University of California at San Diego – Department of Physics – Andy Wan

Physics 210B Non-equilibrium Fall 2025
Assignment 1 – Solutions

Due 11:59pm Monday, October 6, 2025

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1. Is there a topic for which you would be disappointed if you didn't learn about it in 210B?

   I would love to see some fluid dynamics. I also really like the idea to work on those simulations.

2. **Brain-warmer: A random walk in $d > 1$ dimensions.**

   (a) Consider a random walk in two dimensions where at each time increment the walker takes a step by $\vec{x}_i = \ell(\cos\theta_i, \sin\theta_i)$ for some fixed length ℓ, where θ_i is chosen from the uniform distribution on $[0, 2\pi)$. Define the mean-square displacement $\left\langle \vec{X}^2 \right\rangle$, where $\vec{X} \equiv \sum_{i=1}^{t} \vec{x}_i$. Show that $\left\langle \vec{X}^2 \right\rangle$ grows linearly with t.

$$\vec{X} = \ell \sum_{i=1}^{t} (\cos\theta_i, \sin\theta_i) \tag{1}$$

$$\left\langle \vec{X}^2 \right\rangle = \ell^2 \left(\sum_{i=1}^{t} \sum_{j=1}^{t} (\cos\theta_i, \sin\theta_i) \cdot (\cos\theta_j, \sin\theta_j) \right) \tag{2}$$

$$= \ell^2 t + \ell^2 \sum_{i \neq j} (\cos\theta_i, \sin\theta_i) \cdot (\cos\theta_j, \sin\theta_j) = \ell^2 t + 2\ell^2 \sum_{1 \le i < j \le t} (\cos\theta_i, \sin\theta_i) \cdot (\cos\theta_j, \sin\theta_j) \tag{3}$$

$$= \ell^2 t + 2\ell^2 \sum_{1 \le i < j \le t} \cos(\theta_i - \theta_j) = \ell^2 t \tag{4}$$

   Clearly, the cross terms vanish because θ_i and θ_j are independent and uniformly distributed on $[0, 2\pi)$, so $\langle \cos(\theta_i - \theta_j) \rangle = 0$. $\left\langle \vec{X}^2 \right\rangle$ grows linerly with t.

   (b) Convince me that the same idea works in any number of dimensions, where now the directions are chosen uniformly on the unit sphere in $d - 1$ dimensions, that is, the increment is $\vec{x}_i = \ell \hat{n}_i$.

$$\vec{X} = \ell \sum_{i=1}^{t} \hat{n}_i \tag{5}$$

$$\left\langle \vec{X}^2 \right\rangle = \ell^2 \left(\sum_{i=1}^{t} \sum_{j=1}^{t} \hat{n}_i \cdot \hat{n}_j \right) \tag{6}$$

$$= \ell^2 t + \ell^2 \sum_{i \neq j} \hat{n}_i \cdot \hat{n}_j = \ell^2 t + 2\ell^2 \sum_{1 \leq i < j \leq t} \hat{n}_i \cdot \hat{n}_j \tag{7}$$

$$= \ell^2 t + 2\ell^2 \sum_{1 \leq i < j \leq t} \cos(\theta_{ij}) = \ell^2 t \tag{8}$$

where θ_{ij} is the angle between $\hat{n}_i$ and $\hat{n}_j$. The cross terms vanish because $\hat{n}_i$ and $\hat{n}_j$ are independent and uniformly distributed on the unit sphere in $d-1$ dimensions, so $\langle \cos(\theta_{ij}) \rangle = 0$. $\left\langle \vec{X}^2 \right\rangle$ grows linerly with t.

