Exercise 1.
Let $\mathbf{f} = (f_x, f_y, f_z)$ be a vector function in three dimensions. Compute the quantity

$$\text{curl}\,(\text{curl}\,\mathbf{f}) \tag{1}$$

in components.

Solution 1.
We recall the formula for the curl in Cartesian coordinates. If $\mathbf{f} = (f_x, f_y, f_z)$, then the curl of $\mathbf{f}$ in components is given by

$$\text{curl}\,(\mathbf{f}) = \hat{\mathbf{i}}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right). \tag{2}$$

For convenience, let's define $\text{curl}\,\mathbf{f} = \mathbf{A}$ as a new vector $\mathbf{A}$. The components of A are then

$$\begin{aligned} A_x &= \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \\ A_y &= \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \\ A_z &= \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}. \end{aligned} \tag{3}$$

Now we wish to compute $\text{curl}\,(\text{curl}\,\mathbf{f})$, which is just $\text{curl}\,\mathbf{A}$. Using (2), the $\hat{\mathbf{i}}$ component of this vector is

$$(\text{curl}\,\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}. \tag{4}$$

To compute each of these derivatives, we must use the components of $\mathbf{A}$ given in (3). The first of these

$$\begin{aligned} \frac{\partial A_z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \\ &= \frac{\partial^2 f_y}{\partial x \partial y} - \frac{\partial^2 f_x}{\partial y^2}. \end{aligned} \tag{5}$$