University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 210B Non-equilibrium Fall 2025
Assignment 4 – Solutions

Due 11:59pm Monday, October 27, 2025

---

1. **The diffusion equation in a domain**

We can use the Neumann boundary conditions:

$$c(x,t) = X(x)\,T(t) \tag{1}$$

$$\partial_t c = D\,\partial_x^2 c \quad \Rightarrow \quad X(x)\,T'(t) = D\,X''(x)\,T(t) \tag{2}$$

$$\frac{T'(t)}{D\,T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{3}$$

$$T'(t) + D\lambda\,T(t) = 0, \qquad X''(x) + \lambda\,X(x) = 0 \tag{4}$$

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots \tag{5}$$

$$T_n(t) = \exp\big(-D\lambda_n t\big) = \begin{cases} 1, & n = 0, \\ \exp\Big[-D\left(\frac{n\pi}{L}\right)^2 t\Big], & n \geq 1. \end{cases} \tag{6}$$

$$c(x,t) = A_0 X_0 + \sum_{n=1}^{\infty} A_n\,X_n(x)\,T_n(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-D(n\pi/L)^2 t}. \tag{7}$$

Now solving the coefficients:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n \geq 1) \tag{8}$$

$$A_0 = c_0\left(1 - \frac{a}{L}\right) \tag{9}$$

$$A_n = -\frac{2c_0}{n\pi}\sin\left(\frac{n\pi a}{L}\right) \tag{10}$$

Plugging into the above general solution:

$$c(x,t) = c_0\left(1 - \frac{a}{L}\right) - 2c_0 \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{L}\right)}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \exp\left[-D\left(\frac{n\pi}{L}\right)^2 t\right]. \tag{11}$$

Therefore, to plot this solution, we observe the following:

$$c(x,t) \to A_0 = c_0\left(1 - \frac{a}{L}\right), \quad \text{as } t \to \infty. \tag{12}$$

$$c(0,t) = c_0\Big(1-\frac{a}{L}\Big) - 2c_0\sum_{n=1}^{\infty}\frac{\sin\!\left(\frac{n\pi a}{L}\right)}{n\pi}\,e^{-D(n\pi/L)^2 t} \tag{13}$$

$$c(\pm L,t) = c_0\Big(1-\frac{a}{L}\Big) - 2c_0\sum_{n=1}^{\infty}\frac{\sin\!\left(\frac{n\pi a}{L}\right)}{n\pi}\,(-1)^n e^{-D(n\pi/L)^2 t}. \tag{14}$$

2. **Active matter**

   (a) Compute $\langle r_\alpha(t)\rangle$ From the problem:

$$\dot{\mathbf{r}}(t) = v_0\,\hat{\mathbf{n}}(t) + \boldsymbol{\eta}^T(t), \qquad \hat{\mathbf{n}}(t) = (\cos\theta(t),\,\sin\theta(t)), \tag{15}$$

$$\langle\boldsymbol{\eta}^T(t)\rangle = \mathbf{0}, \qquad \dot\theta(t) = \eta^R(t), \qquad \langle\eta^R(t)\eta^R(t')\rangle = D_R\,\delta(t-t'). \tag{16}$$

$$\hat{\mathbf{n}}_0 = (\cos\theta_0, \sin\theta_0). \tag{17}$$

Compute for cosine and sine separately:

$$\langle\cos\theta(t)\rangle = \cos\theta_0\langle\cos\Delta\theta\rangle = \cos\theta_0\,e^{-D_R t/2}, \tag{18}$$

$$\langle\sin\theta(t)\rangle = \sin\theta_0\,e^{-D_R t/2}. \tag{19}$$

Need to deal with $\Delta\theta(t)$:

$$\Delta\theta(t) \equiv \theta(t) - \theta_0 = \int_0^t \eta^R(s)\,ds. \tag{20}$$

$$\langle\Delta\theta(t)\rangle = \int_0^t \langle\eta^R(s)\rangle\,ds = 0. \tag{21}$$

$$\begin{aligned}
\langle[\Delta\theta(t)]^2\rangle &= \left\langle \int_0^t ds\int_0^t ds'\;\eta^R(s)\eta^R(s')\right\rangle && (22)\\
&= \int_0^t ds\int_0^t ds'\;\langle\eta^R(s)\eta^R(s')\rangle && (23)\\
&= \int_0^t ds\int_0^t ds'\;D_R\,\delta(s-s') && (24)\\
&= D_R\int_0^t ds = D_R t. && (25)
\end{aligned}$$

Therefore,

$$\langle\hat{\mathbf{n}}(t)\rangle = e^{-D_R t/2}\,\hat{\mathbf{n}}_0, \tag{26}$$

$$\langle\mathbf{r}(t)\rangle = v_0\int_0^t\langle\hat{\mathbf{n}}(s)\rangle\,ds = v_0\hat{\mathbf{n}}_0\int_0^t e^{-D_R s/2}\,ds = \frac{2v_0}{D_R}\left(1-e^{-D_R t/2}\right)\hat{\mathbf{n}}_0. \tag{27}$$

$$\langle r_\alpha(t)\rangle = \frac{2v_0}{D_R}\left(1-e^{-D_R t/2}\right)n_{0,\alpha}, \qquad \alpha\in\{x,y\}. \tag{28}$$

