$ or $0, b_d \in \mathcal{B}_1$. Therefore, from the proof of claim 5, we have:

$$|\mathcal{A}_1| |\mathcal{B}_1| \leq 2^d, \quad |\mathcal{A}_0| (|\mathcal{B}_0| + 2) \leq 2^d$$

Thus, from inequality 2 and claim 5, we get:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (d-1) 2^d + 2 \cdot 2^d - 2 |\mathcal{A}_0| \leq (d+1) 2^d - |\mathcal{A}| < (d+1) 2^d$$

3. $\dim U_0 = d - 1$. In this case, assuming $0, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \emptyset$. We consider two subcases:

- a) $\mathcal{B}_* \neq \emptyset$. Then inequality 1 is only tight when $|\mathcal{A}_0| = |\mathcal{A}_1|$, and inequality 2 is only tight when $|\mathcal{A}_0| |\pi(\mathcal{B})| = d 2^{d-1}$. By the induction hypothesis, the latter is possible in one of two cases:

- i) \mathcal{A}_0 is affinely isomorphic to $\{0, 1\}^{d-1}$. Then, $|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = 2^d$, which is only possible if \mathcal{A} is affinely isomorphic to $\{0, 1\}^d$, and \mathcal{B} can consist only of a basis and zero vector.
- ii) $|\mathcal{A}_0| = d$. Then, $|\mathcal{B}| \leq |\mathcal{A}| = 2d$, and $|\mathcal{B}| \cdot |\mathcal{A}| \leq 4d^2$, which is less than $(d+1)2^d$ for $d \geq 4$. For $d = 3$, the inequality $|\mathcal{B}| \cdot |\mathcal{A}| \leq 32$ cannot yield equality since $|\mathcal{A}| = 6$. Finally, for $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

- b) $\mathcal{B}_* = \emptyset$. Then, $\mathcal{B}_1 = \mathcal{B}$ and, consequently, $\dim(\text{span}(\mathcal{B}_1)) = d$. In this case:

$$(\forall b \in \mathcal{B}_1 \exists \xi : \forall a \in \mathcal{A}_1 \langle a, b \rangle = \xi) \Rightarrow \dim(\mathcal{A}_1) \leq d - \dim(\text{span}(\mathcal{B}_1)) = 0 \Rightarrow |\mathcal{A}_1| = 1$$

Similarly to case a), inequality 1 is only tight in one of the following cases:

- i) $|\mathcal{A}_0| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1)^2 < (d+1) 2^d$.
- ii) $|\mathcal{A}_0| = 2^{d-1}$, $|\pi(\mathcal{B})| = d$. Then, $|\mathcal{A}| \cdot |\mathcal{B}| = 2d(2^{d-1} + 1)$, which is less than $(d+1)2^d$ for $d > 2$. For $d = 2$, we have $|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 = 9 < 3 \cdot 2^2$.

□

We will improve the bound on $|\mathcal{A}| \cdot |\mathcal{B}|$ for families that differ from the extremal example. To do this, we will use an auxiliary

Inequality 3. *For an integer $2 \leq f \leq d$, we have:*

$$(d + f)(2^{d-1} + 2^{d-f}) \leq d2^d + 2d.$$

A short but technical proof of this inequality can be found in Appendix A.

Theorem 5. *Let both \mathcal{A} and $\mathcal{B} \subseteq \mathbb{R}^d$ contain a basis for \mathbb{R}^d , and $\langle a, b \rangle \in \{0, 1\}$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$. If, in addition, both families are inclusion-wise maximal with respect to this property and have size of at least $d + 2$, then $|\mathcal{A}| \cdot |\mathcal{B}| \leq (d + \lambda(d)) 2^d$, where $0 < \lambda(d) \leq 1$ is a (non-strictly) decreasing function.*

Proof. We will denote $\lambda(d)$ as λ_d . As in the proof of Theorem 4, we will use induction on d . For the base case, we can choose $\lambda = 1$. Assuming validity for smaller dimensions, let us prove the statement for dimension d . We consider possible values of $\dim U_0$:

1. $\dim U_0 < d - 2$. Then $|\mathcal{A}| \cdot |\mathcal{B}| \leq (\dim U_0 + 3) 2^d \leq d2^d$.
2. $\dim U_0 = d - 2$. Applying the induction hypothesis and Theorem 4 to the families $\tau(\pi(\mathcal{B}))$ and \mathcal{A}_0 , we have three cases:
 - a) \mathcal{A}_0 is affine isomorphic to $\{0, 1\}^{d-2}$, and $\tau(\pi(\mathcal{B}))$ consists of zero and the basis of U_0 . From claim 5 and the assumption that $0, b_d \in \mathcal{B}_1$, it follows that $|\mathcal{B}_0| \leq 2$. Since $|\mathcal{B}_0|$ is even, we have two scenarios:
 - i) $|\mathcal{B}_0| = 0$. Then, from inequality 1 and statement 5, we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4(d - 1) 2^{d-2} + 2^d = d2^d$$

- ii) $|\mathcal{B}_0| = 2$. Let $k + 1$ vectors have two pre-images under the action of τ ($k \geq 0$, as $\mathcal{B}_0 \subset U_0^\perp$ is not empty). Among these $k + 1$, let t_2 be the number of those vectors with both pre-images in $\pi(\mathcal{B}_1)$, and let t_1 be the number of those with exactly one pre-image in $\pi(\mathcal{B}_1)$. The remaining $k - t_1 - t_2$ have both pre-images in $\pi(\mathcal{B}_*)$. Suppose that the vectors in $\tau(\pi(\mathcal{B}))$ with a single pre-image under τ consist of q projections onto $\pi(\mathcal{B}_1)$ and $d - 2 - k - q$ projections onto $\pi(\mathcal{B}_*)$. We then have:

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_*| + |\mathcal{B}_0| + |\mathcal{B}_1| \\ &= (k - t_1 - 2t_2 + d - 2 - q) + 2 + (2 + 4t_2 + 2t_1 + 2q) \\ &= d + k + q + t_1 + 2t_2 + 2 \end{aligned}$$

First, consider the case when $t_2 > 0$. Then, $U_0^\perp \subset \text{span}(\mathcal{B}_1)$, which implies:

$$\begin{aligned} \dim(\text{span}(\mathcal{B}_1)) &= t_1 + t_2 + q + 2 \implies |\mathcal{A}_1| \leq 2^{d-t_1-t_2-q-2}, \\ |\mathcal{A}| &= |\mathcal{A}_0| + |\mathcal{A}_1| \leq 2^{d-2} + 2^{d-2-t_1-t_2-q} \end{aligned}$$

$$\begin{aligned}
|\mathcal{A}| \cdot |\mathcal{B}| &\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right) (d + k + q + t_1 + 2t_2 + 2) \\
&\leq \left(2^{d-2} + 2^{d-2-t_1-t_2-q}\right) (2d + t_1 + 2t_2) \tag{4}
\end{aligned}$$

$$\leq \left(2^{d-1} + 2^{d-1-t_1-t_2-q}\right) (d + t_1 + t_2) \tag{5}$$

$$\begin{aligned}
&\leq \left(2^{d-1} + 2^{d-1-t_1-t_2}\right) (d + t_1 + t_2 + 1) \\
&\leq d2^d + 2d \tag{6}
\end{aligned}$$

