Multidimensional Disjunctive Inequalities for Chance-Constrained Stochastic Problems with Finite Support

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(Mixed-Integer) Linear Chance-Constrained Problem (LCCP)

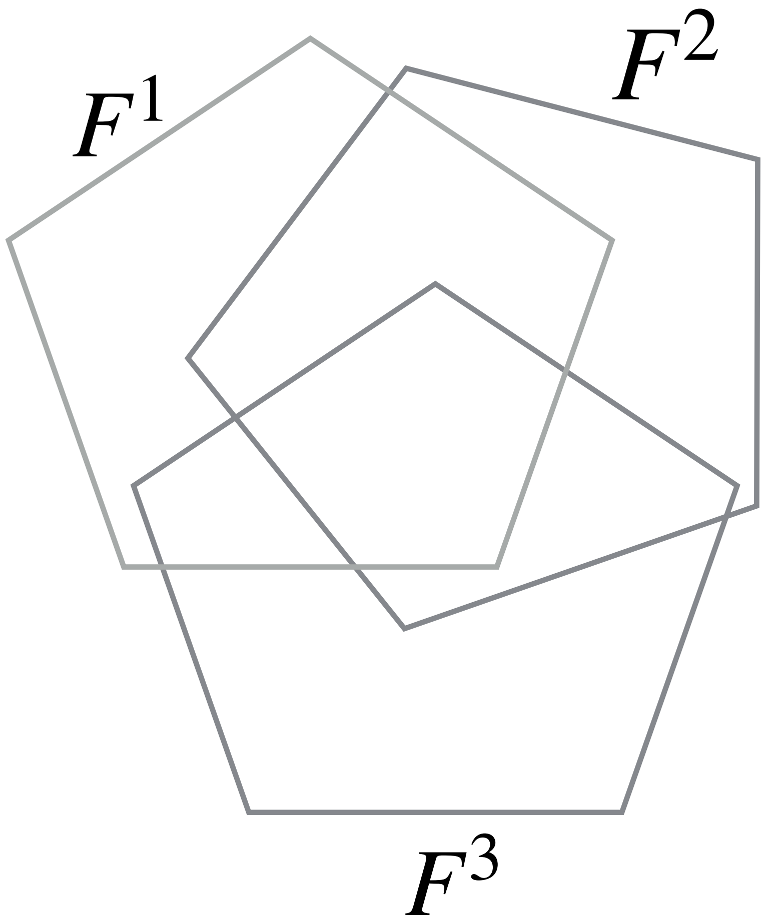
$$
\begin{aligned}
\min_{x} \quad & c^\top x \\
\text{s.t.} \quad & x \in X, \\
& \mathbb{P}(A^\omega x - b^\omega \geq 0) \geq 1 - \tau
\end{aligned}
$$

ω : random variable with discrete support

LCCP: applications

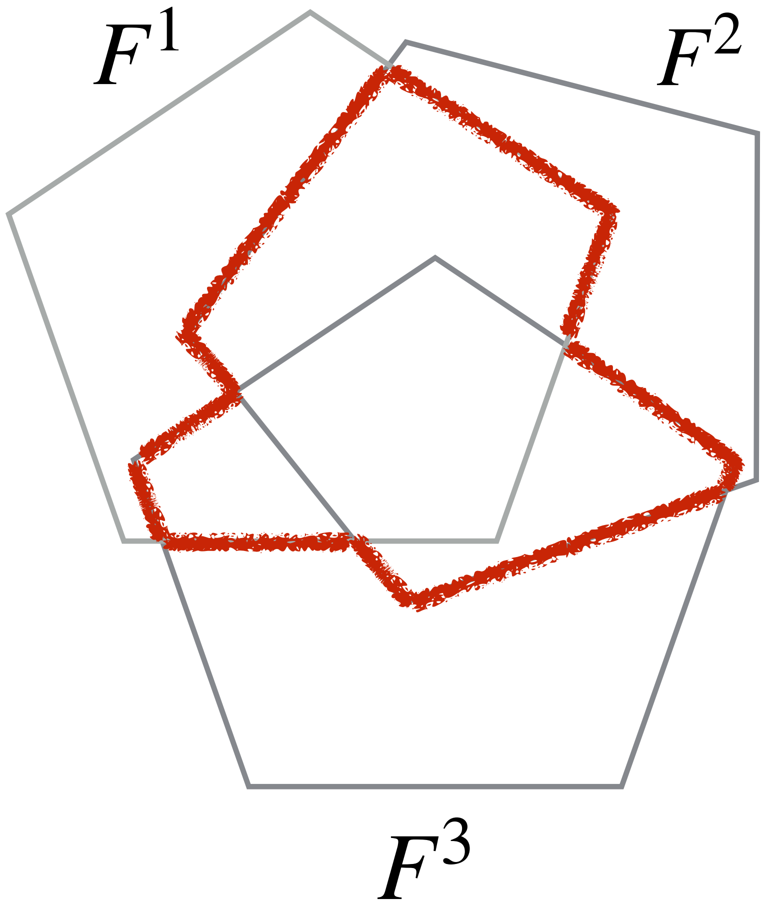
- Energy: optimal power flow
- Finance: portfolio optimisation
- Communication: reliable network design
- Chemical processing
- Production planning
- Water resources management

