Scenario Dominance Cuts for Risk-Averse Multi-Stage Stochastic Mixed-Integer Programs

Esra Büyüktahtakın Toy, Ph.D.

Grado Department of Industrial and Systems Engineering, Virginia Tech

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Multi-Stage Stochastic Optimization

- We consider a dynamic decision-making problem over multiple stages, with uncertainty revealed sequentially.
- We assume a finite number of decision stages, and a discrete stochastic process.

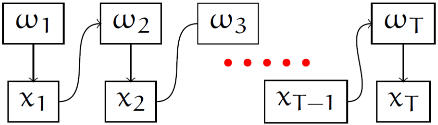


Figure: Decisions (x_t) in stages, in between which uncertainty (ω_t) is revealed to us.

- Formulated as a multi-stage stochastic mixed-integer program (M-SMIP), involving both discrete and continuous decisions.

Motivation and Background

- **Challenge:** M-SMIPs are non-convex, involve integer decisions and sequential uncertainty–making them notoriously hard to solve (Shapiro et al., 2009; Schultz and Tiedemann, 2003; Heitsch and Römisch, 2003).
- **Cutting planes:** Benders (Benders, 2005), locally valid cuts (Abgottspon et al., 2014), McCormick relaxations (Cerisola et al., 2012), single-scenario cuts (Küçükyavuz and Sen, 2017; Dey et al., 2018; Chen and Luedtke, 2022), pseudo-valid cuts (Romeijnders and van der Laan, 2020)
- **Decomposition methods:** Lagrangian and dual decomposition (Schultz and Tiedemann, 2006); L-Shaped method (Linderoth and Wright, 2003); Scenario decomposition and branch-and-fix coordination (Heinze and Schultz, 2008); Progressive Hedging (PH) (Watson and Woodruff, 2011); Stochastic Dual Dynamic Integer Programming (SDDiP) (Zou et al., 2019); Risk-averse stochastic programming methods (Luedtke, 2008; Bayraksan and Morton, 2011; Sen and Higle, 2014; Liu et al., 2017; Eckstein et al., 2016; Dentcheva and Ruszczyński, 2024; Sandıkçı and Özaltın, 2017)
- **Learning for Stochastic Programming:** (Yilmaz and Büyüktahtakın, 2024a,c; Bushaj and Büyüktahtakın, 2024)

Research Contributions

- We present a new and general method of “scenario dominance" to effectively solve the risk-averse M-SMIPs.
- Our definition of scenario dominance is similar to the ordering of scenarios that was discussed in Ruszczyński (2002); but the way we use the scenario ordering concept is fundamentally different than the approach in Ruszczyński (2002).
- To our knowledge, this study is the first to use the dominance relations among scenarios in the context of risk-averse multi-stage stochastic programs (Büyüktahtakın, 2022)*.
- For the first time, we define the notion of “stage-t scenario dominance."

*İ. E. Büyüktahtakın, “*Stage-t scenario dominance for risk-averse multi-stage stochastic mixed-integer programs*,” *Annals of Operations Research*, 309: 1–35, 2022.

Risk Neutral Multi-stage Stochastic MIPs

$$\min \quad f_1(x_1) + \mathbb{E}_{\tilde{\xi}^2}\left[\min_{x_2} f_2(x_2, \tilde{\xi}^2) + \mathbb{E}_{\tilde{\xi}^3|\xi_{[2]}}\left[\ldots + \mathbb{E}_{\tilde{\xi}^T|\xi_{[T-1]}}\left[\min_{x_T} f_T(x_T, \tilde{\xi}^T)\right]\right]\right] \tag{1a}$$

$$\text{s.t.} \quad A_1 x_1 \geq b_1, \tag{1b}$$

$$A_2\left(\tilde{\xi}^1\right) x_1 + H_2\left(\tilde{\xi}_{[2]}\right) x_2\left(\tilde{\xi}_{[2]}\right) \geq b_2\left(\tilde{\xi}_{[2]}\right), \tag{1c}$$

$$A_t\left(\tilde{\xi}_{[t-1]}\right) x_{t-1}\left(\tilde{\xi}_{[t-1]}\right) + H_t\left(\tilde{\xi}_{[t]}\right) x_t\left(\tilde{\xi}_{[t]}\right) \geq b_t\left(\tilde{\xi}_{[t]}\right) \quad \forall t \in \mathcal{T} \backslash \{1, 2\}, \tag{1d}$$

$$x_1 \in R_+^{n_1 - q_1} \times Z_+^{q_1}; x_t\left(\tilde{\xi}_{[t]}\right) \in R_+^{n_t - q_t} \times Z_+^{q_t} \quad \forall t \in \mathcal{T}. \tag{1e}$$

Example Multi-Stage Scenario Tree

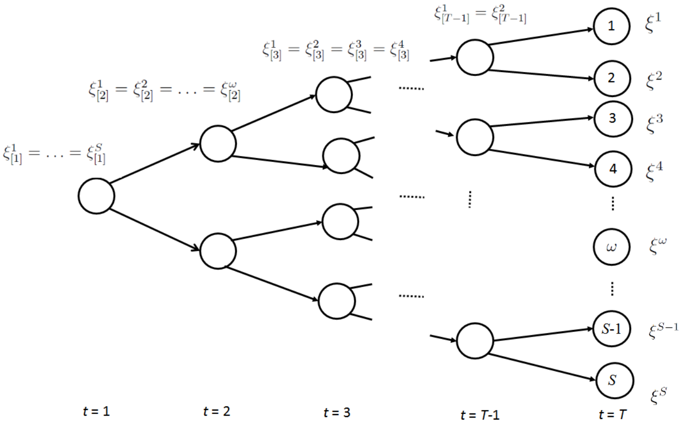


Figure: A multi-stage scenario tree and non-anticipativity relations.

Mean-Risk Formulation

$$\min_{x \in X} \{\mathbb{E}[f(x,\omega)] + \lambda \rho [f(x,\omega)]\}, \tag{2}$$

$\lambda \in \mathbb{R}_+$ is a trade-off coefficient that represents the relative weight of the risk objective with respect to the expectation function.

