Some preprocessing and polyhedral results on facility location with preferences

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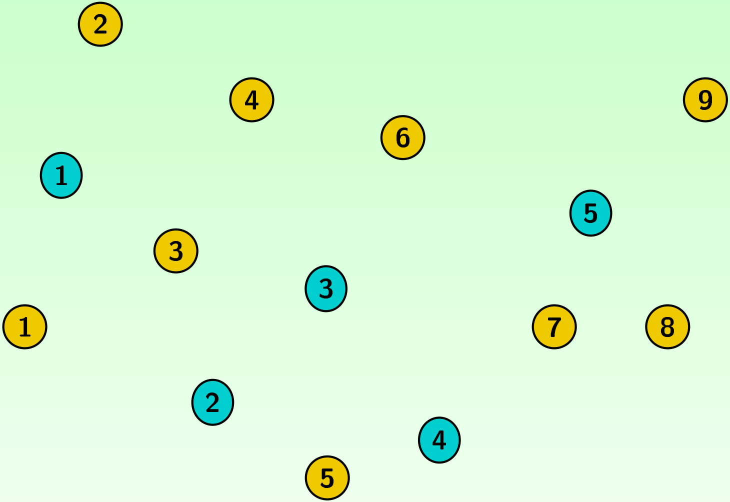
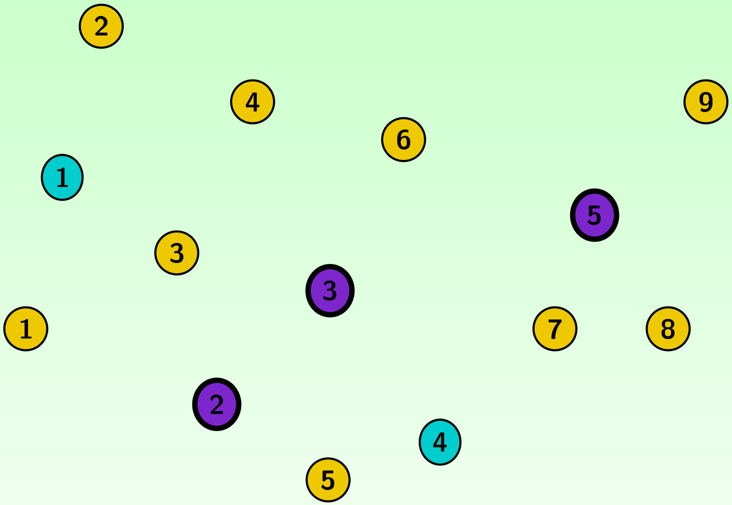
3 June 2025
Mixed Integer Programming Workshop 2025

SPLP example



SPLP example



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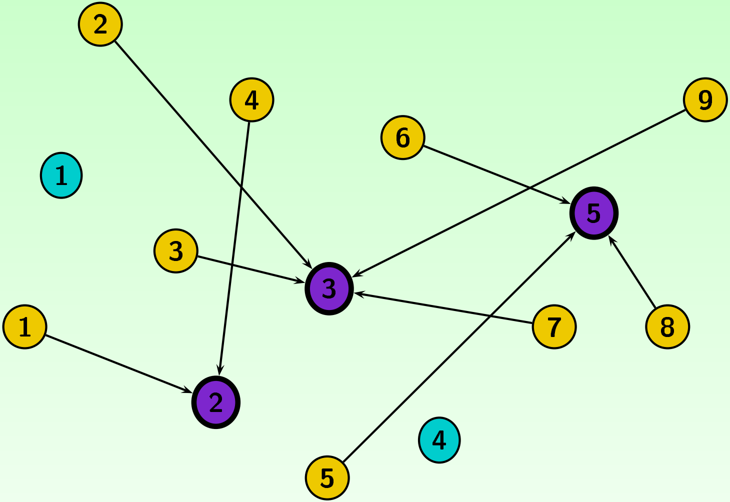

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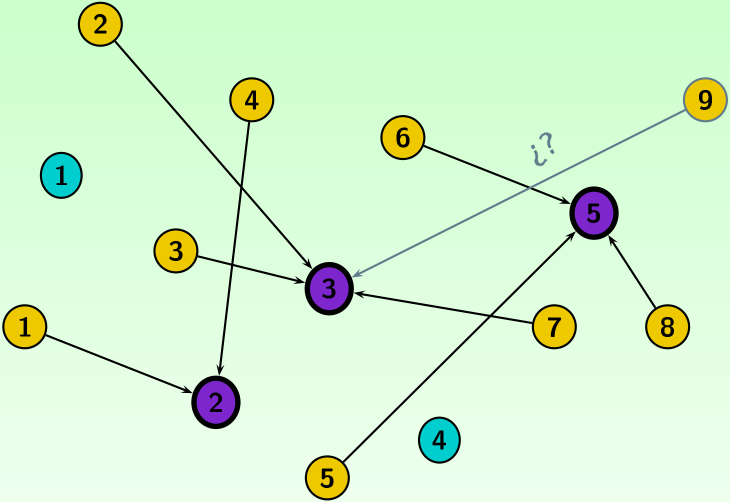

