Rayleigh Fading Sinusoid

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Under H_1: $x(n) = A\cos(2\pi f_0 + \phi) + w(n) \quad n = 0, \ldots, N-1$

Unpredictable amplitude and phase due to time varying multipath typical of underwater acoustic channels or the tropo scatter channel.

If observation time T is short, then can model the received signal as a pure sinusoid with unknown amplitude and phase. Reasonable to assume that they are independent random variables.

Rather than assign a pdf to A and ϕ, more convenient to use

$$s(n) = A\cos(2\pi f_0 n + \phi) = a\cos(2\pi f_0 n) + b\sin(2\pi f_0 n)$$

Where: $a = A\cos\phi$

$b = -A\sin\phi$

and assign a pdf to $\begin{bmatrix} a \\ b \end{bmatrix}$

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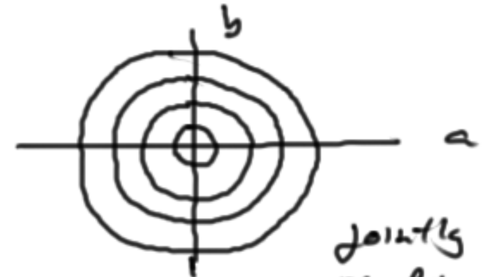
Note that signal model is now linear in the random parameters a & b

$$\underline{\theta} = \begin{bmatrix} a \\ b \end{bmatrix} \sim N(0, \sigma_s^2 I)$$



jointly Gaussian random variables

Thus $A = \sqrt{a^2+b^2}$ is Rayleigh

$\phi = \tan^{-1}\left(-\frac{b}{a}\right)$ is uniform

and A and ϕ are independent

Bayesian linear model $\underline{x} = H\underline{\theta} + \underline{w}$

where:

$$H = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 (1) & \sin 2\pi f_0 (1) \\ \vdots & \vdots \\ \cos 2\pi f_0 (N-1) & \sin 2\pi f_0 (N-1) \end{bmatrix}$$

$\underline{\theta} \sim N(0, \sigma_s^2 I)$ and $\underline{w} \sim N(0, \sigma^2 I)$

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$H_0: \underline{x} = \underline{w}$

$H_1: \underline{x} = H\underline{\theta} + \underline{w}$

$$T(\underline{x}) = \underline{x}^T H C_\theta H^T (H C_\theta H^T + \sigma^2 I)^{-1} \underline{x} \qquad \text{Eq. 5.18}$$

$$= \sigma_s^2 \underline{x}^T H H^T (\sigma_s^2 H H^T + \sigma^2 I)^{-1} \underline{x}$$

Making use of the matrix inversion lemma

$$= \sigma_s^2 \underline{x}^T H H^T \Big\{ \frac{1}{\sigma^2} \overset{N\times N}{I} - \frac{1}{\sigma^4} \sigma_s^2 H \Big(\sigma_s^2 \overset{2\times 2}{\frac{H^T H}{\sigma^2}} + I \Big)^{-1} H^T \Big\} \underline{x}$$

Assuming N large and $0 < f_0 < 1/2$, $H^T H \simeq \frac{N}{2} \overset{2\times 2}{I}$

$$= \sigma_s^2 \underline{x}^T H H^T \Big\{ \frac{1}{\sigma^2} I - \frac{\sigma_s^2}{\sigma^4} H \frac{1}{\frac{N\sigma_s^2}{2\sigma^2} + 1} I H^T \Big\} \underline{x}$$

$$= \frac{\sigma_s^2}{\sigma^2} \underline{x}^T H H^T \underline{x} - \frac{\frac{N\sigma_s^4}{2\sigma^4}}{\frac{N\sigma_s^2}{2\sigma^2} + 1} \underline{x}^T H H^T \underline{x}$$

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$$= \frac{c}{N} \underline{x}^T H H^T \underline{x}$$

Where: $c = \dfrac{N\sigma_s^2}{\frac{N}{2}\sigma_s^2 + \sigma^2}$

Thus

$$T'(\underline{x}) = \frac{1}{N} \underline{x}^T H H^T \underline{x} = \frac{1}{N} \| H^T \underline{x} \|^2$$

$$= \frac{1}{N} \left\| \begin{matrix} \sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n) \\ \sum_{n=0}^{N-1} x(n) \sin(2\pi f_0 n) \end{matrix} \right\|^2$$

$$= \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j 2\pi f_0 n} \right|^2$$