Here, the second inequality follows from $k + q \leq d - 2$, and the last one follows from inequality 3. If $t_2 = 0$, we get a slightly weaker bound:

$$\dim(\text{span}(\mathcal{B}_1)) \geq t_1 + t_2 + q + 1$$

This means (5) becomes $(2^{d-1} + 2^{d-t_1})(d + t_1)$, which is still less than (6) when $t_1 \geq 2$ according to inequality 3. Finally, when $t_2 = 0$ and $t_1 = 0, 1$, expression (4) yields estimates of $d2^d$ and $(2^{d-2} + 2^{d-3})(2d + 1) = d2^d - (d - \frac{3}{2})2^{d-2} \leq d2^d$, respectively.

b) \mathcal{A}_0 consists of zero and the basis of U_0 . Then:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq |\mathcal{A}|^2 \leq 4(d-1)^2 \leq d2^d + 2d$$

c) $|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq (d-2 + \lambda_{d-2})2^{d-2}$. Using inequalities 1, 2, and statement 3, we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d-2 + \lambda_{d-2})2^{d-2} + 2 \cdot 2^d = (d + \lambda_{d-2})2^d$$

3. $\dim U_0 = d - 1$. Again, applying the induction hypothesis to $\pi(\mathcal{B})$ and \mathcal{A}_0 , we have three cases (recall that from the assumption $0, b_d \in \mathcal{B}_1$, we have $\mathcal{B}_0 = \emptyset$):

a) \mathcal{A}_0 is isomorphic to $\{0, 1\}^{d-1}$, and $\pi(\mathcal{B})$ is the basis with the zero vector.

i) $\dim \mathcal{B}_1 = 1$. In this case, $|\mathcal{B}| = d + 1$, which does not satisfy the condition in the theorem's statement.

ii) $\dim \mathcal{B}_1 = k \geq 2$. Then $|\mathcal{B}_1| = 2k$, $|\mathcal{A}_1| \leq 2^{d-k}$, and we have:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq (2^{d-1} + 2^{d-k})(d + k) \leq d2^d + 2d$$

based on inequality 3.

b) $|\mathcal{A}_0| = d$. Then $|\mathcal{A}|^2 \leq 4d^2$, which is not larger than $d2^d + 2d$ for $d > 3$.

c) $|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1 + \lambda_{d-1})2^{d-1}$. For the final time, from inequalities 1 and 2 and statement 5, we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2 \cdot (d-1 + \lambda_{d-1})2^{d-1} + 2^d = (d + \lambda_{d-1})2^d.$$

□

Now, let us find the optimal value of $\lambda(d)$ from Lemma 5:

Theorem 6. *Suppose both \mathcal{A} and $\mathcal{B} \subseteq \mathbb{R}^d$ contain a basis for \mathbb{R}^d , and $\langle a, b \rangle \in \{0, 1\}$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$. If both families are maximal with respect to this property and have size of at least $d + 2$, then $|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d$.*

Proof. We will again proceed with induction on d , and without loss of generality, assume that $|\mathcal{A}| \geq |\mathcal{B}|$. For $d < 3$, the estimate coincides with Theorem 1. The desired estimate has already been obtained in all subcases of the proof of Lemma 5, except for two induction steps, 2c) and 3c). Thus, it suffices to obtain the required estimate in these cases:

2c') $|\mathcal{A}_0| \cdot |\tau(\pi(\mathcal{B}))| \leq 2(d-2)(2^{d-3} + 1)$. Using inequalities 1, claim 3, and (3), we have

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4 \cdot (d-2) \left(2^{d-2} + 2\right) + 2 \cdot 2^d - 2|\mathcal{A}_0| = 2d(2^{d-1} + 1) + 2(3d - 8 - |\mathcal{A}_0|)$$

This completes the proof when $|\mathcal{A}_0| \geq 3d - 8$. Otherwise:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 4|\mathcal{A}_0|^2 \leq 4(3d - 8)^2$$

This is less than $d2^d + 2d$ for all d except $d = 5, 6$, for which the desired estimate can be obtained by computer enumeration.

3c') Both $|\mathcal{A}_0|$ and $|\pi(\mathcal{B})|$ are at least $d + 1$. By the induction hypothesis,

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1) \left(2^{d-1} + 2\right).$$

Then, from claims 2, 5, and the fact that $\mathcal{B}_0 = \emptyset$, we have

$$|\mathcal{A}| \cdot |\mathcal{B}| = 2|\mathcal{A}_0| |\pi(\mathcal{B})| + |\mathcal{A}_1| |\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \quad (7)$$

$$\leq 2(d-1) \left(2^{d-1} + 2\right) + |\mathcal{A}_1| |\mathcal{B}_1| - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \quad (8)$$

$$\leq 2(d-1) \left(2^{d-1} + 2\right) + 2^d - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*|$$

$$= d2^d + 2d - (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| + (2d - 4) \quad (9)$$

Thus, it suffices to show, for example, that $(|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \geq 2d - 4$.

Consider the case where $\dim A_1 = d - 1$: then $\mathcal{B}_1 = \{0, b_d\}$, and using

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}| |\pi(\mathcal{B})| + |\mathcal{A}| \cdot \frac{1}{2} |\mathcal{B}_1| \leq d2^d + 2d - 2^d + |\mathcal{A}| + (2d - 4)$$

we obtain the desired inequality when $|\mathcal{A}| \leq 2^d - 2d + 4$. Note that $|\mathcal{A}| > 2^d - 2d + 4$, is indeed impossible, as that would imply

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| > (2^{d-1} - d + 2) \cdot (d + 1) \geq (d - 1)(2^{d-1} + 2)$$

which contradicts the induction hypothesis. We may thus now assume $\dim A_1 < d - 1$. Note that due to this, we can also assume that $|\mathcal{A}_0| > |\mathcal{A}_1|$, since in the case that $|\mathcal{A}_0| = |\mathcal{A}_1|$ we can start by shifting the family \mathcal{A} and changing the signs of some vectors in \mathcal{B} so that all conditions remain in force, and \mathcal{A}_0 and \mathcal{A}_1 switch places, reducing the situation to the case where $\dim U_0 < d - 1$.

