Exercise 1.
Let $\mathbf{f} = (f_x, f_y, f_z)$ be a vector function in three dimensions. Compute the quantity

$$\text{curl}\,(\text{curl}\,\mathbf{f}) \tag{1}$$

in components.

Solution 1.
We recall the formula for the curl in Cartesian coordinates. If $\mathbf{f} = (f_x, f_y, f_z)$, then the curl of $\mathbf{f}$ in components is given by

$$\text{curl}\,(\mathbf{f}) = \hat{\mathbf{i}}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right). \tag{2}$$

For convenience, let's define $\text{curl}\,\mathbf{f} = \mathbf{A}$ as a new vector $\mathbf{A}$. The components of A are then

$$\begin{aligned} A_x &= \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \\ A_y &= \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \\ A_z &= \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}. \end{aligned} \tag{3}$$

Now we wish to compute $\text{curl}\,(\text{curl}\,\mathbf{f})$, which is just $\text{curl}\,\mathbf{A}$. Using (2), the $\hat{\mathbf{i}}$ component of this vector is

$$(\text{curl}\,\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}. \tag{4}$$

To compute each of these derivatives, we must use the components of $\mathbf{A}$ given in (3). The first of these

$$\begin{aligned} \frac{\partial A_z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \\ &= \frac{\partial^2 f_y}{\partial x \partial y} - \frac{\partial^2 f_x}{\partial y^2}. \end{aligned} \tag{5}$$

Next, we compute the second term of the $\hat{\mathbf{i}}$ component of $\text{curl}\,(\text{curl}\,\mathbf{f})$ by using the x component in (3):

$$\begin{aligned} \frac{\partial A_y}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) \\ &= \frac{\partial^2 f_x}{\partial z^2} - \frac{\partial^2 f_z}{\partial z \partial x} \end{aligned} \tag{6}$$

To finish finding $\text{curl}\,(\text{curl}\,\mathbf{f})$, we must find the terms of the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components. The full $\hat{\mathbf{j}}$ component is

$$(\text{curl}\,\mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}. \tag{7}$$

The first term of this component is

$$\begin{aligned} \frac{\partial A_x}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) \\ &= \frac{\partial^2 f_z}{\partial z \partial y} - \frac{\partial^2 f_y}{\partial z^2}. \end{aligned} \tag{8}$$

The second term is

$$\begin{aligned} \frac{\partial A_z}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \\ &= \frac{\partial^2 f_y}{\partial x^2} - \frac{\partial^2 f_x}{\partial x \partial y}. \end{aligned} \tag{9}$$

The final component, the $\hat{\mathbf{k}}$ component, is

$$(\operatorname{curl}\mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{10}$$

The first term is

$$\begin{aligned}\frac{\partial A_y}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\\ &= \frac{\partial^2 f_x}{\partial x \partial z} - \frac{\partial^2 f_z}{\partial x^2}.\end{aligned} \tag{11}$$

The final term is

$$\begin{aligned}\frac{\partial A_x}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\\ &= \frac{\partial^2 f_z}{\partial y^2} - \frac{\partial^2 f_y}{\partial y \partial z}.\end{aligned} \tag{12}$$

Assembling the various components computed above, we conclude that

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{f})_x &= \frac{\partial^2 f_y}{\partial x \partial y} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2} + \frac{\partial^2 f_z}{\partial z \partial x},\\ \operatorname{curl}(\operatorname{curl}\mathbf{f})_y &= \frac{\partial^2 f_z}{\partial z \partial y} - \frac{\partial^2 f_y}{\partial z^2} - \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_x}{\partial x \partial y},\\ \operatorname{curl}(\operatorname{curl}\mathbf{f})_z &= \frac{\partial^2 f_x}{\partial x \partial z} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_y}{\partial y \partial z}.\end{aligned} \tag{13}$$

Exercise 2.
Show that the components of the vector which you computed in exercise 1 can be written as

$$\begin{aligned} [\operatorname{curl}(\operatorname{curl}\mathbf{f})]_x &= \frac{\partial}{\partial x}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}\right), \\ [\operatorname{curl}(\operatorname{curl}\mathbf{f})]_y &= \frac{\partial}{\partial y}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}\right), \\ [\operatorname{curl}(\operatorname{curl}\mathbf{f})]_z &= \frac{\partial}{\partial z}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right). \end{aligned} \tag{14}$$

This identity can be re-written as

$$\operatorname{curl}(\operatorname{curl}\mathbf{f}) = \operatorname{grad}(\operatorname{div}\mathbf{f}) - \operatorname{lap}\mathbf{f}, \tag{15}$$

where the gradient and divergence have their usual meanings, and the Laplacian of a vector function is another vector function whose components are given by

$$\begin{aligned} [\operatorname{lap}\mathbf{f}]_x &= \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}, \\ [\operatorname{lap}\mathbf{f}]_y &= \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}, \\ [\operatorname{lap}\mathbf{f}]_z &= \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}. \end{aligned} \tag{16}$$

Solution 2.
First recall the definition of the divergence:

$$\operatorname{div}(\mathbf{f}) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}. \tag{17}$$

Using this, we can now compute the right side of each of the three equations in (14). First,

$$\begin{aligned} \frac{\partial}{\partial x}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) - \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}\right) \\ &= \frac{\partial^2 f_y}{\partial x \partial y} + \frac{\partial^2 f_z}{\partial x \partial z} - \frac{\partial^2 f_x}{\partial y^2} - \frac{\partial^2 f_x}{\partial z^2}. \end{aligned} \tag{18}$$

This matches the x component of $\operatorname{curl}(\operatorname{curl}\mathbf{f})$ computed above!

We repeat for the y component:

$$\begin{aligned} \frac{\partial}{\partial y}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) - \left(\frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}\right) \\ &= \frac{\partial^2 f_x}{\partial x \partial y} + \frac{\partial^2 f_z}{\partial y \partial z} - \frac{\partial^2 f_y}{\partial x^2} - \frac{\partial^2 f_y}{\partial z^2}. \end{aligned} \tag{19}$$

Likewise, this matches the y component of $\operatorname{curl}(\operatorname{curl}\mathbf{f})$. Finally, we compute the z component of (14):

$$\begin{aligned} \frac{\partial}{\partial z}(\operatorname{div}\mathbf{f}) - \left(\frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right) &= \frac{\partial}{\partial z}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) - \left(\frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right) \\ &= \frac{\partial^2 f_x}{\partial x \partial z} + \frac{\partial^2 f_y}{\partial y \partial z} - \frac{\partial^2 f_z}{\partial x^2} - \frac{\partial^2 f_z}{\partial y^2}. \end{aligned} \tag{20}$$

As expected, this matches the z component of $\operatorname{curl}(\operatorname{curl}\mathbf{f})$. Finally, we conclude that

$$\operatorname{curl}(\operatorname{curl}\mathbf{f}) = \operatorname{grad}(\operatorname{div}\mathbf{f}) - \operatorname{lap}\mathbf{f}. \tag{21}$$