Ingeniería Industrial
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS UNIVERSIDAD DE CHILE

Tightening Convex Relaxations of Trained Neural Networks

A unified approach for convex and S-shaped activations

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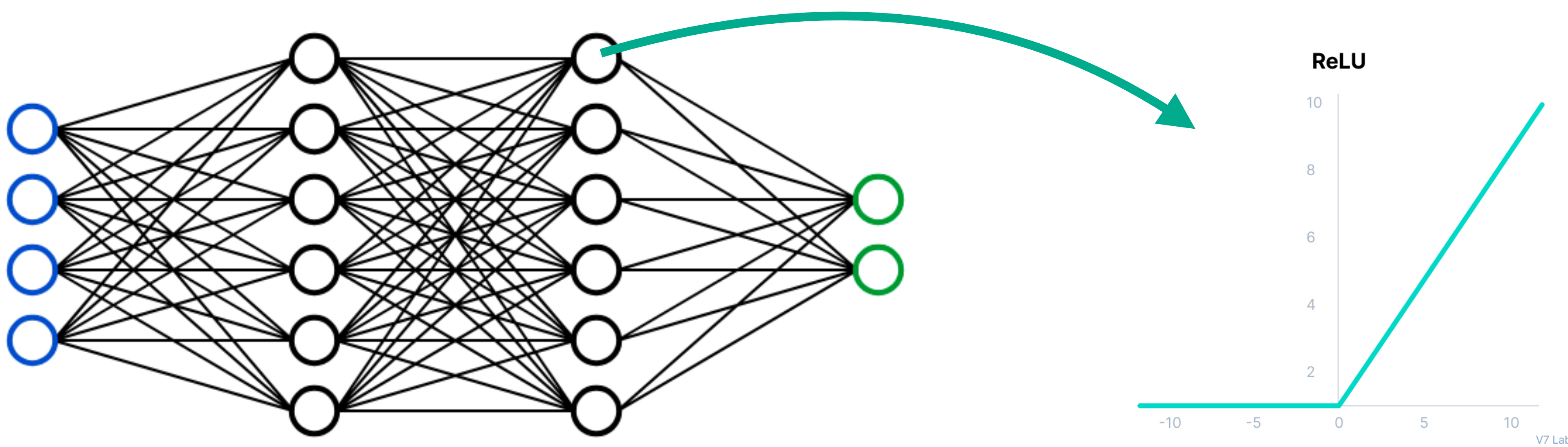
Gonzalo Muñoz (Universidad de Chile)

Mixed-Integer Programming Workshop 2025

Optimization using trained neural networks

High level goal: to convexity $y = \mathrm{NN}(x)$



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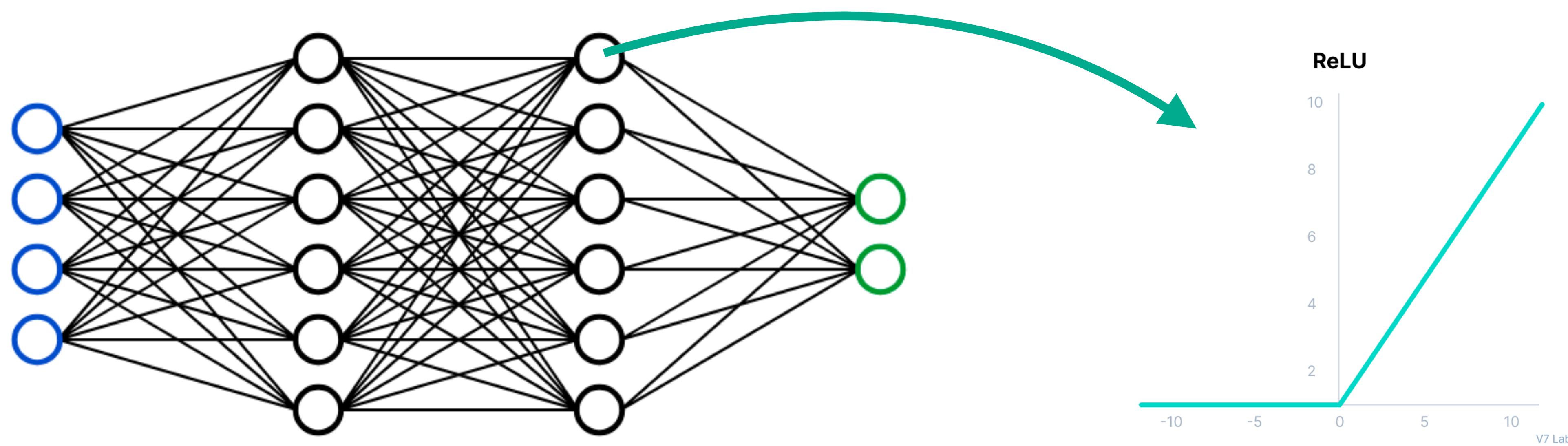
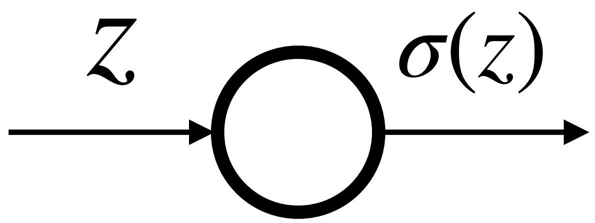
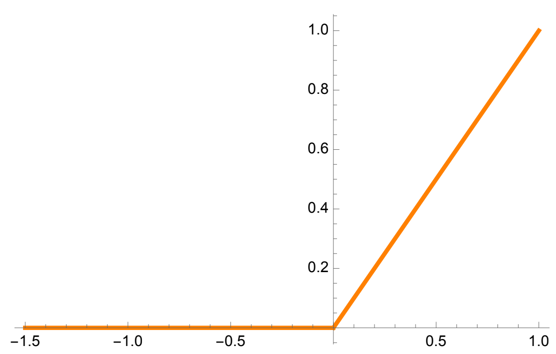
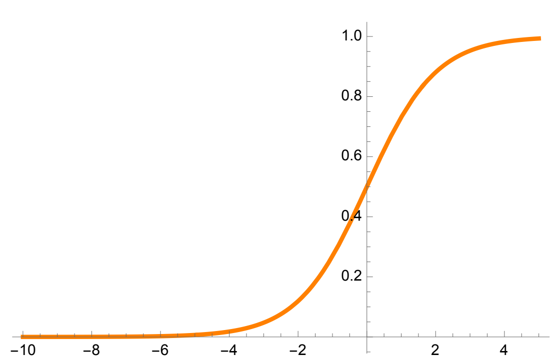


This can be useful in either using or evaluating a trained neural network via an optimization model

Single neuron convexifications







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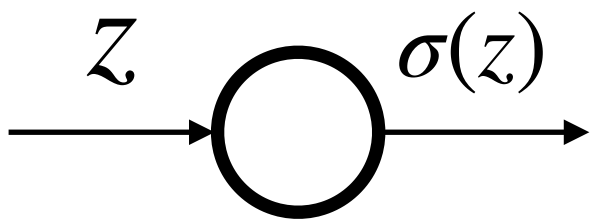
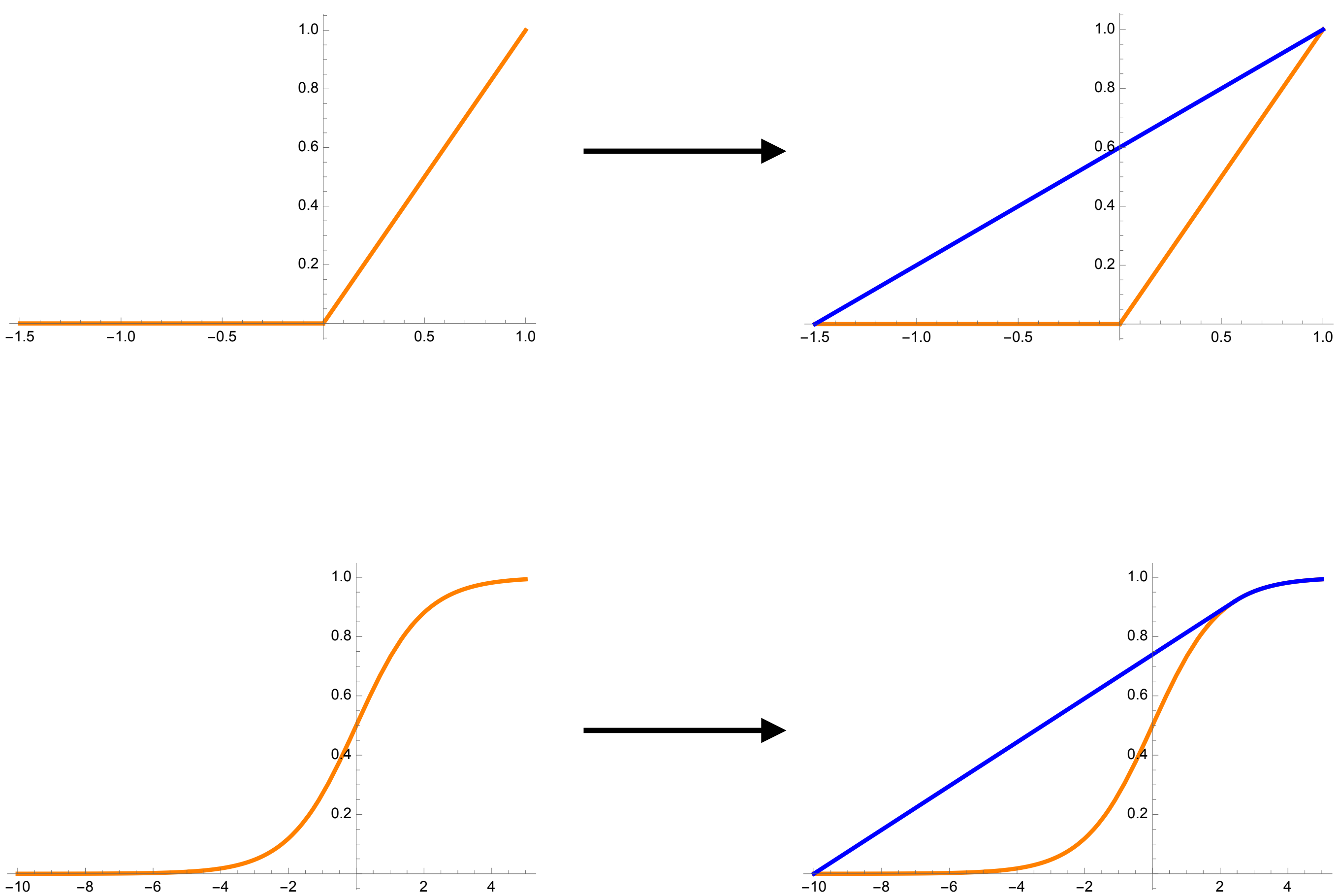


Function envelopes



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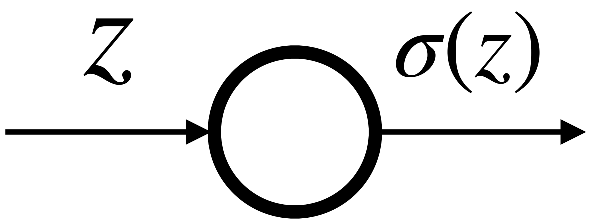

