University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 210B Non-equilibrium Fall 2025
Assignment 3 – Solutions

Due 11:59pm Monday, October 20, 2025

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1. **Level ratios for Poisson processes.** Suppose we have a random process that produces a discrete set of real numbers $\{E_i\}$ (such as the spectrum of a random matrix, or decay times of radioactive atoms, the arrival times of buses ...). Consider the observable $\min(r, 1/r)$ where $r = \frac{E_{i+1}-E_i}{E_i-E_{i-1}}$, which I'll call the level ratio statistic. The average of this quantity is a nice measure of the level spacing, since any overall scales cancel out.

   Compute the expected value of this quantity for a Poisson process, where the events occur at a constant rate per unit E and are independent of each other.

   [Hint: the independence of the events means that the spacings between events are independent and distributed according to the exponential distribution $p(x) = \frac{1}{\mu}e^{-x/\mu}$.]

   Let S_i be the spacing between levels, i.e. $S_i = E_i - E_{i-1}$. Then $r = S_{i+1}/S_i$.

   $$p_s(s) = \frac{1}{\mu}e^{-s/\mu} \tag{1}$$

   Let r be the ratio of two independent spacings $\frac{S_{i+1}}{S_i}$. Then let $\tilde{r} = \min\left(r, \frac{1}{r}\right)$. The probability distribution of $\tilde{r}$ is given by

   $$p_{\tilde{r}}(\tilde{r}) = p_r(\tilde{r}) + p_r\left(\frac{1}{\tilde{r}}\right)\left|\frac{d\left(\frac{1}{\tilde{r}}\right)}{d\tilde{r}}\right| = p_r(\tilde{r}) + \frac{1}{\tilde{r}^2}\,p_r\left(\frac{1}{\tilde{r}}\right) \tag{2}$$

   To find $p_r(r)$, we have

   $$p_r(r) = \int_0^\infty p_s(s)\,p_s(rs)\,s\,ds = \int_0^\infty \frac{1}{\mu}e^{-s/\mu}\frac{1}{\mu}e^{-rs/\mu}s\,ds = \frac{1}{\mu^2}\int_0^\infty s\,e^{-s(1+r)/\mu}ds \tag{3}$$

   Using $\displaystyle\int_0^\infty xe^{-ax}dx = \frac{1}{a^2}$ gives

   $$p_r(r) = \frac{1}{\mu^2}\cdot\frac{\mu^2}{(1+r)^2} = \frac{1}{(1+r)^2} \tag{4}$$

   $$\langle\tilde{r}\rangle = \int_0^\infty \tilde{r}\,p_{\tilde{r}}(\tilde{r})\,d\tilde{r} = \int_0^1 \frac{\tilde{r}}{(1+\tilde{r})^2}d\tilde{r} + \int_1^\infty \frac{1}{\tilde{r}(1+\tilde{r})^2}d\tilde{r} \tag{5}$$

$$\langle \tilde{r} \rangle = 2\left(\ln 2 + \tfrac{1}{2} - 1 \right) = 2\ln 2 - 1 \approx 0.3863. \tag{6}$$

Bonus problem: estimate the expected value of r for the GOE ensemble of random matrices either (a) using the Wigner surmise, or (b) [super bonus problem] from scratch. Do the integral numerically if you must.

2. **Wigner surmise for $N = 2$.** Consider a Gaussian random real symmetric matrix H that is 2×2:

$$H = \begin{pmatrix} x & z \\ z & y \end{pmatrix}, \tag{7}$$

with $P(H) \propto e^{-\frac{\mathrm{tr} H^2}{2\sigma^2}}$. We're going to find the resulting distribution for the level spacing, $P(s)$.

(a) Find the difference s between the larger and smaller eigenvalue of H as a function of the random variables x, y, z, i.e. $s(x, y, z)$.

The eigenvalues of H are given by

$$\lambda_\pm = \frac{x+y}{2} \pm \sqrt{\left(\frac{x-y}{2}\right)^2 + z^2} \tag{8}$$

Thus, the level spacing is

$$s = \lambda_+ - \lambda_- = 2\sqrt{\left(\frac{x-y}{2}\right)^2 + z^2} = \sqrt{(x-y)^2 + 4z^2} \tag{9}$$

(b) The distribution for H says that x, y, z are Gaussian random variables (though z has a different width). What is the distribution for $\delta \equiv x - y$?

