Lab Things to know (2021-12-13 14:16):

Lab 1

- Norms
- order of accuracy:

$$E_{order} = \frac{\log\left(\frac{e_{old}}{e_{new}}\right)}{\log\left(\frac{N_{new}}{N_{old}}\right)}$$

- Fourier series

Lab 2

- Fourier series
- assumptions required for Fourier expansion
  - C_1 piecewise
    * continuous in the first derivative, piecewise
    * finitely many discontinuities over the interval
  - L_2 (square integrable)
    * if C_1 piecewise, that is a stricted condition than L_2
  - periodic
- Bessel's equality
  - sum of the square of the Fourier coefficients is the norm
- ℓ_2 error - $\|\cdot\|_2$ of the difference

Lab 3

- difference operators D_0, D_-, D_+
- general central divided differences formula

$$D_0^k f(x) = (2h)^{-k} \sum_{j \in \mathbb{N}_k} (-1)^j \binom{k}{j} f(x + (k-2j)h) \qquad \forall\ k \in \mathbb{Z}_{\geq 0}$$

Lab 4

- CFL
  - CFL is the numerical wavespeed
- amplification factor Q
  - plug Fourier modes into scheme
  - Q has the CFL baked into it
- determining bounds on the CFL
  - $|Q| \leq 1$ gives condition on CFL
  - if $|Q| = 1$, then conserving energy
- energy analysis
  - **probably will be on exam**
  - usually defined with 2-norm

  - take the derivative, work through until you get the boundary terms
  - put inside the governing equation??
  - energy conserving means derivative of energy will be 0

energy analysis example for $u_t + au_x = 0$

$$u \cdot u_t = -a \cdot u_x \cdot u$$
$$\frac{du^2}{dt} = 2u \cdot u_t$$
$$\frac{1}{2}\int_\Omega \frac{d|u|^2}{dt}dx$$

Lab 5

- dissapative, dispersive, diffusive errors
  - use modified equation
    - take scheme and taylor expand everything
    - take governing equation and shove it to the left, everything else on the right
    - what's left on the right is truncation error
      - truncation error is before you take Δ_x, Δ_t to λ
    - to get to the modified equation, put all the Δ_t in terms of λ using the CFL definition (depends on eqn)
    - then, use the governing equation to replace all time derivs with spatial ones (everything in terms of Δ_x) **this is the modified equation**
- look at leading error term in terms of Δ_x, look at order of spatial derivatives
  - if you have even derivatives, error is dissapative (melting)
  - odd derivatives, dispersive (waves shifting)
  - both is diffusive
- she may ask us to look at a plot and say things about the plot
  - for example, what is the error?
  - shifts means dispersive
  - decays means dissapative
  - both means diffusive
- taking a second order and splitting it up into a system
- decoupling a system
  - looking for the eigen modes?
  - define 2 variables: u which you solve for, and v defined by $v_t = u_x$
  - **joel** upload notes

Lab 6

- implicit schemes
  - make matrix operators, take the inverse

$$Au^{n+1} = Bu^n \qquad \Longrightarrow \qquad u^{n+1} = A^{-1}Bu^n$$

- amplification factor (matrix) for multiple timestep schemes (leapfrog)
  - see equation 5
  - want to keep the vector $\begin{bmatrix} \hat{u}^{n+1} \\ \hat{u}^n \end{bmatrix}$ to have norm constant or decreasing
  - do this by making sure the matrix is a contracting map (eigenvals bounded by 1 in magnitude)

Lab 7

stability regions :(

- temporal stability plot:
  - form the differential equation $\frac{\partial v}{\partial t} = \gamma \cdot v$
  - approximate the time derivative using the time deriv approx from the scheme
  - find where the solution is stable or decays in time (be constant or go to zero, we dont want it to blow up)
    - either solve the scheme explicitly, or use the amplification factor type argument
  - set $\gamma = 1$, plot Δ_t in the complex plane
- spatial stability plot
  - take the spatial discretization that we have in the scheme and build a matrix out of it
  - inside of that matrix, set $\Delta_x, \Delta_y = 1$
  - determine Δ_t from the CFL condition (will have to try multiple)
  - compute the eigenvalues for some number of gridpoints N and plot them on top of the temporal stability (dots)
  - if you have multiple time levels on top of the scheme:
    - **good luck**
    - block matrix system?????????

Lab 8

how to make a first order system out of a second order equation (example is wave equation)

- set

  $$v = u_t, \qquad w = u_x$$

  > note: this works for second order linear, but idk about others

- substitute one of v, w into the governing equation

- other one is the "compatibility condition"

  - Lax pairs? $v_x = w_t$ (not important)

- take those two equations and shove them into the linear system

- $$\vec{v}_t = A\vec{v}_x$$

- $$\vec{v} = \begin{bmatrix} v \\ w \end{bmatrix}$$
- take the eigenvals, eigenvects of A
- ????
- look at the lab, it explains things

Lab 9

block matrices? idk its hard

when you have a 2D system, you get block matrices when you discretize it

Lab 10, Lab 11

spectral stuff, not on exam!

Homeworks 2021-12-13 16:47

- taylor expand
- temportal as spatial
- drawing stencil
- truncation errors
- modified equation

  Note: "using Fourier analysis" or "taking a Fourier transform" just means plug in a Fourier mode

Well-posedness (hw2)

3 conditions:

- existence
- uniqueness
- varies continuously with initial condition
  - bound on the Fourier mode
  - $$\|u(x,t)\|_2 \leq Ae^{at}\|u(x,0)\|_2$$

idk what this part is

need to restrict the coefficient of the highest order even derivative - consider the case when that is zero? - go to the next highest order?

Debugging

- CFL:
  - funny oscillations (Gibbs phenomena)
  - if you take Δ_t, it should work (this removes the CFL as a problem)
    * if setting Δ_t to be small fixes the problem, then the issue is with the CFL
    * if setting it to be small does **not** fix the problem, then the issue is with the method
- Boundary conditions
  - if the solution is seeming to "leak" out, then there might be an issue with the boundary conditions
  - if it looks fine in the middle, but not on the edges, also check BCs
- Initial conditions:
  - evolve a really small amount, see if it works
  - if wrong in early timestep, then probably an IC issue
- check indexing

Exam

CFL:

- ratio of exact wavespeed to numerical wavespeed
- if greater than one, then the method is unstable (usually..,)

Lax-Richmeyer equivalence theorem

- stability:
  -
    $$|Q|^{n+1} \leq K(T_f) \qquad \text{as} \qquad \sup_n, \quad \lim_{\Delta_t, \Delta_x \to 0}$$
- consistency: $\lim_{\Delta_t, \Delta_x \to 0}$ of modified equation is the actual equation (truncation error goes to zero)
- converges: $\lim_{\Delta_t, \Delta_x \to 0}$ of the grid function, is the actual function (pointwise)

theorem: A linear scheme that is both *consistent* and *stable* converges.