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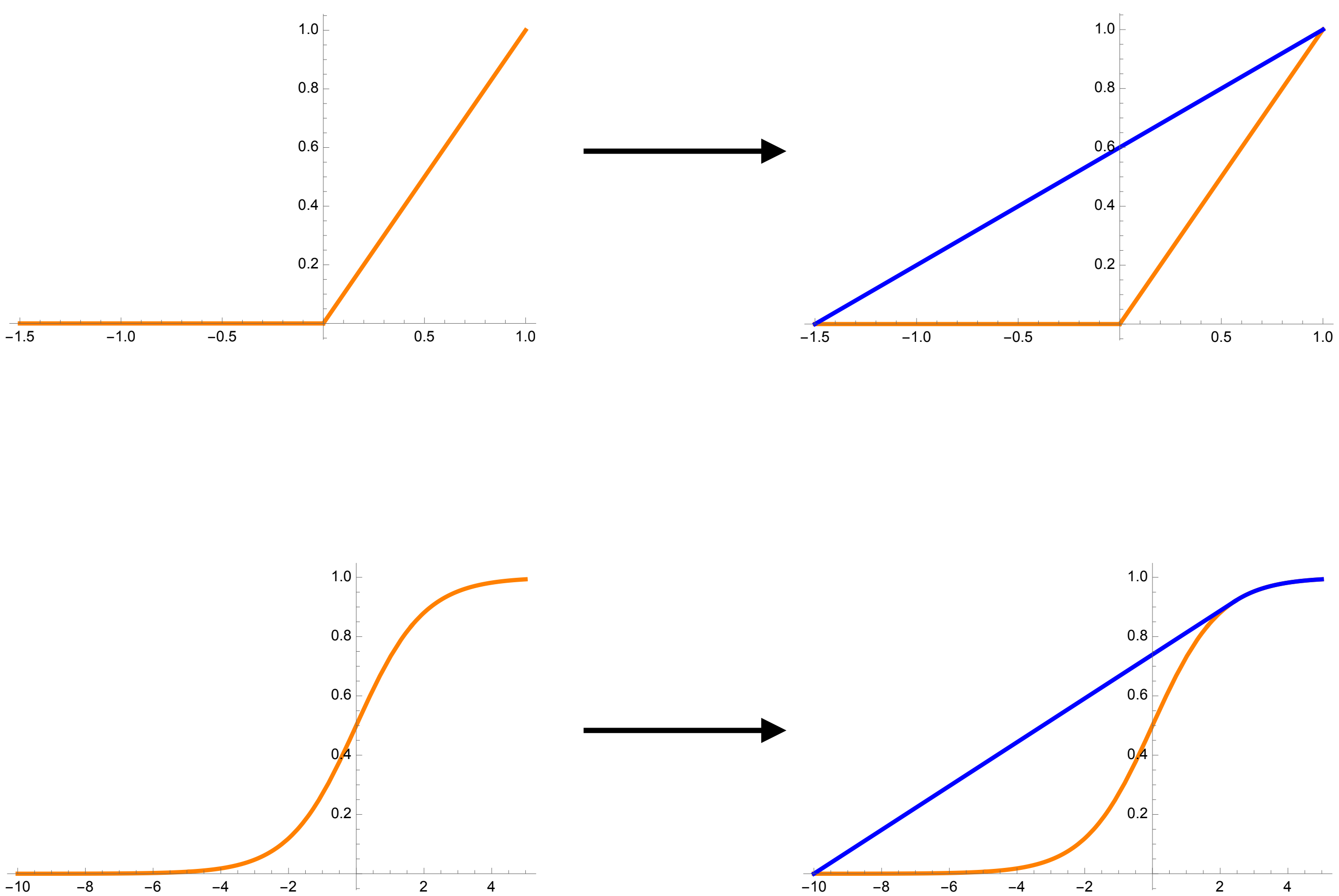

(Locally)

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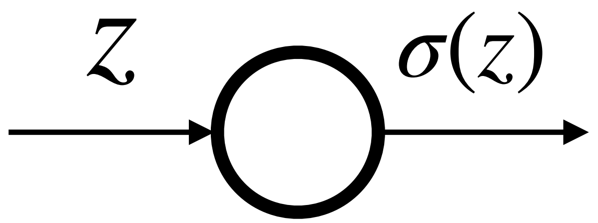

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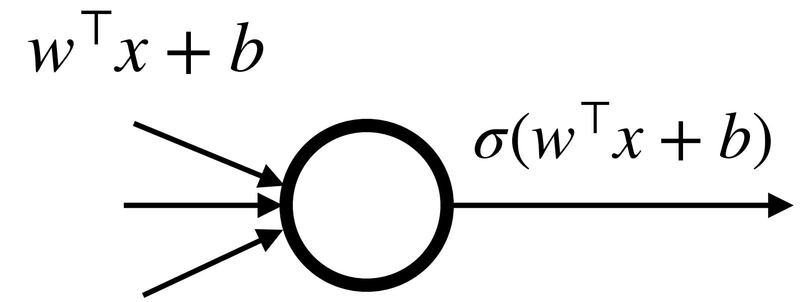

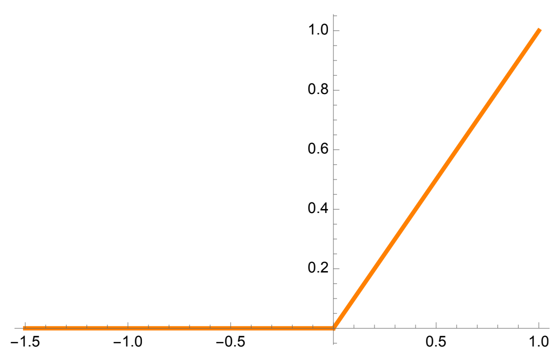

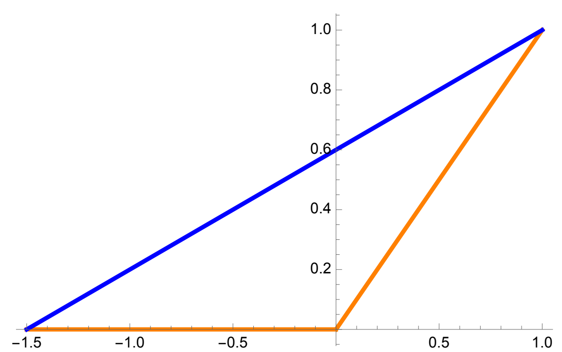

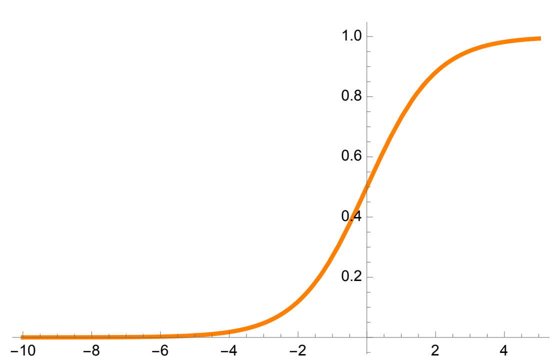

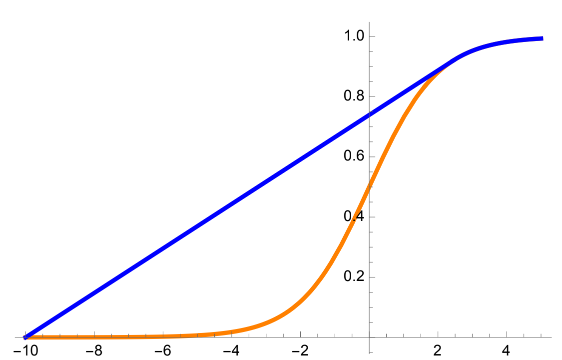

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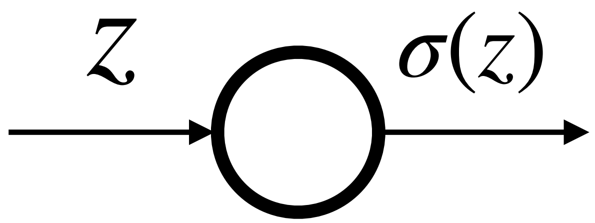

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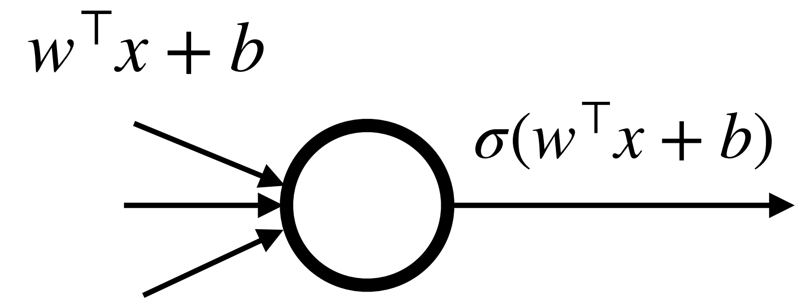

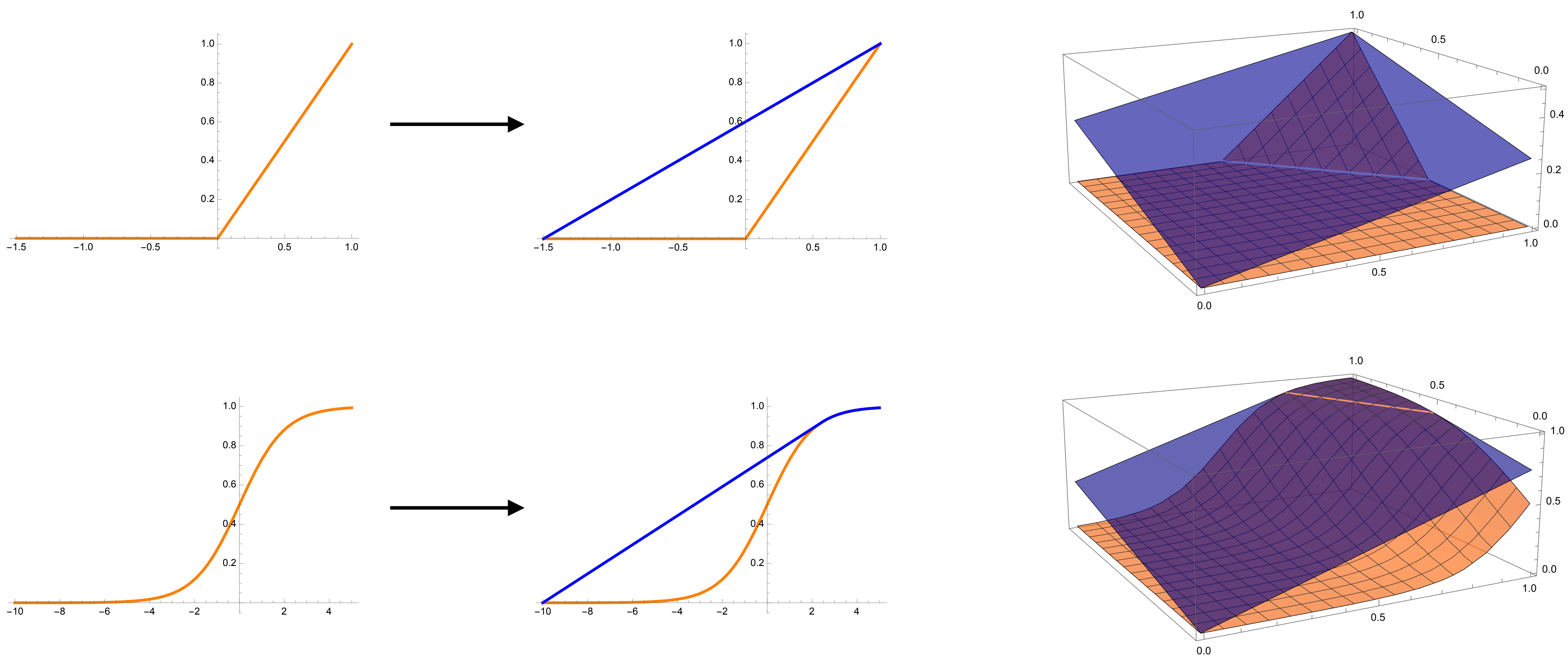

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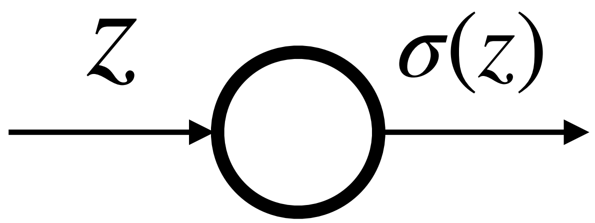