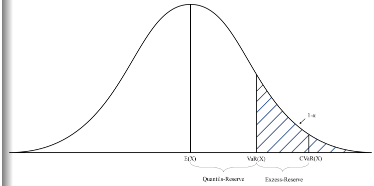
Definition

The Conditional Value-at-Risk (i.e., average or tail value-at-risk) of a random variable Z, at confidence level $\alpha \in [0, 1)$ can be expressed as the optimal value of the following optimization problem (Rockafellar and Uryasev, 2000):

$$CVaR_\alpha [Z] = \min_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{1-\alpha} \mathbb{E}\left[(Z - \eta)_+ \right] \right\}. \tag{3}$$

where $(a)_+ := max\{a, 0\}$. The minimum value in (3) is obtained at the α-quantile, which is known as the Value-at-Risk (VaR) at confidence level α:

$$VaR_\alpha [Z] = \min \{\eta \in \mathbb{R} : P(Z \leq \eta) \geq \alpha\}. \tag{4}$$



Mean-CVaR M-SMIP Formulation

$$
\begin{aligned}
(P)\quad \min\ & \sum_{\omega\in\Omega} p^\omega \sum_{t\in\mathcal{T}} c_t^\omega x_t^\omega + \lambda \sum_{\omega\in\Omega} p^\omega \sum_{t\in\mathcal{T}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} v_t^\omega \right) && \text{(5a)}\\
\text{s.t. }\ & A_1 x_1^\omega = b_1,\ \omega\in\Omega && \text{(5b)}\\
& A_t^\omega x_{t-1}^\omega + H_t^\omega x_t^\omega = b_t^\omega \quad \forall t\in\mathcal{T}\setminus\{1\},\ \omega\in\Omega, && \text{(5c)}\\
& v_t^\omega \geq c_t^\omega x_t^\omega - \eta_t^\omega, \quad \forall t\in\mathcal{T},\ \omega\in\Omega, && \text{(5d)}\\
& x_t^\omega = x_t^{\omega'}, \eta_t^\omega = \eta_t^{\omega'}, v_t^\omega = v_t^{\omega'}, \quad \forall t\in\mathcal{T},\ \omega,\omega'\in\Omega\ \ s.t.\ \xi_{[t]}^\omega = \xi_{[t]}^{\omega'}, && \text{(5e)}\\
& x_t^\omega \in \mathbb{R}_+^{n_t-q_t}\times\mathbb{Z}_+^{q_t} \quad \forall t\in\mathcal{T}\ \ \omega\in\Omega, && \text{(5f)}\\
& \eta_t^\omega, v_t^\omega \geq 0, \quad \forall t\in\mathcal{T},\ \omega\in\Omega. && \text{(5g)}
\end{aligned}
$$

Applications of Risk-Averse M-SMIPs

- **Financial portfolio optimization** under return uncertainty (Krokhmal et al., 2002)
- **Energy systems planning and smart grids** under demand and price uncertainty (Huang and Ahmed, 2009; Zhang et al., 2020)
- **Disaster relief network design** with uncertain demand and supply availability (Noyan, 2012)
- **Water resource allocation** under growing population and demand uncertainty (Zhang et al., 2016)
- **Forestry invasive insect management** under ecological uncertainty–funded by USDA Forest Service (Kibis et al., 2021; Bushaj et al., 2022)
- **Epidemic control and vaccine allocation** under disease evolution–funded by NSF (Yin and Büyüktahtakın, 2021, 2022; Yin et al., 2023)

Scenario Dominance Algorithm

- We present the Scenario Dominance Algorithm to solve the problem more efficiently.
- Steps we follow to decompose and define cuts:
  1. Formulate and solve new scenario sub-problems.
  2. Define lower and upper bounds using the scenario sub-problem.
  3. Define the scenario dominance relations and create sets.
  4. Define cuts based on the scenario dominance sets.

New Scenario Sub-Problem

Definition

New Single Scenario Sub-Problem. The scenario group sub-problem for the set consisting the single scenario, ξ^ω, is denoted as the scenario-ξ^ω problem ($P^{\{\omega\}}$) and formulated as follows:

$$Z^{\{\omega\}} = \min_{(x,\eta,v)\in\mathcal{X}} p^\omega \sum_{t\in\mathcal{T}} c_t^\omega x_t^\omega + \lambda p^\omega \sum_{t\in\mathcal{T}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} v_t^\omega\right). \tag{6}$$

Definition

New Scenario Group Sub-Problem. Let $S \subseteq \Omega$. The scenario group sub-problem involving each $\omega \in S$, (P^S), is formulated as follows:

$$Z^S := \min_{(x,\eta,v)\in\mathcal{X}} \sum_{\omega\in S} \left(p^\omega \sum_{t\in\mathcal{T}} {c_t^\omega}^\top x_t^\omega + \lambda p^\omega \sum_{t\in\mathcal{T}\setminus\{1\}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} v_t^\omega\right)\right), \tag{7}$$

where

$$\mathcal{X} := \{(x,\eta,v) \in \mathbb{R}_+^{(n_1-q_1)\bar{\Omega}} \times \mathbb{Z}_+^{q_1\bar{\Omega}} \times \ldots \times \mathbb{R}_+^{(n_T-q_T)\bar{\Omega}} \times \mathbb{Z}_+^{q_T\bar{\Omega}} \times \mathbb{R}^{T\bar{\Omega}} \times \mathbb{R}_+^{T\bar{\Omega}} : \ \textit{Const}.\ (5b)-(5g)\}.$$

Definition

Classical Scenario Group Sub-Problem (Madansky, 1960; Ahmed, 2013) The relaxed scenario group sub-problem involving each $\omega \in S$, (P_R^S), is formulated as follows:

$$Z_R^S = \min_{(x,\eta,v)\in\mathcal{X}^S} \sum_{\omega\in S} \left(p^\omega \sum_{t\in\mathcal{T}} c_t^\omega x_t^\omega + \lambda p^\omega \sum_{t\in\mathcal{T}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} v_t^\omega \right) \right), \tag{8}$$

where

$$\mathcal{X}^S = \left\{ \left(x^S, \eta^S, v^S \right) \in \mathbb{R}_+^{n-q} \times \mathbb{Z}_+^q \times \mathbb{R}_+^s \times \mathbb{R}_+^s : \text{ Constraints (5b)} - \text{(5g) only for } \omega \in S \subseteq \Omega \right\}.$$

Proposition

P_R^S is a relaxation of P^S; that is $Z^S \geq Z_R^S \ \forall S \subseteq \Omega$.