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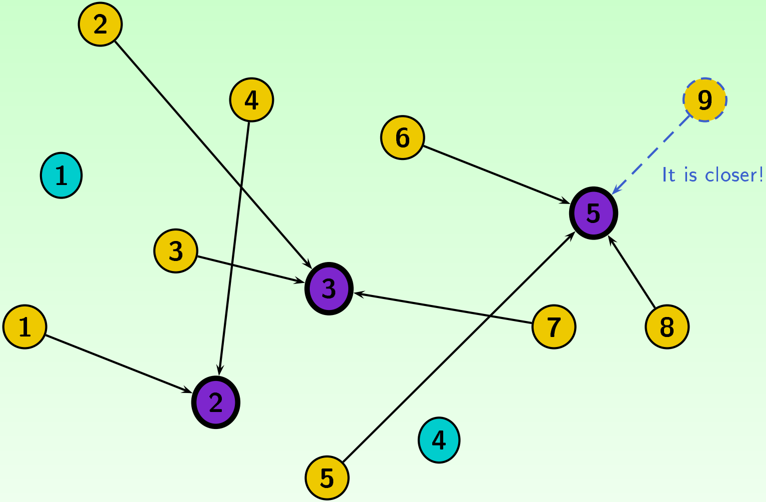

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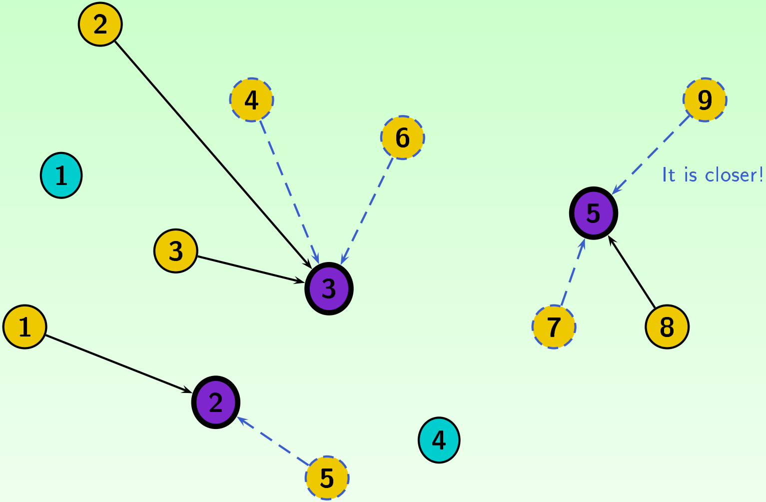
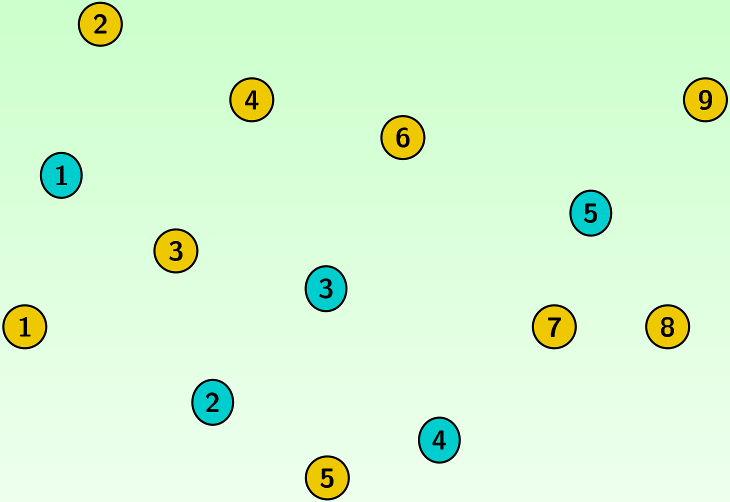


Closest assignment (CA)