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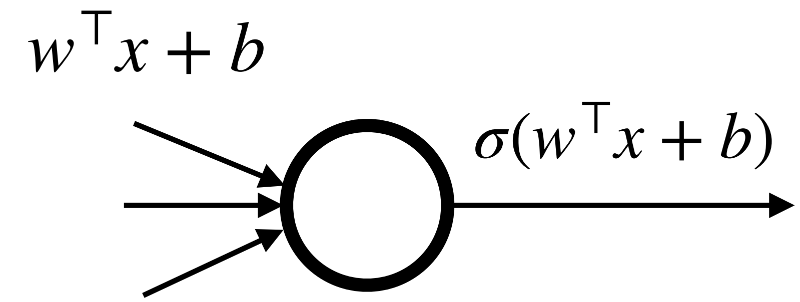

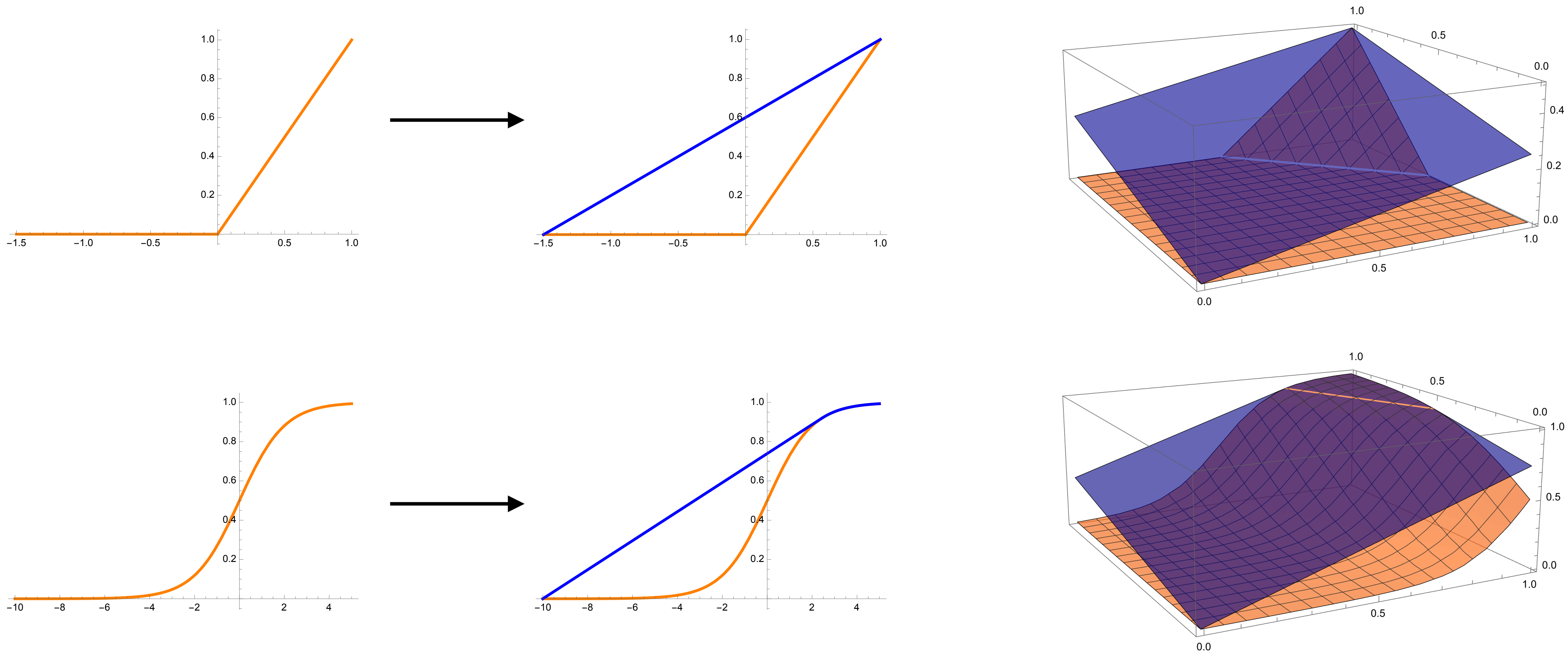

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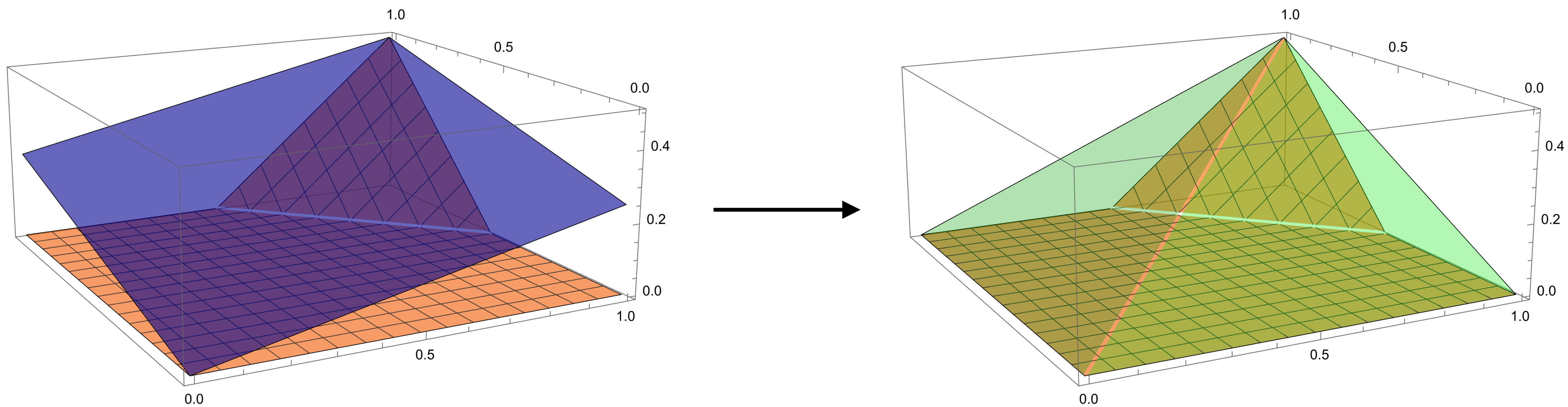
A path forward for the ReLU

Anderson et al. (2020) and Tjandraatmadja et al. (2020) showed how to tightly convexify

$$y = \sigma(w^\top x + b)$$



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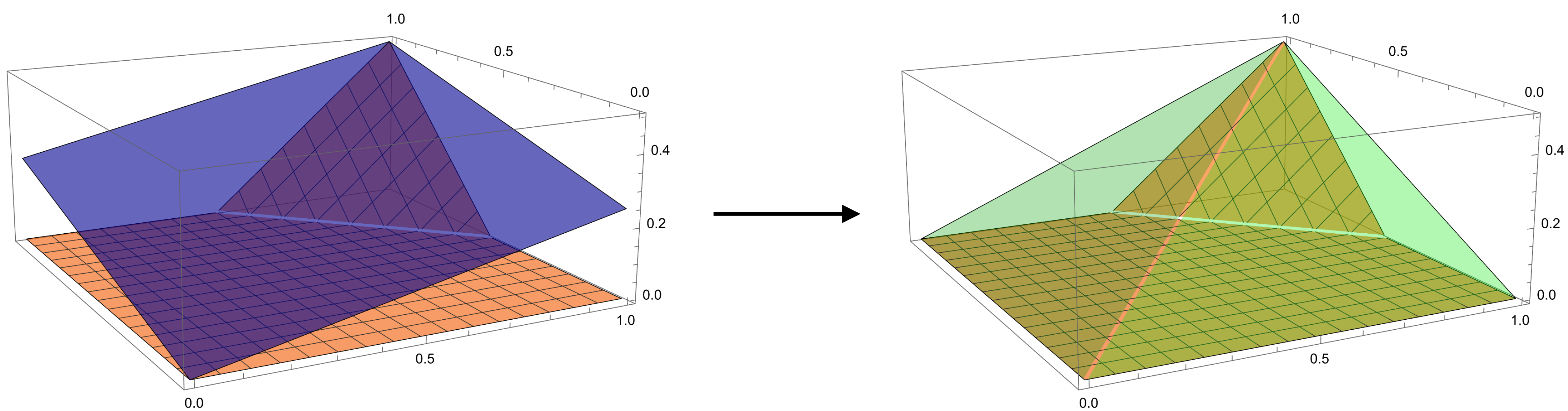
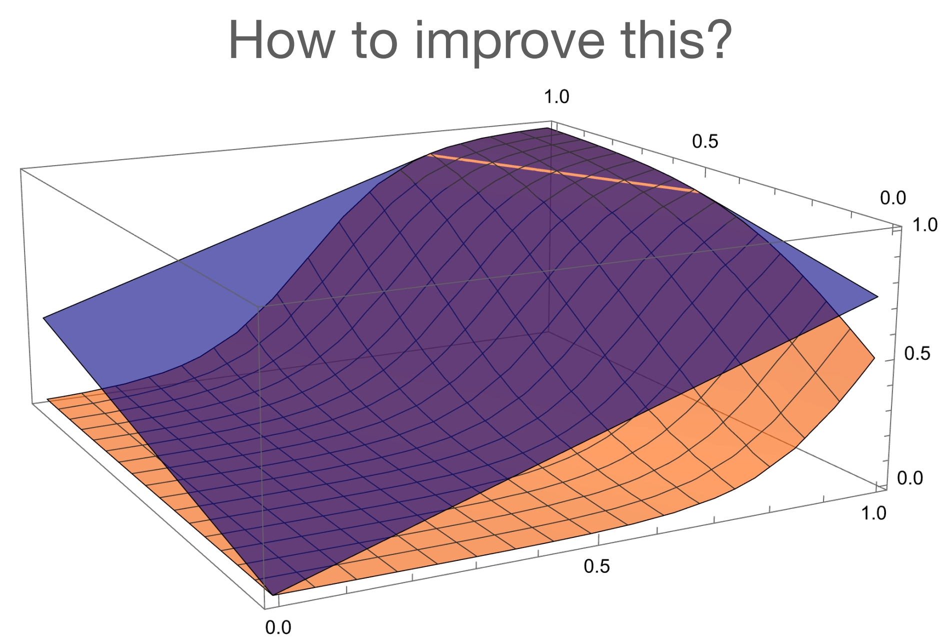