Scenario Group Sub-Problem is a Relaxation of the Original Problem

Proposition (Relaxation via Scenario Grouping)

Assume that the objective function for each scenario is non-negative. Let Z be the optimal objective value of the original problem P (5). Then, for any $S \subseteq \Omega$, the scenario group sub-problem P^S is a relaxation:

$$Z \geq Z^S.$$

Sketch of the proof:

Introduce an auxiliary variable $H \in \mathbb{R}$.
Reformulate both problems as:

$$Z = \min_{(x,\eta,v,H)\in\tilde{\mathcal{X}}} H, \quad Z^S = \min_{(x,\eta,v,H)\in\hat{\mathcal{X}}} H,$$

where the feasible regions differ by:

$$\tilde{\mathcal{X}} := \mathcal{X} \cap \left\{ H \geq \sum_{\omega\in\Omega} p^\omega f^\omega \right\}, \quad \hat{\mathcal{X}} := \mathcal{X} \cap \left\{ H \geq \sum_{\omega\in S} p^\omega f^\omega \right\}.$$

Since $S \subseteq \Omega$, we have $\tilde{\mathcal{X}} \subseteq \hat{\mathcal{X}}$, implying $Z \geq Z^S$. □

Superadditivity and Lower Bound

Definition

A function $\Gamma : \mathbb{R}^m \to \mathbb{R}$ is **superadditive** on $D \subseteq \mathbb{R}^m$ if

$$\Gamma(a+b) \geq \Gamma(a) + \Gamma(b) \quad \text{for all } a, b \in D \text{ such that } a+b \in D.$$

Proposition (Superadditivity of Scenario Costs)

For any $\omega, \omega' \in S \subseteq \Omega$, we have:

$$Z^{\{\omega,\omega'\}} \geq Z^{\{\omega\}} + Z^{\{\omega'\}}.$$

Proposition (Lower Bound)

If scenario objectives are non-negative, then:

$$Z \geq Z^S \geq \sum_{\omega \in S} Z^{\{\omega\}}. \tag{9}$$

The bound is tightest when $S = \Omega$. Non-negativity, convexity, and continuity are not required for the second inequality.

Upper Bound

Proposition (Upper Bound)

Let $\dot{x}^{\omega}$ be optimal solutions to the scenario-ξ^{ω} problem $P^{\{\omega\}}$; and let $\bar{Z}^{\omega}$ be the objective function value of P at solution $\dot{x}^{\omega}$. Then,

$$Z \leq \min_{\omega \in \Omega} \bar{Z}^{\omega}. \tag{10}$$

Defining Scenario Dominance

Definition (Scenario Dominance)

Let $\xi_t^{\omega} = (c_t^{\omega}, b_t^{\omega}, A_t^{\omega}, H_t^{\omega})$ denote the scenario realization at time t for scenario $\omega \in \Omega$.
We say that scenario ξ^{ω} **dominates** $\xi^{\omega'}$, denoted $\xi^{\omega'} \preceq \xi^{\omega}$, with respect to problem P, if for all $t \in \mathcal{T}$:

$$
\begin{aligned}
p^{\omega} &\geq p^{\omega'} \\
c_t^{\omega} &\geq c_t^{\omega'} \\
b_t^{\omega} &\geq b_t^{\omega'} \\
A_t^{\omega} &\leq A_t^{\omega'}, \quad H_t^{\omega} \leq H_t^{\omega'}
\end{aligned}
$$

All comparisons are elementwise. If, for example, $c_t^{\hat{\omega}} := \left(c_{1,t}^{\hat{\omega}}, \ldots, c_{n_t,t}^{\hat{\omega}}\right)$ and $A_t^{\hat{\omega}} := \left(A_{1,1,t}^{\hat{\omega}}, \ldots, A_{m_t,n_t,t}^{\hat{\omega}}\right)$, then the scenario domination is determined based on an element-wise comparison of the realizations of $c_t^{\hat{\omega}}$ and $A_t^{\hat{\omega}}$ for $\hat{\omega} = \omega, \omega'$, such as:

$$c_{j,t}^{\omega} \geq c_{j,t}^{\omega'} \quad \forall j = 1, \ldots, n_t, \quad \forall t = 1, \ldots, T,$$

$$A_{i,j,t}^{\omega} \leq A_{i,j,t}^{\omega'} \quad \forall i = 1, \ldots, m_t, \quad \forall j = 1, \ldots, n_t, \quad \forall t = 1, \ldots, T.$$

Stage-t Scenario Dominance

Definition

Stage-t Scenario Dominance. Given two scenarios ξ^{ω} and $\xi^{\omega'}$, scenario ξ^{ω} stage-t dominates scenario $\xi^{\omega'}$ for $t \in \mathcal{T}$, denoted by $\xi_t^{\omega'} \preceq \xi_t^{\omega}$, with respect to the problem P (5) if

$$\left(p^{\omega} \geq p^{\omega'}\right) \wedge \left(c_j^{\omega} \geq c_j^{\omega'}\right) \wedge \left(b_j^{\omega} \geq b_j^{\omega'}\right) \wedge \left(A_j^{\omega} \leq A_j^{\omega'}\right) \wedge \left(H_j^{\omega} \leq H_j^{\omega'}\right) \quad \text{for} \quad j = 1, \ldots, t,$$

and the objective function (5a) portion for a single scenario index $\hat{\omega} = \omega, \omega'$ $\left(p^{\hat{\omega}} \sum_{j=1}^{t} c_j^{\hat{\omega}\top} x_j^{\hat{\omega}} + \lambda p^{\hat{\omega}} \sum_{j=2}^{t} \left(\eta_j^{\hat{\omega}} + \frac{1}{1-\alpha_j} v_j^{\hat{\omega}}\right)\right)$ is a non-decreasing function of $\left(x^{\hat{\omega}}, \eta^{\hat{\omega}}, v^{\hat{\omega}}\right)$.