Consider the orthogonal projection $\pi_{\mathcal{B}_1} : \mathbb{R}^d \rightarrow \text{span}(\mathcal{B}_1)$. By the definition of \mathcal{A}_1 , we have $|\pi_{\mathcal{B}_1}(\mathcal{A}_1)| = 1$. Let $k = \dim(\text{span}(\mathcal{B}_1))$. Since \mathcal{B} contains a basis of \mathbb{R}^d , we have:

$$|\mathcal{B}_*| \geq d - k, (|\mathcal{A}_0| - |\mathcal{A}_1|) |\mathcal{B}_*| \geq d - k \quad (10)$$

Before proceeding to a slightly more systematic case analysis, let us understand that we can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$: first, $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k$ because $0 \in \mathcal{A}_0$, and $\text{span}(\pi_{\mathcal{B}_1}(\mathcal{A}_0)) = \text{span}(\pi_{\mathcal{B}_1}(\text{span}(\mathcal{A}_0))) = \text{span}(\mathcal{B}_1) \cap b_d^\perp$, which means $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ contains 0 and a basis of an $(k - 1)$ -dimensional space. Second, if $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| \geq k + 1$, then applying Theorem 1 to \mathcal{B}_1 and $\pi_{\mathcal{B}_1}(\mathcal{A})$ yields:

$$\begin{aligned} |\mathcal{B}_1| \cdot |\pi_{\mathcal{B}_1}(\mathcal{A})| &\leq (k + 1) 2^k, |\pi_{\mathcal{B}_1}(\mathcal{A})| \geq k + 2 \Rightarrow |\mathcal{B}_1| \leq 2^k \left(1 - \frac{1}{k + 2}\right) \Rightarrow \\ |\mathcal{B}_1| |\mathcal{A}_1| &\leq 2^d \left(1 - \frac{1}{k + 2}\right) \Rightarrow |\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4) - \frac{2^d}{k + 2} - (d - k) \end{aligned}$$

This proves the required estimate for all $d \notin \{3, 4, 5\}$, because for $d \geq 6$

$$d + k - 4 - \frac{2^d}{k + 2} \leq 2d - 4 - \frac{2^d}{d + 2} = -\frac{2}{d + 2} \left(2^{d-1} - (d + 2)(d - 2)\right) \leq 0$$

For $d \in \{3, 4, 5\}$, computer enumeration is carried out.

Now we can assume $|\pi_{\mathcal{B}_1}(\mathcal{A}_0)| = k$, meaning $\pi_{\mathcal{B}_1}(\mathcal{A}_0)$ consists of zero and a basis of $\text{span}(\mathcal{B}_1) \cap b_d^\perp$, while $\pi_{\mathcal{B}_1}(\mathcal{A})$ consists of zero and a basis of $\text{span}(\mathcal{B}_1)$. We will now deal with possible values of k :

- i) $k = 1$, which means $\mathcal{B}_1 = \{0, b_d\}$. Since $\dim \mathcal{A}_1 < d - 1$, from the proof of claim 5, it follows that $|\mathcal{A}_1| \leq 2^{d-2}$. Substituting this into (8), we obtain:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq d2^d + 2d + (2d - 4 - 2^{d-1}) \leq d2^d + 2d$$

- ii) $k = 2$. From the proof of claim 5, it follows that $|\mathcal{B}_1| \leq 4$, and $|\mathcal{A}_1| \leq 2^{d-2}$. Due to (10), $|\mathcal{B}_*| \geq d - 2$, so if $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2$, (9) yields the desired estimate. Similarly, (9) completes the proof if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| \geq 2d - 4$. Finally, if $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$ and $|\mathcal{B}_*| < 2d - 4$, then:

$$|\mathcal{A}| \cdot |\mathcal{B}| = (2|\mathcal{A}_1| + 1) \cdot (|\mathcal{B}_*| + |\mathcal{B}_1|) < (2^{d-1} + 1) \cdot (2d - 4 + 4) = d2^d + 2d$$

- iii) $k = d$ and $\mathcal{B}_* = \emptyset$. Note that, due to (10), $\mathcal{B}_* = \emptyset$ is not possible for other values of k . The definition of \mathcal{B}_1 and our assumption that it has full dimension imply that \mathcal{A}_1 consists of only one point. Hence, (8) becomes:

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2(d - 1) (2^{d-1} + 2) + |\mathcal{B}|$$

which completes the proof when $|\mathcal{B}| \leq 2^d - 2d + 4$. The opposite is indeed impossible, as it would contradict Theorem 1:

$$|\mathcal{A}| \cdot |\mathcal{B}| \geq |\mathcal{B}|^2 \geq (2^d - 2d + 4)^2 > (d + 1) 2^d$$

iv) $2 < k \leq d$ and $\mathcal{B}_* \neq \emptyset$. Let's denote the elements of $\pi_{\mathcal{B}_1}(\mathcal{A})$ as $a_0 = 0, a_1, \dots, a_k$, and their preimages in \mathcal{A} as $\mathbb{A}_j = \pi_{\mathcal{B}_1}^{-1}(a_j) \cap \mathcal{A}$. We'll choose the numbering such that $\mathbb{A}_1 = \mathcal{A}_1$. Let $b_{11}, b_{12}, \dots, b_{1k}$ be a basis of \mathcal{B}_1 that is dual to a_1, \dots, a_k . For example, according to our choice of numbering, $b_{11} = b_d$. Note that, due to \mathcal{B} being inclusion-wise maximal, all b_{1j} must belong to \mathcal{B}_1 (otherwise, they, along with $b_{1j} + b_d$ for $j > 1$, could be added to \mathcal{B}). If $\dim \mathcal{A}_1 < d - k$, we can follow a similar argument as in part i) to obtain $|\mathcal{A}_1| \leq 2^{d-2}$ and the desired estimate. Consequently, we can now assume that $\dim \mathcal{A}_1 = d - k$.

Our further plan is to write \mathcal{A} in a particular basis to see that, due to $\dim \mathcal{A}_1 = d - k$, any of the b_{1j} could be initially chosen as b_d , and that a suitable choice would lead to the desired bound.