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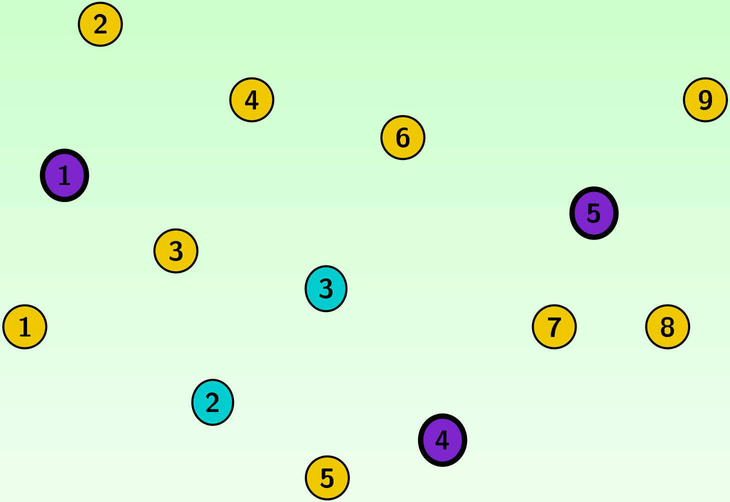
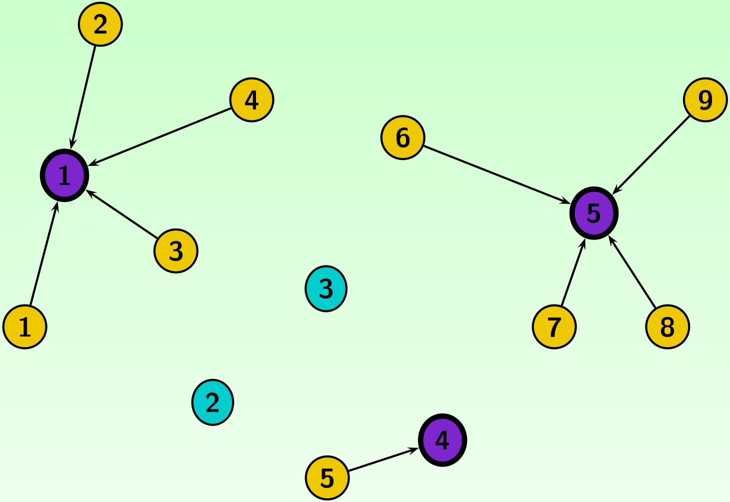


Closest assignment (CA)



CA situations

Sometimes CA comes with the model itself:

- classical p-median facility location problem: minimize distance traveled by the customers (Hakimi, 1964).

In many other situations (i.e. public sector models), CA must be stated explicitly:

- emergency response vehicles;
- schools (school districting);
- polling stations (political districting).

CA situations

- Locator: determines the facility to open.
- Allocator: decides which customers attend which facilities.

They may coincide... or may not.

Preferences

Each customer:

1. ranks their preferences for the facilities, and
2. they will attend their most preferred open facility.

Notation (1)

I = set of customers, J = set of facilities.

Variables:

$$x_{ij} = \text{fraction of demand of node } j \text{ served by node } i$$

$$y_k = \begin{cases} 1 & \text{if facility } k \text{ is open,} \\ 0 & \text{otherwise.} \end{cases}$$

Constants:

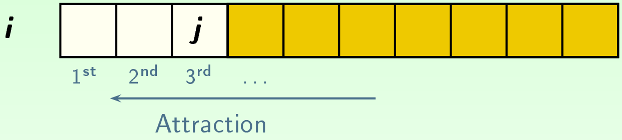
$$c_{ij} = \text{cost of serving the whole demand of customer } i \text{ from facility } j;$$

$$f_j = \text{cost of opening a facility at node } j.$$

Notation (2)

Let $i \in I$ and $j, k \in J$.
Facility k is i-worse than facility j, $k <_i j$, if customer i prefers j to k.

$$W(i,j) = \{k \in J \;/\; k <_i j\}$$



Formulation

$$
(SPLPO)\left\{
\begin{array}{llcrr}
\text{Min.} & \displaystyle\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{k\in J} f_k y_k & & & \\
\text{s.t.} & \displaystyle\sum_{j\in J} x_{ij} & = & 1 & \forall i \in I, \\
& x_{ij} & \leq & y_j & \forall i \in I, j \in J, \\
& \displaystyle\sum_{k\in W(i,j)} x_{ik} + y_j & \leq & 1 & \forall i \in I, j \in J, \\
& x_{ij} & \geq & 0 & \forall i \in I, j \in J, \\
& y_j & \in & \{0,1\} & \forall j \in J.
\end{array}
\right.
$$

Strict preferences are assumed.