Scenario-Dominance Sets

Definition

Stage-*t* Scenario-Dominance Sets. Θ^+_{t,ξ^ω} is the index set of scenarios, which are stage-t dominated by scenario $\xi^\omega \in \Xi$, Θ^-_{t,ξ^ω} is the index set of scenarios which stage-t dominate scenario $\xi^\omega \in \Xi$, and N_{t,ξ^ω} is the index set of scenarios, which neither stage-t dominate nor are stage-t dominated by $\xi^\omega \in \Xi$ for $t \in \mathcal{T}$ as defined below:

$$\Theta^+_{t,\xi^\omega} := \left\{\omega' \in \Omega, \omega' \neq \omega : \xi_t^{\omega'} \preceq \xi_t^{\omega}\right\},$$

$$\Theta^-_{t,\xi^\omega} := \left\{\omega' \in \Omega : \xi_t^{\omega} \preceq \xi_t^{\omega'}\right\},$$

$$N_{t,\xi^\omega} := \left\{\omega' \in \Omega : \xi_t^{\omega} \npreceq \xi_t^{\omega'} \text{ and } \xi_t^{\omega'} \npreceq \xi_t^{\omega}\right\}.$$

Example of Scenario Dominance Sets

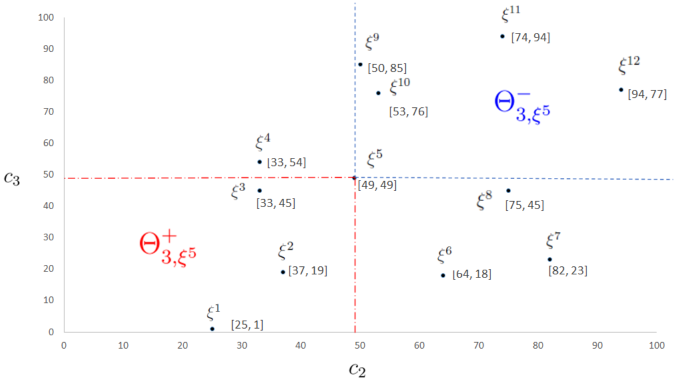


Figure: Example of 12 scenario realizations of the c_t variable in two stages, where each black point represents a scenario $\xi^i := [c_2^i, c_3^i]$ for $i = 1, \ldots, 12$. The Θ^+_{3,ξ^5} represents the index set of scenarios that are stage-3 dominated by ξ^5, Θ^-_{3,ξ^5} represents the index set of scenarios that stage-3 dominate ξ^5, and N_{3,ξ^5} represents scenarios neither stage-3 dominate nor are stage-3 dominated by ξ^5.

Lemmas Motivating the Main Theorem

Lemma (Scenario Subproblem Lower Bound)

Let (x^*, η^*, v^*) be the optimal solution to the original problem (5) and $(\dot{x}^\omega, \dot{\eta}^\omega, \dot{v}^\omega)$ be the optimal solution of the scenario-ξ^ω problem (6). Then, the portion of the optimal solution for the original problem (5) that corresponds to scenario-ξ^ω, $(x^{\omega *}, \eta^{\omega *}, v^{\omega *})$, satisfies the following inequality:

$$p^\omega \sum_{t \in \mathcal{T}} c_t^{\omega \top} x_t^\omega + \lambda p^\omega \sum_{t \in \mathcal{T} \setminus \{1\}} \left(\eta_t^\omega + \frac{1}{1 - \alpha_t} v_t^\omega \right) \geq Z^\omega \quad \forall \omega \in \Omega. \tag{11}$$

Lemma (Dominance Implies Lower Bound)

Let ξ^ω be a scenario such that $\omega \in \Theta^-_{T,\xi^{\omega'}} \subseteq \Omega$ for a given scenario $\xi^{\omega'}$. Let $(\ddot{x}^\omega, \ddot{\eta}^\omega, \ddot{v}^\omega)$ and $(\dot{x}^{\omega'}, \dot{\eta}^{\omega'}, \dot{v}^{\omega'})$ be the optimal solution of the scenario-ξ^ω and the scenario-$\xi^{\omega'}$ problems, Z^ω and $Z^{\omega'}$ be the corresponding objective values, respectively. Then the following inequality holds:

$$Z^\omega \geq Z^{\omega'} \quad \forall \omega \in \Theta^-_{T,\xi^{\omega'}}. \tag{12}$$

That is, dominated scenarios have lower or equal subproblem cost than that of their dominators.

Logical Flow of the Second Lemma's Proof †

1. **Construct Modified Problems:** Define $\tilde{P}^{\omega}$ and $\tilde{P}^{\omega'}$ by adding non-anticipativity constraints (shared decisions across scenarios).
2. **Define Feasible Shared Solution:** Let $(\tilde{x}, \tilde{\eta}, \tilde{v})$ be a feasible solution for both $\tilde{P}^{\omega}$ and $\tilde{P}^{\omega'}$.
3. **Define Objective Values:** Let $\tilde{Z}^{\omega}$ and $\tilde{Z}^{\omega'}$ be the optimal values of $\tilde{P}^{\omega}$ and $\tilde{P}^{\omega'}$.
4. **Compare Objective Values:**
   - $\tilde{Z}^{\omega} = Z^{\omega}$: Modified problem is equivalent to original for ω.
   - $\tilde{Z}^{\omega} \geq \tilde{Z}^{\omega'}$: Due to cost dominance and shared decisions.
   - $\tilde{Z}^{\omega'} \geq Z^{\omega'}$: Additional constraints may increase cost.
5. **Conclude:**

$$Z^{\omega} = \tilde{Z}^{\omega} \geq \tilde{Z}^{\omega'} \geq Z^{\omega'}.$$

† İ. E. Büyüktahtakın, "*Stage-t scenario dominance for risk-averse multi-stage stochastic mixed-integer programs*," *Annals of Operations Research*, 309: 1–35, 2022.

