**ECE 254 Mid-term**

**Correlated Noise**

**Name: Mingxuan Wang**

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- Title: Correlated Noise for Mid-term
- Objective:

With reference to HW #3 ("Correlated Noise"), consider the following six signal types: (1) s(n) as defined in HW #3 with $f_c = 1/16$, (2) s(n) as in (1) except $f_c = 1/4$, (3) s(n) as in (1) except $f_c = 3/8$, (4) s(n) a FM chirp with the form:

$$s_{FM}(n) = A\sin\left\{2\pi(f_c + \frac{f'}{2}n)n\right\} \; ; \; f_c = \frac{1}{16}, \; f' = \frac{3}{8 \bullet 16} \quad \text{(chirp rate)}, \; n = 0, \ldots, 15$$
$$= 0 \; ; \text{otherwise},$$

(5) s(n) as in (4) except $f_c = 1/8$ and $f' = 1/(4 \bullet 16)$, and (6) s(n) the optimum waveform for the correlated noise covariance matrix C (i.e. the s(n) that maximizes $d^2$ for the optimum correlated noise processor) where the energy is kept the same as in signal types (1)-(3).

Plot the time series and corresponding spectrum $|S(f)|^2$ (dB) (just the positive frequencies) of each waveform s(n). Also, for each waveform s(n), overplot $|S(f)|^2$ (dB) and $|H(f)|^2$ (dB) (just the positive frequencies) as in Part II in HW #3.

Carry out Part III in HW #3 for the six signal types. In addition to your ROC curves, please include a table with the SNR and $d^2$ values for each of your cases. Compare the performance of the three processors for the different signal types and comment on the reasons for the differences seen. Note that the ROC curve for the optimum waveform is off the Pd/Pf scale. Please include grid lines on your ROC plots in order to more easily determine the values you are obtaining for Pd and Pf.

Include the theoretical expressions for $d^2$ and summarize their derivations along with how the noise covariance matrix C is obtained. Comment on how the optimum waveform is obtained.

Show analytically why the optimum colored noise processor and the mismatched processor have identical performance for the optimum signal waveform. Recall the expressions for $d^2$ and the solution for the optimum waveform in terms of an eigenvalue/eigenvector decomposition of the colored noise covariance matrix.

You should use the same amplitude for the FM chirps that you are using for the sinusoidal signals. Note that the correct expression for FM chirp energy is $s^Ts$ not $N(A^2/2)$ as it is for a sinusoid with an integer number of cycles over N points. You should use the actual chirp energy $s^Ts$ in your calculations for SNR and $d^2$. Don't be concerned that the SNR's for the two chirps and the sinusoids are different (in fact, they are quite close but not exactly the same). The part that is of interest here is the comparison across the three processors for a given signal and not specifically the absolute performance comparison between the six signal types.

1. Let h(n) take on the following structure:

   a) h(n) = {1}

   b) h(n) = $(1/1.81)^{1/2}\{1, 0.9\}$

2. Compute and plot:

   a) $|S(f)|^2$(dB)

   b) $|H(f)|^2$(dB)

3. Determine the ROC performance of the following process

   a) SKE with h(n) as in 1a above. What is the processor input SNR?

      What is $d^2$?

   b) SKE with h(n) as in 1b above. What is the processor input SNR?

      What is $d^2$

   c) Mismatched matched filter with the actual data as in 1b above but the processor assuming the data was from 1a above. Thus, the processor is the conventional matched filter. What is the processor input SNR? What is is $d^2$

- **Approach:**

**1. Define the two h(n):**

a) h(n) = {1} means that h(0) = 1, which is actually an impulse signal.

b) h(n) = $(1/1.81)^{1/2}\{1, 0.9\}$ means that h(0) = $(1/1.81)^{1/2}$ and h(0) = $(1/1.81)^{1/2} * 0.9$. Actually there are two impulses at n = 0 and n = 1. Note that h(n) is (1/1.81)^(1/2)δ(n)+(1/1.81)^(1/2)*0.9δ(n − 1).

**2. Compute $|\boldsymbol{S(f)}|^2(\boldsymbol{dB})$ and $|\boldsymbol{H(f)}|^2(\boldsymbol{dB})$:**

I do Fourier Transform on s(n) and h(n), calculate square of them and then transfer them to dB

$$10 * log10(|S(f)|^2)$$

Note that we only care about positive frequency range, so I only plot s(0:128) and h(0:128).

**3. Compute $\boldsymbol{d^2, SNR\ and\ C}$:**

a) For the first condition: white noise and matched filter

i. For sinusoid signal: (1-3)

First, we can easily compute with for signal 1-3

$$SNR1 = \frac{A^2}{2\sigma^2} = \frac{1}{4}$$

From the lecture, we know that:

$$d^2 = \frac{\left(E(T|H0) - E(T|H1)\right)^2}{Var(T|H0)}$$

For T above:

$$T(x) = x^T C^{-1} s$$

In this case, C is diagonal matrix $\sigma^2$I. So that $T(x) = x^T s$.

We can obtain:

$$E(T|H0) = E[x^T s] = E[n^T s] = 0$$

$$E(T|H1) = E[x^T s] = E[(s+n)^T s] = E[s^T s] = \frac{NA^2}{2}$$

$$Var(T|H0) = E[(x^T s)^T (x^T s)] = E[s^T C s] = \frac{NA^2\sigma^2}{2}$$

So that d1 = 2 for all signal 1-3

ii. For Fm chirp signal: (4-5)

For fm chirp signal, the energy of signal is $s^T s$. And the noise power is $\sigma^2$=1. So that SNR for fm chirp signal is $\frac{s^T s}{N\sigma^2}$

So for signal 4: SNR4 = 0.2409

For signal 5: SNR5 = 0.2486

We can obtains:

$$E(T|H0) = E[x^T s] = E[n^T s] = 0$$

$$E(T|H1) = E[x^T s] = E[(s+n)^T s] = E[s^T s]$$

$$Var(T|H0) = E[(x^T s)^T (x^T s)] = E[s^T C s]$$

So we can calculate d:

d4 = $s^T s$ = 3.8549

d5 = $s^T s$ = 3.9773

b) For the second case: correlated noise and general matched filter

First, SNR is the same as before.

Second, Compute C:

We have

$$n(n) = \left(\frac{1}{1.81}^{0.5}\right) w(n) + \left(\frac{1}{1.81}^{0.5}\right) * 0.9 * w(n-1), w(n)\ is\ white\ Gaussian\ noise$$

C can be computed by:

$$C_{mn} = E\big(n(m)n(n)\big)$$

$$= \frac{1}{1.81}\big(1.81 r_{ww}(m-n) + 0.9 r_{ww}(m-n-1) + 0.9 r_{ww}(m-n+1)\big),$$

$$where\ r_{ww}(n) = 1\ when\ n = 0;\ r_{ww}(n) = 0\ otherwise$$

So, Cmn = 1 when m =n,

Cmn = 0.9/1.81 when |m-n| = 1,

Cmn = 0 otherwise.

Then we can obtain:

$$E(T|H0) = E[n^T C^{-1} s] = \ 0$$

$$E(T|H1) = E[(s+n)^T C^{-1} s] = E(s^T C^{-1} s + n^T C^{-1} s) = \ s^T C^{-1} s$$

$$Var(T|H0) = E\big((x^T C^{-1} s - 0)^T (x^T C^{-1} s - 0)\big)$$

$$= E(s^T C^{-1} n n^T C^{-1} s) = \ s^T C^{-1} s$$

Then

$d^2 \ = \frac{\left(E(T|H0)-E(T|H1)\right)^2}{Var(T|H0)} = s^T C^{-1} s$

Plug in each signal, we have:

$d1^2 \ = 2.0935$

$d2^2 \ = 4.2223$

$d3^2 \ = 14.7386$

$d4^2 \ = 7.4790$

$d5^2 \ = 5.1418$

c) For the third case: correlated noise and matched filter

First, SNR is computed as before.

Second, C is the same as b).