LCCP



$$\mathbb{P}(F^i) = \frac{1}{3}, i = 1,2,3 \qquad 1 - \tau = \frac{2}{3}$$

LCCP



Feasible domain

=

union of exponential number of convex sets

LCCP: Complexity

LCCP is strongly NP-hard even if
all scenanios have equal probability and

there is only one constraint per scenario
(Amaldi & Kann 1995)

only the RHS vector is random (Luedtke et al. 2010)

LCCP: MIP formulation

$$
\begin{aligned}
\min_{x,y} \quad & c^\top x \\
\text{s.t.} \quad & x \in X, \\
& A^s x \geq \bar{b}^s + (b^s - \bar{b}^s) y^s, \quad s \in S, \\
& \sum_{s \in S} p^s y^s \geq 1 - \tau, \\
& y^s \in \{0, 1\}, \quad s \in S.
\end{aligned}
$$

Performance of CPLEX
(from Qiu et al. 2014)

20 variables, 200 scenarios, 1 constraint per scenario

Table 1 Performance of CPLEX on CKVLP

	Dense data			Sparse data			Random objective		
k	Gap closed (%)	Nodes	Time	Gap closed (%)	Nodes	Time	Gap closed (%)	Nodes	Time
15	2	3,537,864	2,454	7	158,039	83	17	1,777	1
20	2	43,296,679	25,948	6	1,769,574	917	21	6,227	2

Some related papers

Qiu, F., Ahmed, S., Dey, S. S., & Wolsey, L. A. (2014). Covering linear programming with violations. *INFORMS Journal on Computing*, *26*(3), 531-546.

Luedtke, J. (2014). A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support. *Mathematical Programming*, *146*(1-2), 219-244.

Ahmed, S., Luedtke, J., Song, Y., & Xie, W. (2017). Nonanticipative duality, relaxations, and formulations for chance-constrained stochastic programs. *Mathematical Programming*, *162*, 51-81.