Theorem

Scenario Dominance Cuts. *Assume that uncertainty in problem* (5) *is observed in the objective function* c_t^ω*, in the right-hand side coefficient* b_t^ω*, and the left-hand side coefficient* (A_t^ω, H_t^ω)*. Let* ξ^ω *and* $\xi^{\omega'}$ *be two scenarios such that* $\xi^{\omega'} \preceq \xi^\omega$, $\omega, \omega' \in \Omega$ *and* $\omega \neq \omega'$*, i.e.,* $p^\omega \geq p^{\omega'}$, $b_t^\omega \geq b_t^{\omega'}$, $A_t^\omega \leq A_t^{\omega'}$ *and* $H_t^\omega \leq H_t^{\omega'}$*, and* $c_t^\omega \geq c_t^{\omega'}$ *for all* $t \in \mathcal{T}$*. Then, the portion of the optimal solution for the original problem* (5) *that corresponds to scenario-*ξ^ω*,* $(x^{\omega *}, \eta^{\omega *}, v^{\omega *})$*, satisfies the following inequality:*

$$p^\omega \sum_{t \in \mathcal{T}} c_t^{\omega\top} x_t^\omega + \lambda p^\omega \sum_{t \in \mathcal{T} \setminus \{1\}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} v_t^\omega \right) \geq Z^{\omega'} \quad \forall \omega \in \Theta^-_{T, \xi^{\omega'}}. \tag{13}$$

Proof.

The first inequality below follows from Lemma 1, while the second follows from Lemma 2.

$$p^\omega \sum_{t \in \mathcal{T}} c_t^{\omega\top} x_t^{\omega *} + \lambda p^\omega \sum_{t \in \mathcal{T} \setminus \{1\}} \left(\eta_t^{\omega *} + \frac{1}{1-\alpha_t} v_t^{\omega *} \right) \geq Z^\omega \geq Z^{\omega'}.$$

□

Corollary

Stage-t Scenario Dominance Cuts. *Let ξ^{ω} and $\xi^{\omega'}$ be two scenarios $l, \omega \in \Omega$ such that $\xi_t^{\omega'} \preceq \xi_t^{\omega}$ for $t \in \hat{\mathcal{T}} = \left\{1, \ldots, \hat{t}\right\} \subseteq \mathcal{T}$, where $\hat{t} \in \mathcal{T}$. Then, the portion of the optimal solution for the original problem* (5) *that corresponds to scenario-ξ^{ω} up to stage t, $(x_t^{\omega *}, \eta_t^{\omega *}, v_t^{\omega *})$, satisfies the following inequality:*

$$p^{\omega} \sum_{j=1}^{t} c_j^{\omega \top} x_j^{\omega} + \lambda p^{\omega} \sum_{j=2}^{t} \left(\eta_j^{\omega} + \frac{1}{1-\alpha_j} v_j^{\omega} \right) \geq Z_t^{\omega'} \quad \forall \omega \in \Theta^{-}_{t, \xi^{\omega'}}, \; t \in \hat{\mathcal{T}} \subseteq \mathcal{T}. \tag{14}$$

Mean-CVaR Multi-Stage Stochastic Knapsack Formulation (CVaR-SMKP)

$$\min \sum_{\omega\in\Omega} p^\omega \sum_{t\in\mathcal{T}} \left(\sum_{i\in\mathcal{I}} c_{it} x_{it}^\omega + d_t z_t^\omega + q_t^\omega y_t^\omega \right) + \lambda \sum_{\omega\in\Omega} p^\omega \sum_{t\in\mathcal{T}} \left(\eta_t^\omega + \frac{1}{1-\alpha_t} \rho_t^\omega \right) \tag{15a}$$

$$\text{s.t.} \sum_{j=1}^{t} \sum_{i\in\mathcal{I}} a_{ij} x_{ij}^\omega + r_t^\omega z_t^\omega \geq b_t^\omega \qquad \forall t\in\mathcal{T},\ \forall\omega\in\Omega \tag{15b}$$

$$\sum_{i\in\mathcal{I}} v_{it} x_{it}^\omega + w_t y_t^\omega \geq h_t \qquad \forall t\in\mathcal{T},\ \forall\omega\in\Omega \tag{15c}$$

$$\rho_t^\omega \geq \sum_{i\in\mathcal{I}} c_{it} x_{it}^\omega + d_t z_t^\omega + q_t^\omega y_t^\omega - \eta_t^\omega \quad \forall t\in\mathcal{T},\ \forall\omega\in\Omega \tag{15d}$$

$$x_{it}^\omega = x_{it}^{\omega'} \quad \forall t\in\mathcal{T},\ i\in\mathcal{I},\ \omega,\omega'\in\Omega \ s.t.\ \xi_{[t]}^\omega = \xi_{[t]}^{\omega'} \tag{15e}$$

$$z_t^\omega = z_t^{\omega'},\ y_t^\omega = y_t^{\omega'} \quad \forall t\in\mathcal{T},\ \omega,\omega'\in\Omega \ s.t.\ \xi_{[t]}^\omega = \xi_{[t]}^{\omega'} \tag{15f}$$

$$\eta_t^\omega = \eta_t^{\omega'},\ \rho_t^\omega = v_t^{\omega'} \quad \forall t\in\mathcal{T},\ \omega,\omega'\in\Omega \ s.t.\ \xi_{[t]}^\omega = \xi_{[t]}^{\omega'} \tag{15g}$$

$$x_{it}^\omega, y_t^\omega \in \{0,1\} \qquad \forall t\in\mathcal{T},\ \forall i\in\mathcal{I},\ \forall\omega\in\Omega \tag{15h}$$

$$z_t^\omega, \eta_t^\omega, \rho_t^\omega \geq 0 \qquad \forall t\in\mathcal{T},\ \forall\omega\in\Omega. \tag{15i}$$

Implementation Specifications

- All computational testing was conducted on a desktop computer running with Intel i7 CPU and 64.0 GB of memory and using CPLEX 12.71.
- A time limit of 7200 CPU seconds was imposed for all test instances.
- We solve a variety of CVaR-SMKP instances with the following approaches:
  - **cpx**: Direct solution of the CVaR-SMKP model (15) by CPLEX using its default settings.
  - **bc**: Lower-bound (9) and upper-bound inequalities (5).
  - **scut**: Stage-*t* dominance cuts (14).
  - **sdcut**: Stage-*t* dominance cuts (14) + strong dominance cuts.