Then we compute d:

$$E(T|H0) = 0$$

$$E(T|H1) = E(s^T C^{-1} s + n^T C^{-1} s) = \ s^T s$$

$$Var(T|H0) = s^T C s$$

Then

$d^2 \ = \frac{\left(E(T|H0)-E(T|H1)\right)^2}{Var(T|H0)} = \frac{(s^T s)^2}{s^T C s}$

Plug in each signal, we have:

$d1^2 \ = 2.0847$

$d2^2 \ = 4$

$d3^2 \ = 13.4771$

$d4^2 = 3.8549$

$d5^2 = 3.9773$

4. The optimal signal is chosen by maximize $s^T C^{-1} s$ subject to the fixed energy constraint $s^T s = E$. Using Lagrangian multipliers I seek to maximize: $F = s^T C^{-1} s + \lambda(E - s^T s)$

   We have:

$$\frac{\partial F}{\partial \mathbf{s}} = 2\mathbf{C}^{-1}\mathbf{s} - 2\lambda\mathbf{s} = 0$$

$$\mathbf{C}^{-1}\mathbf{s} = \lambda\mathbf{s}.$$

   Hence, s is an eigenvector of $C^{-1}$, which corresponding eigenvalue is maximum.

   Because the energy is the same as 1-3 so SNR6 = SNR1 = 0.25.

   For the first case (white noise, matched filter), d of optimal signal is the same as before hence $d^2$ is 4.

   For the second case (correlated noise, general matched filter), the computation of d is the same as 3b) so that $d^2$ is 178.1127. (Note that for this case, to keep the energy to be same as before, we have to multiply the eigenvector by 2).

   For the third case (correlated noise, matched filter), the computation of d is the same as 3c) so that $d^2$ is 178.1127. (Note that for this case, to keep the energy to be same as before, we have to multiply the eigenvector by 2).

   We know that s is chosen as an eigenvector of $C^{-1}$ which corresponding eigenvalue is maximum. This is the case as second case. At the same time, for the third case, $d^2 = \frac{(E(T|H0) - E(T|H1))^2}{Var(T|H0)} = \frac{(s^T s)^2}{s^T C s}$, we can find that s is chosen as an eigenvector of C which corresponding eigenvalue is minimum. Because C is symmetric so that it is obvious that eigenvalue of C is 1/eigenvalue of $C^{-1}$. So that the second and third case, $d^2$ are the same.

- Results(including plots):

Plots:

