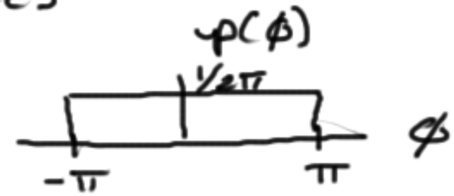
Signal Known Except for Phase (SKEP)

SKEP (Signal Known Except for Phase)

$$s(n) = A\cos(2\pi f_0 n + \phi)$$



Where ϕ is a random variable distributed uniformly between $-\pi$ and π

$$\Lambda(\underline{x}) = \frac{p(\underline{x} \mid H_1)}{p(\underline{x} \mid H_0)} = \frac{\int_{-\pi}^{\pi} p(\underline{x} \mid \phi, H_1)\, p(\phi)\, d\phi}{p(\underline{x} \mid H_0)}$$

$$= \int_{-\pi}^{\pi} \Lambda(\underline{x} \mid \phi)\, p(\phi)\, d\phi \quad \text{where } \Lambda(\underline{x} \mid \phi) = \frac{p(\underline{x} \mid \phi, H_1)}{p(\underline{x} \mid H_0)}$$

$$= \underbrace{\frac{1}{2\pi}}_{p(\phi)} \int_{-\pi}^{\pi} \underbrace{\exp\left[-\frac{E}{N_0} + \frac{A}{\sigma^2} \sum_{n=0}^{N-1} x(n)\cos(2\pi f_0 n + \phi)\right]}_{\Lambda(\underline{x} \mid \phi)} d\phi$$

$$= \exp\left[-\frac{E}{N_0}\right] I_0\left[\frac{A}{\sigma^2} \underbrace{(a^2+b^2)}_{\text{magnitude}^2}{}^{1/2}\right]$$

Note
$X(k) = a + jb$
DFT/FFT of $\underline{x}$
k is an integer "bin" index

Recall $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

I_0 is the modified Bessel function of order 0

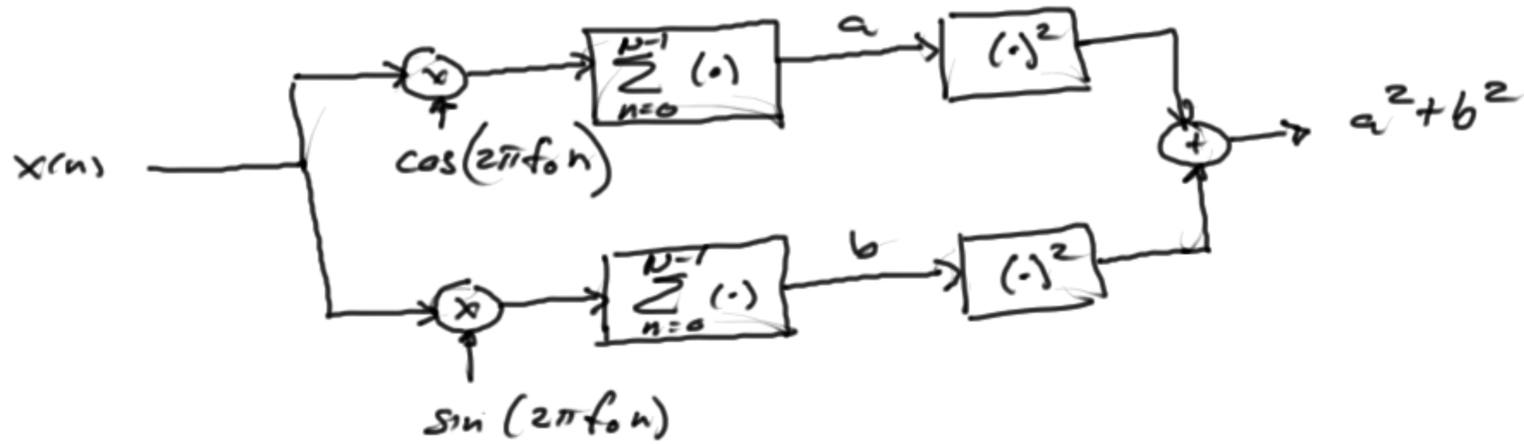
Signal Known Except for Phase (SKEP)

$$\Lambda(\underline{x}) = \exp\left[-\frac{E}{N_0}\right] I_0\left[\frac{A}{\sqrt{2}}\left(a^2+b^2\right)^{1/2}\right]$$

$$X(k) = a + jb \qquad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

k is an integer frequency index $\quad \omega_k = 2\pi f_k = \left(\frac{2\pi}{N}\right)k$

$$\Lambda(\underline{x}) \propto \left(a^2+b^2\right) = \left|X(k)\right|^2$$



Signal Known Except for Phase (SKEP)

$$d^2_{SKEP} = \frac{E[I_N(k)|H_1] - E[I_N(k)|H_0]}{(\mathrm{var}\ I_N(k)|H_0)^{1/2}}$$

Where: $I_N(k) = \frac{1}{N}|X(k)|^2 = \frac{1}{N}(a^2+b^2)$

$E[I_N(k)|H_0] = \sigma^2$ $\qquad$ $E[I_N(k)|H_1] = \sigma^2 + \frac{A^2N}{4}$

$\mathrm{var}[I_N(k)|H_0] = \sigma^4$ $\qquad$ $\mathrm{var}[I_N(k)|H_1] = \sigma^4 + \sigma^2\frac{A^2}{2}N$