Knapsack Instance Generation (CVaR-SMKP)

- Test instances follow Angulo et al. (2016); 60 total across $(T, I) \in \{(5, 120), \ldots, (10, 10)\}$.
- Parameters $c_{it}, v_{it}, d_t, a_{ij}, b_t, w_t, h_t \sim U\{1, \ldots, R\}$, where $R = 100$.
- Uncertainty in:
  - q_t^ω, r_t^ω (objective/coefficients)
  - b_t^ω (RHS capacity)
- Two-level scenario tree: low/high values split at $R/2$.
- Dominance defined as:

$$p^k \geq p^l,\ q_j^k \geq q_j^l,\ r_j^k \leq r_j^l,\ b_j^k \geq b_j^l \quad \forall j \leq t.$$

- Problem size reaches up to 71K variables (binary and continuous) and 67K constraints.
- Uniform scenario probability: $p^\omega = 1/|\Omega|$.
- We used $\frac{2^{T-2}}{T-1}$ to determine the number of scenarios to be used for cut generation.

Results on Bounds and Scut

Table: Optimality gap results due to **bc** and **scut** for CVaR-SMKP instances over 5–10 stages.

	cpx	**bc+scut (2^{T-2})**				**bc+scut (2^{T-1})**			
(T, I)	**IGap (%)**	**cut**	**Ctime**	**RGap (%)**	**GImp (%)**	**cut**	**Ctime**	**RGap (%)**	**GImp (%)**
(5, 120)	0.19	10	7	0.10	45.7	18	16	0.02	89.7
(6, 50)	0.71	18	10	0.40	44.5	34	17	0.05	93.6
(7, 40)	0.93	34	22	0.52	46.5	66	42	0.05	94.9
(8, 20)	3.31	66	47	1.74	47.5	130	76	0.07	97.6
(9, 15)	3.41	130	106	1.84	46.1	258	239	0.21	93.6
(10,10)	5.65	258	292	2.92	48.2	514	758	0.19	96.7
Average	2.37	86	80	1.25	46.4	170	191	0.10	94.3

Results on Sdcut

Table: Experiments for CVaR-SMKP instances with $\lambda = 1$ and $\alpha_t = 0.95$ over 5–10 stages.

(T, I)	exp	time	Ctime	Ttime	Tfac	cut	node (100K)	gap[1] (%)	gap[2] (%)	unsol
(5, 120)	**cpx**	6,534	0	6,534		0	14.2	0.02		9
	sdcut	6	53	60	110	21	0.2		0.01	0
(6, 50)	**cpx**	7,287	0	7,287		0	5.8	0.08		10
	sdcut	12	56	68	108	35	0.2		0.02	0
(7, 40)	**cpx**	6,053	0	6,053		0	3.6	0.07		9
	sdcut	15	75	90	67	135	0.0		0.02	0
(8, 20)	**cpx**	7,207	0	7,207		0	6.3	0.07		8
	sdcut	41	70	111	65	255	0.3		0.00	0
(9, 15)	**cpx**	6,492	0	6,493		0	4.2	0.12		9
	sdcut	1	112	114	57	961	0.1		0.05	0
(10, 10)	**cpx**	5,774	0	5,780		0	3.5	0.15		8
	sdcut	1	220	227	25	1,705	0.1		0.08	0
Average	**cpx**	6,558	0	6,559		0	6.3	0.08		9
	sdcut	13	98	111	72	519	0.1		0.03	0
Total	**cpx**	39,348	0	39,354	0	0	37.6	0.59	0	53
	sdcut	76	586	669	432	3112	22.2		0.19	0

Generalization to Multiple Random Variables

Table: Results for instances with $T = 10$, $I = 10$, and a varying number of random variables **RV**.

exp	$\hat{I}$	RV	Sd ($t = 2$)	Sd ($t = T$)	Sd (total)	cut	Ttime	gap^1(%)	gap^2(%)
cpx	0	0	262,144	262,144	2,359,296	0	1,460	0.03	
sdcut						140,800	216		0.00
cpx	1	9	196,608	19,683	727,383	0	7,223	0.09	
sdcut						17,319	242		0.03
cpx	2	18	157,286	3,889	399,414	0	5,394	0.07	
sdcut						4,867	226		0.03
cpx	3	27	150,733	1,563	328,533	0	5,321	0.09	
sdcut						3,225	232		0.05
cpx	4	36	131,072	742	274,202	0	5,843	0.13	
sdcut						2,196	237		0.07
cpx	5	45	137,626	710	277,011	0	5,787	0.09	
sdcut						2,142	231		0.04
cpx	6	54	137,626	538	274,714	0	6,510	0.20	
sdcut						1,876	231		0.14
cpx	7	63	137,626	563	275,507	0	6,500	0.17	
sdcut						1,779	230		0.12
cpx	8	72	137,626	627	281,997	0	5,781	0.13	
sdcut						2,046	227		0.11
cpx	9	81	131,072	512	261,632	0	5,842	0.11	
sdcut						1,705	232		0.05
cpx	10	90	131,072	512	261,632	0	7,216	0.19	
sdcut						1,705	236		0.11
cpx	Ave	45	155,499	26,499	520,120	0	5,716	0.12	
sdcut						16,333	231		0.07

Key Contributions and Takeaways

- Introduced the **scenario dominance concept and the algorithm** for solving general risk-averse multi-stage stochastic mixed-integer programs (M-SMIPs).
- Developed **stage-t scenario dominance** to identify structural implications among scenarios via pairwise coefficient comparisons.
- Scenario dominance can close the initial integrality gap by 94.3% when all scenario sub-problems are solved.
- Achieved up to **100x speedups** in computational time on large CVaR-SMKP instances with up to **90 random variables** and 512 scenarios.
- Demonstrated effectiveness of the cuts even when dominance relations are sparse.