We will augment $\{b_{11}, \dots, b_{1k}\}$ with elements from \mathcal{B}_* to form a basis for \mathbb{R}^d and represent \mathcal{A} in the dual basis. Then vectors of \mathcal{A} , arranged as column-vectors, form a matrix of the following form:

$$\mathcal{A} = \left(\begin{array}{c|c|c|c|c|c} \mathbb{A}_0 & \mathbb{A}_1 & \mathbb{A}_2 & \dots & \mathbb{A}_k & \\ \hline \mathbf{0} & \begin{array}{c} 1 \ 1 \ \dots \ 1 \ 1 \\ \mathbf{0} \end{array} & \begin{array}{c} 0 \ 0 \ \dots \ 0 \ 0 \\ 1 \ 1 \ \dots \ 1 \ 1 \end{array} & & & \\ \hline \dots & \dots & \dots & \dots & \dots & \\ \hline \ddots & \ddots & \ddots & & \ddots & \end{array} \right) \begin{array}{l} k \\ d - k \end{array}$$

$\underbrace{\hspace{10em}}_{\dim = d - k}$

The rank of the highlighted block coincides with the affine dimension of $\mathbb{A}_1 = \mathcal{A}_1$, which is $d - k$. Therefore,

$$\forall j > 1: d - 1 = \dim(\text{span}(\mathcal{A} \setminus \mathbb{A}_j)) = \dim(\mathcal{A} \cap b_{1j}^\perp)$$

Which means that, indeed, any of the b_{1j} could be set as b_d from the start. Choose b_{1j} with the smallest dimension of \mathbb{A}_j , and repeat all the same reasoning with it as b_d . Note that in this case, $|\mathcal{A} \setminus \mathbb{A}_j| > |\mathbb{A}_j|$, so there will be no need for translation of \mathcal{A} that swaps \mathcal{A}_0 and \mathcal{A}_1 in claim 1, and we can thus safely assume

$$\begin{aligned} \forall j > 1: |\mathbb{A}_1| \leq |\mathbb{A}_j| &\implies \\ |\mathcal{A}_0| - |\mathcal{A}_1| = |\mathbb{A}_0| + \sum_{j>1} |\mathbb{A}_j| &\geq (k - 1) |\mathcal{A}_1| \geq 2 |\mathcal{A}_1| \end{aligned} \quad (11)$$

If $|\mathcal{A}_0| - |\mathcal{A}_1| \geq 2d - 4$, non-emptiness of \mathcal{B}_* and (9) imply the desired estimate. Otherwise

$$\begin{aligned} |\mathcal{A}_0| - |\mathcal{A}_1| < 2d - 4 &\stackrel{(11)}{\implies} |\mathcal{A}| < 2 \cdot (2d - 4) \implies \\ |\mathcal{A}| \cdot |\mathcal{B}| &\leq |\mathcal{A}|^2 < (4d - 8)^2 < d2^d + 2d, \end{aligned}$$

concluding the proof. □

Two examples that demonstrate tightness of the bound in Theorem 6 are

Example 1. Let $\{e_i\}$ be the standard basis of \mathbb{R}^d ,

$$\mathcal{A} = \left\{ \sum_{i=2}^d \delta_i e_i \right\} \cup \{e_1\}, \mathcal{B} = \{\delta_1 e_1 + e_j\} \cup \{e_1, 0\}, \text{ where } \delta_i \text{ range over } \{0, 1\} \text{ and } j > 1.$$

Here $|\mathcal{A}| = 2^{d-1} + 1$ and $|\mathcal{B}| = 2d$.

Example 2. Let $\{e_i\}$ be the standard basis of \mathbb{R}^d ,

$$\mathcal{A} = \left\{ e_d + \sum_{i=1}^{d-1} \varepsilon_i e_i \right\} \cup \{0\}, \mathcal{B} = \left\{ \frac{1}{2} (e_d + \varepsilon_i e_i) \right\}, \text{ where } \varepsilon_i \text{ range over } \{-1, 1\}.$$

Just like in example 1, $|\mathcal{A}| = 2^{d-1} + 1$ and $|\mathcal{B}| = 2d$.

3 Application to 2-level polytopes

Our main application of Theorem 6 is the following

Theorem 3. For $d > 1$ let P be a d -dimensional 2-level polytope that is not affinely isomorphic to the cube or the cross-polytope. Then $f_0(P)f_{d-1}(P) \leq (d-1)2^{d+1} + 8(d-1)$.

Before following with the proof let us make a simple observation, proof of which is given in Appendix A for completeness:

Lemma 1. Let \mathcal{S} be a family of subsets of $[d-1]$ such that $|\mathcal{S}| = d$ and

$$\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1.$$

Then either $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$.

Proof of Theorem 3. The statement is trivial on the plane, so we assume $d > 2$. Let us denote $V = f_0(P)$ and $F = f_{d-1}(P)$ for conciseness. Shift P so that 0 is among its vertices and let \mathcal{A} denote the vertex set of P and \mathcal{B}' denote the minimal set of vectors such that every facet of P lies in a hyperplane $\{x : \langle x, b \rangle = \delta\}$ for some $\delta \in \{0, 1\}$ and $b \in \mathcal{B}'$. Let $\mathcal{B} = \mathcal{B}' \cup \{0\}$. If every vector in \mathcal{B}' defines one facet of P , we are done by Theorem 1:

$$V \cdot F < |\mathcal{A}| \cdot |\mathcal{B}| \leq (d+1)2^d < (d-1)2^{d+1} + 8(d-1).$$

Otherwise, let $b_d \in \mathcal{B}'$ define two facets of P and consider the setting of the proof of Theorem 6. Note that we may assume $|\mathcal{A}_0| \geq |\mathcal{A}_1|$ if appropriate translation of P was made, so there will be no need for translation of \mathcal{A} or inversions of vectors in \mathcal{B} . Since $\dim(\mathcal{A}_1) = d-1$, we have $\mathcal{B}_1 = \{0, b_d\}$ and $|\pi(\mathcal{B})| = |\mathcal{B}_*| + 1$, which means

$$|\mathcal{A}| \cdot |\mathcal{B}| = |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})| + |\mathcal{A}|. \quad (12)$$

Since every vector in \mathcal{B}' defines at most two facets of P , $|\mathcal{B}| \geq \frac{F}{2} + 1$, thus from (12) we conclude

$$V \cdot F \leq 2(|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| + |\mathcal{A}_1| \cdot |\pi(\mathcal{B})|) \leq 4 \cdot |\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \quad (13)$$

Consider three cases:

1. $|\mathcal{A}_0| > d$ and $|\pi(\mathcal{B})| > d$. By Theorem 6 we have

$$|\mathcal{A}_0| \cdot |\pi(\mathcal{B})| \leq (d-1)2^{d-1} + 2(d-1)$$

and with (13) we are done.