**Signal 1: sn as defined in HW#3 with fc = 1/16**

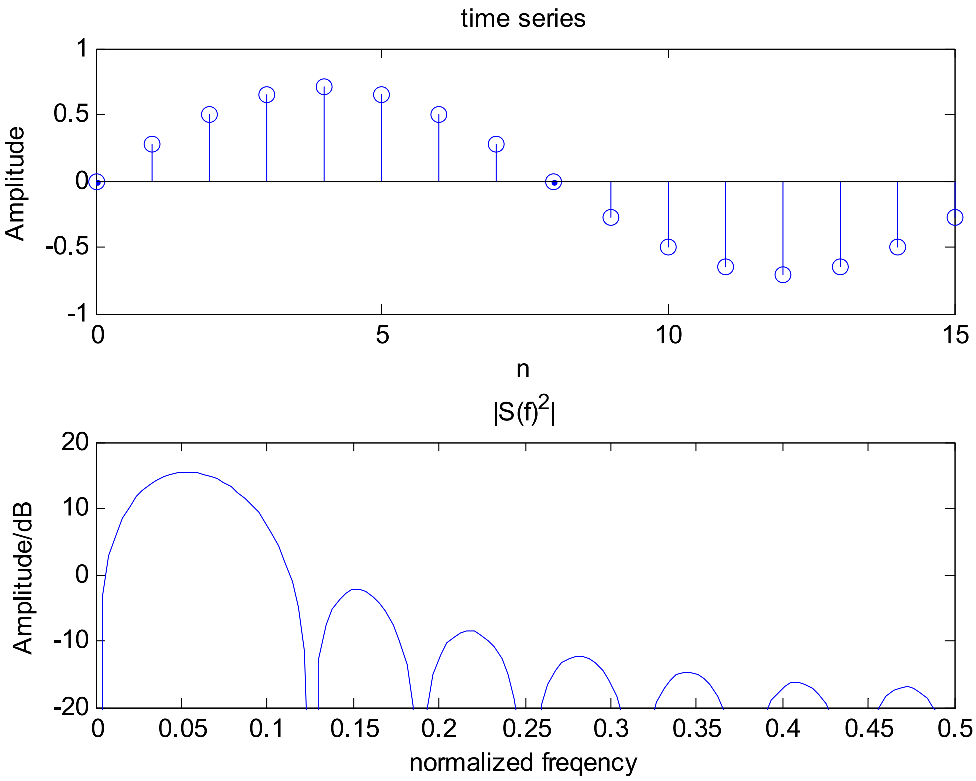


Figure 1 Signal 1-time series and sf

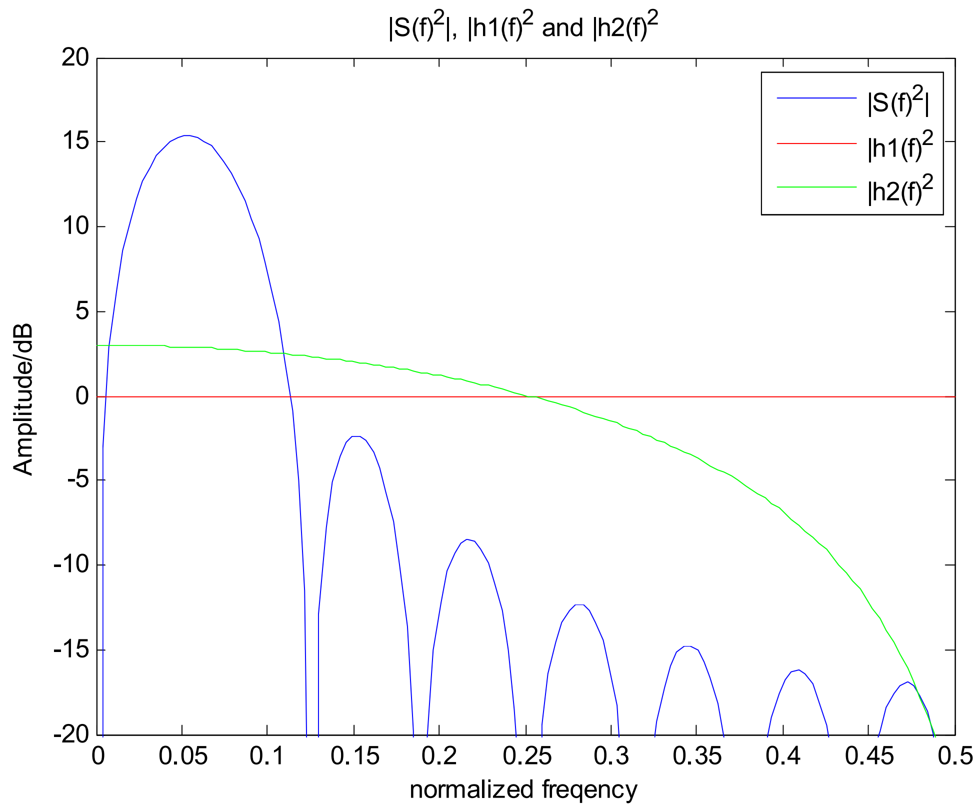


Figure 2 Signal 1 - sf h1f and h2f

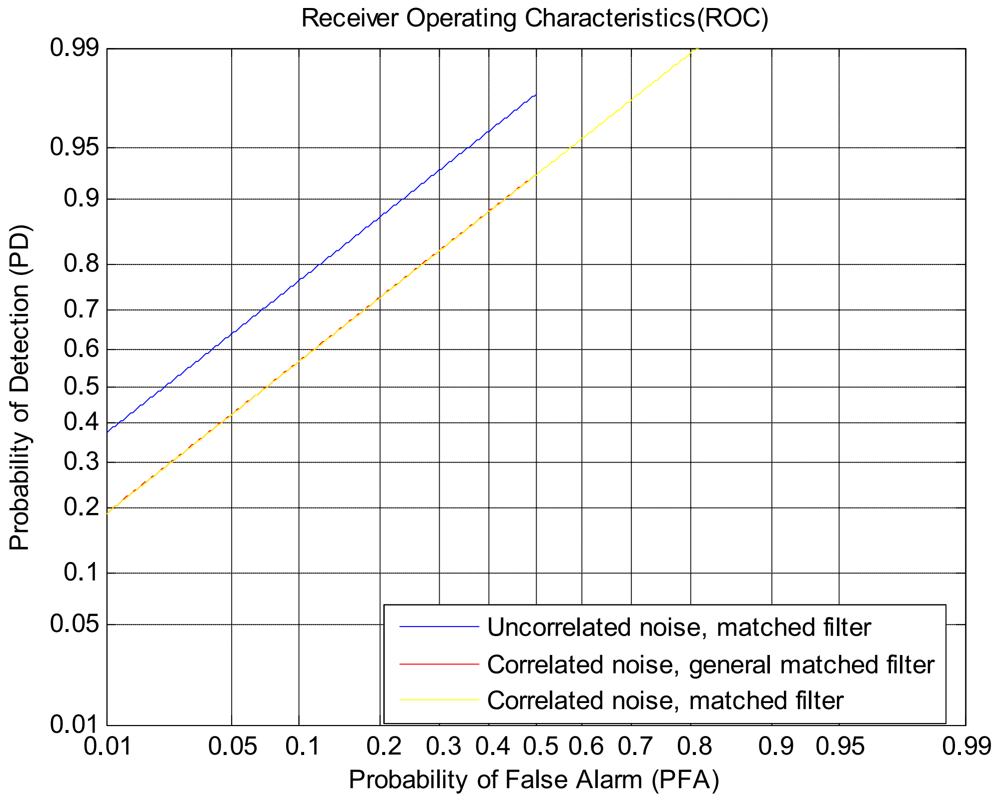


Figure 3 Signal 1 - ROC curves

Table 1 Signal 1 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 1/4 | 4 |
| Correlated noise and general matched filter | 1/4 | 2.0935 |
| Correlated noise and matched filter | 1/4 | 2.0847 |

**Signal 2: sn as defined in HW#3 with fc = 1/4**

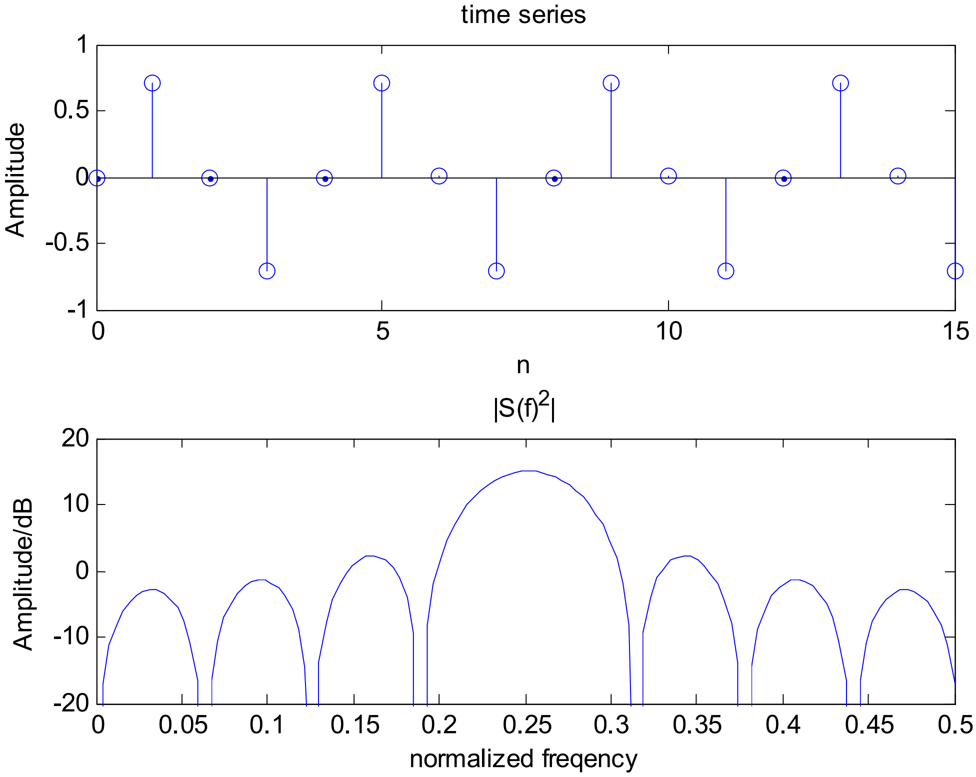


Figure 4 Signal 2 - time series and sf

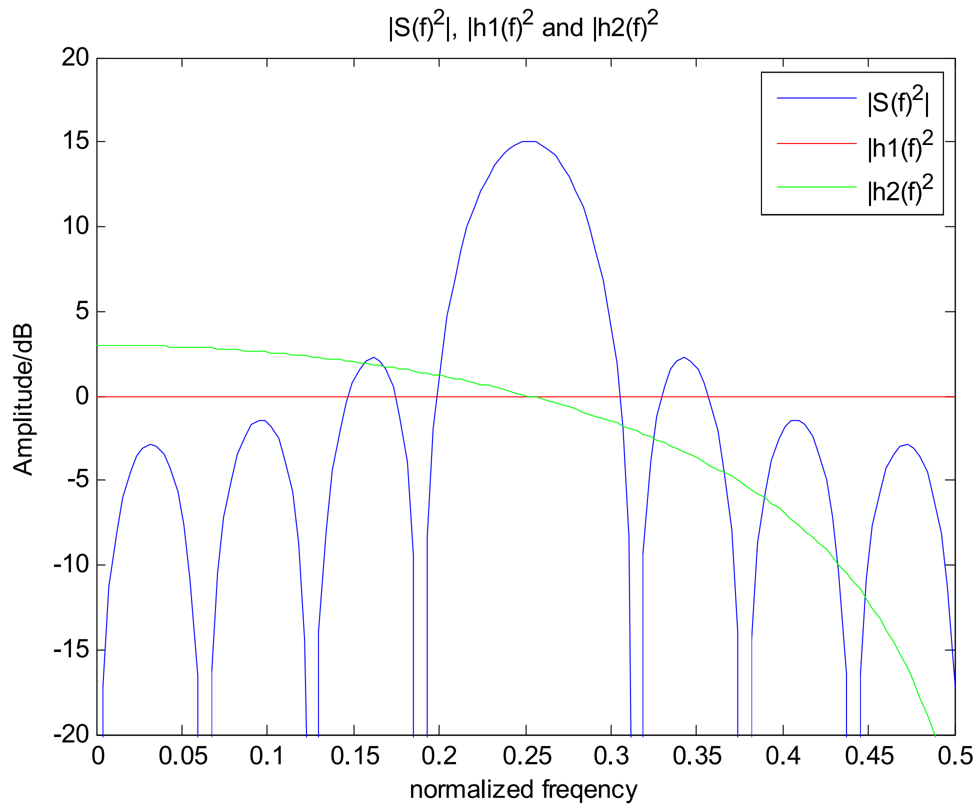


Figure 5 Signal 2 - sf, h1f and h2f

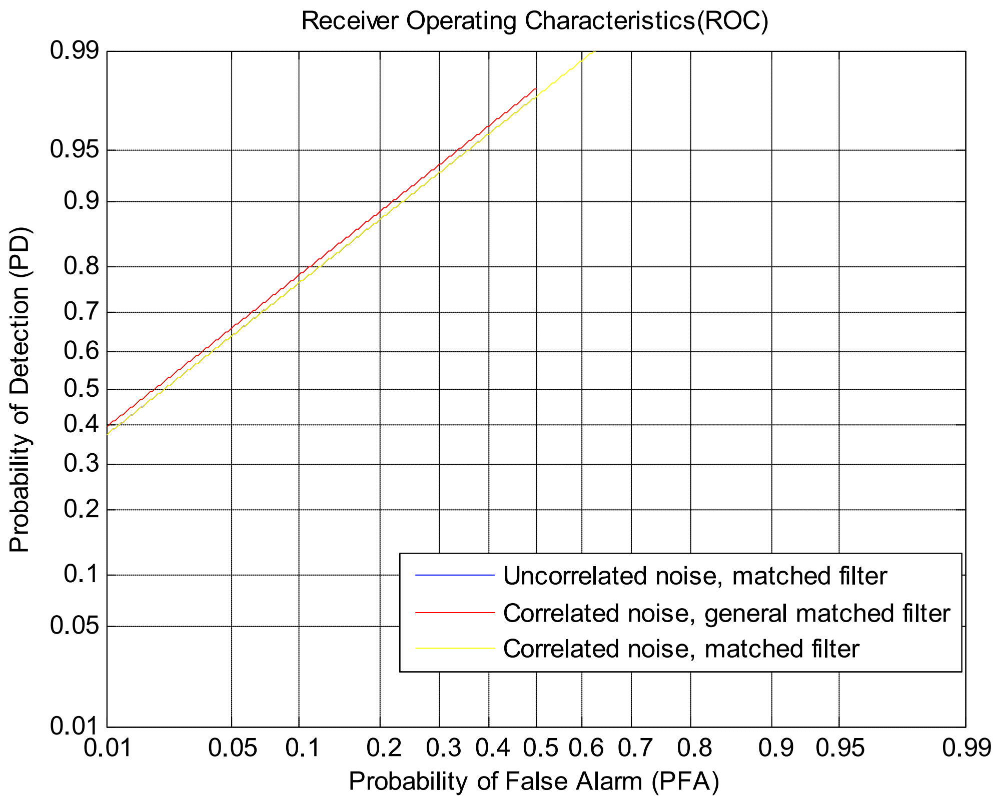


Figure 6 Signal 2 - ROC curves

Table 2 Signal 2 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 1/4 | 4 |
| Correlated noise and general matched filter | 1/4 | 4.2223 |
| Correlated noise and matched filter | 1/4 | 4 |

