University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 210B Non-equilibrium Fall 2025
Assignment 5 – Solutions

Due 11:59pm Monday, November 3, 2025

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1. **The adjoint and the FP operator.** Write the Fokker-Planck equation $\partial_t P + \vec{\nabla} \cdot \vec{J} = 0$ for an overdamped particle in a potential $U(x)$ as

$$\partial_t P(x,t) = -\hat{L}P(x,t). \tag{1}$$

Assume the fluctuation-dissipation relation $D = \mu T$, so that

$$P_{\rm eq}(x) = e^{-U(x)/T}/Z \tag{2}$$

is a stationary solution of the FP equation.

(a) Show that the linear operator

$$\hat{H} = \frac{1}{\sqrt{P_{\rm eq}}}\hat{L}\sqrt{P_{\rm eq}} \tag{3}$$

is hermitian with respect to the inner product $\langle f|g\rangle \equiv \int_{-\infty}^{\infty} dx f(x)g(x)$ (the usual L^2 inner product on real functions on $\mathbb{R}$). ($\hat{H}$ is the linear operator in the FP equation for $P/\sqrt{P_{\rm eq}}$.)

For one thing, this shows that the eigenvalues of $\hat{L}$ are real, so that, in this problem, there are no oscillations in the approach to equilibrium.

$$\hat{H} = \frac{1}{\sqrt{P_{\rm eq}}}\hat{L}\sqrt{P_{\rm eq}} \tag{4}$$

let $\psi \equiv \frac{P}{\sqrt{P_{eq}}}$

$$\partial_t \psi = -\hat{H}\psi \tag{5}$$

$$\partial_x \sqrt{P_{eq}} = -\frac{1}{2T}\sqrt{P_{eq}}U' \tag{6}$$

$$\partial_x^2 \sqrt{P_{eq}} = \left(\frac{1}{4T^2}U'^2 - \frac{1}{2T}U''\right)\sqrt{P_{eq}} \tag{7}$$

Let's apply to a test function $\phi(x)$:

$$\hat{L}(\sqrt{P_{eq}}\psi) = \sqrt{P_{eq}}\left[-\mu T\partial_x^2\phi + \mu(\frac{U'^2}{4T} - \frac{U''}{2})\phi\right] \tag{8}$$

So,

$$\hat{H} = D\left[-\partial_x^2 + \frac{U'^2}{4T^2} - \frac{U''}{2T}\right] = D\left[-\partial_x^2 + W^2 - W'\right] \tag{9}$$

This can be factorized into

$$\hat{H} = D\left(-\partial_x + W\right)\left(\partial_x + W\right) \tag{10}$$

where $W = \frac{U'}{2T}$. Now to show it, we do

$$\langle f|\hat{H}g\rangle = D\int dx f\left(-\partial_x + W\right)\left(\partial_x + W\right) g = D\int dx f\left(-g'' + Vg\right) \tag{11}$$

$$\langle f|\hat{H}g\rangle = D\int dx\left(f'g' + fVg\right) = D\int dx\left(-f'' + Vf\right) g = \langle \hat{H}f|g\rangle \tag{12}$$

(b) [Bonus] Show that the adjoint of $\hat{L}$ with respect to this inner product is the operator appearing in the *backwards* FP equation describing the evolution of the *initial* time

$$\partial_{t'} P(x,t|x',t') = -\hat{L}^\dagger P(x,t|x't'). \tag{13}$$

2. **Stochastic topology.** Suppose you tie your dog to a flagpole (by an infinitely-extendible, massless leash that can pass through itself) in the center of a big round field and go run an errand[1]. Your dog performs Brownian motion with diffusion constant D. We would like to find (analytically!) the distribution for the *winding angle*

$$\theta(t) = \int_0^t dt\dot{\theta} = \int_0^t dt\partial_t \arctan(y/x) = \int_0^t dt\frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \tag{14}$$

of your dog's motion around the flagpole.