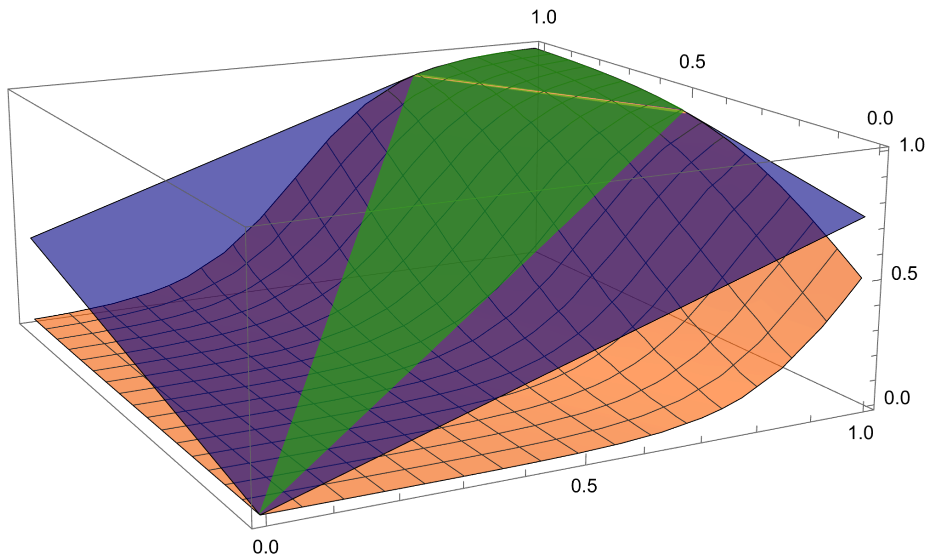
Their approach can handle polyhedral domains, piecewise linear convex activations, and they develop a polynomial time separation

Beyond the ReLU



Beyond the ReLU

How to improve this?



The green part is OK

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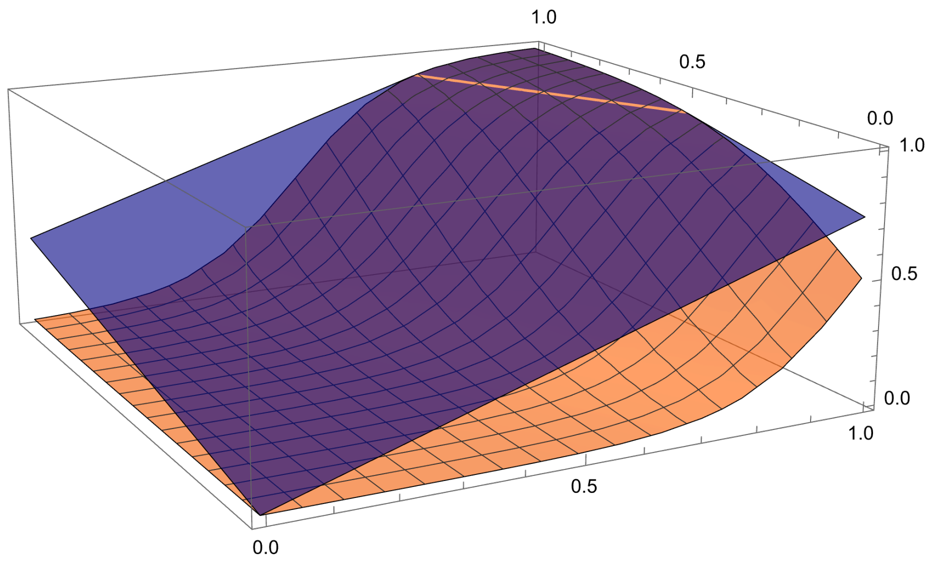

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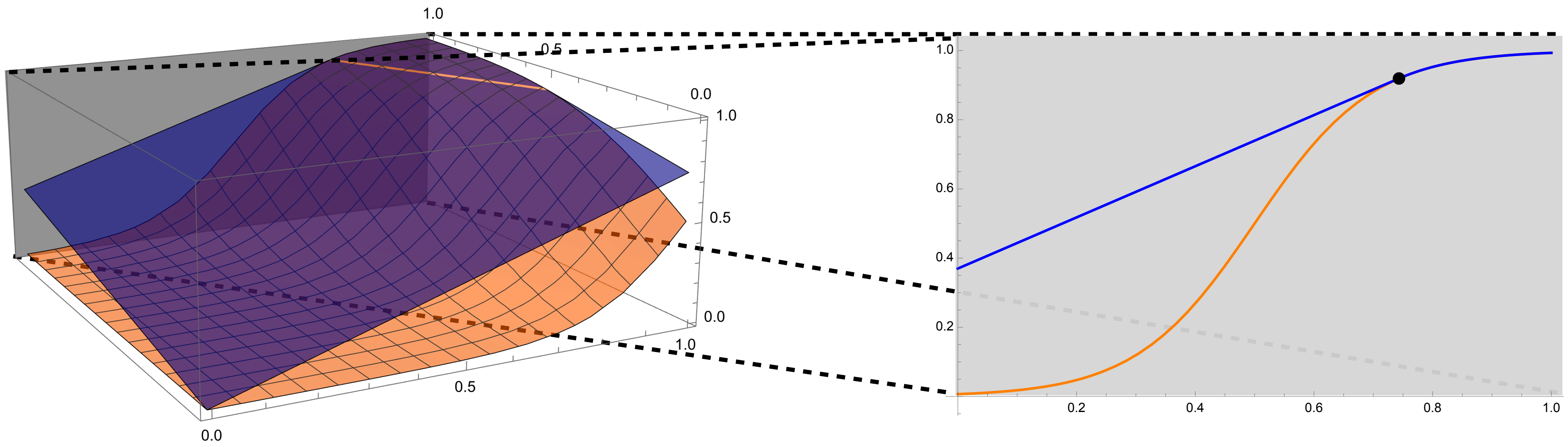

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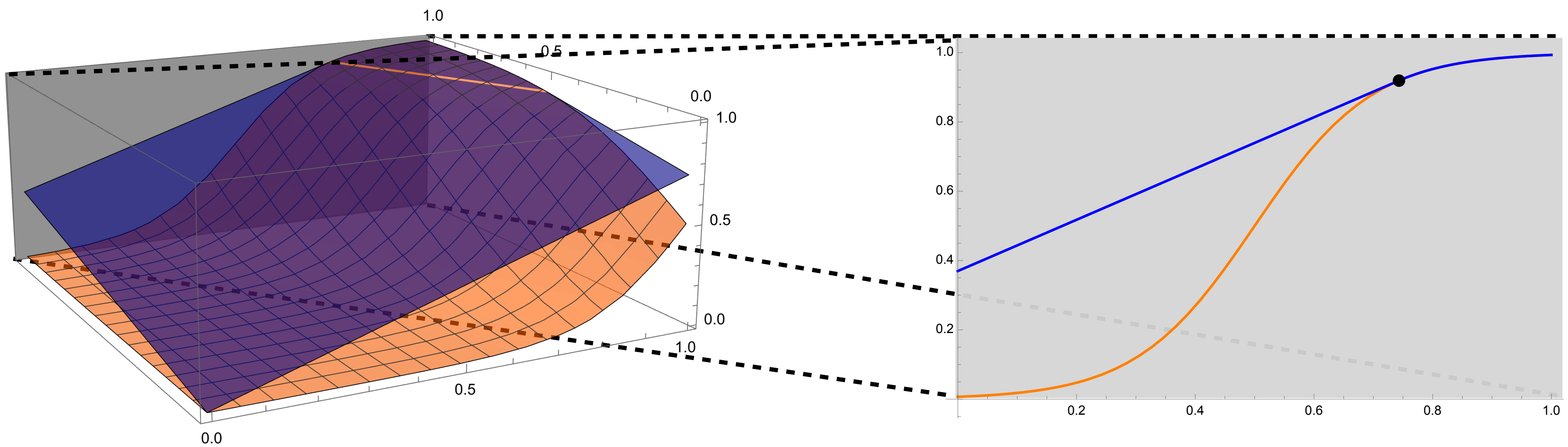
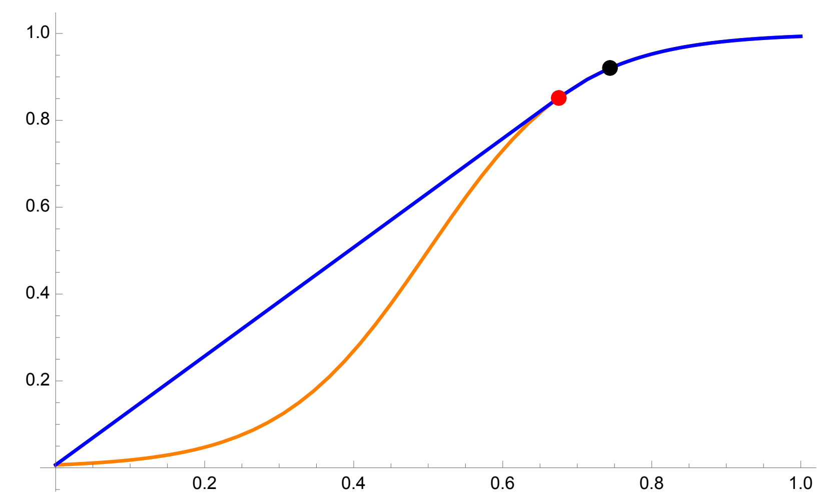


Tawarmalani and Sahinidis (2002) showed that, on the boundary, the envelope should look like this



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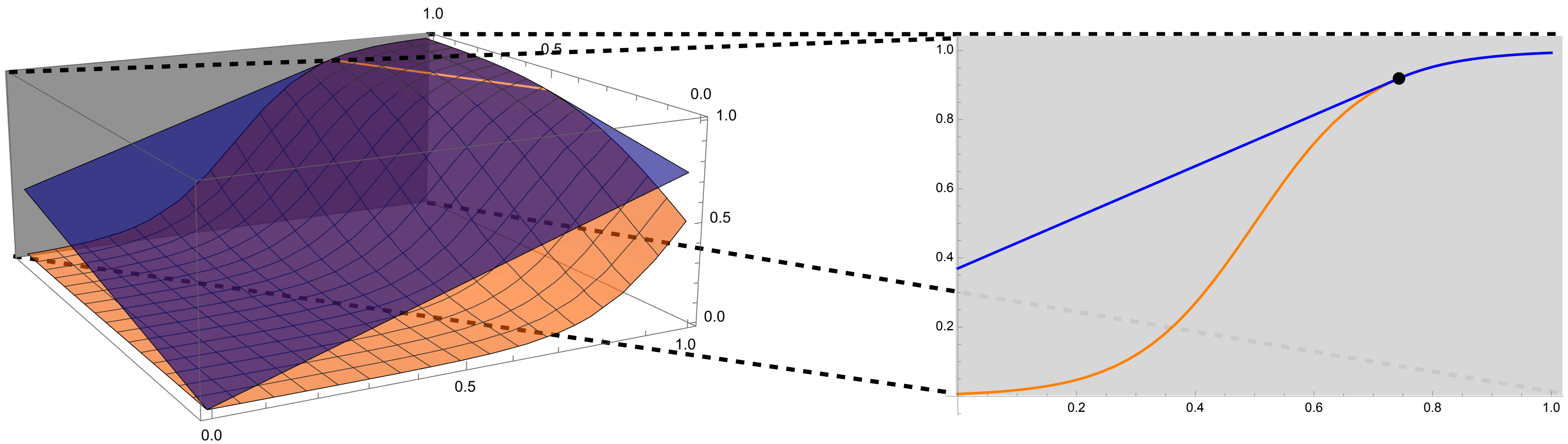