**Signal 3: sn as defined in HW#3 with fc = 3/8**

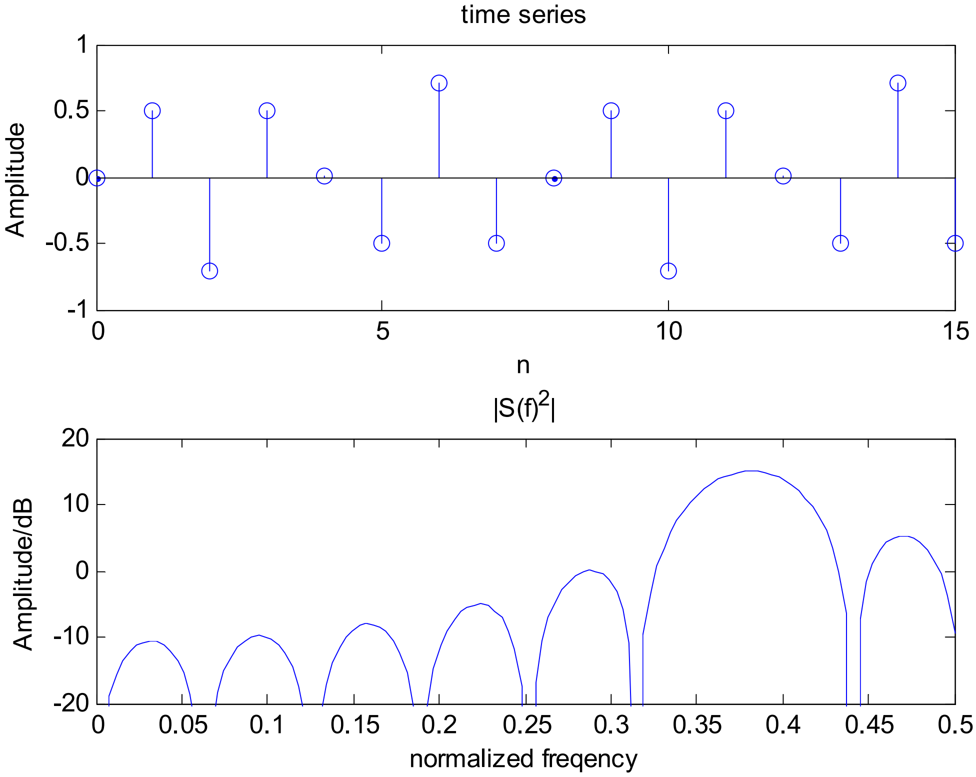


Figure 7 Signal 3 - time series and sf

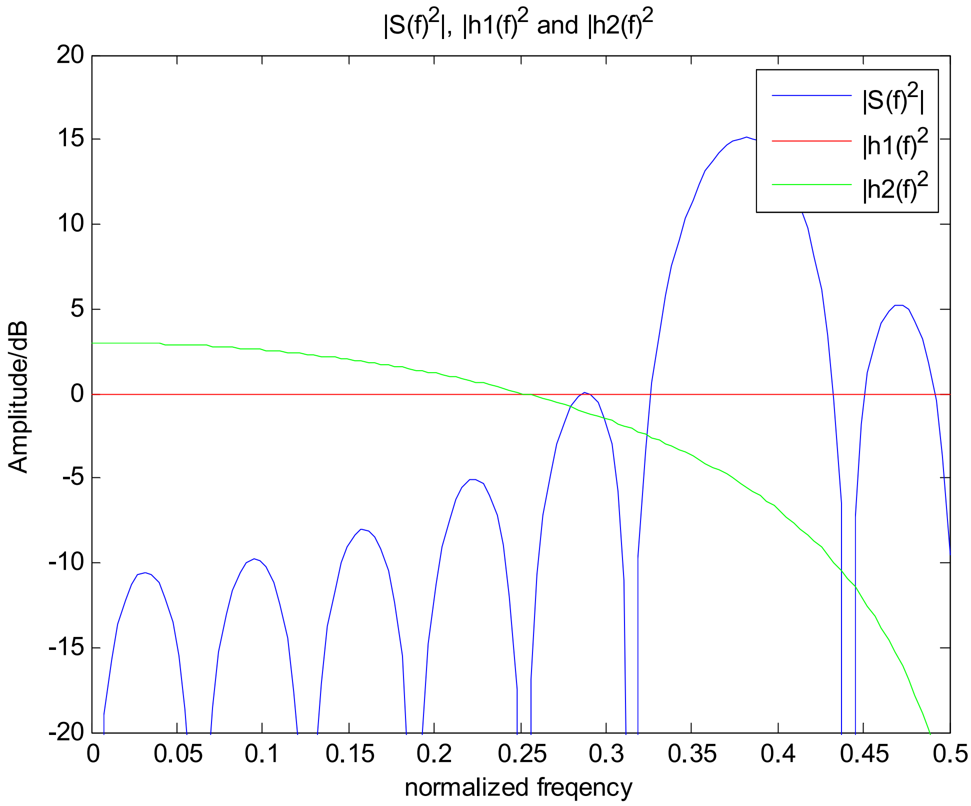


Figure 8 Signal 3 - sf, h1f and h2f

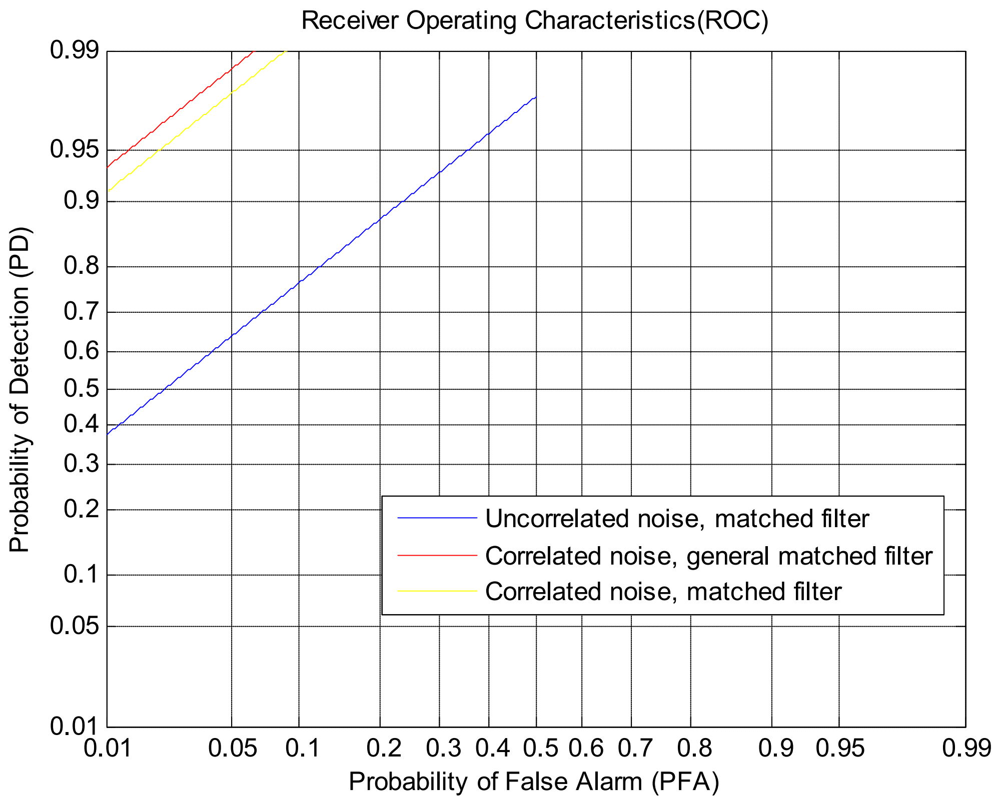


Figure 9 Signal 3 - ROC curves

Table 3 Signal 3 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 1/4 | 4 |
| Correlated noise and general matched filter | 1/4 | 14.7386 |
| Correlated noise and matched filter | 1/4 | 13.4771 |