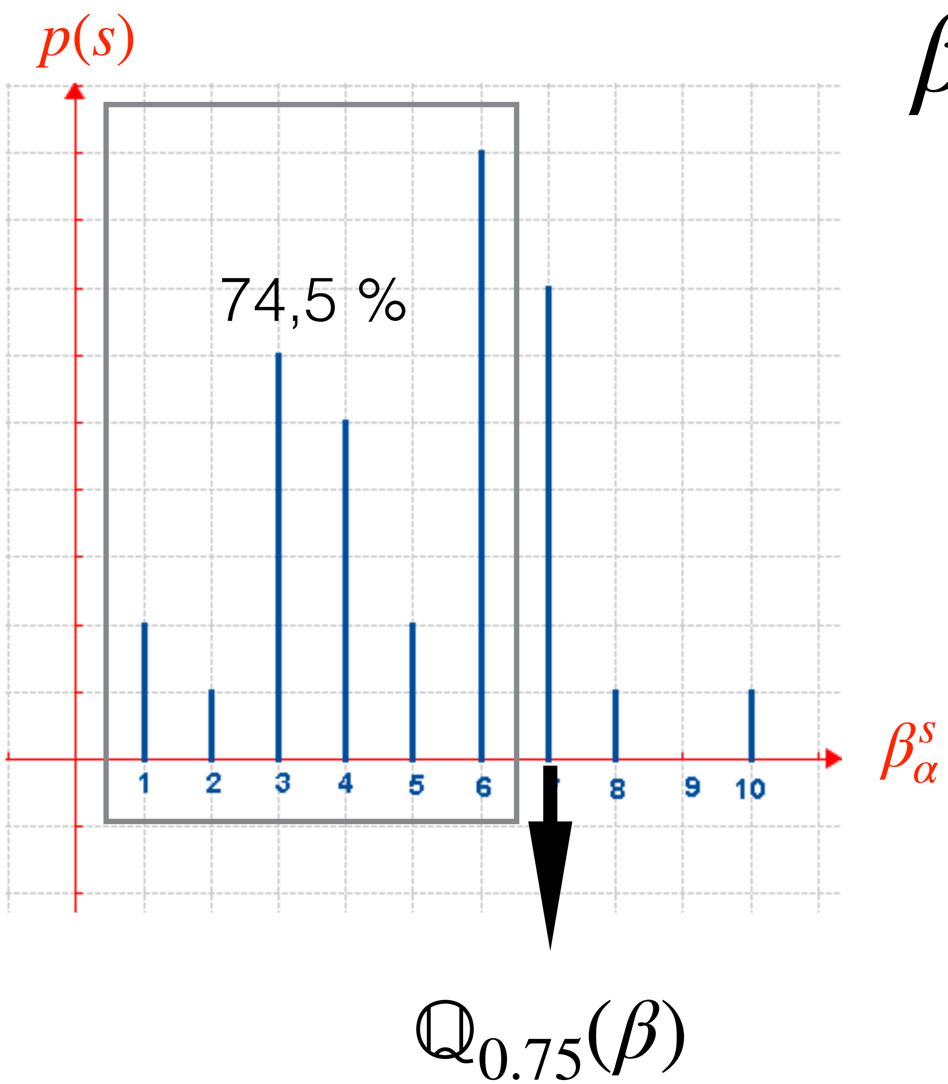
Xie, W., & Ahmed, S. (2018). On quantile cuts and their closure for chance constrained optimization problems. *Mathematical Programming*, *172*, 621-646.

Song, Y., & Luedtke, J. R. (2013). Branch-and-cut approaches for chance-constrained formulations of reliable network design problems. *Mathematical Programming Computation*, *5*(4), 397-432.

Song, Y., Luedtke, J. R., & Küçükyavuz, S. (2014). Chance-constrained binary packing problems. *INFORMS Journal on Computing*, *26*(4), 735-747.

Cattaruzza, D., Labbé, M., Petris, M., Roland, M., & Schmidt, M. (2024). Exact and Heuristic Solution Techniques for Mixed-Integer Quantile Minimization Problems. INFORMS Journal on Computing.

Quantile cut



$$\beta_\alpha^s = \min_x \{\alpha^\top x : A^s x \geq b^s, x \geq 0\}$$

$$\beta^\alpha = \mathbb{Q}_{1-\tau}(\beta_\alpha^s)$$

$$\boxed{\alpha^\top x \geq \beta_\alpha}$$

Mixing set inequality (Günlük and Pochet 2001)

$$\boxed{\alpha^{\mathsf{T}} x \geq \beta_\alpha + (\beta_k - \beta_\alpha) y_k}$$

for $\beta_k > \beta_\alpha$

Relaxed Problem

$$\lambda^s \geq 0, \quad e^\top \lambda^s = 1, \quad s \in S$$

$$(\lambda^s)^\top A^s x \geq (\lambda^s)^\top \bar{b}^s + (\lambda^s)^\top (b^s - \bar{b}^s) y^s, \quad s \in S,$$