Open Research Directions

- **Generalizability:** Extend dominance cuts to non-linear and non-convex stochastic MIPs; explore limitations.
- **Decomposition:** Embed **sdcut** in L-shaped and scenario decomposition schemes for large-scale instances.
- **Scalability:** Enable parallel and distributed dominance evaluation at scale; integrate ML for automated cut selection Yilmaz and Büyüktahtakın (2024b).
- **Theory:** Connect scenario dominance to stochastic dominance and advance risk-averse optimization theory.
- **Applications:** Apply to finance, logistics, and epidemic control, where risk and uncertainty are strategic drivers.

Related Publication and Data Resources

Related Peer-Reviewed Publication:

Büyüktahtakın, I. Esra (2022). Stage-t scenario dominance for risk-averse multi-stage stochastic mixed-integer programs. Annals of Operations Research, ***309****(1), 1–35. Springer.*
`https://doi.org/10.1007/s10479-021-04388-3`

Benchmark Library:

Büyüktahtakın, I. Esra (2023). SysOptiMaL SMIPLIB: Systems Optimization and Machine Learning Laboratory Multi-Stage Stochastic Multi-Dimensional Mixed 0-1 Knapsack (S-MKP) and Multi-Stage Stochastic Multi-item Capacitated Lot-Sizing (S-MCLSP) Test Problem Library.
`https://soml.ise.vt.edu/people.html`

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Abgottspon, H., K. Njálsson, M. A. Bucher, and G. Andersson (2014). Risk-averse medium-term hydro optimization considering provision of spinning reserves. In *Probabilistic Methods Applied to Power Systems (PMAPS), 2014 International Conference on*, pp. 1–6. IEEE.

Ahmed, S. (2013). A scenario decomposition algorithm for 0–1 stochastic programs. *Operations Research Letters 41*(6), 565–569.

Angulo, G., S. Ahmed, and S. S. Dey (2016). Improving the integer L-shaped method. *INFORMS Journal on Computing 28*(3), 483–499.

Bayraksan, G. and D. P. Morton (2011). Sequential sampling for stochastic programming: Complete problem statement and two-stage analysis. *Operations Research 59*(4), 898–913.

Benders, J. F. (2005). Partitioning procedures for solving mixed-variables programming problems. *Computational Management Science 2*(1), 3–19.

Bushaj, S. and İ. E. Büyüktahtakın (2024). A K-means supported reinforcement learning framework to multi-dimensional knapsack. *Journal of Global Optimization*, 1–31.

Bushaj, S., İ. E. Büyüktahtakın, and R. G. Haight (2022). Risk-averse multi-stage stochastic optimization for surveillance and operations planning of a forest insect infestation. *European Journal of Operational Research 299*(3), 1094–1110.

Büyüktahtakın, İ. E. (2022). Stage-*t* scenario dominance for risk-averse multi-stage stochastic mixed-integer programs. *Annals of Operations Research 309*, 1–35.

Cerisola, S., J. M. Latorre, and A. Ramos (2012). Stochastic dual dynamic programming applied to nonconvex hydrothermal models. *European Journal of Operational Research 218*(3), 687–697.

Chen, R. and J. Luedtke (2022). On generating lagrangian cuts for two-stage stochastic integer programs. *INFORMS Journal on Computing 34*(4), 2332–2349.

Dentcheva, D. and A. Ruszczyński (2024). *Risk-Averse Optimization and Control: Theory and Methods*. Springer Nature.

Dey, S. S., M. Molinaro, and Q. Wang (2018). Analysis of sparse cutting planes for sparse milps with applications to stochastic milps. *Mathematics of Operations Research 43*(1), 304–332.

Eckstein, J., D. Eskandani, and J. Fan (2016). Multilevel optimization modeling for risk-averse stochastic programming. *INFORMS Journal on Computing 28*(1), 112–128.

Heinze, T. and R. Schultz (2008). A branch-and-bound method for multistage stochastic integer programs with risk objectives. *Optimization 57*(2), 277–293.

Heitsch, H. and W. Römisch (2003). Scenario reduction algorithms in stochastic programming. *Computational Optimization and Applications 24*(2-3), 187–206.

Huang, K. and S. Ahmed (2009). The value of multistage stochastic programming in capacity planning under uncertainty. *Operations Research 57*(4), 893–904.

Kibis, E., İ. Büyüktahtakın, R. G. Haight, N. Akhundov, K. Knight, and C. Flower (2021). A new multi-stage stochastic programming model and cutting planes for the optimal surveillance and control of emerald ash borer in cities. *INFORMS Journal on Computing 33*(2), 808–834.

Krokhmal, P., J. Palmquist, and S. Uryasev (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of Risk 4*, 43–68.

Küçükyavuz, S. and S. Sen (2017). An introduction to two-stage stochastic mixed-integer programming. In *Leading Developments from INFORMS Communities*, pp. 1–27. INFORMS.

Linderoth, J. and S. Wright (2003). Decomposition algorithms for stochastic programming on a computational grid. *Computational Optimization and Applications 24*, 207–250.

Liu, X., S. Küçükyavuz, and N. Noyan (2017). Robust multicriteria risk-averse stochastic programming models. *Annals of Operations Research 259*, 259–294.

Luedtke, J. (2008). New formulations for optimization under stochastic dominance constraints. *SIAM Journal on Optimization 19*(3), 1433–1450.

Madansky, A. (1960). Inequalities for stochastic linear programming problems. *Management Science 6*(2), 197–204.

Noyan, N. (2012). Risk-averse two-stage stochastic programming with an application to disaster management. *Computers & Operations Research 39*(3), 541–559.

Rockafellar, T. and S. Uryasev (2000). Optimization of conditional value-at-risk. *Journal of Risk 2*, 21–42.

Romeijnders, W. and N. van der Laan (2020). Pseudo-valid cutting planes for two-stage mixed-integer stochastic programs with right-hand-side uncertainty. *Operations Research 68*(4), 1199–1217.

Ruszczyński, A. (2002). Probabilistic programming with discrete distributions and precedence constrained knapsack polyhedra. *Mathematical Programming 93*(2), 195–215.

Sandıkçı, B. and O. Özaltın (2017). A scalable bounding method for multistage stochastic programs. *SIAM Journal on Optimization 27*(3), 1772–1800.