The joint distribution for x and y is given by

$$P(x, y) \propto e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{10}$$

Changing variables to $\delta = x - y$ and $X = \frac{x+y}{2}$, we have

$$P(\delta, X) \propto e^{-\frac{(X+\delta/2)^2 + (X-\delta/2)^2}{2\sigma^2}} = e^{-\frac{2X^2+\delta^2/2}{2\sigma^2}} = e^{-\frac{X^2}{\sigma^2}} e^{-\frac{\delta^2}{4\sigma^2}} \tag{11}$$

Integrating out X gives

$$P(\delta) \propto e^{-\frac{\delta^2}{4\sigma^2}} \tag{12}$$

Thus, δ is a Gaussian random variable with mean 0 and variance $2\sigma^2$.

(c) Find the distribution for the level spacing,

$$P(s) = \int dxdydzp(x,y,z)\delta(s - s(x,y,z)). \tag{13}$$

The joint distribution for δ and z is given by

$$P(\delta, z) \propto e^{-\frac{\delta^2}{4\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \tag{14}$$

Using the expression for s from part (a), we have

$$P(s) = \int d\delta\, dz\, P(\delta, z)\ \delta\left(s - \sqrt{\delta^2 + 4z^2}\right) \tag{15}$$

[Hint: redefine variables so that s is a sum in quadrature of two iid gaussian random variables. Then go to polar coordinates.]

(d) Finally, choose σ so that the mean level spacing is 1 and write the final expression for $P(s)$.

Changing variables to polar coordinates with $\delta = r\cos\theta$ and $2z = r\sin\theta$, we have

$$P(s) = \int_0^\infty dr \int_0^{2\pi} d\theta\, r\, P(r,\theta)\ \delta(s - r) \tag{16}$$

where

$$P(r,\theta) \propto \exp\left[-\frac{r^2\left(\frac{\cos^2\theta}{4} + \frac{\sin^2\theta}{8}\right)}{\sigma^2}\right] \tag{17}$$

Integrating over θ gives

$$P(s) \propto s\, e^{-\frac{s^2}{8\sigma^2}} \int_0^{2\pi} d\theta\, e^{-frac s^2 8\sigma^2 \cos 2\theta} = 2\pi s\, e^{-\frac{s^2}{8\sigma^2}} I_0\left(\frac{s^2}{8\sigma^2}\right) \tag{18}$$

To normalize $P(s)$ and set the mean level spacing to 1, we compute

$$\langle s \rangle = \int_0^\infty s\, P(s)\, ds \tag{19}$$

$$\langle s \rangle = 4\sigma\sqrt{2\pi} \cdot {}_2F_1\left(\frac{3}{2}, -\frac{1}{2}; 1; 1\right) = 4\sigma\sqrt{2\pi} \cdot \frac{4}{3\pi} = \frac{16\sigma}{3}\sqrt{\frac{2}{\pi}} \tag{20}$$

Setting $\langle s \rangle = 1$ gives $\sigma = \frac{3}{16}\sqrt{\frac{\pi}{2}}$. Thus, the final expression for $P(s)$ is

$$P(s) = \frac{32}{3\pi} s\, e^{-\frac{2s^2}{9\pi}} I_0\left(\frac{s^2}{9\pi}\right) \tag{21}$$

(e) [Super bonus problem] **Wigner surmise for the joint distribution of 3 successive levels?** You may have noticed when doing the previous problem that assuming that $x \equiv E_{i+1} - E_i$ and $y \equiv E_i - E_{i-1}$ are independent and both governed by the Wigner surmise gives the wrong answer for the ratio statistic on the previous problem. We can do better by approximating the joint distribution for x and y. A not bad approximation can be found by imitating what we did in this problem: just take $N = 3$ and find the distribution for the two differences between the three eigenvalues. The claim is that the answer will be of the form

$$P(x, y) = Cxye^{-B(x^2+y^2)-\alpha xy} \tag{22}$$

where the constants are fixed by normalization, unit level spacing, and the correct value of the correlation between x and y, which is $\langle xy \rangle - \langle x \rangle \langle y \rangle = \rho\sigma_x\sigma_y$ with $\rho \approx -0.27$. Use this distribution to compute the level ratio statistic. [Warning: I haven't found a way to do this that doesn't involve horrible calculations.]

