Composite Hypothesis Testing

<u>Composite Hypothesis Testing</u> Ch 6 (and from Ch. 7)

Composite hypothesis - we need to accommodate unknown parameters

PDFs under H_0 and H_1 may include unknown parameters and are parameterized by them i.e. $\underline{\theta}_0$ and $\underline{\theta}_1$

Two principal approaches which philosophically are different (thus, a direct comparison is <u>not</u> possible

- Bayesian
  - unknown parameters are realizations of random variables with known a priori densities
  - troublesome unknown parameters are "integrated" out
  - difficulty in actually specifying the prior pdf and in carrying out the integration
    - detector cannot be claimed to be optimal if the unknown parameters are in fact deterministic or random with a different prior pdf than assumed.

Composite Hypothesis Testing

- GLRT (Generalized Likelihood Ratio Test)
  - unknown parameters are modeled as deterministic
  - GLRT (a suboptimum processor) will usually produce good detection results
  - Detection loss incurred using a GLRT can be bounded by comparing its performance to that of the clairvoyant detector (i.e. the detector with perfect knowledge of the unknown parameter).

Uncertain Amplitude (Sect. 6.3, 6.4, and 7.4)

Detection of a deterministic signal known except for amplitude in WGN

$H_0: \; x(n) = w(n) \qquad n = 0, \dots, N-1$

$H_1: \; x(n) = A\, s(n) + w(n)$

Where: $s(n)$ is known, A is unknown

$w(n)$ is uncorrelated with $w(n) \sim \mathcal{N}(0, \sigma^2)$

Conditional likelihood ratio:

$$L(\underline{x} \mid A) = \frac{p(\underline{x} \mid A, H_1)}{p(\underline{x} \mid H_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x(n) - A s(n))^2 \right\}}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2(n) \right\}}$$

Composite Hypothesis Testing

Take logarithms and simplifying

$$-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(-2A\,s(n)\,x(n) + A^2 s^2(n)\right) > \ln\gamma$$

or

$$A\sum_{n=0}^{N-1} x(n)\,s(n) > \sigma^2 \ln\gamma + \frac{A^2}{2}\sum_{n=0}^{N-1} s^2(n) = \gamma'$$

If $A > 0$, then decide H_1 if

$$\sum_{n=0}^{N-1} x(n)\,s(n) > \frac{\gamma'}{A} = \gamma''$$

If $A < 0$, then decide H_1 if

$$\sum_{n=0}^{N-1} x(n)\,s(n) < \frac{\gamma'}{A} = \gamma''$$

"uniformly most powerful"

If sign of A is known, then a UMP test exists (i.e. yields highest P_D for a given P_F). If sign is not known, then a UMP does not exist.

Composite Hypothesis Testing

Bayesian (Sect 6.4.1 and 7.4.2)

Assign prior PDF to $\underline{\Theta}_0$ and $\underline{\Theta}_1$ modeling the unknown parameters as realizations of a vector of random variables

$$L(\underline{x}) = \frac{p(\underline{x}|H_1)}{p(\underline{x}|H_0)} = \frac{\int p(\underline{x}|\underline{\Theta}_1, H_1)\, p(\underline{\Theta}_1)\, d\underline{\Theta}_1}{\int p(\underline{x}|\underline{\Theta}_0, H_0)\, p(\underline{\Theta}_0)\, d\underline{\Theta}_0}$$

For the uncertain amplitude problem $\underline{\Theta}_0 = 0$ and $\underline{\Theta}_1 = A$

Assume $A \sim N(0, \sigma_A^2)$

thus, $$L(\underline{x}) = \frac{p(\underline{x}|H_1)}{p(\underline{x}|H_0)} = \frac{\int_{-\infty}^{\infty} p(\underline{x}|A, H_1)\, p(A)\, dA}{p(\underline{x}|H_0)} > \gamma$$

or

$$L(\underline{x}) = \int_{-\infty}^{\infty} L(\underline{x}|A)\, p(A)\, dA$$

Composite Hypothesis Testing

$$L(\underline{x}) = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(-2A\, s(n)\, x(n) + A^2 s^2(n)\right)\right\}\left\{\frac{1}{2\pi\sigma_A^2}\exp\left(-\frac{1}{2\sigma_A^2}A^2\right)\right\}dA$$

worked out for DC signal signal on pp. 199-201

leading to $T(\underline{x}) = (\bar{x})^2$

<u>Note</u> Assume $A \sim N(\mu_A, \sigma_A^2)$. Then, in the form of the Bayesian linear model (see Sect 5.6, p. 168)

under H_1: $\underline{x} = \underline{s} A + \underline{w}$ where $H = \underline{s}$ and $\underline{\theta} = A$