(c) Now suppose that the lengths ℓ of the steps are sampled from a distribution $g(\ell)$, say $g(\ell) = \frac{1}{\lambda} e^{-\ell/\lambda}$. Show that the result still holds.

$$\left\langle \vec{X}^2 \right\rangle = \left(\sum_{i=1}^{t} \sum_{j=1}^{t} l_i \hat{n}_i \cdot l_j \hat{n}_j \right) \tag{9}$$

$$= \sum_{i=1}^{t} l_i^2 + \sum_{i \neq j} l_i l_j \hat{n}_i \cdot \hat{n}_j = \sum_{i=1}^{t} l_i^2 + 2 \sum_{1 \leq i < j \leq t} l_i l_j \hat{n}_i \cdot \hat{n}_j \tag{10}$$

$$= \sum_{i=1}^{t} l_i^2 + 2 \sum_{1 \leq i < j \leq t} l_i l_j \cos(\theta_{ij}) = \sum_{i=1}^{t} l_i^2 \tag{11}$$

$$\left\langle \vec{X}^2 \right\rangle = t \left\langle l^2 \right\rangle = 2t\lambda^2 \tag{12}$$

Comment on simulations: One of the great joys of being away from equilibrium is that things depend on time and we can watch them move around by simulating their dynamics. So we are going to do some simulations in this class. You can use whatever software you like. I prefer to use Julia (download here) via the very convenient interface VS Code (download here). Please email me if you have any trouble that isn't easily answered by asking the internets.

3. **Some simulations of random walks.**

(a) Simulate an ordinary 1d random walk with fixed step size. Get enough statistics to see that the mean-square displacements grow with time $\langle \Delta X^2 \rangle \propto t$.

Check the prediction of the Central Limit Theorem, i.e. that the distribution for the mean displacement is Gaussian. Here is one way to check this numerically: divide up a long walk into a collection of blocks whose size is larger than the correlation time. Then plot a histogram of the block lengths and fit it to a Gaussian.

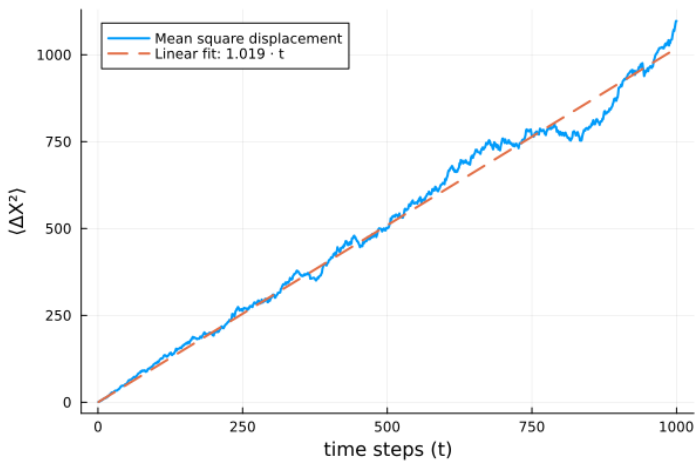

Figure 1: 1D random walk with 1000 steps and 100 trials.

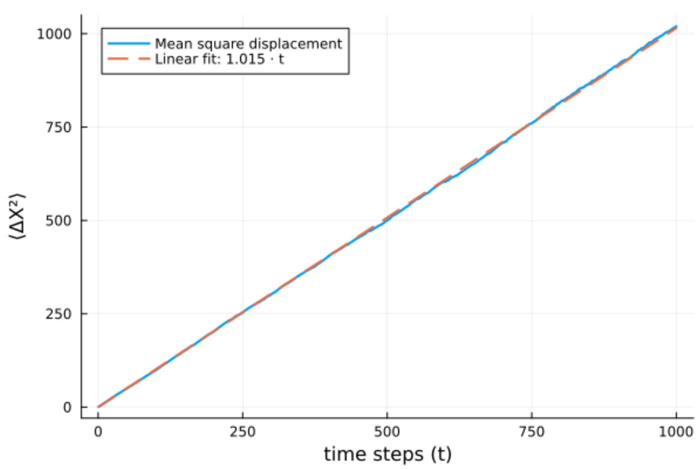

Figure 2: 1D random walk with 1000 steps and 10000 trials.