**Signal 4: sn as FM chirp with fc = 1/16, f’ = 3/(8*16)**

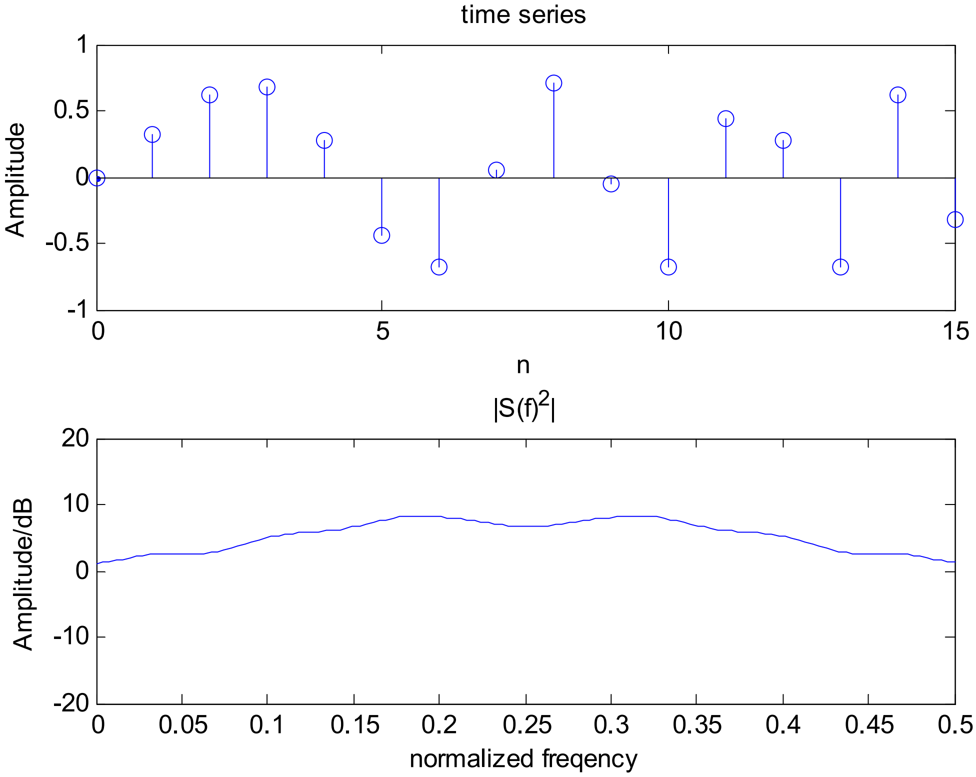


Figure 10 Signal 4 - time series and sf

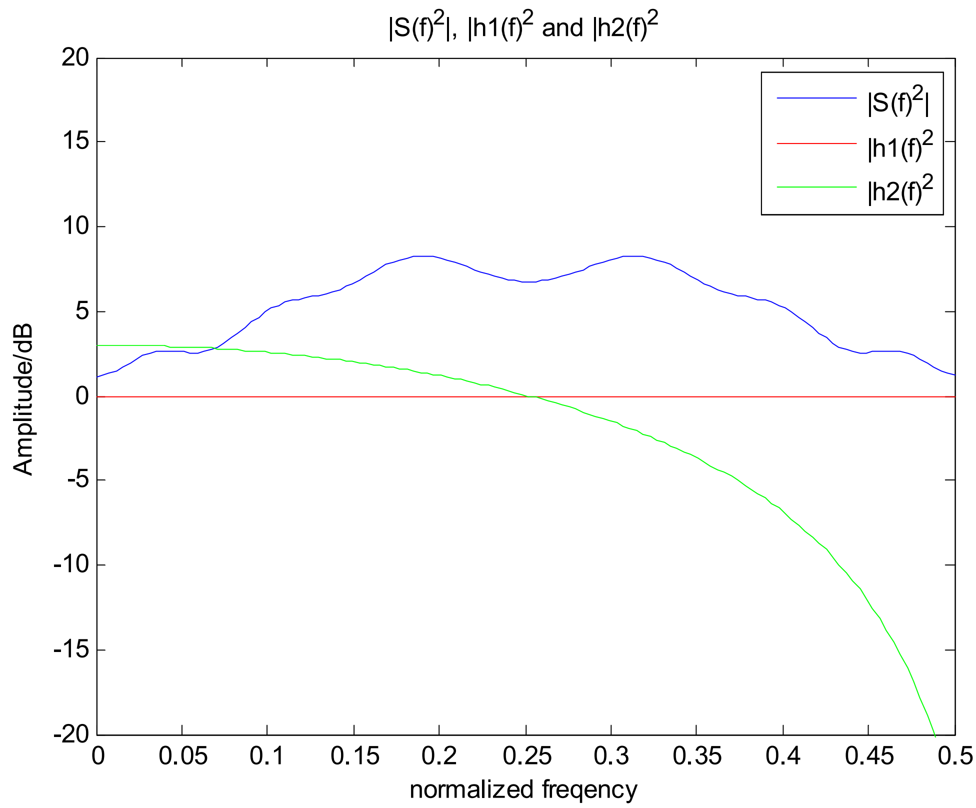


Figure 11 Signal 4 - sf, h1f and h2f

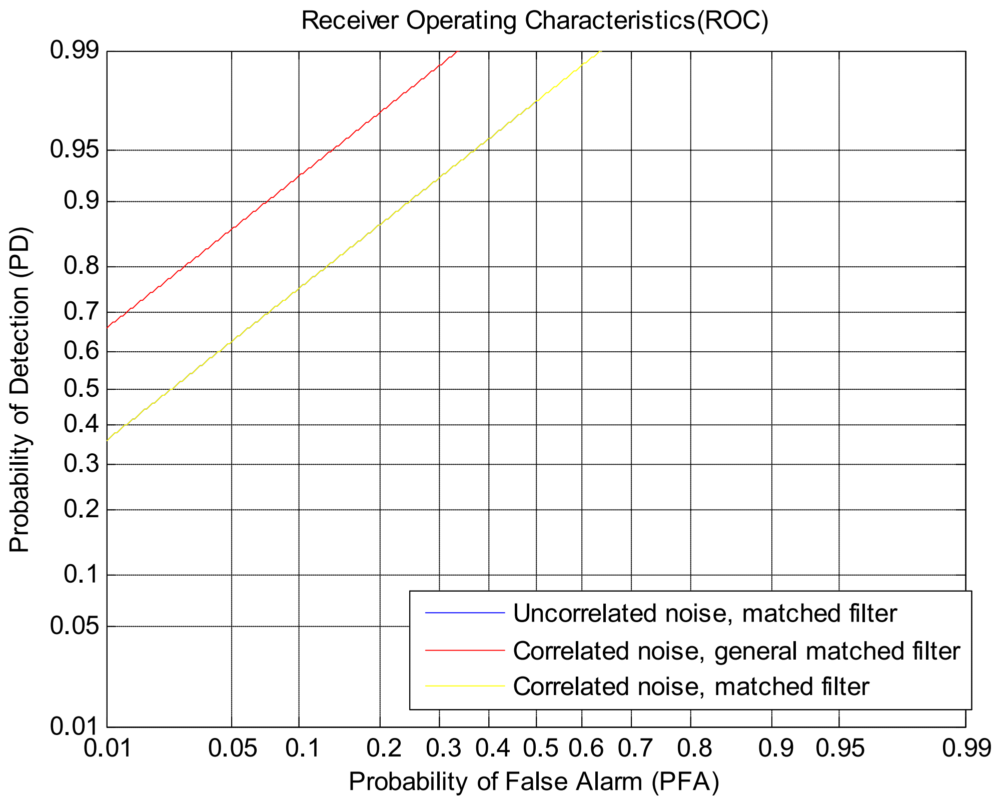


Figure 12 Signal 4 - ROC curves

Table 4 Signal 4 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 0.2409 | 3.8549 |
| Correlated noise and general matched filter | 0.2409 | 7.4790 |
| Correlated noise and matched filter | 0.2409 | 3.8549 |