$$T'(\underline{x}) = \underline{x}^T (H C_\theta H^T + C_w)^{-1} H \mu_\theta + \frac{1}{2}\underline{x}^T C_w^{-1} H C_\theta H^T (H C_\theta H^T + C_w)^{-1}\underline{x} > \gamma'$$

eq. 7.17

$\mu_\theta = \mu_A$, $C_\theta = \sigma_A^2$, and $C_w = \sigma^2 I$

Composite Hypothesis Testing

$$T'(\underline{y}) = \underline{x}^T(\sigma_A^2 \underline{s}\underline{s}^T + \sigma^2 \underline{I})^{-1} \underline{s}\, \mu_A + \frac{1}{2\sigma^2} \underline{x}^T \sigma_A^2 \underline{s}\underline{s}^T (\sigma_A^2 \underline{s}\underline{s}^T + \sigma^2 \underline{I})^{-1} \underline{x}$$

$$= \underline{x}^T(\sigma_A^2 \underline{s}\underline{s}^T + \sigma^2 \underline{I})^{-1} \underline{s}\, \mu_A + \frac{\sigma_A^2}{2\sigma^2} \underline{x}^T \underline{s}\, \underline{x}^T (\sigma_A^2 \underline{s}\underline{s}^T + \sigma^2 \underline{I})^{-1} \underline{s}$$

Using Woodbury's identity (special case of matrix inversion lemma)

$$(\sigma_A^2 \underline{s}\underline{s}^T + \sigma^2 \underline{I})^{-1} \underline{s} = \left(\frac{1}{\sigma^2} \underline{I} - \frac{\sigma_A^2}{\sigma^4} \frac{\underline{s}\underline{s}^T}{1 + \frac{\sigma_A^2 \underline{s}^T\underline{s}}{\sigma^2}} \right) \underline{s}$$

$$= \frac{1}{\sigma^2} \left(\underline{s} - \frac{\sigma_A^2 \underline{s}^T\underline{s}\, \underline{s}}{\sigma^2 + \sigma_A^2 \underline{s}^T\underline{s}} \right) = \frac{1}{\sigma^2 + \sigma_A^2 \underline{s}^T\underline{s}} \underline{s}$$

Thus,

$$T'(\underline{y}) = \frac{\mu_A}{\sigma^2 + \sigma_A^2 \underline{s}^T\underline{s}} \underline{x}^T\underline{s} + \frac{\sigma_A^2}{2\sigma^2(\sigma^2 + \sigma_A^2 \underline{s}^T\underline{s})} (\underline{x}^T\underline{s})^2$$

Note $\sigma_A^2 = 0$ the second term $= 0$ and we have the known amplitude case (correlator)

$\mu_A = 0$ then first term $= 0$ and we the Gaussian amplitude case (squared correlator) i.e. energy detector after matched filter

Composite Hypothesis Testing

$$T'(\underline{x}) = \frac{\mu_A}{\sigma^2 + \sigma_A^2\,\underline{s}^T\underline{s}}\,\underline{x}^T\underline{s} + \frac{\sigma_A^2}{2\sigma^2(\sigma^2 + \sigma_A^2\,\underline{s}^T\underline{s})}\,(\underline{x}^T\underline{s})^2$$

Performance of Bayesian detector (Prob #6.7 and #7.2)

Assume $\mu_A = 0$ and $A \sim N(0, \sigma_A^2)$

$$T'(\underline{x}) = \frac{\sigma_A^2}{2\sigma^2(\sigma^2 + \sigma_A^2\,\underline{s}^T\underline{s})}\left(\underline{x}^T\underline{s}\right)^2$$

$H_0:\ \underline{x} \sim N(0, \sigma^2 I)$

$H_1:\ \underline{x} \sim N(0, \sigma_A^2\,\underline{s}\,\underline{s}^T + \sigma^2 I)$

Composite Hypothesis Testing

$$P_D = P_r\{\, |\underline{x}^T\underline{s}| > \gamma'' \mid H_1 \}$$

$$= 2 P_r\{\, \underline{x}^T\underline{s} > \gamma'' \mid H_1 \}$$

$$= 2 Q\left(\frac{\gamma''}{\sigma_1}\right) \qquad \sigma_1^2 = \sigma_A^2 (\underline{s}^T\underline{s})^2 + \sigma^2 \underline{s}^T\underline{s}$$

Similarly, letting $\sigma_A^2 = 0$

$$P_F = 2 Q\left(\frac{\gamma''}{\sigma_0}\right) \qquad \sigma_0^2 = \sigma^2 \underline{s}^T\underline{s}$$