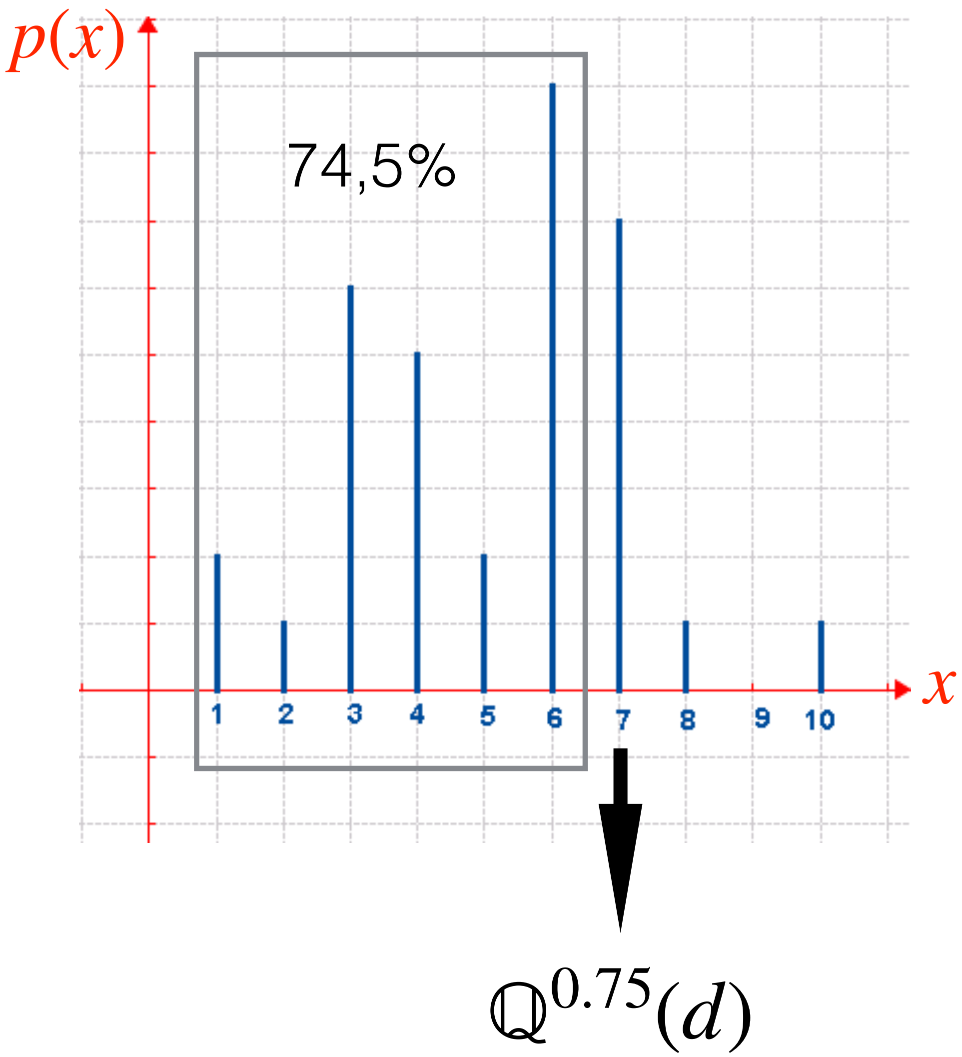
$$\sum_{s \in S} p^s y^s \geq 1 - \tau,$$

$$y^s \in \{0, 1\}, \quad s \in S.$$

$$\Updownarrow$$

$$\boxed{\mathbb{Q}_{1-\tau}[(\lambda^s)^\top (b^s - A^s x)] \leq 0}$$

Quantile



$$\max_{u,q} \quad (1-\tau)q - \sum_{s\in\mathcal{S}} p^s u^s$$
$$\text{s.t.} \quad q - u^s \leq d^s, \quad u^s \geq 0, \quad s \in \mathcal{S},$$
$$q \in \mathbb{R}.$$

$$\min_{w} \quad \sum_{s\in\mathcal{S}} d^s w^s$$
$$\text{s.t.} \quad \sum_{s\in\mathcal{S}} w^s = 1 - \tau,$$
$$0 \leq w^s \leq p^s, \quad s \in \mathcal{S}.$$

