Rayleigh Fading Signal

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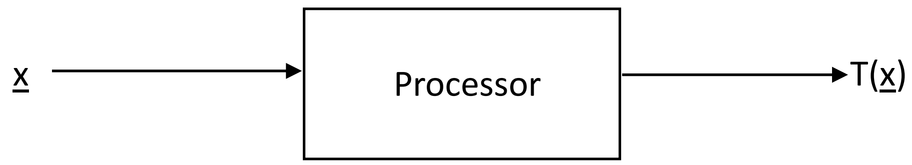
Title :

Rayleigh Fading Signal

Objective (purpose) :

To decide presence or absence of a signal buried in uncorrelated Gaussian noise where:
H1:x(n)=s(n)+w(n),n=0,1,…,N-1
H0:x(n)=w(n),n=0,1,…,N-1
w(n) is an uncorrelated, Gaussian noise sequence ~N(0,σ^2).
The processor structure is below:



x(n) equals that s(n) adds noise w(n). And we use matlab to realize all of these tasks and plot what we get.

Background (Introduction):

Rayleigh fading is a statistical model for the effect of a propagation environment on a radio signal, such as that used by wireless devices. Rayleigh fading models assume that the magnitude of a signal that has passed through such a transmission medium (also called a communications channel) will vary randomly, or fade, according to a Rayleigh distribution — the radial component of the sum of two uncorrelated Gaussian random variables.Rayleigh fading is viewed as a reasonable model for tropospheric and ionospheric signal propagation as well as the effect of heavily built-up urban environments on radio signals. Rayleigh fading is most applicable when there is no dominant propagation along a line of sight between the transmitter and receiver. If there is a dominant line of sight, Rician fading may be more applicable.

Approach(Procedure):

1. Consider three different classes of signals:

A.s(n)=Asin(2π$f_c n + \varphi$),$f_c = \frac{1}{16}$

A known and φ uniformly distributed.

B.s(n)=Asin(2π$f_c + \varphi$), $f_c = \frac{1}{16}$

A Rayleigh distributed and φ uniformly distributed.
C.s(n)=w_s(n)

Uncorrelated Gaussian signal~N(0,$\sigma_s{}^2$).

2. Summarize briefly the analytical derivation of the test statistic and performance for the following optimum detection receivers:

A.SKEP (N=128)

B. Rayleigh fading sinusoid (N=128)

C. Energy detector (N=128 and N=16).

Express P_D in terms of P_F for the SKEP and Rayleigh fading sinusoid processors.

Solution:

(1).In terms of SKEP,s(n)= Asin(2π$f_c n+\varphi$),when φ is a random variable distributed uniformly between –π andπ.

$\Lambda(\mathrm{x}) = \frac{p(x|H1)}{p(x|H0)} = \frac{\int_{-\pi}^{\pi} p(x|\varphi 1H1)p(\varphi)d\varphi}{p(x|H0)} = \int_{-\pi}^{\pi}\Lambda(x|\varphi)p(\varphi)d\varphi$ (where $\Lambda(x|\varphi) = \frac{p(x|\varphi 1H1)}{p(x|H0)}$)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-\frac{E}{N_0}+\frac{A}{\sigma^2}\sum_{N=0}^{N-1}x(n)\,\mathrm{Asin}(2\pi f_c n+\varphi)]\mathrm{d}\varphi$$

=$\exp\left[-\frac{E}{N_0}\right]I_0[\frac{A}{\sigma^2}(a^2+b^2)^{\frac{1}{2}}]$ $(\sin(2\pi f_c n+\varphi) = \sin(2\pi f_c n)\cos(\varphi)+$ $cos(2\pi f_c n)\sin(\varphi)$ and I_0 is the modified Bessel function of order 0.)

$\Lambda(\mathrm{x}) = \exp\left[-\frac{E}{N_0}\right]I_0[\frac{A}{\sigma^2}\left(a^2+b^2\right)^{\frac{1}{2}}]$ $\Lambda(\mathrm{x}) \propto (a^2+b^2)$

a= $\sum_{n=0}^{N-1}x(n)*cos(2\pi f_c n)$, b= $\sum_{n=0}^{N-1}x(n)*sin(2\pi f_c n)$

where test statistic T(x)=$I_N(k) = \frac{1}{N}|X(k)|^2 = \frac{1}{N}(a^2+b^2)$

E$[I_N(k)|H0] = \sigma^2$,var$[I_N(k)|H0]$=σ^4; E$[I_N(k)|H1] = \sigma^2+\frac{A^2N}{4}$, var$[I_N(k)|H1] =$ $\sigma^4+\frac{A^2N}{2}\sigma^2$

And we know that this Test statistic T(x) is subject to the Chi square distribution according to K text in chapter 2. It has v=2 degrees of freedom and on condition H0 is central Chi square and on condition H1 is non-central with noncentrality parameter λ=$\sum_{i=1}^{v}\mu_i{}^2 = \frac{A^2N}{2\sigma^2}$.