**Signal 5: sn as FM chirp with fc = 1/8, f' = 1/(4*16)**

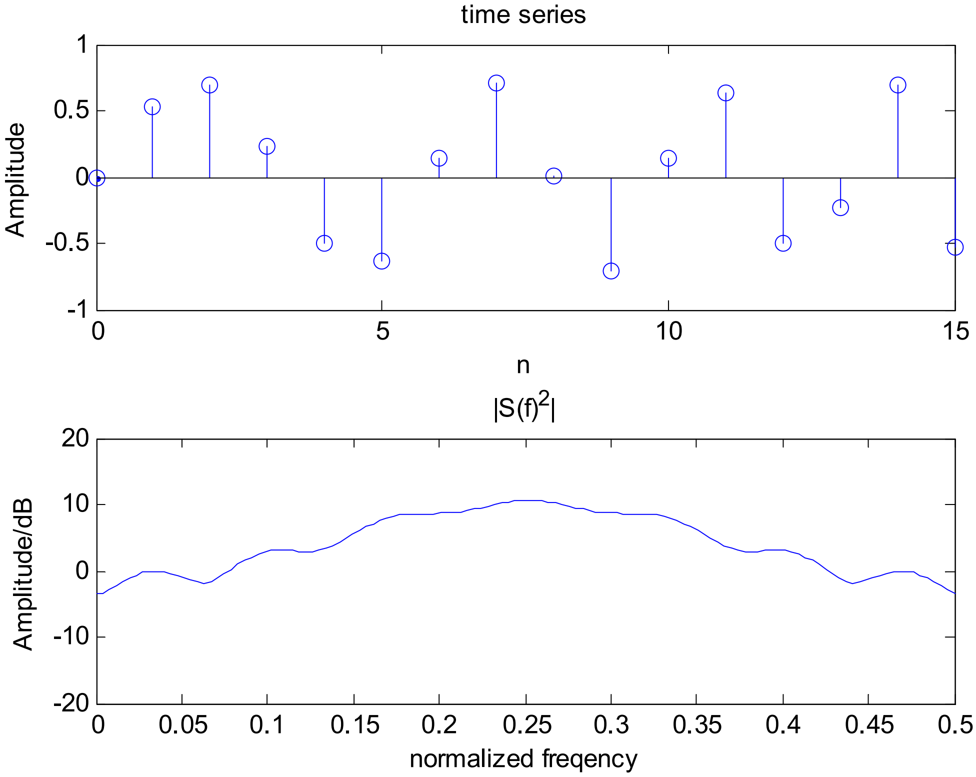


Figure 13 Signal 5 - time series and sf

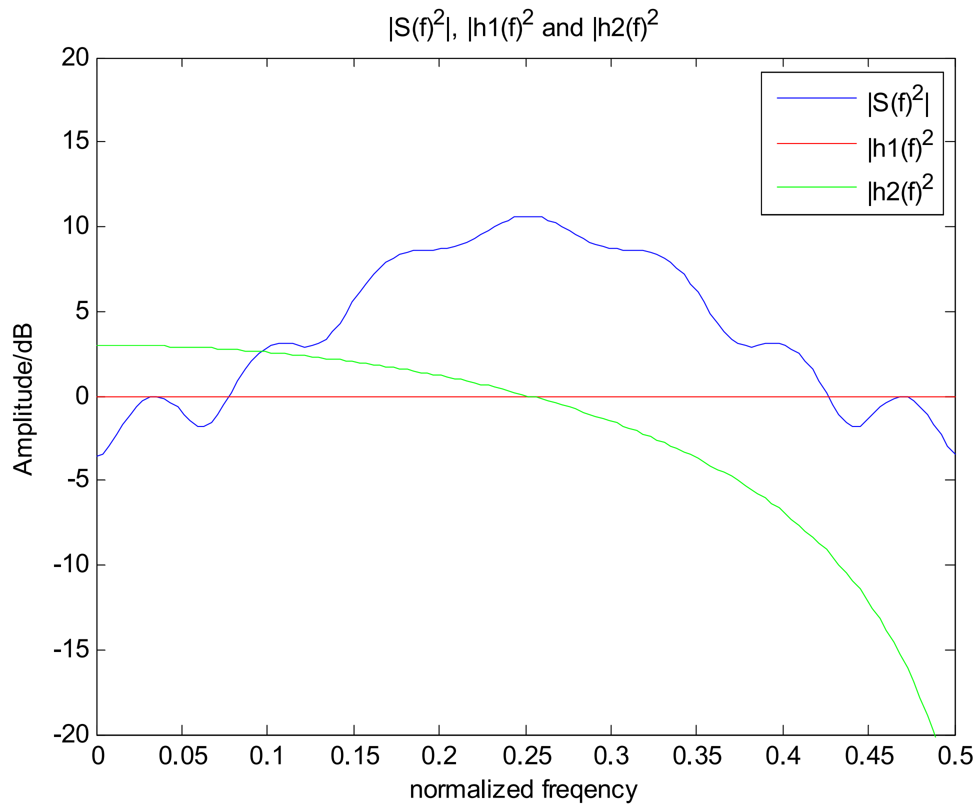


Figure 14 Signal 5 - sf, h1f and h2f

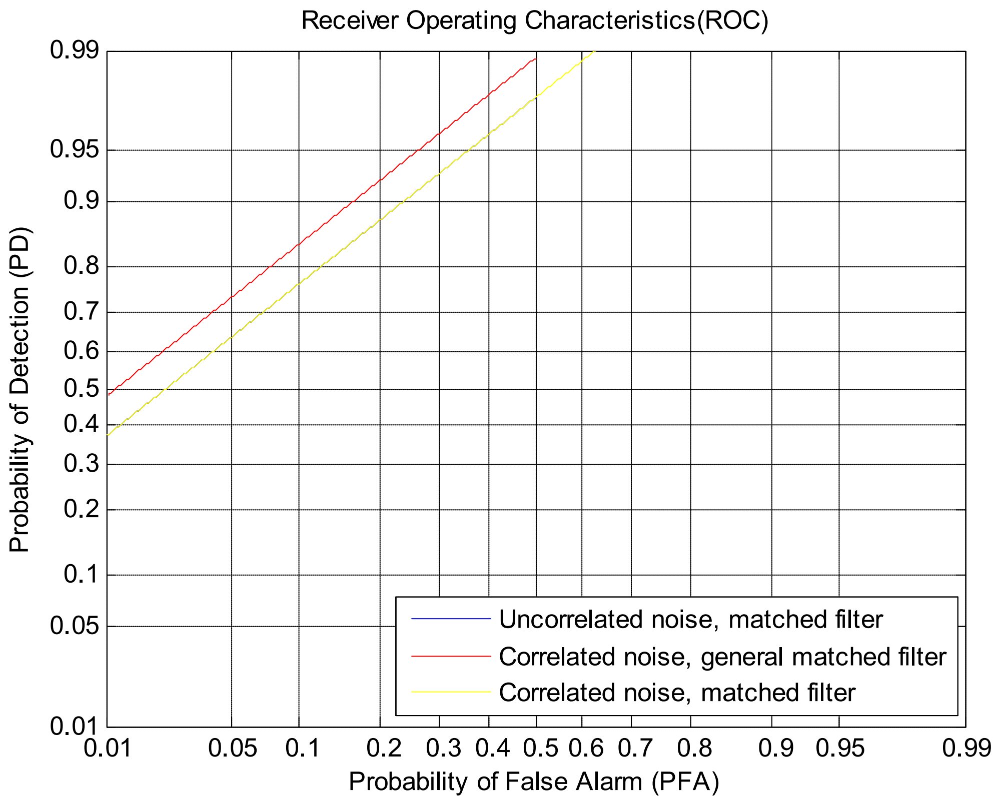


Figure 15 Signal 5 - ROC curves

Table 5 Signal 5 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 0.2486 | 3.9773 |
| Correlated noise and general matched filter | 0.2486 | 5.1418 |
| Correlated noise and matched filter | 0.2486 | 3.9773 |