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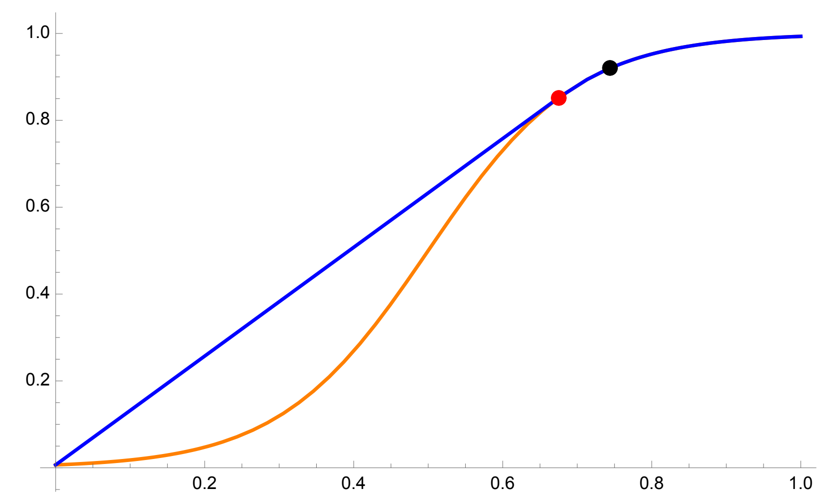

How to build from this?

Beyond the ReLU

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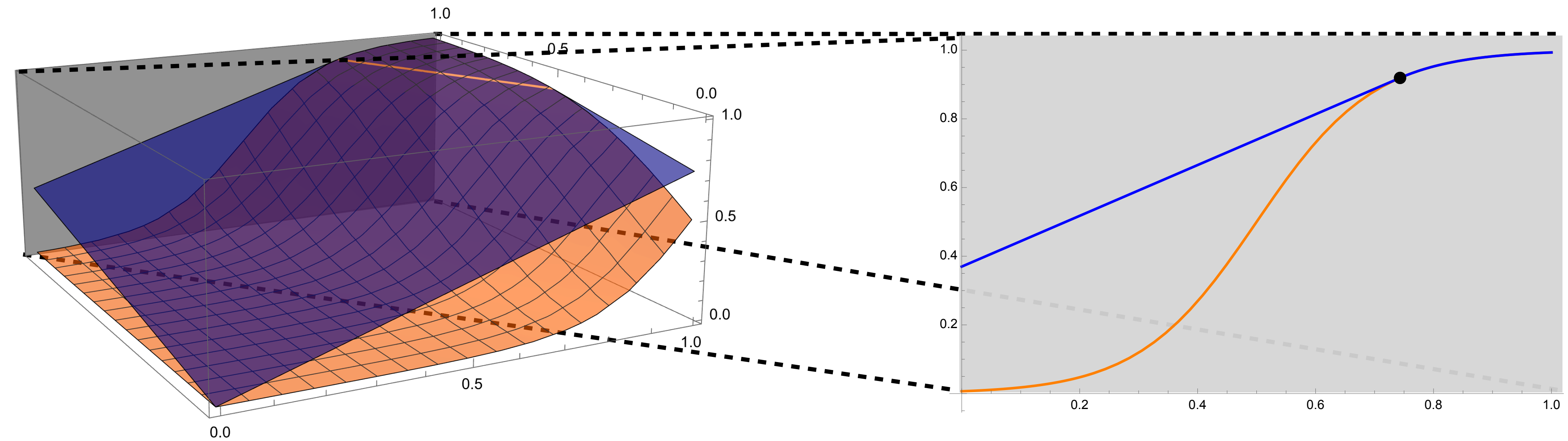

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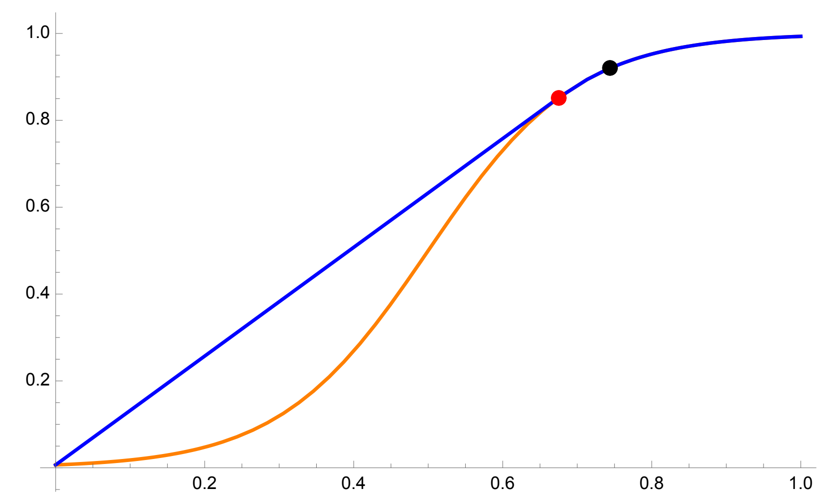

How to build from this?

Taking perspective from the boundary!

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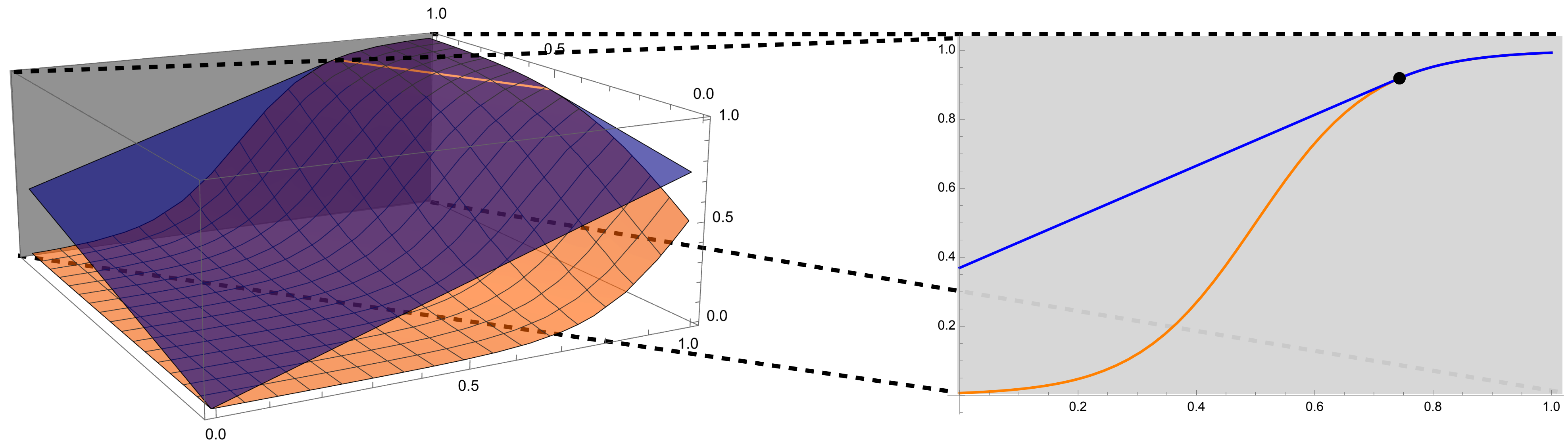

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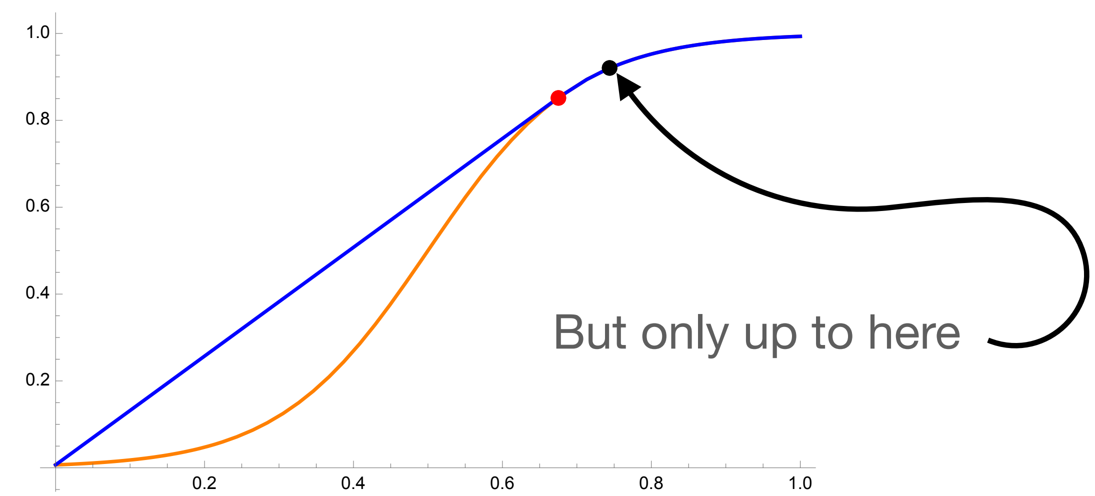

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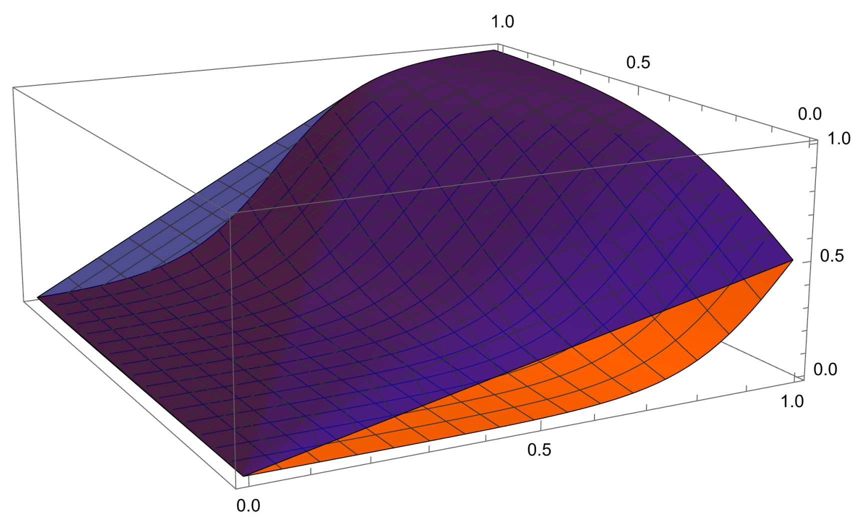
Taking perspective from the boundary!