Threshold x'=$\frac{I_N(k)}{\frac{\sigma^2}{2}}$ on condition H1

$P_F = Q_{\chi_2{}^2}(x) = \exp\left(-\frac{x}{2}\right)$ *and x is the threshold for expression of* P_F .We can get that threshold $\mathrm{x} = 2\ln\frac{1}{P_F}$ and $P_D = Q_{\chi'_2(\lambda)^2}(2\ln\frac{1}{P_F})$.

(2)Raleigh fading sinusoid: s(n)=Asin(2π$f_c + \varphi$), $f_c = \frac{1}{16}$ where A Rayleigh distributed and φ uniformly distributed.

s(n)= Asin(2π$f_c + \varphi$)=A*[sin(2π$f_c n$) cos(φ) + cos(2π$f_c n$)sin(φ)] = $acos(2\pi f_c n) + b\sin(2\pi f_c n)$ where a=A sin(φ),b=Acos(φ).

Random variable $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$ is subject to Gaussian distribution N(0,$\sigma_s^2 I$)

Then A=$\sqrt{a^2+b^2}$ is Rayleigh $\varphi = \arctan(-\frac{b}{a})$ is uniform and A andφ are independent variables.Now with the assumption we have the Bayesian linear model

$x = H\theta + w$ where H=$\begin{bmatrix} 1 & 0 \\ cos(2\pi f_c) & sin(2\pi f_c) \\ \vdots & \vdots \\ cos(2\pi f_c(N-1)) & sin(2\pi f_c(N-1)) \end{bmatrix}$ $\theta \sim N(0, \sigma_s^2 I)$ and w~ N(0,$\sigma^2 I$).

T(x)=$\sigma_s^2 x^T HH^T\ (\sigma_s^2 HH^T\ + \sigma^2 I)^{-1} x$=$\frac{c}{N} x^T HH^T\ x$ where c=$\frac{N\sigma_s^2}{\frac{N\sigma_s^2}{2}+\sigma^2}$.T(x)'=$\frac{1}{N}\,|H^T x|^2 =$

$\frac{1}{N}\left\|\begin{matrix} \sum_{n=0}^{N-1} x(n)cos(2\pi f_c) \\ \sum_{n=0}^{N-1} x(n)sin(2\pi f_c) \end{matrix}\right\|^2$=$\frac{1}{N}\,|\sum_{n=0}^{N-1} x[n]\exp(-j2\,\pi f_c n)|^2$.

$C_s = HC_\theta H^T = H(\sigma_s^2 I)H^T = \sigma_s^2 HH^T$ (H=[h0 h1])=$\frac{N}{2}\sigma_s^2 \frac{h0}{\sqrt{N/2}}\frac{h0}{\sqrt{N/2}}^T + \frac{N}{2}\sigma_s^2 \frac{h1}{\sqrt{N/2}}\frac{h1}{\sqrt{N/2}}^T$

Let $\lambda_{s0} = \lambda_{s1} = \frac{N}{2}\sigma_s^2, e_0 = \frac{h0}{\sqrt{N/2}}$, $e_1 = \frac{h1}{\sqrt{N/2}}$ then $C_s = \lambda_{s0} e_0 {e_0}^T + \lambda_{s1} e_1 {e_1}^T$

P_{FA}=Pr{ T(x)'>$\gamma'''|H0$}= Pr{T(x)'>γ''}=exp(-$\frac{\gamma'''}{\sigma^2}$) where $\gamma'' = c\gamma'''$ and $c = \frac{N\sigma_s^2}{N\sigma_s^2/2+\sigma^2}$.