$$\boxed{(1-\tau)q - \sum_{s\in S} p^s u^s \geq \sum_{s\in S} d^s w^s}$$

Primal-Dual (PD) inequality

$$\boxed{\sum_{s\in\hat{S}} p^s(\lambda^s)^\top(A^s x - b^s) - \sum_{i=1}^{n} U_i(\lambda)x_i + L(\lambda) \leq 0}$$

$$U_i(\lambda) = \max_w\{\sum_{s\in S}(\lambda^s)^\top A^s_{.i} w^s : \sum_{s\in S} w^s = 1-\tau,\ 0 \leq w^s \leq p^s,\ s\in S\}$$

and

$$L(\lambda) = \min_w\{\sum_{w\in S}(\lambda^s)^\top b^s w^s : \sum_{s\in S} w^s = 1-\tau,\ 0 \leq w^s \leq p^s,\ s\in S\}$$

multi-disjunctive (multi-D) inequality

Set $\bar{S}$ of scenarios s.t. $p(\bar{S}) \geq \tau$

$$(\lambda^s)^\top (A^s x - b^s) \geq 0, \quad \text{for} \quad s \in \hat{S} \subseteq \bar{S} : p(\hat{S}) \geq p(\bar{S}) - \tau$$

$$\boxed{\sum_{s \in \hat{S}} w^s (\lambda^s)^\top A^s x \geq \sum_{s \in \hat{S}} w^s (\lambda^s)^\top b^s}$$

multi-disjunctive (multi-D) inequality

$$\boxed{\sum_{i=1}^{n} U_i(\lambda, \bar{S})x_i \geq L(\lambda, \bar{S}), \quad \bar{S} \subseteq \mathcal{S},\ p(\bar{S}) > \tau,}$$

with

$$U_i(\lambda, \bar{S}) = \max_w \{\sum_{s \in \bar{S}} (\lambda^s)^\top A^s_{.i} w^s : \sum_{s \in \bar{S}} w^s = p(\bar{S}) - \tau,\ 0 \leq w^s \leq p^s,\ \ s \in \bar{S}\}$$

and

$$L(\lambda, \bar{S}) = \min_w \{\sum_{w \in \bar{S}} (\lambda^s)^\top b^s w^s : \sum_{s \in \bar{S}} w^s = p(\bar{S}) - \tau,\ 0 \leq w^s \leq p^s,\ \ s \in \bar{S}\}$$

dominates P-D inequality

Separation

- All VIs are hard to separate: Determine $\bar{S}$ and $\lambda^s, s \in S$

- For given $\lambda^s, s \in S$, PD inequalities are easy to separate

- For given $\lambda^s, s \in S$, multi-D inequalities are hard to separate

- For given $\bar{S}$, PD and multi-D inequalities are easy to separate (LPs)

Chance Constraint- Covering LP

$$
\begin{aligned}
\min_{x,y} \quad & c^\top x \\
\text{s.t.} \quad & A^s x \geq y^s 1, \quad s \in S, \\
& x \geq 0, \\
& \sum_{s \in S} p^s y^s \geq 1 - \tau, \\
& y^s \in \{0, 1\}, \quad s \in S
\end{aligned}
$$

Closures for CC-Covering-LP

- $\mathscr{Q} = \cap$ all quantile cuts
- multi-$\mathscr{D} = \cap$ all multi-disjunctive inequalities
- single-$\mathscr{D} = \cap$ all single-disjunctive inequalities

  ($p(\bar{S}) > \tau$ and $p(\bar{S}\setminus\{s\}) < \tau$, for all $s \in \bar{S}$)

$$\boxed{\text{Theorem: multi - } \mathscr{D} \subseteq \text{single - } \mathscr{D} = \mathscr{Q}}$$

- There exists instances where the inclusion is strict

Cut impact: single-covering

TABLE 1. Results of the CP-tests with MD-VIs, mixing-set inequalities and quantile cuts on the instances of the CKVLP.

