

Lab Things to know (2021-12-13 14:16):

Lab 1

- Norms
- order of accuracy:

$$E_{order} = \frac{\log\left(\frac{e_{old}}{e_{new}}\right)}{\log\left(\frac{N_{new}}{N_{old}}\right)}$$

- Fourier series

Lab 2

- Fourier series
- assumptions required for Fourier expansion
 - C_1 piecewise
 - * continuous in the first derivative, piecewise
 - * finitely many discontinuities over the interval
 - L_2 (square integrable)
 - * if C_1 piecewise, that is a strict condition than L_2
 - periodic
- Bessel's equality
 - sum of the square of the Fourier coefficients is the norm
- ℓ_2 error - $\|\cdot\|_2$ of the difference

Lab 3

- difference operators D_0, D_-, D_+
- general central divided differences formula

$$D_0^k f(x) = (2h)^{-k} \sum_{j \in \mathbb{N}_k} (-1)^j \binom{k}{j} f(x + (k-2j)h) \quad \forall k \in \mathbb{Z}_{\geq 0}$$

Lab 4

- CFL
 - CFL is the ratio of exact wavespeed over numerical wavespeed
- amplification factor Q
 - plug Fourier modes into scheme
 - Q has the CFL baked into it
- determining bounds on the CFL
 - $|Q| \leq 1$ gives condition on CFL
 - if $|Q| = 1$, then conserving energy
- energy analysis
 - **probably will be on exam**
 - usually defined with 2-norm

- take the derivative, work through until you get the boundary terms
- put inside the governing equation??
- energy conserving means derivative of energy will be 0

energy analysis example for $u_t + au_x = 0$

$$u \cdot u_t = -a \cdot u_x \cdot u$$

$$\frac{du^2}{dt} = 2u \cdot u_t$$

$$\frac{1}{2} \int_{\Omega} \frac{d|u|^2}{dt} dx$$

Lab 5

- dissipative, dispersive, diffusive errors
 - use modified equation
 - * take scheme and Taylor expand everything
 - * take governing equation and shove it to the left, everything else on the right
 - * what's left on the right is truncation error
 - truncation error is before you take Δ_x, Δ_t to λ
 - * to get to the modified equation, put all the Δ_t in terms of λ using the CFL definition (depends on eqn)
 - * then, use the governing equation to replace all time derivs with spatial ones (everything in terms of Δ_x) **this is the modified equation**
- look at leading error term in terms of Δ_x , look at order of spatial derivatives
 - if you have even derivatives, error is dissipative (melting)
 - odd derivatives, dispersive (waves shifting)
 - both is diffusive
- she may ask us to look at a plot and say things about the plot
 - for example, what is the error?
 - shifts means dispersive
 - decays means dissipative
 - both means diffusive
- taking a second order and splitting it up into a system
- decoupling a system
 - looking for the eigen modes?
 - define 2 variables: u which you solve for, and v defined by $v_t = u_x$
 - **joel** upload notes

Lab 6

- implicit schemes
 - make matrix operators, take the inverse

$$Au^{n+1} = Bu^n \quad \implies \quad u^{n+1} = A^{-1}Bu^n$$

- amplification factor (matrix) for multiple timestep schemes (leapfrog)
 - see equation 5
 - want to keep the vector $\begin{bmatrix} \hat{u}^{n+1} \\ \hat{u}^n \end{bmatrix}$ to have norm constant or decreasing
 - do this by making sure the matrix is a contracting map (eigenvals bounded by 1 in magnitude)

Lab 7

stability regions : (

- temporal stability plot:
 - form the differential equation $\frac{\partial v}{\partial t} = \gamma \cdot v$
 - approximate the time derivative using the time deriv approx from the scheme
 - find where the solution is stable or decays in time (be constant or go to zero, we dont want it to blow up)
 - * either solve the scheme explicitly, or use the amplification factor type argument
 - set $\gamma = 1$, plot Δ_t in the complex plane
- spatial stability plot
 - take the spatial discretization that we have in the scheme and build a matrix out of it
 - inside of that matrix, set $\Delta_x, \Delta_y = 1$
 - determine Δ_t from the CFL condition (will have to try multiple)
 - compute the eigenvalues for some number of gridpoints N and plot them on top of the temporal stability (dots)
 - if you have multiple time levels on top of the scheme:
 - * **good luck**
 - * block matrix system????????

Lab 8

how to make a first order system out of a second order equation (example is wave equation)

- set

$$v = u_t, \quad w = u_x$$

> note: this works for second order linear, but idk about others

- substitute one of v, w into the governing equation
- other one is the “compatibility condition”
 - Lax pairs? $v_x = w_t$ (not important)
- take those two equations and shove them into the linear system
-

$$\vec{v}_t = A \vec{v}_x$$

-

$$\vec{v} = \begin{bmatrix} v \\ w \end{bmatrix}$$

- take the eigenvals, eigenvects of A
- ????
- look at the lab, it explains things

Lab 9

block matrices? idk its hard

when you have a 2D system, you get block matrices when you discretize it

Lab 10, Lab 11

spectral stuff, not on exam!

Homeworks 2021-12-13 16:47

- taylor expand
- temporal as spatial
- drawing stencil
- truncation errors
- modified equation

Note: “using Fourier analysis” or “taking a Fourier transform” just means plug in a Fourier mode

Well-posedness (hw2)

3 conditions:

- existence
- uniqueness
- varies continuously with initial condition
 - bound on the Fourier mode
 -

$$\|u(x, t)\|_2 \leq Ae^{at} \|u(x, 0)\|_2$$

idk what this part is

need to restrict the coefficient of the highest order even derivative - consider the case when that is zero? - go to the next highest order?

Debugging

- CFL:
 - funny oscillations (Gibbs phenomena)
 - if you take Δ_t , it should work (this removes the CFL as a problem)
 - * if setting Δ_t to be small fixes the problem, then the issue is with the CFL
 - * if setting it to be small does **not** fix the problem, then the issue is with the method
- Boundary conditions
 - if the solution is seeming to “leak” out, then there might be an issue with the boundary conditions
 - if it looks fine in the middle, but not on the edges, also check BCs
- Initial conditions:
 - evolve a really small amount, see if it works
 - if wrong in early timestep, then probably an IC issue
- check indexing

Exam

CFL:

- ratio of exact wavespeed to numerical wavespeed
- if greater than one, then the method is unstable (usually..)

Lax-Richtmyer equivalence theorem

- stability:

$$|Q|^{n+1} \leq K(T_f) \quad \text{as} \quad \sup_n, \lim_{\Delta_t, \Delta_x \rightarrow 0}$$

- consistency: $\lim_{\Delta_t, \Delta_x \rightarrow 0}$ of modified equation is the actual equation (truncation error goes to zero)
- converges: $\lim_{\Delta_t, \Delta_x \rightarrow 0}$ of the grid function, is the actual function (point-wise)

theorem: A linear scheme that is both *consistent* and *stable* converges.