P_D=exp(-$\frac{\gamma''}{2\lambda_{s0}}$) $= \exp\left(-\frac{\gamma''}{N\sigma_s^2}\right) = \exp\left(-\frac{\gamma'''}{N\sigma_s^2/2+\sigma^2}\right)$.

$$\gamma''' = \sigma^2 ln\frac{1}{P_{FA}}$$

Then we obtain that

P_D=${P_{FA}}^{\frac{1}{1+N\sigma_s^2/2\sigma^2}}$=${P_{FA}}^{\frac{1}{1+\eta/2}}$ η = average ENR = $\frac{N\sigma_s^2}{\sigma^2}$

(3) s(n)=w_s(n) Uncorrelated Gaussian signal ~ N(0,${\sigma_s}^2$):

Energy detector:

H0:x(n)=w(n) n=0,1,…,N-1

H1:x(n)=s(n)+w(n)

Where:w(n)~ N(0,$\sigma^2 I$)

s(n)~ N(0,${\sigma_s}^2 I$)

thus x(n)~ N(0,$\sigma^2 I$) under H0

$\sim N(0,({\sigma_s}^2+\sigma^2)I)$ under H1

$$\Lambda(\mathrm{x})=\frac{p(x|H1)}{p(x|H0)}=\frac{1/(2\pi({\sigma_s}^2+\sigma^2))^{N/2}\exp\{-\frac{1}{2({\sigma_s}^2+\sigma^2)}\sum_{n=0}^{N-1}x^2(n)\}}{1/(2\pi\sigma^2)^{\frac{N}{2}}\exp\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2(n)\}}$$

$\ln\Lambda(\mathrm{x})=\frac{N}{2}\ln\left(\frac{\sigma^2}{{\sigma_s}^2+\sigma^2}\right)+\frac{1}{2}\frac{{\sigma_s}^2}{\sigma^2({\sigma_s}^2+\sigma^2)}\sum_{n=0}^{N-1}x^2(n)$

Test statistic : $T(x)=\sum_{n=0}^{N-1}x^2(n)$

Alternatively $T(x)'=\frac{1}{N}\sum_{n=0}^{N-1}x^2(n)$ is an estimate of the variance and the threshold recognition that the variance under H0: σ^2 and under H1: ${\sigma_s}^2+\sigma^2$

We know that test statistic T(x) are χ_N^2under H0 and H1.

$T(x)=\sum_{n=0}^{N-1}x^2(n)>\gamma'$

The detection performance can be found according to Chapter2

$\frac{T(x)}{\sigma^2}\sim\chi_N^2$ under H0;

$\frac{T(x)}{{\sigma_s}^2+\sigma^2}\sim\chi_N^2$ under H1.

$P_{FA}=\Pr\{T(x)>\gamma'|H0\}=\Pr\{\frac{T(x)}{\sigma^2}>\frac{\gamma'}{\sigma^2}|H0\}=Q_{\chi_N^2}(\frac{\gamma'}{\sigma^2})$

$P_D=\Pr\{T(x)>\gamma'|H1\}=\Pr\{\frac{T(x)}{{\sigma_s}^2+\sigma^2}>\frac{\gamma'}{{\sigma_s}^2+\sigma^2}|H0\}=Q_{\chi_N^2}(\frac{\gamma'}{{\sigma_s}^2+\sigma^2})$

SNR=$\frac{{\sigma_s}^2}{\sigma^2}$ letting $\gamma''=\frac{\gamma'}{\sigma^2}$

$P_{FA}=Q_{\chi_N^2}(\gamma'')$

$P_D=Q_{\chi_N^2}(\frac{\gamma''}{\frac{{\sigma_s}^2}{\sigma^2}+1})=Q_{\chi_N^2}(\frac{\gamma''}{SNR+1})$

3. Plot the performance of the processors in 2. Above as:

A. PD vs PF on normal probability paper for 10log(ENR)=10dB.

B. PD (linear) vs ENR(dB) for PF=10^{-1},10^{-2},and 10^{-3} and ENR from 0 to 30dB.

ENR is the expected energy-to-noise ratio.

(1).The solution to problem 3.A:

In order to obtain the plot of PD vs PF on normal probability we need to make ENR fixed and ENR=10^(10/10)=10.The relationship between PD and PF (SKEP (N=128)): $P_D=Q_{{\chi'}_2(\lambda)^2}(2\ln\frac{1}{P_F})$.The result of SKEP is the solid curve in the figure.1.

Similarly, Rayleigh fading sinusoid (N=128): $P_D={P_{FA}}^{\frac{1}{1+N\sigma_s^2/2\sigma^2}}={P_{FA}}^{\frac{1}{1+\eta/2}}$ $\eta=$ average $\mathrm{ENR}=\frac{N\sigma_s^2}{\sigma^2}$,the result of Rayleigh fading sinusoid is the dash curve in the figure.1.For energy detector we need to get inverse Q function of PF so we are supposed to calculate the threshold for a given PF. according to prob.5.1 in Kay text (pp.176-177) for an iterative formula.