(b) Introduce weak correlations between the steps and check that the CLT still holds.

Here are two ways to introduce short-time correlations in the steps s_n of our random walk $X(t) = \sum_{n=1}^{t} s_n$:

(1) Choose a set of iid increments $\eta_i, i = 1 : N$ all at once. Then define the

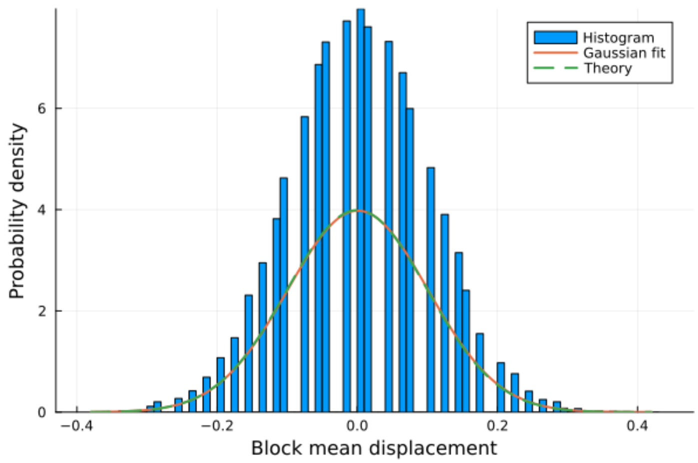

Figure 3: CLT with uncorrelated increments.

step to be

$$s_n = \sum_{j=1}^{q} a_j \eta_{n-j} \tag{13}$$

where a_j is a set of q numbers you pick (and $\eta_i = 0$ for $i < 1$ or $i > N$). This is called 'finite-memory moving average', and will induce correlations over a time q.

(2) Sample the increments s_n from an "Ornstein-Uhlenbeck process", that is:

$$s_{n+1} = \rho s_n + \eta_n \tag{14}$$

where ρ (with $|\rho| < 1$) is a parameter and η_n is some iid Gaussian noise with variance $\sigma_\eta = \sqrt{1-\rho^2}$. Here are the key steps in Julia:

```
s = zeros(N)
s[1] = randn()
for n in 2:N
 s[n] = rho * s[n-1] + randn() * sigma_eta
end
```

Is the distribution still Gaussian? Convince me.

Yes, the distribution is still Gaussian. See figure 4.

(c) [Bonus problem] Simulate a self-avoiding walk. Keep track of the whole history of the walk. Define some small distance of minimal approach. At each candidate step, make sure that it is not getting too close to a previous location before you accept. [continued on next page]

Can you see a difference in the RMS displacement versus number of steps?

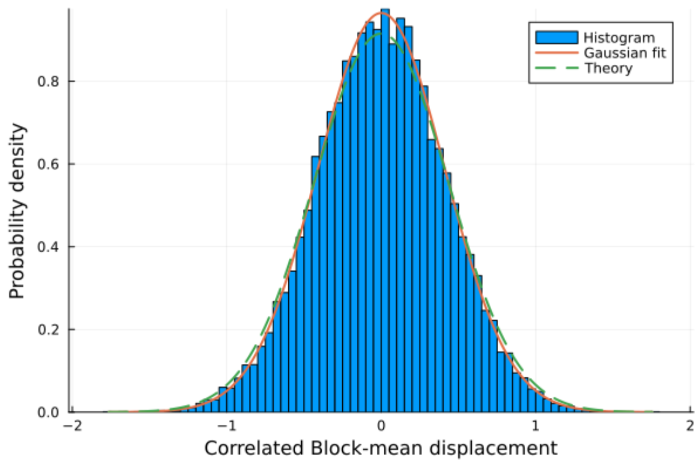

Figure 4: CLT with correlated increments.

[Note that this may not be (is not) the best way to simulate such a correlated walk. If you think of a better way, such as keeping track of a grid of visited sites, please feel free to do that instead.]