2-customer preference inequalities (1)

Let $i_1, i_2 \in I$ and $j \in J$.

$$\sum_{k \in W(i_1,j)} x_{i_1 k} + \sum_{k \in W(i_2,j) \setminus W(i_1,j)} x_{i_2 k} + y_j \leq 1$$

This inequality is stronger than

$$\sum_{k \in W(i_1,j)} x_{i_1 k} + y_j \leq 1$$

2-customer preference inequalities (2)

Particularly, if

$$W(i_1, j) \cap W(i_2, j) = \emptyset,$$

then

$$\sum_{k \in W(i_1, j)} x_{i_1 k} + \sum_{k \in W(i_2, j)} x_{i_2 k} + y_j \leq 1.$$

One new constraint is stronger than two starting constraints.

Dominance inequalities

Let $i_1, i_2 \in I$ and $j \in J$. If

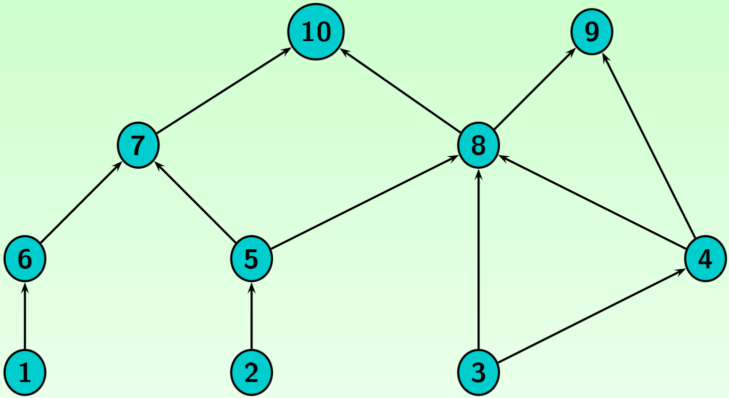
$$W(i_1, j) \subseteq W(i_2, j),$$

then

$$x_{i_1 j} \leq x_{i_2 j}.$$

Inequality $x_{ij} \leq y_j$ may not be a facet.

A "tree" of valid inequalities



A "tree" of valid inequalities

$$\sum_{i=1}^{8} |\text{predecessors}(i)| = 3+5+4+3+4+2+1+2 = 24.$$

$$\begin{array}{cccc}
x_{1j} \le x_{6j}, & x_{2j} \le x_{9j}, & x_{4j} \le x_{8j}, & x_{5j} \le x_{10\,j}, \\
x_{1j} \le x_{7j}, & x_{2j} \le x_{10\,j}, & x_{4j} \le x_{9j}, & x_{6j} \le x_{7j}, \\
x_{1j} \le x_{10\,j}, & x_{3j} \le x_{4j}, & x_{4j} \le x_{10\,j}, & x_{6j} \le x_{10\,j}, \\
x_{2j} \le x_{5j}, & x_{3j} \le x_{8j}, & x_{5j} \le x_{7j}, & x_{7j} \le x_{10\,j}, \\
x_{2j} \le x_{7j}, & x_{3j} \le x_{9j}, & x_{5j} \le x_{8j}, & x_{8j} \le x_{9j}, \\
x_{2j} \le x_{8j}, & x_{3j} \le x_{10\,j}, & x_{5j} \le x_{9j}, & x_{8j} \le x_{10\,j}.
\end{array}$$

A "tree" of valid inequalities

- However, more than half of these inequalities are dominated by other inequalities belonging to this set.
- The non-dominated inequalities are those represented with arcs connecting two nodes which do not accept a longest path.
- There are up to ten such inequalities:

$$\begin{array}{cc} x_{1j} \leq x_{6j}, & x_{5j} \leq x_{8j}, \\ x_{2j} \leq x_{5j}, & x_{6j} \leq x_{7j}, \\ x_{3j} \leq x_{4j}, & x_{7j} \leq x_{10\,j}, \\ x_{4j} \leq x_{8j}, & x_{8j} \leq x_{9j}, \\ x_{5j} \leq x_{7j}, & x_{8j} \leq x_{10\,j}. \end{array}$$

- Thus, 14 out of 24 inequalities are redundant.