PFA=exp($-\frac{\gamma'}{2\sigma^2}$)$[1+\sum_{r=1}^{\frac{N-1}{2}}\frac{(\frac{\gamma'}{2\sigma^2})^r}{r!}]$

Let $\gamma''=\gamma'/2\sigma^2$and rearranging terms we have

$\gamma''=-ln\mathrm{PFA}+\ln[\sum_{r=1}^{\frac{N-1}{2}}\frac{(\gamma'')^r}{r!}]$.

To slove for γ'' we can use the fixed point iteration

$\gamma''_{k+1}=-ln\mathrm{PFA}+\ln[\sum_{r=1}^{\frac{N-1}{2}}\frac{(\gamma''_k)^r}{r!}]$. (the threshold γ' by iteration with $\gamma''_0=1$.

After doing the calculation we can get the result of plot of Energy detector (N=128 and N=16). The dot and dash curve is Energy detector on condition N=16 and the dot curve is Energy detector on condition N=128 in the figure.1.

(2) Similar to the previous solution, in problem 3.B we only make the PF fixed and get the relationship between PD and ENR.

Result and discussion:

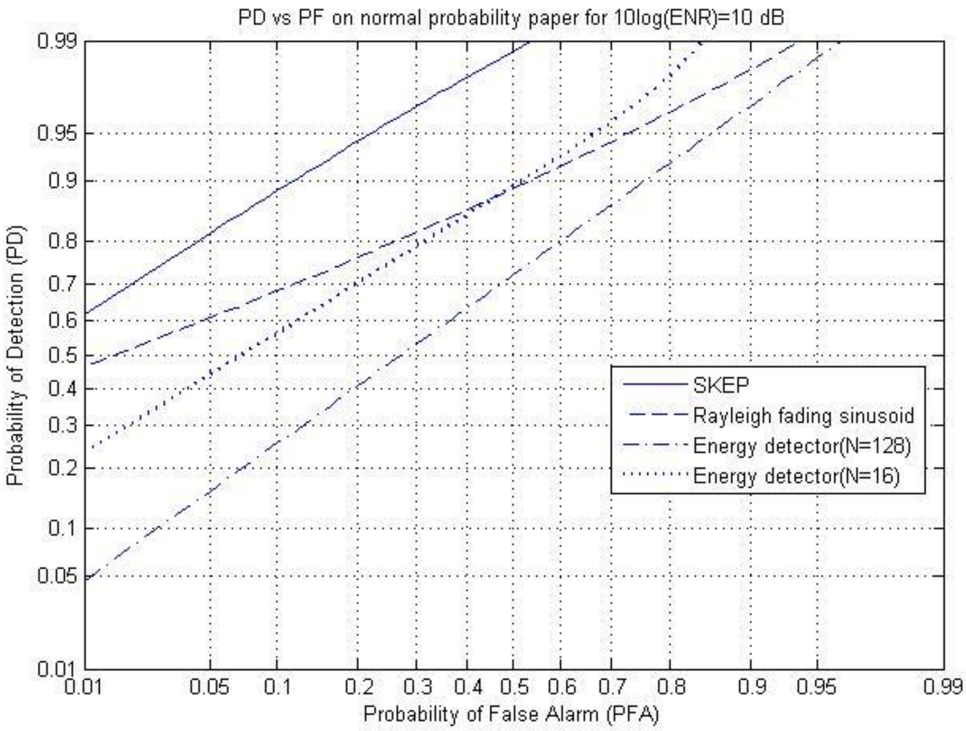


figure.1 PD vs PF on normal probability paper

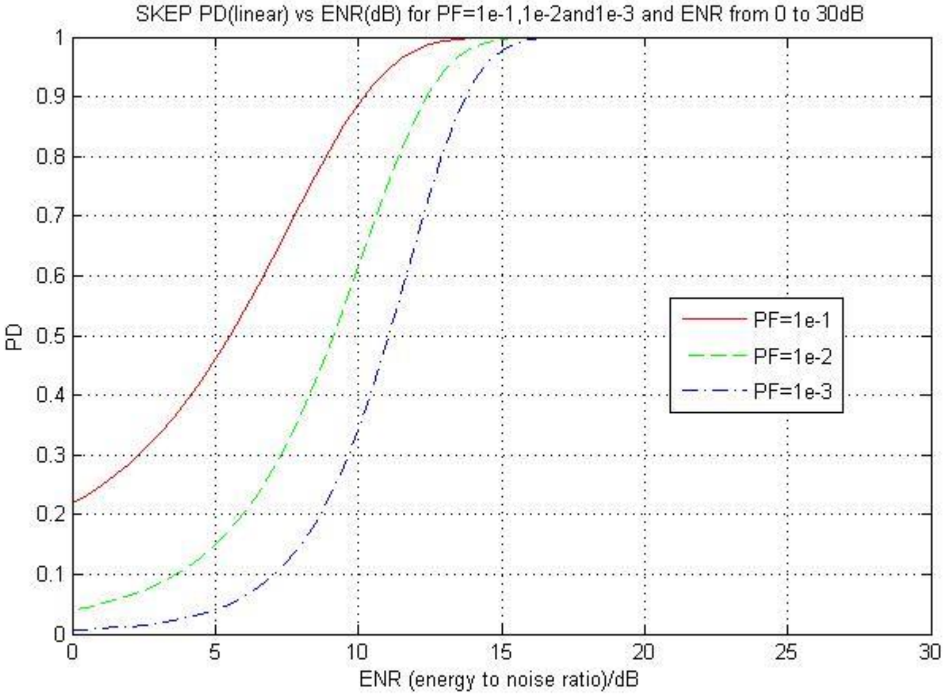


figure.2.PD vs ENR of SKEP in linear paper

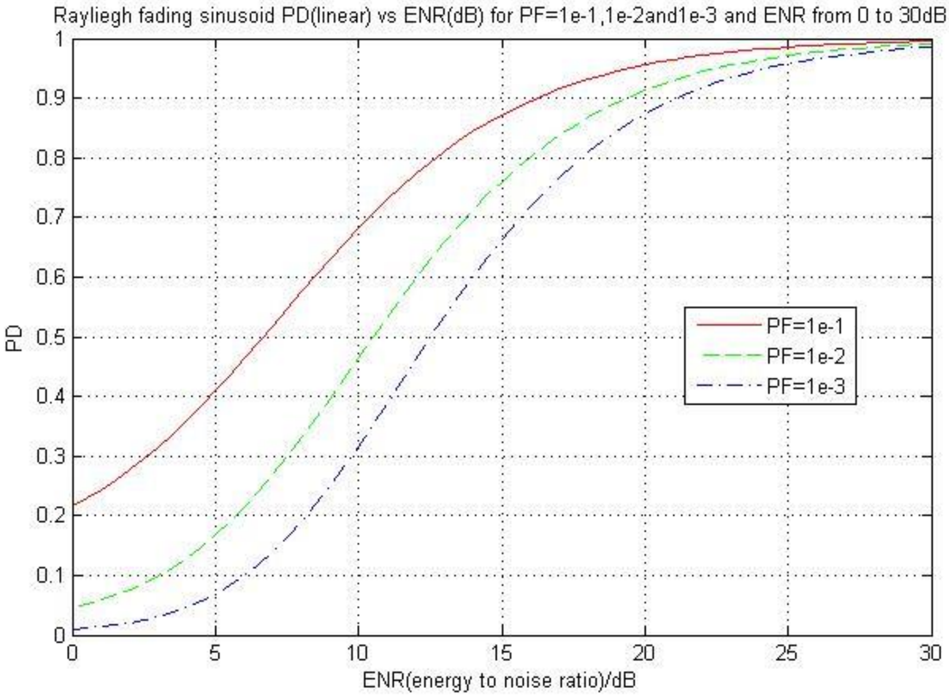


figure.2.PD vs ENR of Rayleigh fading sinusoid in linear paper

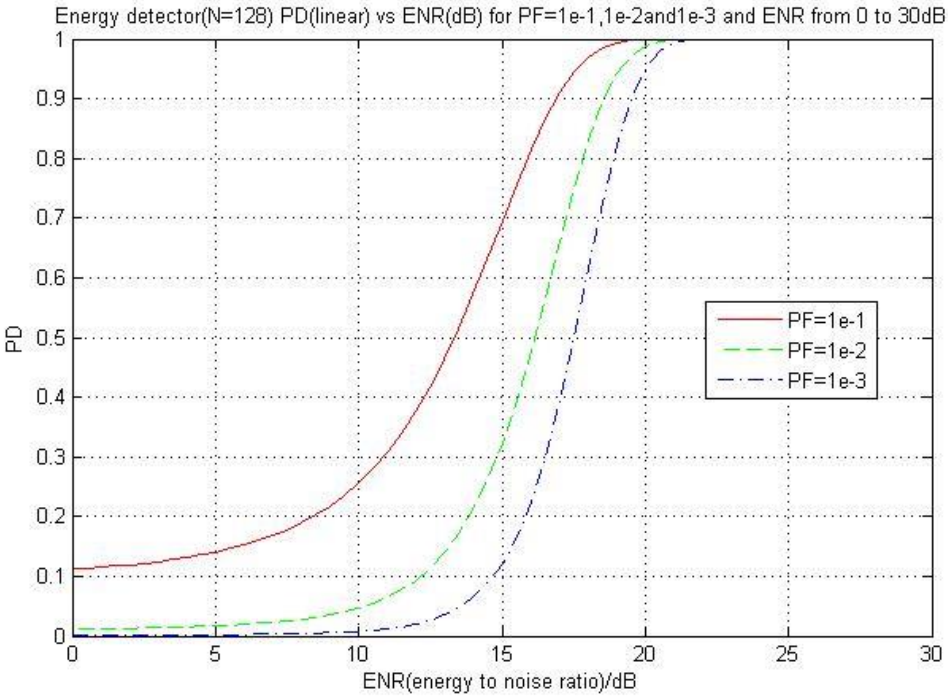