2. $|\pi(\mathcal{B})| = d$. Together with $\mathcal{B}_1 = \{0, b_d\}$, this means that \mathcal{B}' is a basis of \mathbb{R}^d . Every vector in \mathcal{B}' then has to define two facets of P , since otherwise P is unbounded. Thus P is affinely isomorphic to the cube.
3. $|\mathcal{A}_0| = d$. Note that as $|\mathcal{A}_1| \leq |\mathcal{A}_0|$ and $\dim(\mathcal{A}_1) = d-1$, we also have $|\mathcal{A}_1| = d$. If $|\pi(\mathcal{B})| \leq \frac{3}{4} \cdot 2^{d-1}$, (13) implies $V \cdot F \leq \frac{3}{4}d \cdot 2^{d+1} < (d-1)2^{d+1} + 8d$, so we may further assume

$$|\pi(\mathcal{B})| > \frac{3}{4} \cdot 2^{d-1}. \quad (14)$$

We will now make several observations about the structure of \mathcal{A} and \mathcal{B} , after which it will become clear that P is affinely isomorphic to the cross-polytope. Let $a_0 = 0, a_1, \dots, a_{d-1}$ be the elements of \mathcal{A}_0 and $\{u_1, \dots, u_{d-1}\}$ be the basis of $\text{span}(\mathcal{A}_0)$, dual to $\{a_1, \dots, a_{d-1}\}$. Note that for every $j \in [d-1]$ there is $b_{\{j\}} \in \mathcal{B}$ such that $\pi(b_{\{j\}}) = u_j$: $b_{\{j\}}$ is the vector orthogonal to the facet of P that contains vertices $\{a_0, \dots, a_{d-1}\} \setminus \{a_j\}$ and differs from \mathcal{A}_0 . Given $S \subseteq [d-1]$ let us denote by b_S an element of \mathcal{B} for which $\pi(b_S) = \sum_{j \in S} u_j$, if there is one, with $b_\emptyset = 0$ to avoid ambiguity. Observe that the basis of \mathbb{R}^d dual to $\{b_{\{1\}}, b_{\{2\}}, \dots, b_{\{d-1\}}, b_d\}$ is $\{a_1, a_2, \dots, a_{d-1}, v\}$ for v that satisfies

$$\langle v, b_d \rangle = 1 \text{ and } \forall j \in [d-1] : \langle v, b_{\{j\}} \rangle = 0.$$

This means that

$$\mathcal{A}_1 = \{v + \sum_{j \in S} a_j : S \in \mathcal{S}\} \quad (15)$$

for some family \mathcal{S} of subsets of $[d-1]$ with $|\mathcal{S}| = d$. Our goal is to show that $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$, as then P is affinely isomorphic to the cross-polytope, and we would be done. For $T \subseteq [d-1]$ denote $\sigma_T = \sum_{j \in T} a_j$ and note that, given $b_S \in \mathcal{B}$,

$$\langle \sigma_T, b_S \rangle = \langle \sigma_T, \pi(b_S) \rangle = \left\langle \sigma_T, \sum_{j \in S} \pi(b_{\{j\}}) \right\rangle = \left\langle \sum_{j \in T} a_j, \sum_{j \in S} b_{\{j\}} \right\rangle = |T \cap S|.$$

Now assume, looking for a contradiction, that $\exists S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| > 1$. (14) means there exists $b_S \in \mathcal{B}$ such that $|S \cap (S_2 \setminus S_1)| > 1$. (15) means that

$$\begin{aligned} \{-1, 0, 1\} &\ni \langle v + \sum_{j \in S_2} a_j, b_S \rangle - \langle v + \sum_{j \in S_1} a_j, b_S \rangle = \langle \sigma_{S_2} - \sigma_{S_1}, b_S \rangle \\ &= |S_2 \cap S| - |S_1 \cap S| = |(S_2 \setminus S_1) \cap S| > 1, \end{aligned}$$

a contradiction. Therefore, $\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1$, which by Lemma 1 implies that either $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$. The latter is, however, impossible, since then P only has $d+1$ facets and $|\pi(\mathcal{B})| = d \leq \frac{3}{4}2^{d-1}$. (15) now shows that P is affinely isomorphic to the cross-polytope, and we are done. \square

Two examples demonstrate tightness of the bound in Theorem 3:

Example 3 (Cross-polytope \times segment). *Let $\{e_i\}$ be the standard basis of \mathbb{R}^d ,*

$$P = \text{Conv}(\{\varepsilon_i e_i + \varepsilon_d e_d\}_{i \leq d-1}), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\} \text{ for } i \in [d].$$

Here $f_0(P) = 4(d-1)$ and $f_{d-1}(P) = 2 + 2^{d-1}$.

Example 4 (Suspension of a cube). *Let $\{e_i\}$ be the standard basis of \mathbb{R}^d ,*

$$P = \text{Conv}\left(\left\{\sum_{i=1}^{d-1} \varepsilon_i e_i\right\} \cup \{e_d, -e_d\}\right), \text{ where } \varepsilon_i \text{ range over } \{-1, 1\}.$$

This is (up to coordinate scaling) the dual of the polytope in the previous example, in particular $f_0(P) = 2 + 2^{d-1}$ and $f_{d-1}(P) = 4(d-1)$.

4 Proofs of claims

In this section, we provide the proofs of all previous claims, whose statements we repeat here. We begin by explaining how the initial transformation in the proof of Theorem ?? can be performed.

Claim 1. *We may translate \mathcal{A} and replace some points in \mathcal{B} by their negatives such that the following holds.*

(i) *We can write $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, where $\mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\}$ for $i = 0, 1$ such that*

$$|\mathcal{A}_0| \geq |\mathcal{A}_1|. \quad (1)$$

(ii) *We still have*

$$\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \quad (2)$$

(iii) *The set $\pi(\mathcal{B})$ does not contain opposite points.*

Proof. If $|\{a \in \mathcal{A} : \langle a, b_d \rangle = 0\}| \leq |\{a \in \mathcal{A} : \langle a, b_d \rangle = 1\}|$, then we can choose any $a_* \in \mathcal{A}$ with $\langle a_*, b_d \rangle = 1$ (which exists since \mathcal{A} spans \mathbb{R}^d) and replace \mathcal{A} by $\mathcal{A} - a_*$, \mathcal{B} by $(\mathcal{B} \setminus \{b_d\}) \cup \{-b_d\}$, and b_d by $-b_d$. This yields (i).

After this replacement, for each $b \in \mathcal{B}$ there is some $\varepsilon_b \in \{\pm 1\}$ such that $\langle a, b \rangle \in \{0, \varepsilon_b\}$ holds for all $a \in \mathcal{A}$. Each b with $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, -1\}$ is replaced by $-b$, which yields (ii).

Let \mathcal{A}'_1 be a translate of \mathcal{A}_1 such that $\mathbf{0} \in \mathcal{A}'_1$. Note that, for each $b \in \mathcal{B}$ we now have $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, 1\}$ or $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0\}$. In the second case, we replace b by $-b$ if $\{\langle a, b \rangle : a \in \mathcal{A}'_1\} = \{0, -1\}$, otherwise we leave it as it is.

It remains to show that $\pi(\mathcal{B})$ does not contain opposite points after this transformation. To this end, let $b, b' \in \mathcal{B}$ such that $\pi(b) = \beta\pi(b')$ for some $\beta \neq 0$, where $\pi(b), \pi(b') \neq \mathbf{0}$. We have to show that $\beta = 1$. Note that for every $a \in \mathcal{A}_0 \cup \mathcal{A}'_1 \subseteq U$ we have

$$\langle a, b \rangle = \langle a, \pi(b) \rangle = \beta \langle a, \pi(b') \rangle = \beta \langle a, b' \rangle.$$

Suppose first that $\{\langle a, b \rangle : a \in \mathcal{A}_0\} \neq \{0\}$. By (2) there exists some $a \in \mathcal{A}_0$ with $1 = \langle a, b \rangle = \beta \langle a, b' \rangle$. Thus, we have $\langle a, b' \rangle \neq 0$ and hence $\langle a, b' \rangle = 1$, again by (2). This yields $\beta = 1$.