The main construction



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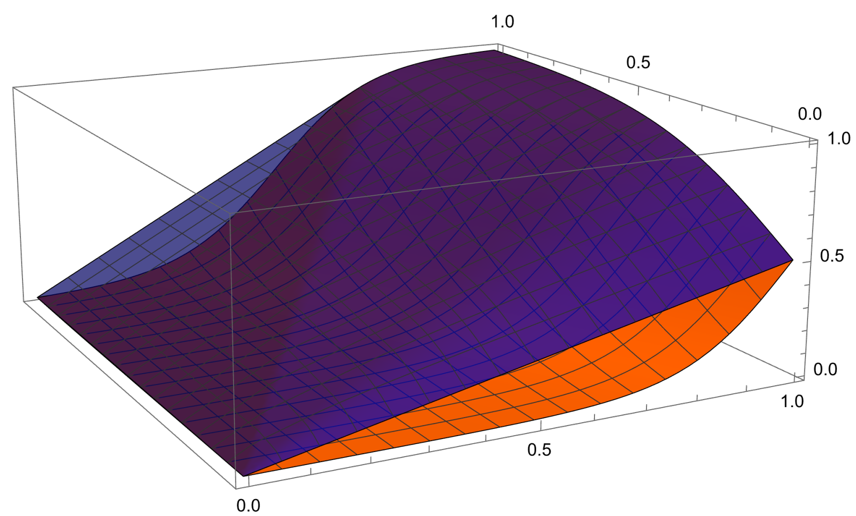

Theorem 1 *Consider* $w \in \mathbb{R}^n_+$, $b \in \mathbb{R}$ *and let* $f : [0,1]^n \to \mathbb{R}$ *be a function of the form* $f(x) = \sigma(w^\top x + b)$ *where* σ *satisfies the STFE property. Then,*

$$conc(f,[0,1]^n)(x) = \begin{cases} f(x) & \text{if } x \in R_f \\ \sigma(b) + \frac{\sigma(\hat{z})-\sigma(b)}{\hat{z}-b}(w^\top x) & \text{if } x \in R_l \\ \sigma(b) + x_i\, conc(f_{-i} - \sigma(b), [0,1]^{n-1})\left(\frac{x_{-i}}{x_i}\right) & \text{if } x \in R_i \end{cases}$$

where R_f, R_l *and* R_i *are defined using Definition 3 with* $\hat{z}$ *as the tie point of* σ *in* $[b, w^\top \mathbf{1} + b]$.

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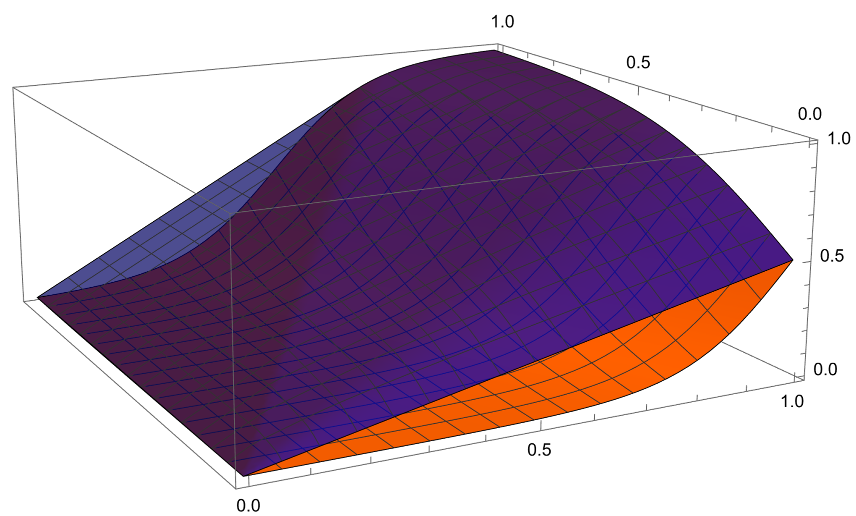

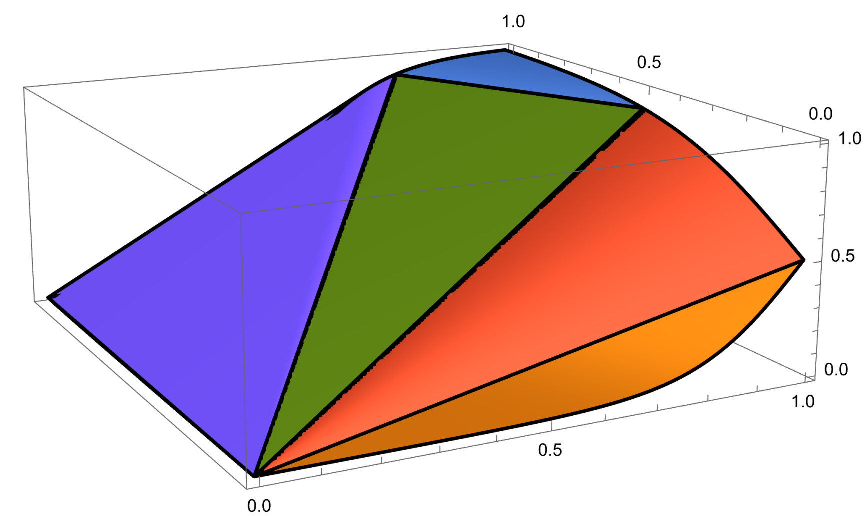

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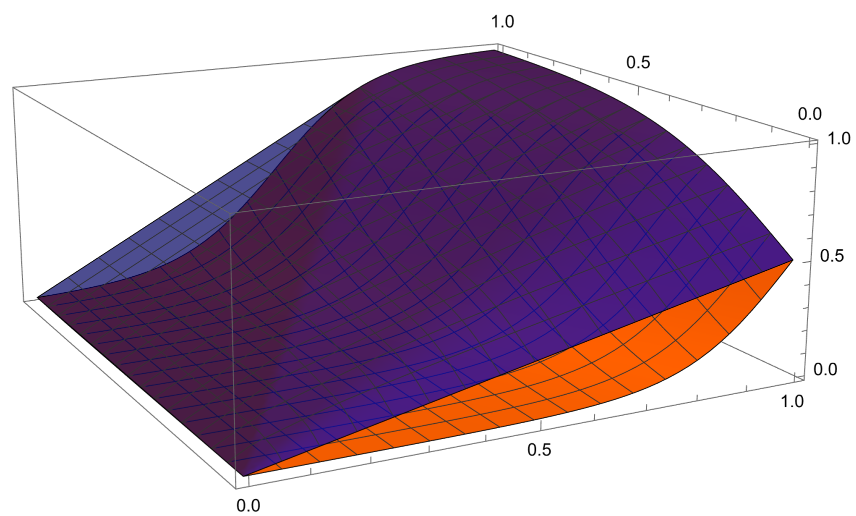

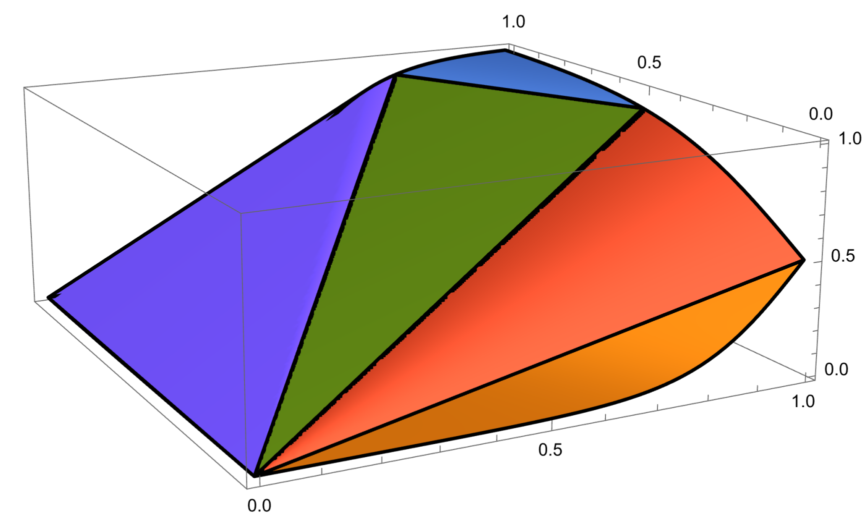

The envelope evaluation is either

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where R_f, R_l and R_i are defined using Definition 3 with $\hat{z}$ as the tie point of σ in $[b, w^\top \mathbf{1} + b]$.

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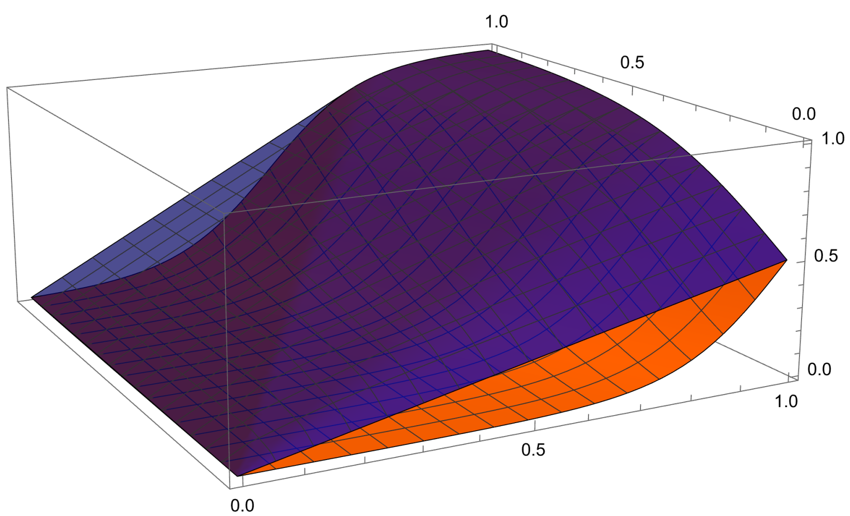

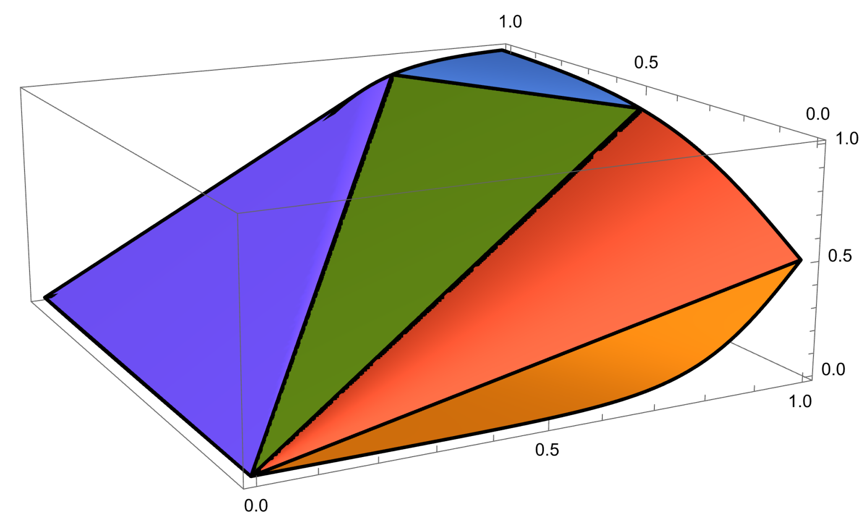

Th… …*of*

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the … *property. Then,*

The function itself

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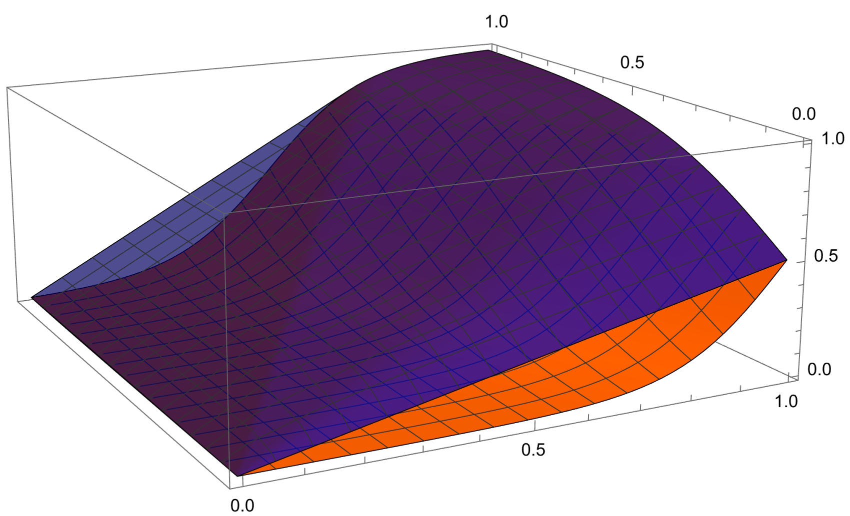