figure.3.PD vs ENR of Energy detector N=128 in linear paper

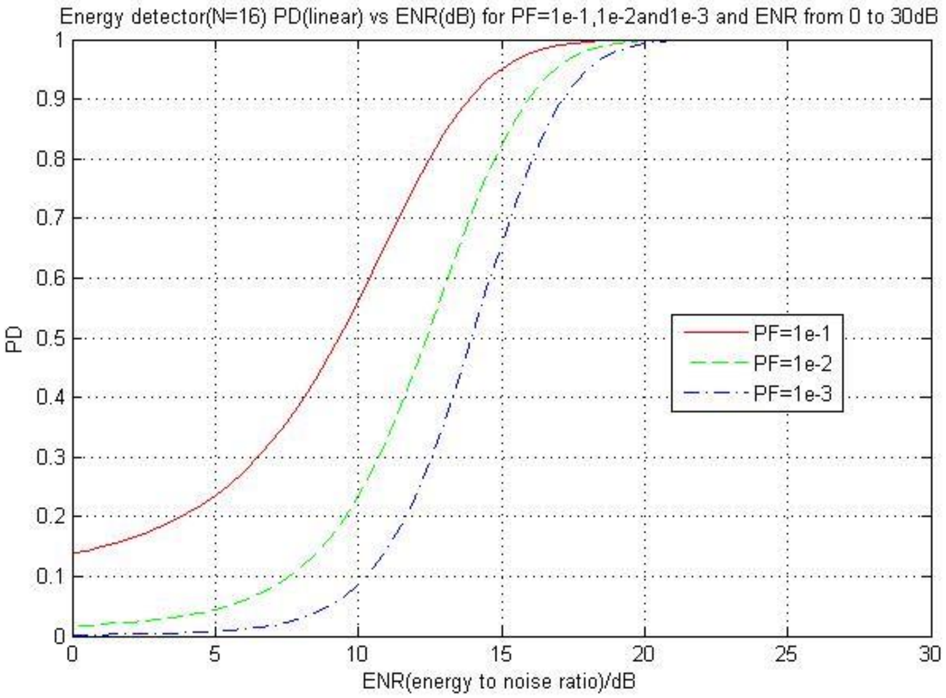


figure.4. PD vs ENR of Energy detector N=16 in linear paper

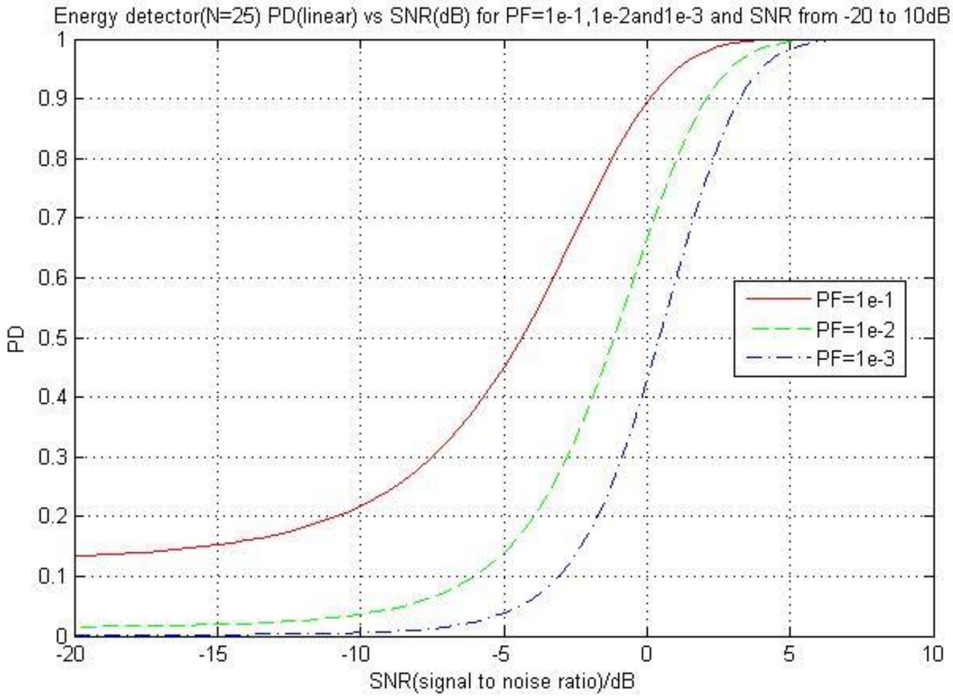


figure.5. PD vs sNR of Energy detector N=25 in linear paper

1. In figure.1 we can know that SKEP has the best performance and the performance of Rayleigh is better than the energy detector(N=16) approximately when PF<0.45 but when PF>0.45 the condition is inverse. Energy detector(N=128) is worse than both SKEP and Rayleigh fading sinusoid.
2. In figure.1 we can learn that when N increase the performance of energy detector becomes worse .When N =128 the start point of the curve is (0,0.05) and the curve is just close to the positive ratio of curve.According to the formula :$P_D = Q_{\chi_N^2}\left(\frac{\gamma''}{\frac{\sigma_s{}^2}{\sigma^2}+1}\right) = Q_{\chi_N^2}\left(\frac{\gamma''}{SNR+1}\right)$ When N $\rightarrow \infty$,and ENR is fixed the SNR is $\rightarrow 0$.
3. From figure.2 to figure.4 we learn that when PF decrease 10^{-1} to 10^{-3} the curve of PD vs PF of SKEP, Rayleigh fading sinusoid and energy detector is shifted to the right. When PF decrease, the slop of curve of PD vs ENR becomes steeper and larger.