Instances			MD-VIs			Mixing-set			Quantile		
n-m	Gen.	$\|\mathcal{S}\|$	T	it.	v (%)	T	it.	v(%)	T	it.	v (%)
20-1	lit	100	0.00	3.80	**58.90**	0.00	2.40	17.32	0.00	2.40	15.82
		500	0.01	3.40	**50.97**	0.01	2.00	23.10	0.01	2.00	22.30
		1000	0.02	3.40	**34.43**	0.01	3.60	19.87	0.01	2.40	19.03
		3000	0.08	3.20	**36.53**	0.05	6.40	20.92	0.04	3.40	17.88
	new	100	0.00	3.00	**32.61**	0.00	2.40	30.57	0.00	2.20	30.11
		500	0.01	3.00	37.24	0.01	2.20	**41.82**	0.01	2.00	41.72
		1000	0.02	3.00	34.01	0.01	3.20	**48.76**	0.01	2.00	48.76
		3000	0.09	3.00	31.72	0.05	6.20	**37.67**	0.04	2.20	37.46
80-1	lit	100	0.00	4.40	**35.99**	0.00	3.00	6.14	0.00	2.40	3.51
		500	0.01	4.00	**33.03**	0.01	3.60	7.81	0.01	2.60	5.88
		1000	0.03	4.40	**22.91**	0.02	3.80	6.38	0.02	3.20	3.77
		3000	0.10	3.20	**42.22**	0.07	4.00	8.36	0.07	2.20	7.21
	new	100	0.00	3.20	**27.04**	0.00	3.40	17.37	0.00	3.20	15.39
		500	0.01	3.00	**31.13**	0.01	3.00	26.55	0.01	2.60	24.79
		1000	0.03	3.00	**30.64**	0.02	4.40	28.29	0.02	3.40	27.46
		3000	0.12	3.00	**33.62**	0.08	6.00	30.99	0.07	2.60	30.80

Cut impact: multi- knapsack

TABLE 2. Results of the CP with MD-VIs, Mixing-set inequalities and quantile cuts on the instances of the CCMKP.

Instances			MD-VIs												Quantile		
			ADM				C-wise			Mixing-set							
n-m	Gen.	$\|\mathcal{S}\|$	$s(\lambda)$	T	it.	v (%)	T	it.	v (%)	T	it.	v(%)			T	it.	v (%)
20-10	lit	100	0.87	0.18	18.80	**30.21**	0.01	4.00	28.05	0.11	4.80	25.07			0.11	4.20	21.18
		500	0.73	0.42	13.80	**25.50**	0.04	4.00	23.47	0.52	6.20	22.10			0.51	3.80	19.84
		1000	0.89	0.81	15.00	**27.00**	0.07	3.80	25.14	1.09	9.00	24.47			1.04	5.00	21.89
		3000	0.88	3.27	11.80	**25.80**	0.26	3.40	24.77	3.88	17.80	22.99			3.23	4.80	20.82
	new	100	0.33	0.23	21.60	**24.03**	0.01	3.60	4.30	0.12	5.20	16.30			0.12	2.40	0.92
		500	0.33	0.43	30.80	**27.27**	0.04	4.20	10.97	0.60	8.00	18.60			0.58	4.00	1.55
		1000	0.29	0.79	15.80	**25.10**	0.09	3.00	11.31	1.17	8.40	15.92			1.09	4.20	2.65
		3000	0.31	3.04	22.20	**25.20**	0.37	3.80	5.21	3.72	18.20	16.67			3.40	3.80	2.11
40-30	lit	100	0.77	0.62	39.80	**27.65**	0.05	4.20	25.05	1.36	6.40	22.78			1.37	5.60	15.81
		500	0.79	1.27	21.00	**26.13**	0.19	3.80	23.66	6.72	7.40	19.72			6.67	4.60	16.04
		1000	0.73	3.17	47.40	**28.19**	0.50	4.20	25.46	13.55	12.40	22.74			11.83	6.20	18.94
		3000	0.79	8.96	20.80	**25.59**	1.66	3.60	23.25	41.06	17.20	19.05			38.16	5.20	15.85
	new	100	0.23	0.76	63.80	**26.08**	0.07	2.60	4.18	1.25	6.80	17.78			1.37	3.00	0.99
		500	0.26	1.15	31.80	**23.30**	0.28	3.20	6.42	6.89	6.80	14.27			6.74	4.00	1.67
		1000	0.22	2.64	39.80	**24.47**	0.67	3.00	4.40	13.18	10.80	15.73			12.66	4.20	2.14
		3000	0.25	8.75	32.20	**25.08**	2.80	3.80	7.13	40.25	19.80	14.93			36.87	4.00	1.77

MIP tests: single-covering

TABLE 3. Results of the MIP with MD-VIs, the mixing-set inequalities, and the quantile cuts for CKVLP instances.