There are some choices to make in modeling this situation (which actually make a difference for the late time behavior):

- The flagpole is pointlike or has a finite radius a.
- The field is bounded by $r < R$ (and your dog can't leave) or goes on forever.

Pick an option for both items. The case $a = 0$ and an unbounded region is the most interesting. As a bonus problem, compare multiple options for the boundary conditions.

[1] I've never had a dog, so I don't know if this is an OK thing to do, but let's say that the dog doesn't mind.

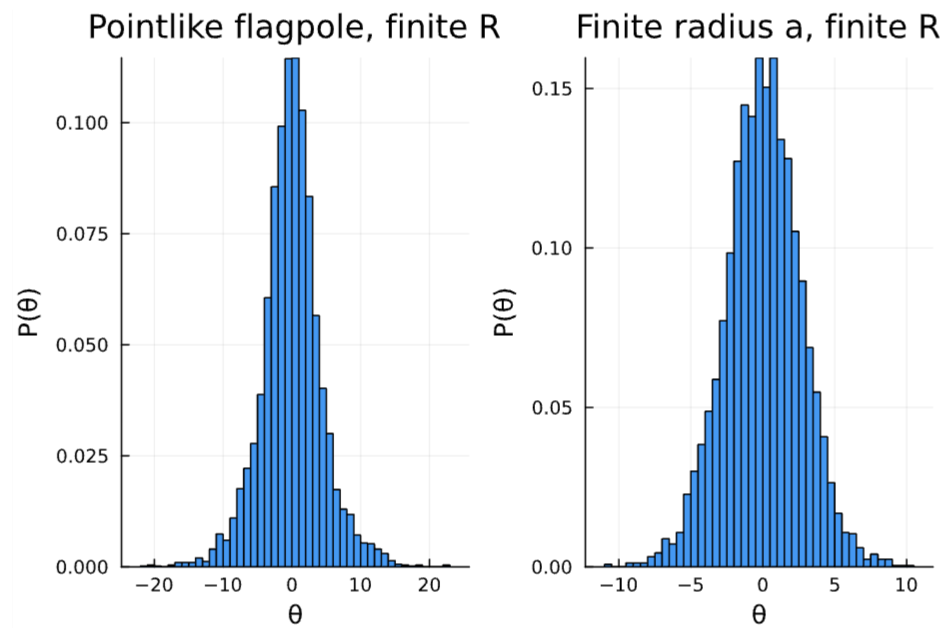

Figure 1: Winding angle distribution at $t = 1$ for pointlike flagpole with finite R (left) and finite radius a with finite R (right).

(a) Simulate the problem with the various choices mentioned above and make histograms of the winding angle distribution at late times. Warning: the case where the region is unbounded requires some care.
Please see the figures 1 and 2. In the case with a pointlike flagpole, the dog is able to jump through and make large angles.

(b) Here is a much easier but much less interesting version of the problem that may help get you started on the next part. Suppose we have a particle diffusing on a circle, S^1 (or a torus $\equiv (S^1)^d$). What is the probability distribution for the winding angle $\theta(t)$ as a function of time? By the winding angle, I mean we should keep track of the total change in angle from its initial position, not just mod 2π:

$$\theta \equiv \int_0^t ds\dot{\theta} \tag{15}$$

(or the discrete version of this formula).
Angular diffusion with constant D

$$d\psi(t) = \sqrt{2D}dW_t, \theta(0) = 0 \tag{16}$$

Now this is equivalent to a 1D Brownian motion with diffusion constant D.