**Signal 6: sn as optimal waveform**

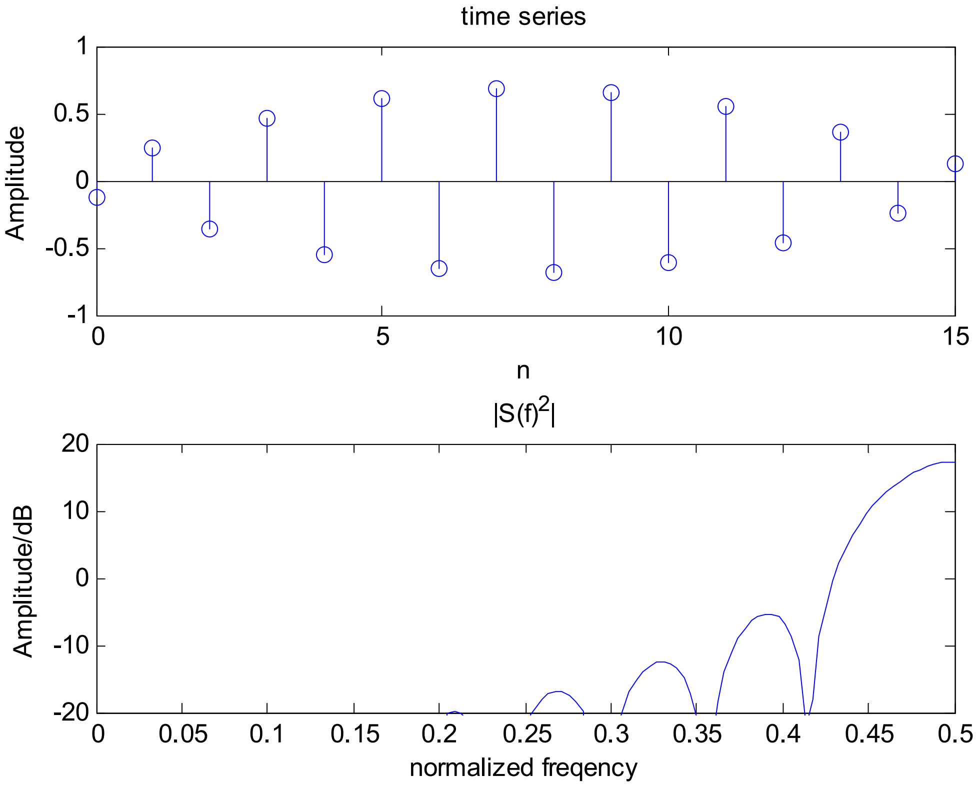


Figure 16 Signal 6 - time series and sf

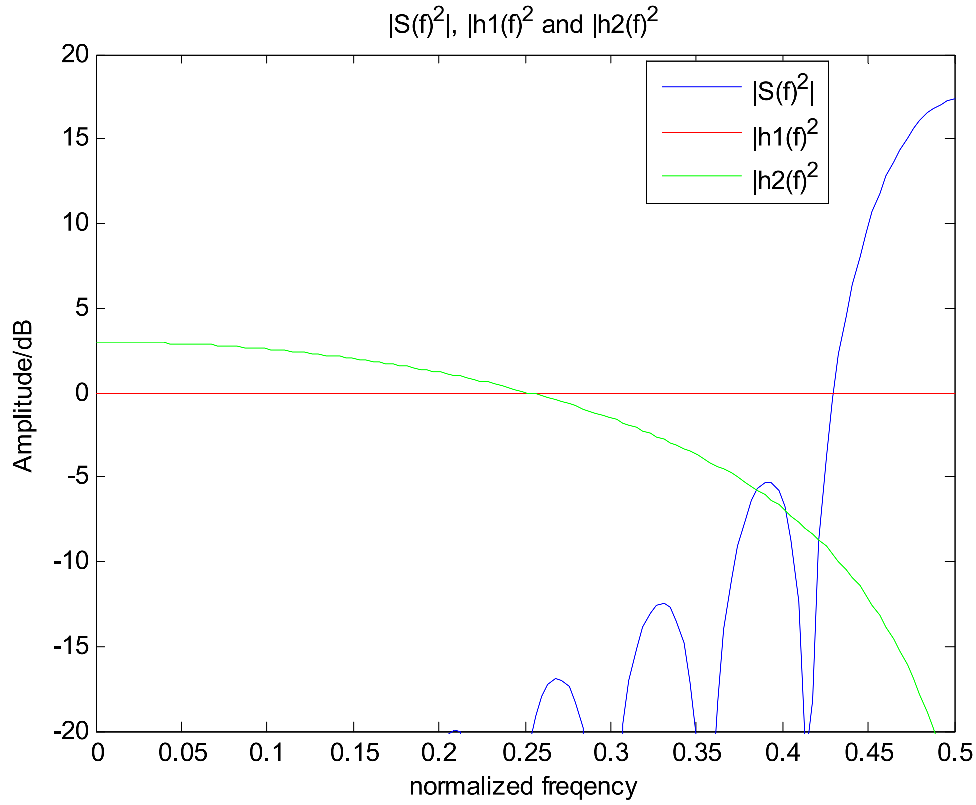


Figure 17 Signal 7 - sf, h1f and h2f

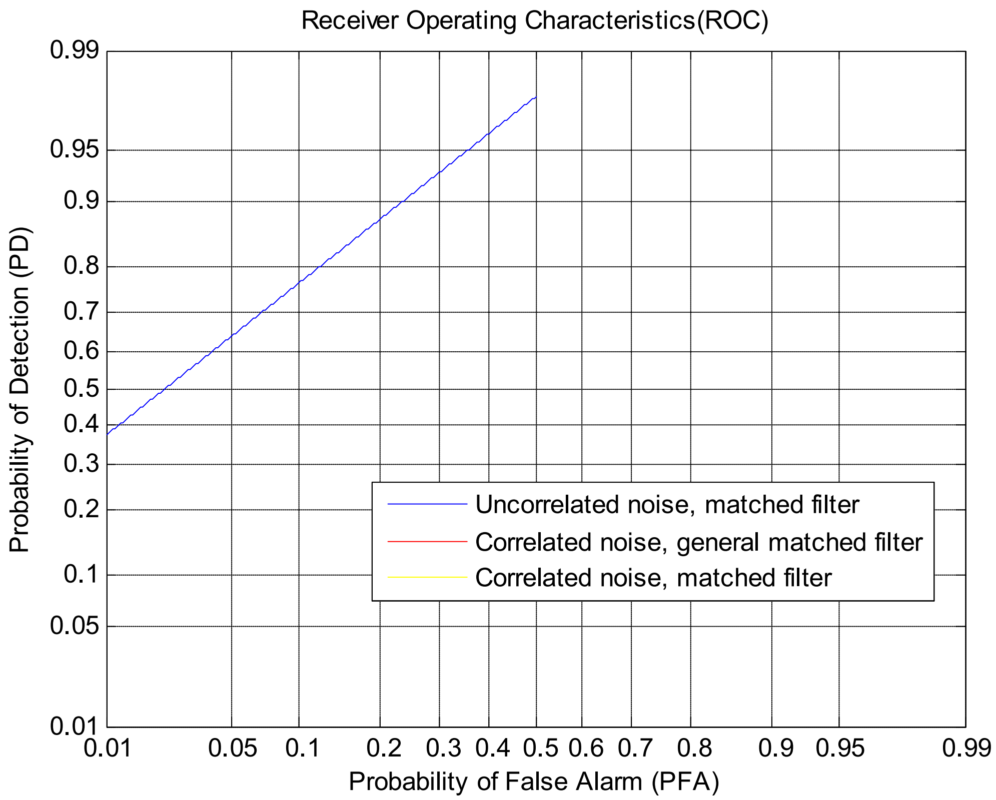


Figure 18 Signal 6 - ROC curves

Table 6 Signal 6 - SNR, d^2 for each case

| Noise type and filter | SNR | $d^2$ |
|---|---|---|
| White noise and matched filter | 0.25 | 4 |
| Correlated noise and general matched filter | 0.25 | 178.1127 |
| Correlated noise and matched filter | 0.25 | 178.1127 |

Table 7 SNR, d^2 for each signal and each case

| Signal | SNR | $d^2$(first case) | $d^2$(second case) | $d^2$(third case) |
|---|---|---|---|---|
| 1 | 0.25 | 4 | 2.0935 | 2.0847 |
| 2 | 0.25 | 4 | 4.2223 | 4 |
| 3 | 0.25 | 4 | 14.7386 | 13.4771 |
| 4 | 0.2409 | 3.8549 | 7.4790 | 3.8549 |
| 5 | 0.2409 | 3.9773 | 5.1418 | 3.9773 |
| 6 | 0.25 | 4 | 178.1127 | 178.1127 |

Discussion:

Note: I use the first case to represent 'uncorrelated noise with matched filter'. The second case to represent 'correlated noise with general matched filter' and the third case to represent 'correlated noise with matched filter'.