Suppose now that $\{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0\}$. Note that this implies $\{\langle a, b' \rangle : a \in \mathcal{A}_0\} = \{0\}$. As $\mathcal{A}_0 \cup \mathcal{A}'_1$ spans U , we must have $\{\langle a, b \rangle : a \in \mathcal{A}'_1\} \neq \{0\}$ and hence there is some $a \in \mathcal{A}'_1$ with $\langle a, b \rangle = 1$. Moreover, we have $\beta \langle a, b' \rangle = 1$, and in particular $\langle a, b' \rangle \neq 0$. This implies $\langle a, b' \rangle = 1$ and hence $\beta = 1$. \square

As in the previous proof, let \mathcal{A}'_1 be a translate of \mathcal{A}_1 such that $\mathbf{0} \in \mathcal{A}'_1$. Note that for each $b \in \mathcal{B}$ there are $\varepsilon_b, \gamma_b \in \{\pm 1\}$ such that

$$\langle a, b \rangle \in \{0, \varepsilon_b\} \text{ for each } a \in \mathcal{A} \text{ and} \quad (16)$$

$$\langle a, b \rangle \in \{0, \gamma_b\} \text{ for each } a \in \mathcal{A}'_1. \quad (17)$$

The proofs of the subsequent claims rely on the following two lemmas.

Lemma 2. *Suppose that $X \subseteq \{0, 1\}^d \cup \{0, -1\}^d$ does not contain opposite points. Then we have $|X| \leq 2^{\dim X}$.*

Proof. We prove the statement by induction on $d \geq 1$, and observe that it is true for $d = 1$. Now let $d \geq 2$. If $\dim X = d$, then we are also done. It remains to consider to case where X is contained in an affine hyperplane $H \subseteq \mathbb{R}^d$. Let $c = (c_1, \dots, c_d) \in \mathbb{R}^d$, $\delta \in \{0, 1\}$ such that

$$H = \{x \in \mathbb{R}^d : \langle c, x \rangle = \delta\}.$$

For each $i \in \{1, \dots, d\}$ let $\pi_i : H \rightarrow \mathbb{R}^{d-1}$ denote the projection that forgets the i -th coordinate, and let $e_i \in \mathbb{R}^d$ denote the i -th standard unit vector. Note that $\pi_{i^*}(X) \subseteq \{0, 1\}^{d-1} \cup \{0, -1\}^{d-1}$.

Suppose there is some $i^* \in \{1, \dots, d\}$ such that $\langle c, e_{i^*} \rangle \neq 0$ and $\pi_{i^*}(X)$ does not contain opposite points. By the induction hypothesis we obtain

$$|X| = |\pi_{i^*}(X)| \leq 2^{\dim \pi_{i^*}(X)} = 2^{\dim X},$$

where the first equality and the last inequality hold since π_{i^*} is injective (due to $\langle c, e_{i^*} \rangle \neq 0$).

It remains to consider the case in which there is no such i^* . Consider any $i \in \{1, \dots, d\}$. If $\langle c, e_i \rangle \neq 0$, then there exist $x = (x_1, \dots, x_d), x' = (x'_1, \dots, x'_d) \in X$, $x \neq x'$ such that $\pi_i(x) = -\pi_i(x')$. We may assume that $\pi_i(x) \in \{0, 1\}^{d-1}$ and hence $\pi_i(x') \in \{0, -1\}^{d-1}$. As X does not contain opposite points, we must have $x_i = 1$ and $x'_i = 0$, or $x_i = 0$ and $x'_i = -1$. In the first case we obtain

$$\begin{aligned} 2\delta &= \langle c, x \rangle + \langle c, x' \rangle = [\langle \pi_i(c), \pi_i(x) \rangle + c_i x_i] + [\langle \pi_i(c), \pi_i(x') \rangle + c_i x'_i] \\ &= [\langle \pi_i(c), \pi_i(x) \rangle + c_i] + [\langle \pi_i(c), \pi_i(x') \rangle] \\ &= c_i. \end{aligned}$$

Similarly, in the second case we obtain $2\delta = -c_i$.

If $\delta = 0$, this would imply that $c = \mathbf{0}$, a contradiction to the fact that $H \neq \mathbb{R}^d$. Otherwise, $\delta = 1$ and hence every nonzero coordinate of c is ± 2 . Thus, for every $x \in \mathbb{Z}^d$ we see that $\langle c, x \rangle$ is an even number, in particular $\langle c, x \rangle \neq \delta$. This means that $X \subseteq \mathbb{Z}^d \cap H = \emptyset$, and we are done. \square

Lemma 3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ such that \mathcal{A} spans \mathbb{R}^d , \mathcal{B} does not contain opposite points, and for every $b \in \mathcal{B}$ there is some $\varepsilon_b \in \{\pm 1\}$ such that $\{\langle a, b \rangle : a \in \mathcal{A}\} \subseteq \{0, \varepsilon_b\}$. Then we have $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$.*

Proof. Let $a_1, \dots, a_d \in \mathcal{A}$ be a basis of \mathbb{R}^d , and let $M \in \mathbb{R}^{d \times d}$ such that $a_i = M^\top e_i$ for $i = 1, \dots, d$. For every $b \in \mathcal{B}$ we obtain

$$\langle e_i, Mb \rangle = \langle a_i, M^{-1}Mb \rangle = \langle a_i, b \rangle \in \{0, \varepsilon_b\}$$

for $i = 1, \dots, d$ and hence $Mb \in \{0, \varepsilon_b\}^d$. Thus, the set $X := \{Mb : b \in \mathcal{B}\}$ is contained in $\{0, 1\}^d \cup \{0, -1\}^d$. As \mathcal{B} does not contain opposite points, we see that also X does not contain opposite points. Lemma 2 now implies $|\mathcal{B}| = |X| \leq 2^{\dim X} = 2^{\dim \mathcal{B}}$. \square

We are ready to continue with the proofs of the remaining claims.

Claim 2. *Every point in $\pi(\mathcal{B})$ has at most two preimages in \mathcal{B} .*

Proof. Let $y := \pi(b)$ for some $b \in \mathcal{B}$ and observe that $\pi^{-1}(y) = \{x \in \mathbb{R}^d : \pi(x) = \pi(y)\}$ is a one-dimensional affine subspace. By (16) and Lemma 3 we obtain $|\mathcal{B} \cap \pi^{-1}(y)| \leq 2$. \square

Claim 3. $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0} |\tau(\pi(\mathcal{B}))|$.