$$\partial_t P = D\frac{\partial^2 P}{\partial \theta^2} \tag{17}$$

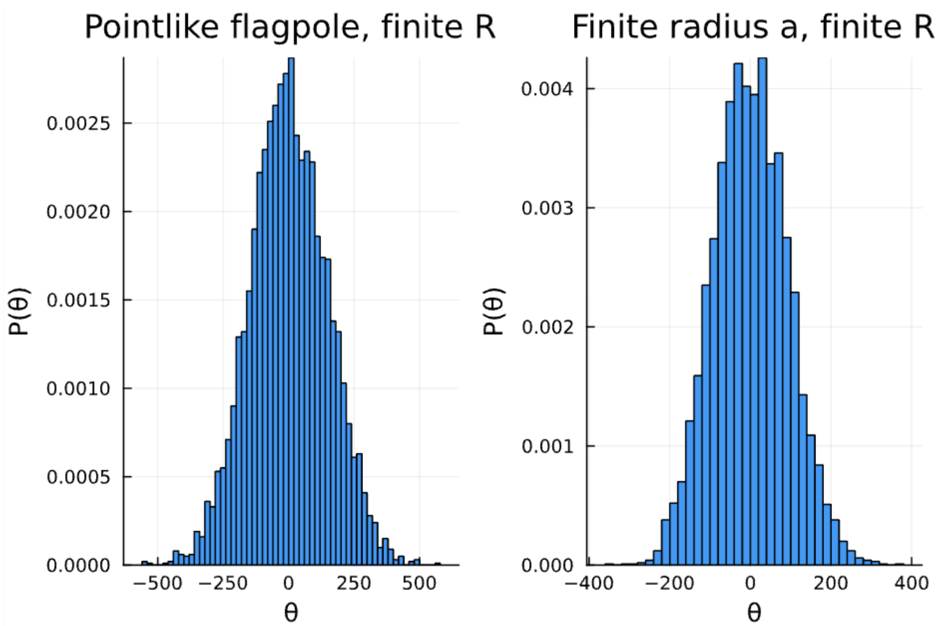

Figure 2: Winding angle distribution at $t = 1200$ for pointlike flagpole with finite R (left) and finite radius a with finite R (right).

$$P(\theta, 0) = \delta(\theta) \tag{18}$$

The solution is

$$P(\theta, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{\theta^2}{4Dt}} \tag{19}$$

Thus, the distribution of the winding angle is gaussian.

(c) Returning to the 2d story of the dog, here is the real problem. Find analytically the probability distribution $P(\theta, t)$ for the winding angle at late times. Suppose the initial distribution is localized to $r = r_0$.

Here are some hints for the case with unbounded domain.

- The following integral is useful:

$$\int_0^\infty dk k J_{|m|}(kr_0) J_{|m|}(kr) e^{-k^2 t} = \frac{1}{2t} e^{-\frac{r^2+r_0^2}{4t}} I_{|m|}\left(\frac{rr_0}{2t}\right). \tag{20}$$

- if you try to do the r integral under the integral over eigenvalues, you'll get nonsense.

Here are some hints for the cases with a bounded domain $R < \infty$:

- Let $j_{m,n}$ be the location of the nth zero of the ordinary Bessel function $J_m(x)$. For small $m \neq 0$, it has the Taylor expansion: $(j_{|m|,1})^2 \approx cm^2 + \mathcal{O}(m^4)$ for some constant c.

- Similarly, let $j'_{m,n}$ be the location of the nth zero of $\partial_x J_m(x)$. For small $m \neq 0$, it has the Taylor expansion: $(j'_{|m|,1})^2 \approx c'm^2 + \mathcal{O}(m^4)$ for some constant c'.

(d) [Bonus] Here is another wrinkle to consider: suppose that $r = R$ is an absorbing barrier (say that if $r > R$, the leash breaks and your dog runs home). What's the distribution of the winding angle *given* that the leash didn't break?

(e) [Bonus] Suppose that there is a wind spiraling around the flagpole counter-clockwise, which applies a drift force on your dog of the form $\vec{F} = r\Omega(r)\hat{\theta}$ with $\Omega(r) = \frac{\beta}{r^2}$. Show that the FP equation is instead

$$\partial_t P = -\Omega(r)\partial_\theta P + D\vec{\nabla}^2 P. \tag{21}$$

Now what is the late-time distribution of θ?