Instances			MIP	MIP-MD		MIP-MS		MIP-Q	
n-m	Gen.	$\lvert\mathcal{S}\rvert$	gap/time	cuts	gap/time	cuts	gap/time	cuts	gap/time
20-1	lit	100	**0.24s**	2.80	0.73s	24.80	0.48s	1.40	0.37s
		500	3.35s	2.40	**1.01s**	100.00	4.74s	1.00	2.49s
		1000	0.03%(4)	2.40	0.10%(4)	199.40	0.06%(4)	1.40	**272.61s**
		3000	3.18%(2)	2.20	**1.61%(3)**	501.00	4.43%(1)	2.40	4.75%(0)
	new	100	**0.13s**	2.00	0.28s	24.40	0.34s	1.20	0.28s
		500	10.01s	2.00	1.18s	104.00	2.07s	1.00	**0.91s**
		1000	20.57s	2.00	1.67s	151.60	1.65s	1.00	**1.22s**
		3000	**0.72%(4)**	2.00	0.97%(4)	466.00	0.97%(3)	1.20	1.25%(4)
80-1	lit	100	0.98s	3.40	**0.91s**	30.60	1.31s	1.40	0.97s
		500	0.58%(4)	3.00	**0.03%(4)**	213.60	0.61%(4)	1.60	0.58%(4)
		1000	3.01%(1)	3.40	**1.94%(1)**	252.80	2.62%(2)	2.20	3.29%(0)
		3000	0.90%(1)	2.20	**0.13%(4)**	271.20	2.14%(2)	1.20	0.52%(3)
	new	100	**0.59s**	2.20	**0.59s**	44.00	1.19s	2.20	0.68s
		500	12.42s	2.00	**4.95s**	161.60	22.85s	1.60	6.06s
		1000	0.01%(4)	2.00	**472.05s**	281.80	0.21%(4)	2.40	0.34%(4)
		3000	1.02%(3)	2.00	0.76%(4)	459.60	1.55%(2)	1.60	**0.53%(4)**

MIP tests: multi- knapsack

Table 5. Results of the MIP with MD-VIs, the mixing-set inequalities, and the quantile cuts for the CCMKP instances, where the x-variables are continuous.