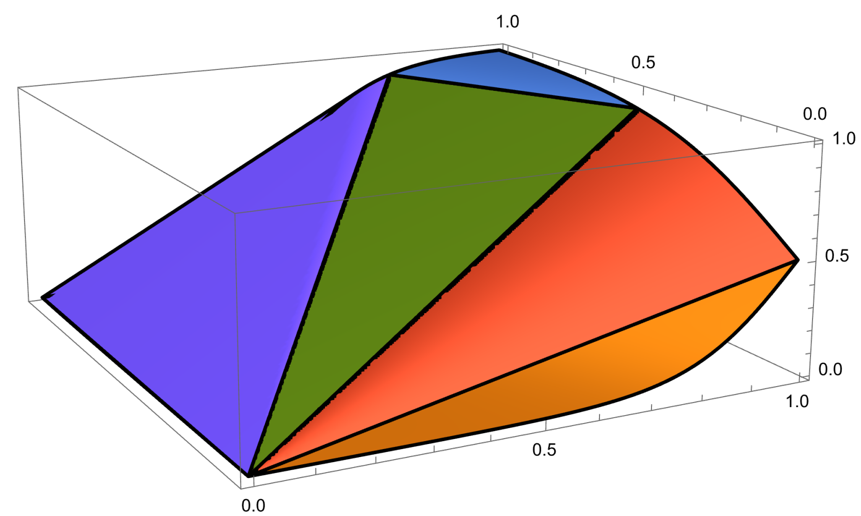

Th... *of the* ...

The envelope evaluation is either

$$conc(f, [0,1]^n)(x) = \begin{cases} \text{The function itself} & f \\ \text{A linear function} & l \\ \sigma(b) + x_i\, conc(f_{-i} - \sigma(b), [0,1]^{n-1}) \left(\frac{x_{-i}}{x_i}\right) & \text{if } x \in R_i \end{cases}$$

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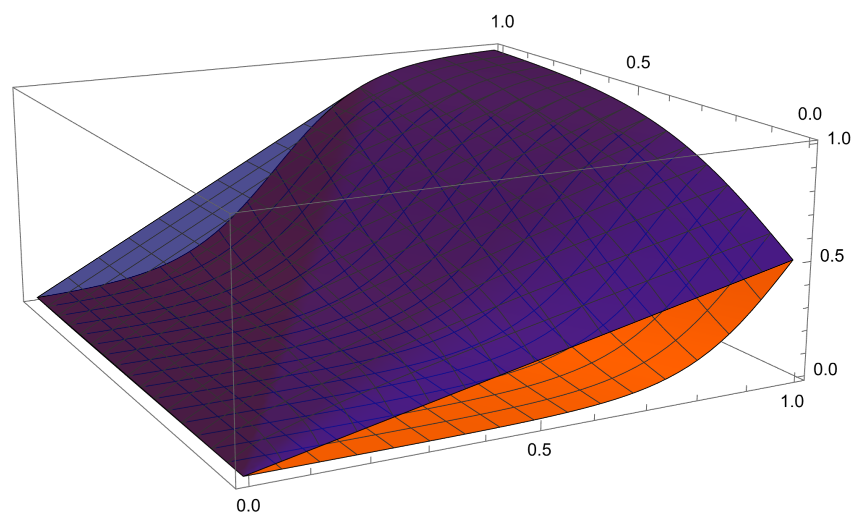

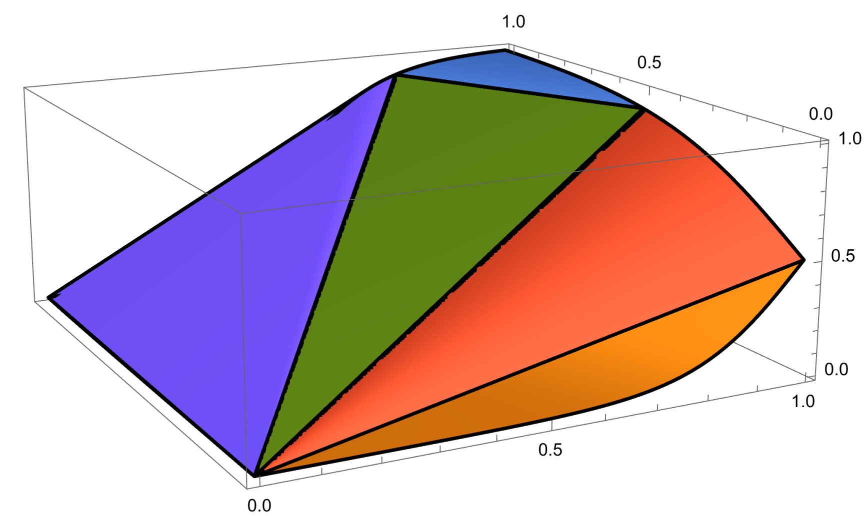

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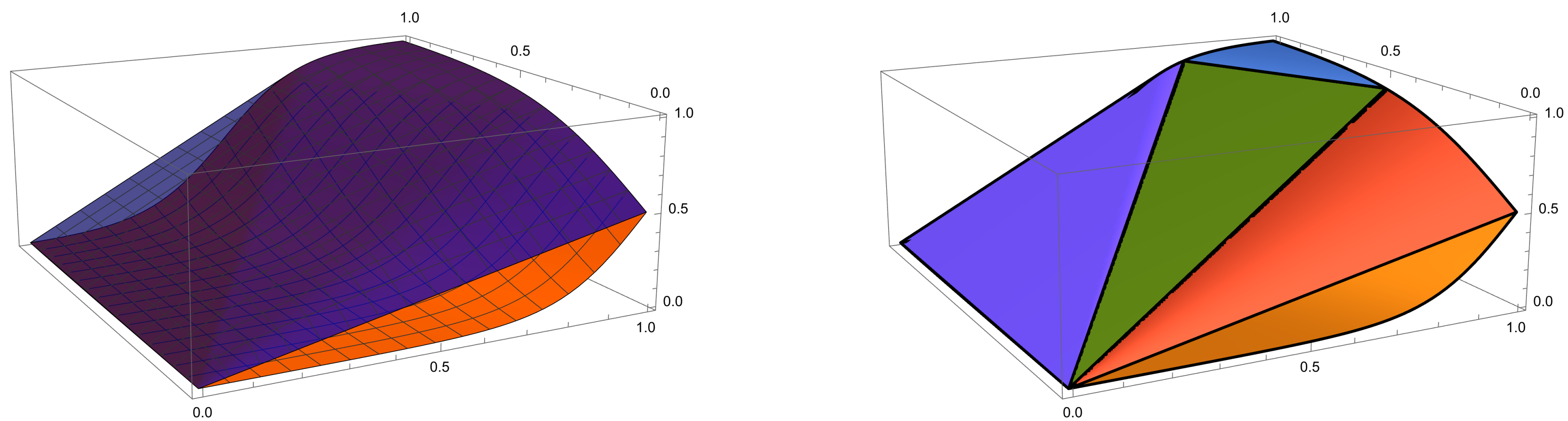

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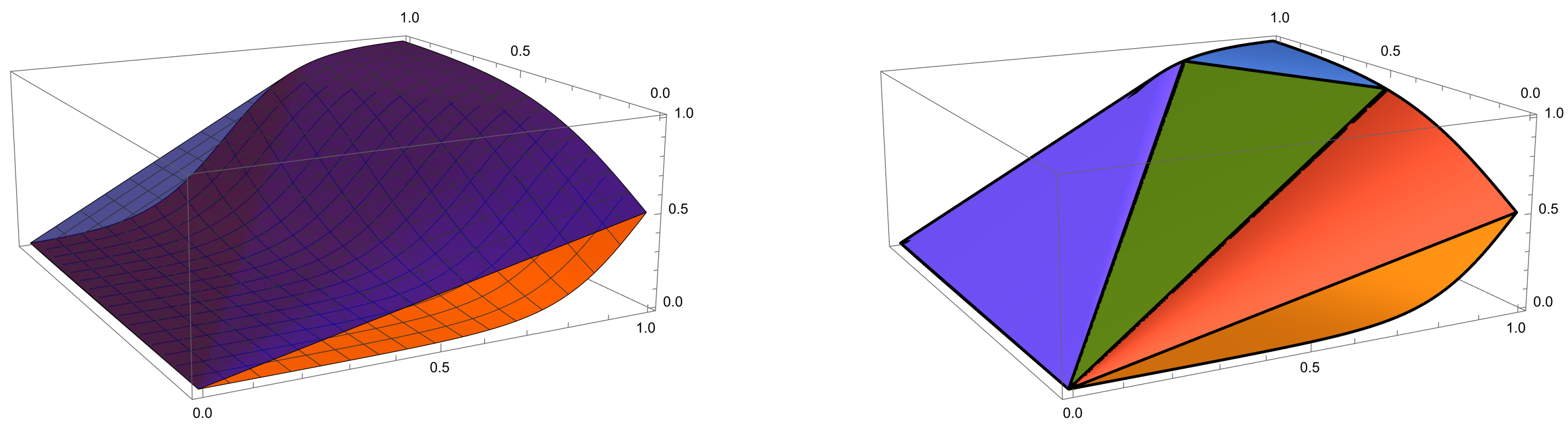
This works for any convex or S-shaped activation function on a box,
can construct supporting hyperplanes of the convex hull of the graph,
and requires at most n 1-dimensional convexifications

The key ingredients



An envelope construction like this does not work in general, here we heavily rely on