3. **Diffusion of a non-spherical particle.** Consider the Brownian motion of an ellipsoid in 2 dimensions. The variables that describe the microstate of such an object are the position of its center of mass denoted by a 2 D vector $\mathbf{r}$ and a unit vector characterizing the orientation of the ellipsoid denoted here by $\hat{\mathbf{u}}$ (see figure). The collisions with the fluid molecules will cause both the center of mass and the orientation to fluctuate. The center of mass fluctuations result in translational diffusion while the orientation fluctuations lead to rotational diffusion. The center of mass coordinate obeys an overdamped Langevin equation

$$\partial_t r_\alpha = \eta_\alpha^T(t),$$

where the index α denotes the cartesian coordinates in 2d, i.e., $\alpha = \{x, y\}$ and η_α^T is a stochastic force that is Markovian in that

$$\left\langle \eta_\alpha^T \right\rangle_c = 0; \left\langle \eta_\alpha^T(t)\, \eta_\beta^T(t') \right\rangle_c = 2D_{\alpha\beta}\delta(t - t')$$

and all higher cumulants are zero and where

$$D_{\alpha\beta} = D_\parallel \hat{u}_\alpha \hat{u}_\beta + D_\perp (\delta_{\alpha\beta} - \hat{u}_\alpha \hat{u}_\beta)$$

This anisotropic diffusion tensor reflects the fact that the particle feels less friction when it moves along its long axis when compared to the case when it moves perpendicular to the long axis.

(a) Use the above form of the noise show that the mean square displacement can be given by

$$\langle \delta r_\alpha (t)\, \delta r_\beta (t) \rangle_{\eta^T} = 2t\overline{D}\delta_{\alpha\beta} + \frac{1}{2}\Delta D \int_0^t dt' M_{\alpha\beta} (\theta (t'))$$

where θ is the angle paramaterizing the unit vector $\hat{\mathbf{u}}$, i.e., $\hat{\mathbf{u}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$ and

$$M_{\alpha\beta} (\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}_{\alpha\beta}$$

Be sure to identify $\overline{D}$ and ΔD in terms of $D_\parallel$ and $D_\perp$.

We have

$$\delta r_\alpha(t) = \int_0^t dt' \eta_\alpha^T(t') \tag{23}$$

Thus,

$$\langle \delta r_\alpha(t)\delta r_\beta(t)\rangle_{\eta^T} = \int_0^t dt' \int_0^t dt'' \langle \eta_\alpha^T(t')\eta_\beta^T(t'')\rangle_c = \int_0^t dt' \int_0^t dt'' 2D_{\alpha\beta}\delta(t'-t'') \tag{24}$$

$$= 2\int_0^t dt' D_{\alpha\beta} = 2tD_{\alpha\beta} \tag{25}$$

Using the expression for $D_{\alpha\beta}$, we have

$$D_{\alpha\beta} = D_\perp\delta_{\alpha\beta} + (D_\parallel - D_\perp)\hat{u}_\alpha\hat{u}_\beta = \overline{D}\delta_{\alpha\beta} + \frac{1}{2}\Delta D M_{\alpha\beta}(\theta) \tag{26}$$

where $\overline{D} = \frac{D_\parallel + D_\perp}{2}$ and $\Delta D = D_\parallel - D_\perp$. Thus,

$$\langle \delta r_\alpha(t)\delta r_\beta(t)\rangle_{\eta^T} = 2t\overline{D}\delta_{\alpha\beta} + \frac{1}{2}\Delta D \int_0^t dt' M_{\alpha\beta}(\theta(t')) \tag{27}$$

Now over and beyond the translational diffusion described above, the orientation of the ellipsoid fluctuates as well. The angle θ parameterizing $\hat{\mathbf{u}}$ obeys a Langevin equation of its own

$$\partial_t\theta = \eta^R(t)$$

where

$$\left\langle \eta^R(t)\, \eta^R(t') \right\rangle = 2D_R\delta(t-t')$$

In order to get the true mean square displacement we need to calculate $\langle r_\alpha (t)\, r_\beta (t)\rangle_{\eta^T\eta^R}$. This involves us calculating $\langle M_{ij}(\theta)\rangle_{\eta^R}$. We do this in two steps.