Constraint aggregation

$$W(i_1,j)\cap W(i_2,j)=\emptyset \quad \Rightarrow \quad \sum_{k\in W(i_1,j)} x_{i_1 k} + \sum_{k\in W(i_2,j)} x_{i_2 k} + y_j \leq 1$$

Constraint aggregation

$$W(i_1,j) \cap W(i_2,j) = \emptyset \quad \Rightarrow \quad \sum_{k \in W(i_1,j)} x_{i_1 k} + \sum_{k \in W(i_2,j)} x_{i_2 k} + y_j \leq 1$$

$$\#9: \sum_{k \in W(i_t,j)} x_{i_t k} + y_j \leq 1 \quad \rightsquigarrow \quad \#4: \sum_{i \in A_t} \sum_{k \in W(i,j)} x_{ik} + y_j \leq 1$$

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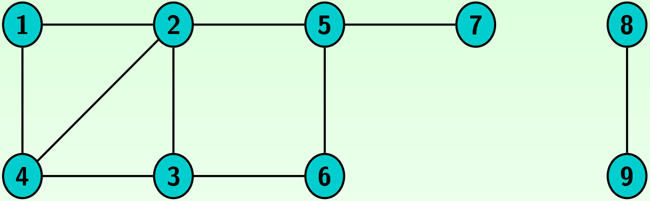
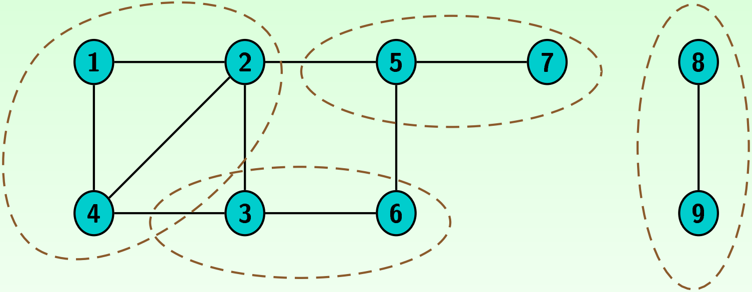


Constraint aggregation

$$W(i_1,j)\cap W(i_2,j)=\emptyset \quad \Rightarrow \quad \sum_{k\in W(i_1,j)} x_{i_1k} + \sum_{k\in W(i_2,j)} x_{i_2k} + y_j \le 1$$

$$\#9: \sum_{k\in W(i_t,j)} x_{i_tk} + y_j \le 1 \quad \rightsquigarrow \quad \#4: \sum_{i\in A_t}\sum_{k\in W(i,j)} x_{ik} + y_j \le 1$$



Computational experience

m =100, n =75										
	No preprocessing			Preprocessing			%			
Instance	Time	Nodes	Gap	Time	Nodes	Gap	TR	RPR	NZE	RI
a_100_75_1	2639	1892	21.27	989	554	18.76	63	15.39	55.07	90.53
a_100_75_2	5116	4304	27.19	3577	3192	25.16	30	15.53	55.07	88.92
a_100_75_3	5813	4889	25.97	3375	2855	24.06	42	15.05	55.08	92.19
a_100_75_4	5891	5080	24.57	2438	2079	22.24	59	15.37	55.07	92.80
b_100_75_1	29511	41971	31.80	12671	21744	28.49	57	15.15	55.08	92.20
b_100_75_2	33593	52725	33.08	12986	22439	30.40	61	15.31	55.07	89.20
b_100_75_3	47627	76295	34.47	17477	26785	30.88	63	15.00	55.08	91.95
b_100_75_4	13466	18497	28.56	4607	6361	26.94	66	15.53	55.07	91.23
c_100_75_1	18760	26384	32.15	7682	13462	27.74	59	15.40	55.07	89.16
c_100_75_2	24096	41874	31.49	9625	18870	28.19	60	15.93	55.06	88.63
c_100_75_3	15708	27268	32.30	4682	8107	28.86	70	15.45	55.07	92.31
c_100_75_4	25775	44457	32.36	8756	15617	29.33	66	15.69	55.06	92.43
Average	19000	28811	29.60	7405	11839	26.75	58	15.40	55.07	90.96

Table legend

- *RPR (Reduced Preference Rows)*: Reduction in the number of preference constraints (in percentage).
- *NZE (Non-Zero Elements)*: Percentage of non-zero elements in the preprocessed formulation compared to the number of non-zero elements in the non-preprocessed one.
- *RI (Reinforced Inequalities)*: Percentage of inequalities $x_{ij} \leq y_j$ from the non-preprocessed formulation which are replaced with a tighter dominance inequality.
- *Time*: CPU time in seconds.
- *Nodes*: Number of nodes of the branching tree;
- *Gap*: $100(1 - z_L/z_I)$, where z_L and z_I are, respectively, the optimal values of the linear relaxation and the integer problem.
- *TR (Time Reduction)*: Reduction (in percentage) in computational time.