Proof. Fix any $b \in \mathcal{B}$ and let $v := \pi(b^*)$. Consider the orthogonal complement $W \subseteq U$ of U_0 in U . As $\tau^{-1}(\tau(v)) = v + W$, it suffices to show that

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}$$

holds. To this end, consider the linear subspace $\Pi \subseteq U$ spanned by v and W and let $\sigma : U \rightarrow \Pi$ denote the orthogonal projection on Π .

First, suppose that $\sigma(\mathcal{A}'_1)$ spans Π . For every $a \in \mathcal{A}'_1 \subseteq U$ and every $b \in \mathcal{B}$ with $\pi(b) \in v + W \subseteq \Pi$ we have

$$\langle \sigma(a), \pi(b) \rangle = \langle a, \pi(b) \rangle = \langle a, b \rangle \in \{0, \gamma_b\}$$

by (17). Moreover, recall that $\pi(\mathcal{B})$ does not contain opposite points by Claim ?? (iii). Thus, the pair $\sigma(\mathcal{A}'_1)$ and $(v + W) \cap \pi(\mathcal{B})$ satisfies the requirements of Lemma 3 (in U), and hence we obtain

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{\dim(v+W)} = 2^{\dim W} = 2^{\dim U - \dim U_0} = 2^{d-1-\dim U_0}.$$

It remains to consider the case in which $\sigma(\mathcal{A}'_1)$ does not span Π . Unless $|(v + W) \cap \pi(\mathcal{B})| = 1$, we will identify points $b_1, b_2 \in \mathcal{B}$ with $\max\{\varphi(b_1), \varphi(b_2)\} > \varphi(b_d)$, a contradiction to the choice of b_d .

As $\mathcal{A}_0 \cup \mathcal{A}'_1$ spans U , we know that $\sigma(\mathcal{A}_0 \cup \mathcal{A}'_1)$ spans Π . Since \mathcal{A}_0 is orthogonal to W , this means that $\sigma(\mathcal{A}_0)$ spans a line, and $\sigma(\mathcal{A}'_1)$ spans a hyperplane H in Π . Note that we

have $v \notin W$ (otherwise $W = \Pi$ and so $\sigma(\mathcal{A}'_1)$ spans Π). Thus, every nonzero point in $\sigma(\mathcal{A}_0)$ has nonzero scalar product with v . Moreover, for every $a \in \mathcal{A}_0$ with $\sigma(a) \neq \mathbf{0}$ we have $\langle \sigma(a), v \rangle = \langle a, v \rangle = \langle a, b \rangle \in \{0, 1\}$ by (2). Thus, since the nonzero vectors in $\sigma(\mathcal{A}_0)$ are collinear, we obtain

$$\sigma(\mathcal{A}_0) \subseteq \{\mathbf{0}, \sigma(a_0)\}$$

for some $a_0 \in \mathcal{A}_0$. Since $\mathbf{0} \in H$, we have $\sigma(\mathcal{A}_0) \setminus H \subseteq \{\sigma(a_0)\}$ and further, since $\sigma(\mathcal{A}_0 \cup \mathcal{A}'_1)$ spans Π , we have $\sigma(\mathcal{A}_0) \setminus H = \{\sigma(a_0)\}$. Let $c \in \Pi$ be a normal vector of H . As $\sigma(a_0) \notin H$, we may scale c so that $\langle \sigma(a_0), c \rangle = 1$. Let $a_* \in \mathcal{A}_1$ such that $\mathcal{A}'_1 = \mathcal{A}_1 - a_*$. We define

$$b_1 := c - \delta_1 b_d \neq \mathbf{0},$$

where $\delta_1 := \langle a_*, c \rangle$. For every $a \in \mathcal{A}_0$ we have

$$\langle a, b_1 \rangle = \langle a, c \rangle = \langle \sigma(a), c \rangle \in \{\langle \mathbf{0}, c \rangle, \langle \sigma(a_0), c \rangle\} = \{0, 1\},$$

and for every $a \in \mathcal{A}_1$ we have

$$\begin{aligned} \langle a, b_1 \rangle &= \underbrace{\langle a - a_*, b_1 \rangle}_{\in \mathcal{A}'_1} + \langle a_*, b_1 \rangle = \langle a - a_*, c \rangle + \langle a_*, b_1 \rangle = \underbrace{\langle \sigma(a - a_*), c \rangle}_{\in H} + \langle a_*, b_1 \rangle \\ &= \langle a_*, b_1 \rangle = \langle a_*, c \rangle - \delta_1 \langle a_*, b_d \rangle = \langle a_*, c \rangle - \delta_1 = 0. \end{aligned}$$

Thus, by the maximality of \mathcal{B} , (a scaling of) the vector b_1 is contained in \mathcal{B} . Since we assumed $\mathbf{0} \in \mathcal{A}_0$, we have $\varphi(b_1) \geq \dim(\mathcal{A}_1) + 1$.

In order to construct b_2 , let us suppose that there is another point $b' \in \mathcal{B}$ with $v' := \pi(b') \neq v$ and $v' \in (v + W)$. If there is no such point, then the statement of the claim is true. Recall that $\sigma(a_0)$ is orthogonal to W , and let

$$\xi := \langle \sigma(a_0), v \rangle = \langle \sigma(a_0), \underbrace{v - v'}_{\in W} \rangle + \langle \sigma(a_0), v' \rangle = \langle \sigma(a_0), v' \rangle.$$

Choose $v'' \in \{v, v'\}$ such that $\xi c \neq v''$, and let $b'' \in \{b, b'\}$ such that $\pi(b'') = v''$. Define $\delta_2 := \langle a_*, v'' - \xi c \rangle$ and note that

$$b_2 := v'' - \xi c - \delta_2 b_d$$

is nonzero since $v'' - \xi c \in U \setminus \{\mathbf{0}\}$. For every $a \in \mathcal{A}_0$ we have

$$\langle a, b_2 \rangle = \langle a, \underbrace{v'' - \xi c}_{\in \Pi} \rangle = \langle \sigma(a), v'' - \xi c \rangle,$$

which is zero if $\sigma(a) = \mathbf{0}$. Otherwise, $\sigma(a) = \sigma(a_0)$ and we obtain

$$\langle a, b_2 \rangle = \langle \sigma(a_0), v'' \rangle - \xi \langle \sigma(a_0), c \rangle = \langle \sigma(a_0), v'' \rangle - \xi = 0.$$

Thus, b_2 is orthogonal to \mathcal{A}_0 . Moreover, note that

$$\langle a_*, b_2 \rangle = \langle a_*, v'' - \xi c \rangle - \delta_2 \underbrace{\langle a_*, b_d \rangle}_{=1} = 0.$$