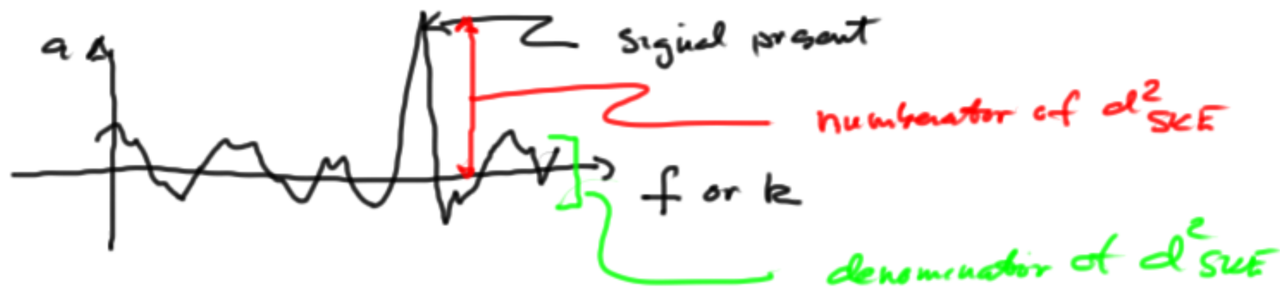
$$d^2_{SKEP} = \frac{\frac{A^2}{4}N}{\sigma^2} \qquad \left(= \frac{N}{2}\cdot SNR\right)$$

$$= \frac{E}{N_0}$$

Note Factor of 2 smaller than d^2_{SKE}

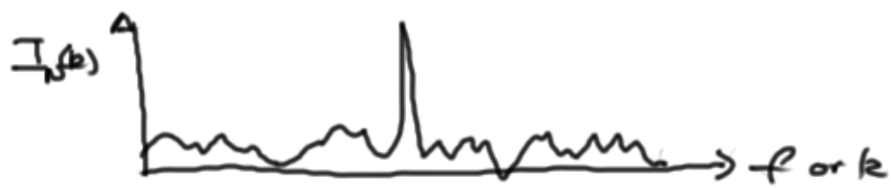
Signal Known Except for Phase (SKEP)

SKE



$$d^2_{SKE} = \frac{\left(E[a|H_1] - E[a|H_0]\right)^2}{var(a|H_0)}$$

SKEP



$$d^2_{SKEP} = \frac{E[I_N(k)|H_1] - E[I_N(k)|H_0]}{\left(var(I_N(k)|H_0)\right)^{1/2}}$$

Signal Known Except for Phase (SKEP)

Ref: Ch 2 Kay

Defn of Chi square pdf

$x_i \sim N(0,1)$ or more generally $x_i \sim N(\mu_i, 1)$

$$x = \sum_{i=1}^{\nu} x_i^2$$

Central $E[x] = \nu$

$\operatorname{var}[x] = 2\nu$

Non-Central $E[x] = \nu + \lambda$ $\qquad \lambda = \sum_{i=1}^{\nu} \mu_i^2$

$\operatorname{var}[x] = 2\nu + 4\lambda$

Signal Known Except for Phase (SKEP)

SKEP

Kay Sect. 7.6.2 Amplitude and Phase Unknown Problem

Eq (7.2.8) p.267

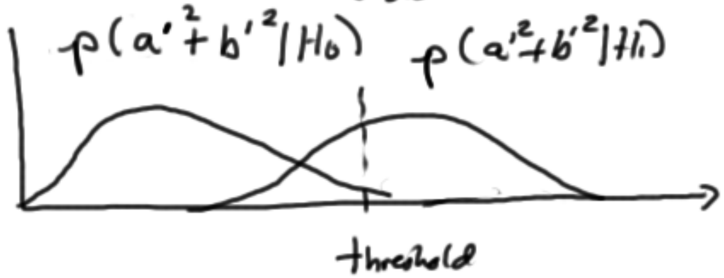
$$P_D = Q_{\chi_2'^2(\lambda)}\left(2 \ln \frac{1}{P_F}\right) \qquad \lambda = \frac{N A^2}{2\sigma^2} = \frac{N\, A^2/2}{\sigma^2}$$

Kay p.25

$$\nu = 2 \qquad P_F = Q_{\chi_2^2}(x) = \exp\left(\frac{-x}{2}\right)$$

← expression for P_F for threshold x

threshold



Signal Known Except for Phase (SKEP)

Transformation of SKEP test statistic $I_N(k)$ to χ^2 normalized form

$$I_N(k) = \frac{1}{N}\left|X(k)\right|^2 = \frac{1}{N}(a^2+b^2)$$

$$E[I_N(k)|H_0] = \sigma^2 \qquad \mathrm{var}(I_N(k)|H_0) = \sigma^4$$

$$E[I_N(k)|H_1] = \sigma^2 + \frac{A^2N}{4} \qquad \mathrm{var}(I_N(k)|H_1) = \sigma^4 + \sigma^2\frac{A^2N}{2}$$

Under H_0, we need to transform $I_N(k)$ so that

$$E[x'] = \nu = 2 \quad \text{and} \quad \mathrm{var}[x'] = 2\nu = 4$$

$$x' = \frac{I_N(k)}{(\sigma^2/2)} \qquad E[x'|H_0] = 2 \qquad \mathrm{var}[x'|H_0] = 4$$

$$E[x'|H_1] = 2 + \frac{A^2N}{2\sigma^2}$$

λ = noncentrality parameter

$$\mathrm{var}[x'|H_1] = 4 + 2\frac{A^2N}{\sigma^2}$$

$\nu = 2$ = number of degrees of freedom