1. Signal 1: sn as defined in HW#3 with fc = 1/16
   a) Figure 1 is the time series of the signal and spectrum of S(n). For the time series, it appears to be a sin wave. With the fc = 1/16, the peak part of the spectrum is near 1/16 for normalized frequency.
   b) Figure 2 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.
   c) Figure 3 and Table 1 are the comparison of ROC performance of three different cases. It is obvious that d is larger, the ROC performance is better. The best one is uncorrelated noise with matched filter. The second best one is correlated noise with general matched filter. The worst is uncorrelated noise with matched filter. The reason is that for the correlated noise, it has more parts overlap with the signal (low frequency). But we can also find that the third case is only a little bit worse than the second one, this is because in the low-frequency, where signal concentrates, the correlated noise is flat and really close to the uncorrelated noise.

2. Signal 2: sn as defined in HW#3 with fc = 1/4
   a) Figure 4 is the time series of the signal and spectrum of S(n). For the time series, it appears to be a sin wave. With the fc = 1/4, the peak part of the spectrum is near 1/4 for normalized frequency.
   b) Figure 5 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.
   c) Figure 6 and Table 2 are the comparison of ROC performance of three different cases. For this signal, the second case is a little bit better than the first and third case and the first and third case are the same. The correlated noise is symmetric with the center of 0.25 in linear axis so $S^TCS$ is nearly equal to $S^TS$. In this case, the first and the third are the same with ROC performance.

3. Signal 3: sn as defined in HW#3 with fc = 3/8
   a) Figure 7 is the time series of the signal and spectrum of S(n). For the time series, it appears to be a sin wave. With the fc = 3/8, the peak part of the spectrum is near 1/4 for normalized frequency.
   b) Figure 8 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.
   c) Figure 9 and Table 3 are the comparison of ROC performance of three different cases. For this signal, the first case is worse than the second and third case. This is because the correlated noise is relatively weak at the high frequency while the signal is strong is this frequency, which

means noise and signal have less overlap frequency. Actually the second case is better than the third case because the general matched filter is designed for the correlated noise and perform better in relatively high frequency, where the noise is low.

4. Signal 4: sn as FM chirp with fc = 1/16, f' = 3/(8*16)
   a) Figure 10 is the time series of the signal and spectrum of S(n). For the time series, it appears to be a FM chirp wave. With the fc = 1/16, f' = 3/(8*16), the signal is symmetric with the center of f = 0.25.
   b) Figure 11 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.
   c) Figure 12 and Table 4 are the comparison of ROC performance of three different cases. For this signal, the second case is the best, the first and third cases are the same. This is because general matched filter which has better performance for relatively high frequency is designed for correlated noise and the correlated noise is relatively low in high frequency. The reason that the first case and third cases are the same is similar to Signal 2.

5. Signal 5: sn as FM chirp with fc = 1/8, f' = 1/(4*16)
   a) Figure 13 is the time series of the signal and spectrum of S(n). For the time series, it appears to be a FM chirp wave. With the fc = 1/16, f' = 3/(8*16), the signal is symmetric with the center of f = 0.25.
   b) Figure 14 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.
   c) Figure 15 and Table 5 are the comparison of ROC performance of three different cases. For this signal, it performs similar to Signal 4. The analysis is also similar. The difference is it performs worse than Signal 4 because the signal energy in Signal 5 is high frequency is relatively low compared to Signal 4.

6. Signal 6: sn as optimal waveform
   a) Figure 16 is the time series of the signal and spectrum of S(n). For the time series, the signal mainly concentrate on the high frequency.
   b) Figure 17 is the overplot of $|S(f)|^2$, $|H1(f)|^2$ and $|H2(f)|^2$. We can see that $|H2(f)|^2$ is low-pass and $|H1(f)|^2$ is an all-pass filter.

c) Figure 18 and Table 6 are the comparison of ROC performance of three different cases. For this signal, the second and third cases is out of the plot with pd = 1. They are much better than the first case. This is due to for the correlated noise, the noise energy is very low in the high frequency, where the signal energy mainly concentrates on.
The proof why the second and third cases perform the same is given in Approach part.

- Conclusion:

  As stated above, we can find we cannot say for which processor the ROC performance is better or for which signal the performance is better. It depends on the noise type, signal type and also the processor type. Be specific:

  a) The ROC performance is determined by the overlap part of signal and noise. It is obvious that the more the overlap part, the worse of the performance.
  b) The ROC performance is also determined by the energy of noise at the frequency that signal energy concentrates on. If the noise is relatively low in the frequency where signal mainly concentrates on, the ROC performance is better.
  c) For the optimal waveform, the general matched filter performs the same as mismatched filter.

- **Appendix:**

```
clear;clc;

C=eye(16);

for m=1:16

    for n=1:16

        if abs(m-n)==1

            C(m,n)=0.9/1.81;

            C(n,m)=0.9/1.81;

        end

    end

end

n = 0:15;

N= 256;

fc = 1/16;

fp = 3/(8*16);

d1=4;

A = (1/2)^(1/2);

sn = A*sin(2*pi*fc*n);

% sn = A*sin(2*pi*(fc+fp*n/2).*n);

hn1 =1;

hn2 = (1/1.81)^(1/2)*[1, 0.9];

omega = 0:0.5/127:0.5;

%optimal
```

```
% [V, D] = eig(inv(C));

% ei = V(:, 16) * 2;

% dmax = ei'/C*ei;

% sn = ei';


sf = 10*log10((abs(fft(sn,N))).^2);

hf1 = 10*log10((abs(fft(hn1, N))).^2);

hf2 = 10*log10((abs(fft(hn2, N))).^2);

figure(1)

subplot(211)

stem(n, sn)

title('time series')

xlabel('n')

ylabel('Amplitude')

subplot(212)

plot(omega,sf(1:128));

title('|S(f)^2|')

xlabel('normalized freqency')

ylabel('Amplitude/dB')

axis([0 0.5 -20 20])

figure(2);

plot(omega,sf(1:128), 'b');
```

```
hold on

plot(omega,hf1(1:128), 'r')

hold on

plot(omega,hf2(1:128), 'g')

title('|S(f)^2|, |h1(f)^2 and |h2(f)^2');

xlabel('normalized freqency');

ylabel('Amplitude/dB')

legend('|S(f)^2|','|h1(f)^2','|h2(f)^2');

axis([0 0.5 -20 20])

% subplot(4,1,2)

% plot(omega,sf(1:128));

% title('|S(f)^2|')

% xlabel('w/rad')

% ylabel('Amplitude/dB')

% axis([0 pi -30 30])

% subplot(4,1,3)

% plot(omega,hf1(1:128))

% title('|h1(f)^2|')

% xlabel('w/rad')

% ylabel('Amplitude/dB')

% subplot(4,1,4)

% plot(omega,hf2(1:128))
```

```
% title('|h2(f)^2|')

% xlabel('w/rad')

% ylabel('Amplitude/dB')

% axis([0 pi -10 5])



%1st


PF1=0:0.0001:0.5;

a1=Qinv(PF1);

PD1=Q(a1-d1^(1/2));


%2nd


d2=sn/C*sn';

PF2=0:0.0001:0.5;

a2=Qinv(PF2);

PD2=Q(a2-d2^(1/2));


% 3rd

ETH1=sn*sn';

varTH0=sn*C*sn';
```

```
d3=ETH1^2/varTH0;

PF3=0:0.0001:0.95;

a3=Qinv(PF3);

PD3=Q(a3-d3^(1/2));


figure(3)

probpaper(PF1,PD1, 'b')

hold on

probpaper(PF2,PD2, 'r')

probpaper(PF3,PD3, 'y')

legend('Uncorrelated noise, matched filter', 'Correlated noise, general matched filter',

'Correlated noise, matched filter');
```