Thus, for every $a \in \mathcal{A}_1$ we have

$$\begin{aligned}\langle a, b_2 \rangle &= \langle a - a_*, b_2 \rangle + \langle a_*, b_2 \rangle = \langle a - a_*, b_2 \rangle = \langle a - a_*, v'' \rangle - \underbrace{\xi \langle a - a_*, c \rangle}_{=0} - \underbrace{\delta_2 \langle a - a_*, b_d \rangle}_{=0} \\ &= \langle a - a_*, v'' \rangle = \langle a - a_*, b'' \rangle \in \{0, \gamma b''\}\end{aligned}$$

by (17). Thus, again by the maximality of \mathcal{B} , (a scaling of) the vector b_2 is contained in \mathcal{B} , and since b_2 is orthogonal to \mathcal{A}_0 and $a_* \in \mathcal{A}_1$, we have $\varphi(b_2) \geq \dim(\mathcal{A}_0) + 1$. However, by the choice of b_d we must have

$$\max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\} + 1 \leq \max\{\varphi(b_1), \varphi(b_2)\} \leq \varphi(b_d) = \max\{\dim(\mathcal{A}_0), \dim(\mathcal{A}_1)\},$$

a contradiction. \square

Claim 4. $\mathcal{B} \setminus \mathcal{B}_* = \mathcal{B}_0 \sqcup \mathcal{B}_1$ holds with $i = 0, 1$ satisfying:

$$\forall b \in \mathcal{B}_i : |\{\langle a, b \rangle : a \in \mathcal{A}_i\}| = 1$$

Proof. Let $b \in \mathcal{B} \setminus \mathcal{B}_*$ and, for the sake of contradiction, suppose that $|\{\langle a, b \rangle : a \in \mathcal{A}_0\}| = |\{\langle a, b \rangle : a \in \mathcal{A}_1\}| = 2$. Let $b' \in \mathcal{B} \setminus \{b\}$ such that $\pi(b) = \pi(b')$. In other words, we have $b' = b + \gamma b_d$ for some $\gamma \neq 0$. Then, by (2) we have

$$\{\langle a, b' \rangle : a \in \mathcal{A}_0\} = \{\langle a, b \rangle : a \in \mathcal{A}_0\} = \{0, 1\}$$

and hence we obtain $\varepsilon_b = \varepsilon_{b'} = 1$ by (16). Again by (16) we see

$$\{0, 1\} \supseteq \{\langle a, b' \rangle : a \in \mathcal{A}_1\} = \{\langle a, b \rangle : a \in \mathcal{A}_1\} + \gamma = \{0, 1\} + \gamma = \{\gamma, 1 + \gamma\},$$

which implies $\gamma = 0$, a contradiction. \square

Claim 5. For $i = 0, 1$, we have $|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^d$.

Proof. First, note that by (16) there is an invertible matrix $M \in \mathbb{R}^{d \times d}$ such that $M(\mathcal{A}) := \{Ma : a \in \mathcal{A}\} \subseteq \{0, 1\}^d$. Now let $i \in \{0, 1\}$. Denote by V the span of \mathcal{B}_i and let $k := \dim V$. Clearly, we have $\mathcal{B}_i \subseteq \mathcal{B} \cap V$ and hence by (16) and Lemma 3 we obtain $|\mathcal{B}_i| \leq |\mathcal{B} \cap V| \leq 2^k$. Recall that for each $b \in \mathcal{B}_i$ there some ξ_b such that $\langle a, b \rangle = \xi_b$ holds for all $a \in \mathcal{A}_i$. Thus, \mathcal{A}_i is a subset of $\mathcal{A} \cap V'$, where $V' := \{x \in \mathbb{R}^d : \langle x, b \rangle = \xi_b \text{ for all } b \in \mathcal{B}_i\}$, and hence we obtain

$$\begin{aligned}|\mathcal{A}_i| &= |M(\mathcal{A}_i)| \leq |M(\mathcal{A} \cap M(V'))| \leq |\{0, 1\}^d \cap M(V')| \leq 2^{\dim M(V')} \\ &= 2^{\dim V'} \leq 2^{d - \dim V} = 2^{d-k},\end{aligned}$$

where the third inequality follows from Lemma 2. We conclude that $|\mathcal{A}_i| |\mathcal{B}_i| \leq 2^{d-k} 2^k = 2^d$. \square

A Appendix

Inequality 3. For an integer $2 \leq f \leq d$, we have:

$$(d+f)(2^{d-1} + 2^{d-f}) \leq d2^d + 2d.$$

Proof. We will prove this by induction on d : when $d = k$, the equality is satisfied. Let's perform the induction step from d to $d+1$. Denoting the left and right sides of the inequality as $l(d, f)$ and $r(d, f)$, respectively, we have

$$\begin{aligned} r(d+1, f) - l(d+1, f) &\geq (r(d+1, f) - r(d, f)) - (l(d+1, f) - l(d, f)) \\ &= \left(d2^d + 2^{d+1} + 2\right) - (d+f+2) \left(2^{d-1} + 2^{d-f}\right) \\ &= 2^{d-f} (d-f+2) \left(2^{f-1} - 1 - \frac{2f}{d-f+2}\right) + 2 \\ &\geq 2^{d-f} (d-f+2) \left(2^{f-1} - 1 - f\right) \end{aligned}$$

The obtained expression is non-negative for $f > 2$. For $f = 2$ and $d \geq 4$, we have $2^{f-1} - 1 - \frac{2f}{d-f+2} \geq 0$, and for $f = 2$ and $d = 2, 3$, the initial inequality is checked explicitly. \square

Lemma 1. Let \mathcal{S} be a family of subsets of $[d-1]$ such that $|\mathcal{S}| = d$ and

$$\forall S_1, S_2 \in \mathcal{S} : |S_2 \setminus S_1| \leq 1.$$

Then either $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$.

Proof. We assume $d > 2$ as the statement is trivial otherwise. $|\mathcal{S}| > 2$ and clearly \mathcal{S} contains sets of at most two different sizes (that differ by one), so let $U, V \in \mathcal{S}$ both be of size $k \in [d-2]$. Observe that there are now only four options for sets in \mathcal{S} :

- (a) $U \cup V$ of size $k+1$.
- (b) Sets of size k that are contained in $U \cup V$.
- (c) Sets of size k that contain $U \cap V$ as a subset.
- (d) $U \cap V$ of size $k-1$.

(a) and (d) are not possible simultaneously, so are (b) and (c) with the exception of U and V . There are $k+1$ and $d-k$ sets satisfying (b) and (c) respectively, so $|\mathcal{S}| = d$ is only possible if $k = d-2$ or $k = 1$ with $\mathcal{S} = \{S \subseteq [d-1] : |S| \geq d-2\}$ or $\mathcal{S} = \{S \subseteq [d-1] : |S| \leq 1\}$ respectively. \square

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