(b) Show that

$$\langle \hat{\mathbf{n}}(t)\,\hat{\mathbf{n}}(t')\rangle = A\begin{pmatrix} \begin{array}{c}\cos 2\theta_0 \exp\left(-D_R\left[t+t'+2\min(t,t')\right]\right)\\ +\exp\left(-D_R\left[t+t'-2\min(t,t')\right]\right)\end{array} & \sin 2\theta_0 \exp\left(-D_R\left[t+t'-2\min(t,t')\right]\right) \\ \sin 2\theta_0 \exp\left(-D_R\left[t+t'-2\min(t,t')\right]\right) & \begin{array}{c}-\cos 2\theta_0 \exp\left(-D_R\left[t+t'+2\min(t,t')\right]\right)\\ +\exp\left(-D_R\left[t+t'-2\min(t,t')\right]\right)\end{array}\end{pmatrix}$$

and find the constant A.

From part (a), we have calculated $\langle \hat{\mathbf{n}}(t)\rangle = e^{-D_R t/2}\,\hat{\mathbf{n}}_0$, .

$$\left\langle e^{i[\theta(t)-\theta(t')]}\right\rangle = \exp\left[-\frac{D_R}{2}\left(t+t'-2\min(t,t')\right)\right], \tag{29}$$

$$\left\langle e^{i[\theta(t)+\theta(t')]}\right\rangle = e^{i2\theta_0}\,\exp\left[-\frac{D_R}{2}\left(t+t'+2\min(t,t')\right)\right]. \tag{30}$$

$$A = \tfrac{1}{2} \tag{31}$$

(c) Compute $\langle r_\alpha(t) r_\beta(t)\rangle$

$$\dot{\mathbf{r}}(t) = v_0\hat{\mathbf{n}}(t) + \boldsymbol{\eta}^T(t), \qquad \dot\theta(t) = \eta^R(t), \qquad \hat{\mathbf{n}}(t) = (\cos\theta, \sin\theta), \tag{32}$$

$$\langle \eta^T_\alpha(t)\eta^T_\beta(t')\rangle = 2D_T\,\delta_{\alpha\beta}\delta(t-t'), \qquad \langle \eta^R(t)\eta^R(t')\rangle = D_R\,\delta(t-t'). \tag{33}$$

$$r_\alpha(t) = v_0\int_0^t n_\alpha(s)\,ds + \int_0^t \eta^T_\alpha(s)\,ds. \tag{34}$$

$$\langle r_\alpha(t) r_\beta(t)\rangle = v_0^2\int_0^t ds\int_0^t ds'\,\langle n_\alpha(s)n_\beta(s')\rangle + \int_0^t ds\int_0^t ds'\,\langle \eta^T_\alpha(s)\eta^T_\beta(s')\rangle. \tag{35}$$

$$\int_0^t ds\int_0^t ds'\,2D_T\,\delta_{\alpha\beta}\delta(s-s') = 2D_T\,t\,\delta_{\alpha\beta}. \tag{36}$$

$$\langle n_\alpha(s)n_\beta(s')\rangle = \frac{1}{2}\,e^{-\frac{D_R}{2}|s-s'|}\,\delta_{\alpha\beta}. \tag{37}$$

$$\int_0^t ds\int_0^t ds'\,e^{-a|s-s'|} = \frac{2}{a}\left[t - \frac{1-e^{-at}}{a}\right], \qquad a \equiv \frac{D_R}{2}. \tag{38}$$

$$\langle r_\alpha(t) r_\beta(t)\rangle = \delta_{\alpha\beta}\left[2D_T\,t + \frac{2v_0^2}{D_R}\,t - \frac{4v_0^2}{D_R^2}\left(1-e^{-\frac{D_R}{2}t}\right)\right]. \tag{39}$$

$$\langle |\mathbf{r}(t)|^2\rangle = 4D_T\,t + \frac{4v_0^2}{D_R}\,t - \frac{8v_0^2}{D_R^2}\left(1-e^{-\frac{D_R}{2}t}\right). \tag{40}$$

(d) Now analyze the form of the mean square displacement at short times

Short time $t \ll D_R^{-1}$.

$$\langle |\mathbf{r}(t)|^2 \rangle = v_0^2 t^2 + 4 D_T t + O(t^3), \tag{41}$$

Long time $t \gg D_R^{-1}$. Since $\left(e^{-D_R t/2} \to 0\right)$,

$$\langle r_\alpha(t) r_\alpha(t) \rangle \simeq 2 \left(D_T + \frac{v_0^2}{D_R} \right) t, \qquad \langle |\mathbf{r}(t)|^2 \rangle \simeq 4 \left(D_T + \frac{v_0^2}{D_R} \right) t, \tag{42}$$