Since $\gamma'' = \sigma_0 Q^{-1}\left(\frac{P_F}{2}\right)$

$$P_D = 2 Q\left(\frac{\sigma_0}{\sigma_1} Q^{-1}\left(\frac{P_F}{2}\right)\right)$$

$$\frac{\sigma_0^2}{\sigma_1^2} = \frac{\sigma^2 \underline{s}^T\underline{s}}{\sigma_A^2 (\underline{s}^T\underline{s})^2 + \sigma^2 \underline{s}^T\underline{s}} = \frac{1}{\frac{\sigma_A^2}{\sigma^2}\underline{s}^T\underline{s} + 1}$$

Composite Hypothesis Testing

Then,

$$P_D = 2\,Q\left(\frac{1}{\left(\frac{\sigma_A^2}{\sigma^2}\,\underline{s}^T\underline{s} + 1\right)^{1/2}}\; Q^{-1}\left(\frac{P_F}{2}\right) \right)$$

$$\overline{ENR} = \frac{\sigma_A^2}{\sigma^2}\,\underline{s}^T\underline{s}$$

Note for sinusoidal signals $A\,s(n)$

$$s(n) = \cos(2\pi f_0 n + \phi) \quad \text{and} \quad \underline{s}^T\underline{s} = \frac{N}{2}$$

$$\overline{ENR} = \frac{N}{2}\,\frac{\sigma_A^2}{\sigma^2}$$

when $s(n)$ goes through an integer number of cycles in N points

Composite Hypothesis Testing

GLRT - Uncertain Amplitude (Sect. 6.4.2 and 7.4.1 and 7.6.1 - Also Sect. 7.7 Example 7.3 p. 275)

The GLRT replaces the unknown parameters by their maximum likelihood estimates (MLEs). Although the GLRT is not optimal, in practice it appears to work quite well.

$$L_G(\underline{x}) = \frac{p(\underline{x} \mid \hat{\underline{\theta}}_1, H_1)}{p(\underline{x} \mid \hat{\underline{\theta}}_0, H_0)}$$

Where: $\hat{\underline{\theta}}_1$ is MLE of $\underline{\theta}_1$ assuming H_1 true (maximizes $p(\underline{x} \mid \underline{\theta}_1, H_1)$)

$\hat{\underline{\theta}}_0$ is MLE of $\underline{\theta}_0$ assuming H_0 true

$$L_G(\underline{x}) = \frac{\max\limits_{\underline{\theta}_1} p(\underline{x} \mid \underline{\theta}_1, H_1)}{\max\limits_{\underline{\theta}_0} p(\underline{x} \mid \underline{\theta}_0, H_0)}$$

Composite Hypothesis Testing

And when no uncertain parameters under H_0

$$L_G(\underline{x}) = \max_{\underline{\theta}_1} \frac{p(\underline{x} \mid \underline{\theta}_1, H_1)}{p(\underline{x} \mid H_0)} = \max_{\underline{\theta}_1} L(\underline{x} \mid \underline{\theta}_1)$$

For uncertain amplitude problem

$$L_G(\underline{x}) = \frac{p(\underline{x} \mid \hat{A}, H_1)}{p(\underline{x} \mid H_0)} > \gamma$$

Where $\hat{A}$ is the MLE of A under H_1

$$\hat{A} = \frac{\sum_{n=0}^{N-1} x(n)\, s(n)}{\sum_{n=0}^{N-1} s^2(n)}$$

e.g. consider a sinusoid

$$s(n) = \cos(2\pi f_0 n + \phi)$$

Composite Hypothesis Testing

Substitute into the reduced likelihood ratio expression

$$-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(-2\hat{A}\,s(n)\,x(n)+\hat{A}^2 s^2(n)\right) > \ln\gamma$$

or

$$-\frac{1}{2\sigma^2}\left(-2\hat{A}\,\hat{A}\sum_{n=0}^{N-1}s^2(n)+\hat{A}^2\sum_{n=0}^{N-1}s^2(n)\right) > \ln\gamma$$

$$= \sum x(n)\,s(n)$$

or

$$\hat{A}^2 > \frac{2\sigma^2\ln\gamma}{\sum_{n=0}^{N-1}s^2(n)}$$

For noise only, we would expect $\hat{A} \simeq 0$ (since $E(\hat{A})=0$) and so when signal is present $|\hat{A}|$ should depart from 0

Alternatively

$$T(\underline{x}) = \left(\sum_{n=0}^{N-1}x(n)\,s(n)\right)^2 > 2\sigma^2\ln\gamma\left(\sum_{n=0}^{N-1}s^2(n)\right)$$

$$= \gamma'$$

Composite Hypothesis Testing

Detector is just a correlator that account for unknown sign of A by taking the square (i.e. robust to sign of A)

$$x[n] \longrightarrow \underset{\substack{\uparrow\\ s[n]}}{\otimes} \longrightarrow \boxed{\sum_{n=0}^{N-1} (\cdot)} \longrightarrow \boxed{(\cdot)^2} \longrightarrow T(\underline{x})$$