(b) Show that because the stochastic process is Gaussian, $\left\langle e^{in\theta(t)}\right\rangle = e^{in\theta_0}\exp\left(-n^2 D_R t\right)$. One way to do this is to note that

$$\theta\left(t\right) = \theta_0 + \Delta\theta\left(t\right)$$

with

$$\Delta\theta\left(t\right) = \int_0^t dt'\eta^R\left(t'\right)$$

From the properties of the noise,

$$\left\langle \Delta\theta\left(t\right)^2\right\rangle_c = 2D_R t$$

and

$$\left\langle \Delta\theta\left(t\right)^n\right\rangle_c = 0,\ \forall n > 2$$

Now, series expand $e^{in\theta(t)}$ in powers of $\Delta\theta\left(t\right)^n$, then use the relationship between cumulants and moments (Eq. 2.14 in Kardar for example) to calculate $\left\langle e^{in\theta(t)}\right\rangle_{\eta^R}$. The relationship you need to know is that for a gaussian distribution, $\langle x^m\rangle = 0$ for all odd m and $\langle x^m\rangle = \frac{m!}{\left(\frac{m}{2}\right)!2^{m/2}}\left\langle x^2\right\rangle^{m/2}$.

We have

$$\langle e^{in\theta(t)}\rangle = e^{in\theta_0}\langle e^{in\Delta\theta(t)}\rangle \tag{28}$$

Expanding the exponential gives

$$\langle e^{in\Delta\theta(t)}\rangle = \sum_{m=0}^{\infty}\frac{(in)^m}{m!}\langle(\Delta\theta(t))^m\rangle \tag{29}$$

$$\langle(\Delta\theta(t))^m\rangle = \frac{m!}{(m/2)!2^{m/2}}\langle(\Delta\theta(t))^2\rangle^{m/2} = \frac{m!}{(m/2)!2^{m/2}}(2D_R t)^{m/2} \text{ for even } m \tag{30}$$

Thus,

$$\langle e^{in\Delta\theta(t)}\rangle = \sum_{k=0}^{\infty}\frac{(in)^{2k}}{(2k)!}\frac{(2k)!}{k!2^k}(2D_R t)^k = \sum_{k=0}^{\infty}\frac{(-1)^k(n^2 D_R t)^k}{k!} = e^{-n^2 D_R t} \tag{31}$$

Finally,

$$\langle e^{in\theta(t)}\rangle = e^{in\theta_0}e^{-n^2 D_R t} \tag{32}$$

(c) Now use this result to get the general expression for $\left\langle \delta r_\alpha\left(t\right)\delta r_\beta\left(t\right)\right\rangle_{\eta^T\eta^R}$ in terms of θ_0 and the diffusion coefficients $D_\parallel$, $D_\perp$ and D_R.

(d) Suppose initially the particle is along the x direction. Calculate $\left\langle \delta r_x\left(t\right)\delta r_x\left(t\right)\right\rangle_{\eta^T\eta^R}$ in two limits $t \ll 1/D_R$ and $t \gg 1/D_R$.

(e) Do the same for $\langle \delta r_y(t)\, \delta r_y(t) \rangle_{\eta^T \eta^R}$ for the same initial condition.

(f) What did we learn about the diffusion of an aspherical particle when compared to that of a spherical particle?

On short times, the diffusion constant is enhanced in the direction of the long axis, and suppressed in the transverse direction. Eventually, the diffusion in the orientation leads it to forget which direction was which, and the result is just an overall relative in the width between the two directions.

(g) [Bonus problem] Simulate Brownian motion of ellipsoidal particles and compare with your answers.