4. Comparing to others, from figure.2 to figure.4,the variance of curve of Rayleigh fading sinusoid is smoother than others, which means the slope of curve of Rayleigh fading sinusoid changes more slowly than others.
5. From figure.2. to figure.4. we can get that on a condition that PF is fixed, the larger the ENR is the larger the PD is or the better the performance is.
6. On a fixed PF in order to obtain PD=1 we need least ENR for SKEP and the Rayleigh needs more or larger ENR than others.

Summary (Conclusions):

1. SKEP has the best performance and the performance of Rayleigh is better than the energy detector(N=16) approximately when PF<0.45 but when PF>0.45 the condition is inverse. Energy detector(N=128) is worse than both SKEP and Rayleigh fading sinusoid.
2. When N increase the performance of energy detector becomes worse .When N =128 the start point of the curve is (0,0.05) and the curve is just close to the positive ratio of curve. According to the formula :$P_D=Q_{\chi_N^2}\left(\frac{\gamma''}{\frac{{\sigma_s}^2}{\sigma^2}+1}\right)= Q_{\chi_N^2}\left(\frac{\gamma''}{SNR+1}\right)$

   When N $\rightarrow \infty$,and ENR is fixed the SNR is $\rightarrow 0$.
3. When PF decrease 10^{-1} to 10^{-3} the curve of PD vs PF of SKEP, Rayleigh fading sinusoid and energy detector is shifted to the right. When PF decrease, the slop of curve of PD vs ENR becomes steeper and larger.
4. Comparing to others, from figure.2 to figure.4, the variance of curve of Rayleigh fading sinusoid is smoother than others, which means the slope of curve of Rayleigh fading sinusoid changes more slowly than others.

Appendix:

PD vs PF:

```
%SKEP performance and 10*log10(ENR)=10dB
ENR=10.^(10/10);
lamda=ENR;
PFA1=[0.01:0.01:1];
x1=-2*log(PFA1);
PD1=Qchipr2(2,lamda,x1,1e-5);
figure(1)
probpaper(PFA1,PD1)
%Rayleigh fading sinusoid performance
PFA2=[0.01:0.01:1];
PD2=PFA2.^(1/(1+ENR/2));
figure(1)
hold on
probpaper(PFA2,PD2)
%Energy detector(N=128)
N1=128;
PFA3=[0.01:0.01:1];
PD3=zeros(1,100);
for i=1:100
r1=getgama(PFA3(i),N1);
R1=2*r1;
gama1=R1/(ENR/N1+1);
PD3(i)=Qchipr2(N1,0,gama1,1e-5);
end
figure(1)
probpaper(PFA3,PD3)
%Energy detector(N=16)
N2=16;
PFA4=[0.01:0.01:1];
grid
r2=getgama(PFA4,N2);
R2=2*r2;
gama2=R2/(ENR/N2+1);
PD4=Qchipr2(N2,0,gama2,1e-5);
figure(1)
probpaper(PFA4,PD4)
```