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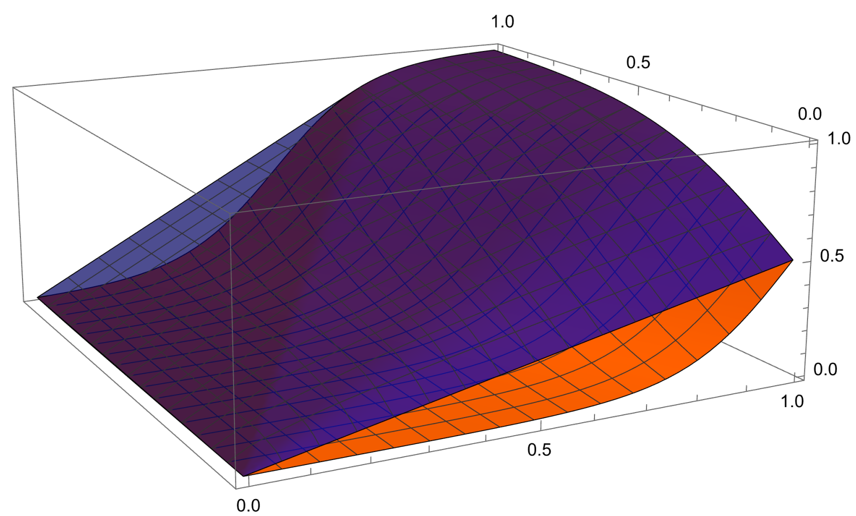

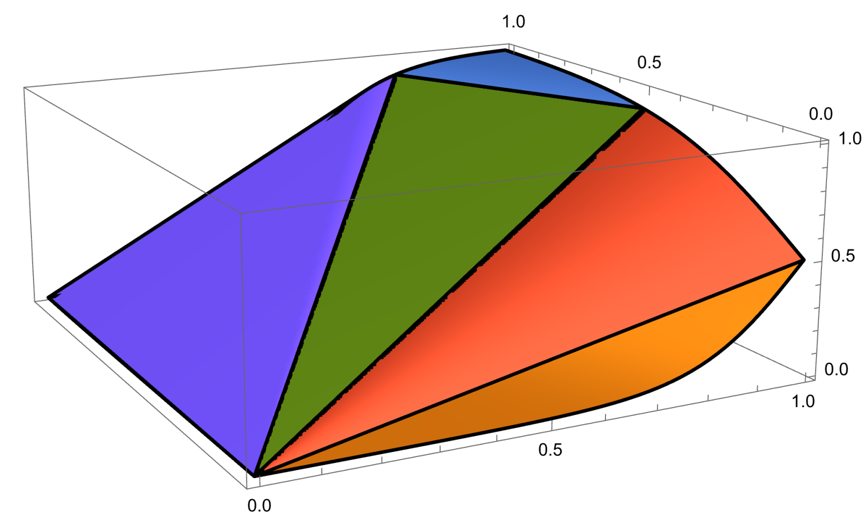

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The one-dimensional envelope is always a secant followed by the function

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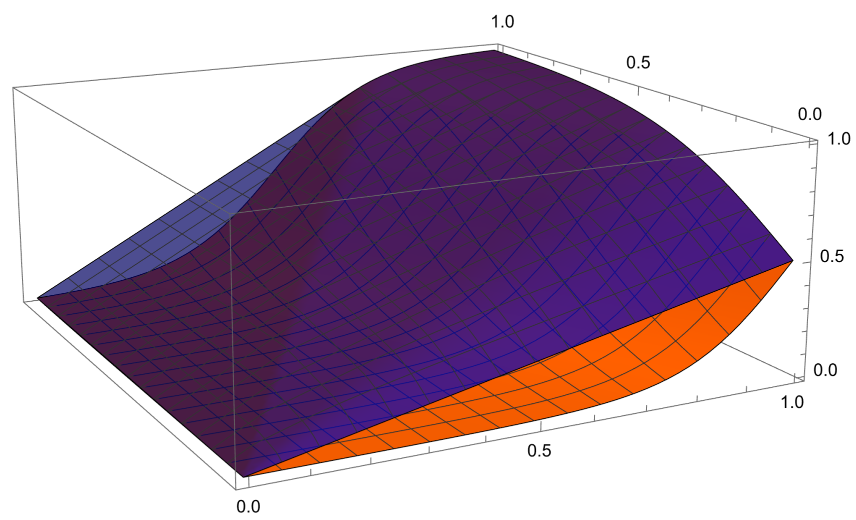

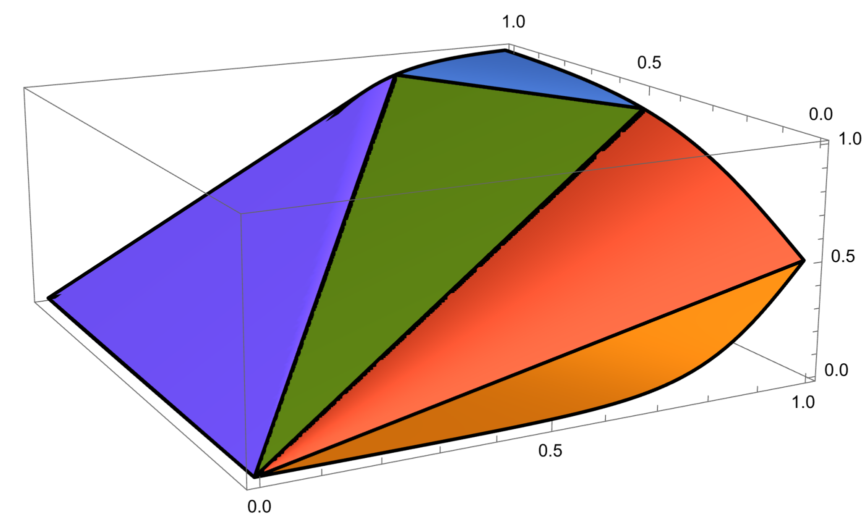

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On the faces, the global structure repeats itself

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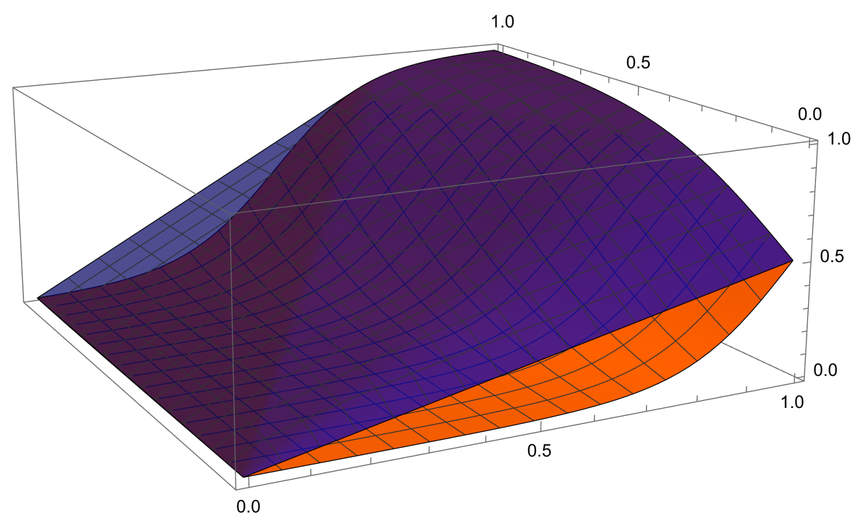

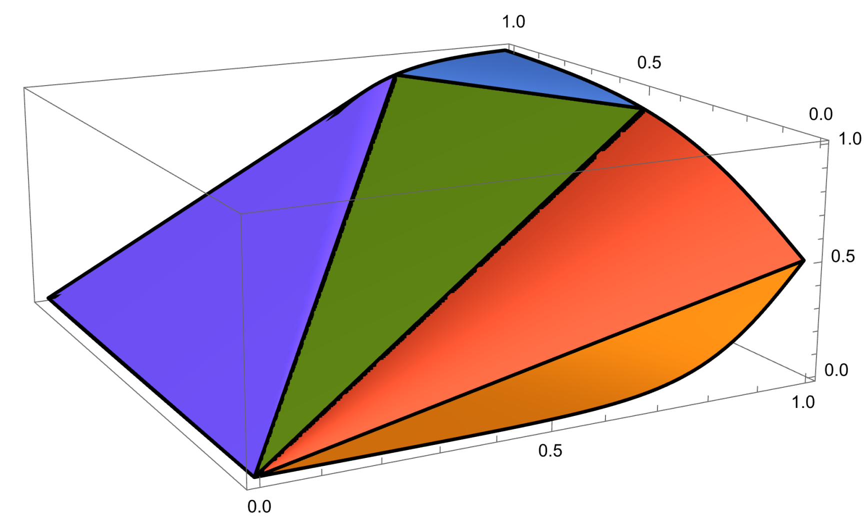

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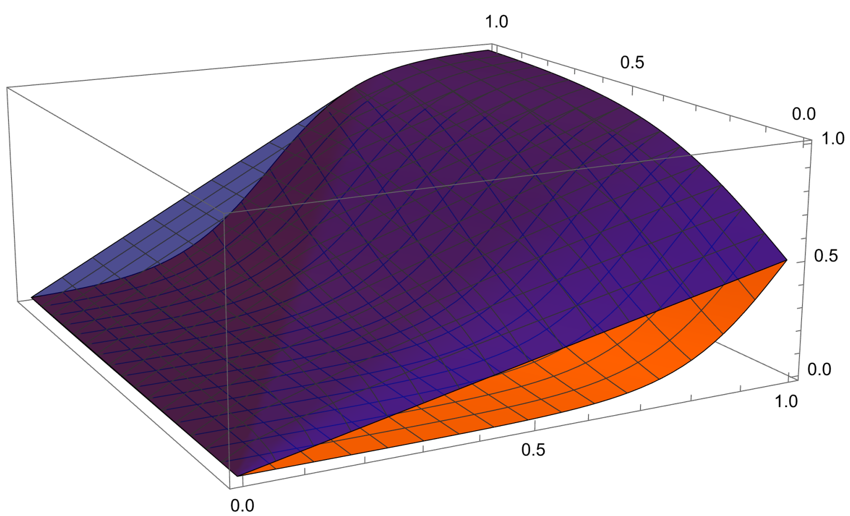
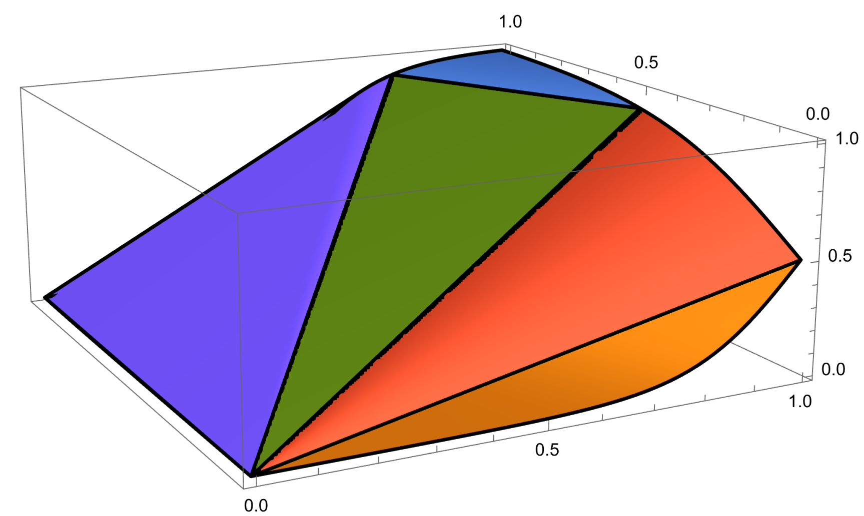
The function $\sigma(w^\top x + b)$ is constant in $w^\perp$

The key ingredients





An envelope construction like this does not work in general, here we heavily rely on

The one-dimensional envelope is always a secant followed by the function

On the faces, the global structure repeats itself

The function $\sigma(w^\top x + b)$ is constant in $w^\perp$

Thank you!