Schultz, R. and S. Tiedemann (2003). Stochastic integer programming: Structure and algorithmic aspects. *Stochastic Programming ESI Special Volume*, 121–162.

Schultz, R. and S. Tiedemann (2006). Conditional value-at-risk in stochastic programs with mixed-integer recourse. *Mathematical Programming 105*(2-3), 365–386.

Sen, S. and J. L. Higle (2014). A note on multistage stochastic programming and dynamic programming. *Operations Research Letters 42*(5), 359–362.

Shapiro, A., D. Dentcheva, and A. Ruszczyński (2009). *Lectures on stochastic programming: modeling and theory*. SIAM.

Watson, J.-P. and D. L. Woodruff (2011). Progressive hedging innovations for a class of stochastic mixed-integer resource allocation problems. *Computational Management Science 8*(4), 355–370.

Yilmaz, D. and İ. E. Büyüktahtakın (2024a). A deep reinforcement learning framework for solving two-stage stochastic programs. *Optimization Letters 18*(9), 1993–2020. https://doi.org/10.1007/s11590-023-02009-5.

Yilmaz, D. and İ. E. Büyüktahtakın (2024b). An expandable machine learning-optimization framework to sequential decision-making. *European Journal of Operational Research 314*(1), 280–296.

Yilmaz, D. and İ. E. Büyüktahtakın (2024c). A non-anticipative learning-optimization framework for solving multi-stage stochastic programs. *Annals of Operations Research*.

Yin, X. and İ. E. Büyüktahtakın (2021). A multi-stage stochastic programming approach to epidemic resource allocation with equity considerations. *Health Care Management Science 24*, 597–622.

Yin, X. and İ. E. Büyüktahtakın (2022). Risk-averse multi-stage stochastic programming to optimizing vaccine allocation and treatment logistics for effective epidemic response. *Industrial and Systems Engineers Transactions on Healthcare Systems Engineering 12*(1), 52–74.

Yin, X., İ. E. Büyüktahtakın, and B. P. Patel (2023). COVID-19: Data-driven optimal allocation of ventilator supply under uncertainty and risk. *European Journal of Operational Research 304*(1), 255–275.

Zhang, W., H. Rahimian, and G. Bayraksan (2016). Decomposition algorithms for risk-averse multistage stochastic programs with application to water allocation under uncertainty. *INFORMS Journal on Computing 28*(3), 385–404.

Zhang, Y., M. Bansal, and A. R. Escobedo (2020). Risk-neutral and risk-averse transmission switching for load shed recovery with uncertain renewable generation and demand. *IET Generation, Transmission & Distribution 14*(21), 4936–4945.

Zou, J., S. Ahmed, and X. A. Sun (2019). Stochastic dual dynamic integer programming. *Mathematical Programming 175*(1-2), 461–502.

THANK YOU!

The Number of Scenario-Dominance Relations

Proposition

Let $(\xi_2, \ldots, \xi_T)$ *be a sequence of independently identically distributed (i.i.d) random vectors of* $k_t = k$ *dimensions each. Each univariate component of a vector* ξ_t *is independently distributed. In a scenario tree with* l *branches at each node in each stage, the minimum total number of scenario-dominance relations,* W*, is equal to:*

$$W = \sum_{t=1}^{T} l^{(2T-t-1)k}. \quad (16)$$

Proof.

In this scenario tree, we have $l^{(T-1)k}$ scenarios. At stage $t' = t$, we have $l^{(t-1)k}$ nodes, and thus $l^{(T-1)k_t}/l^{(t-1)k}$ scenarios share the same node at stage t, resulting in a total of $l^{(t-1)k}\left(l^{(T-1)k}/l^{(t-1)k}\right)^2$ domination relations at stage t. This results in at least a total number of $\sum_{t=1}^{T} l^{(2T-t-1)k}$ scenario-dominance relations over all stages. □

Example 12. Consider a CVaR-SMKP instance in which $T = 4$, $l = 2$, and $k_t = 2$ for $t = 1, \ldots, T$, and each univariate component of the random vector ξ_t is independently distributed. Then by Proposition 6 we have $\bar{\Omega} = 64$ scenarios and a total of at least $W = 5,440$ scenario-dominance relations at the minimum.

Example Scenario Dominance Cuts

Example. Consider a CVaR-SMKP instance in which $T = 4$ and $I = 2$. This instance has $\bar{\Omega} = 8$ scenarios, and the probability of each scenario is set as $1/8$. The index set of scenarios that dominate ξ^3 is defined as $\Theta^-_{2,\xi^3} = \{1,2,3,4\}$, $\Theta^-_{3,\xi^3} = \{3,4\}$, $\Theta^-_{4,\xi^3} = \{3\}$. Solving the stage-$t$ scenario-ξ^3 problem for $t = 2,3,4$, we have $Z_2^3 = 31.62$, $Z_3^3 = 59.89$, and $Z_4^3 = 72.39$.
For $\omega = 4 \in \Theta^-_{t,\xi^3}$, the inequalities corresponding to (14) for $t = 2,3$ can be, respectively, written as:

$$p^4 \left(\sum_{i \in \mathcal{I}} c_{i2} x_{i2}^4 + d_2 z_2^4 + q_2^3 y_2^4 \right) + \lambda p^4 \left(\eta_2^4 + \frac{1}{1-\alpha_2} v_2^4 \right) \geq 31.62.$$

$$p^4 \sum_{t \in \{2,3\}} \left(\sum_{i \in \mathcal{I}} c_{it} x_{it}^4 + d_t z_t^4 + q_t^3 y_t^4 \right) + \lambda p^4 \sum_{t \in \{2,3\}} \left(\eta_t^4 + \frac{1}{1-\alpha_t} v_t^4 \right) \geq 59.89.$$

The optimal objective function value to CVaR-SMKP (15) is 614.08, and its Linear Programming (LP) relaxation is 507.60. By adding inequalities (10), (**??**), and (14) for $S = \{1,2,3,4\}$, the optimal relaxation solution is cut-off, and the LP relaxation objective of (15) increases to 560.61, closing the optimality gap by 50%.