Lack of amplitude knowledge will degrade performance but only slightly from that of the correlator

Performance of the GLRT detector (see Eq. 6.9 and Eq. 7.16 also Example 7.3 p. 275)

$$u(\underline{x}) = \sum_{n=0}^{N-1} x[n]\, s[n] \sim \begin{cases} N\left(0, \sigma^2 \sum_{n=0}^{N-1} s^2[n]\right) & \text{under } H_0 \\ N\left(A \sum_{n=0}^{N-1} s^2[n], \ \sigma^2 \sum_{n=0}^{N-1} s^2[n]\right) & \text{under } H_1 \end{cases}$$

Ref. Eq. 4.11

Composite Hypothesis Testing

$$P_f = P_r\left\{ |u(\underline{x})| > \sqrt{\gamma'} \mid H_0 \right\} = 2 P_r\left\{ u(\underline{x}) > \sqrt{\gamma'} \right\}$$

$$= 2\, Q\left(\frac{\sqrt{\gamma'}}{\left(\sigma^2 \sum_{n=0}^{N-1} s^2(n)\right)^{1/2}} \right)$$

Since $\frac{\sqrt{\gamma'}}{\left(\sigma^2 \sum_{n=0}^{N-1} s^2(n)\right)^{1/2}} = Q^{-1}\left(\frac{P_F}{2}\right)$ and $d^2 = \frac{\varepsilon}{\sigma^2} = \frac{A^2 \sum_{n=0}^{N-1} s^2(n)}{\sigma^2}$

$$P_D = P_r\left\{ |u(\underline{x})| > \sqrt{\gamma'} \mid H_1 \right\}$$

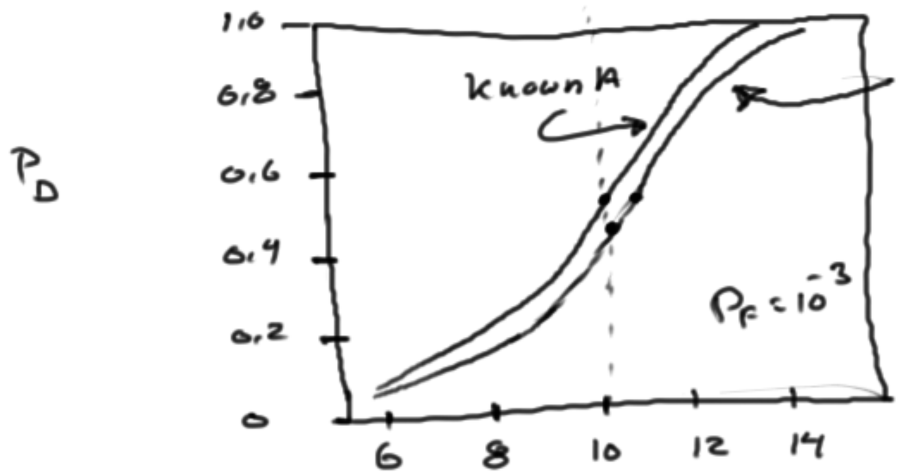
$$= Q\left(Q^{-1}\left(\frac{P_F}{2}\right) - \left(\frac{\varepsilon}{\sigma^2}\right)^{1/2} \right) + Q\left(Q^{-1}\left(\frac{P_F}{2}\right) + \left(\frac{\varepsilon}{\sigma^2}\right)^{1/2} \right)$$

or

$$= Q\left(Q^{-1}\left(\frac{P_F}{2}\right) - (d^2)^{1/2} \right) + Q\left(Q^{-1}\left(\frac{P_F}{2}\right) + (d^2)^{1/2} \right)$$

Eq. 7.16

Composite Hypothesis Testing



<u>Note</u> Fig. 7.6a does <u>not</u> match Fig 7.3 for unknown A (Fig. 7.6a is known A case instead)

$$\text{ENR (dB)} = 10 \log \frac{\mathcal{E}}{\sigma^2}$$

Note degradation is ~0.5 dB for low P_F's

e.g. $\quad$ ENR = 10 dB, $P_F = 10^{-3}$

Known A	$P_D = 0.52$	
GLRT A	$P_D = 0.45$	
GLRT A	$P_D = 0.52$	for ENR = 10.5 dB

Composite Hypothesis Testing

Note for sinusoidal signals $A\, s(n)$

(Sect 7.6.1 pp. 261-262 and Fig. 7.6a)

$$s(n) = \cos(2\pi f_0 n + \phi)$$

MLE A

$$\hat{A} = \frac{\sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} \cos^2(2\pi f_0 n + \phi)}$$

$$= \frac{2}{N} \sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n + \phi)$$

When signal goes through an integer number of cycles in N points

$$\frac{\mathcal{E}}{\sigma^2} = \frac{N A^2}{2\sigma^2}$$