i.e. $|FFT|^2$ for frequency bin at f_0

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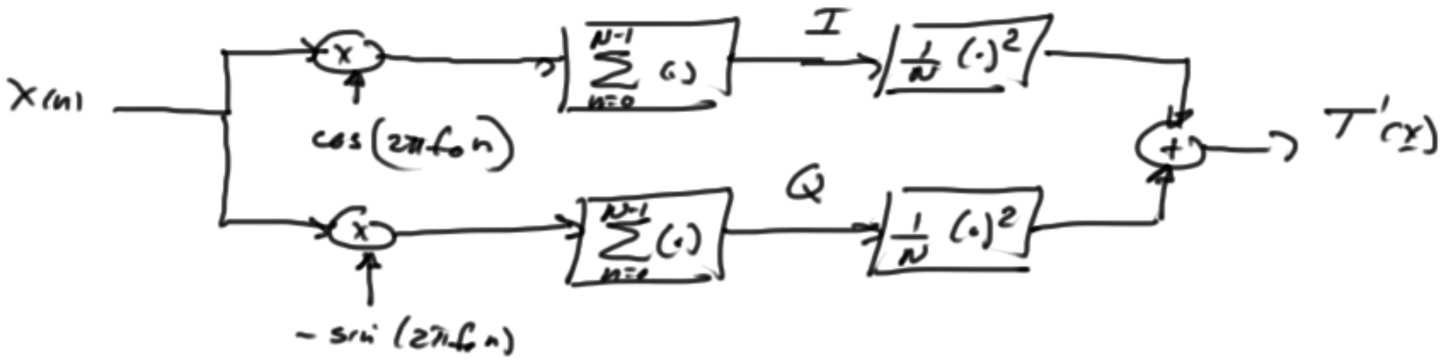

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Performance for Rayleigh Fading Sinusoid in White Noise

Simplified due to the orthogonal representation in H

$$C_s = H C_{\underline{\theta}} H^T = H(\sigma_s^2 I)H^T = \sigma_s^2 H H^T \qquad H = [\underline{h}_0 \ \underline{h}_1]$$

$$= \sigma_s^2 [\underline{h}_0 \ \underline{h}_1] \begin{bmatrix} \underline{h}_0^T \\ \underline{h}_1^T \end{bmatrix}$$

$$= \frac{N}{2} \sigma_s^2 \frac{\underline{h}_0}{\sqrt{N/2}} \frac{\underline{h}_0^T}{\sqrt{N/2}} + \frac{N}{2} \sigma_s^2 \frac{\underline{h}_1}{\sqrt{N/2}} \frac{\underline{h}_1^T}{\sqrt{N/2}}$$

Letting $\lambda_{s_0} = \lambda_{s_1} = \frac{N}{2}\sigma_s^2$, $\underline{e}_0 = \frac{\underline{h}_0}{\sqrt{N/2}}$, $\underline{e}_1 = \frac{\underline{h}_1}{\sqrt{N/2}}$

Then $C_s = \lambda_{s_0} \underline{e}_0 \underline{e}_0^T + \lambda_{s_1} \underline{e}_1 \underline{e}_1^T$

and $\underline{e}_0^T \underline{e}_1 = \frac{2}{N} \sum_{n=0}^{N-1} \cos(2\pi f_0 n) \sin(2\pi f_0 n) \simeq 0$ for large N

Thus, $\underline{e}_0$ and $\underline{e}_1$ are approximately eigenvectors of C_s with eigenvalues λ_{s_0} and λ_{s_1}.

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Performance

The general performance expression (Eq. 5.10 on p.151)

simplifies

$$P_F = \exp\left(-\frac{\gamma'''}{\sigma^2}\right)$$

$$P_D = \exp\left(-\frac{\gamma'''}{\frac{N}{2}\sigma_s^2 + \sigma^2}\right)$$

Note that $T'(\underline{x})$ is sum of squares of two zero mean Gaussian random variables under both H_0 and H_1

To relate P_D and P_F, let $\bar{\eta} = N \frac{E[A^2/2]}{\sigma^2} = N \frac{\sigma_s^2}{\sigma^2}$

average or expected ENR

$$P_D = \exp\left(-\frac{\gamma'''}{\sigma^2\left(\frac{\bar{\eta}}{2}+1\right)}\right)$$

Substituting $\gamma''' = \sigma^2 \ln\left(\frac{1}{P_F}\right)$

$$P_D = P_F^{\left(\frac{1}{\frac{\bar{\eta}}{2}+1}\right)}$$

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Note that P_D increases slowly with increasing $\bar{\mathcal{R}} = \frac{\bar{\mathcal{E}}}{\sigma^2}$

Due to sinusoid amplitude having a Rayleigh PDF.

Even as σ_s^2 increases, there still is a high probability that the sinusoid amplitude will be small and results in the average probability of detection not increasing rapidly with σ_s^2.

$$E[A] = \sqrt{\pi/2}\,\sigma_s$$

$$E[A^2] = 2\sigma_s^2$$