PD vs ENR:

```
PFA1=10^-1;
PFA2=10^-2;
PFA3=10^-3;
ENR=[0:0.5:30];
```

```
%SKEP PDvsENR
lamda=10.^(ENR/10);
x1=2*log(1/PFA1);
x2=2*log(1/PFA2);
x3=2*log(1/PFA3);
PD1=zeros(1,61);
PD2=zeros(1,61);
PD3=zeros(1,61);
for i=1:61
PD1(i)=Qchipr2(2,lamda(i),x1,1e-5);
PD2(i)=Qchipr2(2,lamda(i),x2,1e-5);
PD3(i)=Qchipr2(2,lamda(i),x3,1e-5);
end
figure(2)
plot(ENR,PD1,'r')
hold on
plot(ENR,PD2,'g')
plot(ENR,PD3,'b')
grid;
%Rayleigh fading sinusoid PDvsENR
x=10.^(ENR/10);
y=1./(x/2+1);
PD4=PFA1.^y;
PD5=PFA2.^y;
PD6=PFA3.^y;
figure(3)
plot(ENR,PD4,'r')
hold on
plot(ENR,PD5,'g')
plot(ENR,PD6,'b')
grid;
%Energy detecor(N=128) PDvsENR
N1=128;
PD7=zeros(1,61);
PD8=zeros(1,61);
PD9=zeros(1,61);
r7=getgama(PFA1,N1);
R7=2*r7;
r8=getgama(PFA2,N1);
R8=2*r8;
r9=getgama(PFA3,N1);
R9=2*r9;
for i=1:61
gama7=R7/(x(i)/N1+1);
```

```
PD7(i)=Qchipr2(N1,0,gama7,1e-5);
gama8=R8/(x(i)/N1+1);
PD8(i)=Qchipr2(N1,0,gama8,1e-5);
gama9=R9/(x(i)/N1+1);
PD9(i)=Qchipr2(N1,0,gama9,1e-5);
end
figure(4)
plot(ENR,PD7,'r')
hold on
plot(ENR,PD8,'g')
plot(ENR,PD9,'b')
axis([0 30 0 1])
grid;
%Energy detecor(N=16) PDvsENR
N2=16;
PD10=zeros(1,61);
PD11=zeros(1,61);
PD12=zeros(1,61);
r10=getgama(PFA1,N2);
R10=2*r10;
r11=getgama(PFA2,N2);
R11=2*r11;
r12=getgama(PFA3,N2);
R12=2*r12;
for i=1:61
gama10=R10/(x(i)/N2+1);
PD10(i)=Qchipr2(N2,0,gama10,1e-5);
gama11=R11/(x(i)/N2+1);
PD11(i)=Qchipr2(N2,0,gama11,1e-5);
gama12=R12/(x(i)/N2+1);
PD12(i)=Qchipr2(N2,0,gama12,1e-5);
end
figure(5)
plot(ENR,PD10,'r')
hold on
plot(ENR,PD11,'g')
plot(ENR,PD12,'b')
grid;

%Energy detecor(N=25) PDvsENR
N3=25;
SNR=[-20:0.5:10];
n=10.^(SNR/10);
PD13=zeros(1,61);
```

```
PD14=zeros(1,61);
PD15=zeros(1,61);
r13=getgama(PFA1,N3);
R13=2*r13;
r14=getgama(PFA2,N3);
R14=2*r14;
r15=getgama(PFA3,N3);
R15=2*r15;
for i=1:61
gama13=R13/(n(i)+1);
PD13(i)=Qchipr2(N3,0,gama13,1e-5);
gama14=R14/(n(i)+1);
PD14(i)=Qchipr2(N3,0,gama14,1e-5);
gama15=R15/(n(i)+1);
PD15(i)=Qchipr2(N3,0,gama15,1e-5);
end
figure(6)
plot(SNR,PD13,'r')
hold on
plot(SNR,PD14,'g')
plot(SNR,PD15,'b')
grid;
```

Iteration function:

```
function y=iteration(PFA,r,N)
s=0;
for i=1:(N/2)-1
    s=s+(r.^i)/factorial(i);
end
y=-log(PFA)+log(1+s);
```

Get threshold function:

```
function y=getgama(PFA,N)
x1=1;
x2=0;
while(abs(x1-x2)>1e-9)
    x2=x1;
    x1=iteration(PFA,x1,N);
end
y=x1;
```