Instances			MIP	MIP-MD				MIP-MS		MIP-Q	
				ADM		C-wise					
n-m	Gen.	$\lvert\mathcal{S}\rvert$	gap/time	cuts	gap/time	cuts	gap/time	cuts	gap/time	cuts	gap/time
20-10	lit	100	204.87s	17.80	172.64s	18.60	**114.96s**	202.80	202.04s	15.60	117.55s
		500	3.35%(0)	12.80	2.76%(0)	20.80	**2.20%(0)**	380.80	2.60%(0)	14.60	2.31%(0)
		1000	11.92%(0)	14.00	8.61%(0)	17.20	**6.99%(0)**	674.00	7.12%(0)	18.20	8.85%(0)
		3000	32.96%(0)	10.80	**9.40%(0)**	17.80	10.69%(0)	1538.00	11.27%(0)	17.60	14.21%(0)
	new	100	313.56s	20.60	**227.12s**	10.20	305.02s	166.20	368.67s	3.60	292.92s
		500	10.10%(0)	29.80	10.83%(0)	12.40	**9.47%(0)**	503.60	9.73%(0)	6.60	9.87%(0)
		1000	16.77%(0)	14.80	14.60%(0)	7.20	13.25%(0)	593.40	**13.17%(0)**	6.80	13.21%(0)
		3000	44.58%(0)	21.20	**19.84%(0)**	9.40	26.31%(0)	1552.40	29.96%(0)	7.40	24.86%(0)
40-30	lit	100	0.15%(2)	38.80	0.21%(3)	46.60	0.12%(3)	214.80	0.14%(4)	44.00	**0.08%(4)**
		500	10.35%(0)	20.00	8.49%(0)	41.20	8.67%(0)	430.00	6.95%(0)	35.60	**6.56%(0)**
		1000	24.77%(0)	46.40	**13.21%(0)**	46.80	14.44%(0)	801.00	15.64%(0)	52.60	14.38%(0)
		3000	35.04%(0)	19.80	**11.91%(0)**	38.80	15.44%(0)	1412.20	18.89%(0)	34.80	20.17%(0)
	new	100	1.47%(1)	62.80	**0.80%(1)**	13.60	1.26%(1)	185.00	1.31%(1)	6.60	0.94%(1)
		500	16.61%(0)	30.80	15.14%(0)	11.00	14.36%(0)	387.00	**13.36%(0)**	10.00	13.60%(0)
		1000	30.26%(0)	38.80	**19.85%(0)**	16.20	21.40%(0)	700.40	24.04%(0)	16.80	21.83%(0)
		3000	41.97%(0)	31.20	**18.78%(0)**	17.20	33.37%(0)	1694.80	36.25%(0)	16.00	30.81%(0)

MIP tests: multi- knapsack

Table 4. Results of the MIP-tests with MD-VIs, the mixing-set inequalities, and the quantile cuts for the CCMKP instances, where the x-variables are binary.

				MIP-MD							
Instances			MIP	ADM		C-wise		MIP-MS		MIP-Q	
n-m	Gen.	$\|\mathcal{S}\|$	gap/time	cuts	gap/time	cuts	gap/time	cuts	gap/time	cuts	gap/time
20-10	lit	100	9.28s	17.80	7.47s	18.60	**4.31s**	202.80	6.34s	15.60	5.49s
		500	110.42s	12.80	46.33s	20.80	**21.53s**	380.80	60.40s	14.60	39.13s
		1000	642.60s	14.00	**111.55s**	17.20	120.40s	674.00	260.68s	18.20	164.71s
		3000	12.03%(1)	10.80	2.19%(4)	17.80	**696.69s**	1538.00	4.60%(4)	17.60	1500.19s
	new	100	14.43s	20.60	13.01s	10.20	8.40s	166.20	**8.01s**	3.60	13.11s
		500	91.75s	29.80	**62.04s**	12.40	70.99s	503.60	72.04s	6.60	91.51s
		1000	388.66s	14.80	**254.66s**	7.20	296.24s	593.40	356.02s	6.80	442.39s
		3000	41.46%(0)	21.20	**32.92%(0)**	9.40	33.83%(1)	1552.40	37.82%(0)	7.40	52.73%(0)
40-30	lit	100	10.17%(0)	38.80	9.16%(1)	46.60	**8.32%(1)**	214.80	11.73%(0)	44.00	12.05%(0)
		500	29.22%(0)	20.00	5.73%(2)	41.20	**1.81%(4)**	430.00	14.44%(0)	35.60	15.76%(0)
		1000	52.17%(0)	46.40	**20.85%(0)**	46.80	24.03%(0)	801.00	29.35%(0)	52.60	34.95%(0)
		3000	41.63%(0)	19.80	**15.07%(0)**	38.80	18.14%(0)	1412.20	24.66%(0)	34.80	29.13%(0)
	new	100	19.70%(0)	62.80	22.23%(0)	13.60	**18.79%(0)**	185.00	21.98%(0)	6.60	19.05%(0)
		500	39.24%(0)	30.80	**20.45%(0)**	11.00	32.42%(0)	387.00	27.21%(0)	10.00	34.72%(0)
		1000	48.89%(0)	38.80	**29.81%(0)**	16.20	55.00%(0)	700.40	39.56%(0)	16.80	52.94%(0)
		3000	49.05%(0)	31.20	**27.68%(0)**	17.20	48.18%(0)	1694.80	42.26%(0)	16.00	50.00%(0)